Year: 2021

# 30 Years of space–time covariance functions

Porcu, Emilio ; Furrer, Reinhard ; Nychka, Douglas

DOI: https://doi.org/10.1002/wics.1512



Originally published at:
Porcu, Emilio; Furrer, Reinhard; Nychka, Douglas (2021). 30 Years of space–time covariance functions. Wiley Interdisciplinary Reviews: Computational Statistics, 13(2):e1512.
DOI: https://doi.org/10.1002/wics.1512

Received: 9 December 2019 | Revised: 2 April 2020 | Accepted: 2 April 2020
DOI: 10.1002/wics.1512

ADVANCED REVIEW

WIREs COMPUTATIONAL STATISTICS WILEY

# 30 Years of space–time covariance functions

**Emilio Porcu**[1] | **Reinhard Furrer**[2] | **Douglas Nychka**[3]

[1]School of Computer Science and Statistics, Trinity College Dublin, Dublin, Ireland
[2]Department of Mathematics and Department of Computational Science, University of Zurich, Zürich, Switzerland
[3]Department of Mathematics, Colorado School of Mines, Golden, Colorado

**Correspondence**
Emilio Porcu, School of Computer Science and Statistics Trinity College Dublin, Dublin, Ireland.
Email: porcue@tcd.ie

**Funding information**
Fondo Nacional de Desarrollo Científico y Tecnológico, Grant/Award Number: 1170290; Schweizerischer Nationalfonds zur Förderung der Wissenschaftlichen Forschung, Grant/Award Number: 175529

**Abstract**
In this article, we provide a comprehensive review of space–time covariance functions. As for the spatial domain, we focus on either the $d$-dimensional Euclidean space or on the unit $d$-dimensional sphere. We start by providing background information about (spatial) covariance functions and their properties along with different types of covariance functions. While we focus primarily on Gaussian processes, many of the results are independent of the underlying distribution, as the covariance only depends on second-moment relationships. We discuss properties of space–time covariance functions along with the relevant results associated with spectral representations. Special attention is given to the *Gneiting* class of covariance functions, which has been especially popular in space–time geostatistical modeling. We then discuss some techniques that are useful for constructing new classes of space–time covariance functions. Separate treatment is reserved for spectral models, as well as to what are termed models with *special features*. We also discuss the problem of estimation of parametric classes of space–time covariance functions. An outlook concludes the paper.

This article is categorized under:
Statistical and Graphical Methods of Data Analysis > Analysis of High Dimensional Data
Statistical Learning and Exploratory Methods of the Data Sciences > Modeling Methods
Statistical and Graphical Methods of Data Analysis > Multivariate Analysis

KEYWORDS
dynamical models, Gneiting functions, great-circle distance, scale mixture, spectral representation

## 1 | INTRODUCTION

Covariance functions describe the second-order dependence of random processes. The popularity of covariance functions in spatial and space–time statistics, as well as in probability theory and machine learning is due to the fact that the properties of Gaussian random fields are completely determined by their first- and second-order moment. Thus, covariance functions are crucial to modeling, estimation, and kriging prediction, of Gaussian random fields.

Space–time covariance functions as models for dependence have been central to many branches of applied and theoretical sciences. Applications include climate modeling (Alexeeff, Nychka, Sain, & Tebaldi, 2016; Berliner, Levine, & Shea, 2000; Crippa et al., 2016; Edwards, Castruccio, & Hammerling, 2019; Genton & Kleiber, 2015; Gething et al., 2010; Guinness & Fuentes, 2016; Heaton et al., 2019; Kühl, Gebhardt, Litt, & Hense, 2002; Sang, Jun, & Huang, 2011), environmental statistics (Bevilacqua, Fassò, Gaetan, Porcu, & Velandia, 2016; Calus, Bijma, & Veerkamp, 2004; Cameletti, Lindgren, Simpson, & Rue, 2012; De Cesare, Myers, & Posa, 2001b; De Iaco, Myers, & Posa, 2002a; Fassò, Finazzi, & Ndongo, 2016; Finazzi & Fassò, 2014; Haslett & Raftery, 1989; Legarra, Misztal, & Bertrand, 2004; Meyer, 1998), image analysis (Benali et al., 1997; De Iaco et al., 2002a; Hengl, Heuvelink, Tadić, & Pebesma, 2012; Jain & Jain, 1981; Meiring, Monestiez, Sampson, & Guttorp, 1997), probability forecast (Giebel, Brownsword, Kariniotakis, Denhard, & Draxl, 2011; Gneiting & Katzfuss, 2014; Gneiting, Larson, Westrick, Genton, & Aldrich, 2006; Zhang, Wang, & Wang, 2014), meteorology (Bourotte, Allard, & Porcu, 2016; Gneiting, 2002b; Handcock & Wallis, 1994; Jun & Stein, 2007; Li, Genton, & Sherman, 2007; Reich, Eidsvik, Guindani, Nail, & Schmidt, 2011) oceanography (Bertino, Evensen, & Wackernagel, 2003; Farmer & Clifford, 1986; Halliwell Jr & Mooers, 1979; White & Bernstein, 1979), extremes (Davis, Klüppelberg, & Steinkohl, 2013a; Davis & Mikosch, 2008; Huerta & Sansó, 2007; Huser & Davison, 2014; Kabluchko, 2009), machine learning (Garg, Singh, & Ramos, 2012; Genton, 2001; Sarkka, Solin, & Hartikainen, 2013), demography (De Iaco, Palma, & Posa, 2015), forestry (Buttafuoco & Castrignanò, 2005; Jost, Heuvelink, & Papritz, 2005), atmospheric sciences (Bardossy & Plate, 1992; Brown, Diggle, Lord, & Young, 2001) turbulence (Kraichnan, 1964; Shkarofsky, 1968), and finance (Fernández-Avilés & Montero, 2016; Porcu, Montero, & Schlather, 2012) to mention just a few.

The first formulations of space–time covariance functions trace back to the early 1990s, albeit exploiting simple mathematical structures. For instance, an easy way to build a space–time covariance function is through the product of a spatial and a temporal covariance function. Such covariance functions are called *separable*: They are easy to construct and allow for considerable computational gains (details are deferred to subsequent sections). However, they are very limited in describing the interaction between space and time, in many cases implying unphysical dependence among process variables. Thus, the 1990s saw an increasing number of efforts to construct *non-separable* covariance functions: The first approaches to building nonseparable space–time covariance functions can be found in Christakos (1990, 1992) and Dimitrakopoulos and Luo (1994). More recently, there has been a wealth of contributions based on direct construction in space and time domains (Cressie & Huang, 1999; De Iaco et al., 2002a; Gneiting, 2002b; Porcu, Gregori, & Mateu, 2006), through spectral densities in frequency space (Fuentes, Chen, & Davis, 2008; Stein, 2005b, 2005c), or on the basis of physical principles and dynamic modeling approaches (Baxevani, Podgórski, & Rychlik, 2011; Brown, Karesen, Roberts, & Tonellato, 2000; Christakos, 1990, 1992).

Modeling space–time covariance functions have had a recent resurgence thanks to the increasing interest in modeling global data. Here the spatial domain is taken to be a sphere, and so covariance functions must respect this topology. As for space–time stochastic processes on the sphere, we refer the reader to the recent approaches in Porcu, Bevilacqua, and Genton (2016); Berg and Porcu (2017) and Jeong and Jun (2015). Generalizations to multivariate space–time processes have been considered in Alegría, Porcu, Furrer, and Mateu (2019). The richness in modeling stochastic processes over spheres or spheres cross time is reflected in the diversity of research in this area: Mathematical analysis (Barbosa & Menegatto, 2017; Beatson, Zu Castell, & Xu, 2014; Chen, Menegatto, & Sun, 2003; Gangolli, 1967; Guella, Menegatto, & Peron, 2016a, 2016b, 2017; Hannan, 1970; Menegatto, 1994, 1995; Menegatto, Oliveira, & Peron, 2006; Schoenberg, 1942), probability theory (Baldi & Marinucci, 2006; Clarke, Alegria, & Porcu, 2018; Hansen, Thorarinsdottir, Ovcharov, & Gneiting, 2015; Lang & Schwab, 2013), spatial point processes (Møller, Nielsen, Porcu, & Rubak, 2018), spatial geostatistics (Christakos & Papanicolaou, 2000; Gerber, Mösinger, & Furrer, 2017; Gneiting, 2002b; Hitczenko & Stein, 2012; Huang, Zhang, & Robeson, 2012), space–time geostatistics (Berg & Porcu, 2017; Christakos, 1991a, 2000; Christakos, Hristopulos, & Bogaert, 2000; Porcu et al., 2016) and mathematical physics (Istas, 2005; Leonenko & Sakhno, 2012; Malyarenko, 2013).

This paper provides a review of space–time covariance functions. The plan of the paper is as follows: Section 2 provides the necessary background on different metrics used on different spaces. We provide some classes of functions defined on the positive real line that become building blocks for more complex covariances. Section 3 is devoted to properties of space–time covariance functions through their spectral representations. Section 4 provides several strategies for constructing space–time covariance functions. Sections 5 and 6 discuss covariance functions having special properties or motivated by certain physical principles. Section 8 discusses the estimation problem for space–time covariance functions. The article concludes with a perspective on future developments.

# 2 | BACKGROUND

## 2.1 | Spaces, distances, and covariance functions

We consider random fields $Z=\{Z(\mathbf{s},t),\ \mathbf{s}\in\mathcal{D},\ t\in\mathbb{T}\}$, where $\mathcal{D}$ is the spatial domain, and $\mathbb{T}$ is time. In this article, we shall work (either) with the case $\mathcal{D}=\mathbb{R}^d$ (the $d$-dimensional Euclidean space) or $\mathcal{D}=\mathbb{S}^d$, the unit $d$-dimensional sphere. As for the domain $\mathbb{T}$, time will be considered for most of this article in a continuous fashion ($\mathbb{T}=\mathbb{R}$), unless explicitly stated otherwise.

Henceforth, we assume that $Z$ has finite second-order moment and focus on the covariance function $C:(\mathcal{D}\times\mathbb{T})^2\to\mathbb{R}$, defined as

$$C((\mathbf{s}_1,t_1),(\mathbf{s}_2,t_2))=\mathbb{C}\mathrm{ov}(Z(\mathbf{s}_1,t_1),Z(\mathbf{s}_2,t_2)),\quad (\mathbf{s}_i,t_i)\in\mathcal{D}\times\mathbb{T}, i=1,2.$$

Covariance functions are a linear measure of dependence between the random variables $Z$ at the space–time locations $(\mathbf{s}_1, t_1)$ and $(\mathbf{s}_2, t_2)$. As such, they must be positive-definite functions. Conversely, it is true that any positive definite function is the covariance function associated with a Gaussian process.

Under the assumption of a Gaussian process $Z$, the mean and covariance functions completely characterize its distribution. Moreover, any finite-dimensional sampling will be distributed multivariate Gaussian, with the mean vector and the covariance matrix determined by the mean function $\mu$ and the covariance function $C$.

The choice of the metric will play a crucial role in our exposition. In particular, for the case $\mathcal{D}=\mathbb{R}^d$, we consider the classical Euclidean distance, denoted as $\|\cdot\|$ throughout. When $\mathcal{D}=\mathbb{S}^d$, the natural distance on the sphere is the geodesic or great circle distance, defined as the mapping $\theta:\mathbb{S}^d\times\mathbb{S}^d\to[0,\pi]$ so that

$$\theta(\mathbf{s}_1,\mathbf{s}_2)=\arccos\left(\mathbf{s}_1^\top\mathbf{s}_2\right),$$

with $\top$ denoting transpose. Thus, the geodesic distance describes an arc between any pair of points located on the spherical shell. Throughout, we shall equivalently use $\theta(\mathbf{s}_1, \mathbf{s}_2)$ or its shortcut $\theta$ to denote the geodesic distance, whenever no confusion can arise.

When $\mathcal{D}=\mathbb{R}^d$, $Z$ is called weakly stationary if

$$C((\mathbf{s}_1,t_1),(\mathbf{s}_2,t_2))=K(\mathbf{s}_2-\mathbf{s}_1,t_2-t_1),\quad (\mathbf{s}_i,t_i)\in\mathbb{R}^d\times\mathbb{T}, i=1,2, \tag{1}$$

for some mapping $K:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$. Additionally, $Z$ is weakly stationary and isotropic if there exists a continuous mapping $\varphi:[0,\infty)^2\to\mathbb{R}$ such that $\varphi(0, 0) < \infty$ and

$$K(\mathbf{s}_2-\mathbf{s}_1,t_2-t_1)=\varphi(\|\mathbf{s}_2-\mathbf{s}_1\|,|t_2-t_1|),\quad (\mathbf{s}_i,t_i)\in\mathbb{R}^d\times\mathbb{T}, i=1,2. \tag{2}$$

When $\mathcal{D}=\mathbb{S}^d$, the definition (1) is not meaningful because translations do not make sense on spheres. Thus, for a second-order process $Z$ defined over $\mathbb{S}^d\times\mathbb{T}$, we define weak stationarity and geodesic isotropy when there exists a continuous mapping $\psi:[0,\pi]\times\mathbb{R}\to\mathbb{R}$ such that $\psi(0, 0)<\infty$ and

$$C((\mathbf{s}_1,t_1),(\mathbf{s}_2,t_2))=\psi(\theta(\mathbf{s}_1,\mathbf{s}_2),|t_2-t_1|),\quad (\mathbf{s}_i,t_i)\in\mathbb{R}^d\times\mathbb{T}, i=1,2. \tag{3}$$

When $d = 2$ and $\mathcal{D}=\mathbb{S}^2$, very often (in particular, in the analysis of climate data) the hypothesis of geodesic isotropy is replaced by that of axial symmetry: There exists a continuous mapping $\mathcal{C}:[0,\pi]^2\times[-2\pi,2\pi)\times\mathbb{T}$ such that

$$C((\mathbf{s}_1,t_1),(\mathbf{s}_2,t_2))=\mathcal{C}(\phi_1,\phi_2,\vartheta_1-\vartheta_2,t_1-t_2),\quad (\phi_i,\vartheta_i,t_i)\in[0,\pi]\times[0,2\pi)\times\mathbb{T},\ i=1,2,$$

where $\phi_i$ denote latitudes and $\vartheta_i$, longitudes, and where subtraction of longitudes is intended as modulo $2\pi$.

## 2.2 | Positive-definite functions and related building blocks

We introduce some classes of continuous functions, defined on the positive real line, that will be useful for the construction of parametric classes of space–time covariance functions. A function $f : [0, \infty) \to \mathbb{R}_+$ is called completely monotonic if it is continuous, infinitely differentiable on $(0, \infty)$, satisfying $(-1)^n f^{(n)}(t) \ge 0,\ n \in \mathbb{N}$. Here, $f^{(n)}$ denotes $n$th derivative and we use $f^{(0)}$ for $f$, where $f(0)$ is required to be finite.

A function $f : [0, \infty) \to \mathbb{R}$ is called a Stieltjes function if

$$f(t) = \int_{[0,\infty)} \frac{\mu(\mathrm{d}\xi)}{t+\xi}, \quad t \ge 0, \tag{4}$$

where $\mu$ is a positive and bounded measure. We require throughout $f(0) = 1$, which implies that $\int \xi^{-1}\mu(\mathrm{d}\xi) = 1$. Let us call $\mathcal{S}$ the set of Stieltjes functions. It has been proved that $\mathcal{S}$ is a convex cone (Berg, 2008), with the inclusion relation $\mathcal{S} \subset \mathcal{C}$, where $\mathcal{C}$ is the set of completely monotone functions. The relation (4) shows that the function $f(t) = 1/(1+t)$, $t \ge 0$, is a Stieltjes function. Using the fact that $f \in \mathcal{S}$ if and only if $1/f$ is a completely Bernstein function (for a definition, see Porcu & Schilling, 2011), we can get a wealth of examples of Stieltjes functions, as the book by Schilling, Song, and Vondracek (2012) provides an entire catalogue of completely Bernstein functions. We finally note that completely Bernstein functions are infinitely differentiable over $(0, \infty)$ and have a completely monotonic derivative. Similarly, Bernstein functions have a completely monotone derivative, but a different integral representation (Berg, 2008).

The Matérn class of continuous functions $f(t) = \mathcal{M}_\nu(t),\ t \ge 0,\ \nu > 0$, is defined as

$$\mathcal{M}_\nu(t) = \frac{2^{1-\nu}}{\Gamma(\nu)} t^\nu \mathcal{K}_\nu(t), \tag{5}$$

where $\mathcal{K}_\nu$ is the MacDonald function (Gradshteyn & Ryzhik, 2007). $\mathcal{M}_\nu(\sqrt{\cdot})$ is completely monotonic on the positive real line for all $\nu > 0$ (Miller & Samko, 2001). The appeal of this class is the parameter $\nu$ that governs the smoothness of the covariance function at the origin (Stein, 1999), and thus the smoothness of a Gaussian field on $\mathbb{R}^d$ in the mean square sense. The Matérn family also has a simple form for its spectral densities. Some special cases, for specific values of half-integer $\nu$, are reported in Table 1.

We finish this section by introducing a class of continuous and positive-definite functions that vanish outside the interval $[0, 1]$ (or a suitably rescaled interval).

We introduce the generalized Wendland class (Gneiting, 2002a; Zastavnyi, 2002) $\mathcal{W}_{\mu,\kappa} : [0,\infty) \to \mathbb{R},\ \kappa > 0,\ \mu > 0$, defined as

$$\mathcal{W}_{\mu,\kappa}(t) = \frac{1}{B(2\kappa+1,\mu)} \int_t^\infty \left(u^2 - t^2\right)^\kappa \mathcal{W}_{\mu-1,0}(u)\, \mathrm{d}u, \quad t \ge 0, \tag{6}$$

where $B$ denotes the beta function, and where $\mathcal{W}_{\mu,0}$ denotes the Askey family of functions (Askey, 1973), defined by

$$\mathcal{W}_{\mu,0}(t) = (1-t)_+^\mu, \quad \mu > 0, \tag{7}$$

**TABLE 1** Special cases of Matérn functions $\mathcal{M}_\nu$ and Wendland functions $\mathcal{W}_{\mu,k}$. SP($k$) means that the sample paths of the associated Gaussian field are $k$ times differentiable

| $\nu$ | $\mathcal{M}_\nu(t)$ | $k$ | $\mathcal{W}_{\mu,k}(t)$ | SP($k$) |
|---|---|---|---|---|
| 0.5 | $e^{-t}$ | 0 | $(1-t)_+^\mu$ | 0 |
| 1.5 | $e^{-t}(1+t)$ | 1 | $(1-t)_+^{\mu+1}(1+t(\mu+1))$ | 1 |
| 2.5 | $e^{-t}\left(1+t+\frac{t^2}{3}\right)$ | 2 | $(1-t)_+^{\mu+2}\left(1+t(\mu+2)+\frac{t^2}{3}(\mu^2+4\mu+3)\right)$ | 2 |
| 3.5 | $e^{-t}\left(1+t+\frac{2t^2}{5}+\frac{t^3}{15}\right)$ | 3 | $(1-t)_+^{\mu+3}\left(1+t(\mu+3)+\frac{t^2}{5}(2\mu^2+12\mu+15)+\frac{t^3}{15}(\mu^3+9\mu^2+23\mu+15)\right)$ | 3 |

with $(\cdot)_+$ denoting the positive part. Closed-form solutions of the integral in (6) can be obtained when $\kappa = k$, a nonnegative integer. In this case,

$$\mathcal{W}_{\mu,k}(t) = \mathcal{W}_{\mu+k,0}(t) P_k(t), \quad t \geq 0,$$

where $P_k$ is a polynomial of order $k$, see Table 1 for examples with $k = 0, 1, 2, 3$. These functions, termed (original) Wendland functions, were originally proposed by Wendland (1995).

The latter two classes are, strictly speaking, correlation functions and can be scaled according to a variance parameter $\sigma^2 > 0$ and range parameter $\varrho > 0$ to obtain the covariance functions $\sigma^2 \mathcal{M}_\nu(t/\varrho)$ and $\sigma^2 \mathcal{W}_{\mu,k}(t/\varrho)$, respectively.

## 3 | PROPERTIES OF COVARIANCE FUNCTIONS

### 3.1 | Descriptive properties

Covariance functions are positive-semidefinite. That is, for any finite collection $\{(\mathbf{s}_k, t_k)\}_{k=1}^N \subset \mathcal{D} \times \mathbb{T}$, and $\{c_k\}_{k=1}^N \subset \mathbb{R}$,

$$\sum_{k=1}^N \sum_{h=1}^N c_k c_h C((\mathbf{s}_k, t_k), (\mathbf{s}_h, t_h)) \geq 0. \tag{8}$$

Positive-definite functions are a convex cone that is closed under the topology of pointwise convergence (Berg, 2008). In particular, the product, or the weighted sum (with nonnegative weights), of two covariance functions defined over their respective spaces provides a new covariance function defined over a product space. Thus, *separable* covariance functions can be built through

$$C((\mathbf{s}_1, t_1), (\mathbf{s}_2, t_2)) = C_{\mathcal{D}}(\mathbf{s}_1, \mathbf{s}_2) C_{\mathbb{T}}(t_1, t_2), \quad (\mathbf{s}_i, t_i) \in \mathcal{D} \times \mathbb{T}, \tag{9}$$

or through

$$C((\mathbf{s}_1, t_1), (\mathbf{s}_2, t_2)) = C_{\mathcal{D}}(\mathbf{s}_1, \mathbf{s}_2) + C_{\mathbb{T}}(t_1, t_2), \quad (\mathbf{s}_i, t_i) \in \mathcal{D} \times \mathbb{T}, \tag{10}$$

where $C_{\mathcal{D}}$ and $C_T$ are spatial and temporal covariance functions, respectively. In all the other cases where (9) does not happen, $C$ is called nonseparable. Notably, $C$ in (10) is not strictly positive definite even if both $C_{\mathcal{D}}$ and $C_T$ are.

In settings with temporally collocated data (every spatial location is observed at each time), separable models allow for ease of computation and dimensionality reduction, as the space–time covariance matrix is obtained through the Kronecker product of the marginal spatial and temporal ones. However, separability is an unrealistic assumption for many applications since it implies limited interactions between the spatial and temporal variations. Also, the computational benefits are lost when there is no complete collocation of the observations. Accordingly, various techniques have been introduced for generating different classes of nonseparable spatiotemporal covariance models.

Nonseparability can account for complex interaction between space and time. To fix concepts, Rodrigues and Diggle (2010) define *positive* (*negative*) nonseparability when, respectively, $C((\mathbf{s}_1, t_1), (\mathbf{s}_2, t_2)) \geq C_{\mathcal{D}}(\mathbf{s}_1, \mathbf{s}_2) C_{\mathbb{T}}(t_1, t_2)$ , or $C((\mathbf{s}_1, t_1), (\mathbf{s}_2, t_2)) \leq C_{\mathcal{D}}(\mathbf{s}_1, \mathbf{s}_2) C_{\mathbb{T}}(t_1, t_2)$. If such inequalities hold for all $(\mathbf{s}_i, t_i) \in \mathcal{D} \times \mathbb{T}$, then $C$ is called uniformly (positive or negative) nonseparable. Generalizations of these concepts are included in De Iaco and Posa (2013).

$C$ is called fully symmetric if

$$C((\mathbf{s}_1, t_1), (\mathbf{s}_2, t_2)) = C((\mathbf{s}_2, t_1), (\mathbf{s}_1, t_2)) = C((\mathbf{s}_2, t_2), (\mathbf{s}_1, t_1)) = C((\mathbf{s}_1, t_2), (\mathbf{s}_2, t_1)).$$

When $\mathcal{D} = \mathbb{R}^d$ and $C$ is stationary, then $C$ is defined through the function $K$ in Equation (1), and under full symmetry, we have $K(\mathbf{h}, u) = K(-\mathbf{h}, u) = K(-\mathbf{h}, -u) = K(\mathbf{h}, -u)$, $(\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{T}$. Obviously, isotropy in space (whatever the space, $\mathcal{D}$) and symmetry in time imply full symmetry. Separable covariance functions are also fully symmetric, and tests for separability can be used to test for full symmetry (Gneiting, Genton, & Guttorp, 2007). A direct way to build a nonseparable covariance function is by considering a product-sum model (De Cesare, Myers, & Posa, 2001a; De Iaco, Myers, & Posa, 2001, 2011): For three positive weights $a_i$, $i = 1, 2, 3$, such a model is obtained through

$$C((\mathbf{s}_1,t_1),(\mathbf{s}_2,t_2)) = a_1 C_{\mathcal{D}}(\mathbf{s}_1,\mathbf{s}_2) + a_2 C_{\mathbb{T}}(t_1,t_2) + a_3 C_{\mathcal{D}}(\mathbf{s}_1,\mathbf{s}_2) C_{\mathbb{T}}(t_1,t_2),.$$

Several generalizations of this construction have been considered by De Iaco et al. (2001, 2002a), De Iaco, Myers, and Posa (2002b) and Gregori, Porcu, Mateu, and Sasvári (2008), to mention a few. More constructions will be discussed in subsequent sections.

A covariance function is compactly supported, with spatial radius $a$ and temporal radius $b$, if it is identically equal to zero outside the finite space–time range, $(a, b)$. Let $\mathcal{A}_{a,b} := \{(\mathbf{s}_i,t_i) \in \mathcal{D}\times\mathbb{T}, i = 1,2: \ \text{distance}(\mathbf{s}_1,\mathbf{s}_2) \le a \text{ and } |t_1 - t_2| \le b\}$. Then, $C((\mathbf{s}_1, t_1), (\mathbf{s}_2, t_2))$ is identically equal to zero whenever $(\mathbf{s}_i, t_i) \notin \mathcal{A}_{a,b}$. We thus say that $C$ is compactly supported over $\mathcal{A}_{a,b}$.

When $\mathcal{D} = \mathbb{R}^d$, then a scalar compact support can be attained if $C((\mathbf{s}_1,t_1),(\mathbf{s}_2,t_2)) \equiv C_{\mathcal{D}\mathbb{T}}\left(\sqrt{a_1^2\|\mathbf{s}_1-\mathbf{s}_2\|^2 + a_2^2\|t_1-t_2\|^2}\right)$, where $a_1$ and $a_2$ are positive scaling factors. Such a situation is termed geometric anisotropy, and space–time models of this type have been proposed by Dimitrakopoulos and Luo (1994). Finally, let $C$ be a space–time covariance function that is spatially isotropic (either in $\mathbb{R}^d$ or in $\mathbb{S}^d$). We call a temporally dynamical radius, $h$, the continuous mapping from $[0, \infty)$ to $(0, \infty)$ such that for each $u_o \in [0, \infty)$, the margin $C(\cdot, u_o)$ is compactly supported on a ball embedded in $\mathbb{R}^d$ with radius $h(u_o)$. Clearly, both Askey (7) and generalized Wendland (6) classes are special cases of dynamical compact support, when $h \equiv b > 0$ is the constant function. We call functions $C$ with such a property dynamically supported (Porcu, Bevilacqua, & Genton, 2019). Clearly, if $C$ is compactly supported over $\mathcal{A}_{a,b}$, then it is also dynamically supported with dynamical radius being identically equal to $b$.

A fully symmetric covariance $C$ has a dimple if $Z(\mathbf{s}_{\text{here}}, t_{\text{now}})$ is more correlated with $Z(\mathbf{s}_{\text{there}}, t_{\text{then}})$ than with $Z(\mathbf{s}_{\text{there}}, t_{\text{now}})$. For isotropic covariance functions, this implies that, for a fixed $r_o > 0$, the functions $\varphi(r_o, \cdot)$ in (2) and $\psi(r_o, \cdot)$ in (3) are no longer monotonically decreasing (for this last case, we obviously require $r_o \le \pi$), thus resulting in a possibly counterintuitive property. A first description of a dimple is due to Kent, Mohammadzadeh, and Mosammam (2011). More recently, a description of dimples through contour curves has been provided by Cuevas, Porcu, and Bevilacqua (2017).

A recent review in De Iaco, Posa, Cappello, and Maggio (2019) digs into other descriptive properties: Explicit distinction is made between partial, additive and total separability, as well as the concepts of axial, full and quadrant symmetries on the plane. Some tests on separability of space–time covariance functions can be found in Scaccia and Martin (2005, 2002, 2011), Fuentes (2006); Bevilacqua, Mateu, Porcu, Zhang, and Zini (2010), Mitchell, Genton, and Gumpertz (2006), Constantinou, Kokoszka, and Reimherr (2017), Lu and Zimmerman (2005), Aston, Pigoli, and Tavakoli (2017), Li et al. (2007), De Iaco, Posa, & Myers, 2013, De Iaco, Palma, & Posa, 2016), and Cappello, De Iaco, and Posa (2018). Tests for axial symmetry are provided by Scaccia and Martin (2002, 2005). For a further review, the reader is deferred to Kyriakidis and Journel (1999).

### 3.2 | Spectral representations

When $\mathcal{D} = \mathbb{R}^d$, $\mathbb{T} = \mathbb{R}$, and the covariance function is weakly stationary and continuous, Bochner's theorem (Bochner, 1955) establishes a one-to-one correspondence between positive-definite functions and the Fourier transforms of positive and bounded measures:

$$K(\mathbf{h},u) = \int_{\mathbb{R}^d\times\mathbb{R}} \mathrm{e}^{\mathrm{i}\mathbf{h}^\top\omega + \mathrm{i}u\tau} F(\mathrm{d}(\omega,\tau)), \qquad \omega\in\mathbb{R}^d, u\in\mathbb{R},$$

where i is the imaginary unit. If $F$ is absolutely continuous with respect to the Lebesgue measure, then

$$K(\mathbf{h},u) = \int_{\mathbb{R}^d\times\mathbb{R}} \mathrm{e}^{\mathrm{i}\mathbf{h}^\top\omega + \mathrm{i}u\tau} f(\omega,\tau)\mathrm{d}\omega\mathrm{d}\tau, \qquad \omega\in\mathbb{R}^d, u\in\mathbb{R},$$

for $f$ being nonnegative and integrable in $\mathbb{R}^d$. The function $f$ is called the *spectral density* for the covariance function $K$.

A very useful criterion is obtained when $K$ is additionally absolutely integrable in $\mathbb{R}^d$ for any fixed value of $u \in \mathbb{R}$. In this case, $K$ is positive definite if and only if the function $C_\omega : \mathbb{R} \to \mathbb{R}$, defined as

$$C_\omega(u) = \int_{\mathbb{R}^d} K(\mathbf{h}, u) e^{i\mathbf{h}^\top \omega} d\mathbf{h}, \quad u \in \mathbb{R}, \tag{11}$$

is positive definite almost for every $\omega \in \mathbb{R}^d$ (see Gneiting, 2002b, theorem 1). This generalizes the criterion provided by Cressie and Huang (1999).

If, additionally, the covariance function is isotropic, then Porcu et al. (2006) have shown that the function $\varphi$ in (2) has the representation

$$\varphi(r,t) = \int_{[0,\infty)^2} \Omega_d(r\xi)\cos(t\zeta)\mu(d(\xi,\zeta)), \quad r,t \geq 0,$$

where $\mu$ is a positive and bounded measure, and $\Omega_d$ involves modified Bessel functions (see Daley & Porcu, 2013, for a detailed account).

The case $\mathcal{D} = \mathbb{S}^d$ and $\mathbb{T} = \mathbb{R}$ has only been elucidated recently. Berg and Porcu (2017) have shown that the continuous function $\psi$ in (3) is positive definite if and only if

$$\psi(\theta,u) = \sum_{k=0}^{\infty} C_{k,\mathcal{T}}(u) P_k^{(d-1)/2}(\cos\theta), \quad (\theta,u) \in [0,\pi] \times \mathbb{R}, \tag{12}$$

where $\{C_{k,\mathcal{T}}(\cdot)\}_{k=0}^{\infty}$ is a sequence of temporal covariance functions with the additional requirement that $\sum_{k=0}^{\infty} C_{k,\mathcal{T}}(0) < \infty$ in order to guarantee the variance of $Z$, $\sigma^2 = \psi(0, 0)$, to be finite. Here, $P_k^\lambda$ denotes the Gegenbauer polynomial with exponent $\lambda > -1/2$ and order $k = 0, 1, \ldots$ (Dai & Xu, 2013).

Finding the analogues of Gneiting's criterion (11) for this general case requires an additional technicality that is outlined below. White and Porcu (2019b) have considered the Gegenbauer transform, defined, for a fixed $d \in \mathbb{N}$, as

$$C_{k,\mathbb{T}}(u) = \int_0^\pi \psi(\theta,u) P_k^{(d-1)/2}(\cos\theta) \sin\theta^{d-1} d\theta, \quad u \in \mathbb{R}, k = 0,1,\ldots \tag{13}$$

with the additional condition that $C_{k,\mathbb{T}} : \mathbb{R} \to \mathbb{R}$, satisfies $\sum_{k=0}^{\infty} \int_{\mathbb{R}} |b_{k,d}(u)| du < \infty$. Then, the following assertions are equivalent:

1 $\psi(\theta, u)$ is the covariance function of a random field on $\mathbb{S}^d \times \mathbb{R}$;
2 The function $C_\tau : [0, \pi] \to \mathbb{R}$, defined as

$$C_\tau(\theta) = \int_{-\infty}^{+\infty} e^{-iu\tau} \psi(\theta,u) du$$

is the covariance function of a random field on $\mathbb{S}^d$ for almost every $\tau \in \mathbb{R}$;
3 For all $k = 0, 1, 2, \ldots$, the functions $C_{k,\mathbb{T}} : \mathbb{R} \to \mathbb{R}$, defined through (13) are continuous, positive definite on $\mathbb{R}$, and $\sum_k b_{k,\,d}(0) < \infty$.

Spectral representations for the case of axial symmetry are available as well, and can be made explicit by using the arguments in Berg and Porcu (2017) in concert with the spectral expansions in Jones (1963). For a thorough account of axial symmetry, see Porcu, Castruccio, Alegria, and Crippa (2019).

## 4 | CLASSES OF COVARIANCE FUNCTIONS

### 4.1 | The persistent value of the Gneiting functions

The Gneiting class has been proposed in a wealth of applications involving space–time geostatistics (Diggle, 2013; Gelfand, Schmidt, Banerjee, & Sirmans, 2004), extreme events (Huser & Davison, 2014), applications to radar–rain gauge merging (Sideris, Gabella, Erdin, & Germann, 2014), solar irradiance forecasting (Yang et al., 2013), meteorology (Spadavecchia & Williams, 2009), particulate matter (Cameletti, Ignaccolo, & Bande, 2011), bubonic plague epidemics

(Christakos, Olea, & Yu, 2007), ground-level ozone (Gilleland & Nychka, 2005), cellular traffic at city scales (Chen, Jin, Qiang, Hu, & Jiang, 2015), and pricing in financial markets (Espen & Jurate, 2012), to cite a few. Beyond these specific applications, the Gneiting functions are also the building blocks for more sophisticated covariance functions: From extreme space–time modeling (Huser & Davison, 2014) to a deeper study of directional properties of space–time random fields (Sherman, 2011), dynamical models (Wikle & Hooten, 2010), dynamic factor analysis (Lopes, Salazar, & Gamerman, 2008), predictive modeling (De Luna & Genton, 2005), projective space–time processes (Wang & Gelfand, 2014), covariate-dependent space–time modeling (Reich et al., 2011), and anisotropic and nonstationary covariance functions (Porcu, Gregori, & Mateu, 2006; Schlather, 2010).

The Gneiting function can be historically traced back to space–time processes that are spatially isotropic where the reference domain for space is $\mathcal{D}=\mathbb{R}^d$. We provide a different exposition for the function here: We consider the family of functions $G_\alpha:[0,\infty)\times[0,\infty)\to\mathbb{R}$, defined through

$$G_\alpha(x,t)=\frac{1}{h(t)^\alpha}f\left(\frac{x}{h(t)}\right),\quad x,t\geq 0, \tag{14}$$

where $\alpha$ is positive, $f$ is continuous and nonnegative, and $h$ is continuous and strictly positive.

## 4.2 | Gneiting class across different metric spaces

We now list relevant findings related to the Gneiting function representation (14).

(A) $[\mathcal{D}=\mathbb{R}^d]$ We have the following cases.

A.1 If $f$ is completely monotonic and $\alpha\geq d/2$, then, $G_\alpha(\|\cdot\|^2,|\cdot|^2)$ is positive definite if and only if $\exp(-ch(|\cdot|^2))$ is positive definite on the real line for all $c>0$. The sufficiency has been proved by Gneiting (2002b). The necessary part of the assertion has been proved by Zastavnyi and Porcu (2011).

A.2 If $f$ is a Stieltjies function and $h$ a Bernstein function, then $G_\alpha(\|\cdot\|^2,|\cdot|^2)$ is positive definite for all $\alpha>0$ and for all $d=1,2,\ldots$. This result was recently proved by Menegatto, Oliveira, and Porcu (2019).

A.3 Assume $f$ is a generalized Wendland function, $\mathcal{W}$ in Equation (6). Let $h$ be continuous and positive function on the positive real line, with $h(0)=1$ and such that $1/h(\cdot)$ is increasing and concave on the positive real line, with $\lim_{t\to\infty}\psi(t)=0$. Then, $G_{-\alpha}(\|\cdot\|,|\cdot|)$ is positive definite provided $\nu\geq(d+5)/2+\kappa$ and $\alpha\leq(d+3)/2+2\kappa$. This result was proved by Porcu, Bevilacqua, and Genton (2019).

(B) $[\mathcal{D}=\mathbb{S}^d]$ For the sphere, we have the following cases.

B.1 If $f$ is completely monotonic and $h$ a Bernstein function, then, $G_\alpha(|\cdot|^2,\theta)$ is positive definite provided $\alpha\geq d/2$ (Porcu, Bevilacqua, & Genton, 2016b).

B.2 If $f$ is completely monotonic, $h$ positive, increasing and concave, then, $G_\alpha(\theta h^2(|\cdot|^2),|\cdot|^2)$ is positive-definite (Porcu et al., 2016b).

B.3 If $f$ is a Stieltjies function and $h$ a Bernstein function, then $G_\alpha(\theta,|\cdot|^2)$ is positive definite for all $\alpha>0$ and for all $d=1,2,\ldots$ (White & Porcu, 2019b).

B.4 $f$ is a generalized Wendland function $\mathcal{W}$, $h$ is positive, decreasing and convex (Porcu, Bevilacqua, & Genton, 2019).

Some comments are in order. All the Gneiting functions listed above are strictly positive over their domain, with the exception of the solutions (A.3) and (B.4), where the combination

$$G_\alpha(x,t)=h(t)^\alpha\mathcal{W}_{\mu,k}\left(\frac{x}{h(t)}\right),\quad x,t\geq 0,$$

clearly shows that in this case $h$ is a dynamical radius for the function $G_\alpha$. If $f=\mathcal{M}_\nu$, with $\mathcal{M}_\nu$ being the Matérn function defined at (5), then

$$\frac{1}{h(u^2)^{\alpha}}\mathcal{M}_{\nu}\left(\frac{\|\mathbf{h}\|}{h(u^2)}\right) \text{ is allowed, } \frac{1}{h(u^2)^{\alpha}}\mathcal{M}_{\nu}\left(\frac{\theta}{h(u^2)}\right) \text{ is allowed if } \nu \le 1/2.$$

A bridge between Gneiting functions and semi-metric spaces has been recently provided by Menegatto et al. (2019).

### 4.3 | Final remarks on the Gneiting class

According to Rodrigues and Diggle (2010), the Gneiting class is always negative nonseparable. Also, Kent et al. (2011) and more recently Cuevas et al. (2017) show conditions on the functions $f$ and $h$ such that a dimple can happen. In particular, Cuevas et al. (2017) offer a dual view of the dimple problem related to space–time correlation functions in terms of their contours. They find that the dimple property in the Gneiting class of correlations is in one-to-one correspondence with nonmonotonicity of the parametric curve describing the associated contour lines. Further, they show that given such a nonmonotonic parametric curve associated with a given level set, all the other parametric curves at smaller levels inherit the nonmonotonicity.

On an historical note, Gneiting's criterion (11) was established as a generalization of a criterion proposed by Cressie and Huang (1999):

$$K(\mathbf{h},u) = \int_{\mathbb{R}^d} e^{i\mathbf{h}^{\top}\omega}\rho(\omega;u)k(\omega)d\omega, \quad (\mathbf{h},u)\in\mathbb{R}^d\times\mathbb{R}, \tag{15}$$

where $\rho(\omega;\cdot)$ is positive definite on the real line for every fixed $\omega\in\mathbb{R}^d$, with $\int\rho(\omega;u)du<\infty$, and $k$ is positive and integrable in $\mathbb{R}^d$.

### 4.4 | Scale mixtures: A smart trick

Scale mixtures have been widely used to obtain new classes of nonseparable covariance functions. The principle can be illustrated as follows: Let $(\Omega,\mathbb{F},\mathbb{P})$ be a measure space. Let $C_{\mathcal{D}}:\mathcal{D}^d\times\Omega\to\mathbb{R}$ and $C_{\mathbb{T}}:\mathbb{R}\times\Omega\to\mathbb{R}$ be continuous mappings such that, for any fixed $\xi\in\Omega$, $C_{\mathcal{D}}(\cdot,\xi)$ and $C_{\mathbb{T}}(\cdot,\xi)$ are stationary (if $\mathcal{D}=\mathbb{S}^d$, then stationary and isotropic) covariance functions in their respective spaces. Let

$$K(x,u) = \int_{\Omega} C_{\mathcal{D}}(x,\xi)C_{\mathbb{T}}(u,\xi)\mathbb{P}(d\xi), \quad u\in\mathbb{T}, \tag{16}$$

with $x$ either $\|\mathbf{h}\|$, $\mathbf{h}\in\mathbb{R}^d$, or $\theta\in[0,\pi]$, is a covariance function, provided the integral above is well defined (Porcu & Zastavnyi, 2011).

A wealth of examples is available thanks to such a construction. For $\mathcal{D}=\mathbb{R}^d$, points (A.1) and (A.2) in Section 4.1 are proved through scale mixtures. The *quasi-arithmetic* class (Porcu, Mateu, & Christakos, 2010) is obtained through scale mixtures as well. The nonseparable models proposed by Fonseca and Steel (2011), Schlather (2010); Apanasovich and Genton (2010), Porcu et al. (2006); Porcu, Mateu, and Bevilacqua (2007), Porcu and Mateu (2007), and Alegría et al. (2019) are all obtained through scale mixture techniques.

The criterion in Cressie and Huang (1999) is a scale mixture as well: Compare (16) with (15) and let $(\Omega,\mathbb{F},\mathbb{P})=\left(\mathbb{R}^d,\mathbb{B}_d,k(\cdot)d\cdot\right)$, with $\mathbb{B}_d$ being the Borel sigma-algebra in $\mathbb{R}^d$. Also, let $\rho(\xi;\cdot)=C_{\mathbb{T}}(\cdot,\xi)$ in (16) and, finally, $C_{\mathcal{D}}(\cdot,\xi)=\exp\left(i\xi^{\top}\cdot\right)$. Similarly, the so-called half spectral approach proposed by Stein (2005c) is a special case of scale mixtures.

The product-sum model of De Iaco et al. (2002a, 2002b), De Iaco and Posa (2012), Myers, De Iaco, Posa, and De Cesare (2002), and De Iaco et al. (2011) is not a scale mixture, but is instead based on the properties of positive-definite functions seen as convex cone. Notably, some weights in the linear combinations in the product-sum model can be negative while preserving positive definiteness. This permits covariance function with oscillatory behavior. Such a challenge has been faced by De Iaco et al. (2001) and Gregori et al. (2008) and has provided general results in a similar context. Peron, Porcu, and Emery (2018) have proposed linear combinations with negative weights for the case $\mathcal{D}=\mathbb{S}^d$.

### 4.5 | Stein's spectral densities and related approaches

Stein (2005b) considers modeling the spectral density directly instead of proposing a closed form for a parametric class of covariance functions. Understanding the rate of decay of the density away from the origin is equivalent to continuity or differentiability of a Gaussian process having a given spectrum. Also, Stein (2005b) emphasizes fact that space–time covariance functions should be more differentiable away from the origin than at the origin. Otherwise, such a lack of smoothness can cause *discontinuities* in the sense that small changes in sampling locations might cause big changes in the correlations between linear combinations of observations. This happens, for instance, with separable covariance functions. Stein (2005b) considers the following family of spectral densities: For $\alpha = (\alpha_1, \alpha_2, \alpha_3, a_1, a_2)^\top$, a vector of strictly positive components, we define

$$f(\omega, \tau; \alpha) = \left( \left( a_1^2 + \|\omega\|^2 \right)^{\alpha_1} + \left( a_2^2 + |\tau|^2 \right)^{\alpha_2} \right)^{-\alpha_3}, \quad \omega \in \mathbb{R}^d, \tau \in \mathbb{R}. \tag{17}$$

The function $f$ is clearly strictly positive, but to make it a spectral density, according to (1), $f$ must be integrable in $\mathbb{R}^d \times \mathbb{R}$. Conditions for integrability are given by Stein (2005b), who then provides the geometric properties of a Gaussian random field having this parametric families of spectral densities. Finding closed-form solutions of the covariance function through Fourier inversion from (17), however, is challenging. Stein (2005b) gives some special cases that provide spatial or temporal margins of the Matérn type. A similar philosophy is followed by Fuentes et al. (2008). Porcu, Gregori, and Mateu (2009) consider Archimedean functionals, that allow to compose spatial and temporal marginal spectral densities to build new classes of space–time spectra. Stein's approach (17) becomes a special case of Archimedean compositions, for a specific choice of the Archimedean functional. Extensions of Stein's approach to nonstationary cases has been attained through spatial adaptation of the parameter vector, $\alpha$, which becomes a function of the spatial coordinates. Pintore and Holmes (2004) consider the simple case of the (square of the) product of two adapted spectral densities. Such an approach is then extended by Porcu et al. (2009) to more sophisticated compositions that are substantially similar to the Archimedean functionals.

When $\mathcal{D} = \mathbb{S}^d$, spectral modeling becomes much more challenging. One starting point, however, is the Berg–Porcu spectral representation in Equation (12). A spectral approach requires a sequence $\{ f_{k,\mathbb{T}}(\cdot) \}_{k=0}^{\infty}$ of temporal spectral densities. To ensure that $\sum_k C_{k,\mathbb{T}}(0) < \infty$ as required in the Berg–Porcu characterization, one can make use of Parseval's theorem, which provides the spectral condition $\sum_k \int f_{k,\mathbb{T}}(\tau) \mathrm{d}\tau < \infty$. Spectral models on the sphere cross time are still very much an open area of research.

## 5 | LAGRANGIAN REFERENCE FRAME AND TRANSPORT EFFECT

Environmental, atmospheric, and geophysical processes are often influenced by prevailing winds or ocean currents (Gneiting et al., 2007). Thus, the covariance function is no longer fully symmetric. In this situation, the idea of a Lagrangian reference frame is useful. Gneiting et al. (2007) summarize the physical justification of such a framework when $\mathcal{D} = \mathbb{R}^d$. For instance, the random rotation might represent a prevailing wind as in Gupta and Waymire (1987). It might be a westerly wind considered by Haslett and Raftery (1989), or again, it might be updated dynamically according to the current state of the atmosphere.

Lagrangian approaches have been proposed by Rodriguez-Iturbe, Cox, and Isham (1987). Covariance functions that are not fully symmetric can also be constructed based on diffusion equations or stochastic partial differential equations, and we refer the reader to Gneiting et al. (2007) for related references. The basic idea behind Lagrangian construction models is to take a $d$-dimensional velocity vector, $\mathbf{V}$, that is distributed according to a probability distribution and a spatial Gaussian random field that is weakly stationary with stationary spatial covariance, $K_{\mathcal{D}}$. Then, the resulting space–time covariance (that is weakly stationary, but not fully symmetric) is obtained through

$$K(\mathbf{h}, u) = \mathbb{E}_{\mathrm{V}}(K_{\mathcal{D}}(\mathbf{h} - \mathbf{V}u)), \quad (\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{T}.$$

The analogue of this construction when $\mathcal{D} = \mathbb{S}^d$ has been recently tackled in Alegría and Porcu (2017): Take a random orthogonal $(d \times d)$ matrix $\mathscr{R}$. Let $Z$ be a Gaussian process on $\mathbb{S}^d$ with geodesically isotropic covariance $C_{\mathbb{S}}(\mathbf{s}_1, \mathbf{s}_2) = \psi_{\mathbb{S}}(\theta(\mathbf{s}_1, \mathbf{s}_2))$. Define

$$Y(\mathbf{s},t) = Z(\mathscr{R}^t\mathbf{s}), \quad \mathbf{s}\in\mathbb{S}^d, t\in\mathbb{T}, \tag{18}$$

with $\mathscr{R}^t$ denoting the $t$-th power of $\mathscr{R}$. Then, the space–time covariance function with transport effect can be expressed as

$$C(\mathbf{s}_1,\mathbf{s}_2,u) = \mathbb{E}_{\mathscr{R}}(C_{\mathbb{S}}(\theta(\mathscr{R}^u\mathbf{s}_1,\mathbf{s}_2))), \quad \mathbf{s}_1,\mathbf{s}_2\in\mathbb{S}^2, \;\; u\in\mathbb{T}.$$

The fact that the resulting covariance is still geodesically isotropic in the spatial component is a nontrivial property and is shown formally, for some specific choice of the random rotation $\mathscr{R}$, in Alegría and Porcu (2017), at least for the case of the sphere $\mathbb{S}^d$. Some comments are in order. The resulting field $Y$ in Equation (18) however, is not Gaussian (it is Gaussian conditional on $\mathscr{R}$). Also, obtaining closed forms for the associated covariance is generally difficult. Alegría and Porcu (2017) provide some special cases.

## 6 | CLASSES WITH SPECIAL FEATURES

Covariance functions through a dynamic model have a long history that can be traced back to Yaglom (1948), Gandin and Boltenkov (1967) and Monin and Yaglom (1967). Representing the space–time process through stochastic integrals allows one to take into account physical properties of the process and adapt to specific applied problems. A wealth of examples and innovative ideas are provided by Christakos (1990, 1991b, 1992, 2000), and Christakos and Hristopoulos (1998). More recently, such dynamical representations have been considered by Brown et al. (2000) through the concept of *blurring*.

Storvik, Frigessi, and Hirst (2002) consider autoregressive representations, under stationarity, of the type

$$Z(\mathbf{s},t) = \int_{\mathbb{R}^2} H(\mathbf{v})Z(\mathbf{s}+\mathbf{v},t-1)\mathrm{d}\mathbf{v} + \varepsilon(\mathbf{s},t), \quad \mathbf{s}\in\mathbb{R}^2, t\in\mathbb{Z},$$

where $\varepsilon$ is temporally uncorrelated and with a given spatial covariance function. Conditions on $H$ (and on its spectrum) that ensure the process to be well defined are provided therein, along with some specific examples. The idea of stochastic integrals and dynamical models has been pursued by Kolovos, Christakos, Hrisopoulos, and Serre (2004).

Very interesting examples have been considered (under discrete-time setting) by Baxevani, Podgórski, and Rychlik (2003), and a wealth of sophisticated examples is provided by Baxevani, Caires, and Rychlik (2009), Baxevani et al. (2011), and Ailliot, Baxevani, Cuzol, Monbet, and Raillard (2011). Other models with special features have been recently proposed by Hristopulos and Tsantili (2016) and Hristopulos and Agou (2019).

Stationary models of covariance functions have been used as building blocks to create more complex models for dependence, for instance, to take into account nonstationarity in space. For just spatial processes, this idea was first proposed by Paciorek and Schervish (2006). The most general version of these models is provided in Porcu et al. (2010), who also extend the Paciorek–Shervish approach to space–time and in turn generalize Stein (2005a). Another path to nonstationarity has been pursued through spatial adaptation, for example, the parameters of a given family of covariance functions are allowed to vary smoothly with the spatial location (Kleiber & Nychka, 2012; Nychka, Wikle, & Royle, 2002). A nonstationary version of the Gneiting class $G_\alpha$ has been provided by Porcu, Mateu, and Bevilacqua (2007) and more recently by Schlather (2010). Another strategy to construct nonstationary covariance functions is through convolutions, as in Rodrigues and Diggle (2010), but surprisingly, we have not found a similar extension in the space–time framework. Finally, we note that the strategy of spatially adapting through spectral approaches has been adopted by Pintore and Holmes (2004), Fuentes et al. (2008), and Porcu et al. (2009).

We are not aware of any extensions of the type above when the spatial domain $\mathcal{D}$ is the unit sphere embedded in a three-dimensional Euclidean space. Certainly, the spatial adaptation of parameters could play an important role, and indeed a first attempt has been made in the spatial setting by Alegría, Cuevas, Diggle, and Porcu (2018) with the so-called $\mathcal{F}$ class that replaces the Matérn covariance function for processes defined over the sphere. Nonstationary models through spectral representations have been characterized by Estrade, Fariñas, and Porcu (2019). Finally, other models based on differential operators, but coupled with the chordal distance, have been proposed by Jun (2011), Jun and Stein (2007), and Hitczenko and Stein (2012).

# 7 | MULTIVARIATE SPACE–TIME COVARIANCE FUNCTIONS

The literature on multivariate covariance functions has become ubiquitous and we refer the reader to Genton and Kleiber (2015) for a comprehensive review. Here, we focus on multivariate covariance functions that are isotropic in the spatial component and symmetric in the temporal one. Throughout, the argument $x \geq 0$ will denote either the Euclidean or the geodesic distance (in this last case, $x = \theta \in [0, \pi]$), depending on the domain $\mathcal{D}$ where the process is defined. We consider an $m$-variate space–time random field $\mathbf{Z}(\mathbf{x},t) = (Z_1(\mathbf{x},t),...,Z_m(\mathbf{x},t))^\top$, for $\mathbf{x}\in\mathcal{D}$ and $t\in\mathbb{R}$, that is isotropic in space and stationary and symmetric in time. Let $\mathrm{C} : X_{\mathcal{D}} \times \mathbb{T} \to \mathbb{R}^{m\times m}$ be a continuous matrix-valued mapping, whose elements are defined as $\mathrm{C}_{ij}(x, u) = \mathbb{C}\mathrm{ov}(Z_i(\mathbf{s}_1, t+u), Z_j(\mathbf{s}_2, t))$, where $X_{\mathcal{D}}$ is either $[0, \infty)$ (with $x = \|\mathbf{s}_1 - \mathbf{s}_2\|$) or $[0, \pi]$ (and $x = \theta(\mathbf{s}_1, \mathbf{s}_2)$). According to that, $\mathbf{C}$ is isotropic (respectively geodesically isotropic, if $\mathcal{D} = \mathbb{S}^d$) in space and stationary in time (Porcu et al., 2016b). The diagonal elements of $\mathbf{C}$, denoted as $\mathrm{C}_{ii}$, are called marginal covariances, whereas the off-diagonal members $\mathrm{C}_{ij}$ are called cross-covariances. Observe that the marginal covariance functions are positive definite, while the cross-covariances, in general, are not. Certainly, any parametric representation of C must respect the non-negative definite condition analogue to Equation (8).

Appendix A in Alegría et al. (2019) contains rich material about the spectral representations associated with $\mathbf{C}$, for both cases $\mathcal{D} = \mathbb{R}^d$ and $\mathcal{D} = \mathbb{S}^d$, with $x = \|\cdot\|$ and $x = \theta$, respectively. For $\mathcal{D} = \mathbb{R}^d$, such spectral representations have been presented by Alonso-Malaver, Porcu, and Giraldo (2015), while the case $\mathcal{D} = \mathbb{S}^d$ has been challenged in Appendix A of Alegría et al. (2019).

Alegría et al. (2019) call C space–time m-separable if there exists two mappings $\mathrm{C}_{\mathcal{D}} : [0,\pi] \to \mathbb{R}^{m\times m}$ and $\mathrm{C}_{\mathbb{T}} : \mathbb{T} \to \mathbb{R}^{m\times m}$, being merely spatial and temporal matrix-valued covariances, respectively, such that

$$\mathrm{C}(\theta,u) = \mathrm{C}_{\mathcal{D}}(\theta) \circ \mathrm{C}_{\mathbb{T}}(u), \quad (\theta,u) \in [0,\pi] \times \mathbb{T},$$

where $\circ$ denotes the Hadamard product. Such a space–time $m$-separability property is called *complete* if

$$\mathrm{C}(\theta,u) = \mathbf{A} C_{\mathcal{D}}(\theta) \mathrm{C}_{\mathbb{T}}(u), \quad (\theta,u) \in [0,\pi] \times \mathbb{T},$$

where $\mathbf{A} \in \mathbb{R}^{m\times m}$ is positive definite, $\mathrm{C}_{\mathcal{D}} : [0,\pi] \to \mathbb{R}$ is a univariate spatial covariance, and $\mathrm{C}_{\mathbb{T}} : \mathbb{R} \to \mathbb{R}$ a univariate temporal covariance function. Finally, Alegría et al. (2019) call the mapping C *m-separable* if

$$\mathrm{C}(\theta,u) = \mathbf{A} C(\theta,u), \quad (\theta,u) \in [0,\pi] \times \mathbb{T},$$

for a univariate space–time covariance $\mathrm{C}_{\mathcal{D}\mathbb{T}} : [0,\pi] \times \mathbb{T} \to \mathbb{R}$, and a matrix $\mathbf{A}$, as previously defined. Clearly, the special case $\mathrm{C}(\theta,u) = \mathrm{C}_{\mathcal{D}}(\theta)\mathrm{C}_{\mathbb{T}}(u)$ offers complete space–time $m$-separability as previously discussed.

## 7.1 | Building nonseparable multivariate space–time covariance functions

The construction principles for multivariate space–time covariance functions are nicely summarized in Alegría et al. (2019), and we report here the essential content thereof.

The linear model of coregionalization (Goulard & Voltz, 1992; Wackernagel, 2003) has been used for decades to model spatial data. The extension to space–time has been considered by Rouhani and Wackernagel (1990), De Iaco, Myers, and Posa (2003), De Iaco, Myers, Palma, and Posa (2013), De Iaco, Palma, and Posa (2005), Sang and Gelfand (2009), Berrocal, Gelfand, and Holland (2010), Finazzi, Scott, and Fassò (2013), and Finazzi and Fassò (2014), to mention a few. The principle is the following: Let $q$ be a positive integer. Given matrices $\mathbf{A}_k$, $k = 1, ..., q$, and univariate space–time covariances $\mathrm{C}_k$, the linear model of coregionalization (LMC) can be written as

$$\mathrm{C}(x,u) = \sum_{k=1}^{q} \mathbf{A}_k \mathrm{C}_k(x,u),$$

where a simplification of the type $\mathrm{C}_k(x,u) = \mathrm{C}_{k,\mathcal{D}}(x)\mathrm{C}_{\mathbb{T}}(u)$ is often imposed. There has been substantial criticism about this model as reported by Gneiting, Kleiber, and Schlather (2010) and Daley, Porcu, and Bevilacqua (2015). For example, the smoothness of any component of the multivariate field is restricted to that of the roughest underlying univariate process. Moreover, the number of parameters can quickly become massive as the number of components increases.

Lagrangian construction frameworks can be adapted to the space–time case. We illustrate for the case $\mathcal{D}=\mathbb{S}^d$. The other case can be obtained analogously using transactions instead of rotations. Let $\boldsymbol{Z}$ be an $m$-variate Gaussian field on $\mathbb{S}^d$ with covariance $\mathrm{C}_{\mathcal{D}}:[0,\pi]\to\mathbb{R}^{m\times m}$. Let $\mathscr{R}$ be a random orthogonal $(d\times d)$ matrix with a given probability law. Let

$$\mathbf{Y}(\mathbf{s},t)=\mathbf{Z}(\mathscr{R}^t\mathbf{s}),\quad (\mathbf{s},t)\in\mathbb{S}^{d-1}\times\mathbb{R},$$

where $\mathscr{R}^t$ denotes the power of $t$ of the orthogonal matrix $\mathscr{R}$. Then, $\mathbf{Y}$ is a random field with transport effect over the sphere. Clearly, $\mathbf{Y}$ is not Gaussian and an evaluation of the corresponding covariance might be cumbersome, as shown in Alegría and Porcu (2017).

Alegría et al. (2019) discuss *multivariate parametric adaptation* as a fundamental construction principle. Let $\mathrm{C}(\cdot,\cdot\,;\boldsymbol{\lambda})$ be a univariate space–time covariance parameterized by $\boldsymbol{\lambda}\in\mathbb{R}^p$. Let $\boldsymbol{\lambda}_{ij}\in\mathbb{R}^p$, for $i, j = 1, ..., m$. For $|\rho_{ij}|\le 1$ and $\rho_{ii} = 1$, and $\sigma_{ii}>0$, find the parametric restrictions such that $\mathbf{C}:X\times\mathbb{R}\to\mathbb{R}^{m\times m}$, defined through

$$\mathrm{C}_{ij}(x,u)=\sigma_{ii}\sigma_{jj}\rho_{ij}\mathrm{C}\left(x,u;\boldsymbol{\lambda}_{ij}\right),\quad (x,u)\in X\times\mathbb{R},$$

is a valid covariance. In Euclidean spaces, this strategy has been adopted by Gneiting et al. (2010) and Apanasovich, Genton, and Sun (2012) for the Matérn model, and by Daley et al. (2015) for models with compact support. For $\mathcal{D}=\mathbb{S}^d$, Alegría et al. (2019) consider the case

$$\mathrm{C}_{ij}(\theta,u)=\frac{\sigma_{ii}\sigma_{jj}\rho_{ij}}{f(|u|)^{n+2}}G_{n+1}\left(\theta h^2(|u|),c_{ij}h(|u|)\right)$$

where $n\le 3$ is a positive integer. Here, $f:[0,\infty)\to[0,\infty)$ in Gneting's composition (14) is completely monotonic, and $h:[0,\infty)\to(0,\infty)$ is strictly increasing and concave. Also, $\sigma_{ii}>0$, $|\rho_{ij}|\le 1$ and $c_{ij}>0$, $i, j = 1, ..., m$, are constants yielding the additional condition specified in Theorem 1 of Alegría et al. (2019). This covariance function is valid on $\mathbb{S}^d$ for $d\le n+1$. Other examples and generalizations are provided by Alegría et al. (2019). For $\mathcal{D}=\mathbb{R}^d$ and $x = \|\cdot\|$, multivariate Gneiting functions have been proposed by Bourotte et al. (2016), where the functions $f_{ij}$ in the Gneiting's compositions (14) belong either to the Matérn or to the generalized Cauchy class.

Scale mixture techniques can be adapted to the multivariate space–time setting by using the construction principle in Porcu and Zastavnyi (2011). Latent dimension approaches have been proposed by Porcu et al. (2006), Apanasovich and Genton (2010), and Porcu and Zastavnyi (2011) for the case $\mathcal{D}=\mathbb{R}^d$. The case $\mathcal{D}=\mathbb{S}^d$ has been challenged in Alegría et al. (2019).

## 8 | ESTIMATION OF SPACE–TIME DEPENDENCIES

Most of the approaches and estimation techniques proposed in the last 10 years are motivated by reaching a compromise between statistical efficiency and computational complexity. This last became an important issue, given the availability of massive (and multivariate) data sets that are often defined over large portions of the globe and repeatedly measured over time. A beautiful illustration of geostatistical estimation techniques for massive spatial data sets is provided by Sun, Li, and Genton (2012). This section departs from their treatment in two directions: We update the approaches described in Sun et al. (2012) with extensions to the space–time setting. Also, we briefly discuss estimation techniques that have been proposed for the spheres cross time problem (Porcu, Alegría, & Furrer, 2018). To set up the discussion we first update the spatial methods with space–time approaches related to separability of covariance functions, covariance matrix tapering, composite likelihoods, spectral techniques, and approximating the random field with a Gaussian Markov random field.

Estimation methods for space–time covariances for large data are conceptually similar to classical approaches. However, they do differ in the way statistical information (read: *Full* maximum likelihood) is sacrificed in favor of computational gains. It is statistically difficult to judge which method is best: The question depends on many critical aspects. Approximating covariance functions with computationally tractable ones is a convenient form of deliberate misspecification of the true underlying covariance structure. Classical examples are covariance tapering where the misspecification consists of using a direct product of the true covariance function and a compactly supported correlation function and the use of compactly supported covariance functions. Spectral methods, rely on truncation of the spectral expansion and so lose information on the geometric properties and the sample paths of the associated random field. Finally, Markov random fields coupled with SPDEs approximate a continuous process with a process defined over a lattice for which the conditional distributions only depend on nearby neighbors, leading to sparseness of the precision matrix, the inverse of the covariance matrix.

A review on composite likelihood methods has been provided by Varin, Reid, and Firth (2011). A specific challenge on composite likelihood for space–time data is instead taken by Bevilacqua, Gaetan, Mateu, and Porcu (2012b), and a numerical comparison of composite likelihoods for space–time is provided by Bevilacqua and Gaetan (2015). Extensions of composite likelihood to multivariate space–time have been proposed by Bourotte et al. (2016). Also, there has been a number of extensions to space–time wrapped Gaussian fields (Alegría, Bevilacqua, & Porcu, 2016), space–time multivariate Markov models (Gao & Song, 2011), Bayesian versions of space–time composite likelihood (Benoit, Allard, & Mariethoz, 2018; Pauli, Racugno, & Ventura, 2011; Ribatet, Cooley, & Davison, 2012), and hidden Markov models (Ranalli, Lagona, Picone, & Zambianchi, 2018). Composite likelihood for space–time extremes has been adopted by Huser and Davison (2014), Davison, Padoan, and Ribatet (2012), Padoan, Ribatet, and Sisson (2010), Davis, Klüppelberg, and Steinkohl (2013b), Castruccio, Huser, and Genton (2016), and Genton, Ma, and Sang (2011). The review by Varin et al. (2011) presents Vecchia's block likelihood (Caragea & Smith, 2006; Katzfuss & Guinness, 2017; Stein, Chi, & Welty, 2004; Vecchia, 1988) as a composite likelihood approach. A recent space–time composite likelihood approach has been proposed by Bai, Song, and Raghunathan (2012). Finally, there have been also some work on tests based on composite likelihood (Bevilacqua et al., 2010). Computational aspects related to space–time covariance functions have been provided by De Cesare, Myers, and Posa (2002), De Iaco, Myers, Palma, and Posa (2010), and De Iaco and Posa (2012).

Tapering of covariance functions has been well understood in the spatial setting. The literature can be separated into tapering with ultimate focus on prediction (e.g., Furrer, Genton, & Nychka, 2006) or on estimation (e.g., Furrer, Bachoc, & Du, 2016; Kaufman, Schervish, & Nychka, 2008) and asymptotic results based on infill-domain asymptotics and increasing-domain asymptotics, most mainly in the framework of purely spatial processes. Extensions to Bayesian versions have been provided by Shaby and Ruppert (2012) and Sang and Genton (2014) studied covariance tapering for max-stable processes. The extension to space–time tapering has been studied only to a limited extent (Fassò, Finazzi, & Bevilacqua, 2011; Finazzi & Fassò, 2014) and some preliminary ideas of covariance tapering can be found in Guerci (2014).

In the framework of theoretical considerations for maximum likelihood estimation, there are two different schools on how the sampling locations are considered. On the one hand, there is the classical increasing-domain asymptotics school where the density of the locations does not change. On the other hand, there is the infill asymptotics where the spatial domain is fixed and the sampling locations are sampled within that specific domain. In this setting, not all parameters are estimable. Infill asymptotics seems natural for sampling within a particular environmental framework (sediment samples of a lake, global meteorological variables), but the theory is rather cumbersome and still lacks elements for multivariate processes. It is not clear what the optimal asymptotic framework for space–time processes should be. Considering a space–time increasing domain asymptotic setting amounts to considering a process that is defined over the $(d + 1)$-dimensional Euclidean space. Thus, the general results provided by Mardia and Marshall (1984) in terms of consistency and normality of the ML estimator of the parameters of a given family of covariance functions still hold. This path is taken, in space–time, by Bevilacqua, Gaetan, Mateu, and Porcu (2012a). A natural way to do space–time asymptotics would be to take an infill asymptotic approach for space while adopting an increasing domain approach for the temporal component. We are not aware of any contributions of this type in the literature. Space–time infill asymptotics with the space–time Matérn covariance function has been considered by Ip and Li (2017), and very recently by Faouzi, Porcu, and Bevilacqua (2020), who considered a class of space–time covariance functions having dynamical radii.

Dynamical approaches have been especially popular for the analysis of climate data, where climate models are generated over a regular grid generated on the sphere and repeatedly across time. As noted by Porcu et al. (2018), a very popular approach to drastically reduce the complexity is to separate the spatial and temporal components and to describe the dynamics of the process by specifying its evolution as a function of the past. Variability is then achieved by assuming a random spatial innovation. For climate data, temporal dynamics have been modeled through covariates only (Furrer, Sain, Nychka, & Meehl, 2007; Geinitz, Furrer, & Sain, 2015). Other relevant references are Cressie and Wikle (2011); Castruccio and Stein (2013); Fassò et al. (2016) and Finazzi and Fassò (2014). Recent work on satellite data has proposed to couple the dynamical approach dimension reduction techniques, and in particular, *fixed rank* *kriging* (FRK; see Cressie & Johannesson, 2008; Nguyen, Katzfuss, Cressie, & Braverman, 2014) to further reduce the parameter dimensionality and to achieve a fit for very large data sets (fixed rank filtering, see Kang, Cressie, & Shi, 2010; Cressie, Shi, & Kang, 2010).

Spectral methods for space–time data have also been quite popular. Castruccio and Stein (2013) have considered the partial Fourier transform cross longitude of an axially symmetric process, that is

$$\varepsilon((\phi,\vartheta),t)=\sum_{k=0}^{\infty}e^{\mathrm{i}\vartheta k}\mathfrak{f}(k;\phi)\tilde{\varepsilon}_t(k;\phi),$$
$$\mathrm{corr}(\tilde{\varepsilon}_t(k;\phi),\tilde{\varepsilon}_t(k';\phi'))=\delta_{k=k'}\,\rho(k;\phi,\phi'),\qquad \phi,\phi'\in[0,\pi],\vartheta\in[0,2\pi),$$

with $\tilde{\varepsilon}_t(k;\phi)$ the Fourier process for wavenumber $k$ and latitude $\phi$, $\mathfrak{f}(k;\phi)$ the spectrum. For any pair of latitudes $(\phi,\phi')$, the function $\rho(k;\phi,\phi')$ defines a spectral correlation (also called *coherence*). The computational aspects of these approaches have been analyzed by Jun and Stein (2008) and Castruccio and Stein (2013). Recent extensions have been provided by Castruccio and Genton (2014, 2016), Castruccio and Guinness (2017), Jeong, Castruccio, Crippa, and Genton (2017), and Horrell and Stein (2015).

### 8.1 | Implicit models

Relations between Gaussian Markov random fields and the solution of stochastic partial differential equations (SPDEs) have been studied by Lindgren, Rue, and Lindstroem (2011) and Bolin and Lindgren (2011). The solution is to consider the SPDE having a solution as Gaussian field with Matérn covariance function of smoothness $\nu$: For a merely spatial process $X$ on $\mathcal{D}$, the authors study the SPDE defined through

$$\left(\kappa^2-\Delta\right)^{\alpha/2}X(\mathbf{s})=\mathcal{W}(\mathbf{s}),\qquad \mathbf{s}\in\mathcal{D},\ \ \alpha=\nu+d/2, \tag{19}$$

where $\kappa>0$, $\Delta$ is the Laplace–Beltrami operator and $\mathcal{W}$ is a Gaussian white noise process on $\mathcal{D}$. Clearly, specific definitions and assumptions are needed depending on whether $\mathcal{D}$ is a planar surface, a sphere, or in general a manifold.

In order to provide a computationally convenient approximation of (19) for integer-valued $\alpha$, Lindgren et al. (2011) find a very ingenious computationally efficient Hilbert space approximation. Namely, the weak solution to (19) is found in some approximation space spanned by some basis functions. The computational efficiency is then attained by imposing local basis functions, that is, basis functions which are compactly supported. This all boils down to approximating the field $X$ with a Gaussian Markov field with the highly sparse precision matrix. This idea is then generalized in Bolin and Lindgren (2011) through nested SPDE models and Bolin and Kirchner (2019) to arbitrary $\alpha>d/2$. This approach has then been coupled with the Bayesian framework by Cameletti et al. (2012) to provide a space–time model. A direct space–time formulation of the SPDE approach is also suggested in Lindgren et al. (2011) and elaborated in Krainski et al. (2018). Extension of the SPDE approach to space–time has been considered recently by Vergara, Allard, and Desassis (2018). We note throughout all the SPDE models that there is an inherent discretization imposed on the problem and a precise covariance function of the approximation might not be available.

## 9 | OUTLOOK

Although we present a comprehensive review of space–time covariance functions, we foresee much new, creative work and many open problems. A list of open problems related to space–time modeling is provided by Porcu et al. (2018).

A promising field of research is represented by the statistical analysis of processes that exhibit cyclic behaviors over time and or/space. This has been advocated in recent papers: Random fields defined over $\mathcal{D}\times\mathbb{S}^1$, where $\mathcal{D}$ is the spatial domain (a path or a planar surface) and $\mathbb{S}^1$ is time wrapped over the circle, have been considered by Benigni and Furrer (2012) to analyze improvised explosive device attacks along a main supply route in Baghdad, or by Shirota and Gelfand (2017) to analyze daily crime events in San Francisco. Similar approaches are then adopted by Mastrantonio et al. (2019), who consider Bayesian hierarchical modeling where seasonality is modeled through conditioning sets. A similar approach under the Bayesian framework has been adopted by White and Porcu (2019a). Very recently, Porcu, Cleanthous, Georgiadis, White, and Alegría (2019) have considered random fields defined over the hypertorus, which is in turn obtained through the product of hyperspheres of possibly different dimensions. The work opens for many questions related to the statistical analysis of such processes.

Many applications are concerned about predictions of the spatial process, and the modeling and estimation of the covariance function is just a means to an end. Hence, the covariance function and the parameters are not necessarily needed for interpretation, thus approximations such as misspecifications are suitable provided that the computational gain justifies the predictive loss. As argued by, for example, Furrer et al. (2016), it is important that the misspecification is the same for the estimation and prediction. But, other than that, it is virtually impossible to provide guidelines for useful approximation routes. Quite often, available computing resources determine the maximum possible flexibility of covariance functions (e.g., dimension of the parameter space) or estimation precision (e.g., the number likelihood evaluations that can be carried out). The parametric covariance of Gerber et al. (2017) and Heaton et al. (2019) are mere attempts to capture the overall dependency structure. If excessive smoothing is not acceptable, nonparametric models are a promising option (Gerber, de Jong, Schaepman, Schaepman-Strub, & Furrer, 2018), but such models are beyond the scope of this review.

## ACKNOWLEDGMENTS

We thank the Associate Editor and two Referees for their work that have improved earlier versions of the manuscript. Partial support was provided by FONDECYT grant 1130647, Chile by the Millennium Science Initiative of the Ministry of Economy, Development, and Tourism, grant “Millennium Nucleus Center for the Discovery of Structures in Complex Data” for Emilio Porcu. This work was supported by the Swiss National Science Foundation (Grant 175529) for Reinhard Furrer.

## CONFLICT OF INTEREST

The authors have declared no conflicts of interest for this article.

## AUTHOR CONTRIBUTIONS

**Emilio Porcu:** Conceptualization; investigation; methodology; supervision; visualization; writing-original draft; writing-review and editing. **Reinhard Furrer:** Conceptualization; investigation; methodology; supervision; visualization; writing-review and editing. **Douglas Nychka:** Conceptualization; investigation; methodology; supervision; visualization; writing-review and editing.

## ORCID

*Emilio Porcu* https://orcid.org/0000-0001-9934-4409
*Reinhard Furrer* https://orcid.org/0000-0002-6319-2332
*Douglas Nychka* https://orcid.org/0000-0003-1387-3356

## RELATED WIRES ARTICLE

Covariance structure of spatial and spatiotemporal processes

## REFERENCES

Ailliot, P., Baxevani, A., Cuzol, A., Monbet, V., & Raillard, N. (2011). Space-time models for moving fields with an application to significant wave height fields. *Environmetrics*, *22*(3), 354–369.

Alegría, A., Bevilacqua, M., & Porcu, E. (2016). Likelihood-based inference for multivariate space-time wrapped-Gaussian fields. *Journal of Statistical Computation and Simulation*, *86*(13), 2583–2597.

Alegría, A., Cuevas, F., Diggle, P., & Porcu, E. (2018). *A new class of covariance functions of random fields on spheres [Technical Report]*. Aalborg, Denmark: University of Aalborg.

Alegría, A., & Porcu, E. (2017). The dimple problem related to space–time modeling under the Lagrangian framework. *Journal of Multivariate Analysis*, *162*, 110–121.

Alegría, A., Porcu, E., Furrer, R., & Mateu, J. (2019). Covariance functions for multivariate Gaussian fields evolving temporally over planet earth. *Stochastic Environmental Research and Risk Assessment*, *33*(8–9), 1593–1608.

Alexeeff, S. E., Nychka, D., Sain, S. R., & Tebaldi, C. (2016). Emulating mean patterns and variability of temperature across and within scenarios in anthropogenic climate change experiments. *Climatic Change*, *146*(3), 319–333.

Alonso-Malaver, C., Porcu, E., & Giraldo, R. (2015). Multivariate and multiradial Schoenberg measures with their dimension walks. *Journal of Multivariate Analysis*, *133*, 251–265.

Apanasovich, T. V., & Genton, M. G. (2010). Cross-covariance functions for multivariate random fields based on latent dimensions. *Biometrika*, *97*, 15–30.

Apanasovich, T. V., Genton, M. G., & Sun, Y. (2012). A valid Matérn class of cross-covariance functions for multivariate random fields with any number of components. *Journal of the American Statistical Association*, *107*(497), 180–193.

Askey, R. (1973). Radial characteristic functions. Technical report. Madison, WI: Research Center, University of Wisconsin.

Aston, J. A., Pigoli, D., & Tavakoli, S. (2017). Tests for separability in nonparametric covariance operators of random surfaces. *The Annals of Statistics*, *45*(4), 1431–1461.

Bai, Y., Song, P. X.-K., & Raghunathan, T. (2012). Joint composite estimating functions in spatiotemporal models. *Journal of the Royal Statistical Society, Series B (Statistical Methodology)*, *74*(5), 799–824.

Baldi, P., & Marinucci, D. (2006). Some characterizations of the spherical harmonics coefficients for isotropic random fields. *Statistics & Probability Letters*, *77*, 490–496.

Barbosa, V. S., & Menegatto, V. A. (2017). Strict positive definiteness on products of compact two–point homogeneous spaces. *Integral Transforms and Special Functions*, *28*(1), 56–73.

Bardossy, A., & Plate, E. J. (1992). Space-time model for daily rainfall using atmospheric circulation patterns. *Water Resources Research*, *28* (5), 1247–1259.

Baxevani, A., Caires, S., & Rychlik, I. (2009). Spatio-temporal statistical modelling of significant wave height. *Environmetrics*, *20*(1), 14–31.

Baxevani, A., Podgórski, K., & Rychlik, I. (2003). Velocities for moving random surfaces. *Probabilistic Engineering Mechanics*, *18*(3), 251–271.

Baxevani, A., Podgórski, K., & Rychlik, I. (2011). Dynamically evolving Gaussian spatial fields. *Extremes*, *14*(2), 223–251.

Beatson, R. K., Zu Castell, W., & Xu, Y. (2014). Pólya criterion for (strict) positive definiteness on the sphere. *IMA Journal of Numerical Analysis*, *34*, 550–568.

Benali, H., Buvat, I., Anton, J.-L., Pélégrini, M., Di Paola, M., Bittoun, J., Burnod, Y., and Di Paola, R. (1997). *Space-time statistical model for functional mri image sequences*. In Biennial International Conference on Information Processing in Medical Imaging. Springer. pp. 285–298

Benigni, M., & Furrer, R. (2012). Spatio-temporal improvised explosive device monitoring: Improving detection to minimise attacks. *Journal of Applied Statistics*, *39*(11), 2493–2508.

Benoit, L., Allard, D., & Mariethoz, G. (2018). Stochastic rainfall modeling at sub-kilometer scale. *Water Resources Research*, *54*(6), 4108–4130.

Berg, C. (2008). Stieltjes–pick–Bernstein–Schoenberg and their connection to complete monotonicity. In S. Mateu & E. Porcu (Eds.), *Positive definite functions: From Schoenberg to space-time challenges* (pp. 15–45). Castellón de la Plana, Spain: Department of Mathematics, University Jaume I.

Berg, C., & Porcu, E. (2017). From Schoenberg coefficients to Schoenberg functions. *Constructive Approximation*, *45*(2), 217–241.

Berliner, L. M., Levine, R. A., & Shea, D. J. (2000). Bayesian climate change assessment. *Journal of Climate*, *13*(21), 3805–3820.

Berrocal, V. J., Gelfand, A. E., & Holland, D. M. (2010). A spatio-temporal downscaler for output from numerical models. *Journal of Agricultural, Biological, and Environmental Statistics*, *15*(2), 176–197.

Bertino, L., Evensen, G., & Wackernagel, H. (2003). Sequential data assimilation techniques in oceanography. *International Statistical Review*, *71*(2), 223–241.

Bevilacqua, M., Fassò, A., Gaetan, C., Porcu, E., & Velandia, D. (2016). Covariance tapering for multivariate Gaussian random fields estimation. *Statistical Methods and Applications*, *25*(1), 21–37.

Bevilacqua, M., & Gaetan, C. (2015). Comparing composite likelihood methods based on pairs for spatial Gaussian random fields. *Statistics and Computing*, *25*(5), 877–892.

Bevilacqua, M., Gaetan, C., Mateu, J., & Porcu, E. (2012a). Estimating space and space-time covariance functions: A weighted composite likelihood approach. *Journal of the American Statistical Association*, *107*, 268–280.

Bevilacqua, M., Gaetan, C., Mateu, J., & Porcu, E. (2012b). Estimating space and space-time covariance functions for large data sets: A weighted composite likelihood approach. *Journal of the American Statistical Association*, *107*(497), 268–280.

Bevilacqua, M., Mateu, J., Porcu, E., Zhang, H., & Zini, A. (2010). Weighted composite likelihood-based tests for space-time separability of covariance functions. *Statistics and Computing*, *20*(3), 283–293.

Bochner, S. (1955). *Harmonic analysis and the theory of probability. California Monographs in mathematical sciences*. Berkeley, CA: University of California Press.

Bolin, D., & Kirchner, K. (2019). The rational SPDE approach for Gaussian random fields with general smoothness. *Journal of Computational and Graphical Statistics*, 1–12. https://doi.org/10.1080/10618600.2019.1665537

Bolin, D., & Lindgren, F. (2011). Spatial models generated by nested stochastic partial differential equations, with an application to global ozone mapping. *The Annals of Applied Statistics*, *5*(1), 523–550.

Bourotte, M., Allard, D., & Porcu, E. (2016). A flexible class of non-separable cross-covariance functions for multivariate space–time data. *Spatial Statistics*, *18*, 125–146.

Brown, P. E., Diggle, P. J., Lord, M. E., & Young, P. C. (2001). Space–time calibration of radar rainfall data. *Journal of the Royal Statistical Society, Series C (Applied Statistics)*, *50*(2), 221–241.

Brown, P. E., Karesen, K. F., Roberts, G. O., & Tonellato, S. (2000). Blur-generated non-separable space-time models. *Journal of the Royal Statistical Society, Series B (Statistical Methodology)*, *62*, 847–860.

Buttafuoco, G., & Castrignanò, A. (2005). Study of the spatio-temporal variation of soil moisture under forest using intrinsic random functions of order k. *Geoderma*, *128*(3–4), 208–220.

Calus, M. P., Bijma, P., & Veerkamp, R. F. (2004). Effects of data structure on the estimation of covariance functions to describe genotype by environment interactions in a reaction norm model. *Genetics Selection Evolution*, *36*(5), 489–507.

Cameletti, M., Ignaccolo, R., & Bande, S. (2011). Comparing spatio-temporal models for particulate matter in piemonte. *Environmetrics*, *22*(8), 985–996.

Cameletti, M., Lindgren, F., Simpson, D., & Rue, H. (2012). Spatio-temporal modeling of particulate matter concentration through the SPDE approach. *Advances in Statistical Analysis*, *97*, 109–113.

Cappello, C., De Iaco, S., & Posa, D. (2018). Testing the type of non-separability and some classes of space-time covariance function models. *Stochastic Environmental Research and Risk Assessment*, *32*(1), 17–35.

Caragea, P., & Smith, R. L. (2006). *Approximate likelihoods for spatial processes*. Joint Statistical Meetings - Section on Statistics & the Environment, Boston, MA. pp. 385–390.

Castruccio, S., & Genton, M. G. (2014). Beyond axial symmetry: An improved class of models for global data. *Stat*, *3*(1), 48–55.

Castruccio, S., & Genton, M. G. (2016). Compressing an ensemble with statistical models: An algorithm for global 3D Spatio-temporal temperature. *Technometrics*, *58*(3), 319–328.

Castruccio, S., & Guinness, J. (2017). An evolutionary Spectrum approach to incorporate large-scale geographical descriptors on global processes. *Journal of the Royal Statistical Society, Series C (Applied Statistics)*, *66*(2), 329–344.

Castruccio, S., Huser, R., & Genton, M. G. (2016). High-order composite likelihood inference for max-stable distributions and processes. *Journal of Computational and Graphical Statistics*, *25*(4), 1212–1229.

Castruccio, S., & Stein, M. L. (2013). Global space-time models for climate ensembles. *The Annals of Applied Statistics*, *7*(3), 1593–1611.

Chen, D., Menegatto, V. A., & Sun, X. (2003). A necessary and sufficient condition for strictly positive definite functions on spheres. *Proceedings of the American Mathematical Society*, *131*, 2733–2740.

Chen, X., Jin, Y., Qiang, S., Hu, W., and Jiang, K. (2015). *Analyzing and modeling spatio-temporal dependence of cellular traffic at city scale*. In 2015 IEEE International Conference on Communications (ICC). pp. 3585–3591.

Christakos, G. (1990). *Random field modelling and its applications in stochastic data processing [PhD Thesis]*. Cambridge, MA: Division of Applied Sciences, Harvard University.

Christakos, G. (1991a). On certain classes of spatiotemporal random fields with application to space-time data processing. *Systems, Man, and Cybernetics*, *21*(4), 861–875.

Christakos, G. (1991b). A theory of spatiotemporal random fields and its application to space–time data processing. *IEEE Transactions on Systems, Man, and Cybernetics*, *21*, 861–875.

Christakos, G. (1992). *Random field models in earth sciences*. San Diego, CA: Academic Press.

Christakos, G. (2000). *Modern spatiotemporal geostatistics*. Oxford: Oxford University Press.

Christakos, G., & Hristopoulos, D. (1998). *Spatiotemporal environmental health modeling: A Tractatus Stochasticus*. Boston: Kluwer.

Christakos, G., Hristopulos, D. T., & Bogaert, P. (2000). On the physical geometry hypotheses at the basis of spatiotemporal analysis of hydrologic geostatistics. *Advances in Water Resources*, *23*, 799–810.

Christakos, G., Olea, R. A., & Yu, H.-L. (2007). Recent results on the spatiotemporal modelling and comparative analysis of black death and bubonic plague epidemics. *Public Health*, *121*(9), 700–720.

Christakos, G., & Papanicolaou, V. (2000). Norm-dependent covariance permissibility of weakly homogeneous spatial random fields. *Stochastic Environmental Research and Risk Assessment*, *14*(6), 1–8.

Clarke, J., Alegria, A., & Porcu, E. (2018). Regularity properties and simulations of Gaussian random fields on the sphere cross time. *Electronic Journal of Statistics*, *12*, 399–426.

Constantinou, P., Kokoszka, P., & Reimherr, M. (2017). Testing separability of space-time functional processes. *Biometrika*, *104*(2), 425–437.

Cressie, N., & Huang, H. (1999). Classes of nonseparable, spatiotemporal stationary covariance functions. *Journal of the American Statistical Association*, *94*, 1330–1340.

Cressie, N., & Johannesson, G. (2008). Fixed rank kriging for very large spatial data sets. *Journal of the Royal Statistical Society, Series B (Statistical Methodology)*, *70*(1), 209–226.

Cressie, N., Shi, T., & Kang, E. (2010). Fixed rank filtering for spatio-temporal data. *Journal of Computational and Graphical Statistics*, *19*(3), 724–745.

Cressie, N., & Wikle, C. K. (2011). *Statistics for spatio-temporal data*. Hoboken, NJ: Wiley & Sons.

Crippa, P., Castruccio, S., Archer-Nicholls, G. B., Lebron, M., Kuwata, A., Thota, S., Sumin, E., Butt, C., Wiedinmyer, W., and Spracklen, D. V. (2016). Population exposure to hazardous air quality due to the 2015 fires in eequatorial asia. *Scientific Reports*, 6:"Article number: 37074".

Cuevas, F., Porcu, E., & Bevilacqua, M. (2017). Contours and dimple for the Gneiting class of space-time correlation functions. *Biometrika*, *104*(4), 995–1001.

Dai, F., & Xu, Y. (2013). *Approximation theory and harmonic analysis on spheres and balls*. Hoboken, NJ: Springer.

Daley, D., Porcu, E., & Bevilacqua, M. (2015). Classes of compactly supported covariance functions for multivariate random fields. *Stochastic Environmental Research and Risk Assessment*, *29*(4), 1249–1263.

Daley, D. J., & Porcu, E. (2013). Dimension walks and Schoenberg spectral measures. *Proceedings of American Mathematical Society*, *141*, 1813–1824.

Davis, R. A., Klüppelberg, C., & Steinkohl, C. (2013a). Max-stable processes for modeling extremes observed in space and time. *Journal of the Korean Statistical Society*, *42*(3), 399–414.
Davis, R. A., Klüppelberg, C., & Steinkohl, C. (2013b). Statistical inference for max-stable processes in space and time. *Journal of the Royal Statistical Society, Series B (Statistical Methodology)*, *75*(5), 791–819.
Davis, R. A., & Mikosch, T. (2008). Extreme value theory for space–time processes with heavy-tailed distributions. *Stochastic Processes and their Applications*, *118*(4), 560–584.
Davison, A. C., Padoan, S. A., & Ribatet, M. (2012). Statistical modeling of spatial extremes. *Statistical Science*, *27*(2), 161–186.
De Cesare, L., Myers, D., & Posa, D. (2001a). Estimating and modeling space–time correlation structures. *Statistics & Probability Letters*, *51*(1), 9–14.
De Cesare, L., Myers, D., & Posa, D. (2001b). Product-sum covariance for space-time modeling: An environmental application. *Environmetrics*, *12*(1), 11–23.
De Cesare, L., Myers, D., & Posa, D. (2002). Fortran programs for space-time modeling. *Computers & Geosciences*, *28*(2), 205–212.
De Iaco, S., Myers, D., Palma, M., & Posa, D. (2010). Fortran programs for space–time multivariate modeling and prediction. *Computers & Geosciences*, *36*(5), 636–646.
De Iaco, S., Myers, D., Palma, M., & Posa, D. (2013). Using simultaneous diagonalization to identify a space–time linear coregionalization model. *Mathematical Geosciences*, *45*(1), 69–86.
De Iaco, S., Myers, D., & Posa, D. (2002a). Space–time variograms and a functional form for total air pollution measurements. *Computational Statistics & Data Analysis*, *41*(2), 311–328.
De Iaco, S., Myers, D., & Posa, D. (2003). The linear coregionalization model and the product–sum space–time variogram. *Mathematical Geology*, *35*(1), 25–38.
De Iaco, S., Myers, D., & Posa, D. (2011). On strict positive definiteness of product and product–sum covariance models. *Journal of Statistical Planning and Inference*, *141*(3), 1132–1140.
De Iaco, S., Myers, D. E., & Posa, D. (2001). Space–time analysis using a general product–sum model. *Statistics & Probability Letters*, *52*(1), 21–28.
De Iaco, S., Myers, D. E., & Posa, D. (2002b). Nonseparable space-time covariance models: Some parametric families. *Mathematical Geology*, *34*(1), 23–42.
De Iaco, S., Palma, M., & Posa, D. (2005). Modeling and prediction of multivariate space–time random fields. *Computational Statistics & Data Analysis*, *48*(3), 525–547.
De Iaco, S., Palma, M., & Posa, D. (2015). Spatio-temporal geostatistical modeling for french fertility predictions. *Spatial Statistics*, *14*, 546–562.
De Iaco, S., Palma, M., & Posa, D. (2016). A general procedure for selecting a class of fully symmetric space-time covariance functions. *Environmetrics*, *27*(4), 212–224.
De Iaco, S., & Posa, D. (2012). Predicting spatio-temporal random fields: Some computational aspects. *Computers & Geosciences*, *41*, 12–24.
De Iaco, S., & Posa, D. (2013). Positive and negative non-separability for space–time covariance models. *Journal of Statistical Planning and Inference*, *143*(2), 378–391.
De Iaco, S., Posa, D., Cappello, C., & Maggio, S. (2019). Isotropy, symmetry, separability and strict positive definiteness for covariance functions: A critical review. *Spatial Statistics*, *29*, 89–108.
De Iaco, S., Posa, D., & Myers, D. (2013). Characteristics of some classes of space–time covariance functions. *Journal of Statistical Planning and Inference*, *143*(11), 2002–2015.
De Luna, X., & Genton, M. G. (2005). Predictive spatio-temporal models for spatially sparse enviromental data. *Statistica Sinica*, *15*, 547–568.
Diggle, P. J. (2013). *Statistical analysis of spatial and spatio-temporal point patterns*. Boca Raton, FL: Chapman and Hall/CRC.
Dimitrakopoulos, R., & Luo, X. (1994). Spatiotemporal modelling: Covariances and ordinary kriging systems. In R. Dimitrakopoulos (Ed.), *Geostatistics for the next century* (pp. 88–93). Hoboken, NJ: Springer.
Edwards, M., Castruccio, S., & Hammerling, D. (2019). A multivariate global spatio-temporal stochastic generator for climate ensembles. *Journal of Agricultural, Biological and Environmental Sciences*, *24*, 464–483.
Espen, B. F., & Jurate, S.-B. (2012). *Modeling and pricing in financial markets for weather derivatives* (Vol. 17). London: World Scientific.
Estrade, A., Fariñas, A., & Porcu, E. (2019). Covariance functions on spheres cross time: Beyond spatial isotropy and temporal stationarity. *Statistics & Probability Letters*, *151*, 1–7.
Faouzi, T., Porcu, E., and Bevilacqua, M. (2020, Submitted). Space-time estimation and prediction under infill asymptotics with compactly supported covariance functions.
Farmer, D., & Clifford, S. (1986). Space-time acoustic scintillation analysis: A new technique for probing ocean flows. *IEEE Journal of Oceanic Engineering*, *11*(1), 42–50.
Fassò, A., Finazzi, F., and Bevilacqua, M. (2011). *Tapering spatio temporal models*. In Spatial2 Conference: Spatial Data Methods for Environmental and Ecological Processes, Foggia (IT), 1–2 September 2011.
Fassò, A., Finazzi, F., & Ndongo, F. (2016). European population exposure to airborne pollutants based on a multivariate spatio-temporal model. *Journal of Agricultural, Biological, and Environmental Statistics*, *21*(3), 492–511.
Fernández-Avilés, G., & Montero, J.-M. (2016). Spatio-temporal modeling of financial maps from a joint multidimensional scaling-geostatistical perspective. *Expert Systems with Applications*, *60*, 280–293.

Finazzi, F., & Fassò, A. (2014). D-STEM: A software for the analysis and mapping of environmental space-time variables. *Journal of Statistical Software*, *62*(6), 1–29.

Finazzi, F., Scott, E. M., & Fassò, A. (2013). A model-based framework for air quality indices and population risk evaluation, with an application to the analysis of scottish air quality data. *Journal of the Royal Statistical Society, Series C (Applied Statistics)*, *62*, 287–308.

Fonseca, T. C. O., & Steel, M. F. J. (2011). A general class of nonseparable space-time covariance models. *Environmetrics*, *22*(2), 224–242.

Fuentes, M. (2006). Testing for separability of spatial–temporal covariance functions. *Journal of Statistical Planning and Inference*, *136*(2), 447–466.

Fuentes, M., Chen, L., & Davis, J. M. (2008). A class of nonseparable and nonstationary spatial temporal covariance functions. *Environmetrics*, *19*(5), 487–507.

Furrer, R., Bachoc, F., & Du, J. (2016). Asymptotic properties of multivariate tapering for estimation and prediction. *Journal of Multivariate Analysis*, *149*(C), 177–191.

Furrer, R., Genton, M. G., & Nychka, D. (2006). Covariance tapering for interpolation of large spatial datasets. *Journal of Computational and Graphical Statistics*, *15*(3), 502–523.

Furrer, R., Sain, S. R., Nychka, D. W., & Meehl, G. A. (2007). Multivariate Bayesian analysis of atmosphere-ocean general circulation models. *Environmental and Ecological Statistics*, *14*(3), 249–266.

Gandin, L. and Boltenkov, V. (1967). On the methods of investigation of three-dimensional macrostructure of meteorological fields. Trudy MGO 165.

Gangolli, R. (1967). Positive definite kernels on homogeneous spaces and certain stochastic processes related to Levy's Brownian motion of several parameters. *Annales Henri Poincaré*, *3*, 121–226.

Gao, X., & Song, P. X.-K. (2011). Composite likelihood EM algorithm with applications to multivariate hidden Markov model. *Statistica Sinica*, *21*, 165–185.

Garg, S., Singh, A., and Ramos, F. (2012). *Learning non-stationary space-time models for environmental monitoring*. In Twenty-Sixth AAAI Conference on Artificial Intelligence.

Geinitz, S., Furrer, R., & Sain, S. R. (2015). Bayesian multilevel analysis of variance for relative comparison across sources of global climate model variability. *International Journal of Climatology*, *35*(3), 433–443.

Gelfand, A. E., Schmidt, A. M., Banerjee, S., & Sirmans, C. (2004). Nonstationary multivariate process modeling through spatially varying coregionalization. *Test*, *13*(2), 263–312.

Genton, M. G. (2001). Classes of kernels for machine learning: A statistics perspective. *Journal of Machine Learning Research*, *2*(December), 299–312.

Genton, M. G., & Kleiber, W. (2015). Cross-covariance functions for multivariate geostatistics. *Statistical Science*, *30*, 147–163.

Genton, M. G., Ma, Y., & Sang, H. (2011). On the likelihood function of Gaussian max-stable processes. *Biometrika*, *98*, 481–488.

Gerber, F., de Jong, R., Schaepman, M. E., Schaepman-Strub, G., & Furrer, R. (2018). Predicting missing values in spatio-temporal remote sensing data. *IEEE Transactions on Geoscience and Remote Sensing*, *56*(5), 2841–2853.

Gerber, F., Mösinger, L., & Furrer, R. (2017). Extending R packages to support 64-bit compiled code: An illustration with spam64 and GIMMS $\mathrm{NDVI}_{3g}$ data. *Computational Geosciences*, *104*, 107–119.

Gething, P. W., Smith, D. L., Patil, A. P., Tatem, A. J., Snow, R. W., & Hay, S. I. (2010). Climate change and the global malaria recession. *Nature*, *465*(7296), 342–345.

Giebel, G., Brownsword, R., Kariniotakis, G., Denhard, M., & Draxl, C. (2011). *The state-of-the-art in short-term prediction of wind power: A literature overview*. ANEMOS.plus. https://doi.org/10.11581/DTU:00000017

Gilleland, E., & Nychka, D. (2005). Statistical models for monitoring and regulating ground-level ozone. *Environmetrics*, *16*(5), 535–546.

Gneiting, T. (2002a). Compactly supported correlation functions. *Jourrnal of Multivariate Analysis*, *83*, 493–508.

Gneiting, T. (2002b). Nonseparable, stationary covariance functions for space-time data. *Journal of the American Statistical Association*, *97*, 590–600.

Gneiting, T., Genton, M. G., & Guttorp, P. (2007). Geostatistical space-time models, Stationarity, Separability and full symmetry. In B. Finkenstadt, L. Held, & V. Isham (Eds.), *Statistical methods for Spatio-temporal systems* (pp. 151–175). Boca Raton: Chapman & Hall/CRC.

Gneiting, T., & Katzfuss, M. (2014). Probabilistic forecasting. *Annual Review of Statistics and Its Application*, *1*, 125–151.

Gneiting, T., Kleiber, W., & Schlather, M. (2010). Matérn cross-covariance functions for multivariate random fields. *Journal of the American Statistical Association*, *105*, 1167–1177.

Gneiting, T., Larson, K., Westrick, K., Genton, M. G., & Aldrich, E. (2006). Calibrated probabilistic forecasting at the stateline wind energy center: The regime-switching space–time method. *Journal of the American Statistical Association*, *101*(475), 968–979.

Goulard, M., & Voltz, M. (1992). Linear coregionalization model: Tools for estimation and choice of cross-variogram matrix. *Mathematical Geology*, *24*(3), 269–286.

Gradshteyn, I. S., & Ryzhik, I. M. (2007). *Tables of integrals, series, and products* (7th ed.). Amsterdam: Academic Press.

Gregori, P., Porcu, E., Mateu, J., & Sasvári, Z. (2008). On potentially negative space time covariances obtained as sum of products of marginal ones. *Annals of the Institute of Statistical Mathematics*, *60*(4), 865–882.

Guella, J. C., Menegatto, V. A., & Peron, A. P. (2016a). An extension of a theorem of Schoenberg to a product of spheres. *Banach Journal of Mathematical Analysis*, *10*(4), 671–685.

Guella, J. C., Menegatto, V. A., & Peron, A. P. (2016b). Strictly positive definite kernels on a product of spheres ii. *Sigma*, *12*(103), 286–301.

Guella, J. C., Menegatto, V. A., & Peron, A. P. (2017). Strictly positive definite kernels on a product of circles. *Positivity*, *21*(1), 329–342.

Guerci, J. R. (2014). *Space-time adaptive processing for radar*. Norwood, MA: Artech House.

Guinness, J., & Fuentes, M. (2016). Isotropic covariance functions on spheres: Some properties and modeling considerations. *Journal of Multivariate Analysis*, *143*, 143–152.

Gupta, V. K., & Waymire, E. (1987). On Taylor's hypothesis and dissipation in rainfall. *Journal of Geophysical Research*, *92*(3), 9657–9660.

Halliwell, G. R., Jr., & Mooers, C. N. (1979). The space-time structure and variability of the shelf water-slope water and gulf stream surface temperature fronts and associated warm-core eddies. *Journal of Geophysical Research, Oceans*, *84*(C12), 7707–7725.

Handcock, M. S., & Wallis, J. R. (1994). An approach to statistical spatial-temporal modeling of meteorological fields. *Journal of the American Statistical Association*, *89*(426), 368–378.

Hannan, E. J. (1970). *Multiple Time Series*. New York: Wiley.

Hansen, L. V., Thorarinsdottir, T. L., Ovcharov, E., & Gneiting, T. (2015). Gaussian random particles with flexible Hausdorff dimension. *Advances in Applied Probability*, *47*, 307–327.

Haslett, J., & Raftery, A. E. (1989). Space-time modelling with long-memory dependence: Assessing Ireland's wind-power resource. *Applied Statistics*, *38*, 1–50.

Heaton, M. J., Datta, A., Finley, A. O., Furrer, R., Guinness, J., Guhaniyogi, R., … Zammit-Mangion, A. (2019). A case study competition among methods for analyzing large spatial data. *Journal of Agricultural, Biological, and Environmental Statistics*, *24*(3), 398–425.

Hengl, T., Heuvelink, G. B., Tadić, M. P., & Pebesma, E. J. (2012). Spatio-temporal prediction of daily temperatures using time-series of modis lst images. *Theoretical and Applied Climatology*, *107*(1–2), 265–277.

Hitczenko, M., & Stein, M. L. (2012). Some theory for anisotropic processes on the sphere. *Statistics Methodology*, *9*, 211–227.

Horrell, M. T., & Stein, M. L. (2015). A covariance parameter estimation method for polar-orbiting satellite data. *Statistica Sinica*, *25*(1), 41–59.

Hristopulos, D. T., & Tsantili, I. C. (2016). Space-time models based on random fields with local interactions. *International Journal of Modern Physics B*, *30*(15), 1541007.

Hristopulos, D. T., & Agou, V. D. (2019). Stochastic local interaction model with sparse precision matrix for space-time interpolation. *Spatial Statistics*, 100403 (in Press).

Huang, C., Zhang, H., & Robeson, S. (2012). A simplified representation of the covariance structure of axially symmetric processes on the sphere. *Statistics & Probability Letters*, *82*, 1346–1351.

Huerta, G., & Sansó, B. (2007). Time-varying models for extreme values. *Environmental and Ecological Statistics*, *14*(3), 285–299.

Huser, R., & Davison, A. (2014). Space–time modelling of extreme events. *Journal of the Royal Statistical Society, Series B (Statistical Methodology)*, *76*(2), 439–461.

Ip, R. H., & Li, W. (2017). On some Matérn covariance functions for spatio-temporal random fields. *Statistica Sinica*, *27*, 805–822.

Istas, J. (2005). Spherical and hyperbolic fractional Brownian motion. *Electronic Communications in Probability*, *10*, 254–262.

Jain, J., & Jain, A. (1981). Displacement measurement and its application in interframe image coding. *IEEE Transactions on Communications*, *29*(12), 1799–1808.

Jeong J., Castruccio S., Crippa P. and Genton M.G. (2018). Reducing storage of global wind ensembles with stochastic generators. *Annals of Applied Statistics*, *12*(1), 490–509.

Jeong, J., & Jun, M. (2015). A class of Matern-like covariance functions for smooth processes on a sphere. *Spatial Statistics*, *11*, 1–18.

Jones, R. H. (1963). Stochastic processes on a sphere. *Annals of Mathematical Statistics*, *34*, 213–218.

Jost, G., Heuvelink, G., & Papritz, A. (2005). Analysing the space–time distribution of soil water storage of a forest ecosystem using spatio-temporal kriging. *Geoderma*, *128*(3–4), 258–273.

Jun, M. (2011). Non–stationary cross–covariance models for multivariate processes on a globe. *Scandinavian Journal of Statistics*, *38*, 726–747.

Jun, M., & Stein, M. L. (2007). An approach to producing space-time covariance functions on spheres. *Technometrics*, *49*, 468–479.

Jun, M., & Stein, M. L. (2008). Nonstationary covariance models for global data. *The Annals of Applied Statistics*, *2*(4), 1271–1289.

Kabluchko, Z. (2009). Extremes of space–time Gaussian processes. *Stochastic Processes and their Applications*, *119*(11), 3962–3980.

Kang, E. L., Cressie, N., & Shi, T. (2010). Using temporal variability to improve spatial mapping with application to satellite data. *The Canadian Journal of Statistics*, *38*(2), 271–289.

Katzfuss, M. and Guinness, J. (2017). A general framework for Veccia approximations of Gaussian processes. *arXiv preprint arXiv: 1708.06302*.

Kaufman, C. G., Schervish, M. J., & Nychka, D. W. (2008). Covariance tapering for likelihood-based estimation in large spatial data sets. *Journal of the American Statistical Association*, *103*(484), 1545–1555.

Kent, J. T., Mohammadzadeh, M., & Mosammam, A. M. (2011). The dimple in Gneiting's spatial-temporal covariance model. *Biometrika*, *98*, 489–494.

Kleiber, W., & Nychka, D. (2012). Nonstationary modeling for multivariate spatial processes. *Journal of Multivariate Analysis*, *112*, 76–91.

Kolovos, G., Christakos, G., Hrisopoulos, D., & Serre, M. (2004). Methods for generating non-separable spatiotemporal covariance models with potential environmental applications. *Advances in Water Resources*, *27*, 815–830.

Kraichnan, R. H. (1964). Diagonalizing approximation for inhomogeneous turbulence. *The Physics of Fluids*, *7*(8), 1169–1177.

Krainski, E. T., Gómez-Rubio, V., Bakka, H., Lenzi, A., Castro-Camilio, D., Simpson, D., … Rue, H. (2018). *Advanced spatial modeling with stochastic partial differential equations using R and INLA*. Boca Raton, FL: CRC Press Github version. Retrieved from www.r-inla.org/spde-book

Kühl, N., Gebhardt, C., Litt, T., & Hense, A. (2002). Probability density functions as botanical-climatological transfer functions for climate reconstruction. *Quaternary Research*, *58*(3), 381–392.

Kyriakidis, P. C., & Journel, A. G. (1999). Geostatistical space–time models: A review. *Mathematical Geology*, *31*(6), 651–684.

Lang, A., & Schwab, C. (2013). Isotropic random fields on the sphere: Regularity, fast simulation and stochastic partial differential equations. *The Annals of Applied Probability*, *25*, 3047–3094.

Legarra, A., Misztal, I., & Bertrand, J. (2004). Constructing covariance functions for random regression models for growth in gelbvieh beef cattle. *Journal of Animal Science*, *82*(6), 1564–1571.

Leonenko, N., & Sakhno, L. (2012). On spectral representation of tensor random fields on the sphere. *Stochastic Analysis and Applications*, *31*, 167–182.

Li, B., Genton, M. G., & Sherman, M. (2007). A nonparametric assessment of properties of space–time covariance functions. *Journal of the American Statistical Association*, *102*(478), 736–744.

Lindgren, F., Rue, H., & Lindstroem, J. (2011). An explicit link between Gaussian fields and Gaussian Markov random fields: The stochastic partial differential equation approach. *Journal of the Royal Statistical Society, Series B (Statistical Methodology)*, *73*(4), 423–498.

Lopes, H. F., Salazar, E., & Gamerman, D. (2008). Spatial dynamic factor analysis. *Bayesian Analysis*, *3*(4), 759–792.

Lu, N., & Zimmerman, D. L. (2005). The likelihood ratio test for a separable covariance matrix. *Statistics & Probability Letters*, *73*(4), 449–457.

Malyarenko, A. (2013). *Invariant random fields on spaces with a group action*. New York, NY: Springer.

Mardia, K. V., & Marshall, R. J. (1984). Maximum likelihood estimation of models for residual covariance in spatial regression. *Biometrika*, *71*(1), 135–146.

Mastrantonio, G., Jona Lasinio, G., Pollice, A., Capotorti, G., Teodonio, L., Genova, G., & Blasi, C. (2019). A hierarchical multivariate spatio-temporal model for clustered climate data with annual cycles. *The Annals of Applied Statistics*, *13*(2), 797–823.

Meiring, W., Monestiez, P., Sampson, P., & Guttorp, P. (1997). Developments in the modelling of nonstationary spatial covariance structure from space-time monitoring data. *Geostatistics Wollongong*, *96*(1), 162–173.

Menegatto, A., Oliveira, C., and Porcu, E. (2019, Submitted). Gneiting class, semi-metric spaces, and isometric embeddings.

Menegatto, V. A. (1994). Strictly positive definite kernels on the Hilbert sphere. *Applied Analysis*, *55*, 91–101.

Menegatto, V. A. (1995). Strictly positive definite kernels on the circle. *The Rocky Mountain Journal of Mathematics*, *25*, 1149–1163.

Menegatto, V. A., Oliveira, C. P., & Peron, A. P. (2006). Strictly positive definite kernels on subsets of the complex plane. *Computers & Mathematcs with Applications*, *51*, 1233–1250.

Meyer, K. (1998). *"Dxmrr"—A program to estimate covariance functions for longitudinal data by restricted maximum likelihood*. In Proceedings of the 6th World Congress on Genetics Applied to Livestock Production, Vol. 27. University of New England, Armidale, NSW, Australia. pp. 465–466.

Miller, K. S., & Samko, S. G. (2001). Completely monotonic functions. *Integral Transforms and Special Functions*, *12*(4), 389–402.

Mitchell, M. W., Genton, M. G., & Gumpertz, M. L. (2006). A likelihood ratio test for separability of covariances. *Journal of Multivariate Analysis*, *97*(5), 1025–1043.

Møller, J., Nielsen, M., Porcu, E., & Rubak, E. (2018). Determinantal point process models on the sphere. *Bernoulli*, *24*(2), 1171–1201.

Monin, A., & Yaglom, A. (1967). *Statistical hydrodynamics (II)*. Moscow, Russia: Science (Nauka) Press.

Myers, D., De Iaco, S., Posa, D., & De Cesare, L. (2002). Space-time radial basis functions. *Computers & Mathematics with Applications*, *43*(3–5), 539–549.

Nguyen, H., Katzfuss, M., Cressie, N., & Braverman, A. (2014). Spatio-temporal data fusion for very large remote sensing datasets. *Technometrics*, *56*(2), 174–185.

Nychka, D., Wikle, C., & Royle, J. A. (2002). Multiresolution models for nonstationary spatial covariance functions. *Statistical Modelling*, *2*(4), 315–331.

Paciorek, C. J., & Schervish, M. J. (2006). Spatial modelling using a new class of nonstationary covariance functions. *Environmetrics*, *17*(5), 483–506.

Padoan, S. A., Ribatet, M., & Sisson, S. A. (2010). Likelihood-based inference for max-stable processes. *Journal of the American Statistical Association*, *105*(489), 263–277.

Pauli, F., Racugno, W., & Ventura, L. (2011). Bayesian composite marginal likelihoods. *Statistica Sinica*, *21*, 149–164.

Peron, A., Porcu, E., & Emery, X. (2018). Admissible nested covariance models over spheres cross time. *Stochastic Environmental Research and Risk Assessment*, *32*(11), 3053–3066.

Pintore, A., & Holmes, C. (2004). Non-stationary covariance functions via spatially adaptive spectra. *Technical Report*.

Porcu, E., Alegría, A., & Furrer, R. (2018). Modeling spatially global and temporally evolving data. *International Statistical Review*, *86*, 344–377.

Porcu, E., Bevilacqua, M., & Genton, M. G. (2016). Spatio-temporal covariance and cross-covariance functions of the great circle distance on a sphere. *Journal of the American Statistical Association*, *11*, 888–898.

Porcu, E., Bevilacqua, M., & Genton, M. G. (2020). Nonseparable, space-time covariance functions with dynamical compact supports. *Statistica Sinica*.

Porcu, E., Castruccio, S., Alegria, A., & Crippa, P. (2019). Axially symmetric models for global data: A journey between geostatistics and stochastic generators. *Environmetrics*, *30*(1), e2555.

Porcu, E., Cleanthous, G., Georgiadis, A., White, P., & Alegría, A. (2019, Submitted). Random fields on the hypertorus: Covariance modeling, regularities, and approximations. Technical Report.

Porcu, E., Gregori, P., & Mateu, J. (2006). Nonseparable stationary anisotropic space–time covariance functions. *Stochastic Environmental Research and Risk Assessment*, *21*(2), 113–122.

Porcu, E., Gregori, P., & Mateu, J. (2009). Archimedean spectral densities for nonstationary space-time geostatistics. *Statistica Sinica*, *19*, 273–286.

Porcu, E., & Mateu, J. (2007). Mixture-based modeling for space-time data. *Environmetrics*, *18*, 285–302.

Porcu, E., Mateu, J., & Bevilacqua, M. (2007). Covariance functions which are stationary or nonstationary in space and stationary in time. *Statistica Neerlandica*, *61*(3), 358–382.

Porcu, E., Mateu, J., & Christakos, G. (2010). Quasi-arithmetic means of covariance functions with potential applications to space-time data. *Journal of Multivariate Analysis*, *100*(8), 1830–1844.

Porcu, E., Montero, J.-M., & Schlather, M. (2012). *Advances and challenges in space-time modelling of natural events* (Vol. 207). Berlin: Springer.

Porcu, E., & Schilling, R. (2011). From Schoenberg to pick-Nevanlinna: Towards a complete picture of the Variogram class. *Bernoulli*, *17*(1), 441–455.

Porcu, E., & Zastavnyi, V. (2011). Characterization theorems for some classes of covariance functions associated to vector valued random fields. *Journal of Multivariate Analysis*, *102*(9), 1293–1301.

Ranalli, M., Lagona, F., Picone, M., & Zambianchi, E. (2018). Segmentation of sea current fields by cylindrical hidden markov models: A composite likelihood approach. *Journal of the Royal Statistical Society, Series C (Applied Statistics)*, *67*(3), 575–598.

Reich, B. J., Eidsvik, J., Guindani, M., Nail, A. J., & Schmidt, A. M. (2011). A class of covariate-dependent spatiotemporal covariance functions. *The Annals of Applied Statistics*, *5*(4), 2265–2687.

Ribatet, M., Cooley, D., & Davison, A. C. (2012). Bayesian inference from composite likelihoods, with an application to spatial extremes. *Statistica Sinica*, *22*, 813–845.

Rodrigues, A., & Diggle, P. (2010). A class of convolution-based models for Spatio-temporal processes with non-separable covariance structure. *Scandinavian Journal of Statistics*, *37*, 553–567.

Rodriguez-Iturbe, I., Cox, D. R., & Isham, V. (1987). Some models for rainfall based on stochastic point processes. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, *410*(1839), 269–288.

Rouhani, S., & Wackernagel, H. (1990). Multivariate geostatistical approach to space-time data analysis. *Water Resources Research*, *26*(4), 585–591.

Sang, H., & Gelfand, A. E. (2009). Hierarchical modeling for extreme values observed over space and time. *Environmental and Ecological Statistics*, *16*(3), 407–426.

Sang, H., & Genton, M. G. (2014). Tapered composite likelihood for spatial max-stable models. *Spatial Statistics*, *8*, 86–103.

Sang, H., Jun, M., & Huang, J. Z. (2011). Covariance approximation for large multivariate spatial data sets with an application to multiple climate model errors. *The Annals of Applied Statistics*, *5*(4), 2519–2548.

Sarkka, S., Solin, A., & Hartikainen, J. (2013). Spatiotemporal learning via infinite-dimensional bayesian filtering and smoothing: A look at Gaussian process regression through Kalman filtering. *IEEE Signal Processing Magazine*, *30*(4), 51–61.

Scaccia, L., & Martin, R. (2002). Testing for simplification in spatial models. In *Compstat* (pp. 581–586). Berlin: Springer.

Scaccia, L., & Martin, R. (2005). Testing axial symmetry and separability of lattice processes. *Journal of Statistical Planning and Inference*, *131*(1), 19–39.

Scaccia, L., & Martin, R. (2011). Model-based tests for simplification of lattice processes. *Journal of Statistical Computation and Simulation*, *81*(1), 89–107.

Schilling, R., Song, R., & Vondracek, Z. (2012). *Bernstein Functions: Theory and Applications*. Berlin: De Gruyter.

Schlather, M. (2010). Some covariance models based on normal scale mixtures. *Bernoulli*, *16*(3), 780–797.

Schoenberg, I. J. (1942). Positive definite functions on spheres. *Duke Mathematical Journal*, *9*, 96–108.

Shaby, B., & Ruppert, D. (2012). Tapered covariance: Bayesian estimation and asymptotics. *Journal of Computational and Graphical Statistics*, *21*(2), 433–452.

Sherman, M. (2011). *Spatial statistics and spatio-temporal data: Covariance functions and directional properties*. Hoboken, NJ: John Wiley & Sons.

Shirota, S., & Gelfand, A. (2017). Space and circular time log gaussian cox processes with application to crime event data. *The Annals of Applied Statistics*, *11*(2), 481–503.

Shkarofsky, I. (1968). Generalized turbulence space-correlation and wave-number spectrum-function pairs. *Canadian Journal of Physics*, *46*(19), 2133–2153.

Sideris, I., Gabella, M., Erdin, R., & Germann, U. (2014). Real-time radar–rain-gauge merging using spatio-temporal co-kriging with external drift in the alpine terrain of Switzerland. *Quarterly Journal of the Royal Meteorological Society*, *140*(680), 1097–1111.

Spadavecchia, L., & Williams, M. (2009). Can spatio-temporal geostatistical methods improve high resolution regionalisation of meteorological variables? *Agricultural and Forest Meteorology*, *149*(6–7), 1105–1117.

Stein, M. L. (1999). *Statistical interpolation of spatial data: Some theory for Kriging*. New York: Springer.

Stein, M. L. (2005a). Nonstationary spatial covariance functions. Unpublished technical report.

Stein, M. L. (2005b). Space-time covariance functions. *Journal of the American Statistical Association*, *100*(469), 310–321.

Stein, M. L. (2005c). Statistical methods for regular monitoring data. *Journal of the Royal Statistical Society, Series B (Statistical Methodology)*, *67*, 667–687.

Stein, M. L., Chi, Z., & Welty, L. J. (2004). Approximating likelihoods for large spatial data sets. *Journal of the Royal Statistical Society, Series B (Statistical Methodology)*, *66*(2), 275–296.

Storvik, G., Frigessi, A., & Hirst, D. (2002). Stationary space-time Gaussian fields and their time autoregressive representation. *Statistical Modelling*, *2*(2), 139–161.

Sun, Y., Li, B., & Genton, M. G. (2012). Geostatistics for large datasets. In E. Porcu, J. Montero, & M. Schlather (Eds.), *Advances and challenges in space-time modelling of natural events. Lecture Notes in Statistics* (Vol. 207, pp. 55–77). Berlin: Springer.

Varin, C., Reid, N., & Firth, D. (2011). An overview of composite likelihood methods. *Statistica Sinica*, *21*, 5–42.

Vecchia, A. V. (1988). Estimation and model identification for continuous spatial processes. *Journal of the Royal Statistical Society, Series B (Statistical Methodology)*, *50*(2), 297–312.

Vergara, R. C., Allard, D., and Desassis, N. (2018). A general framework for SPDE-based stationary random fields. *arXiv preprint arXiv: 1806.04999*.

Wackernagel, H. (2003). *Multivariate geostatistics: An introduction with applications* (3rd ed.). New York: Springer.

Wang, F., & Gelfand, A. E. (2014). Modeling space and space-time directional data using projected Gaussian processes. *Journal of the American Statistical Association*, *109*(508), 1565–1580.

Wendland, H. (1995). Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree. *Advances in Computational Mathematics*, *4*, 389–396.

White, P., & Porcu, E. (2019a). Nonseparable covariance models on circles cross time: A study of Mexico City ozone. *Environmetrics*, *30*, e2558.

White, P., & Porcu, E. (2019b). Towards a complete etc. *Electronic Journal of Statistics*, *11*, 111–111.

White, W., & Bernstein, R. (1979). Design of an oceanographic network in the midlatitude north pacific. *Journal of Physical Oceanography*, *9* (3), 592–606.

Wikle, C. K., & Hooten, M. B. (2010). A general science-based framework for dynamical spatio-temporal models. *Test*, *19*(3), 417–451.

Yaglom, A. (1948). *Homogeneous and isotropic turbulence in a viscous compressible fluid*. Moscow, Russia: USSR Academy of Sciences, Geographic and Geophysics Service.

Yang, D., Gu, C., Dong, Z., Jirutitijaroen, P., Chen, N., & Walsh, W. M. (2013). Solar irradiance forecasting using spatial-temporal covariance structures and time-forward kriging. *Renewable Energy*, *60*, 235–245.

Zastavnyi, V. (2002). Positive-definite radial functions and splines. *Doklady Mathematics*, *66*, 213–216.

Zastavnyi, V. P., & Porcu, E. (2011). Characterization theorems for the Gneiting class of space–time covariances. *Bernoulli*, *17*(1), 456–465.

Zhang, Y., Wang, J., & Wang, X. (2014). Review on probabilistic forecasting of wind power generation. *Renewable and Sustainable Energy Reviews*, *32*, 255–270.

**How to cite this article:** Porcu E, Furrer R, Nychka D. 30 Years of space–time covariance functions. *WIREs Comput Stat*. 2020;e1512. https://doi.org/10.1002/wics.1512