On the geometry of Deligne-Mumford stacks

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ON THE GEOMETRY OF DELIGNE–MUMFORD STACKS

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ABSTRACT. General structure results about Deligne–Mumford stacks are summarized, applicable to stacks of finite type over a field. When the base field has characteristic 0, a class of "(quasi-)projective" Deligne–Mumford stacks is identified, defined to be those that embed as a (locally) closed substack of a smooth proper Deligne–Mumford stack having projective coarse moduli space. These conditions are shown to be equivalent to some well-studied hypotheses.

1. INTRODUCTION

This note summarizes some general structure results about Deligne–Mumford (DM) stacks, which can be found in the literature. Algebraic stacks are objects which generalize schemes, principally by allowing "points" to be endowed with stabilizer groups, a feature that makes stacks especially well adapted to the study of moduli problems. There are two kinds of algebraic stacks, the simpler DM stacks which allow only finite, reduced stabilizer groups, and the more general Artin stacks. This article focuses on DM stacks, and only on stacks of finite type over a field. Many ideas and results are only valid when the base field is the field of complex numbers, or is at least of characteristic 0. Over a field of characteristic $p > 0$, extra difficulties arise when there are stabilizer groups of order divisible by p, and the best structure results apply only to *tame* DM stacks[1], meaning those having only stabilizer groups of order prime to p.

This article also contains the suggestion that a DM stack over a field of characteristic 0 should be called (quasi-)projective if it admits a (locally) closed embedding to a smooth proper DM stack having projective coarse moduli space. Certainly, for a stack to merit such a description it should at least itself be separated and possess a (quasi-)projective coarse moduli space. We show that for a separated DM stack with quasi-projective coarse moduli space over a field of characteristic 0 to admit an embedding as indicated is equivalent to several well-studied hypotheses: being a quotient stack, satisfying the resolution property, admitting a finite flat covering by a scheme, and possessing a generating sheaf.

The organization of this article reflects the spirit of the presentation given in Seattle. A short discussion of stacks in general is followed by a concrete description of smooth DM stacks in dimension 1, specifically orbifold curves over $\mathbb{C}$. Then the discussion turns to smooth DM stacks of arbitrary dimension. Finally, concerning possibly singular DM stacks, results are presented characterizing those which can be embedded into a smooth proper DM stack having projective coarse moduli space, when the base field is of characteristic 0.

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[1] Abramovich, Olsson, and Vistoli identify a class of tame Artin stacks [AOV].

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2. Generalities on stacks

We start with some remarks on the general nature of stacks. Stack quotients of schemes by algebraic groups are an important class of stacks. If Y is a scheme, which we will take to be of finite type over a field k, and if G is an algebraic group over k, where the important examples for this exposition are finite groups and linear algebraic groups, then an action of G on Y determines an algebraic stack $[Y/G]$.

In some situations there is also a quotient scheme Y/G. Generally, we will view a variety or scheme X via its functor of points as a *fibered category.* This will be the category whose objects are pairs (S,φ) where S is *any* scheme and φ is a morphism of schemes $S \to X$. A morphism from (S,φ) to (S',φ') is by definition a morphism of schemes $f\colon S \to S'$ such that $\varphi' \circ f = \varphi$. The sense in which this category is "fibered" over schemes is that one can forget the data involving X and obtain a functor to the base category of schemes. This sends an object (S,φ) to S and a morphism $f\colon (S,\varphi) \to (S',\varphi')$ to $f\colon S \to S'$. This way we can speak of objects, respectively morphisms, "over" a scheme S, respectively a morphism of schemes f.

The essential feature of stacks in general is that they are allowed to have many morphisms over a single morphism in the base category. This feature is not present in the fibered category associated with a scheme, above. But it is generally present in the case of $[Y/G]$, in which an object is an étale locally trivial G-torsor (principal homogeneous G-space) $E \to S$ together with a G-equivariant morphism $E \to Y$. If another object is specified by $E' \to S'$ and $E' \to Y$, then a morphism in $[Y/G]$ between these objects consists of $f\colon S \to S'$ together with a G-equivariant morphism $E \to E'$ that fits into a commutative square with S and S' and into a commutative triangle with Y. The case $Y = \operatorname{Spec} k$ is important; then $[Y/G]$ is the category of G-torsors, denoted BG and known as a *classifying stack.*

This feature allows stacks to encode more information than schemes. For instance, if k is algebraically closed and G is a finite group acting on quasi-projective Y with quotient scheme X, then we can describe concretely how the quotient stack $[Y/G]$ compares with X. Both the k-points of $[Y/G]$ and the k-points of X can be canonically identified with the G-orbits of k-points of Y, where k-points of a stack are understood to be isomorphism classes of objects over $\operatorname{Spec} k$. But each k-point of $[Y/G]$ will have a stabilizer group, namely the group of automorphisms lying over the identity morphism of $\operatorname{Spec} k$. This will be the stabilizer group for the G action of a representative point of Y. In an example like $G = \mathbb{Z}/2\mathbb{Z}$ acting on $Y = \mathbb{A}^2$ with quotient the quadric cone (we take k to be of characteristic different from 2, and the action to be that given by $(x,y) \mapsto (-x,-y)$), one could picture $[Y/G]$ as a quadric cone where the vertex is endowed with the nontrivial stabilizer. The stack $[Y/G]$ is in fact smooth, while the quotient variety is singular.

The definition of *Deligne–Mumford stack* is quite involved (see [DM] or the Appendix to [V] for the full definition), but an important property is that the stack $\mathcal{X}$ must admit an étale cover by a scheme

$$U \to \mathcal{X}. \tag{2.1}$$

There will then be an "atlas"

$$\mathcal{X} \cong [R \rightrightarrows U], \tag{2.2}$$

meaning that a DM stack can be specified by a pair of schemes, with some structural morphisms.

A morphism of stacks is a functor which sends objects over S to objects over S and morphisms over f to morphisms over f, for every scheme S, respectively morphism of schemes f. A morphism from a scheme to a stack, or from a stack to a scheme, is understood by viewing the scheme as a stack by its functor of points (importantly, by Yoneda's Lemma, morphisms from one scheme to another in this sense reproduce the usual notion of morphisms of schemes). This is the sense of (2.1); an example is $Y \to [Y/G]$ where G is a finite group acting on a scheme Y. In this case the corresponding atlas will be $[Y \times G \rightrightarrows Y]$, where the two morphisms indicated by the notation (2.2) are projection and group action, and other structural morphisms are "identity", "inverse", and "multiplication". Generally, for some given $R \rightrightarrows U$, when the structural maps from R to U are étale and the map $R \to U \times U$ determined by them is quasi-compact and separated, the associated stack will be a Deligne–Mumford stack. There are "slice" theorems that assert, for instance, that when G is an algebraic group acting on a scheme Y with finite reduced geometric stabilizer group schemes, then $[Y/G]$ is a DM stack ([DM, Thm. 4.21], [LM, Thm. 8.1], but for more specific results when Y is affine and G is reductive, see [Lu], [BR]).

3. Orbifold curves

In this section we work over the base field $k = \mathbb{C}$. A *complex orbifold curve* is, by definition, a smooth separated irreducible DM stack $\mathcal{X}$ of dimension 1 and finite type over $\mathbb{C}$ with trivial generic stabilizer. An orbifold curve is specified uniquely up to isomorphism by its coarse moduli space X, which will be a smooth quasi-projective curve over $\mathbb{C}$, together with a finite set of points $x_1, \ldots, x_r$ and integers $n_1, \ldots, n_r \geq 2$. The local model for an orbifold point of $\mathcal{X}$ is the unit disc $\Delta = \{\, z \in \mathbb{C} : |z| < 1 \,\}$ with multiplicative action of the group μ_n of n-th roots of unity. Working in the analytic category, $\mathcal{X}$ can be constructed by gluing a copy of $[\Delta/\mu_{n_i}]$, for each i, to $X \smallsetminus \{x_1, \ldots, x_r\}$. Standard results concerning coverings of curves imply the existence of an atlas consisting of algebraic curves.

3.1. **Vector bundles on curves.** One application of orbifold curves is to give a convenient language for expressing some classical results on vector bundles on algebraic curves. Let X be a compact Riemann surface of genus ≥ 1. By a classical result of Weil [W] (see also Grothendieck's exposition [Gr]), a vector bundle on X arises from a representation of $\pi_1(X)$ if and only if each of its indecomposable components has degree 0. Inequivalent representations may give rise to isomorphic vector bundles. Narasimhan and Seshadri have shown [NS] that a vector bundle arises from a *unitary* representation of $\pi_1(X)$ if and only if each of its indecomposable components is *stable* of degree 0, and two such vector bundles are isomorphic if and only if the respective representations are equivalent. In other words,

$$\mathrm{Rep}(\pi_1(X), U(n))/\!\sim \;\;\longleftrightarrow\;\; M(n,0)$$

where $M(n,0)$ denotes the moduli space of semistable bundles of rank n and degree 0. They characterize stable bundles of arbitrary degree as follows:

$$\mathrm{Rep}^d(\pi_1^{(n)}(X), U(n))/\!\sim \;\;\longleftrightarrow\;\; M(n,-d),$$

where $\pi_1^{(n)}(X)$ is a discrete group, acting effectively and properly on a simply connected Riemann surface Y with quotient X such that $Y \to X$ is ramified over a single point $x_0 \in X$, with order of ramification n. Concretely, $\pi_1^{(n)}(X)$ is generated by a_i, b_i, $i = 1, \ldots, g$ (where g is the genus of X) and c with relations

$$a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}c = \mathrm{id} \qquad \text{and} \qquad c^n = \mathrm{id},$$

and admits a surjective homomorphism to $\pi_1(X)$ whose kernel is the normal closure of the element c. Here a_i and b_i are (lifts of) standard generators, and the element c stabilizes a chosen pre-image $y_0 \in Y$ of x_0 and acts by multiplication by a chosen primitive n-th root of unity ζ on a local holomorphic coordinate near y_0. Now $\mathrm{Rep}^d(\pi_1^{(n)}(X), U(n))$ by definition consists of representations which send c to scalar multiplication by ζ^d.

For more general actions of discrete subgroups $\pi \subset \mathrm{Aut}(Y)$, the language of π-bundles [Se] and parabolic bundles [MS] can be used to relate unitary representations of π to semi-stable bundles (where stability is defined with the aid of appropriate notions of degree) and obtain generalizations of the results just discussed. Crucial notions in each case can be traced back to Weil. An article by Biswas [Bi] explains the link between the different kinds of bundles. In the context of these results, π can be identified with the *orbifold fundamental group* $\pi_1^{\mathrm{orb}}(\mathcal{X})$ of the orbifold quotient $\mathcal{X}$. When $\mathcal{X}$ is compact, there is a notion of orbifold degree of a vector bundle on $\mathcal{X}$, and in the language of orbifolds, Weil's result states that a vector bundle on $\mathcal{X}$ comes from a representation of $\pi_1^{\mathrm{orb}}(\mathcal{X})$ if and only if every indecomposable component has orbifold degree 0.

The orbifold fundamental group, or fundamental group of a DM stack, can be defined in algebraic geometry by imitating the treatment of algebraic fundamental group given by Grothendieck [SGA1]. For details, see [Z] and [No]. The concept of orbifold fundamental group has long been familiar to topologists, especially through the work Thurston; for a description, see [Sco].

3.2. **Uniformization.** A key property that extends to the orbifold setting is uniformization. The simply connected compact orbifold curves are:

- $\mathbb{P}^1$,
- "teardrops", i.e., orbifold $\mathbb{P}^1$'s with $r = 1$ and $n_1 \in \mathbb{Z}$, $n_1 \geq 2$,
- "footballs"[2], i.e., orbifold $\mathbb{P}^1$'s with $r = 2$ and n_1, $n_2 \in \mathbb{Z}$, n_1, $n_2 \geq 2$, $\gcd(n_1, n_2) = 1$.

A compact orbifold curve $\mathcal{X}$, with underlying coarse moduli space X and orders of ramification $n_1, \ldots, n_r$, is *spherical*, respectively *Euclidean*, respectively *hyperbolic*, when the quantity

$$2 - 2g - \sum_{i=1}^{r} \frac{n_i - 1}{n_i}$$

is positive, respectively zero, respectively negative, where g is the genus of X. In the spherical case, $\mathcal{X}$ has compact universal cover, which will be one of the orbifolds listed above. Otherwise $\mathcal{X}$ is uniformized by Euclidean space in the Euclidean case and the complex upper half-plane in the hyperbolic case. These results are well known; a complete discussion with proof and extension to the case of Deligne–Mumford curves with nontrivial generic stabilizer is given by Behrend and Noohi [BN].

[2]The American variant is envisaged.

For example, when $\mathcal{X}$ is given by $X = \mathbb{P}^1$ with $r = 2$ and $n_1 = n_2 = n$, then $\pi_1^{\mathrm{orb}}(\mathcal{X}) = \mathbb{Z}/n\mathbb{Z}$, and the universal cover is $\mathbb{P}^1$. In this case we have $\mathcal{X} \cong [\mathbb{P}^1/(\mathbb{Z}/n\mathbb{Z})]$.

4. Smooth DM stacks

In this section we consider a DM stack $\mathcal{X}$ that is smooth and of finite type over a base field k. We need to make the further hypothesis that $\mathcal{X}$ has *finite stabilizer*, meaning that projection from the *inertia stack* $\mathcal{I}(\mathcal{X})$ (the fiber product of the diagonal with itself $\mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$) to $\mathcal{X}$ is a finite morphism (this condition is weaker than requiring $\mathcal{X}$ to be separated). Given these hypotheses there will be a coarse moduli space X associated with $\mathcal{X}$ [KMo]. In general, X might not be a scheme, but only an algebraic space [Kn]. These hypotheses will be in effect for this entire section, except in Definition 4.1 and Remarks 4.2 and 4.3.

4.1. **Orbifolds.** We wish to discuss first the case of orbifolds. Different authors use the term differently, but usually an algebraic orbifold denotes a smooth DM stack that has a dense open subset isomorphic to an algebraic variety. In other words, we want the stack $\mathcal{X}$ to have *trivial generic stabilizer*. (One does not need to require $\mathcal{X}$ to be irreducible; then every irreducible component of $\mathcal{X}$ should have trivial generic stabilizer.)

The first general structure result that we quote here is that when $\mathcal{X}$ has trivial generic stabilizer we have $\mathcal{X} \cong [P/G]$ for some algebraic space P and linear algebraic group G. This result is well known when the base field has characteristic 0, for then having trivial generic stabilizer implies that at any geometric point the stabilizer group acts faithfully on the tangent space, and we can then take P to be the frame bundle associated with the tangent bundle of $\mathcal{X}$ and take G to be GL_r, where $r = \dim \mathcal{X}$ [Sa, §1.5]. The argument in positive characteristic is similar, using jet bundles [EHKV, Thm. 2.18].

4.2. **Quotient stacks.** Let $\mathcal{X}$ be a DM stack of finite type over the given base field.

Definition 4.1. We say that $\mathcal{X}$ is a *quotient stack* or is a *global quotient* when $\mathcal{X} \cong [P/G]$ for some algebraic space P and linear algebraic group G.

So the result of Section 4.1 can be informally stated: "Every orbifold is a global quotient." (Caution: some authors use the term "global quotient" to denote quotients by finite groups.)

Remark 4.2. Let $\mathcal{X}$ be a DM stack of finite type over the base field. If we have $\mathcal{X} \cong [P/G]$, where P is an algebraic space and G is a linear algebraic group, and if we represent G as an algebraic subgroup of GL_n for some n, then $[P/G] \cong [P'/GL_n]$ where P' is the (algebraic space) quotient $P' = P \times GL_n/G$ with G acting by the given action on P and by translation on GL_n (cf. [EHKV, Remark 2.11]). So there is no loss of generality in taking G to be GL_n for some n in the quotient presentation of $\mathcal{X}$.

Remark 4.3. Let $\mathcal{X} \cong [P/G]$ as in Remark 4.2, and assume that $\mathcal{X}$ has finite stabilizer. Let X be the coarse moduli space. Then $P \to X$ is affine. In particular, if X is a scheme then so is P, and if X is quasi-projective then so is P. To see this, it suffices to perform a base change to an affine scheme X' étale over X and show that $P \times_X X'$ is affine. Since formation of the coarse moduli space commutes with flat base change, we are reduced to showing that P is affine when X is affine. There exists a scheme

Z and a finite surjective morphism $Z \to \mathcal{X}$ [V, Prop. 2.6] [LM, Thm. 16.6] [EHKV, Thm. 2.7]. Since the morphism $\mathcal{X} \to X$ is proper and quasi-finite, Z is finite over X and hence is affine. The morphism $P \to \mathcal{X}$ is affine, so $W := Z \times_{\mathcal{X}} P$ is also affine. But $W \to P$ is finite and surjective, so P is affine by Chevalley's theorem for algebraic spaces [Kn, Thm. III.4.1].

A DM stack (over a field) is called *tame* when the characteristic of the base field does not divide the order of any geometric stabilizer group, and is called *generically tame* when it has a dense open substack that is tame. The next result says, in particular, that when the base field has characteristic 0, every smooth separated DM stack with quasi-projective coarse moduli space is a quotient stack.

Theorem 4.4. *Let $\mathcal{X}$ be a smooth separated generically tame DM stack of finite type over a field, and assume that the coarse moduli space of $\mathcal{X}$ is quasi-projective. Then $\mathcal{X} \cong [Y/G]$ for some quasi-projective scheme Y and linear algebraic group G.*

The proof proceeds in several steps; we outline it here. We may assume that $\mathcal{X}$ is irreducible.

Step 1. There is a smooth DM stack $\mathcal{X}_0$ with trivial generic stabilizer and an étale morphism $\mathcal{X} \to \mathcal{X}_0$ such that $\mathcal{X}$ and $\mathcal{X}_0$ have the same coarse moduli space, and such that there exists an étale cover $U \to \mathcal{X}_0$ with $U \times_{\mathcal{X}_0} \mathcal{X} \cong U \times BG$ for some finite group G whose order divides the orders of all the geometric stabilizer groups of $\mathcal{X}$. In short, "$\mathcal{X}$ is a gerbe over an orbifold."

Step 2. By a Bertini-type argument, there exists a smooth scheme Z and a finite flat surjective morphism $Z \to \mathcal{X}_0$.

Now the fiber product $Z \times_{\mathcal{X}_0} \mathcal{X}$ is a gerbe for the group G (étale locally a product with BG) over Z. Notice that Z is finite over the coarse moduli space of $\mathcal{X}$ and hence is quasi-projective.

Step 3. If $Z \times_{\mathcal{X}_0} \mathcal{X}$ is a quotient stack then so is $\mathcal{X}$. Hence we are reduced to showing that every tame gerbe over a smooth quasi-projective scheme is a quotient stack.

Step 4. A further reduction to the case of a gerbe for a group μ_n of roots of unity (where the characteristic of the base field is assumed not to divide n) over a smooth quasi-projective scheme.

Step 5. The case of μ_n-gerbes in Step 4 is implied by the statement that for any smooth quasi-projective scheme Z, the map from the (Azumaya) Brauer group to the cohomological Brauer group

$$\mathrm{Br}(Z) \to \mathrm{Br}'(Z) \tag{4.1}$$

is surjective.

Brauer groups of schemes were introduced by Grothendieck [GB], generalizing the classical definition for fields. Two variants were introduced: the Brauer group $\mathrm{Br}(Z)$, which is the group of equivalence classes of sheaves of Azumaya algebras (a generalization of the classical notion of central simple algebra over a field, and Brauer equivalence), and the cohomological Brauer group $\mathrm{Br}'(Z)$, which is the torsion subgroup of the étale cohomology group $H^2(Z, \mathbb{G}_m)$. They are related (provided that Z is quasi-compact) by the Brauer map (4.1), which is injective.

The fact quoted in Step 1 is well known and can be found in [BN, Prop. 4.6], [O, Prop. 2.1]. The construction of $\mathcal{X}_0$ from $\mathcal{X}$ is an example of *rigidification*, that is, killing off a part of the stabilizer of $\mathcal{X}$. The inertia stack $\mathcal{I}(\mathcal{X})$ is a group object in the category whose objects are representable morphisms of DM stacks to $\mathcal{X}$ and in which a morphism from $u\colon \mathcal{S} \to \mathcal{X}$ to $v\colon \mathcal{T} \to \mathcal{X}$ is an equivalence class of pairs (f, α) where $f\colon \mathcal{S} \to \mathcal{T}$ is a morphism of stacks and $\alpha\colon v \circ f \Rightarrow u$ is a 2-morphism (natural isomorphism of functors), with $(f, \alpha) \sim (g, \beta)$ if there exists a (necessarily unique) 2-morphism $\gamma\colon f \Rightarrow g$ such that the composition with β is equal to α. The union of components of $\mathcal{I}(\mathcal{X})$ which are flat (hence, étale) over $\mathcal{X}$ forms a subgroup object of $\mathcal{I}(\mathcal{X})$. The rigidification construction of [ACV, Thm. 5.1.5], which produces a quotient of $\mathcal{X}$ by a central flat subgroup of inertia, generalizes to the setting of a general flat subgroup object [AOV, App. A]. The resulting stack $\mathcal{X}_0$ has the same coarse moduli space as $\mathcal{X}$.

Step 2 requires the observation, discussed in Section 4.1, that every orbifold is a quotient stack. The result is then an application of [KV, Thm. 1]. The idea is to use the global quotient structure to cover $\mathcal{X}_0$ by a projectivized vector bundle which is a scheme away from a locus of high codimension. By taking enough general slices we obtain a finite flat scheme cover.

The reduction step, Step 3, uses the fact [EHKV, Lemma 2.13] that to show that $\mathcal{X}$ is a quotient stack, it suffices to exhibit a quotient stack which covers $\mathcal{X}$ by a finite flat representable morphism.

Step 4 is outlined in [EHKV, §3]. General facts about gerbes imply that a given gerbe for some finite group G can be covered, via a finite flat representable morphism, by a gerbe for the center of G. We are then easily reduced to considering gerbes for cyclic groups, which, after passing to a finite extension to the base field and using tameness, we may identify with groups of roots of unity.

Then Step 5 is [EHKV, Thm. 3.6], which is the statement that a μ_n-gerbe over a scheme is a quotient stack if and only if the corresponding n-torsion element of the cohomological Brauer group is in the image of the Brauer map. (More precisely, there is a notion of gerbe "banded by μ_n"; such a gerbe will cover, via a finite flat representable morphism, the given gerbe, and it is to a μ_n-banded gerbe that there is an associated Brauer group element.) The link with stacks comes from the fact that μ_n-banded gerbes are classified by a second étale cohomology group which is closely related to the cohomological Brauer group.

Steps 1 through 5 are stated as [KV, Thm. 2], which asserts that the validity of Theorem 4.4 is equivalent to the surjectivity of the Brauer map.

Surjectivity of the Brauer map for quasi-projective schemes over an arbitrary affine base scheme (and more generally for schemes possessing an ample invertible sheaf) is a result due to Gabber (unpublished). De Jong has given a new proof of this result [deJ]. Gabber's result completes the proof of Theorem 4.4.

A non-separated scheme for which the Brauer map fails to be surjective has been exhibited [EHKV, Exa. 2.17]. The scheme in question is the union of two quadric cones $\operatorname{Spec} k[x, y, z]/(xy - z^2)$, glued along the complement of the vertex (we require k to have characteristic different from 2). A manifestation of the non-surjectivity of the Brauer map is that a particular stack, the union of two copies of $\operatorname{Spec} k[x, y, z]/(xy - z^2) \times B(\mathbb{Z}/2\mathbb{Z})$ glued via a nontrivial automorphism of the product of the cone minus the vertex with $B(\mathbb{Z}/2\mathbb{Z})$, is not a quotient stack (in fact, has no nontrivial vector

bundles at all). The surjectivity of the Brauer map for separated but non-quasi-projective schemes remains a tantalizing open question. The case of geometrically normal surfaces has been settled affirmatively by Schröer [Sch].

5. Finite-type DM stacks

We now drop the hypothesis that $\mathcal{X}$ should be smooth. In this section, $\mathcal{X}$ always denotes a DM stack of finite type over the base field k which has finite stabilizer.

Two fundamental questions that we can raise about such a stack $\mathcal{X}$ are:

Question 1. Under what hypotheses must $\mathcal{X}$ be a quotient stack?

Question 2. When does $\mathcal{X}$ deserve to be called "(quasi-)projective"?

5.1. **Criteria to be a quotient stack.** Question 1 remains difficult, and we can at present only give a partial answer. Theorem 4.4 identifies a large class of smooth DM stacks that are quotient stacks. A DM stack that admits a representable morphism to a quotient stack must be a quotient stack. By [EHKV, Lemma 2.13], a DM stack that admits a projective (which implies representable) flat covering map from a quotient stack is a global quotient. A DM stack is a quotient stack if and only if it has a vector bundle for which the actions of geometric stabilizer groups on fibers are faithful [EHKV, Lemma 2.12]. Another criterion relates to the finite cover by a scheme which exists for any $\mathcal{X}$ (see Remark 4.3). Letting $f\colon Z \to \mathcal{X}$ be a finite scheme cover, according to [EHKV, Thm. 2.14] $\mathcal{X}$ is a global quotient if the coherent sheaf $f_*\mathcal{O}_Z$ admits a surjective morphism from a locally free coherent sheaf on $\mathcal{X}$.

Being a quotient stack is closely related to the *resolution property*, which is the statement that every coherent sheaf admits a surjective morphism from a locally free coherent sheaf. This condition, for algebraic stacks, was studied by Totaro [To]. (Totaro's results are interesting and new even for schemes, implying for instance that a normal Noetherian scheme that has the resolution property must be the quotient of a quasi-affine scheme by GL_n for some n.)

Proposition 5.1. *Assume that $\mathcal{X}$ (a DM stack of finite type over a field k with finite stabilizer) has quasi-projective coarse moduli space. Then the following are equivalent.*

(i) *$\mathcal{X}$ is a quotient stack.*
(ii) *$\mathcal{X}$ is isomorphic to the stack quotient of a quasi-projective scheme by a reductive algebraic group acting linearly.*
(iii) *There exists a locally free coherent sheaf on $\mathcal{X}$ for which the geometric stabilizer group actions on fibers are faithful.*
(iv) *The resolution property holds for $\mathcal{X}$.*
(v) *There exists a scheme Z and a finite flat surjective morphism $Z \to \mathcal{X}$, such that the smooth locus of Z is the pre-image of the smooth locus of $\mathcal{X}$.*

Proof. Assume that $\mathcal{X}$ is a quotient stack. Then we have seen, by Remark 4.3, that $\mathcal{X}$ must be isomorphic to a stack of the form $[Y/G]$ where G is a linear algebraic group and Y is a quasi-projective scheme, affine over X. By Remark 4.2, there is no loss of generality in taking G to be GL_n for some n. By pulling back an ample line bundle from X we get a canonical linearization of the G-action on Y. So, (i) is equivalent to (ii). As already noted, (i) is equivalent to (iii).

Given a linear algebraic group over a field, the resolution property for a linearized action on a quasi-projective scheme is known classically; see [Th, Lemma 2.6] for a proof. So, (ii) implies (iv). We already know that (iv) implies (i). Lastly, (i) implies

(v) by [KV, Thm. 1], which uses a Bertini-type argument as outlined above, and we already know that (v) implies (i). □

We have seen, in the orbifold curve case, that to be a global quotient by a *finite* group is too much to hope for. But, if X denotes the coarse moduli space of the DM stack $\mathcal{X}$, then a well-known consequence of the construction of the coarse moduli space in [KMo] is the existence, for any geometric point of $\mathcal{X}$ with image $x \in X$, of an étale neighborhood $U \to X$ of x such that $U \times_X \mathcal{X}$ is isomorphic to a stack of the form $[Y/G]$ (for some scheme Y) where G is the stabilizer group of the given geometric point. It is natural to ask whether, in fact, every DM stack admits a Zariski open covering by finite group quotients. The first remark in this direction is that we must allow the finite groups to be larger than the geometric stabilizer groups. For instance, if n is a positive integer and X is a nonsingular variety over an algebraically closed field k of characteristic not dividing n, and if $\mathcal{X}$ is a μ_n-gerbe associated with some nonzero n-torsion element $\alpha \in \mathrm{Br}(X)$, then $\mathcal{X}$ will never have an open substack of the form $[Y/\mu_n]$ (the map $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$ is injective, so in particular for any nonempty open $U \subset X$, the map $\mathrm{Br}(X) \to \mathrm{Br}(U)$ is injective). One result that we can give is the following.

Proposition 5.2. *Let $\mathcal{X}$ be a DM stack (of finite type over a field k with finite stabilizer) whose coarse moduli space X is a scheme. Then $\mathcal{X}$ admits a Zariski open covering by quotient stacks if and only if it admits a Zariski open covering by stack quotients of schemes by finite groups.*

Proof. One implication is obvious. The reverse implication can be established by imitating the Bertini-type argument used in the proof of [KV, Thm. 1]. Replacing $\mathcal{X}$ by an open substack which is a global quotient and has quasi-projective coarse moduli space, the projectivized vector bundle over $\mathcal{X}$ can be cut down to a finite flat cover, maintaining smoothness of the scheme locus of the fiber over a chosen closed point $x \in X$. Then $Z \to \mathcal{X}$ will be étale after restricting to an open subset $U \subset X$ containing x. Let us replace X by U and $\mathcal{X}$ by the corresponding open substack. Having $Z \to \mathcal{X}$ finite, étale, and surjective implies, by [LM, Thm. 6.1], that there is a scheme, finite and étale over Z, such that $\mathcal{X}$ is isomorphic to the stack quotient of this scheme by a finite group (supposing $Z \to \mathcal{X}$ to be of constant degree d, this can be taken to be the complement of the diagonals of the d-fold fiber product $Z \times_{\mathcal{X}} Z \times_{\mathcal{X}} \cdots Z$, with action of the symmetric group S_d). □

5.2. **Generating sheaves and quasi-projective stacks.** A closely related condition is the existence of a *generating sheaf* on a tame DM stack, a notion introduced by Olsson and Starr [OS]. Let $\mathcal{X}$ be a tame DM stack, with $\pi\colon \mathcal{X} \to X$ the morphism to the coarse moduli space. Then a locally free coherent sheaf $\mathcal{V}$ is a generating sheaf for $\mathcal{X}$ if, for any quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$, the morphism

$$\pi^*(\pi_* \mathit{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{V}, \mathcal{F})) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{V} \to \mathcal{F} \tag{5.1}$$

is surjective. By [OS, Thm. 5.5], a generating sheaf exists for $\mathcal{X}$ if and only if $\mathcal{X}$ is a quotient stack. Another result [OS, Thm. 5.2] states that $\mathcal{V}$ is a generating sheaf if and only if, for every geometric point of $\mathcal{X}$, with G the geometric stabilizer group, the representation of G on the fiber of $\mathcal{V}$ at the geometric point contains every irreducible representation of G.

We saw, in Section 5.1, the weaker condition that the geometric stabilizer groups should act faithfully on fibers (equivalent to $\mathcal{X}$ being a quotient stack). Let $\mathcal{V}$ be

a vector bundle with faithful geometric stabilizer group actions on fibers, so $\mathcal{X}$ is a quotient stack and hence must possess a generating sheaf. By a theorem of Burnside which states that the sum of tensor powers of a faithful representation contains every irreducible representation [Bu, Thm. XV.IV], we have explicitly that $\bigoplus_{i=1}^{r} \mathcal{V}^{\otimes i}$ is a generating sheaf for sufficiently large r (in fact, r greater than or equal to the number of conjugacy classes of any geometric stabilizer group of $\mathcal{X}$ is sufficient; see [Br]).

The surjectivity in (5.1) can be compared with the standard condition for an invertible sheaf $\mathcal{L}$ on a separated quasi-compact scheme X to be ample, which can be stated:

$$\bigoplus_{n\geq 1} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}^{\otimes -n}, \mathcal{F}) \otimes_{\mathbb{Z}} \mathcal{L}^{\otimes -n} \to \mathcal{F} \tag{5.2}$$

is surjective for any quasi-coherent sheaf $\mathcal{F}$. From (5.1) and (5.2), we see that on $\mathcal{X}$, the sheaves $\mathcal{V} \otimes \mathcal{L}^{\otimes -n}$ have the property that their global morphisms into $\mathcal{F}$ will generate $\mathcal{F}$, for any quasi-coherent sheaf $\mathcal{F}$. Combining these observations, we may assert that if $\mathcal{V}$ is a locally free coherent sheaf on $\mathcal{X}$ that has faithful actions of geometric stabilizer groups on fibers and $\mathcal{L}$ is an ample invertible sheaf on X, then the combination of tensor powers of $\mathcal{V}$ and tensor powers of the pull-back of $\mathcal{L}$ are sufficient to generate an arbitrary quasi-coherent sheaf on $\mathcal{X}$.

We have the following result when the base field has characteristic 0.

Theorem 5.3. *Assume that the base field k has characteristic 0. Then the following are equivalent, for a DM stack $\mathcal{X}$ (of finite type over k and having finite stabilizer).*

(i) *$\mathcal{X}$ has a quasi-projective coarse moduli space and is a quotient stack (i.e., satisfies any of the equivalent conditions of Proposition 5.1).*
(ii) *$\mathcal{X}$ has a quasi-projective coarse moduli space and possesses a generating sheaf.*
(iii) *There exists a locally closed embedding $\mathcal{X} \to \mathcal{W}$ where $\mathcal{W}$ is a smooth DM stack, proper over* $\mathrm{Spec}\, k$ *with projective coarse moduli space.*

Proof. The equivalence of (i) and (ii) follows from Proposition 5.1 and the above discussion. The stack $\mathcal{W}$ in (iii) has to be a quotient stack by Theorem 4.4, so (iii) implies (i).

Suppose $\mathcal{X}$ is a quotient stack and has a quasi-projective moduli space, so by Proposition 5.1 is isomorphic to a stack of the form $[Y/G]$, where Y is quasi-projective and G is a reductive algebraic group acting linearly on Y. Then for some N we have a linear action of G on $\mathbb{P}^N$ and an equivariant embedding $Y \to \mathbb{P}^N$, such that the embedding factors through the stable locus $(\mathbb{P}^N)^{\mathrm{s}}$ [MFK, Ampl. 1.8 and Conv. 1.12]. We would be done if we had $(\mathbb{P}^N)^{\mathrm{ss}} = (\mathbb{P}^N)^{\mathrm{s}}$, i.e., the strictly semi-stable locus is empty, for then the stack quotient of the stable locus by G would be a smooth proper DM stack into which $\mathcal{X}$ embeds. In general, this situation can be brought about by applying Kirwan's blow-up construction [Ki], or the stable resolution construction in [R]: then we have a sequence of blow-ups of nonsingular G-invariant subvarieties, leaving $(\mathbb{P}^N)^{\mathrm{s}}$ unchanged and producing a variety whose strictly semi-stable locus is empty. □

Corollary 5.4. *Let $\mathcal{X}$ be a DM stack, proper over a base field k of characteristic 0. Then the following are equivalent.*

(i) *$\mathcal{X}$ has a projective coarse moduli space and is a quotient stack.*
(ii) *$\mathcal{X}$ has a projective coarse moduli space and possesses a generating sheaf.*
(iii) *$\mathcal{X}$ admits a closed embedding to a smooth proper DM stack with projective coarse moduli space.*

It would be interesting to know what sorts of results are true (under suitable tameness hypotheses) when the base field has positive characteristic. Keeping to the case of characteristic 0, the results just stated and the discussion preceding them motivate the following answer to Question 2.

Definition 5.5. Suppose that the base field k has characteristic 0. Then we say that a DM stack $\mathcal{X}$ over k is *quasi-projective*, respectively *projective*, if $\mathcal{X}$ admits a locally closed embedding, respectively a closed embedding, to a smooth DM stack which is proper over $\operatorname{Spec} k$ and has projective coarse moduli space.

Many well-known moduli stacks are projective. We continue to assume that k has characteristic 0. There are the moduli stacks $\overline{\mathcal{M}}_{g,n}(V,d)$ of n-pointed genus g stable maps to a projective target variety V (of degree d, with respect to a given polarization on V), introduced by Kontsevich (see [KMa]) and a crucial ingredient in Gromov–Witten theory. These stacks are proper and admit embeddings into smooth DM stacks that are quotient stacks according to the construction of [FP, Sec. 2], which avoids stack language; [GP, App. A] provides a stack-based treatment and also points out that the embeddings are into not necessarily separated DM stacks. Now by Theorem 5.3 there exist, in fact, embeddings into smooth proper DM stacks having projective coarse moduli spaces. So, we may assert that the stacks $\overline{\mathcal{M}}_{g,n}(V,d)$ are projective DM stacks.

More generally there are moduli stacks $\mathcal{K}_{g,n}(\mathcal{X},d)$ of twisted stable maps [AV], where the target is now a proper DM stack $\mathcal{X}$ having projective coarse moduli space. These stacks play a central role in orbifold Gromov–Witten theory; see [AGV] and [CR] for an algebraic respectively symplectic approach to the theory. By [AGOT], under the further assumption that $\mathcal{X}$ is a quotient stack, the stacks $\mathcal{K}_{g,n}(\mathcal{X},d)$ admit embeddings into smooth DM stacks that are quotient stacks. So, the hypothesis that $\mathcal{X}$ is a projective DM stack implies that the stacks $\mathcal{K}_{g,n}(\mathcal{X},d)$ are also projective DM stacks.

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