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Limited liability and the demand for coinsurance by individuals and corporations*

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Abstract

Within the context of expected utility and in a discrete loss setting, we provide a complete account of the demand for insurance by strictly-risk averse agents and risk-neutral firms when they enjoy limited liability. When exposed to a bankrupting, binary loss and under actuarially fair prices, individuals and firms will either fully insure or not insure at all. The decision to insure will depend on whether the benefits the insuree derives from insurance after having compensated the damaged party are sufficiently attractive to justify the premium paid. When the loss is nonbinary, even when prices are actuarially fair, any amount of coinsurance can be optimal depending on the nature of the loss.

Keywords: insurance; risk-averse agent; risk-neutral firm; franchise value; limited liability.

JEL classification: D21; D81; G22; G32; G33.

1 Introduction

Liability insurance protects the insuree against the claims from a third party who has suffered a loss for which the insuree is liable. Typical examples include general liability insurance, e.g., commercial general liability and business liability, professional liability insurance, providing coverage against, e.g., negligence or misrepresentation, and workers' compensation and employer liability insurance. When individuals and firms have limited liability, insurance may be the only means to ensure that the damaged party is compensated for the loss. Thus, the purpose of liability insurance is not only to

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protect the insuree, but also to protect the damaged party. As a result, understanding the demand for insurance by individuals and firms who enjoy limited liability is particularly important from a policymaking perspective. Indeed, as pointed out by [Sinn \(1982\)](#), if there is no intrinsic incentive to buy liability insurance, it may be desirable from a societal perspective to make it compulsory.¹ This paper revisits the impact of limited liability on the demand for coinsurance by strictly risk-averse individuals and by risk-neutral corporations. Within an expected utility framework and in a discrete setting, we provide a comprehensive analysis of the problem, thereby extending, unifying and clarifying results found in the literature.

Limited liability and strictly risk-averse individuals

The demand for coinsurance in an expected utility setting is well understood for a strictly risk-averse agent with deterministic initial wealth and an exposure to a random loss. In a competitive market, i.e., in a market where insurance sells at actuarially fair prices, such an agent will always opt for full insurance.² This result is often referred to as Mossin's theorem and goes back to [Mossin \(1968\)](#) and [Smith \(1968\)](#). Mossin's theorem has been extended in several directions ranging from allowing for random initial wealth ([Doherty and Schlesinger \(1983\)](#), [Doherty and Schlesinger \(1985\)](#), [Hong et al. \(2011\)](#)), through considering other types of contract, e.g., deductible insurance ([Schlesinger \(1981\)](#) and [Schlesinger \(2000\)](#)) or upper-limit insurance ([Schlesinger \(2006\)](#)), to making behavioral assumptions beyond the expected-utility paradigm (e.g., [Schlesinger \(1997\)](#)). In this paper we are concerned with a different line of inquiry that was pursued in [Sinn \(1982\)](#), [Shavell \(1986\)](#), [Konrad and Skaperdas \(1993\)](#), and [Gollier et al. \(1997\)](#). In this literature, it is assumed that the agent has limited liability. This is a highly realistic feature since, typically, an individual cannot lose more than the value of what he or she owns, but may very well be exposed to "bankrupting" losses that exceed this value. Such losses may arise, for example, from claims due to damages to the property or health of third parties. The key question is whether agents have an intrinsic incentive to buy liability insurance. As observed in the cited literature, in the presence of a bankrupting loss agents may choose not to insure at all.

[Sinn \(1982\)](#), [Shavell \(1986\)](#), [Konrad and Skaperdas \(1993\)](#) work in a simple setting in which the individual is exposed to a binary loss, i.e., a loss of a fixed size either occurs or not. The main observation is that, even though prices are actuarially fair, the agent may choose not to insure at all. The reason for that is that having the third party absorb the loss may be more attractive than

¹Several papers have investigated the social and economic implications of limited liability and compulsory liability insurance. Early studies on the interaction between liability rules and insurance are [Shavell \(1982\)](#) and [Shavell \(1986\)](#). For further studies on the welfare impact of mandatory liability insurance for individuals and corporations, references include [Keeton and Kwerel \(1984\)](#), [Han and MacMinn \(1990\)](#), [Jost \(1996\)](#), and [Boomhower \(2019\)](#). For a general review of theoretical models and empirical applications related to personal and corporate liability insurance, see [Baker and Siegelman \(2013\)](#).

²If premia are actuarially unfavourable, strictly risk-averse agents will never opt for full insurance and may possibly not insure at all.

paying the premium to insure. However, if the agent chooses to insure, he or she will insure fully. The setting of [Gollier et al. \(1997\)](#) is more general and not limited to insurance.³ In that paper, the main focus is on showing that an agent with limited liability will display greater willingness to accept risk than an agent with unlimited liability. Moreover, they show that if the wealth level is sufficiently low, the agent will choose a maximal risk exposure, i.e., the agent will not insure. [Gollier et al. \(1997\)](#) do not provide any results on whether it is sometimes optimal to have less than full risk exposure or even no risk exposure at all.

In this paper we provide a complete account of the demand for insurance when the loss is binary: Under actuarially fair premia, the agent will insure fully if and only if full insurance increases expected utility. Otherwise, the agent will not insure at all. A similar result holds when premia are actuarially unfavorable but with partial insurance replacing full insurance. The cited literature contains part of these results, which, in particular, show that Mossin's theorem does not hold in the presence of limited liability. For a loss that is not binary, Mossin's theorem fails even more dramatically. As we show, even when prices are actuarially fair, the amount of insurance demanded by a strictly risk-averse agent may be anywhere between no insurance and full insurance and is case dependent. This invalidates the claim in [Sinn \(1982\)](#),⁴ that all results obtained for binary losses remain true when losses are not binary. It also reinforces the need for caution when extrapolating from models with binary losses to models with more general types of losses. In any case, the main implication of the results for the binary loss model, namely that a strictly risk-averse agent may not have an incentive to protect damaged or injured third parties by insuring, remains true and the same policy implications remain valid.

Limited liability and risk-neutral firms

A good case can be made for assuming that publicly-held corporations are risk neutral because their owners can effectively diversify idiosyncratic risk; see, e.g., [Mayers and Smith \(1982\)](#). As a result, the firm's owners will maximize firm value. In a setting in which the firm is not exposed to financial market risk, this amounts to maximizing the expected cash flows to shareholders discounted at the risk-free rate. Similarly to [Froot et al. \(1993\)](#), we think of firm value as being decomposed into two components. While *net asset value* corresponds to the value of the firm assuming it stopped operating at the end of the period, *franchise value* corresponds to the value associated with the firm's ability to exploit future business opportunities in the subsequent periods.⁵ We note that, in the corporate finance literature, [Myers \(1977\)](#) was among the first to explicitly recognize the value of future investment opportunities as the market value of the firm over and above the value of assets

³[Gollier et al. \(1997\)](#) address the problem from an optimal "risk taking" rather than an optimal "insurance" perspective. Moreover, translated into our setting, their proofs are given for losses whose distributions admit a continuous density function.

⁴See footnote 5 in [Sinn \(1982\)](#).

⁵As in [Froot et al. \(1993\)](#) and [Froot and Stein \(1998\)](#), summarizing future expected profits in the end-of-period franchise value is a way to capture multi-periodicity in a single-period model.

in place. Publicly-held firms always have limited liability, i.e., the firm's owners have right to walk away from any obligations that exceed the value of the firm's assets. As in the case of individual agents, the firm may lose more than the value of the assets it owns when it is exposed to liability claims resulting from injuries or losses caused by the products or services they sell; see, e.g., [Han and MacMinn \(1990\)](#). The value of limited liability can be made explicit by decomposing net asset value into *default-free capital*, i.e., the value of end-of-period capital assuming unlimited liability, and *default option value*, i.e., the value of the claims on the firm that need not be settled due to limited liability. Hence, we have

$$\text{Firm Value} = \text{Default-Free Capital} + \text{Default Option Value} + \text{Franchise Value}.$$

To exploit its franchise, the firm has two sources of financing. It can use whatever capital it has generated internally by the end of the period, but it can also raise external capital. In line with the literature on optimal financial contracting in the presence of asymmetric information, see, e.g., [Townsend \(1979\)](#), [Gale and Hellwig \(1985\)](#), [Bolton and Scharfstein \(1990\)](#), we assume that the firm has private information about its future prospects that investors do not have. In such a setting, [Gale and Hellwig \(1985\)](#) show that the optimal way for the firm to raise capital is through the issuance of a standard debt contract.

The crucial question is: Assuming the availability of insurance for the liability related losses at actuarially fair prices, will the firm purchase insurance and, if so, how much? The risk neutrality of the firm suggests that the firm should be indifferent between buying and not buying insurance. However, there are several factors that may induce a risk-neutral firm to behave *as if* it were risk averse. For example, as pointed out in [Mayers and Smith \(1982\)](#), the presence of frictions can make carrying risk costly and the firm may choose to insure to avoid incurring these costs. These frictions include taxation, costs of financial distress, agency costs, and regulatory costs; see [Mayers and Smith \(1982\)](#) for an overview.⁶ When the access to external capital is costless, a firm is not constrained in exploiting future business opportunities. However, when there is a cost to raising external capital, the firm may choose to insure so as to reduce its reliance on costly external financing. This has been highlighted, for instance, in [Froot et al. \(1993\)](#) whose main focus is on discussing the benefits of hedging to protect the franchise value of the firm. While insurance may be beneficial to the firm either to avoid frictional costs or to protect its franchise, insurance also decreases the value of the option to default. As a result, retaining risk can have a positive impact on firm value; see, e.g., [MacMinn \(1987\)](#) and [Han and MacMinn \(1990\)](#). Consequently, whether or not the firm chooses to buy insurance will ultimately depend on tradeoffs that need to be made.

⁶Examples of papers that focus on specific frictions are [Mayers and Smith Jr \(1987\)](#), [MacMinn \(1987\)](#), [Han and MacMinn \(1990\)](#), [Garven and MacMinn \(1993\)](#), [Grace and Rebello \(1993\)](#), [Garven and MacMinn \(2013\)](#), [Mayers and Smith Jr \(1987\)](#), [MacMinn \(1987\)](#), [Han and MacMinn \(1990\)](#), [Garven and MacMinn \(1993\)](#), [Grace and Rebello \(1993\)](#), and [Garven and MacMinn \(2013\)](#).

A critical tradeoff is that between increasing the value of the default option and protecting the firm's franchise value in an environment where raising capital is costly. The relevance of this tradeoff for value-maximizing firms has been recognized and investigated extensively in other streams of literature. In the banking literature, [Marcus \(1984\)](#) and [Keeley \(1990\)](#) showed that in banks with a high franchise value,⁷ the incentive to seek financial market risk induced by deposit insurance⁸ was typically not exploited to the full because of the potential adverse effects on franchise value. In the context of insurance firms, [Staking and Babbel \(1995\)](#) and [Babbel and Merrill \(2005\)](#) argue that insurance firms with low franchise values, i.e., with poor prospects of attracting profitable new business, have an incentive to take excessive risk, whereas firms with a high franchise value to protect will tend to behave in a more risk-averse manner.

In this paper we provide a thorough analysis of how the “default option - franchise value” tradeoff drives the demand for insurance of a risk-neutral limited-liability firm. To the best of our knowledge, this tradeoff has not been systematically considered in the literature on the corporate demand for insurance. Several papers, e.g., [MacMinn \(1987\)](#), [Mayers and Smith Jr \(1987\)](#), [Han and MacMinn \(1990\)](#), and [Garven and MacMinn \(2013\)](#), have discussed the effect of limited liability, but have not considered the interplay with the firm's franchise value. The main reference for a discussion of the impact of risk on franchise value is [Froot et al. \(1993\)](#), who, however, ignore the impact on the default option. Hence, our paper complements the existent literature by addressing this tradeoff in an explicit and rigorous manner.

Underlying the entire literature on why otherwise risk-neutral corporations may seek to reduce risk is the following key insight: Taking into account the various frictions and the franchise value in a suitable manner, the firms “objective function” is formally equivalent to the expected utility of a strictly risk-averse agent. Hence, the behaviour of the firm is “observationally equivalent” to the behaviour of a strictly risk-averse agent. In the absence of limited liability, this means that the firm's demand for insurance is fully described by the classical version of Mossin's theorem for a strictly risk-averse individual. In the case of limited liability, one has to apply the corresponding results for strictly risk-averse agents with limited liability. Though the corporate demand for insurance can thus be reduced to the problem of the demand for insurance by a strictly risk-averse individual, it is worthwhile to provide explicit results for firms. This is because the additional structure of the problem allows to make plain the tradeoffs that are at work behind the decision to insure or not.

Structure of the paper

The paper is organized as follows. In Section 2, we present a brief account of the binary case that reproduces the results discussed above from a slightly different perspective. We show that, in this case, a version of Mossin's theorem under limited liability still holds. In Section 3, we highlight the

⁷In the banking literature, franchise value is usually referred to as charter value.

⁸Which, in that context, plays a similar role to the firm's default option.

tradeoffs involved in a limited-liability firm's demand for insurance. Finally, in Section 4, we show that a limited-liability version of Mossin's theorem is not possible beyond the case of binary losses. Section 5 concludes.

2 The optimal insurance problem for a strictly risk-averse agent

In this section we provide a complete account of the demand for insurance by a strictly risk-averse agent with limited liability in the case of a binary loss and in the context of expected utility. The agent's utility function $u : \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be strictly increasing and strictly concave.⁹ Without loss of generality we assume that $u(0) = 0$. All random variables will be assumed to be defined on a nonatomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$.¹⁰ The expected value of a random variable X is denoted by $\mathbb{E}[X]$. For every scalar $x \in \mathbb{R}$ we define $x_+ := \max\{x, 0\}$ and $x_- := \max\{-x, 0\}$. We start by recalling Mossin's theorem in the form that we need it.

2.1 Mossin's theorem

An agent with initial wealth $w > 0$ may suffer a random loss L . The agent can alter his or her terminal wealth by purchasing a coinsurance contract that we identify with the contractual level of coinsurance $\alpha \in [0, 1]$. The contractual indemnity for contract α is therefore given by αL and the associated premium equals

$$\pi_\lambda(\alpha) := (1 + \lambda)\alpha\mathbb{E}[L],$$

where $\lambda \geq 0$ is the insurer's profit loading. The case where $\lambda = 0$, the "actuarially fair" case, corresponds to the case in which insurance markets are perfectly competitive and the insurer has zero expected profit. After purchasing insurance, the agent's terminal wealth is

$$W_\lambda(\alpha) := w - L - \pi_\lambda(\alpha) + \alpha L.$$

The agent's expected utility is given by

$$U_\lambda(\alpha) := \mathbb{E}[u(W_\lambda(\alpha))].$$

An optimal coinsurance level is a contract $\alpha^* \in [0, 1]$ that maximizes expected utility, i.e., such that

$$U_\lambda(\alpha^*) = \max_{\alpha \in [0, 1]} U_\lambda(\alpha). \tag{1}$$

⁹We do not require that u is differentiable, but, being strictly concave, we know that u must be continuous and that the one-sided derivatives exist at every point.

¹⁰If $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic, we find for every $p \in [0, 1]$ an event $A \in \mathcal{F}$ such that $\mathbb{P}(A) = p$. We will need this flexibility when constructing a special random loss in Section 4.

By our assumptions on u , the function U_λ is strictly concave. Hence, problem (1) admits a unique optimal insurance level which we denote by $\alpha_\lambda^* \in [0, 1]$. The following result goes back to Mossin (1968) and Smith (1968), and describes the qualitative behaviour of the optimal insurance level in dependence of the profit loading.

Theorem 2.1 (Mossin). *There exists a critical insurance profit loading $\bar{\lambda} > 0$ such that:*

$$(i) \quad \alpha_\lambda^* = 1 \text{ for } \lambda = 0.$$

$$(ii) \quad 0 < \alpha_\lambda^* < 1 \text{ for } 0 < \lambda < \bar{\lambda}.$$

$$(iii) \quad \alpha_\lambda^* = 0 \text{ for } \lambda \geq \bar{\lambda}.$$

Moreover, $\{\alpha_\lambda^* ; 0 \leq \lambda \leq \bar{\lambda}\} = [0, 1]$.

2.2 Liability insurance with a binary loss

For the rest of this section we consider a binary loss L , i.e.

$$L = \begin{cases} \ell & \text{with probability } p, \\ 0 & \text{with probability } 1 - p \end{cases}$$

for some $\ell > 0$ and $0 < p < 1$. The premium for the contract α is then given by

$$\pi_\lambda(\alpha) = (1 + \lambda)\alpha p \ell,$$

and the agent's terminal wealth by

$$W_\lambda(\alpha) := \begin{cases} w - \pi_\lambda(\alpha) & \text{in case of no loss,} \\ w - \pi_\lambda(\alpha) - (1 - \alpha)\ell & \text{in case of loss.} \end{cases}$$

We assume that the loss arises from a claim for compensation against damage to a third party, so that α represents a liability insurance contract. Since the agent cannot lose more than his or her initial wealth, it is natural to assume limited liability, i.e., the agent will compensate the damaged party only up to the amount w . As a result, from the agent's perspective, the “relevant” wealth is not $W_\lambda(\alpha)$, which represents wealth “before limited liability”, but wealth “after limited liability”

$$W_\lambda^+(\alpha) := [W_\lambda(\alpha)]_+ = \max\{W_\lambda(\alpha), 0\}.$$

The agent's expected utility “after limited liability” is then given by

$$V_\lambda(\alpha) := \mathbb{E}[u(W_\lambda^+(\alpha))].$$

In this case, an optimal level of insurance is any $\alpha^* \in [0, 1]$ satisfying

$$V_\lambda(\alpha^*) = \max_{\alpha \in [0, 1]} V_\lambda(\alpha). \quad (2)$$

Since V_λ is continuous and $[0, 1]$ is compact, the above maximization problem always admits a solution. However, in contrast to U_λ , the objective function V_λ fails to be concave in general because terminal wealth enters as $W_\lambda^+(\alpha)$ and not as $W_\lambda(\alpha)$. As a result, it is unclear whether the optimal solution is unique or not. We are interested in understanding when optimal insurance levels are unique, when partial or full insurance are optimal, and, in particular, when full insurance is not optimal. As pointed out in the introduction, knowing when full insurance is not optimal is important because the loss primarily affects the damaged party rather than the prospective policyholder. As a result, if agents do not have an incentive to protect the damaged party, compulsory insurance may be desirable from a societal perspective.

2.3 Optimal coinsurance level

A first obvious requirement for purchasing insurance to make sense at all is that the premium paid does not exceed the compensation received in the loss state, i.e. that $\pi_\lambda(\alpha) < \alpha\ell$ for some, and hence for every, $\alpha > 0$. This is easily seen to be equivalent to

$$0 \leq \lambda < \frac{1-p}{p}. \quad (3)$$

To analyze the maximization problem (2) it is useful to distinguish *bankrupting* losses ($\ell > w$) from *nonbankrupting* losses ($\ell \leq w$). When the loss is nonbankrupting and (3) holds, wealth after limited liability always coincides with wealth before limited liability, i.e., $W_\lambda^+(\alpha) = W_\lambda(\alpha)$ for every $\alpha \in [0, 1]$. As a result, for nonbankrupting losses, the problems with and without limited liability coincide and the solution to the maximization problem (2) is already fully described by Theorem 2.1. We summarize the preceding discussion in the following proposition.

Proposition 2.2. *If ℓ is nonbankrupting, then $W_\lambda^+(\alpha) = W_\lambda(\alpha)$ for every $\alpha \in [0, 1]$ and the optimization problems (2) and (1) coincide. In particular, the optimal level of coinsurance is fully described by Theorem 2.1.*

Assuming that the loss is bankrupting, we first note that if wealth *before* limited liability in the loss state is negative for some contract $\alpha \in (0, 1]$, then the agent will not want to buy the contract because it is “ineffective”. Indeed, wealth in the no-loss state would be strictly reduced by the premium and, in the loss state, any benefit obtained from insurance would be used to pay the claimants, so that wealth *after* limited liability would remain 0. Furthermore, if wealth before limited liability in the loss state is negative for every possible contract $\alpha \in (0, 1]$, then the agent will

never contemplate buying insurance. It is easy to see that this is the case if and only if $w \leq \pi_\lambda(1)$ or, equivalently, $\lambda \geq \frac{w-p\ell}{p\ell}$. Thus, we immediately obtain the following result.

Proposition 2.3. *If ℓ is bankrupting and $\lambda \geq \frac{w-p\ell}{p\ell}$, then $\alpha = 0$ is the unique optimal coinsurance level.*

In light of Proposition 2.3, henceforth we may focus on bankrupting losses satisfying

$$0 \leq \lambda \leq \frac{w-p\ell}{p\ell}, \quad (4)$$

where, for convenience, we allow for $\lambda = \frac{w-p\ell}{p\ell}$ even though for this λ insurance is not attractive. Note that, since the loss is bankrupting, this condition implies that for every contract the premium is strictly less than the compensation in case of loss, i.e., condition (4) implies condition (3).

Under condition (4), we define the *effectiveness threshold* by

$$\tilde{\alpha}_\lambda := \min\{\alpha \in [0, 1] ; W_\lambda(\alpha) \geq 0\} = \frac{\ell - w}{\ell - (1 + \lambda)p\ell} \in (0, 1).$$

This threshold corresponds to the lowest coinsurance level such that wealth before limited liability in the loss state is positive. From the discussion above we infer that contracts in the interval $(0, \tilde{\alpha}_\lambda]$ are ineffective and thus undesirable. Hence, the optimal contract on the interval $[0, \tilde{\alpha}_\lambda]$ is $\alpha = 0$. By contrast, contracts in the interval $[\tilde{\alpha}_\lambda, 1]$ are effective and we have $W_\lambda^+(\alpha) = W_\lambda(\alpha)$. This implies that the expected utility derived from contracts in that range is the same whether the agent has limited or unlimited liability. In particular, V_λ is strictly concave and, hence, admits a unique maximum β_λ^* on that subinterval. As a result, the optimal solution of the optimization problem (2) under limited liability is either no insurance or coinsurance at the level β_λ^* . Noting that $V_\lambda(0) = V_0(0)$, we can establish which of these contracts is optimal by simply comparing $V_0(0)$ and $V_\lambda(\beta_\lambda^*)$. We record this conclusion in the following proposition.

Proposition 2.4. *If ℓ is bankrupting and $0 \leq \lambda \leq \frac{w-p\ell}{p\ell}$, then:*

- (i) *If $V_0(0) > V_\lambda(\beta_\lambda^*)$, then $\alpha = 0$ is the unique optimal coinsurance level.*
- (ii) *If $V_0(0) = V_\lambda(\beta_\lambda^*)$, then $\alpha = 0$ and $\alpha = \beta_\lambda^*$ are the only optimal coinsurance levels.*
- (iii) *If $V_0(0) < V_\lambda(\beta_\lambda^*)$, then $\alpha = \beta_\lambda^*$ is the unique optimal coinsurance level.*

Remark 2.5. *Proposition 2.4 implies that the optimal level of coinsurance in the limited-liability case is always smaller than that in the classical unlimited-liability case. Indeed, in both cases optimal coinsurance results from maximizing the same objective function, but in the limited liability case over a smaller interval. This point was also made in Gollier et al. (1997), who show that introducing limited liability increases the optimal exposure to risk of risk-averse agents.*

Together with Proposition 2.3 and Proposition 2.4, the following result provides a full characterization of the optimal demand for insurance when the agent has limited liability and is exposed to a binary loss. The theorem below distinguishes cases depending on the attractiveness of full insurance at actuarially fair prices.

Theorem 2.6 (Individual demand with binary losses and limited liability). *If ℓ is bankrupting and $w > \pi_0(1)$, then:*

- (i) *If $V_0(1) < V_0(0)$, then $\alpha = 0$ is the unique optimal coinsurance level for every $\lambda \geq 0$.*
- (ii) *If $V_0(1) = V_0(0)$, then $\alpha = 0$ and $\alpha = 1$ are the only optimal coinsurance levels for $\lambda = 0$ whereas $\alpha = 0$ is the unique optimal coinsurance level for $\lambda > 0$.*
- (iii) *If $V_0(1) > V_0(0)$, then $\alpha = 1$ is the unique optimal coinsurance level for $\lambda = 0$ and there exists a critical insurance profit loading $0 < \bar{\lambda} < \frac{w-p\ell}{p\ell}$ such that:*
 - (a) *$\alpha = \beta_\lambda^* \in (0, 1)$ is the unique optimal coinsurance level for $0 < \lambda < \bar{\lambda}$.*
 - (b) *$\alpha = 0$ and $\alpha = \beta_\lambda^* \in (0, 1)$ are the only optimal coinsurance levels for $\lambda = \bar{\lambda}$.*
 - (c) *$\alpha = 0$ is the unique optimal coinsurance levels for $\lambda > \bar{\lambda}$.*

Remark 2.7. *It is clear how to generalize Theorem 2.6 to the case where the agent has a strictly positive minimum guaranteed wealth as in Sinn (1982). Note that our result encompasses the results in Sinn (1982), Shavell (1986), Konrad and Skaperdas (1993), who show that insurance may not be attractive to a risk-averse decision maker with limited liability even when prices are actuarially fair. However, these references provide neither a complete treatment nor a formal derivation of the agent's optimal demand for coinsurance. In addition, Sinn (1982) claims that his results can be generalized to settings with arbitrary losses. In Section 4 we show that this is, in fact, not possible.*

One of the main points highlighted in the literature on insurance demand under limited liability is the fact that, for small levels of wealth, a strictly risk-averse agent will choose to buy no insurance. Sinn (1982), Shavell (1986), and Konrad and Skaperdas (1993) show this in the case of binary losses and Gollier et al. (1997) for general loss profiles. This type of behavior is often referred to as “betting for resurrection”. We conclude this section by showing that in our setting such a behavior is implied by Theorem 2.6. We do this assuming a competitive insurance market, i.e., under actuarially fair prices. We complement the result by highlighting the impact of loss size on demand for insurance.

Proposition 2.8. *If $\lambda = 0$, then:*

- (i) *There exists a critical wealth level $\tilde{w} > 0$ such that:*
 - (a) *$\alpha = 1$ is the unique optimal coinsurance level if $w > \tilde{w}$.*

- (b) $\alpha = 0$ and $\alpha = 1$ are the only optimal coinsurance levels if $w = \tilde{w}$.
 - (c) $\alpha = 0$ is the unique optimal coinsurance level if $w < \tilde{w}$.
- (ii) There exists a critical loss level $\tilde{\ell} > 0$ such that:
- (a) $\alpha = 1$ is the unique optimal coinsurance level if $\ell < \tilde{\ell}$.
 - (b) $\alpha = 0$ and $\alpha = 1$ are the only optimal coinsurance levels if $\ell = \tilde{\ell}$.
 - (c) $\alpha = 0$ is the unique optimal coinsurance level if $\ell > \tilde{\ell}$.

Figure 1 illustrates part (i) of the preceding proposition by displaying the outcomes for high and low levels of wealth. The model used for both panels differs only in the amount of initial wealth. Panel (a), based on a larger wealth, shows a situation in which full insurance is optimal. In Panel (b), reduced wealth leads to a higher premium making full insurance less attractive than no insurance. Not shown in the figure but clear from continuity considerations is that for some intermediate level of wealth the agent will be indifferent between no insurance and full insurance.

Figure 1: Expected utility $V_\lambda(\alpha)$ as a function of the coinsurance level α for actuarially fair prices ($\lambda = 0$). Panel (a) shows the case in which $V_\lambda(1) > V_\lambda(0)$, whereas Panel (b) shows the case in which $V_\lambda(1) < V_\lambda(0)$. The effectiveness threshold $\tilde{\alpha}_\lambda$ represents the coinsurance level below which no insurance is optimal and above which full insurance is preferred.

3 The optimal insurance problem for a risk-neutral firm

We now investigate the demand for insurance by firms. On one extreme, when a firm is owned by a sole strictly risk-averse individual, the firm will purchase the level of insurance that maximizes

the owner's expected utility, i.e., the problem fits into the setting of the previous section. When a firm is closely held by more than one owner, each with a different utility function, there will, in general, be no unanimity as to the objective of the firm so that the optimal insurance problem is not well posed. On the other extreme, when a firm is publicly traded with a broad ownership base and its profits are not exposed to financial market risk, its shareholders are assumed to hold well diversified portfolios that eliminate any exposure to idiosyncratic risk, so that the firm is considered to behave as if it were risk neutral; see, e.g., [Mayers and Smith \(1982\)](#). This is the case we focus on. As we point out in Remark 3.13, it is well known that when borrowing is costly, the costs of external financing can induce the firm to act in a risk-averse manner in spite of the assumed risk neutrality. In such cases, the insurance problem for a firm can be embedded into the model treated in the previous section.¹¹ Our aim, however, is to make explicit the tradeoff that plays out between the impact of risk on the limited liability option, on the one side, and on the ability to exploit future business opportunities, on the other. To be able to focus on the main message, we deal exclusively with the case of competitive insurance markets, i.e., we assume that insurance is sold at actuarially fair rates.

We continue to work in a one-period model with two dates $t = 0$ and $t = 1$ and consider a risk-neutral firm with initial capital $\kappa > 0$, which it uses to generate a deterministic return $\rho > 0$. For convenience, we assume that the risk-free rate is equal to zero. At date $t = 1$, the firm is exposed to a random loss given for $\ell > 0$ and $0 < p < 1$ by

$$L = \begin{cases} \ell & \text{with probability } p, \\ 0 & \text{with probability } 1 - p. \end{cases}$$

As in the case of a strictly-risk averse individual, we allow for the loss to exceed the value of the firm's assets. Such a possibility arises, for instance, when a firm is exposed to claims resulting from injuries or losses caused by the products or services it sells; see, e.g., [Han and MacMinn \(1990\)](#). We assume that the firm is able to purchase insurance at actuarially fair prices, i.e., the premium for the contract α is

$$\pi(\alpha) := \alpha \mathbb{E}[L] = \alpha p \ell.$$

The firm's end-of-period capital after purchasing the contract α and before accounting for a possible default is given by the random variable

$$W(\alpha) := \begin{cases} \kappa(1 + \rho) - \pi(\alpha) & \text{in case of no loss,} \\ \kappa(1 + \rho) - \pi(\alpha) - (1 - \alpha)\ell & \text{in case of loss.} \end{cases}$$

¹¹Induced risk aversion refers to a behavior akin to those of risk-averse agents observed in otherwise risk-neutral individuals or firms. This phenomenon has been studied extensively within the literature on the determinants of corporate risk management, especially hedging and insurance policies. These studies include, but are not limited to, [Smith and Stulz \(1985\)](#), [Nance et al. \(1993\)](#), [Froot et al. \(1993\)](#), [Froot and Stein \(1998\)](#), and [Purnanandam \(2008\)](#).

In view of limited liability, end-of-period capital after buying insurance is given by

$$W^+(\alpha) := [W(\alpha)]_+ = \max\{W(\alpha), 0\}.$$

As in the preceding section, we distinguish between *bankrupting* losses ($\ell > \kappa(1 + \rho)$) and *non-bankrupting* losses ($\ell \leq \kappa(1 + \rho)$).

3.1 Franchise value and optimal borrowing

In addition to terminal equity, there is another source of value at date $t = 1$ to which shareholders are entitled: the value of the firm's ability to exploit future business opportunities. To finance these future business opportunities, we allow the firm to raise capital at the end of the period. We assume that the firm has private information about its future prospects that investors do not have. We can thus rely on the literature on optimal financial contracting in the presence of asymmetric information; see, e.g., [Townsend \(1979\)](#), [Gale and Hellwig \(1985\)](#), and [Bolton and Scharfstein \(1990\)](#). In such a setting, [Gale and Hellwig \(1985\)](#) have shown that the firm finds it optimal to finance itself by issuing a standard debt contract. Consequently, when beneficial, the firm will raise external capital by accessing the debt market, which we assume to be competitive and, without loss of generality, to offer an interest rate of zero. We also assume that borrowing the amount $e \geq 0$ requires the firm to incur costs $C(e) \geq 0$. As pointed out, e.g., in [Froot et al. \(1993\)](#), to account for financial frictions such as monitoring costs and the costs of potential cash diversion, the cost function C can be assumed to be increasing and convex. We summarize our assumptions on the cost of borrowing, which we also allow to be costless, as follows:

Assumption 3.1. *The function $C : [0, \infty) \rightarrow [0, \infty)$ is either identically zero or twice continuously differentiable, strictly increasing, strictly convex, and satisfies $C(0) = 0$ and $C(e) \rightarrow \infty$ as $e \rightarrow \infty$.*

If, at date $t = 1$, the firm has at its disposal funds amounting to $d \geq 0$, it can exploit investment opportunities with present value $d + F(d)$, i.e., $F(d)$ represents the net present value, or the value added, when investing d . We shall call $F : [0, \infty) \rightarrow \mathbb{R}$ the firm's *growth-opportunities function* at date $t = 1$. We impose some fairly natural conditions on F , which are similar in spirit to those stipulated in [Froot et al. \(1993\)](#):¹²

Assumption 3.2. *The function $F : [0, \infty) \rightarrow \mathbb{R}$ is continuously differentiable, strictly increasing, strictly concave, and $F(0) = 0$. Moreover, we assume there exists $F_\infty > 0$ such that*

$$F(d) \rightarrow F_\infty \quad \text{as } d \rightarrow \infty. \tag{5}$$

¹²Compared to [Froot et al. \(1993\)](#), we impose the additional intuitive condition that the function F exhibits *strongly diminishing returns*, which requires that for sufficiently large levels of capital, the marginal value from an additional unit of funds becomes negligible; see, e.g., [Morrison and Siegel \(1997\)](#) and [Basu and Fernald \(1997\)](#).

This assumption reflects the intuition that the firm cannot create value if no funds are available ($F(0) = 0$), that the net present value of future investments increases with the amount of capital invested (F is strictly increasing), and that there are diminishing returns to new business opportunities as the firm grows in size (F is strictly concave). In fact, we assume strongly diminishing returns, i.e., that for sufficiently large levels of capital, the marginal value from an additional unit of funds becomes negligible ($F(d) \rightarrow F_\infty$ as $d \rightarrow \infty$). Figure 2 shows the qualitative shape of F under Assumption 3.2.

Figure 2: Qualitative shape of the growth-opportunities function F .

When the firm has internal funds amounting to $\kappa \geq 0$ and borrows optimally, the actual value of the firm's growth opportunities at date $t = 1$ is

$$F_C(\kappa) := \sup_{e \geq 0} [F(\kappa + e) - C(e)] . \quad (6)$$

When borrowing is costless, the firm can always fully exploit future growth opportunities by accessing debt markets. In fact, in this situation, it is an immediate consequence of Condition (5) that the value of future growth opportunities is always equal to F_∞ regardless of the amount of internal funds κ that is available at time $t = 1$. We record this as a proposition.

Proposition 3.3 (Growth-Opportunities Function with Costless Borrowing). *If C is identically zero, then $F_C(\kappa) = F_\infty$ for every $\kappa \geq 0$.*

By Assumption (3.1), it is possible to show that when borrowing is costly, there is always an optimal

level of borrowing. To see this, consider the function $H_\kappa : [0, \infty) \rightarrow \mathbb{R}$ defined by

$$H_\kappa(e) := F(\kappa + e) - C(e).$$

This function represents the present value of future growth opportunities at time $t = 1$ if, at that time, the firm has internal funds amounting to κ and chooses to borrow the amount $e \geq 0$. By our assumptions on F and C , H_κ is strictly concave. Moreover, since F is bounded from above, we have

$$\lim_{e \rightarrow \infty} H_\kappa(e) = -\infty.$$

Consequently, for every $\kappa \geq 0$, there exists a unique optimal level of borrowing $e^*(\kappa) \geq 0$ such that

$$H_\kappa(e^*(\kappa)) = \max_{e \geq 0} H_\kappa(e).$$

The following result characterizes when borrowing is affordable for the firm, depending on the firm's initial level of capitalization. In particular, the next lemma shows that a necessary condition for the firm to find it optimal to borrow external funds is that the value of growth opportunities outweighs at the margin the associated increment in borrowing costs when financing is most needed ($\kappa = 0$).

Lemma 3.4 (Optimal Costly Borrowing). *The following statements hold:*

- (i) *If $F'(0) \leq C'(0)$, then $e^*(\kappa) = 0$ for every $\kappa \geq 0$.*
- (ii) *If $F'(0) > C'(0)$, then there exists $\bar{\kappa} > 0$ such that $e^*(\kappa) > 0$ whenever $\kappa < \bar{\kappa}$ and $e^*(\kappa) = 0$ whenever $\kappa \geq \bar{\kappa}$.*
- (iii) *The function $e^* : [0, \infty) \rightarrow \mathbb{R}$ is continuously differentiable and strictly decreasing on $[0, \bar{\kappa}]$.*

The actual value of the firm's growth opportunities at date $t = 1$ introduced in (6) can be equivalently expressed as

$$F_C(\kappa) = F(\kappa + e^*(\kappa)) - C(e^*(\kappa))$$

upon adjusting for optimal external borrowing. We collect some basic properties of F_C .

Proposition 3.5. *The function $F_C : [0, \infty) \rightarrow \mathbb{R}$ is continuously differentiable and strictly concave. Moreover, the following statements hold:*

- (i) *If $F'(0) \leq C'(0)$, then $F_C(\kappa) = F(\kappa)$ for every $\kappa \geq 0$.*
- (ii) *If $F'(0) > C'(0)$, then $F_C(\kappa) > F(\kappa)$ for every $0 \leq \kappa < \bar{\kappa}$ and $F_C(\kappa) = F(\kappa)$ for every $\kappa \geq \bar{\kappa}$.*

3.2 Firm value and its decomposition

After purchasing the contract α , the value of the firm at time $t = 1$ is given by the random variable

$$W^+(\alpha) + F_C(W^+(\alpha)),$$

which captures terminal equity and the value of future growth opportunities assuming the firm borrows optimally. The risk neutrality of the firm implies that firm value at time $t = 0$ is determined by taking the expected value of firm value at time $t = 1$ and discounting it at the risk-free rate, which we have assumed to be 0, i.e. firm value at date $t = 0$ after buying the contract α is

$$V(\alpha) := \mathbb{E}[W^+(\alpha)] + FV(\alpha) \tag{7}$$

where $\mathbb{E}[W^+(\alpha)]$ is the value of terminal equity and

$$FV(\alpha) := \mathbb{E}[F_C(W^+(\alpha))]$$

is the *franchise value*.¹³ Note that we can rewrite terminal equity at date $t = 1$ as

$$W^+(\alpha) = W(\alpha) + W^-(\alpha),$$

where $W(\alpha)$ corresponds to *default-free capital* at time $t = 1$, i.e., capital assuming unlimited liability, which is given as above by

$$W(\alpha) := \kappa(1 + \rho) - L - \pi(\alpha) + \alpha L,$$

and $W^-(\alpha)$ corresponds to the firm's *default option*, i.e., the amount by which the firm defaults at date $t = 1$, which is given by¹⁴

$$W^-(\alpha) := [W(\alpha)]_- = \max\{-W(\alpha), 0\}.$$

¹³Note that terminology used to refer to the present value of future growth opportunities is not uniform across the literature. While the term “franchise value” is often used in the literature on insurance, see, e.g., [Babbel \(1999\)](#), [Hancock et al. \(2001\)](#), [Babbel and Merrill \(2005\)](#), the banking literature typically favors the term “charter value”, see, e.g., [Herring and Vankudre \(1987\)](#), [Keeley \(1990\)](#). In the asset pricing and corporate finance literature the expressions “value of growth opportunities” and “value of the growth option” have become customary, see, e.g., [McDonald and Siegel \(1986\)](#), [Berk et al. \(1999\)](#), [Cooper \(2006\)](#), [Da et al. \(2012\)](#).

¹⁴This is, in fact, a problem in [Froot et al. \(1993\)](#) because their mathematical model is based on normal returns for the risky asset, which means that internally generated capital can be negative in some states of the world. As a consequence, their growth-opportunities function cannot be a concave function of internally generated capital and their use of arguments invoking concavity is unjustified.

It follows that firm value at date $t = 0$ can be expressed as

$$V(\alpha) = DFC(\alpha) + DO(\alpha) + FV(\alpha), \quad (8)$$

where

$$DFC(\alpha) := \mathbb{E}[W(\alpha)] \equiv \kappa(1 + \rho) - p\ell, \quad (9)$$

$$DO(\alpha) := \mathbb{E}[W^-(\alpha)] \quad (10)$$

are the value of *default-free capital* and the value of the *default option*, respectively.

3.3 Optimal coinsurance level

We now study optimal levels of coinsurance, i.e. contracts $\alpha^* \in [0, 1]$ such that

$$V(\alpha^*) = \max_{\alpha \in [0, 1]} V(\alpha). \quad (11)$$

As argued in the previous section, we can always ensure the existence of optimal solutions due to the continuity of V but nothing can be said a priori on their multiplicity, because V is not (strictly) concave.

By (9), default-free capital $DFC(\alpha)$ does not depend on $\alpha \in [0, 1]$. Hence, to understand optimal coinsurance levels it suffices to understand the impact of purchasing a contract $\alpha \in [0, 1]$ on the value of the default option $DO(\alpha)$ and on the franchise value $FV(\alpha)$. We will treat the cases of costless and costly borrowing separately, but the following result on the default option is true in both cases.

Lemma 3.6. *The following statements hold:*

(i) *If ℓ is nonbankrupting, then $DO(\alpha) = 0$ for every $\alpha \in [0, 1]$.*

(ii) *If ℓ is bankrupting and $\kappa(1 + \rho) > p\ell$, then*

$$\tilde{\alpha} := \frac{\ell - \kappa(1 + \rho)}{(1 - p)\ell} \in (0, 1) \quad (12)$$

and DO is strictly decreasing on $[0, \tilde{\alpha}]$ and constant and equal to 0 on $[\tilde{\alpha}, 1]$.

(iii) *If ℓ is bankrupting and $\kappa(1 + \rho) \leq p\ell$, then DO is strictly decreasing on $[0, 1]$.*

Note that, when the loss is bankrupting and $\kappa(1 + \rho) > p\ell$, we have

$$\tilde{\alpha} = \min\{\alpha \in [0, 1]; \kappa(1 + \rho) - \ell - \pi(\alpha) + \alpha\ell \geq 0\}$$

so that, similarly to what we did in Section 2, $\tilde{\alpha}$ corresponds to an *effectiveness threshold*, i.e. it is the threshold beyond which wealth after limited liability becomes strictly positive. Thus, this threshold corresponds to the lowest coinsurance level making the firm default free.

Costless borrowing

When borrowing is costless, it follows from Proposition 3.3 that $FV(\alpha)$ does not depend on $\alpha \in [0, 1]$. Hence, whether or not insurance increases firm value depends only on its impact on the value of the default option. According to Lemma 3.6, when the loss is nonbankrupting, the value of the default option is 0 regardless of the level of insurance that is bought. Hence, the firm is indifferent between buying and not buying any coinsurance contract. If the loss is bankrupting, however, Lemma 3.6 implies that $DO(\alpha) < DO(0)$ for every $\alpha \in (0, 1]$. Hence, firm value strictly decreases when buying any nonzero coinsurance contract and not insuring is the only optimal insurance strategy. We summarize this in the following result, which provides a complete description of corporate demand for insurance under costless borrowing.

Theorem 3.7 (Costless borrowing). *If borrowing is costless ($C \equiv 0$), then:*

- (i) *If ℓ is nonbankrupting, then every $\alpha \in [0, 1]$ is an optimal coinsurance level.*
- (ii) *If ℓ is bankrupting, then $\alpha = 0$ is the unique optimal coinsurance level.*

It follows that, for insurance to be valuable for a risk-neutral firm, borrowing must be costly.

Costly borrowing

When borrowing is costly and the loss is nonbankrupting, then the default option has no value regardless of the level of insurance bought. Hence, whether insurance increases firm value depends only on its impact on the firm's franchise value. The following result shows that when the loss is *nonbankrupting*, franchise value is maximized if full insurance is purchased.

Proposition 3.8. *If ℓ is nonbankrupting, then FV is strictly increasing on $[0, 1]$. In particular, $\alpha = 1$ is the unique optimal level of coinsurance.*

As was the case for the strictly risk-averse individual in Section 2, when the loss is bankrupting, it is disadvantageous to buy any insurance if terminal capital after limited liability remains equal to zero for all coinsurance levels, which is the case if and only if $\kappa(1 + \rho) \leq p\ell$. We record this in the next proposition.

Proposition 3.9. *If ℓ is bankrupting and $\kappa(1 + \rho) \leq p\ell$, then FV is strictly decreasing on $[0, 1]$. In particular, and $\alpha = 0$ is the unique optimal coinsurance level.*

By the preceding proposition, from now on we may focus on bankrupting losses for which

$$\kappa(1 + \rho) > p\ell,$$

in which case the effectiveness threshold $\tilde{\alpha}$ defined in (12) is a well defined number in $(0, 1)$. This threshold is useful to understand the impact of buying insurance on the value of the default option and franchise value. In fact, we already know from Lemma 3.6 that DO is strictly decreasing on $[0, \tilde{\alpha}]$ and constant and equal to 0 on $[\tilde{\alpha}, 1]$. The following result describes the behaviour of the franchise value.

Lemma 3.10. *If borrowing is costly ($C \neq 0$), ℓ is bankrupting, and $\kappa(1 + \rho) > p\ell$, then FV is strictly decreasing on $[0, \tilde{\alpha}]$ and strictly increasing on $[\tilde{\alpha}, 1]$.*

Thus, to determine the optimal coinsurance level, a tradeoff between the loss in the value of the default option and the gain in franchise value needs to be resolved. After purchasing a contract α , the *loss* in the value of the default option is given by

$$DO^{loss}(\alpha) := DO(0) - DO(\alpha),$$

which, by Lemma 3.10, is positive, strictly increasing on $[0, \tilde{\alpha}]$, and equal to $DO(0)$ on $[\tilde{\alpha}, 1]$. On the other hand, the *gain* in franchise value is equal to

$$FV^{gain}(\alpha) := FV(\alpha) - FV(0),$$

which, by Lemma 3.10, is positive, strictly decreasing on $[0, \tilde{\alpha}]$, and strictly increasing on $[\tilde{\alpha}, 1]$. It follows that on $[0, \tilde{\alpha}]$ the default option value and the franchise value both decrease. Hence, it is never optimal to buy a contract in the interval $[0, \tilde{\alpha}]$. On the other hand, $DO^{loss}(\alpha)$ is constant and $FV^{gain}(\alpha)$ is increasing on $[\tilde{\alpha}, 1]$. Hence, it is always optimal to buy a contract in the interval $[\tilde{\alpha}, 1]$. As a consequence, the only potential optimal coinsurance levels are $\alpha = 0$ and $\alpha = 1$, with full insurance being the optimal strategy if and only if

$$FV(1) - FV(0) = FV^{gain}(1) \geq DO^{loss}(1) = DO(0),$$

i.e., if and only if the loss in the value of the default option is offset by a sufficiently large gain in franchise value. The tradeoffs we have just described are illustrated in Figure 3 for the case in which full insurance is optimal and Figure 4 for the case in which no insurance is optimal. The preceding discussion yields the following theorem, which is just a recasting of Theorem 2.6 into the corporate setting and, together with Proposition 3.8 and Proposition 3.9, completes the picture on the corporate demand for insurance by a risk-neutral firm under costly borrowing and actuarially fair prices.

Figure 3: The figure plots the firm value V and changes in the firm value components DO , FV as a function of the insurance level α . Panel (a) shows the firm value for the case in which full insurance is optimal, $V(1) > V(0)$. Panel (b) illustrates the loss in the default option value $DO^{loss}(\alpha)$ and the gain in franchise value $FV^{gain}(\alpha)$ corresponding to the value function in Panel (a). The x-axis shows the effectiveness threshold $\tilde{\alpha}$, below which no insurance is optimal and above which full insurance is preferred.

Theorem 3.11 (Costly borrowing). *If borrowing is costly ($C \neq 0$), ℓ is bankrupting, and $\kappa(1+\rho) > p\ell$, then:*

- (i) *If $FV(1) - FV(0) > DO(0)$, then $\alpha = 1$ is the optimal coinsurance level.*
- (ii) *If $FV(1) - FV(0) = DO(0)$, then $\alpha = 0$ and $\alpha = 1$ are the optimal coinsurance levels.*
- (iii) *If $FV(1) - FV(0) < DO(0)$, then $\alpha = 0$ is the optimal coinsurance level.*

Our previous result can be cast in terms of critical levels for the firm's capitalization and for the size of the loss to which the firm's capital is exposed. As is to be expected, all else being equal, when either the initial capital is too low or similarly the loss size is too large, then the value of the default option will be too high to be offset by the gain in franchise when fully insuring. In the opposite situation, the value of the default option will be too low to be worth protecting by not insuring.

Proposition 3.12. *If borrowing is costly ($C \neq 0$), then:*

- (i) *There exists a critical capital level $\tilde{\kappa} > 0$ such that:*
 - (a) *$\alpha = 1$ is the optimal coinsurance level if $\kappa > \tilde{\kappa}$.*
 - (b) *$\alpha = 0$ and $\alpha = 1$ are the optimal coinsurance levels if $\kappa = \tilde{\kappa}$.*
 - (c) *$\alpha = 0$ is the optimal coinsurance level if $\kappa < \tilde{\kappa}$.*

Figure 4: The figure plots the firm value V and changes in the firm value components DO , FV as a function of the insurance level α . Panel (a) shows the firm value for the case in which no insurance is optimal, $V(1) < V(0)$. Panel (b) illustrates the loss in the default option value $DO^{\text{loss}}(\alpha)$ and the gain in franchise value $FV^{\text{gain}}(\alpha)$ corresponding to the value function in Panel (a). The x-axis shows the effectiveness threshold $\tilde{\alpha}$, below which no insurance is optimal and above which full insurance is preferred.

(ii) There exists a critical loss level $\tilde{\ell} > 0$ such that:

- (a) $\alpha = 1$ is the optimal coinsurance level if $\ell < \tilde{\ell}$.
- (b) $\alpha = 0$ and $\alpha = 1$ are the optimal coinsurance levels if $\ell = \tilde{\ell}$.
- (c) $\alpha = 0$ is the optimal coinsurance level if $\ell > \tilde{\ell}$.

Remark 3.13. Note that the firm value V in Equation (8) can be written equivalently as

$$V(\alpha) = \mathbb{E}[u([\kappa(1 + \rho) - L - \pi(\alpha) + \alpha L]_+)],$$

where $u : \mathbb{R} \rightarrow \mathbb{R}$ is any strictly-increasing and concave function satisfying

$$u(x) = x + F_C(x), \quad x \geq 0.$$

If borrowing is costless, u can be taken to be affine, and the firm's problem is formally identical to the problem of a risk-neutral individual with limited liability who maximizes expected utility. However, when borrowing is costly, u is strictly concave by Assumption (3.2), and the objective function of the risk-neutral firm turns out to be of the same form as the objective function of a strictly risk-averse individual with limited liability. Consequently, all results of the previous section could be applied directly. Although this is a straightforward translation, it is instructive to exploit the additional

structure of the firm to disentangle the impact of insurance on the various components of the firm value to expose the tradeoffs influencing the optimal insurance decision of the firm.

4 Demand for coinsurance when the loss is nonbinary

In the binary loss case, Mossin's theorem fails to hold because even at actuarially fair prices it is possible that an agent will not demand insurance. In this section we show that when losses are nonbinary the failure of Mossin's theorem is even more pronounced. More precisely, we prove that a strictly risk-averse individual with limited liability may demand *any* level of insurance depending on the loss.

The setting is as follows. We consider an agent with utility function $u : \mathbb{R} \rightarrow \mathbb{R}$, which is assumed to be strictly increasing and strictly concave, and is normalized to satisfy $u(0) = 0$. We denote the agent's initial wealth by $w > 0$. Given the loss L to which the agent is exposed, her expected utility after buying the contract with coinsurance level α is given by

$$V^L(\alpha) := \mathbb{E}[u([w - L - \alpha\mathbb{E}[L] + \alpha L]_+)].$$

As before, an optimal insurance level for L is any $\alpha^* \in [0, 1]$ such that

$$V^L(\alpha^*) = \max_{\alpha \in [0, 1]} V^L(\alpha).$$

Our aim is to prove the following result:¹⁵

Theorem 4.1 (Failure of Mossin's Theorem). *For every $\alpha^* \in (0, 1)$ there exists a loss L such that α^* is the unique optimal coinsurance level for L .*

Recall from Theorem 2.6 that, if the loss L is binary, then V^L can attain its maximum only at $\alpha = 0$ or $\alpha = 1$. Hence, the loss L in Theorem 4.1 cannot be binary. To emphasize how special binary losses are, we will show that we can always choose L in Theorem 4.1 to take only three values. The remainder of this section is devoted to constructing such an L .

The structural form of L

Consider a loss L taking the three values. Such a loss can always be described as

$$L = \begin{cases} 0 & \text{with probability } (1 - p_2)(1 - p_1) \\ \ell_1 & \text{with probability } (1 - p_2)p_1 \\ \ell_2 & \text{with probability } p_2 \end{cases}$$

¹⁵Clearly, a corresponding version of Theorem 4.1 also holds for a limited-liability firm.

with $0 < \ell_1 < \ell_2$ and $p_1, p_2 \in (0, 1)$. We fix p_1 and ℓ_1 and view p_2 and ℓ_2 as free parameters. Clearly, at least ℓ_2 must be bankrupting since, otherwise, full insurance would be optimal by Theorem 2.1. We assume that only ℓ_2 is bankrupting and that initial wealth is sufficiently large to buy full insurance. More explicitly, we assume that:

$$(A1) \quad 0 < \ell_1 < w < \ell_2.$$

$$(A2) \quad w > \mathbb{E}[L].$$

Similarly to Section 2, we define the *effectiveness threshold* as

$$\tilde{\alpha} := \inf\{\alpha \in [0, 1] ; w - \ell_2 - \alpha\mathbb{E}[L] + \alpha\ell_2 \geq 0\} = \frac{\ell_2 - w}{\ell_2 - \mathbb{E}[L]} \in (0, 1).$$

The idea is to maximize V^L on $[0, \tilde{\alpha}]$ and $[\tilde{\alpha}, 1]$ separately and obtain the solution to the global maximization over $[0, 1]$ by simply comparing the respective solutions to these partial maximization problems.

Maximizing V^L over $[\tilde{\alpha}, 1]$

As a first simple observation, note that V^L coincides with the expected utility of the agent with unlimited liability on the interval $[\tilde{\alpha}, 1]$. Hence, we know from Theorem 2.1 that the maximum of V^L over $[\tilde{\alpha}, 1]$ is attained at $\alpha = 1$.

Maximizing V^L over $[0, \tilde{\alpha}]$

Denote by $\tilde{L}$ a binary loss with loss size ℓ_1 and loss probability p_1 . For a contract $\alpha \in [0, \tilde{\alpha}]$, wealth is always positive when $L = 0$ or $L = \ell_1$, but negative when $L = \ell_2$. Hence, as is easily seen,

$$V^L(\alpha) = (1 - p_2)\mathbb{E}[u(w - \tilde{L} - \alpha\mathbb{E}[L] + \alpha\tilde{L})]. \quad (13)$$

Note that

$$\mathbb{E}[L] = (1 - p_2)\mathbb{E}[\tilde{L}] + p_2\ell_2 = (1 + \lambda(p_2, \ell_2))\mathbb{E}[\tilde{L}], \quad (14)$$

where we have set

$$\lambda(p_2, \ell_2) = p_2 \frac{\ell_2 - \mathbb{E}[\tilde{L}]}{\mathbb{E}[\tilde{L}]} > 0.$$

Hence, we can rewrite (13) as

$$V^L(\alpha) = (1 - p_2)\mathbb{E}[u(w - \tilde{L} - \alpha(1 + \lambda(p_2, \ell_2))\mathbb{E}[\tilde{L}] + \alpha\tilde{L})]. \quad (15)$$

It follows that, on the interval $[0, \tilde{\alpha}]$, the function V^L coincides up to multiplication by $1 - p_2$ with the expected utility of an agent with utility function u , initial wealth w , exposed to the loss $\tilde{L}$, and

buying coinsurance priced with a loading $\lambda(p_2, \ell_2)$.

Choosing p_2 and ℓ_2

Denote by $V_\lambda^{\tilde{L}}$ the expected utility for the strictly risk-averse agent with utility function u , initial wealth w , exposed to the loss $\tilde{L}$, and buying coinsurance with a loading λ . Since ℓ_1 is nonbankrupting we know from Mossin's theorem (Theorem 2.1) that there exists a unique optimal coinsurance level α_λ^* for this agent. We now fix $\alpha^* \in (0, 1)$ and denote by $\lambda > 0$ the unique loading such that $\alpha_\lambda^* = \alpha^*$, which exists again by Theorem 2.1. Note that we must have

$$w > (1 + \lambda)p\ell \quad (16)$$

since otherwise we would have $\alpha^* = \alpha_\lambda^* = 0$ which is not possible by assumption.¹⁶

For every $\ell_2 > \ell_1$ we set

$$p_2(\ell_2) := \frac{\lambda \mathbb{E}[\tilde{L}]}{\ell_2 - \mathbb{E}[\tilde{L}]} \in (0, 1)$$

and note that

$$\lambda(p_2(\ell_2), \ell_2) = \lambda. \quad (17)$$

Moreover, $p_2(\ell_2) \rightarrow 0$ as $\ell_2 \rightarrow \infty$. From now on whenever we choose $\ell_2 > \ell_1$ we will always implicitly assume that we choose $p_2 = p_2(\ell_2)$. Note that, for every choice of $\ell_2 > \ell_1$, we have by (14) that

$$\mathbb{E}[L] = (1 + \lambda)\mathbb{E}[\tilde{L}].$$

Hence, (16) implies that assumption (A2), namely $w \geq \mathbb{E}[L]$, is satisfied for every choice of $\ell_2 > \ell_1$. It remains to choose ℓ_2 so that α^* is the global maximum of V^L . Choose $\ell_2 > \ell_1$ sufficiently large so that

$$w - \ell_2 - \alpha^*\mathbb{E}[L] + \alpha^*\ell_2 < 0.$$

In particular, by definition of $\tilde{\alpha}$, we have $0 < \alpha^* < \tilde{\alpha}$. Using (15), it follows that the maximum of V^L over $[0, \tilde{\alpha}]$ is attained at α^* . As a last step, we may enlarge $\ell_2 > \ell_1$, if necessary, to ensure§ 1 that α^* is also the global maximum of V^L . To this effect, recall that $\alpha^* = \alpha_\lambda^*$ and note that

$$V_\lambda^{\tilde{L}}(\alpha^*) > V_\lambda^{\tilde{L}}(1) = u(w - (1 + \lambda)\mathbb{E}[\tilde{L}]) = u(w - \mathbb{E}[L]) = V^L(1)$$

by optimality of α_λ^* and by (14). Since $p_2(\ell_2) \rightarrow 0$ as $\ell_2 \rightarrow \infty$, we find ℓ_2 large enough such that

$$V^L(\alpha^*) = (1 - p_2(\ell_2))V_\lambda^{\tilde{L}}(\alpha^*) > V^L(1)$$

¹⁶Indeed, if $w \leq (1 + \lambda)p\ell$, the function $\alpha \mapsto w - \ell_1 - (1 + \lambda)\mathbb{E}[\tilde{L}] + \alpha\ell_1$, which corresponds to wealth in the loss state, would be decreasing implying that $\alpha_\lambda^* = 0$.

by (15). For such ℓ_2 , the global maximum of V^L is attained at α^* as desired and concludes the proof of Theorem 4.1.

5 Conclusion

Understanding the insurance demand by individuals or firms exposed to liability claims is important, because whether or not they insure is a matter of public concern. Indeed, liability insurance protects not only the insurance taker but also the damaged party in case the insurance taker does not have sufficient wealth to fully meet obligations arising from a liability case. Within the context of expected utility and in a discrete loss setting, we have provided a complete account of the demand for insurance by strictly risk-averse agents and risk-neutral firms when they enjoy limited liability. When exposed to a bankrupting, binary loss and under actuarially fair prices, individuals and firms will either fully insure or not insure at all. The decision to insure will depend on whether the benefits the insuree derives from insurance after having compensated the damaged party are sufficiently attractive to justify the premium paid. In the case of firms, we provide a full analysis of the tradeoff between the positive impact of risk on the value of the option to default and the negative impact on the firm's franchise value. When the loss is nonbinary, even when prices are actuarially fair, any amount of coinsurance can be optimal depending on the nature of the loss. In particular, this is a strong reminder of how special binary models are and calls for caution when extrapolating from these models to more general situations.

Appendix

This appendix collects the proofs of all results in the paper that are not immediate. Hence, we omit the proofs of Proposition 2.2, Proposition 2.3, Proposition 2.4, Proposition 3.3, Theorem 3.7, and Theorem 3.11.

A.1 Proof of Theorem 2.6

To prove Theorem 2.6 we require the following auxiliary result.

Lemma A.1. *Let $p\ell \leq w < \ell$. The function $\lambda \mapsto \beta_\lambda^*$ is continuous on $[0, \frac{w-p\ell}{p\ell}]$ and $\lambda \mapsto V_\lambda(\beta_\lambda^*)$ is continuous and strictly increasing on $[0, \frac{w-p\ell}{p\ell}]$. Moreover:*

$$(i) \quad \beta_0^* = \beta_{\frac{w-p\ell}{p\ell}}^* = 1.$$

$$(ii) \quad \beta_\lambda^* < 1 \text{ for } 0 < \lambda < \frac{w-p\ell}{p\ell}.$$

Proof. Continuity follows from general theorems on the continuity of solutions of parametric optimization problems but we provide a direct proof for convenience. Let (λ_n) be a sequence in $[0, \frac{w-p\ell}{p\ell}]$

such that $\lambda_n \rightarrow \lambda_0$. We have to show that $\beta_{\lambda_n}^* \rightarrow \beta_{\lambda_0}^*$. Note first that, since $(\beta_{\lambda_n}^*)$ is a sequence in the compact interval $[0, 1]$, we find $\beta^* \in [0, 1]$ such that $\beta_{\lambda_n}^* \rightarrow \beta^*$ (if necessary we can pass to a convenient subsequence). Observe that $\tilde{\alpha}_{\lambda_n} \leq \beta_{\lambda_n}^* \leq 1$ for every n implies that $\tilde{\alpha}_{\lambda_0} \leq \beta^* \leq 1$. Moreover, since $(\lambda, \alpha) \mapsto V_\lambda(\alpha)$ is jointly continuous, we obtain that $V_{\lambda_n}(\beta_{\lambda_n}^*) \rightarrow V_{\lambda_0}(\beta^*)$. Assume that $\beta^* \neq \beta_{\lambda_0}^*$ so that $V_{\lambda_0}(\beta^*) < V_{\lambda_0}(\beta_{\lambda_0}^*)$ by optimality of $\beta_{\lambda_0}^*$. It follows that for sufficiently large n we have $V_{\lambda_n}(\beta_{\lambda_n}^*) < V_{\lambda_n}(\beta_{\lambda_0}^*)$, contradicting the optimality of $\beta_{\lambda_n}^*$. We infer that $\beta^* = \beta_{\lambda_0}^*$ must hold. The fact that $\lambda \mapsto V_\lambda(\beta_\lambda^*)$ is continuous follows from the joint continuity of $(\lambda, \alpha) \mapsto V_\lambda(\alpha)$ and the continuity of $\lambda \mapsto \beta_\lambda^*$ we have just established. Monotonicity of $\lambda \mapsto V_\lambda(\beta_\lambda^*)$ is clear.

To prove (i), note first that U_0 attains its maximum over $[0, 1]$ at $\alpha = 1$. Hence, the maximum of V_0 over $[\tilde{\alpha}_0, 1]$ is also attained at $\alpha = 1$, i.e., $\beta_0^* = 1$. Moreover, since $\tilde{\alpha}_{\frac{w-p\ell}{p\ell}} = 1$, we have $\beta_{\frac{w-p\ell}{p\ell}}^* = 1$.

To prove (ii), let $0 < \lambda < \frac{w-p\ell}{p\ell}$. In this case, U_0 is decreasing in a neighborhood of $\alpha = 1$ and, hence, so is V_λ . It follows that $\alpha = 1$ cannot be a maximum, i.e., $\beta_\lambda^* < 1$. $\square$

We now proceed with the proof of Theorem 2.6. If $V_0(1) < V_0(0)$, then Lemma A.1 implies that $\alpha = 0$ is the only solution for $\lambda = 0$ and, hence, also for $\lambda > 0$. This proves (i). If $V_0(1) = V_0(0)$, then Lemma A.1 implies that $\alpha = 0$ and $\alpha = 1$ are optimal solutions for $\lambda = 0$. Moreover, for $\lambda > 0$ we have $V_\lambda(\beta_\lambda^*) < V_0(1) = V_0(0)$, implying that $\alpha = 0$ is the optimal solution in this case. This establishes (ii). To conclude, observe that the function $\lambda \mapsto V_\lambda(\beta_\lambda^*)$ is continuous and strictly increasing by Lemma A.1. Moreover, $V_0(\beta_0^*) = V_0(1) > V_0(0)$ and $V_{\frac{w-p\ell}{p\ell}}(\beta_{\frac{w-p\ell}{p\ell}}^*) = V_{\frac{w-p\ell}{p\ell}}(1) = 0 < V_0(0)$. It follows that there exists $0 < \tilde{\lambda} < \frac{w-p\ell}{p\ell}$ such that (a)-(c) are satisfied.

A.2 Proof of Proposition 2.8

Since u is strictly concave and $u(0) = 0$, we have

$$(1-p)u(\ell) = (1-p)u(\ell) + pu(0) < u(\ell - p\ell).$$

First, recall that no insurance is optimal if $w < p\ell$ by Proposition 2.3 while full insurance is optimal if $w \geq \ell$ by Proposition 2.2. Now, let $p\ell \leq w < \ell$ and note that $V_0(0) = (1-p)u(w)$ and $V_0(1) = u(w - p\ell)$. For w close to ℓ we have $(1-p)u(w) < u(w - p\ell)$. In this case, full insurance is optimal by Theorem 2.6. Moreover, as w converges to $p\ell$ from above, $(1-p)u(w)$ converges to $(1-p)u(p\ell) > 0$ and $u(w - p\ell)$ converges to 0. As a result, $(1-p)u(w) > u(w - p\ell)$ for w close to $p\ell$. In this case, no insurance is optimal again by Theorem 2.6. Since $u(w - p\ell)$ is increasing in w , there exists exactly one level $\tilde{w}$ satisfying (i).

Similarly, note that full insurance is optimal if $\ell \leq w$ by Proposition 2.2 while no insurance is optimal if $\ell > \frac{w}{p}$ by Proposition 2.3. Now, let $w < \ell \leq \frac{w}{p}$ and note that $V_0(0) = (1-p)u(w)$ and $V_0(1) = u(w - p\ell)$. For ℓ close to w we have $(1-p)u(w) < u(w - p\ell)$. In this case, full insurance is optimal by Theorem 2.6. Moreover, as ℓ converges to $\frac{w}{p}$ from below, $u(w - p\ell)$ converges to 0. As

a result, $(1-p)u(w) > u(w-p\ell)$ for w close to $p\ell$. In this case, no insurance is optimal again by Theorem 2.6. Since $u(w-p\ell)$ is decreasing in ℓ , there exists exactly one level $\tilde{\ell}$ satisfying (ii).

A.3 Proof of Lemma 3.4

Define a function $f : [0, \infty) \rightarrow \mathbb{R}$ by setting

$$f(\kappa) := H'_\kappa(0) = F'(\kappa) - C'(0).$$

Note that f is continuous and strictly decreasing. If $F'(0) \leq C'(0)$, then we have $f(0) \leq 0$. Since f is strictly decreasing, this implies that $f(\kappa) \leq 0$ for every $\kappa \geq 0$. As a result, $e^*(\kappa) = 0$ for every $\kappa \geq 0$. This proves (i).

If $F'(\kappa) > C'(0)$, then $f(0) > 0$. As F is bounded above, we have that $f(\kappa) \rightarrow 0$ as $\kappa \rightarrow \infty$. Together with f being continuous and strictly decreasing, this implies that there exists $\bar{\kappa} > 0$ such that $f(\bar{\kappa}) = 0$ and $f(\kappa) > 0$ whenever $\kappa < \bar{\kappa}$ as well as $f(\kappa) < 0$ whenever $\kappa > \bar{\kappa}$. As a result, we must have $e^*(\kappa) > 0$ for every $\kappa < \bar{\kappa}$ and $e^*(\kappa) = 0$ for every $\kappa \geq \bar{\kappa}$. This establishes (ii).

A direct application of the smooth version of the maximum theorem, see, e.g., Theorem 6.1 in Carter (2001), shows that e^* is continuously differentiable. To prove that e^* is strictly decreasing on $[0, \bar{\kappa}]$, assume that $0 < x < y < \bar{\kappa}$ but $e^*(x) \leq e^*(y)$ (we can discard the extreme cases 0 and $\bar{\kappa}$ by point (ii)). Then, $x + e^*(x) < y + e^*(y)$ and

$$0 = H'_x(e^*(x)) = F'(x + e^*(x)) - C'(e^*(x)) > F'(y + e^*(y)) - C'(e^*(y)) = H'_y(e^*(y)).$$

This contradicts the fact that $e^*(y)$ must be a critical point of H_y and we conclude that $e^*(x) \geq e^*(y)$ holds, delivering the desired monotonicity. This delivers (iii).

A.4 Proof of Proposition 3.5

Since F and C as well as e^* are continuously differentiable, so is F_C . Using the strict concavity of F and the strict convexity of C we have for all $a \in (0, 1)$ and $\kappa_1, \kappa_2 \geq 0$ with $\kappa_1 \neq \kappa_2$ that

$$\begin{aligned} F_C(a\kappa_1 + (1-a)\kappa_2) &= F(a\kappa_1 + (1-a)\kappa_2 + e^*(a\kappa_1 + (1-a)\kappa_2)) - C(e^*(a\kappa_1 + (1-a)\kappa_2)) \\ &\geq F(a\kappa_1 + (1-a)\kappa_2 + ae^*(\kappa_1) + (1-a)e^*(\kappa_2)) - C(ae^*(\kappa_1) + (1-a)e^*(\kappa_2)) \\ &> aF(\kappa_1 + e^*(\kappa_1)) + (1-a)F(\kappa_2 + e^*(\kappa_2)) - aC(e^*(\kappa_1)) + (1-a)C(e^*(\kappa_2)) \\ &= aF_C(\kappa_1) + (1-a)F_C(\kappa_2). \end{aligned}$$

In the first inequality we used that $H_{a\kappa_1+(1-a)\kappa_2}(e^*(a\kappa_1 + (1-a)\kappa_2)) \geq H_{a\kappa_1+(1-a)\kappa_2}(e)$ for every $e \geq 0$ by definition of $e^*(a\kappa_1 + (1-a)\kappa_2)$. This proves strict concavity of F_C . In view of the monotonicity of F and C , the remaining statements follow directly from Lemma 3.4.

A.5 Proof of Lemma 3.6

Note first that $DO(\alpha) = \mathbb{E}[W^-(\alpha)] = p[\kappa(1+\rho) - \ell + (1-p)\alpha\ell]_-$ for every $\alpha \in [0, 1]$.

(i) If $\kappa(1+\rho) \geq \ell$ and $\alpha \in [0, 1]$, then $\kappa(1+\rho) - \ell + (1-p)\alpha\ell \geq 0$ and, thus, $DO(\alpha) = 0$.

(ii) If $\kappa(1+\rho) < \ell$ and $\kappa(1+\rho) > p\ell$, then clearly $\frac{\ell - \kappa(1+\rho)}{(1-p)\ell} \in (0, 1)$ and

$$DO(\alpha) = \begin{cases} -p[\kappa(1+\rho) - \ell + (1-p)\alpha\ell] & \text{if } \alpha \in [0, \frac{\ell - \kappa(1+\rho)}{(1-p)\ell}], \\ 0 & \text{if } \alpha \in [\frac{\ell - \kappa(1+\rho)}{(1-p)\ell}, 1]. \end{cases}$$

In particular, DO is strictly decreasing on $[0, \frac{\ell - \kappa(1+\rho)}{(1-p)\ell}]$.

(iii) If $\kappa(1+\rho) < \ell$ and $\kappa(1+\rho) \leq p\ell$, then $DO(\alpha) = -p[\kappa(1+\rho) - \ell + (1-p)\alpha\ell]$ for every $\alpha \in [0, 1]$.

In particular, DO is strictly decreasing on $[0, 1]$.

A.6 Proof of Proposition 3.8

If $\kappa(1+\rho) \geq \ell$ and $\alpha \in [0, 1]$, then $\kappa(1+\rho) - \ell + (1-p)\alpha\ell \geq 0$ and, hence,

$$FV(\alpha) = pF_C(\kappa(1+\rho) - \ell + (1-p)\alpha\ell) + (1-p)F_C(\kappa(1+\rho) - \alpha p\ell).$$

Denote the left derivative of FV at $\alpha = 1$ by $FV'(1)$ and note that

$$FV'(1) = p(1-p)\ell F'_C(\kappa(1+\rho) - p\ell) - (1-p)p\ell F'_C(\kappa(1+\rho) - p\ell) = 0.$$

Together with the strict concavity of FV , which follows from that of F_C , this implies that FV is strictly increasing on $[0, 1]$.

A.7 Proof of Proposition 3.9

By assumption, for every $0 < \alpha \leq 1$, we have $W^+(\alpha) = 0$ in the loss state. As a result, for every $0 < \alpha \leq 1$

$$FV(\alpha) = (1-p)F_C(\kappa(1+\rho) - \alpha p\ell).$$

Hence, FV is clearly strictly decreasing on $[0, 1]$. Since, by Lemma 3.6, DO is also strictly decreasing on $[0, 1]$, we immediately infer that $\alpha = 0$ is the unique optimal coinsurance level.

A.8 Proof of Lemma 3.10

By definition of $\tilde{\alpha}$, we have

$$FV(\alpha) = \begin{cases} (1-p)F_C(\kappa(1+\rho) - \alpha p\ell) & \text{if } 0 \leq \alpha \leq \tilde{\alpha}, \\ pF_C(\kappa(1+\rho) - \ell + (1-p)\alpha\ell) + (1-p)F_C(\kappa(1+\rho) - \alpha p\ell) & \text{otherwise.} \end{cases}$$

It is clear that FV is strictly decreasing on $[0, \tilde{\alpha}]$. The proof that FV is strictly increasing on $[\tilde{\alpha}, 1]$ is analogous to the proof of Proposition 3.8.

A.9 Proof of Proposition 3.12

It follows from Proposition 3.9 that no insurance is optimal if $\kappa < \frac{p\ell}{1+\rho}$ or equivalently $\ell > \frac{\kappa(1+\rho)}{p}$. We also know from Theorem 3.11 that full insurance is optimal if $\kappa \geq \frac{\ell}{1+\rho}$ or equivalently $\ell \leq \kappa(1+\rho)$. Now, assume that $\frac{p\ell}{1+\rho} \leq \kappa < \frac{\ell}{1+\rho}$. In this case,

$$FV(1) - FV(0) = F_C(\kappa(1+\rho) - p\ell) + (1-p)F_C(\kappa(1+\rho)), \quad DO(0) = p(\ell - \kappa(1+\rho)).$$

If κ tends to $\frac{p\ell}{1+\rho}$ from above, then $DO(0)$ converges to $p(1-p)\ell > 0$ while $FV(1) - FV(0)$ converges to $-(1-p)F_C(p\ell) < 0$. Hence, Theorem 3.11 implies that no insurance is optimal. Similarly, if κ tends to $\frac{\ell}{1+\rho}$ from below, then $DO(0)$ converges to 0 while $FV(1) - FV(0)$ converges to $F_C((1-p)\ell) - (1-p)F_C(\ell)$. Observe that

$$F_C((1-p)\ell) - (1-p)F_C(\ell) > pF_C(0) + (1-p)F_C(\ell) - (1-p)F_C(\ell) = 0$$

by strict concavity of F_C . Hence, Theorem 3.11 implies that full insurance is optimal. Since, on the relevant interval, $DO(0)$ is a strictly-decreasing affine function of κ and $FV(1) - FV(0)$ is a strictly-concave function of κ , we infer the existence of a unique point $\tilde{\kappa}$ satisfying (i).

If ℓ tends to $\frac{\kappa(1+\rho)}{p}$ from below, then $DO(0)$ converges to $(1-p)\kappa(1+\rho) > 0$ while $FV(1) - FV(0)$ converges to $-(1-p)F_C(\kappa(1+\rho)) < 0$. Hence, Theorem 3.11 implies that no insurance is optimal. Similarly, if ℓ tends to $\kappa(1+\rho)$ from above, then $DO(0)$ converges to 0 while $FV(1) - FV(0)$ converges to $F_C((1-p)\kappa(1+\rho)) - (1-p)F_C(\kappa(1+\rho))$. Observe that

$$F_C((1-p)\kappa(1+\rho)) - (1-p)F_C(\kappa(1+\rho)) > pF_C(0) + (1-p)F_C(\kappa(1+\rho)) - (1-p)F_C(\kappa(1+\rho)) = 0$$

by strict concavity of F_C . Hence, Theorem 3.11 implies that full insurance is optimal. Since, on the relevant interval, $DO(0)$ is a strictly-increasing affine function of ℓ and $FV(1) - FV(0)$ is a strictly-concave function of ℓ , we infer the existence of a unique point $\tilde{\ell}$ satisfying (ii).

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