**University of
Zurich**^{UZH}

**Zurich Open Repository and
Archive**

University of Zurich
University Library
Strickhofstrasse 39
CH-8057 Zurich
www.zora.uzh.ch

Year: 2017

The Grothendieck–Teichmüller group action on differential forms and formality morphisms of chains

Willwacher, Thomas

Abstract: It is known that one can associate a Kontsevich-type formality morphism to every Drinfeld associator. In the present paper, we show that this morphism may be extended to a Kontsevich–Shoikhet formality morphism of cochains and chains, by describing the action of the Grothendieck–Teichmüller group on such objects (up to homotopy).

DOI: <https://doi.org/10.1515/crelle-2014-0135>

Posted at the Zurich Open Repository and Archive, University of Zurich

ZORA URL: <https://doi.org/10.5167/uzh-207524>

Journal Article

Published Version

Originally published at:

Willwacher, Thomas (2017). The Grothendieck–Teichmüller group action on differential forms and formality morphisms of chains. *Journal für die Reine und Angewandte Mathematik*, 2017(726):249-265.

DOI: <https://doi.org/10.1515/crelle-2014-0135>

The Grothendieck–Teichmüller group action on differential forms and formality morphisms of chains

By *Thomas Willwacher* at Zurich

Abstract. It is known that one can associate a Kontsevich-type formality morphism to every Drinfeld associator. In the present paper, we show that this morphism may be extended to a Kontsevich–Shoikhet formality morphism of cochains and chains, by describing the action of the Grothendieck–Teichmüller group on such objects (up to homotopy).

1. Introduction

Let T_{poly} be the space of multivector fields on $\mathbb{R}^n$ and let D_{poly} be the space of multidifferential operators on $\mathbb{R}^n$. M. Kontsevich’s formality theorem [6] states that there is a Lie_∞ quasi-isomorphism

$$\mathcal{U} : T_{\text{poly}}[1] \rightarrow D_{\text{poly}}[1].$$

Here we understand $T_{\text{poly}}[1]$ as a Lie algebra endowed with the Schouten bracket and $D_{\text{poly}}[1]$ as a Lie algebra endowed with the Gerstenhaber bracket. The differential forms $\Omega_\bullet$ on $\mathbb{R}^n$, with non-positive grading, form a Lie module over $T_{\text{poly}}[1]$. The action of a k -vector field γ on a differential form α is given by the Lie derivative

$$L_\gamma \alpha = d \iota_\gamma \alpha + (-1)^k \iota_\gamma d \alpha$$

where d is the de Rham differential and ι_γ is the operation of contraction with γ . Similarly, the (topological) Hochschild chain complex $C_\bullet = C_\bullet(C^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^n))$ forms a module over the multidifferential operators D_{poly} , see [12]. B. Shoikhet [10] showed that there is a Lie_∞ quasi-isomorphism of modules

$$\mathcal{V} : C_\bullet \rightarrow \Omega_\bullet$$

thus proving an earlier conjecture of B. Tsygan [12]. Here $C_\bullet$ is considered as a Lie_∞ module over $T_{\text{poly}}[1]$ by pulling back the $D_{\text{poly}}[1]$ module structure along the morphism $\mathcal{U}$.

In fact, the “correct” objects to consider are not formality morphisms $\mathcal{U}$ on $\mathbb{R}^n$ for some fixed n , but *stable* formality morphisms in the sense of [2]. The components of such morphisms are expressed by operations in a suitable operad of graphs, which acts on the pair $(T_{\text{poly}}, D_{\text{poly}})$ for any n , so that from a stable formality morphism one obtains a formality morphism for each n . By the results of V. Dolgushev [2] the space of stable formality morphisms up to homotopy is a torsor for the zeroth cohomology of the graph complex fGC (see below for a definition). The latter object may be identified with the Grothendieck–Teichmüller Lie algebra grt_1 , see [15]. It follows that the space of stable formality morphisms (up to homotopy) may be identified with the space of Drinfeld associators. An explicit construction of a formality morphism given a Drinfeld associator has been described by D. Tamarkin earlier [4, 11]. It was shown in [13] that applying one version of D. Tamarkin’s construction to the Alekseev–Torossian associator [1, 9] yields Kontsevich’s formality morphism. Furthermore, all formality morphisms thus obtained may actually be extended to homotopy Gerstenhaber (instead of just Lie_∞) formality morphisms [13].

In this paper, we extend the above picture to formality morphisms of chains and cochains. Concretely, we will show that one can associate a formality morphism of chains and cochains to every Drinfeld associator.

Theorem 1. *For each Drinfeld associator Φ there is a stable formality morphism $\mathcal{U}_\Phi$ in the homotopy class associated to Φ , together with a formality morphism of chains*

$$\mathcal{V}_\Phi : C_\bullet \rightarrow \Omega_\bullet$$

also given by graphical formulas, where the $T_{\text{poly}}[1]$ action on $C_\bullet$ is obtained by pulling back the action of $D_{\text{poly}}[1]$ on $C_\bullet$ along $\mathcal{U}_\Phi$.

To show the theorem it suffices to lift the action of the Grothendieck–Teichmüller group on the homotopy classes of stable formality morphisms of cochains to homotopy classes of stable formality morphisms of cochains and chains. To this end we will consider a version of the graph complex which we call fEGC acting on $(T_{\text{poly}}[1], \Omega_\bullet)$ by Lie_∞ derivations, and hence on (stable) formality morphisms of chains and cochains. More concretely, as a graded Lie algebra

$$\text{fEGC} \cong \text{fGC} \ltimes \text{fGC}_1$$

where fGC is as before and acts on $T_{\text{poly}}[1]$ while the part fGC_1 to be introduced below acts on $\Omega_\bullet$ considered as Lie_∞ module. There is a projection map $\text{fEGC} \rightarrow \text{fGC}$, but a priori it is not clear that, for example, any cocycle in fGC may be extended to one in fEGC . To describe the cohomology of fEGC , we need to introduce the “divergence” operator ∇ on fGC , which is defined on a graph by summing over all ways to add an edge. Alternatively,

$$\nabla = [\text{ } \circlearrowleft \text{ } , \cdot]$$

is the Lie bracket with a tadpole (short-loop) graph. The operator ∇ commutes with the differential and induces a degree -1 operator on $H(\text{fGC})$. Unfortunately, the precise form of this operator on cohomology is unknown, owed to the fact that most of the cohomology of fGC is unknown. The main result of this paper is the computation of the cohomology of fEGC in terms of that of fGC , and the action of ∇ .

Theorem 2. *One has*

$$H(\mathfrak{fEGC}) \cong \mathbb{K}B \oplus \mathbb{K}B \otimes H(\mathfrak{fGC}) \oplus \mathbb{K}\mathbb{1} \oplus H(\mathfrak{fGC} \oplus \mathfrak{fGC}[-2], \nabla)$$

where B and $\mathbb{1}$ are explicitly known cohomology classes described below (see (2.4)), and the ∇ on the right is understood as a degree 1 map from $\mathfrak{fGC}$ to $\mathfrak{fGC}[-2]$. Furthermore, the map

$$H^0(\mathfrak{EGC}) \rightarrow H^0(\mathfrak{GC}) \cong \text{grt}_1$$

is an isomorphism.

We provide an explicit combinatorial formula for the cocycles in $\mathfrak{fEGC}$ corresponding to “divergence free” (see Section 2.6 below) cocycles in $\mathfrak{fGC}$.

Hence the Grothendieck–Teichmüller group action on the homotopy classes of stable formality morphisms of cochains lifts to an action on (stable) formality morphisms of cochains and chains. In particular, we may associate a stable formality morphism of chains and cochains to each Drinfeld associator Φ . Concretely, to the Alekseev–Torossian associator we associate the Kontsevich–Shoikhet morphism. The stable formality morphisms corresponding to another Drinfeld associators Φ' may be recovered by acting with the unique element of the Grothendieck–Teichmüller group sending Φ to Φ' . This shows Theorem 1.

1.1. Structure of the paper. In Section 2, we briefly recall the definition of the graph complex $\mathfrak{fGC}$, and introduce the graph complex $\mathfrak{fEGC}$, which can be extracted from [13]. Section 3 contains the proof of Theorem 2. Finally, in Section 4 we will derive the explicit formula for the cocycles in $\mathfrak{fEGC}$ corresponding to divergence free cocycles in $\mathfrak{fGC}$.

2. Graph complexes and operads

2.1. Definition of the graph complexes. Let $\text{gra}_{N,k}$ be the set of undirected graphs with vertex set $[N] = \{1, \dots, N\}$ and edge set $[k]$, without short cycles (“tadpoles”). It carries an action of the group $S_N \times S_k$ by renumbering the vertices and renumbering the edges.

Fix a field $\mathbb{K}$ of characteristic zero. One may define an operad of graphs Gra such that the space of N -ary operations is

$$\text{Gra}(N) := \prod_{k \geq 0} (\mathbb{K}\langle \text{gra}_{N,k} \rangle \otimes \mathbb{K}[1]^{\otimes k})_{S_k}$$

where S_k acts diagonally on $\text{gra}_{N,k}$ and on the factors of $\mathbb{K}[1]$ by permutations, with appropriate signs. Elements of Gra are series of undirected graphs with edges of degree -1 , for example the following one.

The operad structure is given by inserting one graph into a vertex of another and reconnecting the incoming edges in all possible ways, see the introductory sections of [15].

Analogously, let $\text{dgra}_{N,k}$ be the set of directed graphs with vertex set $[N] = \{1, \dots, N\}$ and edge set $[k]$, and define an operad dGra whose space of N -ary operations is

$$\text{dGra}(N) := \prod_{k \geq 0} (\mathbb{K} \langle \text{dgra}_{N,k} \rangle \otimes \mathbb{K}[1]^{\otimes k})_{S_k}.$$

There is a map of operads

$$\text{Gra} \rightarrow \text{dGra}$$

sending each undirected edge to the sum of the edges in either direction.

Next consider a similar set $\text{dgra}_{1,N,k}$ whose elements are directed graphs with vertex set $N \cup \{in, out\}$ and edge set $[k]$, such that at the vertex out there are only outgoing edges and at the vertex in there are only incoming edges. We define vector spaces

$$\text{Gra}_1(N) := \prod_{k \geq 0} (\mathbb{K} \langle \text{dgra}_{1,N,k} \rangle \otimes \mathbb{K}[1]^{\otimes k})_{S_k}.$$

Elements are linear combinations of directed graphs with two special vertices in and out , for example the following.

The two spaces $\text{dGra}(N)$ and $\text{Gra}_1(N)$ assemble naturally into a 2-colored operad. Via the embedding $\text{Gra} \rightarrow \text{dGra}$ the spaces $\text{Gra}(N)$ and $\text{Gra}_1(N)$ also assemble into a 2-colored operad which we call EGra . Here $\text{Gra}(N)$ is the space of operations with N inputs and the output in color 1, and $\text{Gra}_1(N)$ is the space of operations with N inputs in color 1, one input in color 2, represented by the vertex in , and the output in color 2, represented by the vertex out . The operadic compositions are obtained by inserting graphs at vertices of others and reconnecting the dangling edges in all possible ways, see [13] or the following examples, from which the procedure should be clear.

$$\begin{array}{c} \textcircled{1} \\ \textcircled{2} \end{array} \textcircled{3} \circ_3 \textcircled{1} \textcircled{2} = \sum_{\text{reconnect edges}} \begin{array}{c} \textcircled{1} \\ \textcircled{2} \end{array} \textcircled{3-4} = \begin{array}{c} \textcircled{1} \textcircled{3} \\ \textcircled{2} \textcircled{4} \end{array} + \begin{array}{c} \textcircled{1} \textcircled{3} \\ \textcircled{2} \textcircled{4} \end{array} + \begin{array}{c} \textcircled{1} \textcircled{3} \\ \textcircled{2} \textcircled{4} \end{array} + \begin{array}{c} \textcircled{1} \textcircled{3} \\ \textcircled{2} \textcircled{4} \end{array}$$

$$\begin{array}{c} \text{out} \\ \textcircled{1} \\ \text{in} \end{array} \circ_1 \textcircled{1} \textcircled{2} = \sum_{\text{reconnect edges}} \begin{array}{c} \text{out} \\ \textcircled{1-2} \\ \text{in} \end{array} = \begin{array}{c} \text{out} \\ \textcircled{1} \textcircled{2} \\ \text{in} \end{array} + \begin{array}{c} \text{out} \\ \textcircled{1} \textcircled{2} \\ \text{in} \end{array} + \begin{array}{c} \text{out} \\ \textcircled{1} \textcircled{2} \\ \text{in} \end{array} + \begin{array}{c} \text{out} \\ \textcircled{1} \textcircled{2} \\ \text{in} \end{array} + \begin{array}{c} \text{out} \\ \textcircled{1} \textcircled{2} \\ \text{in} \end{array}$$

$$\begin{array}{c} \text{out} \\ \textcircled{1} \\ \text{in} \end{array} \circ \begin{array}{c} \text{out} \\ \textcircled{1} \\ \text{in} \end{array} = \sum_{\text{reconnect edges}} \begin{array}{c} \text{out} \\ \textcircled{1} \\ \text{in} \end{array} \textcircled{2} = \begin{array}{c} \text{out} \\ \textcircled{1} \textcircled{2} \\ \text{in} \end{array} + \begin{array}{c} \text{out} \\ \textcircled{1} \textcircled{2} \\ \text{in} \end{array} + \begin{array}{c} \text{out} \\ \textcircled{1} \textcircled{2} \\ \text{in} \end{array}$$

The first line illustrates the composition of two elements of Gra . The second line illustrates the composition of an element of Gra_1 and of dGra . Finally, the third line depicts the composition of two elements of Gra_1 . Here one removes vertex *in* from the first graph and vertex *out* from the second and reconnects the edges, those from the top to the bottom and vice versa. Note that out of the four possible reconnections in the example one produces a graph with a double edge between vertices 1 and 2, which hence vanishes by symmetry.

Denote the Lie operad by Lie , its suspension by ΛLie , and the minimal cofibrant resolution thereof by ΛLie_∞ . We denote the two-colored operad that governs a ΛLie algebra and a representation of that ΛLie algebra by ELie , and its minimal cofibrant resolution by ELie_∞ .

Recall from [13] that there is a natural map of colored operads $\text{ELie}_\infty \rightarrow \text{ELie} \rightarrow \text{EGra}$. We define the full graph complex as the operadic deformation complex

$$\text{fEGC} := \text{Def}(\text{ELie}_\infty \rightarrow \text{EGra}) \cong \prod_{N \geq 1} \text{Gra}(N)^{S_N} [2 - 2N] \oplus \prod_{N \geq 0} \text{Gra}_1(N)^{S_N} [-2N].$$

Concretely, elements of fEGC are just series of graphs as occurring in Gra and Gra_1 , invariant under permutations of the symbols $1, 2, \dots$ decorating the vertices. The abstract definition as a deformation complex has the advantage that it is immediate that one has a differential graded (dg) Lie algebra structure on fEGC .

Let us dissect the dg Lie algebra fEGC again into smaller pieces. First it contains a quotient Lie algebra

$$\text{fGC} := \text{Def}(\Lambda\text{Lie}_\infty \rightarrow \text{Gra}) \cong \prod_{N \geq 1} \text{Gra}(N)^{S_N} [2 - 2N],$$

the full graph complex. It consists of series of graphs as in Gra , invariant under permutations of the symbols $1, 2, \dots$ decorating the vertices.

Similarly, there is a sub-dg Lie algebra $\text{fGC}_1 \subset \text{fEGC}$ consisting of series in graphs as in Gra_1 , invariant under renumbering the vertices,

$$\text{fGC}_1 \cong \prod_{N \geq 0} \text{Gra}_1(N)^{S_N} [-2N].$$

One can write $\text{fEGC} = \text{fGC} \ltimes \text{fGC}_1$.

A similar construction can be carried out using the directed graphs operad dGra . One in particular obtains the directed graph complex

$$\text{dfGC} \cong \text{Def}(\Lambda\text{Lie}_\infty \rightarrow \text{dGra}).$$

It can be checked (see [15, Appendix K]) that the natural map

$$\text{fGC} \rightarrow \text{dfGC}$$

is a quasi-isomorphism.

2.2. The bracket and differential on fEGC . For concreteness, let us give pictorial description of the bracket and differential on the graph complex fEGC . First consider the differential. It is a sum of three pieces, the first piece $\delta : \text{fGC} \rightarrow \text{fGC}$ sums over all vertices and splits the vertex into two in all possible ways. Pictorially,

$$\delta \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} = \sum \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \\ \diagdown \quad \diagup \end{array}.$$

The second piece which we also denote by δ maps $\mathbf{fGC}_1$ to $\mathbf{fGC}_1$ as follows.

$$\begin{aligned} \delta \text{ (vertex)} &= \sum \text{ (two configurations)} \\ \delta \text{ (out box)} &= \sum \text{ (two configurations)} \pm \sum \text{ (two configurations)} \\ \delta \text{ (in box)} &= \sum \text{ (two configurations)} \pm \sum \text{ (two configurations)} \end{aligned}$$

Here the bent edges are supposed to be reconnected to some other vertex of the graph. (One sums over all choices.) Finally, there is a third piece of the differential $\delta_1 : \mathbf{fGC} \rightarrow \mathbf{fGC}_1$, which sends an element $\Gamma \in \mathbf{fGC}$ to

$$\delta_1 \Gamma = \text{cloud } \Gamma - \text{cloud } \Gamma.$$

Next consider the Lie bracket on $\mathbf{fEGC} = \mathbf{fGC} \ltimes \mathbf{fGC}_1$. There will be three possible situations to explain: brackets of two elements of $\mathbf{fGC}$, taking values in $\mathbf{fGC}$, brackets of one element of $\mathbf{fGC}$ and one of $\mathbf{fGC}_1$, taking values in $\mathbf{fGC}_1$, and finally brackets of two elements of $\mathbf{fGC}_1$, taking values in $\mathbf{fGC}_1$. The Lie algebra $\mathbf{fGC}$ in fact carries a pre-Lie bracket

$$(2.1) \quad \bullet : \mathbf{fGC} \times \mathbf{fGC} \rightarrow \mathbf{fGC}, \quad (\gamma, \gamma') \mapsto \gamma \bullet \gamma'$$

such that the differential satisfies

$$[\gamma, \gamma'] = \gamma \bullet \gamma' - (-1)^{|\gamma||\gamma'|} \gamma' \bullet \gamma$$

for homogeneous elements $\gamma, \gamma' \in \mathbf{fGC}$. Combinatorially, $\gamma \bullet \gamma'$ may be defined by inserting graph γ' into vertices of γ in all possible ways,

$$\sum_{\substack{v \\ \text{vertex of } \gamma}} \sum_{\text{reconnect edges}} \gamma \text{ with } \gamma' \text{ at } v.$$

The second piece of the Lie bracket we also denote by $\bullet$, abusing notation a bit,

$$(2.2) \quad \bullet : \mathbf{fGC}_1 \times \mathbf{fGC} \rightarrow \mathbf{fGC}_1, \quad (\Gamma, \gamma) \mapsto \Gamma \bullet \gamma = [\Gamma, \gamma].$$

Combinatorially, this piece is obtained by inserting graph γ into the vertices of Γ in all possible

ways, schematically:

Finally, for $\Gamma, \Gamma' \in \text{fGC}_1$ the bracket is again obtained from a pre-Lie (even associative, in fact) product

$$(2.3) \quad \circ : \text{fGC}_1 \times \text{fGC}_1 \rightarrow \text{fGC}_1, \quad (\Gamma, \Gamma') \mapsto \Gamma \circ \Gamma'$$

such that

$$[\Gamma, \Gamma'] = \Gamma \circ \Gamma' - (-1)^{|\Gamma||\Gamma'|} \Gamma' \circ \Gamma.$$

Concretely, the product $\circ$ is induced the composition on Gra_1 , schematically:

Here vertex *in* is removed from Γ and vertex *out* is removed from Γ' and the edges incident at the removed *in* are reconnected to Γ' (including the new vertex *in*), and the edges incident at the removed vertex *out* are reconnected to Γ (including the new vertex *out*).

2.3. Smaller graph complexes. The graph complexes considered above admit disconnected graphs and vertices of all valences. One often restricts to smaller subcomplexes. For example, there are further dg Lie subalgebras

$$\text{GC} \subset \text{fcGC} \subset \text{fGC}$$

where fcGC consists of the connected graphs only and GC consists of connected graphs with all vertices of valence ≥ 3 . Clearly,

$$\text{fGC} \cong S^+(\text{fcGC}[-2])[2]$$

is a completed symmetric product without unit. Furthermore, the cohomology of fcGC may be expressed in terms of that of GC , cf. [15, Proposition 3.4]. Similarly, we may identify sub-dg Lie algebras

$$\text{GC}_1 \subset \text{fcGC}_1 \subset \text{fGC}_1.$$

Here fcGC_1 consists of graphs that are connected after deleting vertices *in* and *out*. Such graphs will be called *internally connected*. The dg Lie algebra GC_1 consists of graphs that are internally connected and all internal vertices, i.e., vertices other than *in* or *out* are at least two-valent. We finally define the dg Lie subalgebra

$$\text{EGC} := \text{GC} \ltimes \text{GC}_1 \subset \text{fEGC}.$$

2.4. Action on multivector fields, on differential forms, and on formality morphisms.

The operad $\mathbf{dGra}$ (and hence $\mathbf{Gra} \subset \mathbf{dGra}$) acts on the space of multi vector fields T_{poly} . Let us pick coordinates $\{x_1, \dots, x_n\}$ on $\mathbb{R}^n$ and abbreviate $\xi_j = \frac{\partial}{\partial x_j}$, so that one can identify

$$T_{\text{poly}} = C^\infty(\mathbb{R}^n)[\xi_1, \dots, \xi_n].$$

Then a graph $\Gamma \in \mathbf{dGra}(N)$ acts on multivector fields $v_1, \dots, v_N \in T_{\text{poly}}$ as follows:

$$\Gamma(v_1, \dots, v_N) := \mu \left(\left(\prod_{\substack{(i,j) \\ \text{edge}}} \sum_{r=1}^n \frac{\partial}{\partial \xi_r^{(i)}} \frac{\partial}{\partial x_r^{(j)}} \right) (v_1 \otimes \dots \otimes v_N) \right).$$

Here the product is over edges of Γ , the operator $\partial/\partial x_r^{(j)}$ acts as $\partial/\partial x_r$ on the j -th factor of the tensor product (and $\partial/\partial \xi_r^{(i)}$ analogously), and μ multiplies all factors in the tensor product.

Similarly, the colored operad $\mathbf{EGra}$ acts on the pair $(T_{\text{poly}}, \Omega_\bullet)$ (the “colored” vector space), with T_{poly} carrying the first color and $\Omega_\bullet$ the second. Concretely, let $\Gamma_1 \in \mathbf{Gra}_1(N)$ be a graph, let $v_1, \dots, v_N, v \in T_{\text{poly}}$ be multivector fields and $\alpha \in \Omega_\bullet$ a differential form. We may assume that $\alpha = f\alpha_0$ with f a function and α_0 constant. The graph Γ_1 may be understood as an element of $\mathbf{dGra}(N+2)$, say Γ'_1 , by considering vertex *out* as the first, *in* as the last vertex and the others according to their numbering, increased by one. Then the action is defined such that

$$\iota_v \Gamma_1(v_1, \dots, v_N, \alpha) = (-1)^{|\Gamma| |v|} \iota_{\Gamma'_1(v, v_1, \dots, v_N, f)} \alpha_0$$

where on the right-hand side we use the formula for the action of $\Gamma'_1 \in \mathbf{dGra}(N+2)$ on multivector fields from above.

The $\Lambda\mathbf{Lie}$ -algebra structure on T_{poly} factors through the $\mathbf{Gra}$ -algebra structure. In other words, the operad map from $\Lambda\mathbf{Lie}$ into the endomorphism operad $\mathbf{End}_{T_{\text{poly}}}$ factors as

$$\Lambda\mathbf{Lie} \rightarrow \mathbf{Gra} \rightarrow \mathbf{End}_{T_{\text{poly}}}.$$

It follows that degree zero cocycles in the dg Lie algebra $\mathbf{fGC}$, and in particular in the Lie subalgebra $\mathbf{GC}$, act on $T_{\text{poly}}[1]$ by L_∞ derivations.

Similarly, the $\mathbf{ELie}$ algebra structure on $T_{\text{poly}} \oplus \Omega_\bullet$ factors through $\mathbf{EGra}$,

$$\mathbf{ELie} \rightarrow \mathbf{EGra} \rightarrow \mathbf{End}_{T_{\text{poly}} \oplus \Omega_\bullet}.$$

Again, the degree zero cocycles in $\mathbf{fEGC}$ act on $(T_{\text{poly}}[1], \Omega_\bullet)$ by derivations of L_∞ algebras and modules. Concretely, let $x + x_1$ be a closed degree zero element of $\mathbf{fEGC} = \mathbf{fGC} \ltimes \mathbf{fGC}_1$, with $x \in \mathbf{fGC}$, $x_1 \in \mathbf{fGC}_1$. Then x acts, as before, by a $\mathbf{Lie}_\infty$ derivation on $T_{\text{poly}}[1]$. However, x does not necessarily respect the $\mathbf{Lie}_\infty$ module structure on $\Omega_\bullet$, but changes the $\mathbf{Lie}_\infty$ module structure to an “infinitesimally different” one. The element x_1 corrects for this defect by providing an “infinitesimal $\mathbf{Lie}_\infty$ morphism of modules” between $\Omega_\bullet$ and $\Omega_\bullet$ with the modified module structure.

Of course, this action $(T_{\text{poly}}[1], \Omega_\bullet)$ by L_∞ derivations yields an action on the set of formality morphisms of chains and cochains

$$(T_{\text{poly}}[1], \Omega_\bullet) \rightarrow (D_{\text{poly}}[1], C_\bullet)$$

by precomposition. It was shown in [15] that $H^0(\mathbf{GC}) \cong \mathbf{grt}_1$, and hence one recovers the action of the Grothendieck–Teichmüller group on formality morphisms of cochains. The main technical contribution of this work is to extend this action to the formality morphisms on chains $C_\bullet \rightarrow \Omega_\bullet$ by showing that $H^0(\mathbf{fEGC}) \cong \mathbf{grt}_1$.

2.5. Some cocycles in $\mathbf{fGC}_1$. Let us describe some cocycles in $\mathbf{fGC}_1$. The simplest two are the following:

$$(2.4) \quad \mathbb{1} = \begin{array}{c} \boxed{\text{out}} \\ | \\ \boxed{\text{in}} \end{array}, \quad B = \begin{array}{c} \boxed{\text{out}} \\ \downarrow \\ \boxed{\text{in}} \end{array}.$$

Furthermore, let us note that there is a map of complexes $\mathbf{dfGC} \rightarrow \mathbf{fGC}_1$ mapping a graph Γ to $I \bullet \Gamma$ where I is the graph with one internal vertex and no edges, i.e.,

$$I = \begin{array}{c} \boxed{\text{out}} \\ \bullet \\ \boxed{\text{in}} \end{array}$$

and the $\bullet$ denotes insertion of Γ in place of the black vertex. Note that we also have a map $\mathbf{fGC} \rightarrow \mathbf{fGC}_1$ by composition of the previous map with the embedding $\mathbf{fGC} \rightarrow \mathbf{dfGC}$.

2.6. The divergence operation. The graphs occurring in the operads $\mathbf{Gra}$ and $\mathbf{dGra}$ were not allowed to contain tadpoles, i.e., edges connecting a vertex to itself. In fact, we might have allowed them as well, to obtain operads we denote by $\mathbf{Gra}^\circ$ and $\mathbf{dGra}^\circ$, and graph complexes $\mathbf{fGC}^\circ$ and $\mathbf{dfGC}^\circ$. These graph complexes contain their tadpole-free relatives as subcomplexes. It is shown in [15, Proposition 3.4] that the graph complexes with tadpoles are quasi-isomorphic to those without tadpoles except possible for the occurrence of the cohomology class represented by the graph

In fact, one may check that the Lie bracket of this graph with itself vanishes, and hence one may define an additional differential of degree -1 on the graph complex $\mathbf{fGC}^\circ$ as

$$\nabla = \left[\begin{array}{c} \text{tadpole} \\ \bullet \end{array}, \cdot \right].$$

We call this operator the divergence operator. It acts on a graph by adding an additional edge in all possible ways. In fact, the action of ∇ leaves invariant the subspace $\mathbf{fGC} \subset \mathbf{fGC}^\circ$, and hence descends to an operator on that space, that we will also denote by ∇ .

In general, it is not known what the induced action of ∇ on the cohomology $H(\mathbf{fGC})$ is. However, since $H^{<0}(\mathbf{fGC}) = 0$ (see [15]), the subspace $H^0(\mathbf{fGC}) \cong \mathbf{grt}_1$ is sent to 0 by degree reasons. It follows in particular that the graph cohomology classes corresponding to $\mathbf{grt}_1$ -elements may be represented by divergence-free cocycles, cf. [7].

Remark 1. Note that in [8] explicit integral formulas for divergence-free graph cocycles corresponding to all Deligne–Drinfeld elements $\sigma_3, \sigma_5, \dots \in \mathbf{grt}_1$ are given.

Below we will need to consider the divergence free sub-dg Lie algebra $\mathbf{fGC}_{\text{div}} \subset \mathbf{fGC}$ and similarly $\mathbf{dfGC}_{\text{div}} \subset \mathbf{dfGC}$ spanned by the elements of $\mathbf{fGC}$ (respectively of $\mathbf{dfGC}$) closed under ∇ , i.e.,

$$\mathbf{fGC}_{\text{div}} := \{\gamma \in \mathbf{fGC} : \nabla \gamma = 0\}.$$

2.7. The operad $\mathbf{dfGraphs}$. We need one more ingredient from the theory of graphical operads, namely a variant of the operad $\mathbf{dGra}$ introduced by M. Kontsevich in [5]. Concretely, consider the operad $\mathbf{dfGraphs} := \mathbf{Tw\,dGra}$ obtained from $\mathbf{dGra}$ by operadic twisting (see [3]). Elements of $\mathbf{Tw\,dGra}(N)$ are series of directed graphs with N numbered “external” vertices $1, \dots, N$ and an arbitrary number of unlabelled “internal” vertices. An example (for $N = 1$) can be found in the following picture.

There is a suboperad $\mathbf{dGraphs} \subset \mathbf{Tw\,dGra}$ given by restricting to graphs which do not have connected components with only internal vertices. For example, the graph above would not have been admissible. One can check (see [15]) that $H(\mathbf{dGraphs}) \cong e_2$ is the Gerstenhaber operad.

Similarly, we may consider an operad $\mathbf{dfGraphs}^\heartsuit$, defined in the exactly the same manner, except that the graphs occurring may have tadpoles, i.e., edges connecting a vertex to itself. This operad contains an operadic ideal $I \subset \mathbf{dfGraphs}^\heartsuit$ whose elements are series of graphs that contain a tadpole at an internal vertex. We may form the quotient

$$\mathbf{fBVGraphs} := \mathbf{dfGraphs}^\heartsuit / I.$$

In other words, one sets graphs with tadpoles at internal vertices to zero, while tadpoles at external vertices are still admissible. The cohomology of the operad $\mathbf{fBVGraphs}$ may be computed, modulo the graph cohomology.

Proposition 1. *One has*

$$H(\mathbf{fBVGraphs}) \cong \mathbf{BV} \otimes S(H(\mathbf{fcGC})[-2])$$

where $\mathbf{BV}$ is the Batalin–Vilkovisky operad and $S(H(\mathbf{fcGC})[-2])$ denotes the completed symmetric product.

Concretely, the elements of $\mathbf{BV}$ are represented by graphs without internal vertices. The factor $S(H(\mathbf{fcGC})[-2])$ corresponds to additional connected components fully consisting of internal vertices that may be present.

Proof sketch. The graphs contributing to $\mathbf{fBVGraphs}$ are the same as those contributing to $\mathbf{dfGraphs}$, except that there may be one or more tadpoles at some external vertices. One checks that these extra tadpoles are not “seen” by the differential. Hence the cohomology of $\mathbf{fBVGraphs}$ is the same as that of $\mathbf{dfGraphs}$, with the possible addition of tadpoles at the external vertices. $\square$

Remark 2. The operad $\mathbf{fBVGraphs}$ contains a sub-operad $\mathbf{BVGraphs} \subset \mathbf{fBVGraphs}$ formed by graphs which do not have connected components with only internal vertices, that is quasi-isomorphic to the Batalin–Vilkovisky operad.

An important fact about the operad $\mathbf{dfGraphs}$ is that it is acted upon by the dg Lie algebra $\mathbf{dfGC}$ by operadic derivations. This follows directly from the formalism of operadic twisting, see [3].¹⁾ Let us briefly recall how to obtain the action, leaving the discussion of sign and combinatorial prefactor subtleties to [3]. First, there is a right action of $\mathbf{dfGC}$ on $\mathbf{dfGraphs}$ (which is not compatible with the differential)

$$(2.5) \quad \Gamma \bullet \gamma := \sum_v \Gamma \bullet_v \gamma$$

where $\gamma \in \mathbf{dfGC}$, $\Gamma \in \mathbf{dfGraphs}$, the sum is over all internal vertices of Γ , and the notation $\bullet_v$ shall indicate that one inserts γ in place of vertex v and reconnects the edges incident to v in all possible ways to vertices of γ .

Similarly, there is a map of vector spaces $\mathbf{dfGC} \rightarrow \mathbf{dfGraphs}(1)$, sending $\gamma \in \mathbf{dfGC}$ to an element γ_1 obtained by summing over all vertices of γ and declaring the vertex to be external. In any operad $\mathcal{P}$ the unary operations $\mathcal{P}(1)$ form an algebra that acts on the operad by derivations. We denote this action symbolically by

$$x \cdot y = x \circ y \pm y \circ x.$$

Hence we obtain another action of $\mathbf{dfGC}$ on $\mathbf{dfGraphs}$ by operadic derivations, which is not compatible with the differentials. However, it turns out that the sum of the two actions considered above respects the differentials, and yields the desired action.

We note that this action does not readily descend to an action of $\mathbf{dfGC}$ on $\mathbf{fBVGraphs}$, the reason being that the right action $\bullet$ does not respect the operadic ideal I considered above. Inserting at a vertex with a tadpole might remove the tadpole. However, the action descends to an action of the divergence free part $\mathbf{dfGC}_{\text{div}} \subset \mathbf{dfGC}$.

3. Proof of Theorem 2

3.1. An auxiliary map. The key step to the proof of Theorem 2 will be the following result, which allows us to identify $\mathbf{fGC}_1$ with a complex whose cohomology we can compute.

Proposition 2. *There is an isomorphism of complexes*

$$F : \mathbf{fGC}_1 \rightarrow \mathbf{fBVGraphs}(1)$$

such that:

- (1) *F is compatible with the operadic compositions, i.e.,*

$$F(\Gamma_1 \circ \Gamma'_1) = F(\Gamma_1) \circ_1 F(\Gamma'_1)$$

for all $\Gamma_1, \Gamma'_1 \in \mathbf{fGC}_1$ where “ $\circ$ ” is as in (2.3) and “ $\circ_1$ ” denotes the operadic composition.

- (2) *F is compatible with the right $\mathbf{dfGC}_{\text{div}}$ action, i.e.,*

$$F(\Gamma_1 \bullet \Gamma) = F(\Gamma_1) \bullet \Gamma$$

for all $\Gamma_1 \in \mathbf{fGC}_1$ and $\Gamma \in \mathbf{dfGC}_{\text{div}}$ where $\bullet$ on the left is as in (2.2) and $\bullet$ on the right is the right action (2.5).

- (3) *The image of the element $L \in \mathbf{fGC}_1$ under F is the element $F(L)$ depicted in Figure 1.*

¹⁾ In fact, $\mathbf{dfGC}$ can be identified with the zero-ary operations in $\mathbf{dfGraphs}$, and the action extends the adjoint action of $\mathbf{dfGC}$ on itself, cf. Section 2.2.

Figure 1. The element $L \in \mathbf{fGC}_1$ (left) and its image in $\mathbf{fBVGraphs}(1)$ under the map F of Proposition 2 (right).

The proof will occupy the remainder of this subsection. Let Γ_1 be a graph in $\mathbf{fGC}_1$. We define

$$F(\Gamma_1) = (-1)^{n_{out}} \sum_{\Gamma} \Gamma \in \mathbf{fBVGraphs}(1)$$

where n_{out} is the valence of out in $\mathbf{fGC}_1$ and the sum runs over all graphs obtained from Γ_1 by (i) removing vertex out , (ii) reconnecting the edges previously incident to out in some way to vertices in Γ_1 and (iii) renaming vertex in to vertex 1.

Example 1. To give a concrete example:

Note that graphs with double edges that occur in the sum are zero and are not shown here. Similarly, by definition of $\mathbf{fBVGraphs}$ graphs with tadpoles at internal vertices are zero.

We have to show the assertions made in Proposition 2. First, we claim that F is an isomorphism of graded vector spaces. To see this filter $\mathbf{fGC}_1$ by the valence of out and filter $\mathbf{fBVGraphs}(1)$ by the number of outgoing edges at the external vertex 1. It is easy to see that F is compatible with these filtrations. Consider the associated graded $\mathbf{gr} F$. It acts on a graph $\Gamma_1 \in \mathbf{fGC}_1$ (up to sign) by (i) connecting vertices in and out and (ii) renaming the newly formed vertex 1. This is clearly an isomorphism, the inverse map just splits vertex 1 appropriately into in and out .

Remark 3. From this description one can also easily find an explicit formula for the inverse F^{-1} of F . We leave it to the reader. One may use that for an invertible linear map $M = D + N$ with N nilpotent $M^{-1} = D^{-1} - D^{-1}ND^{-1} + D^{-1}ND^{-1}ND^{-1} - \dots$.

Next, statement (3) of the proposition is an easy explicit computation. It is also easy to convince oneself that statement (2) is correct. (One has to require that $\nabla \Gamma = 0$, as otherwise the action on $\mathbf{fBVGraphs}(1)$ is not well-defined.) The most difficult assertion is statement (1).

Fix graphs $\Gamma_1, \Gamma'_1 \in \text{fGC}_1$. We want to show that

$$F(\Gamma_1 \circ \Gamma'_1) = F(\Gamma_1) \circ_1 F(\Gamma'_1).$$

Let us depict Γ_1 and Γ'_1 schematically as

where the thick arrows shall stand for (possibly) multiple arrows connecting vertices of Γ_1, Γ'_1 to *in* and *out*. A general term (graph) in $\Gamma_1 \circ \Gamma'_1$ can be depicted as follows.

Here some subset I of the edges incident at *in* on Γ_1 is connected to *in*, while the remainder J of the edges incident at *in* on Γ_1 is connected to Γ'_1 in some way. Similarly, some subset I' of the edges incident at *out* on Γ'_1 is connected to *out*, while the remainder J' of the edges incident at *out* on Γ'_1 is connected to Γ_1 in some way. Next consider $F(\Gamma_1 \circ \Gamma'_1)$. A general term may be depicted as follows.

Edges K previously connected to *out* of Γ_1 are connected either to vertices of Γ_1, Γ'_1 or to vertex 1. We split accordingly $K = K_1 \sqcup K_2 \sqcup K_3$. Similarly, the subset I' of edges as before is further split into $I' = I'_1 \sqcup I'_2 \sqcup I'_3$, with edges in I'_1 being connected to vertices of Γ_1 , while

edges in I'_2 are connected to vertices of Γ'_1 and edges in I'_3 are connected to vertex 1. Note that the overall sign of such terms is

$$(-1)^{|K|+|I'|} = (-1)^{|K|+|I'_1|+|I'_2|+|I'_3|}.$$

It follows that all terms for which $J' \sqcup I'_1 \neq \emptyset$ cancel out. This is because the same edge may participate in either J' or I'_1 , with opposite signs. Hence we are left with terms for which $I' = J'_1 = \emptyset$. But these terms are exactly those appearing in $F(\Gamma_1) \circ_1 F(\Gamma'_1)$. Note that in particular that all edges in $F(\Gamma_1) \circ_1 F(\Gamma'_1)$ between a vertex in Γ_1 and a vertex in Γ'_1 originate from edges in Γ_1 (and not from edges in Γ'_1).

We have thus shown that F is an isomorphism of graded vector spaces and that assertions (1)–(3) of Proposition 2 hold. It remains to be shown that F is an isomorphism of complexes, i.e., that it is compatible with the differential. For $\Gamma_1 \in \mathbf{fGC}_1$ we want to show that

$$\delta F(\Gamma_1) = F(\delta \Gamma_1).$$

Unraveling the formulas for the differentials this can be seen to be equivalent to

$$\begin{aligned} & (-1)^{|\Gamma_1|} F(\Gamma_1) \bullet \mu + (-1)^{|\Gamma_1|} F(\Gamma_1) \circ_1 \mu_1 + \mu_1 \circ_1 F(\Gamma_1) \\ &= F((-1)^{|\Gamma_1|} \Gamma_1 \bullet \mu + (-1)^{|\Gamma_1|} \Gamma_1 \circ L + L \circ \Gamma_1) \end{aligned}$$

where

$$\mu = \bullet \rightarrow \bullet, \quad \mu_1 = \textcircled{1} \rightarrow \bullet + \bullet \rightarrow \textcircled{1}.$$

By using the second assertion of Proposition 2 (that we already showed) and that $\nabla \mu = 0$, we see that the first terms on both sides are the same. By using the first assertion of the proposition it follows that

$$F(\Gamma_1 \circ L) = F(\Gamma_1) \circ F(L).$$

The element $F(L)$ is depicted in Figure 1. Hence

$$F(\Gamma_1 \circ L) = F(\Gamma_1) \circ_1 F(L) = F(\Gamma_1) \circ \mu.$$

Next compute

$$F(L \circ \Gamma_1) = F(L) \circ_1 F(\Gamma_1) = \mu \circ_1 F(\Gamma_1).$$

Hence the above equation holds and we have shown Proposition 2. $\square$

Remark 4. The motivation behind the definition of F is the following. Let T_{poly} be the multivector fields on $\mathbb{R}^n$, and let $\Omega_\bullet$ be the differential forms. These spaces are isomorphic, up to degree shift, with the isomorphism given by sending $\gamma \in T_{\text{poly}}$ to the differential form $\iota_\gamma \omega$ where $\omega = dx_1 \dots dx_n$ is the standard volume form. The isomorphism $T_{\text{poly}} \rightarrow \Omega_\bullet[-n]$ induces an isomorphism between the algebras of endomorphisms $\text{End}(T_{\text{poly}}) \cong \text{End}(\Omega_\bullet)$. Now $\mathbf{fGC}_1$ may be viewed as a graphical version of $\text{End}(\Omega_\bullet)$, while $\mathbf{dfGraphs}(1)$ can be seen as a graphical version of $\text{End}(T_{\text{poly}})$. The map F defined above is just the graphical version of the identification $\text{End}(T_{\text{poly}}) \cong \text{End}(\Omega_\bullet)$.

3.2. Remainder of the proof of Theorem 2. By Proposition 2 we know that

$$H(\mathbf{fGC}_1) \cong H(\mathbf{fBVGraphs}(1)).$$

The right-hand side is known by Proposition 1 to be

$$H(\mathbf{fBVGraphs}(1)) \cong (\mathbb{K}\mathbb{1} \oplus \mathbb{K}D) \otimes S(H(\mathbf{fGC})[-2])$$

where $\mathbb{1}$ and D correspond to the graphs

$$\mathbb{1} = \textcircled{1}, \quad D = \textcircled{1} \text{ with a loop on the vertex.}$$

Note that $\mathbb{1}$ and D are the image of the elements $\mathbb{1} \in \mathbf{fGC}_1$ and $B \in \mathbf{fGC}_1$ (see (2.4)) under the map F .

Now we can compute $H(\mathbf{fEGC})$. Note that as graded vector spaces

$$\mathbf{fEGC} \cong \mathbf{fGC} \oplus \mathbf{fGC}_1.$$

The parts of the differential are as follows.

$$\begin{array}{ccc} \delta & & \delta \\ \curvearrowright & & \curvearrowright \\ \mathbf{fGC} & \xrightarrow{\delta_1 = L \bullet} & \mathbf{fGC}_1 \end{array}$$

We may consider a very simple spectral convergent sequence whose first page sees only the differentials δ , whose second page sees δ_1 and for which all higher differentials vanish. Taking the cohomology with respect to δ we obtain

$$H(\mathbf{fGC}) \oplus \mathbb{K}\mathbb{1} \oplus \mathbb{K}B \oplus \mathbb{K}\mathbb{1} \otimes H(\mathbf{fGC})[-2] \oplus \mathbb{K}B \otimes H(\mathbf{fGC})[-2]$$

by Propositions 2 and 1. Next, applying δ_1 to a given graph cocycle $\Gamma \in \mathbf{fGC}$ we obtain the linear combination

To determine its cohomology class in $H(\mathbf{fGC}_1)$ let us apply the map F of Proposition 2. We obtain

The first two terms together are exact. The last term determines a possibly nontrivial cohomology class, determined by the divergence $\nabla \Gamma$ of Γ . Hence the first part of Theorem 2 follows. To check that indeed $H^0(\mathbf{fEGC}) \cong \mathbf{grt}_1$, note that $H^{<0}(\mathbf{fGC}) = 0$ by [15], and hence the divergence operator ∇ has to vanish on $H^0(\mathbf{fGC})$. The result then follows since except for $H^0(\mathbf{fGC})$ no other term contributes to the zero-th cohomology of $\mathbf{fEGC}$.

4. The explicit formula for divergence free cocycles

In this section, we describe explicitly the cocycles in fEGC corresponding to divergence free cocycles in fGC. Concretely, we will describe a map of dg Lie algebras

$$\Psi : \text{fGC}_{\text{div}} \rightarrow \text{fEGC}.$$

In particular, if one picks divergence free representatives of the elements in $\text{grt}_1 \cong H^0(\text{fGC})$, one obtains an explicit formula for the corresponding cocycles in fEGC.

For a element $X \in \text{fGC}_{\text{div}}$, let $X_1 \in \text{dfGraphs}(1)$ be the element obtained by declaring one vertex (say the first) external. Then we define the map

$$\Psi : \text{fGC}_{\text{div}} \rightarrow \text{fEGC}, \quad X \mapsto X + F^{-1}(X_1).$$

We claim that the map is a map of dg Lie algebras. To check compatibility with the differential, we have to verify that

$$F^{-1}((\delta X)_1) = L \bullet X + \delta F^{-1}(X_1).$$

Since F is an isomorphism, we may as well apply F to the both sides and check that

$$(\delta X)_1 = F(L \bullet X + \delta F^{-1}(X_1)) = F(L \bullet X) + \delta X_1.$$

Now consider the first term on the right. Since X is divergence free we have by Proposition 2 that

$$F(L \bullet X) = F(L) \bullet X = \mu_1 \bullet X.$$

But it is not hard to check by a small graphical calculation that

$$\mu_1 \bullet X + \delta X_1 = (\delta X)_1$$

and hence compatibility with the differential follows.

Next let us consider compatibility with the Lie bracket. Let Y be another closed element in fGC_{div} and let again Y_1 be the element in $\text{dfGraphs}(1)$ obtained by declaring one vertex external. We assume X and Y are homogeneous of degrees $|X|$ and $|Y|$. Compute

$$\begin{aligned} [X + F^{-1}(X_1), Y + F^{-1}(Y_1)] &= [X, Y] + F^{-1}(X_1) \bullet Y - (-1)^{|X||Y|} F^{-1}(Y_1) \bullet X \\ &\quad + F^{-1}(X_1) \circ F^{-1}(Y_1) \\ &\quad - (-1)^{|X||Y|} F^{-1}(Y_1) \circ F^{-1}(X_1). \end{aligned}$$

Apply F to the part in fGC_1 and use Proposition 2 again to compute

$$\begin{aligned} F(\dots) &= F(F^{-1}(X_1) \bullet Y) - (-1)^{|X||Y|} F(F^{-1}(Y_1) \bullet X) + F(F^{-1}(X_1) \circ F^{-1}(Y_1)) \\ &\quad - (-1)^{|X||Y|} F(F^{-1}(Y_1) \circ F^{-1}(X_1)) \\ &= X_1 \bullet Y - (-1)^{|X||Y|} Y_1 \bullet X + X_1 \circ Y_1 - (-1)^{|X||Y|} Y_1 \circ X_1. \end{aligned}$$

But this is $([X, Y])_1$ and we are done.

Remark 5. Note that the images of Ψ commute with the element B of equation (2.4). This means in particular that the derivation of $\Omega_\bullet$ corresponding to elements of grt_1 is compatible with the de Rham differential. Hence the main result of [14] of the compatibility of the Shoikhet morphism with the de Rham differential may be extended to all formality morphisms obtained by the fGC_{div} action.

References

- [1] *A. Alekseev and C. Torossian*, Kontsevich deformation quantization and flat connections, *Comm. Math. Phys.* **300** (2010), no. 1, 47–64.
- [2] *V. Dolgushev*, Stable formality quasi-isomorphisms for Hochschild cochains I, preprint 2011, <http://arxiv.org/abs/1109.6031>.
- [3] *V. Dolgushev and T. Willwacher*, Operadic twisting – with an application to Deligne’s conjecture, *J. Pure Appl. Algebra* **219** (2015), no. 5, 1349–1428.
- [4] *V. Hinich*, Tamarkin’s proof of Kontsevich formality theorem, *Forum Math.* **15** (2003), no. 4, 591–614.
- [5] *M. Kontsevich*, Operads and motives in deformation quantization, *Lett. Math. Phys.* **48** (1999), no. 1, 35–72.
- [6] *M. Kontsevich*, Deformation quantization of Poisson manifolds, *Lett. Math. Phys.* **66** (2003), no. 3, 157–216.
- [7] *S. Merkulov and T. Willwacher*, Grothendieck–Teichmüller and Batalin–Vilkovisky, *Lett. Math. Phys.* **104** (2014), no. 5, 625–634.
- [8] *C. Rossi and T. Willwacher*, P. Etingof’s conjecture about Drinfeld associators, preprint 2014, <http://arxiv.org/abs/1404.2047>.
- [9] *P. Ševera and T. Willwacher*, Equivalence of formalities of the little discs operad, *Duke Math. J.* **160** (2011), no. 1, 175–206.
- [10] *B. Shoikhet*, A proof of the Tsygan formality conjecture for chains, *Adv. Math.* **179** (2003), no. 1, 7–37.
- [11] *D. Tamarkin*, Another proof of M. Kontsevich formality theorem, preprint 1998, <http://arxiv.org/abs/math/9803025>.
- [12] *B. Tsygan*, Formality conjectures for chains, in: *Differential topology, infinite-dimensional Lie algebras, and applications*, Amer. Math. Soc. Transl. Ser. 2 **194**, American Mathematical Society, Providence (1999), 261–274.
- [13] *T. Willwacher*, A note on Br-infinity and KS-infinity formality, preprint 2011, <http://arxiv.org/abs/1109.3520>.
- [14] *T. Willwacher*, Formality of cyclic chains, *Int. Math. Res. Not. IMRN* **2011** (2011), no. 17, 3939–3956.
- [15] *T. Willwacher*, M. Kontsevich’s graph complex and the Grothendieck–Teichmüller Lie algebra, *Invent. Math.* (2014), DOI 10.1007/s00222-014-0528-x.

Thomas Willwacher, Institute of Mathematics, University of Zurich,
 Winterthurerstrasse 190, 8057 Zurich, Switzerland
 e-mail: thomas.willwacher@math.uzh.ch

Eingegangen 14. Januar 2014, in revidierter Fassung 25. November 2014