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ALGEBRAIC PROPERTIES OF ZIGZAG ALGEBRAS

MICHAEL EHRIG AND DANIEL TUBBENHAUER

Abstract. We give necessary and sufficient conditions for zigzag algebras and certain generalizations of them to be (relative) cellular, quasi-hereditary or Koszul.

Contents

1. Introduction

Throughout, we let Γ denote a finite, connected, simple graph, and we work over an arbitrary field $\Bbbk$. Let $\mathrm{Z}_{\rightleftarrows} = \mathrm{Z}_{\rightleftarrows}(\Gamma)$ be the zigzag algebra associated to Γ, and let further $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows} = \mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}(\Gamma)$ be the zigzag algebra with a vertex-loop condition (vertex condition for short) at some fixed set of vertices $\mathsf{C} \neq \emptyset$.

The main statements. Zigzag algebras are around for many years, see e.g. [Wak80] for an early reference from classical algebra. Further, as shown in e.g. [HK01], they appear in various places in modern mathematics. For example, for a viewpoint from categorical braid group actions see [KS02] and [KMS09], for one from symplectic geometry see [EL17], for one from KLR algebras, associated representations and blocks of symmetric groups see [KM19] and [EK18], for one from geometric group theory see [Lic17], and for one from Soergel bimodules and 2-representation theory see [MT16]. Similarly, the algebra $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ comes from considerations in modular representation theory or representation theory at roots of unity, see e.g. [AT17], [QS16], or versions of category $\mathcal{O}$, see e.g. [Str03], [CK14] or [BS12].

The purpose of this paper is to show the following algebraic properties of $\mathrm{Z}_{\rightleftarrows}$ and $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$.

Theorem A. $\mathrm{Z}_{\rightleftarrows}$ is cellular if and only if Γ is a finite type A graph. $\mathrm{Z}_{\rightleftarrows}$ is relative cellular if and only if Γ is a finite or affine type A graph.

For us e.g. not being cellular always means that there is no choice of a cell datum. Further, in all cases where $\mathrm{Z}_{\rightleftarrows}$ is (relative) cellular, the path length endows it with the structure of a graded (relative) cellular algebra.

Theorem B. $\mathrm{Z}_{\rightleftarrows}$ is never quasi-hereditary.

Theorem C. $\mathrm{Z}_{\rightleftarrows}$ is Koszul if and only if Γ is not a type ADE graph.

Additionally, we give an algorithmic construction for the minimal linear projective resolutions of simple $\mathrm{Z}_{\rightleftarrows}$-modules.

Using the same ideas as for $\mathrm{Z}_{\rightleftarrows}$ we can also prove:

Theorem A′. $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ is cellular if and only if Γ is a finite type A graph and the vertex condition is imposed on one leaf. $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ is relative cellular in exactly the same cases.

Theorem B′. $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ is quasi-hereditary if and only if Γ is a finite type A graph and the vertex condition is imposed on one leaf.

Theorem C′. $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ is always Koszul.

Note that, as follows from our main theorems and their proofs, $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}(\mathsf{A})$ with vertex condition imposed on one leaf is a graded cellular, quasi-hereditary, Koszul algebra, which makes it quite special, since these are the only zigzag algebras having these properties. Indeed, $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}(\mathsf{A})$ has some very nice spectral properties (in the sense of spectral graph theory), and is the version of zigzag algebras which is mostly studied in the literature.

Remark 1.1. Let us note that the restrictions on Γ being finite, connected and simple can be relaxed, and we imposed them for convenience. ▲

Strengthening and collecting known results. Some of these results are actually known and stated implicitly or explicitly in the literature. (This is partially due to the different nomenclatures.) So another point of our paper is to collect these in one document.

To elaborate a bit, the construction of the cellular respectively relative cellular structures for Theorem A are folklore or standard, while the converses appear to be new. Theorem B is very easy (indeed, our proof here is basically one line) and stated for completeness. Theorem C appears in [MV96] or [EE07], where the theorem is proven for the preprojective algebra (which is the Koszul dual of the zigzag algebra in case Γ is bipartite), and in [Dub17] whenever Γ has a circle. However, our methods (inspired by [Ben08]) to prove Theorem C, constructing explicit resolutions using Chebyshev polynomials, seem to be new as well. Having this, let us note that the well-established periodicity of the Chebyshev polynomials for type ADE graphs can be used to reprove the results about almost Koszulness of the preprojective algebra in these cases [BBK02].

Finally, as far as we can tell, Theorems A′, B′ and C′ are new and actually generalizations of the other three theorems. But, again, the constructions of the relevant structures for Theorems A′ and B′ are folklore, and the punchlines are the conserves.

Remark 1.2. Note that we will always assume $\mathsf{C} \neq \emptyset$. The only reason for this is that we want to treat our main statements separately, i.e. there is no problem to allow $\mathsf{C} = \emptyset$ in most of the arguments which we are going to use, and $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ is in fact a strict generalization of $\mathrm{Z}_{\rightleftarrows} = \mathrm{Z}^{\emptyset}_{\rightleftarrows}$. And although we will formulate $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ as a quotient of $\mathrm{Z}_{\rightleftarrows}$, we think of it as a quasi-hereditary cover of $\mathrm{Z}_{\rightleftarrows}$, since this is what happens in the case of type A graphs. ▲

Towards generalizations. A striking question is how zigzag algebras can be generalized, and how to control them algebraically.

There are at least two different generalizations of zigzag algebras for which our methods seems to be applicable: either the one from [Gra17], generalizing the connection to e.g. the preprojective algebras, Iyama's higher representation types and categorical group actions, or the one from [MMMT18, Section 5C], generalizing the connections to e.g. 2-representation theory, affine Hecke algebras and modular representation theory. In both cases the spectral properties of the underlying graphs seem to play a crucial role and we hope that our methods presented here generalize to those algebras. In particular, the generalizations of zigzag algebras from [MMMT18, Section 5C] are usually not connected to preprojective algebras, and their quasi-hereditary covers are similar in spirit to $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ (by adding certain vertex conditions), and our paper might help to understand algebraic properties of these algebras.

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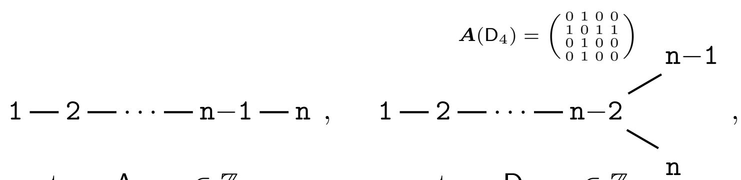
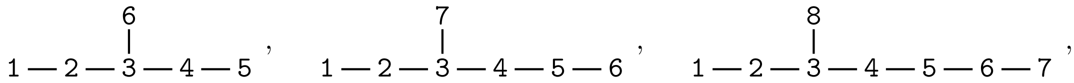
2. Preliminaries

We denote by i, j etc. the vertices of Γ, and i–j means that i and j are connected in Γ by an edge. For each such graph Γ we chose an enumeration of its vertices, and we obtain its adjacency matrix $\boldsymbol{A} = \boldsymbol{A}(\Gamma)$.

Example 2.1. Of paramount importance for us are the finite type ADE graphs

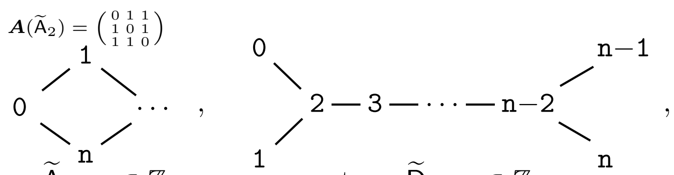


type A_n; $n \in \mathbb{Z}_{\geq 1}$ type D_n; $n \in \mathbb{Z}_{\geq 4}$

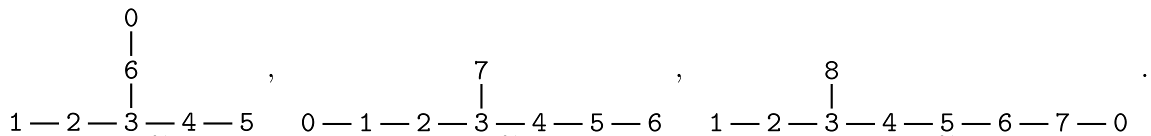


type E_6 type E_7 type E_8

as well as their affine counterparts



type $\widetilde{\mathsf{A}}_n$; $n \in \mathbb{Z}_{\geq 2}$ type $\widetilde{\mathsf{D}}_n$; $n \in \mathbb{Z}_{\geq 4}$



type $\widetilde{\mathsf{E}}_6$ type $\widetilde{\mathsf{E}}_7$ type $\widetilde{\mathsf{E}}_8$

The enumeration of the vertices matters for some calculations, and we always number them as indicated above. (Note that we omit the type $\widetilde{\mathsf{A}}_1$ graph.) ▲

The zigzag algebra. The *double graph* $\Gamma_{\rightleftarrows}$ is the oriented graph obtained from Γ by doubling all edges i–j of Γ into a pair of parallel edges i→j (oriented from i to j) and j→i (oriented from j to i), and by adding two loops $\alpha_{\mathtt{s}} = (\alpha_{\mathtt{s}})_{\mathtt{i}}$ and $\alpha_{\mathtt{t}} = (\alpha_{\mathtt{t}})_{\mathtt{i}}$ per vertex.

Let $\mathrm{R}(\Gamma_{\rightleftarrows})$ denote the path algebra for $\Gamma_{\rightleftarrows}$. It is graded by using the path length, but putting loops in degree 2. We identify its length zero paths with the vertices of Γ (e.g. i also denotes the vertex idempotent), and we let i→j→k = i→j ∘ j→k etc. denote the composition.

Definition 2.2. Let $\mathrm{Z}_{\rightleftarrows} = \mathrm{Z}_{\rightleftarrows}(\Gamma)$, for Γ having at least three vertices, be the quotient of $\mathrm{R}(\Gamma_{\rightleftarrows})$ by the following defining relations.

(2.2.a) **Boundedness.** Any path involving three distinct vertices is zero.

(2.2.b) **The relations of the cohomology ring of the variety of full flags in $\mathbb{C}^2$.** $\alpha_{\mathtt{s}} \circ \alpha_{\mathtt{t}} = \alpha_{\mathtt{t}} \circ \alpha_{\mathtt{s}}$, $\alpha_{\mathtt{s}} + \alpha_{\mathtt{t}} = 0$ and $\alpha_{\mathtt{s}} \circ \alpha_{\mathtt{t}} = 0$.

(2.2.c) **Zigzag.** i→j→i $= \alpha_{\mathtt{s}} - \alpha_{\mathtt{t}}$ for i–j.

In case Γ has one vertex we let $\mathrm{Z}_{\rightleftarrows} = \Bbbk[\alpha_{\mathtt{s}}, \alpha_{\mathtt{t}}]/(\alpha_{\mathtt{s}} + \alpha_{\mathtt{t}}, \alpha_{\mathtt{s}}\alpha_{\mathtt{t}})$, by convention, and in case Γ has two vertices we additionally to (2.2.a), (2.2.b) and (2.2.c) kill paths of length three.

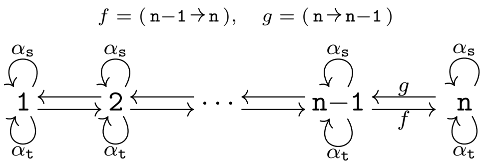
We call $\mathrm{Z}_{\rightleftarrows}$ the *zigzag algebra* associated to Γ. ▲

The relations of $\mathrm{Z}_{\rightleftarrows}$ are homogeneous with respect to the path length grading, which thus endow $\mathrm{Z}_{\rightleftarrows}$ with the structure of a graded algebra. (Throughout, graded means $\mathbb{Z}$-graded.)

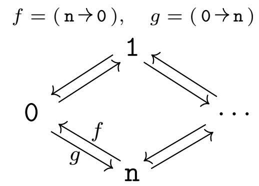
Remark 2.3. Our definition of $Z_{\rightleftarrows}$ is slightly different from the one as e.g. in [HK01, Section 3], but they are equivalent. To see this use the isomorphism from $\Bbbk[\mathtt{x}]/(\mathtt{x}^2)$ to $\Bbbk[\alpha_{\mathtt{s}},\alpha_{\mathtt{t}}]/(\alpha_{\mathtt{s}}+\alpha_{\mathtt{t}},\alpha_{\mathtt{s}}\alpha_{\mathtt{t}})$ given by $\mathtt{x}\mapsto\alpha_{\mathtt{s}}-\alpha_{\mathtt{t}}\in\mathrm{End}_{Z_{\rightleftarrows}}(\mathtt{i})$. (If $\Bbbk$ has characteristic 2, then $\mathtt{x}\mapsto\alpha_{\mathtt{s}}=\alpha_{\mathtt{t}}$ gives the isomorphism.) We prefer the formulation as in Definition 2.2 since it fits to the generalizations of the zigzag algebra from [MMMT18, Section 5C]. ▲

We call $\mathtt{x_i}=\alpha_{\mathtt{s}}-\alpha_{\mathtt{t}}$ (or $\mathtt{x_i}=\alpha_{\mathtt{s}}$ in characteristic 2) the *volume element* (at vertex $\mathtt{i}$).

Example 2.4. The most classical examples of zigzag algebras are the cases where Γ is either a type A_n graph or a type $\widetilde{\mathsf{A}}_n$ graph.

$Z_{\rightleftarrows}(\mathsf{A}_n)=$ 

living on the type A_n graph

$Z_{\rightleftarrows}(\widetilde{\mathsf{A}}_n)=$ 

living on the type $\widetilde{\mathsf{A}}_n$ graph

where we omitted the loops for the type $\widetilde{\mathsf{A}}_n$ graph for illustration purposes only. ▲

Basic properties. We omit the (easy) proofs of the following basics facts.

Lemma 2.5. If Γ has three or more vertices, then $Z_{\rightleftarrows}$ is quadratic, i.e. it is generated in degree 1, and its relations are generated in degree 2. ■

Note hereby that the relation $\alpha_{\mathtt{s}}+\alpha_{\mathtt{t}}=0$ can be omitted by reparametrization of the endomorphisms spaces, cf. Remark 2.3.

Lemma 2.6. The association

$$\mathrm{tr}(\mathtt{i})=0,\quad \mathrm{tr}(\mathtt{x_i})=1,\quad \mathrm{tr}(\mathtt{i}\to\mathtt{j})=0,$$

gives rise to a non-degenerate trace form $\mathrm{tr}\colon Z_{\rightleftarrows}\to\Bbbk$, which endows $Z_{\rightleftarrows}$ with the structure of a (non-commutative) Frobenius algebra. ■

Using Lemma 2.6, the following is also easy to prove, where we recall that elements of $Z_{\rightleftarrows}$ are graded by path length.

Lemma 2.7. We have

$$\mathsf{q}\mathrm{dim}(\mathrm{Hom}_{Z_{\rightleftarrows}}(\mathtt{i},\mathtt{j}))=\begin{cases}2_{\mathsf{q}}, & \text{if } \mathtt{i}=\mathtt{j}, \quad \{\mathtt{i},\mathtt{x_i}\} \text{ is a basis},\\ \mathsf{q}, & \text{if } \mathtt{i}{-}\mathtt{j}, \quad \{\mathtt{i}\to\mathtt{j}\} \text{ is a basis},\\ 0, & \text{else}, \quad \emptyset \text{ is a basis},\end{cases}$$

where $\mathsf{q}\mathrm{dim}(\_)$ denotes the graded dimension, and $2_{\mathsf{q}}=1+\mathsf{q}^2$. ■

Projective and simple modules. For $s\in\mathbb{Z}$ we will denote by q^s degree shifts using the convention that a map of degree d between M and N is of degree $d-s_1+s_2$ seen as a map from $\mathsf{q}^{s_1}\mathrm{M}$ to $\mathsf{q}^{s_2}\mathrm{N}$. Further, for us the action of $Z_{\rightleftarrows}$ on modules will always be given by

left: pre-composition of paths & right: post-composition of paths.

In particular,

$$\mathsf{q}^s\mathrm{P}_{\mathtt{i}}=\{\mathtt{i},\mathtt{j}\to\mathtt{i},\mathtt{x_i}\mid\mathtt{i}{-}\mathtt{j}\},\qquad \mathsf{q}^s{}_{\mathtt{i}}\mathrm{P}=\{\mathtt{i},\mathtt{i}\to\mathtt{j},\mathtt{x_i}\mid\mathtt{i}{-}\mathtt{j}\},$$

are left, respectively right, graded projective $Z_{\rightleftarrows}$-modules with $\mathtt{i}$ being in degree s. Moreover,

$$\{\mathsf{q}^s\mathrm{P}_{\mathtt{i}}\mid\mathtt{i}\in\Gamma,s\in\mathbb{Z}\},\qquad\{\mathsf{q}^s{}_{\mathtt{i}}\mathrm{P}\mid\mathtt{i}\in\Gamma,s\in\mathbb{Z}\},$$

are complete, irredundant sets of indecomposable, graded projective left, respectively right, $Z_{\rightleftarrows}$-modules. In contrast, the simple left $Z_{\rightleftarrows}$-modules $\mathrm{L}_{\mathtt{i}}$ are one-dimensional and only one vertex idempotent of $Z_{\rightleftarrows}$ acts non-trivially on them, and elements of positive degree act as zero. Thus, they can be identified with elements of $Z_{\rightleftarrows}$ and, for example, for $\mathtt{i}{-}\mathtt{j}$

$$\mathrm{L}_{\mathtt{i}}=\{\mathtt{i}\}\cong\mathsf{q}^{-1}\{\mathtt{i}\to\mathtt{j}\}\cong\mathsf{q}^{-2}\{\mathtt{x_i}\}$$

is the same incarnation of the simple $\mathrm{L}_{\mathtt{i}}$ corresponding to $\mathtt{i}$. Similarly, of course, for the right simples ${}_{\mathtt{i}}\mathrm{L}$. Thus, we get the Loewy picture

$$\mathrm{P}_{\mathtt{i}} = \begin{array}{c}\mathtt{i}\\ \mathtt{j}{\to}\mathtt{i}\\ \mathtt{x}_{\mathtt{i}}\end{array} \text{ (for } \mathtt{i}\text{–}\mathtt{j}\text{)}, \qquad {}_{\mathtt{i}}\mathrm{P} = \begin{array}{c}\mathtt{i}\\ \mathtt{i}{\to}\mathtt{j}\\ \mathtt{x}_{\mathtt{i}}\end{array} \text{ (for } \mathtt{i}\text{–}\mathtt{j}\text{)} \tag{2-1}$$

of the projectives, where the vertex idempotent spans the head and the volume element spans the socle.

Having all this, the following easy, but crucial, statement is immediate, where $\boldsymbol{I}$ is the identity matrix. (Recall hereby that the *graded Cartan matrix* encodes the graded filtration of the projectives $\mathrm{P}_{\mathtt{i}}$ or ${}_{\mathtt{i}}\mathrm{P}$ by simples, where we enumerate the rows and columns as given by the enumeration of the vertices.)

Proposition 2.8. The graded Cartan matrix $\boldsymbol{C}_{\mathsf{q}} = \boldsymbol{C}_{\mathsf{q}}(\mathrm{Z}_{\rightleftarrows})$ of $\mathrm{Z}_{\rightleftarrows}$ is $2_{\mathsf{q}}\boldsymbol{I} + \mathsf{q}\boldsymbol{A}$. ■

In particular, forgetting the grading, the *Cartan matrix* $\boldsymbol{C} = \boldsymbol{C}(\mathrm{Z}_{\rightleftarrows})$ of $\mathrm{Z}_{\rightleftarrows}$ is just $2\boldsymbol{I} + \boldsymbol{A}$.

From now on we will focus on the case of left modules (and omit to say so); the case of right modules can be done in the same way.

Vertex conditions. Fix a non-empty set C of vertices of Γ.

Definition 2.9. The *zigzag algebra* $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows} = \mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}(\Gamma)$ *with vertex condition* for C is the quotient of $\mathrm{Z}_{\rightleftarrows}$ obtained by killing the volume elements $\mathtt{x}_{\mathtt{c}}$ for $\mathtt{c} \in \mathsf{C}$. ▲

Clearly, everything done above for $\mathrm{Z}_{\rightleftarrows}$ works, mutatis mutandis, for $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ as well. (However, $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ is always quadratic.) In particular, the Loewy pictures are as in (2-1), but with the projective $\mathrm{P}^{\mathsf{C}}_{\mathtt{c}}$ for $\mathtt{c} \in \mathsf{C}$ having no volume element, and we will use the superscript $^{\mathsf{C}}$ to indicate $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$-modules which are different from their $\mathrm{Z}_{\rightleftarrows}$-counterparts.

The following combinatorial difference will play a key role, where we denote by $\boldsymbol{E}_{\mathsf{C}}$ the matrix with only non-zero entry equal to 1 in the $\mathtt{c}$-$\mathtt{c}$ position for $\mathtt{c} \in \mathsf{C}$.

Proposition 2.10. The graded Cartan matrix $\boldsymbol{C}^{\mathsf{C}}_{\mathsf{q}} = \boldsymbol{C}_{\mathsf{q}}(\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows})$ of $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ is $2_{\mathsf{q}}\boldsymbol{I} - \mathsf{q}^2\boldsymbol{E}_{\mathsf{C}} + \mathsf{q}\boldsymbol{A}$. ■

The Cartan matrix $\boldsymbol{C}^{\mathsf{C}} = \boldsymbol{C}^{\mathsf{C}}(\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows})$ of $\mathrm{Z}_{\rightleftarrows}$ is just the dequantization $\boldsymbol{C}^{\mathsf{C}} = 2\boldsymbol{I} - \boldsymbol{E}_{\mathsf{C}} + \boldsymbol{A}$.

Let $\Gamma - \mathtt{c}$ denote the graph obtained from Γ by removing a fixed vertex $\mathtt{c}$. Further, let us write $\mathrm{Z}_{\rightleftarrows} = \mathrm{Z}^{\emptyset}_{\rightleftarrows}$ for convenience of notation.

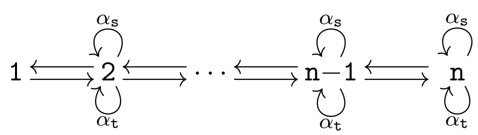
Lemma 2.11. We have $\det(\boldsymbol{C}^{\mathsf{C}}_{\mathsf{q}}) = \det(\boldsymbol{C}^{\mathsf{C}}_{\mathsf{q}}(\mathrm{Z}^{\mathsf{C}-\mathtt{c}}_{\rightleftarrows})) - \det(\boldsymbol{C}^{\mathsf{C}}_{\mathsf{q}}(\mathrm{Z}^{\mathsf{C}-\mathtt{c}}_{\rightleftarrows}(\Gamma - \mathtt{c})))$. □

Note that Lemma 2.11 gives a recursive way to compute the *graded Cartan determinant* $\det(\boldsymbol{C}^{\mathsf{C}}_{\mathsf{q}})$ of $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ from that of $\mathrm{Z}_{\rightleftarrows}$. Explicitly, in case C has just one entry $\mathtt{c}$, then $\det(\boldsymbol{C}^{\mathsf{C}}_{\mathsf{q}}) = \det(\boldsymbol{C}_{\mathsf{q}}) - \det(\boldsymbol{C}_{\mathsf{q}}(\mathrm{Z}_{\rightleftarrows}(\Gamma - \mathtt{c})))$.

Proof. By using Proposition 2.10, this follows directly by row expansion. ■

Example 2.12. Very similar to Example 2.4, the most important example is the case of a type A_n graph with vertex condition imposed on one of its leafs.

$\mathsf{C} = \{1\}$

$\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}(\mathsf{A}_n) =$ 

,

living on the type A_n graph

where we have illustrated the case $\mathsf{C} = \{1\}$. ▲

3. Cellularity

A brief reminder. We briefly recall the definition of a relative cellular algebra as it appears in [ET17, Definition 2.1], sneaking in the graded setting as in [HM10, Definition 2.1].

Definition 3.1. A *relative cellular algebra* is an associative, unital algebra R together with a (relative) cell datum, i.e.

$$(\mathtt{X},\mathtt{M},\mathtt{C},{}^{\star},\mathtt{E},\mathtt{O},\epsilon),$$

such that the following hold.

(3.1.a) $\mathtt{X}$ is a set and $\mathtt{M}=\{\mathtt{M}(\lambda)\mid\lambda\in\mathtt{X}\}$ a collection of finite sets such that

$$\mathtt{C}(\_,\_)\colon \textstyle\coprod_{\lambda\in\mathtt{X}}\mathtt{M}(\lambda)\times\mathtt{M}(\lambda)\to \mathrm{R}$$

is an injective map with image forming a basis of R. For $S,T\in\mathtt{M}(\lambda)$ we write $\mathtt{C}(S,T)=\mathtt{C}^{\lambda}_{S,T}$ from now on.

(3.1.b) * is an anti-involution ${}^{\star}\colon \mathrm{R}\to\mathrm{R}$ such that $(\mathtt{C}^{\lambda}_{S,T})^{\star}=\mathtt{C}^{\lambda}_{T,S}$.

(3.1.c) $\mathtt{E}$ is a set of pairwise orthogonal, non-zero idempotents, all fixed by *, i.e. $\varepsilon^{\star}=\varepsilon$ for all $\varepsilon\in\mathtt{E}$. Further, $\mathtt{O}=\{<_{\varepsilon}\mid\varepsilon\in\mathtt{E}\}$ is a set of partial orders $<_{\varepsilon}$ on $\mathtt{X}$, and ϵ is a map $\epsilon\colon\coprod_{\lambda\in\mathtt{X}}\mathtt{M}(\lambda)\to\mathtt{E}$ sending S to $\epsilon(S)=\varepsilon_S$ such that

$$\text{(3-1)}\qquad \varepsilon\mathrm{R}\varepsilon\,\mathtt{C}^{\lambda}_{S,T}\in\mathrm{R}(\leq_{\varepsilon}\lambda),\qquad\qquad \text{(3-2)}\qquad \varepsilon\mathtt{C}^{\lambda}_{S,T}=\begin{cases}\mathtt{C}^{\lambda}_{S,T}, & \text{if } \varepsilon_S=\varepsilon,\\ 0, & \text{if } \varepsilon_S\neq\varepsilon,\end{cases}$$

for all $\lambda\in\mathtt{X}$, $S,T\in\mathtt{M}(\lambda)$ and $\varepsilon\in\mathtt{E}$.

(3.1.d) For $\lambda\in\mathtt{X}$, $S,T\in\mathtt{M}(\lambda)$ and $a\in\mathrm{R}$ we have

$$a\mathtt{C}^{\lambda}_{S,T}\in\textstyle\sum_{S'\in\mathtt{M}(\lambda)}r_a(S',S)\,\mathtt{C}^{\lambda}_{S',T}+\mathrm{R}(<_{\varepsilon_T}\lambda)\varepsilon_T,$$

with scalars $r_a(S',S)\in\Bbbk$ only depending on a,S,S'.

We call the set $\{\mathtt{C}^{\lambda}_{S,T}\mid\lambda\in\mathtt{X},S,T\in\mathtt{M}(\lambda)\}$ a *relative cellular basis.*

In the case $\mathtt{E}=\{1\}$ we call R a *cellular algebra*, and we write $<=<_1$.

The whole setup is called *graded* if the very same conditions as in [HM10, Definition 2.1] are satisfied (which can be easily adapted to the relative case). ▲

Note that the notion of a cellular algebra in the sense of [GL96, Definition 1.1] coincides with our definition here as one can easily check. In particular, a cellular algebra is relative cellular, but not conversely as will follow e.g. from Example 3.3 combined with Lemma 3.7.

The crucial examples. The following examples partially appeared in [ET17, Section 2E].

Example 3.2. Let Γ be a type A_3 graph. Then $\mathrm{Z}_{\rightleftarrows}$ is cellular and its cell datum is as follows. The anti-involution * is the linear extension of the assignment which swaps source and target of paths. Further, let $\mathtt{X}=\{\mathtt{0}<\mathtt{1}<\mathtt{2}<\mathtt{3}\}$, with $\mathtt{0}$ playing the role of a dummy, and let

$$\mathtt{M}(\mathtt{0})=\{\mathtt{1}{\to}\mathtt{2}\},\quad \mathtt{M}(\mathtt{1})=\{\mathtt{1},\mathtt{2}{\to}\mathtt{1}\},\quad \mathtt{M}(\mathtt{2})=\{\mathtt{2},\mathtt{3}{\to}\mathtt{2}\},\quad \mathtt{M}(\mathtt{3})=\{\mathtt{3}\},$$

which determines the cells $\mathtt{M}(\mathtt{i})\times\mathtt{M}(\mathtt{i})$ by $\mathtt{C}^{\mathtt{i}}_{S,T}=S\circ T^{\star}$ for $(S,T)\in\mathtt{M}(\mathtt{i})\times\mathtt{M}(\mathtt{i})$. Clearly, this is a graded structure. Imposing the vertex condition $\mathsf{C}=\{\mathtt{1}\}$ gives a cellular structure for $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$, but without any dummy. ▲

Example 3.3. Let Γ be a type $\widetilde{\mathsf{A}}_2$ graph. Then $\mathrm{Z}_{\rightleftarrows}$ is relative cellular and its cell datum is almost the same as in Example 3.2. The crucial differences are that there is no dummy cell, and one has two idempotents $\mathtt{E}=\{\mathtt{0},\varepsilon=\mathtt{1}+\mathtt{2}\}$ where the orderings are the same as in the finite case for ε, and the opposite for $\mathtt{0}$. ▲

3A. The case of $\mathrm{Z}_{\rightleftarrows}$.

Construction. For any Γ, let us define an anti-involution $^\star\colon Z_{\rightleftarrows} \to Z_{\rightleftarrows}$ by

$$\mathtt{i}^\star = \mathtt{i}, \qquad \mathtt{i}{\to}\mathtt{j}^\star = \mathtt{j}{\to}\mathtt{i}, \text{ for } \mathtt{i}{-}\mathtt{j},$$

which determines * completely. Further, we will always use the rule

$$\mathtt{C}^{\mathtt{i}}_{S,T} = S \circ T^\star, \text{ for } (S,T) \in \mathtt{M}(\mathtt{i}) \times \mathtt{M}(\mathtt{i}),$$

to give the cells $\mathtt{M}(\mathtt{i}) \times \mathtt{M}(\mathtt{i})$ which we are going to define now in the cases where Γ is a finite or affine type A graph.

For type A_n let the indexing set be $\mathtt{X} = \{\mathtt{0} < \mathtt{1} < \ldots < \mathtt{n}\}$ with the index $\mathtt{0}$ playing the role of a dummy. The cell sets $\mathtt{M}(\mathtt{i})$ are

$$\mathtt{M}(\mathtt{0}) = \{\mathtt{1}{\to}\mathtt{2}\}, \qquad \mathtt{M}(\mathtt{i}) = \{\mathtt{i}, \mathtt{i}{+}\mathtt{1}{\to}\mathtt{i}\}, \text{ for } \mathtt{i} \notin \{\mathtt{0}, \mathtt{n}\}, \qquad \mathtt{M}(\mathtt{n}) = \{\mathtt{n}\}.$$

The path length grading gives a way to view this datum as a graded datum by assigning to each element of $\mathtt{M}(\mathtt{i})$ the corresponding path length degree.

For type $\widetilde{\mathsf{A}}_n$ the cell datum is almost the same. The crucial differences are that the cells are now all of size four, and one has two idempotents $\mathtt{E} = \{\boldsymbol{\varepsilon} = \mathtt{1} + \cdots + \mathtt{n}{-}\mathtt{1}, \mathtt{n}\}$ where the orderings are the same as in the finite case for $\boldsymbol{\varepsilon}$, and the opposite for $\mathtt{n}$.

Proposition 3.4. The above defines the structure of a graded relative cellular algebra on $Z_{\rightleftarrows}$. This structure is the structure of a graded cellular algebra in case Γ is of type A_n. □

Proof. We only prove the claim in case Γ is a type $\widetilde{\mathsf{A}}_n$ graph, the rest follows then directly from the fact that idempotent truncations of relative cellular algebras are relative cellular, cf. [ET17, Proposition 2.8], as well as [ET17, Example 2.3]. First, (3.1.a) follows from Lemma 2.7, while (3.1.b) is immediate. The only statement from (3.1.c) which needs to be checked is (3-1) which follows since e.g. $\mathtt{n}Z_{\rightleftarrows}\mathtt{n} = \Bbbk\{\mathtt{n}, \mathtt{x_n}\}$. The last condition (3.1.d) is verified via a small calculation. The statement about the grading is immediate. ■

Elimination. The crucial lemma, which is a consequence of [KX98, Proposition 3.2] and [ET17, Corollary 3.25], is:

Lemma 3.5. If $Z_{\rightleftarrows}$ is cellular, then its Cartan matrix $\boldsymbol{C}$ is positive definite. If $Z_{\rightleftarrows}$ is relative cellular, then $\boldsymbol{C}$ is positive semidefinite. ■

For any relative cellular algebra, recall that for each $\lambda \in \mathtt{X}$ there is an associated *cell module* Δ_λ. Define the *decomposition numbers* $d_{\mu\lambda}$ as the multiplicity of L_λ in Δ_μ, where $\lambda \in \mathtt{X}_0$ correspond to idempotent cells. In particular, $d_{\mu\lambda} \in \mathbb{Z}_{\geq 0}$. In the case of $Z_{\rightleftarrows}$ we will identify $\mathtt{X}_0 = \{\mathtt{i} \mid \mathtt{i} \text{ vertex of } \Gamma\}$, and if we want to sort these numbers into a matrix $\boldsymbol{D}$, then we will always enumerate columns as indicated by the enumeration of the vertices. Further, we enumerate the rows by first using the elements from $\mathtt{X}_0$ (in the enumeration of the vertices) followed by the remaining elements with a fixed, arbitrary enumeration. By [GL96, Proposition 3.6] or [ET17, Theorem 3.23] we get:

Lemma 3.6. If $Z_{\rightleftarrows}$ is relative cellular, then $\boldsymbol{C} = \boldsymbol{D}^{\mathrm{T}}\boldsymbol{D}$ where $d_{\mu\mathtt{i}} = 0$ unless $\mu <_{\boldsymbol{\varepsilon}(\mathtt{i})} \mathtt{i}$, and $d_{\mathtt{ii}} = 1$, where $\boldsymbol{\varepsilon}(\mathtt{i})$ is the unique idempotent in $\mathtt{E}$ with $\boldsymbol{\varepsilon}(\mathtt{i})\mathtt{i} = \mathtt{i}$. ■

The following lemma and the ideas in its proof, will reappear throughout.

Lemma 3.7. If Γ is a bipartite graph which is not an ADE graph, then $Z_{\rightleftarrows}$ is not cellular. If Γ is a bipartite graph which is not a finite or affine ADE graph, then $Z_{\rightleftarrows}$ is not relative cellular. □

Proof. Let Γ be bipartite. Then we claim the following, where we added the (graded) Cartan determinants for later use, since they can be computed using the numerical data given in the references below.

Claim. $\boldsymbol{C}$ is positive definite if and only if Γ is a type ADE graph. $\boldsymbol{C}$ is positive semidefinite if and only if Γ is a finite or affine type ADE graph. Further, in these cases

(3-3)

	A_n	D_n	E_6	E_7	E_8
det	$n+1$	4	3	2	1
qdet	$(n+1)_{\mathsf{q}}$	$(1+\mathsf{q}^{2n-2})2_{\mathsf{q}}$	$(1+\mathsf{q}^{10})2_{\mathsf{q}}$ $-\mathsf{q}^{6}1_{\mathsf{q}}$	$(1+\mathsf{q}^{12})2_{\mathsf{q}}$ $-\mathsf{q}^{6}2_{\mathsf{q}}$	$(1+\mathsf{q}^{14})2_{\mathsf{q}}$ $-\mathsf{q}^{6}3_{\mathsf{q}}$

$\widetilde{\mathsf{A}}_n$	$\widetilde{\mathsf{D}}_n$	$\widetilde{\mathsf{E}}_6$	$\widetilde{\mathsf{E}}_7$	$\widetilde{\mathsf{E}}_8$
4 or 0	0	0	0	0
$\mathsf{q}^{2n+2}+1$ $+(-1)^n 2\mathsf{q}^{n+1}$	$(1+\mathsf{q}^{2n})2_{\mathsf{q}}$ $-(\mathsf{q}^{4}+\mathsf{q}^{2n-4})2_{\mathsf{q}}$	$(1+\mathsf{q}^{12})2_{\mathsf{q}}$ $-(\mathsf{q}^{6}+\mathsf{q}^{6})2_{\mathsf{q}}$	$(1+\mathsf{q}^{14})2_{\mathsf{q}}$ $-(\mathsf{q}^{6}+\mathsf{q}^{8})2_{\mathsf{q}}$	$(1+\mathsf{q}^{16})2_{\mathsf{q}}$ $-(\mathsf{q}^{6}+\mathsf{q}^{10})2_{\mathsf{q}}$

are the Cartan determinants, respectively the graded Cartan determinants where we let $a_{\mathsf{q}} = 1+\mathsf{q}^2+\dots+\mathsf{q}^{2a-2}+\mathsf{q}^{2a}$ for $a \in \mathbb{Z}_{\geq 0}$.

Proof of the claim. The eigenvalues of the adjacency matrices of finite or affine type ADE graphs are known, cf. [Smi70] or [BH12, Section 3.1.1], and they all are in the interval $]-2,2[$ for the finite types, or in the interval $[-2,2]$ for the affine types. Moreover, by the same references, the converse is also true: If Γ is a graph whose adjacency matrix has eigenvalues contained in $[-2,2]$, then Γ is a finite or affine type ADE graph. In particular, all other graphs have a Perron–Frobenius eigenvalue strictly bigger than 2. Finally, if Γ is bipartite, then they also have an eigenvalue strictly smaller than -2, and the claim follows by Proposition 2.8.

Then the lemma itself follows from this claim and Lemma 3.5. ■

Lemma 3.8. If Γ is a type $\widetilde{\mathsf{A}}$ graph, then $\mathrm{Z}_{\rightleftarrows}$ is not cellular. □

Proof. Assume that $\mathrm{Z}_{\rightleftarrows}$ is cellular. Then, by Proposition 2.8 and Lemma 3.6, we get

$$\begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 1 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 1 & \ddots & \vdots \\ 0 & 0 & 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} = \boldsymbol{C} = \boldsymbol{D}^{\mathrm{T}}\boldsymbol{D} = \begin{pmatrix} 1 & d_{21} & d_{31} & \cdots \\ d_{12} & 1 & d_{32} & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 & d_{12} & \cdots \\ d_{21} & 1 & \ddots \\ d_{31} & d_{32} & \ddots \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

We observe that this implies that each column of $\boldsymbol{D}$ contains precisely two non-zero entries, both of which are equal to 1. One of these is the number $d_{\mathtt{ii}}$, the other we will write as $d_{a(\mathtt{i}),\mathtt{i}}$. If $\mathtt{i},\mathtt{j}$ are non-adjacent vertices of Γ, then it now follows that $a(\mathtt{i}) \neq a(\mathtt{j})$ and $d_{\mathtt{ij}} = 0 = d_{\mathtt{ji}}$. It also follows that for $\mathtt{i}–\mathtt{j}$ we can only have three distinct cases:

$$\text{(i)}\colon d_{\mathtt{ij}} = 1, \qquad \text{(ii)}\colon d_{\mathtt{ji}} = 1, \qquad \text{(iii)}\colon a(\mathtt{i}) = a(\mathtt{j}) \in \mathtt{X} - \mathtt{X}_0.$$

The case (iii) does not give any solution as long as we have four or more vertices. In the other cases the matrix $\boldsymbol{D}$ has to be of either form

$$\text{(i) and (ii)}\colon \boldsymbol{D} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \text{ or } \boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad \text{(iii)}\colon \boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 1 & 1 \end{pmatrix},$$

here exemplified for four vertices for (i) and (ii). However, it is impossible to permute (by simultaneous row and column reenumeration) the two leftmost matrices into upper triangular matrices, contradicting Lemma 3.6, and the claim follows for $n \geq 3$. For type $\widetilde{\mathsf{A}}_2$ we need an extra argument to rule out (iii). In this case we would have four cell modules (corresponding to the rows of $\boldsymbol{D}$), three of which are simple and one of dimension three containing all simples in its filtration. Using (2-1), this gives a contradiction to [GL96, Lemmas 2.9 and 2.10], since there is no way to filter the projectives by these cell modules in any order compatible way, because all three indecomposable projective modules have non-equivalent socles and thus, could not agree with the socle of the three-dimensional cell module. ■

Lemma 3.9. If Γ is not a bipartite graph, then $\mathrm{Z}_{\rightleftarrows}$ is not cellular. □

Proof. A non-bipartite graph has a subgraph of type $\widetilde{\mathsf{A}}_{n=2m}$, and the claim follows by using Lemma 3.8 since idempotent truncations of cellular algebras are cellular by the same arguments as in e.g. [KX99, Section 6]. ■

Lemma 3.10. If Γ is a type D_4 graph, then $\mathrm{Z}_{\rightleftarrows}$ is not relative cellular. □

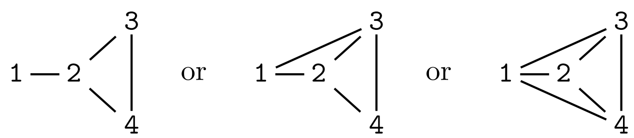
Proof. Assume relative cellularity. Then we get

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} = \boldsymbol{C} = \boldsymbol{D}^{\mathrm{T}}\boldsymbol{D} = \left(\begin{array}{cccc|cc} 1 & d_{21} & d_{31} & d_{41} & d_{51} & \cdots \\ d_{12} & 1 & d_{32} & d_{42} & d_{52} & \cdots \\ d_{13} & d_{23} & 1 & d_{43} & d_{53} & \cdots \\ d_{14} & d_{24} & d_{34} & 1 & d_{54} & \cdots \end{array}\right) \left(\begin{array}{cccc} 1 & d_{12} & d_{13} & d_{14} \\ d_{21} & 1 & d_{23} & d_{24} \\ d_{31} & d_{32} & 1 & d_{34} \\ d_{41} & d_{42} & d_{43} & 1 \\ \hline d_{51} & d_{52} & d_{53} & d_{54} \\ \vdots & \vdots & \vdots & \vdots \end{array}\right)$$

by the same arguments as before in the proof of Lemma 3.8. Actually, we get the very same conditions for the entries of $\boldsymbol{D}$, depending on the connectivity of the vertices only. But this case is easier than the case of type $\widetilde{\mathsf{A}}$ since there is no possible solution. ■

Lemma 3.11. If Γ is a finite or affine type DE graph, then $\mathrm{Z}_{\rightleftarrows}$ is not relative cellular. If Γ is a non-bipartite graph which is not a type $\widetilde{\mathsf{A}}_{n=2m}$ graph, then $\mathrm{Z}_{\rightleftarrows}$ is not relative cellular. □

Proof. Since idempotent truncations of relative cellular algebras would be relative cellular, cf. [ET17, Proposition 2.8], the statement follows from Lemma 3.10, and by observing that the three cases of a trivalent vertex containing no subgraph of type D_4, namely



have -1 as an eigenvalue for their adjacency matrices, contradicting Lemma 3.5. ■

The proof. With the work already done we get:

Proof of Theorem A. Proposition 3.4 shows the existence of a cellular structure for finite or affine type A graphs. Conversely, Lemmas 3.7, 3.9 and 3.11 prove that none of the remaining cases can be (relative) cellular. ■

3B. The case of $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$.

Construction. Let Γ be a type A_n graph, and let $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}(\mathsf{A}_n)$ be the associated zigzag algebra with vertex condition $\mathsf{C} = \{\mathtt{1}\}$ or $\mathsf{C} = \{\mathtt{n}\}$. In this case the construction of the cell datum works verbatim as for $\mathrm{Z}_{\rightleftarrows}(\mathsf{A}_n)$ with the only difference that we do not need a dummy cell. In particular, the following can be proven, mutatis mutandis, as Proposition 3.4.

Proposition 3.12. The above defines a graded cell datum for $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}(\mathsf{A}_n)$. ■

Elimination. For completeness:

Lemma 3.13. If $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ is relative cellular, then $\det(\boldsymbol{C}^{\mathsf{C}}) \geq 0$. ■

Lemma 3.14. Let Γ be of finite or affine ADE graph. If $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ is relative cellular, then Γ is a type A_n graph and $\mathsf{C} = \{\mathtt{1}\}$ or $\mathsf{C} = \{\mathtt{n}\}$. □

Proof. The claim follows from a direct application of Lemmas 3.13 and 2.11: For $\mathsf{C} = \{\mathtt{c}\}$ a small case-by-case check verifies that

type	A_n	$\mathsf{D}, c{=}1$	$\mathsf{D}_4, c{=}2$	$\mathsf{D}_{>4}, c{>}1$	E	$\widetilde{\mathsf{A}}_{2m}$	$\widetilde{\mathsf{A}}_{2m+1}$	$\widetilde{\mathsf{D}}$	$\widetilde{\mathsf{E}}$
Cdet	$n{+}1{-}c(n{-}c{+}1)$	0	<0	<0	<0	$4{-}2m{-}1$	<0	<0	<0

where we have used the data collected in (3-3). Since $n + 1 - c(n - c + 1) \geq 0$ holds only in the cases $\mathsf{C} = \{\mathtt{1}\}$ or $\mathsf{C} = \{\mathtt{n}\}$, with one exception $\mathsf{C} = \{\mathtt{2}\}$ and $n = 3$, and $4 - 2m - 1 \geq 0$ holds only if $m = 1$, it remains to rule out those cases.

Type A_3 *with* $\mathsf{C} = \{\mathtt{2}\}$. Assuming relative cellularity, reciprocity would give

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \boldsymbol{C}^{\mathsf{C}} = \boldsymbol{D}^{\mathrm{T}}\boldsymbol{D} = \left(\begin{array}{ccc|cc} 1 & d_{21} & d_{31} & d_{41} & \cdots \\ d_{21} & 1 & d_{32} & d_{42} & \cdots \\ d_{31} & d_{32} & 1 & d_{43} & \cdots \end{array}\right)\left(\begin{array}{ccc} 1 & d_{12} & d_{13} \\ d_{21} & 1 & d_{23} \\ d_{31} & d_{32} & 1 \\ \hline d_{41} & d_{42} & d_{43} \\ \vdots & \vdots & \vdots \end{array}\right),$$

which is impossible as one can easily check.

Type D_n *with* $\mathsf{C} = \{\mathtt{1}\}$. First assume that $n \geq 5$. Then there would be a type D_4 subgraph without vertex condition and idempotent truncation gives a contradiction to Lemma 3.10. The remaining case, D_4 with vertex on any leaf, can be ruled out with the same matrix comparison as we have used above.

Type $\widetilde{\mathsf{A}}_2$. We first observe that in this case all three possibilities to impose the vertex condition give isomorphic algebras. So let $\mathsf{C} = \{\mathtt{0}\}$. Then reciprocity can only work with

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \boldsymbol{C}^{\mathsf{C}} = \boldsymbol{D}^{\mathrm{T}}\boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as the usual reasoning shows. This would mean that we have three cell modules, with Δ_0 being of dimension three, and the other two being simple. By (2-1), we also have projectives

$$\mathrm{P}_{\mathtt{0}}^{\mathsf{C}} = \begin{array}{c} \mathtt{0} \\ \mathtt{1}{\to}\mathtt{0}\ \ \mathtt{2}{\to}\mathtt{0} \end{array}, \qquad \mathrm{P}_{\mathtt{1}} = \begin{array}{c} \mathtt{1} \\ \mathtt{0}{\to}\mathtt{1}\ \ \mathtt{2}{\to}\mathtt{1} \\ \mathrm{x}_1 \end{array}, \qquad \mathrm{P}_{\mathtt{2}} = \begin{array}{c} \mathtt{2} \\ \mathtt{0}{\to}\mathtt{2}\ \ \mathtt{1}{\to}\mathtt{2} \\ \mathrm{x}_2 \end{array}$$

with head $\mathrm{P}_{\mathtt{0}}^{\mathsf{C}}$ being $\mathrm{L}_{\mathtt{0}}$, and the heads of $\mathrm{P}_{\mathtt{1}}$ and $\mathrm{P}_{\mathtt{2}}$ are $\mathrm{L}_{\mathtt{1}} = \Delta_1$ and $\mathrm{L}_{\mathtt{2}} = \Delta_2$, respectively. However, any indecomposable projective has a filtration by cell modules, see [ET17, Proposition 3.19], but only Δ_0 contains $\mathrm{L}_{\mathtt{0}}$, which gives a contradiction.

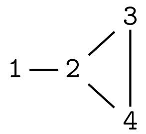
If C has more than one vertex, then, with the same idempotent truncation arguments as above, we only need to rule out the case of Γ being a type A_2 graph with $\mathsf{C} = \{\mathtt{1}, \mathtt{2}\}$ or a type D_4 graph for any C, or a type A_3 graph with $\mathsf{C} = \{\mathtt{2}\}$. Most of these have already been ruled out above. By symmetry, for D_4 we just calculate

type	$\mathsf{D}_4, \mathsf{C} = \{\mathtt{1},\mathtt{2}\}$	$\mathsf{D}_4, \mathsf{C} = \{\mathtt{1},\mathtt{3}\}$	$\mathsf{D}_4, \mathsf{C} = \{\mathtt{1},\mathtt{2},\mathtt{3}\}$	$\mathsf{D}_4, \mathsf{C} = \{\mathtt{1},\mathtt{2},\mathtt{3},\mathtt{4}\}$
C-det	-4	-1	-3	-2

which rules out this case by Lemma 3.13. Finally, the case of a type A_2 graph with $\mathsf{C} = \{\mathtt{1}, \mathtt{2}\}$ one again gets no matrix solution for reciprocity. ■

Lemma 3.15. If Γ is not a finite or affine ADE graph, then $\mathrm{Z}_{\rightleftarrows}^{\mathsf{C}}$ is not relative cellular. □

Proof. Assume that $\mathrm{Z}_{\rightleftarrows}^{\mathsf{C}}$ is relative cellular. By the usual subgraph-truncation argument and by Lemma 3.10 as well as Lemma 3.14, we see that Γ can not contain a type D_4 subgraph (with or without any $\mathtt{c} \in \mathsf{C}$ being on this subgraph). Using the same arguments and Lemma 3.14, we see that Γ can not contain a type $\widetilde{\mathsf{A}}$ subgraph with $\mathsf{c} \in \mathsf{C}$ on it. Since Γ is not a type $\widetilde{\mathsf{A}}$ graph itself, it remains to rule out one case, i.e.



for $\mathsf{C} = \{\mathsf{1}\}$.

(All other configurations of a trivalent vertex containing no D_4 subgraph contain a subgraph of type $\widetilde{\mathsf{A}}_2$ with at least one vertex condition.) The same reciprocity reasoning as before give only one potential solution, namely

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} = \boldsymbol{C}^{\mathsf{C}} = \boldsymbol{D}^{\mathrm{T}}\boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

from where we can read of the cell modules. Again, this gives a contradiction to the filtration of the indecomposable projectives by cell modules. ■

The proof. Nothing remains to be done:

Proof of Theorem A′. Combine Proposition 3.12 with Lemmas 3.14 and 3.15 ■

In fact, by using [AT17, Proposition 3.9], [AST18, Theorem 3.9] and [BT17, Theorem A.13], a non-trivial consequence of Theorem A′ is that zigzag algebras (with or without vertex condition) can only appear as endomorphism algebras of duality stable tilting modules in certain highest weight categories if and only if Γ is a type A graph.

4. Quasi-heredity

A brief reminder. We start by recalling the definition of a quasi-hereditary algebra as it appears e.g. in [CPS88, Below Example 3.3]. To this end, a (two-sided) ideal J in a finite-dimensional algebra R, which is projective as a left R-module, is called *hereditary* if $\mathrm{JJ} = \mathrm{J}$ and $\mathrm{J}\mathrm{Rad}(\mathrm{R})\mathrm{J} = 0$ hold, where $\mathrm{Rad}(\mathrm{R})$ is the *Jacobson radical* of R.

Definition 4.1. A finite-dimensional algebra R is called *quasi-hereditary* if there exists a chain of ideals

$$0 = \mathrm{J}_0 \subset \mathrm{J}_1 \subset \cdots \subset \mathrm{J}_{k-1} \subset \mathrm{J}_k = \mathrm{R},$$

for some $k \in \mathbb{Z}_{\geq 1}$, such that the quotients $\mathrm{J}_l/\mathrm{J}_{l-1}$ are hereditary ideals in $\mathrm{R}/\mathrm{J}_{l-1}$. ▲

A chain as in Definition 4.1 is called a *hereditary chain*.

The crucial example.

Example 4.2. Let Γ be a type A_3 graph, and let $\mathsf{C} = \{\mathsf{1}\}$. Let $\mathrm{J}_1 = \mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}(\mathsf{1})\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$, $\mathrm{J}_2 = \mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}(\mathsf{1}+\mathsf{2})\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ and $\mathrm{J}_3 = \mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}(\mathsf{1}+\mathsf{2}+\mathsf{3})\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$, i.e.

$$\mathrm{J}_1 = \Bbbk\{\mathsf{1}, \mathsf{2}{\to}\mathsf{1}, \mathsf{1}{\to}\mathsf{2}, \mathtt{x}_2\}, \quad \mathrm{J}_2 = \Bbbk\{\mathsf{2}, \mathsf{3}{\to}\mathsf{2}, \mathsf{2}{\to}\mathsf{3}, \mathtt{x}_3\} \oplus \mathrm{J}_1, \quad \mathrm{J}_3 = \Bbbk\{\mathsf{3}\} \oplus \mathrm{J}_1 \oplus \mathrm{J}_2,$$

where we recall that $\mathtt{x}_2 = \mathsf{2}{\to}\mathsf{1}{\to}\mathsf{2}$ and $\mathtt{x}_3 = \mathsf{3}{\to}\mathsf{2}{\to}\mathsf{3}$, while $\mathtt{x}_1 = 0$ because $\mathsf{C} = \{\mathsf{1}\}$. It follows that these form a hereditary chain. ▲

4A. The case of $\mathrm{Z}_{\rightleftarrows}$.

Proof of Theorem B. The discussion about simple $\mathrm{Z}_{\rightleftarrows}$-modules in Section 2 shows that the Jacobson radical $\mathrm{Rad}(\mathrm{Z}_{\rightleftarrows})$ is equal to the span of all paths of positive length. In particular, all volume elements $\mathtt{x}_\mathtt{i}$ are in $\mathrm{Rad}(\mathrm{Z}_{\rightleftarrows})$. Hence, $\mathrm{Z}_{\rightleftarrows}$ can never be quasi-hereditary since J_1 could not contain any idempotent, because any such idempotent would be a sum of vertex idempotents. Thus, we would have $\mathrm{J}_1\mathrm{Rad}(\mathrm{Z}_{\rightleftarrows})\mathrm{J}_1 \neq 0$. ■

4B. **The case of $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$.** Before we start, note that the Jacobson radical $\mathrm{Rad}(\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows})$ of $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ is, as in the case of $\mathrm{Z}_{\rightleftarrows}$, the span of all paths of positive length.

Construction. Let Γ be a type A_n graph, and let $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ the associated zigzag algebra with vertex condition $\mathsf{C}=\{\mathtt{1}\}$. Let $\mathrm{J}_0=0$, and for each $\mathtt{i}\in\{1,\dots,n\}$ we define

$$\mathrm{J}_i=\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}(\mathtt{1}+\dots+\mathtt{i})\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}. \tag{4-1}$$

Similarly, but reversing the summation order, in case $\mathsf{C}=\{\mathtt{n}\}$.

The following is just a summary of Example 4.2.

Proposition 4.3. Assignment (4-1) gives rise to a hereditary chain for $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}(\mathsf{A}_n)$ with vertex condition $\mathsf{C}=\{\mathtt{1}\}$. Similarly in case of $\mathsf{C}=\{\mathtt{n}\}$. □

Proof. By construction, the J_i are ideals in $\mathrm{Z}_{\rightleftarrows}$ and form a chain as in Definition 4.1. Moreover, $\mathrm{J}_i/\mathrm{J}_{i-1}$ contains precisely one idempotent (which we identified with $\mathtt{i}$), paths of length one either starting or ending at $\mathtt{i}$ and the volume element $\mathtt{x}_{\mathtt{i+1}}$. That is, we have

$$\mathrm{J}_1=\Bbbk\{\mathtt{1},\mathtt{2}\to\mathtt{1},\mathtt{1}\to\mathtt{2},\mathtt{x}_{\mathtt{2}}=\mathtt{2}\to\mathtt{1}\to\mathtt{2}\},\quad \mathrm{J}_n/\mathrm{J}_{n-1}=\Bbbk\{\mathtt{n}\},$$
$$\mathrm{J}_i/\mathrm{J}_{i-1}=\Bbbk\{\mathtt{i},\mathtt{i+1}\to\mathtt{i},\mathtt{i}\to\mathtt{i+1},\mathtt{x}_{\mathtt{i+1}}=\mathtt{i+1}\to\mathtt{i}\to\mathtt{i+1}\},\quad i\in\{2,\dots,n-1\},$$

which shows that $\mathrm{J}_i/\mathrm{J}_{i-1}$ is an idempotent ideal. To check the other two conditions we observe that $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}(\mathsf{A}_n)/\mathrm{J}_i\cong\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}(\mathsf{A}_{n-i})$ with vertex condition imposed on its first leaf. This means we only need to check these two conditions for J_1 where we get

$$\mathrm{J}_1\cong\mathrm{P}_{\mathtt{1}}\oplus\mathrm{P}_{\mathtt{1}},\qquad \mathrm{J}_1\mathrm{Rad}(\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows})\mathrm{J}_1=0,$$

cf. (2-1). By symmetry, the same works with the vertex condition $\mathsf{C}=\{\mathtt{n}\}$. ■

Elimination. We recall a consequence of a classical result, cf. [BF89, Proposition 1.3].

Lemma 4.4. If $0=\mathrm{J}_0\subset\mathrm{J}_1\subset\dots\subset\mathrm{J}_{k-1}\subset\mathrm{J}_k=\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$, is a hereditary chain of $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$, then $\det(\boldsymbol{C}^{\mathsf{C}}(\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}/\mathrm{J}_{l-1}))=1$ for all $l\in\{1,\dots,k\}$. ■

Lemma 4.5. Let Γ be of finite or affine ADE graph, and assume that $\mathsf{C}=\{\mathtt{c}\}$. If $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ is quasi-hereditary, then Γ is a type A_n graph and $\mathsf{C}=\{\mathtt{1}\}$ or $\mathsf{C}=\{\mathtt{n}\}$. □

Proof. By Lemmas 4.4 and very similar to the proof of Lemma 3.14 (by calculating determinants), it remains to check the case that Γ is a type $\widetilde{\mathsf{A}}_2$ graph with one vertex condition. To this end, we first observe that in this case all three possibilities to impose the vertex condition give isomorphic algebras. So let $\mathsf{C}=\{\mathtt{0}\}$. Then, by e.g. [BF89, Corollary 1.2], one would need to set

$$\mathrm{J}_1=\Bbbk\{\mathtt{0},\mathtt{1}\to\mathtt{0},\mathtt{0}\to\mathtt{1},\mathtt{x}_{\mathtt{1}},\mathtt{2}\to\mathtt{0},\mathtt{0}\to\mathtt{1},\mathtt{x}_{\mathtt{2}}\}.$$

But we have $\det(\boldsymbol{C}^{\mathsf{C}}(\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}/\mathrm{J}_1))=0$, contradicting Lemmas 4.4. ■

Lemma 4.6. If Γ is not a type A_n graph with $\mathsf{C}=\{\mathtt{1}\}$ or $\mathsf{C}=\{\mathtt{n}\}$, then $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ is not quasi-hereditary. □

Proof. Assume that $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ is quasi-hereditary. Again, by [BF89, Corollary 1.2], the ideal J_1 has to contain a primitive idempotent for some $\mathtt{c}\in\mathsf{C}$, and we get

$$\mathrm{J}_1=\Bbbk\{\mathtt{c},\mathtt{j}\to\mathtt{c},\mathtt{c}\to\mathtt{j},\mathtt{x}_{\mathtt{j}},\mathtt{d}\to\mathtt{c},\mathtt{c}\to\mathtt{d}\mid\mathtt{c}\text{–}\mathtt{j},\mathtt{j}\notin\mathsf{C},\mathtt{c}\text{–}\mathtt{d},\mathtt{d}\in\mathsf{C}\}$$
$$\cong\mathrm{P}^{\mathsf{C}}_{\mathtt{c}}\oplus\textstyle\bigoplus_{\mathtt{c}-\mathtt{j},\mathtt{j}\notin\mathsf{C}}\Bbbk\{\mathtt{c}\to\mathtt{j},\mathtt{x}_{\mathtt{j}}\}\oplus\bigoplus_{\mathtt{c}-\mathtt{d},\mathtt{d}\in\mathsf{C}}\Bbbk\{\mathtt{c}\to\mathtt{d}\},$$

where the decomposition follows from the structure of $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$. If C does not contain a leaf, then each indecomposable projective of $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ is of dimension at least three, cf. (2-1), and J_1 can not be a projective $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$-module. Thus, assume that $\mathtt{c}\in\mathsf{C}$ is a leaf that gives J_1. By the same argument as above, $\mathtt{c}$ could not have a neighbor $\mathtt{c}\in\mathsf{C}$ since each indecomposable projective of $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$ in this case is of dimension at least two. In the remaining case, i.e. $\mathtt{c}\in\mathsf{C}$ is a leaf and its neighbor $\mathtt{c}\text{–}\mathtt{j}$ is not in C, we can consider $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}/\mathrm{J}_1$ which would be quasi-hereditary, cf. [CPS88, Below Example 3.3]. But this would recursively give a contradiction, since we have $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}/\mathrm{J}_1\cong\mathrm{Z}^{\mathsf{C}-\mathtt{c}}_{\rightleftarrows}(\Gamma-\mathtt{c})$ where the vertex condition for $\mathrm{Z}^{\mathsf{C}-\mathtt{c}}_{\rightleftarrows}(\Gamma-\mathtt{c})$ is at $\mathtt{j}$: If $|\mathsf{C}|\geq 2$, then, at one point, one has two neighboring vertex conditions, contradicting the above observation. If $\mathsf{C} = \{\mathtt{c}\}$, then one will have a finite or affine type ADE graph with one vertex condition. However, by Lemma 4.5, the only case where this would not give a contradiction is the case with Γ being a type A_n graph with $\mathsf{C} = \{\mathtt{1}\}$ or $\mathsf{C} = \{\mathtt{n}\}$. ■

The proof. We collect the harvest:

Proof of Theorem B′. Combine Proposition 4.3, which constructs the hereditary chain, with Lemma 4.5, which rules out all other cases. ■

5. Koszulity

A brief reminder. We start one definition of a Koszul algebra, cf. [BGS96, Definition 1.1.2] or [PP05, Section 2.2]. Recall here that a *linear projective resolution* of a graded module M of a finite-dimensional, positively graded algebra R is an exact sequence

$$\cdots \longrightarrow \mathsf{q}^2 \mathrm{Q}_2 \longrightarrow \mathsf{q}\mathrm{Q}_1 \longrightarrow \mathrm{Q}_0 \twoheadrightarrow \mathrm{M}, \tag{5-1}$$

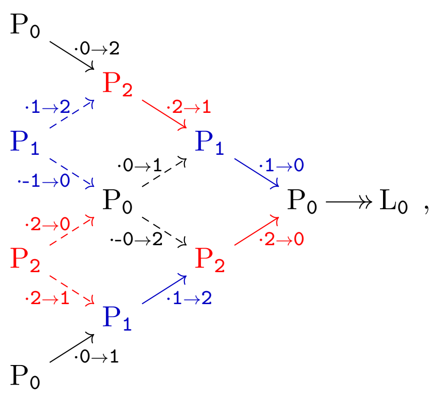
with graded projective R-modules $\mathsf{q}^t \mathrm{Q}_t$ (for us this is the t^{th} part of the resolution) generated in degree t, and R-equivariant maps of degree 0. Using our grading conventions from Section 2, this is the same data as giving an exact sequence of homogeneous, R-equivariant maps of degree 1 between the graded projectives Q_t.

Definition 5.1. A finite-dimensional, positively graded algebra R is called *Koszul* if its degree 0 part is semisimple and each simple R-module admits a linear projective resolution. ▲

For us only the property of having a linear projective resolution will play a role; our algebras are evidently finite-dimensional, positively graded and have a semisimple degree 0 part. Moreover, up to shifts, we can focus on simples which are concentrated in degree 0.

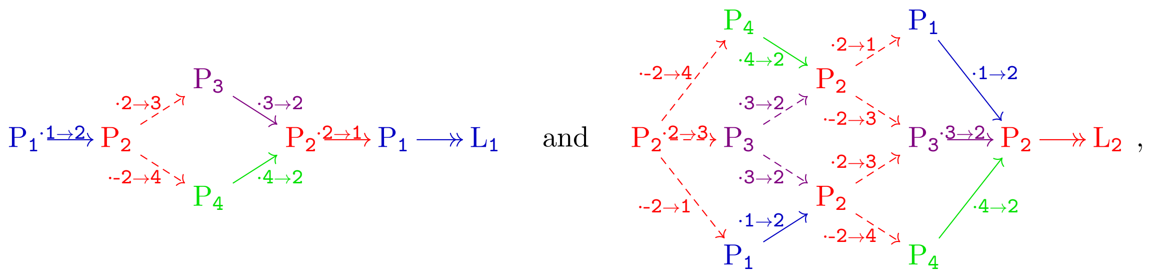
The crucial examples.

Example 5.2. Let Γ be a type $\widetilde{\mathsf{A}}_2$ graph. In this case it is easy to write down a linear projective resolution of L_0 (of course, the others are similar):



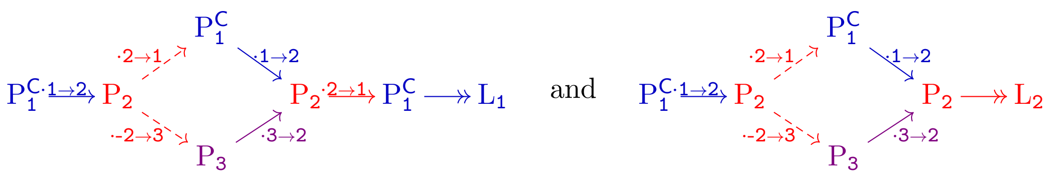
where the rightmost map is the projection, while the other maps are given by post-composition (which commutes with the left action given by pre-composition) with the corresponding paths, where the dashed arrows hit linear combinations in the kernels. For example, because $0{\to}1{\to}0 = \mathtt{x}_0 = 0{\to}2{\to}0$, the element $0{\to}1 - 0{\to}2$ is in the 1^{st} kernel and we use two maps from P_0 to compensate for it. Since these are of degree 1, we get a linear projective resolution by continuation of the sketched pattern. ▲

Example 5.3. Let Γ be a type D_4 graph. We try to resolve L_1 and L_2 linearly by projectives:



where we use the same conventions as in Example 5.2. The 4^{th} parts have kernels supported in degree 2, and we are stuck. (Note that this happens after the same number of steps.) As we will see, the same holds for all type ADE graphs.

In contrast, if we add the vertex condition $\mathsf{C} = \{\mathtt{1}\}$ to the type A graph, then the linear projective resolutions exist and are all finite. This happens since $\mathrm{P}_{\mathtt{1}}^{\mathsf{C}}$ will not contain a volume element, e.g.



$$\text{and} \quad \mathrm{P}_{\mathtt{1}}^{\mathsf{C}} \xrightarrow{\cdot \mathtt{1}\to\mathtt{2}} \mathrm{P}_{\mathtt{2}} \xrightarrow{\cdot \mathtt{2}\to\mathtt{3}} \mathrm{P}_{\mathtt{3}} \twoheadrightarrow \mathrm{L}_{\mathtt{3}} \ ,$$

where we calculate the resolutions for all simple $\mathrm{Z}_{\rightleftarrows}^{\mathsf{C}}(\mathsf{A}_3)$-modules. (In this case all resolutions are finite, but of different length. We will see below that this is in fact always the case for the type A graphs with vertex condition imposed on one leaf.) ▲

5A. The case of $\mathrm{Z}_{\rightleftarrows}$.

Construction. We first show abstractly that $\mathrm{Z}_{\rightleftarrows}$ is Koszul in case Γ is not a type ADE graph. Then we construct the linear projective resolution explicitly using a non-terminating algorithm motivated by Chebyshev polynomials.

Before showing koszulity we make the following observation, which is inspired by [Ben08, Section 2]. Let us write $\mathrm{Q}(\_)$ for the projective cover of a module. If K_0 is a $\mathrm{Z}_{\rightleftarrows}$-module which has a radical filtration of length 2 with multiplicity vectors $\underline{a}_0$ of $\mathrm{K}_0/\mathrm{rad}(\mathrm{K}_0)$, respectively $\underline{b}_0$ of $\mathrm{rad}(\mathrm{K}_0)/\mathrm{rad}^2(\mathrm{K}_0)$, then we have a short exact sequence $\mathrm{K}_1 \hookrightarrow \mathrm{Q}(\mathrm{K}_0) \twoheadrightarrow \mathrm{K}_0$, where $\underline{a}_1 = \boldsymbol{A}\underline{a}_0 - \underline{b}_0$, respectively $\underline{b}_1 = \underline{a}_0$, are the corresponding multiplicity vectors of the radical filtration of K_1. Further, if K_0 is graded, then so is K_1.

Assume that $\underline{a}_1 \neq 0$. If K_0 is generated in degree 0 and its radical filtration is equal to its grading filtration, then K_1 is generated in degree 1, and its radical filtration is also equal to its grading filtration. This holds since $\mathrm{rad}(\mathrm{K}_1)$ lies inside $\mathrm{rad}^2(\mathrm{Q}(\mathrm{K}_0))$, hence is of degree 2, and $\underline{a}_1$ agrees with the multiplicities of the kernel of $\mathrm{Q}(\mathrm{K}_0) \twoheadrightarrow \mathrm{K}_0$ in degree 1. Thus, the filtrations agree and K_1 is generated in degree 1. Hence, as long as $\underline{a}_t \neq 0$, one can produce $\mathrm{Z}_{\rightleftarrows}$-modules K_t for all $t \in \mathbb{Z}_{\geq 0}$ in the same manner having the same properties.

Now come some of our main players in this section, the *Chebyshev polynomials* (of the second kind). They are defined via the recursion

$$U_{-1}(\mathsf{X}) = 0, \quad U_0(\mathsf{X}) = 1, \quad U_t(\mathsf{X}) = \mathsf{X}U_{t-1}(\mathsf{X}) - U_{t-2}(\mathsf{X}), \text{ for } t \in \mathbb{Z}_{\geq 1}. \tag{5-2}$$

Having the polynomials defined, observe that, for $t \in \mathbb{Z}_{\geq 0}$, the t^{th} multiplicity vectors of the radical filtration of K_t are given by

$$\begin{pmatrix} \underline{a}_t \\ \underline{b}_t \end{pmatrix} = \begin{pmatrix} U_t(\boldsymbol{A}) & -U_{t-1}(\boldsymbol{A}) \\ U_{t-1}(\boldsymbol{A}) & -U_{t-2}(\boldsymbol{A}) \end{pmatrix} \begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix}, \tag{5-3}$$

where we let $U_{-2}(\mathsf{X}) = 0$ in case $t = 0$.

Proposition 5.4. If Γ is not a type ADE graph, then $\mathrm{Z}_{\rightleftarrows}$ is Koszul. □

Recall that a resolution as in (5-1) is called *minimal* if no indecomposable summand of Q_t lies in the kernel for all $t \in \mathbb{Z}_{\geq 0}$.

Proof. We first note that the only graphs Γ such that $U_t(\boldsymbol{A}) = 0$ holds for some $t \in \mathbb{Z}_{\geq 0}$ are type ADE graphs. This follows since the roots of the Chebyshev polynomial $U_t(X)$ are known to be all of the form $2\cos(k/(t+1)\pi) \in]-2, 2[$ for $k \in \{1, \dots, t\}$, while, by [Smi70] or [BH12, Section 3.1.1], Γ has a Perron–Frobenius eigenvalue $\lambda \geq 2$. This also implies that $U_t(\boldsymbol{A})$ is irreducible (meaning that, for all $\mathtt{i}, \mathtt{j}$, there exists $N_{\mathtt{ij}} \in \mathbb{Z}_{\geq 0}$ such that the $\mathtt{i}$-$\mathtt{j}$ position of $U_t(\boldsymbol{A})^{N_{\mathtt{ij}}}$ is in $\mathbb{Z}_{>0}$). Further, that $U_t(\boldsymbol{A})$ has non-negative entries follows from categorification, cf. [MT16, End of Section 5.1] (which uses the polynomial $2\boldsymbol{I} + \boldsymbol{A}$). Hence, $U_t(\boldsymbol{A})$ is a symmetric, irreducible, non-negative matrix, and we can apply Perron–Frobenius theory to find an eigenvector of $U_t(\boldsymbol{A})$ with entries and associated eigenvalue from $\mathbb{R}_{>0}$. But having such an eigenvector implies that no columns or rows can be zero. Thus, if we start with $\mathrm{K}_0 = \mathrm{L}_{\mathtt{i}}$, then $\underline{a}_t$ in (5-3) will never be the zero vector. Summarized, to produce a minimal projective resolution of $\mathrm{K}_0 = \mathrm{L}_{\mathtt{i}}$ we successively resolve the module K_t with its projective cover $\mathrm{Q}(\mathrm{K}_t)$ with kernel K_{t+1}. By the arguments above, this minimal projective resolution will be linear, and the decomposition of $\mathrm{Q}(\mathrm{K}_t)$ into indecomposables is given by the $\mathtt{i}^{\text{th}}$ column of $U_t(\boldsymbol{A})$. ■

We now construct the linear projective resolutions explicitly.

Definition 5.5. Fix a graph Γ. A *resolution graph* $\Theta = (V, E)$ associated to Γ is a directed graph, whose vertex set $V = \bigcup_{t \in \mathbb{Z}_{\geq 0}} V_t$ is a disjoint union of finite sets V_t such that each vertex $v(\mathtt{i})$ is labeled by a vertex $\mathtt{i}$ of Γ. Moreover, the edge set $E = \bigcup_{t \in \mathbb{Z}_{\geq 1}} E_t$ is a disjoint union of finite sets E_t such that E_t contains only edges $e(z)\colon v(\mathtt{i}) \to v(\mathtt{j})$ from V_t to V_{t+1} that are labeled by some $z \in \mathbb{k}$. The two sets V_t, E_t are called the t^{th} *level of* Θ.

A level s resolution graph Θ_s is the same data, but $V_t = \emptyset = E_t$ for all $t \in \mathbb{Z}_{\geq s}$.

Further, if we fix $\mathtt{i}$, then we also consider monochrome sets $V_{t-1}(\mathtt{i}) = \{v(\mathtt{i}) \in V_{t-1}\}$ ("colored" only by $\mathtt{i}$), $E_t(\mathtt{i}) = \{e(z)\colon v(\mathtt{i}) \to v(\mathtt{j}) \mid v(\mathtt{i}) \in V_{t-1}\}$ and $V_t(\mathtt{i}) = \{v(\mathtt{j}) \mid v(\mathtt{j})$ is a target of $e(z) \in E_t(\mathtt{i})\}$, and the graph $\Theta_t(\mathtt{i}) = (V_{t-1}(\mathtt{i}) \cup V_t^{\mathtt{i}}, E_t(\mathtt{i}))$. Denote by $\Theta_t(\mathtt{i}) = \bigcup_r \Theta_t(\mathtt{i}, r)$ its decomposition into connected components $\Theta_t(\mathtt{i}, r) = (V_{t-1}(\mathtt{i}, r) \cup V_t(\mathtt{i}, r), E_t(\mathtt{i}, r))$ (seen as an unoriented graph). ▲

A *successor of a vertex* $v(\mathtt{i})$ in a directed graph is a vertex $v(\mathtt{j})$ such that there is a directed path from $v(\mathtt{i})$ to $v(\mathtt{j})$. Similarly in case of successors of a set of vertices.

Definition 5.6. A $\mathbb{k}$-*weighting* of a resolution graph and a fixed set of vertices V' of it are elements $b_{v(\mathtt{j})} \in \mathbb{k}$ for all successors $v(\mathtt{j})$ of V' such that for each vertex $v(\mathtt{i}) \in X$ we have

$$\textstyle\sum_{v(\mathtt{j})} z_{v(\mathtt{j})} b_{v(\mathtt{j})} = 0,$$

where $z_{v(\mathtt{j})} \in \mathbb{k}$ is the label of $e(z)\colon v(\mathtt{i}) \to v(\mathtt{j})$, and the sum is over all successors $v(\mathtt{j})$ of $v(\mathtt{i})$. We denote the vector space of all $\mathbb{k}$-weightings of V' by $\mathrm{We}(V')$. ▲

Algorithm 1 is, by birth, well-defined and depends on some choices, since in each step there are a few systems of linear equations (with unknowns $b_{v(\mathtt{j})}$) one needs to solve and choose a basis for its solution space. However, we will see in Lemma 5.9 that the vertex sets are independent of the involved choices. Note further that Algorithm 1 usually does not terminate, and output is to be understood that we can stop the algorithm at any level and get arbitrary long parts of the resolutions.

Example 5.7. In this example, for readability, we omit the orientation of the edges (we always read left to right and underline the starting vertex). We also write $\mathtt{i}$ short for $v(\mathtt{i})$ in

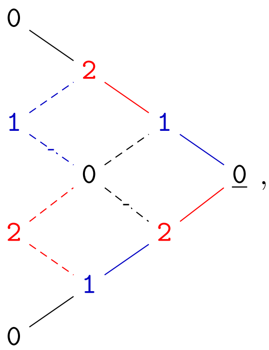
input : a graph Γ and a fixed vertex $\mathtt{i}$ of it;
output : a resolution graph $\Theta = \Theta(\mathtt{i}) = (\bigcup_s V_s, \bigcup_s E_s)$ for Γ;

initialization $(s = 0)$, let $\Theta_0 = (V_0, E_0)$ be the level 0 resolution graph with $V_0 = \{\mathtt{i}\}$, $E_0 = \emptyset$, and set $V_t = \emptyset = E_t$ for all $t \in \mathbb{Z}_{\geq -1}$, $t \neq 0$;
for $s \in \mathbb{Z}_{\geq 1}$ **do**
  `/* recall that the vertices of` Γ `are numbered, say` $\{\mathtt{1}, \dots, \mathtt{n}\}$ `*/;`
  for $\mathtt{j} = \mathtt{1}$ **to** $\mathtt{n}$ **do**
    for $\Theta_s^{\mathtt{i}}(r)$ *connected component* **do**
      `/* the potential solutions */;`
      fix a basis $\mathbb{B}(s, \mathtt{i}, r)$ of $\mathrm{We}(V_{s-1}(\mathtt{i}, r))$;
      `/* add them to` Θ_s`; called fork moves */;`
      for $b = (b_1, \dots, b_l) \in \mathbb{B}(s, \mathtt{i}, r)$ **do**
        add a vertex $v(\mathtt{i})$ to V_{s+1} and edges $e(b_k)\colon v(\mathtt{j}_k) \to v(\mathtt{i})$ to E_s;
      end
    end
  end
  `/* singleton moves */;`
  for $v(\mathtt{j}) \in V_{s-1}$ **do**
    add a vertex $v(\mathtt{k})$ to V_s for all neighbors of $\mathtt{j}$ in Γ which are not neighbors of $v(\mathtt{k})$ in Θ, and an edge $e(1)\colon v(\mathtt{k}) \to v(\mathtt{j})$ to E_s;
  end
end

Algorithm 1: The Chebyshev algorithm a.k.a. producing linear projective resolutions.

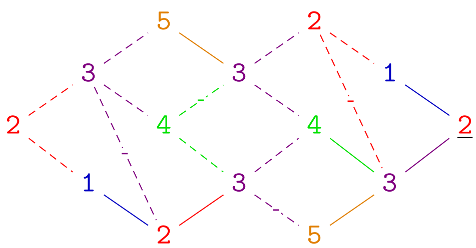
illustrations, and if we do not specify the edge labels, then they are 1, by convention, while we write $-$ instead of -1 for short.

The first example is a type $\widetilde{\mathsf{A}}_2$ graph where we choose the vertex $\mathtt{0}$. Then one gets

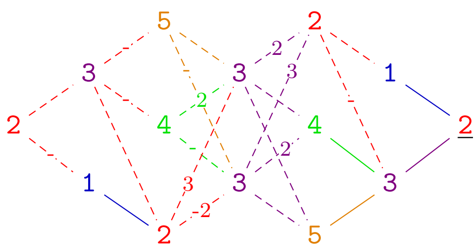


where the dashed lines come from fork moves, while the straight lines come from singleton moves, cf. Example 5.2. Note that all $\Bbbk$-weighting spaces are one-dimensional in this case, and the above is up to scaling unique.

Let us now do an example with two very different choices along the way. So let Γ be a type D_5 graph with starting vertex $\mathtt{2}$, and we get



or



.

Note the following two crucial observation: First, although we made quite different choices above, the multiplicities in each column are equal, and given by the Chebyshev recursion (5-3). Second, Algorithm 1 terminates for the finite type ADE graphs, but not for the affine ones. We will see that both is always the case. ▲

If we have a minimal linear projective resolution as in (5-1) which stops at Q_s for some $s \in \mathbb{Z}_{\geq 0}$, then we say its of length s.

Definition 5.8. Let $s \in \mathbb{Z}_{\geq 0}$, and let $\Theta_s = (\bigcup_t V_t, \bigcup_t E_t)$ be the output of Algorithm 1 of which we assume that it has not terminated. Then we define a complex of $\mathrm{Z}_{\rightleftarrows}$-modules:

(5.8.a) For $0 \leq t \leq s$ we let $\mathrm{Q}_t = \bigoplus_{v(\mathtt{i}) \in V_t} \mathrm{P}_{\mathtt{i}}$, i.e. we identify the vertices $v(\mathtt{i})$ in the t^{th} level with the projectives $\mathrm{Z}_{\rightleftarrows}$-modules $\mathrm{P}_{\mathtt{i}}$, and take direct sums.

(5.8.b) For $0 \leq t \leq s-1$ we let $\phi\colon \mathrm{Q}_{t-1} \to \mathrm{Q}_t = \bigoplus_{e(z)\colon v(\mathtt{i}) \to v(\mathtt{j}) \in E_t} (\cdot z\mathtt{i}{\to}\mathtt{j})$, i.e. we identify the edges $e(z)\colon v(\mathtt{i}) \to v(\mathtt{j})$ in the t^{th} level with the $\mathrm{Z}_{\rightleftarrows}$-equivariant maps given by post-composition with $z\mathtt{i}{\to}\mathtt{j}$, and take matrices.

We write $\mathrm{Reso}(\Theta_s)$ for this complex. ▲

Lemma 5.9. Assume Algorithm 1 does not terminate before the s^{th} level, and $\mathrm{Reso}(\Theta_s)$ is a minimal linear projective resolution of length s with multiplicities given by (5-3). If $U_s(\boldsymbol{A}) \neq 0$, then $\mathrm{Reso}(\Theta_{s+1})$ is minimal linear projective resolution of length $s+1$. □

Proof. Because $U_s(\boldsymbol{A}) \neq 0$ we know by the same arguments as above that the kernel K_s in generated in degree 1. Thus, its degree 1 part K_s^1 is a semisimple $\mathrm{Z}_{\rightleftarrows}$-module. Because K_s^1 is semisimple, the next step of the resolution is determined by choosing a basis for this $\mathrm{Z}_{\rightleftarrows}$-module, which is precisely what Algorithm 1 does. ■

Lemma 5.10. Algorithm 1 terminates if and only if Γ is a type ADE graph. □

Proof. This is a direct consequence of Lemma 5.9 since (as we have already seen above) $U_t(\boldsymbol{A})$ contains a zero column for some $t \in \mathbb{Z}_{\geq 0}$ if and only if $U_t(\boldsymbol{A}) = 0$ for some $t \in \mathbb{Z}_{\geq 0}$ if and only if Γ is a type ADE graph (with $t+1$ being the Coxeter number of Γ). ■

Proposition 5.11. If Algorithm 1 does not terminate, then it produces a minimal linear projective resolution of the $\mathrm{Z}_{\rightleftarrows}$-module $\mathrm{L}_{\mathtt{i}}$ with multiplicities given by (5-3). □

Proof. This is a direct consequence of Lemmas 5.9 and 5.10. ■

Elimination. Here is the numerical condition which we are going to use.

Lemma 5.12. If $\mathrm{Z}_{\rightleftarrows}$ is Koszul, then its graded Cartan matrix $\boldsymbol{C}_{\mathsf{q}}$ is invertible in the ring of matrices with entries from $\mathbb{Z}[[\mathsf{q}]]$. Moreover, the column sums of $\boldsymbol{C}_{\mathsf{q}}^{-1}$ are power series in $\mathbb{Z}[[\mathsf{q}]]$ of the from

$$\textstyle\sum_{s=0}^{\infty} (-1)^s a_s \mathsf{q}^s$$

with coefficients $a_s \in \mathbb{Z}_{\geq 1}$. □

Proof. The statement about the invertibility of the graded Cartan matrix is the usual consequence of koszulity, cf. [BGS96, Theorem 2.11.1]. For the second statement note that the inverse of the graded Cartan matrix encodes the graded multiplicities of the resolution of the simples by projectives. ■

For a formal power series $f = \sum_{i=0}^{\infty} b_i \mathsf{q}^i \in \mathbb{Z}[[\mathsf{q}]]$ we say that f has *gaps of size* $k \in \mathbb{Z}_{\geq 0}$ if there exist $i \leq j \in \mathbb{Z}_{\geq 0}$ such that $j - i + 1 = k$ and $b_i = b_{i+1} = \dots = b_{j-1} = b_j = 0$.

Recall that n denotes the number of vertices of Γ.

Lemma 5.13. If $\mathrm{Z}_{\rightleftarrows}$ is Koszul, then $(\det(\boldsymbol{C}_{\mathsf{q}}))^{-1}$ does not have gaps of size $> 2n-2$. □

Proof. Recall that the matrix of cofactors A^* of a fixed matrix A is determined by $AA^* = \det(A)\boldsymbol{I}$. By construction, the matrix $\boldsymbol{C}^*_{\mathsf{q}}$ has its entries in $\mathbb{Z}[\mathsf{q}]$. In fact, the entries of $\boldsymbol{C}^*_{\mathsf{q}}$ are polynomials in $\mathbb{Z}[\mathsf{q}]$ of degree at most $2n-2$ (this follows from Proposition 2.8), and the statement from Lemma 5.12 boils down to the graded Cartan determinant being invertible with the claimed property. To be precise, if the j^{th} column sum of $\boldsymbol{C}^*_{\mathsf{q}}$ is of the form $\sum_{s=0}^{2n-2} b_s\mathsf{q}^s$, and the inverse of the graded Cartan determinant is $\sum_{s=0}^{\infty} b'_s\mathsf{q}^s$, then $(\sum_{s=0}^{2n-2} b_s\mathsf{q}^s)(\sum_{s=0}^{\infty} b'_s\mathsf{q}^s) = \sum_{s=0}^{\infty}(-1)^s a_s\mathsf{q}^s$ implies $(-1)^k a_k = b_0b'_k + \cdots + b_{2n-2}b'_{k-2n+2}$. Thus, $a_j = 0$ for some $j \in \mathbb{Z}_{\geq 0}$ if there would be a gap of size $> 2n-2$. ∎

Proposition 5.14. If Γ is a type ADE graph, then $\mathrm{Z}_{\rightleftarrows}$ is not Koszul. □

Proof. We want to use Lemma 5.13 to show that $\mathrm{Z}_{\rightleftarrows}$ is not Koszul. To this end, observe that we already know from (3-3) the corresponding graded Cartan determinants. Their formal inverses are not hard to compute:

$$\begin{aligned}
\mathsf{A}_n&\colon (1-\mathsf{q}^2)\textstyle\sum_{s=0}^{\infty}\mathsf{q}^{(2n+2)s}, && \text{gap} = 2n-1,\\
\mathsf{D}_n, n \text{ even}&\colon (1-\mathsf{q}^2\pm\cdots+\mathsf{q}^{2n-4})\textstyle\sum_{s=0}^{\infty}(-1)^s(s+1)\mathsf{q}^{(2n-2)s}, && \text{gap} = 1,\\
\mathsf{D}_n, n \text{ odd}&\colon (1-\mathsf{q}^2\pm\cdots-\mathsf{q}^{2n-4})\textstyle\sum_{s=0}^{\infty}\mathsf{q}^{(4n-4)s}, && \text{gap} = 2n-1,\\
\mathsf{E}_6&\colon (1-\mathsf{q}^2+\mathsf{q}^4-\mathsf{q}^8+\mathsf{q}^{10}-\mathsf{q}^{12})\textstyle\sum_{s=0}^{\infty}\mathsf{q}^{24s}, && \text{gap} = 11,\\
\mathsf{E}_7&\colon (1-\mathsf{q}^2+\mathsf{q}^4)\textstyle\sum_{s=0}^{\infty}(-1)^s\mathsf{q}^{18s}, && \text{gap} = 13,\\
\mathsf{E}_8&\colon (1-\mathsf{q}^2+\mathsf{q}^4+\mathsf{q}^{10}-\mathsf{q}^{12}+\mathsf{q}^{14})\textstyle\sum_{s=0}^{\infty}(-1)^s\mathsf{q}^{30s}, && \text{gap} = 15.
\end{aligned}$$

Hence, except for the type D_n graph with $n = 2m$, the gap equals $2n-1$. To rule out the remaining cases, first recall the various graded Cartan determinants, cf. in the proof of Lemma 3.7, and we make the following claim.

Claim. The graded determinant of every $n-1$ minor of $\boldsymbol{C}_{\mathsf{q}}(\mathrm{Z}_{\rightleftarrows}(\mathsf{D}_{2m}))$ is divisible by 2_{q}.

Before we prove this claim, let us state the consequences. This means that we can simplify

$$\boldsymbol{C}_{\mathsf{q}}\boldsymbol{C}^*_{\mathsf{q}} = (1+\mathsf{q}^{2n-2})2_{\mathsf{q}}\boldsymbol{I} \quad\rightsquigarrow\quad \boldsymbol{C}_{\mathsf{q}}\boldsymbol{c}^*_{\mathsf{q}} = (1+\mathsf{q}^{2n-2})\boldsymbol{I},$$

where $\boldsymbol{c}^*_{\mathsf{q}} = (2_{\mathsf{q}})^{-1}\boldsymbol{C}^*_{\mathsf{q}}$ is a matrix with entries in $\mathbb{Z}[\mathsf{q}]$ of degree at most $2n-4$. But now the argument using gaps applies again since $(1+\mathsf{q}^{2n-2})^{-1} = \sum_{s=0}^{\infty}(-1)^s\mathsf{q}^{(2n-2)s}$ (the gap is of size $2n-3$, which is strictly greater than $2n-4$). Hence, this case is ruled out as well, and it remains to prove the claim.

Proof of the claim. We prove the statement by induction on the number of vertices. For the type D_4 graph the matrix of cofactors is easy to compute, i.e. we have

$$\boldsymbol{C}_{\mathsf{q}}(\mathrm{Z}_{\rightleftarrows}(\mathsf{D}_4))^* = 2_{\mathsf{q}}\begin{pmatrix} 1+\mathsf{q}^4 & -\mathsf{q}2_{\mathsf{q}} & \mathsf{q}^2 & \mathsf{q}^2\\ -\mathsf{q}2_{\mathsf{q}} & (2_{\mathsf{q}})^2 & -\mathsf{q}2_{\mathsf{q}} & -\mathsf{q}2_{\mathsf{q}}\\ \mathsf{q}^2 & -\mathsf{q}2_{\mathsf{q}} & 1+\mathsf{q}^4 & \mathsf{q}^2\\ \mathsf{q}^2 & -\mathsf{q}2_{\mathsf{q}} & \mathsf{q}^2 & 1+\mathsf{q}^4 \end{pmatrix}.$$

So assume that $n = 2m > 4$, and take a i-j minor of $\boldsymbol{C}_{\mathsf{q}}(\mathrm{Z}_{\rightleftarrows}(\mathsf{D}_n))$, obtained by erasing the i^{th} row and the j^{th} column. By symmetry, it suffices to consider the case $i \leq j$. As long as $i \leq n-2$ it turns out the determinant of this minor is equal to the determinant of $\boldsymbol{C}_{\mathsf{q}}(\mathrm{Z}_{\rightleftarrows}(\mathsf{A}_{i-1}))$ times the determinant of a $n-i$ minor of $\boldsymbol{C}_{\mathsf{q}}(\mathrm{Z}_{\rightleftarrows}(\mathsf{D}_{n-i+1}))$. But either $\det(\boldsymbol{C}_{\mathsf{q}}(\mathrm{Z}_{\rightleftarrows}(\mathsf{A}_{i-1}))) = i_{\mathsf{q}}$ is divisible by 2_{q}, in case i is even, or we know by induction that $\det(\boldsymbol{C}_{\mathsf{q}}(\mathrm{Z}_{\rightleftarrows}(\mathsf{D}_{n-i+1})))$, is in case i is odd. Thus, three cases remain, i.e. $i = j = n-1$, $i = j = n$ and $i = n-1$ and $j = n$. The first two cases are easy since the determinant of the minor is equal to $\det(\boldsymbol{C}_{\mathsf{q}}(\mathrm{Z}_{\rightleftarrows}(\mathsf{A}_{n-1}))) = n_{\mathsf{q}}$. For the remaining case one first expands the minor along the last column, followed by an expansion in the last row to determine that this minor is equal to $\pm\det(\boldsymbol{C}_{\mathsf{q}}(\mathrm{Z}_{\rightleftarrows}(\mathsf{A}_{n-3}))) = (n-2)_{\mathsf{q}}$. ∎

The proof. We collect the all statements from above.

Proof of Theorem C. Proposition 5.14 shows that $\mathrm{Z}_{\rightleftarrows}$ is not Koszul for Γ being a type ADE graph, while Proposition 5.11 constructs the linear projective resolution in all other cases, where we use the fact that having a linear projective resolution of length s for all $s \in \mathbb{Z}_{\geq 0}$ implies koszulity. ■

5B. The case of $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$.

Construction – Part I. We first observe again how the successive kernels in a resolution of a $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$-module with a two step radical filtration changes.

For this purpose, we use the same notation and argumentation as above. In particular, we get a short exact sequence $\mathrm{K}^{\mathsf{C}}_1 \hookrightarrow \mathrm{Q}(\mathrm{K}^{\mathsf{C}}_0) \twoheadrightarrow \mathrm{K}^{\mathsf{C}}_0$. The crucial difference is that the multiplicity vectors of the radical filtration of $\mathrm{K}^{\mathsf{C}}_1$ are now $\underline{a}_1 = \boldsymbol{A}\underline{a}_0 - \underline{b}_0$ and $\underline{b}_1 = (\boldsymbol{I} - \boldsymbol{E}_{\mathsf{C}})\underline{a}_0$, where $\boldsymbol{E}_{\mathsf{C}}$ as before denotes the diagonal matrix with only non-zero entries equal to 1 in the c-c position. Hence, as before, we can produce $\mathrm{Z}^{\mathsf{C}}_{\rightleftarrows}$-modules $\mathrm{K}^{\mathsf{C}}_t$ for all $t \in \mathbb{Z}_{\geq 0}$ as long as $\underline{a}_t \neq 0$ which are generated in degree t and have radical and grading filtrations that agree.

Note that, we do not have a pure polynomial recursion due to the occurrence of the matrix $\boldsymbol{E}_{\mathsf{C}}$, and we have to modify our arguments. To this end, we introduce a *recursion of Chebyshev polynomials with matrix coefficients*, namely

$$U^{\mathsf{C}}_{-1}(\boldsymbol{X}) = 0,\ U^{\mathsf{C}}_0(\boldsymbol{X}) = \boldsymbol{I},\ U^{\mathsf{C}}_t(\boldsymbol{X}) = \boldsymbol{X}U^{\mathsf{C}}_{t-1}(\boldsymbol{X}) - (\boldsymbol{I} - \boldsymbol{E}_{\mathsf{C}})U^{\mathsf{C}}_{t-2}(\boldsymbol{X}),\ \text{for } t \in \mathbb{Z}_{\geq 1}. \tag{5-4}$$

Remark 5.15. One could state $U^{\mathsf{C}}_t(\boldsymbol{X})$ in terms of a polynomial in two non-commuting variables $\boldsymbol{X}, \boldsymbol{Y}$. But we only need the version with $\boldsymbol{Y} = (\boldsymbol{I} - \boldsymbol{E}_{\mathsf{C}})$, so we stick with it. ▲

Having these, observe that for $t \in \mathbb{Z}_{\geq 0}$, the t^{th} multiplicity vectors of the radical filtration of $\mathrm{K}^{\mathsf{C}}_t$ are then given by

$$\begin{pmatrix} \underline{a}_t \\ \underline{b}_t \end{pmatrix} = \begin{pmatrix} U^{\mathsf{C}}_t(\boldsymbol{A}) & -U^{\mathsf{C}}_{t-1}(\boldsymbol{A}) \\ (\boldsymbol{I} - \boldsymbol{E}_{\mathsf{C}})U^{\mathsf{C}}_{t-1}(\boldsymbol{A}) & -(\boldsymbol{I} - \boldsymbol{E}_{\mathsf{C}})U^{\mathsf{C}}_{t-2}(\boldsymbol{A}) \end{pmatrix} \begin{pmatrix} \underline{a}_0 \\ \underline{b}_0 \end{pmatrix}. \tag{5-5}$$

We now express the $U^{\mathsf{C}}_t(\boldsymbol{X})$ in term of the usual Chebyshev polynomials $U_t(\boldsymbol{X})$ viewed as polynomials with matrix coefficients. To state it, we need the set of length k compositions $C(t,k) = \{(i_1, \dots, i_k) \mid i_j \in \mathbb{Z}_{\geq 0}, i_1 + \dots + i_k = t + 2 - 2k\}$ for $k, t \in \mathbb{Z}_{\geq 0}$.

Lemma 5.16. For $t \in \mathbb{Z}_{\geq 0}$ it holds

$$U^{\mathsf{C}}_t(\boldsymbol{X}) = \textstyle\sum_{k=1}^{r(t)} \sum_{\underline{i} \in C(t,k)} U_{i_1}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}}U_{i_2}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}} \dots \boldsymbol{E}_{\mathsf{C}}U_{i_k}(\boldsymbol{X}), \tag{5-6}$$

where $r(t) = \lfloor t/2 \rfloor + 1$. □

Note that, in case $\boldsymbol{E}_{\mathsf{C}}$ would be the zero matrix, (5-6) gives $U^{\mathsf{C}}_t(\boldsymbol{X}) = U_t(\boldsymbol{X})$.

Proof. We prove the statement by induction. For $t = 0$ and $t = 1$ the equality holds. (In this case $U^{\mathsf{C}}_t(\boldsymbol{X}) = U_t(\boldsymbol{X})$.) To show equality for $t \geq 2$, denote by $W_t(\boldsymbol{X})$ the right-hand side of (5-6) for fixed t and we verify it satisfies the same recursion as $U^{\mathsf{C}}_t(\boldsymbol{X})$, i.e.

$$W_t(\boldsymbol{X}) = \boldsymbol{X}W_{t-1}(\boldsymbol{X}) - (\boldsymbol{I} - \boldsymbol{E}_{\mathsf{C}})W_{t-2}(\boldsymbol{X}).$$

Next, let $U_{i_1}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}}U_{i_2}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}} \dots \boldsymbol{E}_{\mathsf{C}}U_{i_k}(\boldsymbol{X})$ be a summand of $W_t(\boldsymbol{X})$ with $\underline{i} \in C(t,k)$ for some $1 \leq k \leq r(t)$. We distinguish three cases.

Case $i_1 \geq 2$. For this we have that $U_{i_1-1}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}}U_{i_2}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}} \dots \boldsymbol{E}_{\mathsf{C}}U_{i_k}(\boldsymbol{X})$ is a summand of $W_{t-1}(\boldsymbol{X})$ for the sequence of indices in $C(t-1,k)$ and $U_{i_1-2}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}}U_{i_2}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}} \dots \boldsymbol{E}_{\mathsf{C}}U_{i_k}(\boldsymbol{X})$ is a summand of $W_{t-2}(\boldsymbol{X})$ for the sequence of indices in $C(t-2,k)$. Hence, we have

$$U_{i_1}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}} \dots \boldsymbol{E}_{\mathsf{C}}U_{i_k}(\boldsymbol{X}) = \boldsymbol{X} \cdot (U_{i_1-1}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}} \dots \boldsymbol{E}_{\mathsf{C}}U_{i_k}(\boldsymbol{X})) - U_{i_1-2}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}} \dots \boldsymbol{E}_{\mathsf{C}}U_{i_k}(\boldsymbol{X})$$

by the Chebyshev recursion for the first factor.

Case $i_1 = 1$. In this case we only have that $U_0(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}}U_{i_2}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}}\dots\boldsymbol{E}_{\mathsf{C}}U_{i_k}(\boldsymbol{X})$ is a summand of $W_{t-1}(\boldsymbol{X})$ for the sequence of indices in $C(t-1,k)$. We obtain

$$U_{i_1}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}}\dots\boldsymbol{E}_{\mathsf{C}}U_{i_k}(\boldsymbol{X}) = \boldsymbol{X}\cdot(U_{i_1-1}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}}\dots\boldsymbol{E}_{\mathsf{C}}U_{i_k}(\boldsymbol{X})),$$

by using that $U_1(\boldsymbol{X}) = \boldsymbol{X}U_0(\boldsymbol{X})$.

Case $i_1 = 0$. Note that, if we omit the first factor in this case, then $U_{i_2}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}}\dots\boldsymbol{E}_{\mathsf{C}}U_{i_k}(\boldsymbol{X})$ is a summand of $W_{t-2}(\boldsymbol{X})$ for the sequence of indices in $C(t-2,k-1)$. Thus,

$$U_{i_1}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}}\dots\boldsymbol{E}_{\mathsf{C}}U_{i_k}(\boldsymbol{X}) = \boldsymbol{E}_{\mathsf{C}}\cdot(U_{i_2}(\boldsymbol{X})\boldsymbol{E}_{\mathsf{C}}\dots\boldsymbol{E}_{\mathsf{C}}U_{i_k}(\boldsymbol{X})),$$

since $U_0(\boldsymbol{X}) = \boldsymbol{I}$.

The sum of the $W_t(\boldsymbol{X})$ terms appearing in the first two cases equals the part $\boldsymbol{X}W_{t-1}(\boldsymbol{X}) - W_{t-2}(\boldsymbol{X})$ in the recursion, while the sum of the terms in the third case equals the part $\boldsymbol{E}_{\mathsf{C}}W_{t-2}(\boldsymbol{X})$. Thus, we obtain that $W_t(\boldsymbol{X})$ satisfies (5-5) and is equal to U_t^{C}. ■

Outside of the case of type ADE graphs, koszulity of $\mathrm{Z}_{\rightleftarrows}^{\mathsf{C}}$ is obtained very similarly to the koszulity of $\mathrm{Z}_{\rightleftarrows}$ with Proposition 5.4, i.e.:

Proposition 5.17. If Γ is not a type ADE graph, then $\mathrm{Z}_{\rightleftarrows}^{\mathsf{C}}$ is Koszul. □

Note that, morally speaking, (5-5) and Proposition 5.17 imply that the number of projective indecomposables in the minimal linear projective resolutions grows faster if one increases the number of vertices having a vertex conditions, and also the closer one gets to such vertices, cf. Example 5.3.

Proof. As argued in the proof of Proposition 5.4, since Γ is not a type ADE graph, we know that $U_t(\boldsymbol{A})$ will always be a non-negative integral matrix without zero columns or rows. Thus, by Lemma 5.16, the same is true for $U_t^{\mathsf{C}}(\boldsymbol{A})$, because all summands in (5-6) evaluated at $\boldsymbol{A}$ have non-negative entries with the leading term $U_t(\boldsymbol{A})$ also having non-zero columns and rows. Hence, if we start with $\mathrm{K}_0^{\mathsf{C}} = \mathrm{L}_{\mathtt{i}}$, then we know that $\underline{a}_t = U_t^{\mathsf{C}}(\boldsymbol{A})\underline{a}_0$ will never be zero, and we are done by the same reasoning as in the proof of Proposition 5.4. ■

Construction – Part II. This time there are no cases that need to be eliminated (as already indicated in Example 5.3), since all Γ and all C will give Koszul algebras. Before we can show this, we recall that the Chebyshev polynomials have a closed form, namely

$$U_t(\mathsf{X}) = \textstyle\sum_{k=0}^{s(t)}(-1)^k\binom{t-k}{k}\mathsf{X}^{t-2k},\ \text{for } t\in\mathbb{Z}_{\geq 0}, \tag{5-7}$$

where $s(t) = \lfloor t/2\rfloor$. To get a matrix version of (5-7) we let $\boldsymbol{X}$ and $\boldsymbol{Y}$ denote two non-commuting variables of degrees 1 and 2, respectively. Let $M(t,k)$ be the set of monomials in $\boldsymbol{X}$ and $\boldsymbol{Y}$ of degree t with k different $\boldsymbol{Y}$-factors. (For example, $M(t,0) = \{\boldsymbol{X}^t\}$ and $M(t,1) = \{\boldsymbol{X}^{t-2}\boldsymbol{Y}, \boldsymbol{X}^{t-3}\boldsymbol{Y}\boldsymbol{X},\dots,\boldsymbol{Y}\boldsymbol{X}^{t-2}\}$.)

Lemma 5.18. We have

$$U_t^{\mathsf{C}}(\boldsymbol{X}) = \textstyle\sum_{k=0}^{s(t)}(-1)^k\sum_{m\in M(t,k)} m(\boldsymbol{X}, \boldsymbol{I}-\boldsymbol{E}_{\mathsf{C}}),\ \text{for } t\in\mathbb{Z}_{\geq 0}, \tag{5-8}$$

where $\boldsymbol{X}$ is to be assumed to be a matrix. □

Indeed, (5-8) specializes to (5-7) in case $\boldsymbol{E}_{\mathsf{C}}$ is the zero matrix.

Proof. One immediately checks that $U_0^{\mathsf{C}}(\boldsymbol{X})$ and $U_1^{\mathsf{C}}(\boldsymbol{X})$ satisfy (5-8). Analyzing formula (5-8), one sees that the monomials in $U_t^{\mathsf{C}}(\boldsymbol{X})$ can be obtained by multiplying the ones from $U_{t-1}^{\mathsf{C}}(\boldsymbol{X})$ with $\boldsymbol{X}$ from the left and subtracting the ones from $U_{t-2}^{\mathsf{C}}(\boldsymbol{X})$ multiplied with $(\boldsymbol{I}-\boldsymbol{E}_{\mathsf{C}})$ from the left, which is exactly the recursion (5-4). Thus, the claim follows. ■

To obtain positivity of $U_t^{\mathsf{C}}(\boldsymbol{A})$ we need a combinatorial interpretation of the summands in (5-8) in terms of certain paths, where we recall that e.g. $\mathtt{i}\to\mathtt{j}\to\mathtt{i}$ denotes a path in the double of Γ, which we from now on identify with paths in Γ, by convention.

Definition 5.19. For each vertex $\mathtt{i} \notin \mathsf{C}$ choose an edge of Γ that is incident with $\mathtt{i}$. If every edge of Γ is chosen by at most one vertex, then we call this a *singleton inflow* (outside of C).

For a fixed choice of singleton inflow, we call a path $\mathtt{i}\to\mathtt{j}\to\mathtt{i}$ a *chosen zigzag at* $\mathtt{i} \notin \mathsf{C}$ if the arrow $\mathtt{j}\to\mathtt{i}$ was the choice for the singleton inflow at the vertex $\mathtt{i}$. ▲

Example 5.20. A singleton inflow is not unique and might not exist at all: a type ADE graph possesses at least one singleton inflow as long as $\mathsf{C} \neq \emptyset$, but if we would allow $\mathsf{C} = \emptyset$, then the type A graph has none at all, while a type $\widetilde{\mathsf{A}}$ graph has exactly two. ▲

Lemma 5.21. Assume that Γ has a singleton inflow and let p be a path in Γ. Then two distinct chosen zigzags in p cannot have any edges of the path in common. □

Proof. We can immediately reduce this statement to a path of length 3, i.e. $p = \mathtt{i}\to\mathtt{j}\to\mathtt{k}\to\mathtt{l}$ and assume that $\mathtt{i}\to\mathtt{j}\to\mathtt{k}$ and $\mathtt{j}\to\mathtt{k}\to\mathtt{l}$ are chosen zigzags. This forces $\mathtt{i} = \mathtt{k}$ and $\mathtt{j} = \mathtt{l}$, thus $p = \mathtt{i}\to\mathtt{j}\to\mathtt{i}\to\mathtt{j}$ with the second and third edge both being chosen for the singleton inflow. This is a contradiction, since they are both the edge connecting $\mathtt{i}$ and $\mathtt{j}$. ■

For a fixed choice of a singleton inflow, we associate to a path $p = \mathtt{i}_1\to\mathtt{i}_2\to\dots\to\mathtt{i}_{t-1}\to\mathtt{i}_t$ a monomial $m(p)$ in non-commuting variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ by first substituting any chosen zigzags in p by $\boldsymbol{Y}$, and afterwards all remaining edges by $\boldsymbol{X}$.

Example 5.22. Take the type A_4 graph with $\mathsf{C} = \{\mathtt{1}\}$. Then there is a unique singleton inflow, which we illustrate by orient an edge towards a vertex in case it is the chosen one for that vertex, i.e.

$$\mathsf{1} \to \mathsf{2} \to \mathsf{3} \to \mathsf{4}\ ,\ \text{for } \mathsf{C} = \{\mathtt{1}\}.$$

With this singleton inflow, for example, the path $\mathsf{3}\to\mathsf{2}\to\mathsf{1}\to\mathsf{2}$ in Γ is associated to $\boldsymbol{XY}$, with $\boldsymbol{Y}$ corresponding to $\mathsf{2}\to\mathsf{1}\to\mathsf{2}$. Similarly, in case $\mathsf{C} = \{\mathtt{1},\mathtt{2}\}$, then a singleton inflow would be

$$\mathsf{1} - \mathsf{2} \to \mathsf{3} \to \mathsf{4}\ ,\ \text{for } \mathsf{C} = \{\mathtt{1},\mathtt{2}\},$$

and $\mathsf{3}\to\mathsf{2}\to\mathsf{1}\to\mathsf{2}$ would be associated to $\boldsymbol{X}^3$. ▲

The next lemma yields a combinatorial interpretation of $m(\boldsymbol{A}, \boldsymbol{I} - \boldsymbol{E}_{\mathsf{C}})$ for $m \in M(t,k)$.

Lemma 5.23. Let $m \in M(t,k)$ for $t \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq s(t)$. Then the $\mathtt{i}$-$\mathtt{j}$ position of $m(\boldsymbol{A}, \boldsymbol{I} - \boldsymbol{E}_{\mathsf{C}})$ is equal to the number of paths $p = \mathtt{j}\to\mathtt{i}_1\to\dots\to\mathtt{i}_{t-1}\to\mathtt{i}$ (of length t) from j to i such that $m(p) = m$. □

Proof. Let $m = \boldsymbol{X}^{a_1}\boldsymbol{Y}^{b_1}\boldsymbol{X}^{a_2}\boldsymbol{Y}^{b_2}\dots\boldsymbol{X}^{a_s}\boldsymbol{Y}^{b_s}$ such that the sum of all a_i is $t - 2k$ and the sum of all b_i is k. By definition, the $\mathtt{i}$-$\mathtt{j}$ position of $m(\boldsymbol{A}, \boldsymbol{I} - \boldsymbol{E}_{\mathsf{C}})$ is equal to the number of paths of length $t - 2k$ where the path made of the first $a_1 + \dots + a_i$ edges ends in a vertex outside of C, for all $\mathtt{i}$. Such a path can be uniquely extended to length t, by adding b_1 chosen zigzags after the first a_1 edges, then b_2 chosen zigzags after the next a_2 edges, etc. The resulting path p satisfies $m(p) = m$, by construction, and clearly any such path can be obtained uniquely in such a way, if m is fixed. ■

Note that a path from $\mathtt{j}$ to $\mathtt{i}$ can contribute to multiple $m(\boldsymbol{A}, \boldsymbol{I} - \boldsymbol{E}_{\mathsf{C}})$ for $m \in M(t,k)$ via Lemma 5.23, and of course all path contribute to $m(\boldsymbol{A}) = \boldsymbol{A}^t$.

Proposition 5.24. If Γ is a type ADE graph, then $\mathsf{Z}^{\mathsf{C}}_{\leftrightarrows}$ is Koszul. □

We stress that the proof will use the fact that $\mathsf{C} \neq \emptyset$.

Proof. We first fix a choice of a singleton inflow, which exists by Example 5.20, and we use the description of $U_t^{\mathsf{C}}(\boldsymbol{A})$ from (5-8). Via this and Lemma 5.23 the $\mathtt{i}$-$\mathtt{j}$ position of $U_t^{\mathsf{C}}(\boldsymbol{A})$ is an alternating sum of numbers of certain paths of length t from $\mathtt{j}$ to $\mathtt{i}$ starting with the total number of paths of length t for $k = 0$.

We start with the following claims.

Claim 1. $U_t^{\mathsf{C}}(\boldsymbol{A})$ has only non-negative entries.

Proof of the claim. The strategy is to show that for each path of length t the contributions for each k in the sum now either give 0 or 1. For this purpose, fix a path p of length t. Now let k be such that $m(p) \in M(t,k)$.

We first claim that p does not give a contribution to any other $m \in M(t,l)$ for $l \geq k$. This is clear for $l > k$, since k is exactly the number of chosen zigzags in p. For $l = k$ the chosen zigzags would need to be at different positions, which is also not possible.

Next, we claim that for $l < k$ there are exactly $\binom{k}{l}$ different $m \in M(t,l)$ that count p. These m are obtained from $m(p)$ by replacing $k-l$ of the occurring $\boldsymbol{Y}$ by $\boldsymbol{X}^2$.

Summing all of this up, we see that for $k > 0$ the contribution of p to the $\mathtt{i}$-$\mathtt{j}$ position of $U_t^{\mathsf{C}}(\boldsymbol{A})$ is $1 - \binom{k}{1} + \binom{k}{2} - \ldots + (-1)^k\binom{k}{k} = 0$, while for $k = 0$ the contribution is 1. Thus, in total, we obtain that $U_t^{\mathsf{C}}(\boldsymbol{A})$ only has non-negative entries, which proves Claim 1.

Claim 2. $U_t^{\mathsf{C}}(\boldsymbol{A}) = 0$ if and only if Γ is a type A graph and $\mathsf{C} \neq \{\mathtt{1}\}$, respectively $\mathsf{C} \neq \{\mathtt{n}\}$.

Proof of the claim. Assume that Γ has a trivalent vertex. Then, for any vertex $\mathtt{i}$, we can construct a path of arbitrary length that contains no chosen zigzag for any choice of a singleton inflow starting at that vertex. This is due to the fact that the trivalent vertex has two edges that are not chosen for it, which one can use to construct such paths.

Assume Γ is of type A_n and $\mathsf{C} \neq \{\mathtt{1}\}$, respectively $\mathsf{C} \neq \{\mathtt{n}\}$. Then for any singleton inflow choice there exists at least one of the following cases: Either there exists a vertex with a vertex condition which has two neighbors, again allowing to construct an arbitrary long path without chosen zigzags. Or both leafs have a vertex condition in which case the edge incident to one of them is not chosen, and we can construct again an arbitrary long path not containing a chosen zigzags.

Finally, we consider the case of a type A_n graph with $\mathsf{C} = \{\mathtt{1}\}$, the case $\mathsf{C} = \{\mathtt{n}\}$ follows by symmetry. In this case there is a unique singleton inflow. For this singleton inflow there can not be any path of length $\geq 2n-1$ not containing any chosen zigzag. Hence, $U_t^{\mathsf{C}}(\boldsymbol{A}) = 0$ for $t \geq 2n-1$, which shows the claim.

Altogether this implies that outside of graphs of type A_n with $\mathsf{C} = \{\mathtt{1}\}$, respectively $\mathsf{C} = \{\mathtt{n}\}$, $\mathrm{Z}_{\rightleftarrows}^{\mathsf{C}}$ is Koszul by the same argument as for Proposition 5.17.

The hardest case remains: a type A_n graph with $\mathsf{C} = \{\mathtt{1}\}$, because the case $\mathsf{C} = \{\mathtt{n}\}$ follows again by symmetry. Observe that there is a unique longest path not containing any chosen zigzag, which is of the form $\mathtt{i}\to\ldots\to\mathtt{n}\to\ldots\to\mathtt{1}$, and which is of length $(2n+1)-i$. Thus, the matrix $U_{(2n+1)-i}^{\mathsf{C}}(\boldsymbol{A})$ contains only one non-zero entry in the i^{th} column which is located in the $\mathtt{1}$-$\mathtt{i}$ position, which shows that this case is also Koszul. ■

Remark 5.25. From the proofs of Propositions 5.17 and 5.24 we obtain the length of a minimal projective resolution for all simple modules, which are given by Chebyshev polynomials. This shows that $\mathrm{Z}_{\rightleftarrows}^{\mathsf{C}}$ has infinite global dimension unless Γ is a type A_n graph with $\mathsf{C} = \{\mathtt{1}\}$, respectively $\mathsf{C} = \{\mathtt{n}\}$, in which case it has global dimension $2n-1$. Note that this is known for the type A_n graph with the vertex condition imposed on one leaf due to its connection to category $\mathcal{O}$, but in general this appears to be a new observation. ▲

Example 5.26. If Γ is a type A graph with $\mathsf{C} = \{\mathtt{1}\}$ or $\mathsf{C} = \{\mathtt{n}\}$, then the linear projective resolutions are all finite. This follows from Proposition 5.24 and Theorem B′. To see this explicitly (we do the case $\mathsf{C} = \{\mathtt{1}\}$, the other follows by symmetry), we observe that the columns of the matrices (exemplified in case of the type A_4 graph)

$$\underset{t=2}{\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&0\end{pmatrix}},\quad \underset{t=3}{\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}},\quad \underset{t=4}{\begin{pmatrix}1&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}},\quad \underset{t=5}{\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}},\quad \underset{t=6}{\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}},\quad \underset{t\geq 7}{\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}},$$

together with the starting matrices $\boldsymbol{I}$ for $t=0$ and $\boldsymbol{A}$ for $t=1$, give the summands in the corresponding linear projective resolutions. In fact, having these matrices it is easy to write down the resolutions using the same methods as in Example 5.3. ▲

The proof.

Proof of Theorem C'. We combine Propositions 5.17 and 5.24. ■

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