Year: 2007

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Originally published at:
Loubet, E (2007). Integration of pair flows of the Camassa-Holm hierarchy. In: Pinsky, M; Birnir, B. Probability, geometry and integrable systems. Cambridge: Cambridge Univ. Press, 261-285.

Probability, Geometry and Integrable Systems
MSRI Publications
Volume **55**, 2007

Integration of pair flows of the Camassa–Holm hierarchy

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To Henry McKean with my admiration and respect on the occasion of his seventy-fifth birthday

ABSTRACT. We present the integration of the "pair" flows associated to the Camassa–Holm (CH) hierarchy i.e., an explicit exact formula for the update of the initial velocity profile in terms of initial data when run by the flow associated to a Hamiltonian which (up to a constant factor) is given by the sum of the reciprocals of the squares of any two eigenvalues of the underlying spectral problem. The method stems from the integration of "individual" flows of the CH hierarchy described in [Loubet 2006; McKean 2003], and is seen to be more general in scope in that it may be applied when considering more complex flows (e.g., when the Hamiltonian involves an arbitrary number of eigenvalues of the associated spectral problem) up to when envisaging the full CH flow itself which is nothing but a superposition of commuting individual actions. Indeed, by incorporating piece by piece into the Hamiltonian the distinct eigenvalues describing the spectrum associated to the initial profile, we may recover McKean's Fredholm determinant formulas [McKean 2003] expressing the evolution of initial data when acted upon by the full CH flow. We also give account of the large-time (and limiting remote past and future) asymptotics and obtain (partial) confirmation of the thesis about soliton genesis and soliton interaction raised in [Loubet 2006].

Keywords: integrable systems, soliton traveling waves, spectral theory, Darboux transformations, asymptotic analysis.

Research partially supported by the Swiss National Science Foundation. This is gratefully acknowledged.

1. Introduction

The equation of Camassa and Holm (CH) [1993; 1994] is an approximate one-dimensional description of unidirectional propagation of long waves in shallow water. In dimensionless space-time variables it reads

$$\frac{\partial m}{\partial t} + (mD + Dm)(v) = 0 \tag{1}$$

in which $D = \partial/\partial x = (\,\cdot\,)'$, $m = (1 - D^2)v$ and at any given time t in $\mathbb{R}$, the real valued function $v = v(t, \cdot)$ represents the fluid velocity (or equivalently the height of the water's free surface above flat bottom). It is an infinite dimensional integrable bi-Hamiltonian system i.e., (1) is equivalent to

$$\frac{\partial m}{\partial t} = \mathcal{J}(m)\left(\frac{\partial H_{\mathrm{CH}}}{\partial m}\right) = \{m, H_{\mathrm{CH}}\}_{\mathcal{J}} = \{m, H^{+}_{\mathrm{CH}}\}_{\mathcal{K}} = \mathcal{K}\left(\frac{\partial H^{+}_{\mathrm{CH}}}{\partial m}\right)$$

with Hamiltonians

$$-H_{\mathrm{CH}} := \tfrac{1}{2}\int_{-\infty}^{+\infty} [v^2 + (v')^2] \quad \text{and} \quad -H^{+}_{\mathrm{CH}} := \tfrac{1}{2}\int v[v^2 + (v')^2]$$

linked, via their corresponding functional gradients, by the Lenard raising/lowering rule [McKean 1993] as in $\mathcal{J}(m)(\partial H_{\mathrm{CH}}/\partial m) = \mathcal{K}(\partial H^{+}_{\mathrm{CH}}/\partial m)$. The pair

$$\mathcal{J}(m) := mD + Dm \qquad \text{and} \qquad \mathcal{K} := D(1 - D^2)$$

of *skew* operators (with respect to the H^0-inner product) being employed to define (via the H^0-inner product) a pair of *compatible* Poisson brackets so that, for a suitable class of functionals defined on phase space, a Lie algebra is specified (see [Loubet 2006] for more details.) Moreover, just like most integrable nonlinear evolution equations, CH equation (1) is also equivalent to the compatibility condition of an *overdetermined linear system* comprising the so called Lax pair; an evolution problem

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial v}{\partial x} f - \left(v + \frac{1}{2\lambda}\right)\frac{\partial f}{\partial x} \tag{2}$$

and a spectral problem, the acoustic equation with "potential" or "mass" m,

$$(1/4 - D^2) f = \lambda m f, \tag{3}$$

where λ and f denote, respectively, the eigenvalue and its associated eigenfunction. Here, compatibility means enforcing the matching of mixed derivatives i.e., $(f^{\bullet})'' = (f'')^{\bullet}$ where $\partial/\partial t = (\,\cdot\,)^{\bullet}$. It follows that (1) preserves the spectral characteristics of (3) i.e., CH flow is *isospectral*.

For real summable m the spectrum of (3) is purely discrete and simple i.e., $\operatorname{spec}(m)=\{\lambda_j(m)\in\mathbb{R},\ j=\dots,-1,0,1,\dots\}$ where λ_j is a real value for which there exists a unique normalized solution f_j of (3) in H^1:

$$|f_j|_1^2 := \int \lambda_j m f_j^2 = \int [f_j'^2 + \tfrac{1}{4} f_j^2] = 1.$$

Most significant is that, within the class of summable m, CH flow is nothing but a *superposition of commuting* individual actions. Indeed, this opens the possibility to analyze CH flow via the accumulated effects that each of its constituents entail, the latter being presumed to be simpler to describe. And, indeed, the flows based upon a Hamiltonian of the form $H=1/\lambda$ where λ is any eigenvalue of (3) turned out to be manageable [Loubet 2006]. More specifically, our goal was to elucidate as many qualitative properties of the full CH flow as possible from a direct and detailed analysis of the changes that each of its constitutive components produce when acting on generic initial data. We paid particular attention to how much could be said about the emergence of solitons for the CH flow by tracking down the effects of its individual actions. This investigation was possible from a careful analysis of explicit exact formulas for the updates of a generic initial profile run by any such elementary flow when expressed in terms of its private "Lagrangian" scale. Denoting by $\mathbb{X}_H$ the Hamiltonian vector field associated to the Hamiltonian H, $\phi^t_{\mathbb{X}_H}$ the corresponding flow map describing the updates $m := (\phi^t_{\mathbb{X}_H} m_0)$ at time t of the elements m_0 in phase space — here the class of real valued summable functions — and $\phi^t_{\mathbb{X}_H *}$ the flow that it induces on functionals of m_0, a "Lagrangian" scale is specified by

$$\frac{\partial \mathcal{L}^t_H}{\partial t} = -\left(\phi^t_{\mathbb{X}_H *}\frac{\partial H}{\partial m_0}\right)\circ \mathcal{L}^t_H, \qquad \mathcal{L}^0_H = \mathrm{id}$$

in which $\partial H/\partial m_0$ denotes the functional gradient. We have proved:

THEOREM 1. *The Hamiltonian flow of the CH hierarchy arising from* $H=1/\lambda$ (*where* λ *is an arbitrary eigenvalue of the acoustic equation* $f_0''=(1/4-\lambda m_0)f_0$ *which is associated to the summable initial data* $m_0=v_0-v_0''$) *is integrated explicitly in terms of the latter with the help of its private "Lagrangian" scale specified by*

$$\partial \mathcal{L}^t_H/\partial t = -(\phi^t_{\mathbb{X}_H *}\partial H/\partial m_0)\circ \mathcal{L}^t_H, \qquad \mathcal{L}^0_H = \mathrm{id}$$

and three "theta" functions (*each of which depends on* t *and a spatial variable denoted by* $\cdot$ *if unspecified*, *e.g.*, $\vartheta=\vartheta(t,\cdot;\lambda m_0)$ *and so on*), *namely*

$$\begin{matrix}\vartheta_- \\ \vartheta \\ \vartheta_+\end{matrix} = 1 + (e^t - 1)\int_{-\infty}^{\cdot} \begin{cases} (f_0' - \frac{1}{2} f_0)^2 \\ \lambda m_0 f_0^2 \\ (f_0' + \frac{1}{2} f_0)^2. \end{cases} \tag{4}$$

To wit,

$$\begin{aligned}(\phi^t_{\mathbb{X}_H} v_0) \circ \mathscr{L}^t_H &= \frac{\vartheta_+}{\vartheta_-}\Big(\frac{v_0 - v_0'}{2}\Big) + \frac{\vartheta_-}{\vartheta_+}\Big(\frac{v_0 + v_0'}{2}\Big) \\ &\quad + \frac{\sqrt{\vartheta_-'\vartheta_+'}}{\lambda\vartheta_-\vartheta_+}(\vartheta_+ - \vartheta_-) + \frac{\vartheta}{2\lambda\vartheta_-\vartheta_+}(\vartheta_+ - \vartheta_-)', \\ \frac{\partial(\phi^t_{\mathbb{X}_H} v_0) \circ \mathscr{L}^t_H}{\partial \mathscr{L}^t_H} &= -\frac{\vartheta_+}{\vartheta_-}\Big(\frac{v_0 - v_0'}{2}\Big) + \frac{\vartheta_-}{\vartheta_+}\Big(\frac{v_0 + v_0'}{2}\Big) \\ &\quad + \frac{\sqrt{\vartheta_-'\vartheta_+'}}{\lambda\vartheta_-\vartheta_+}(\vartheta_+ - \vartheta_-)' + \frac{\vartheta}{2\lambda\vartheta_-\vartheta_+}(\vartheta_+ + \vartheta_-)' \\ &\quad + \frac{\vartheta_-'\vartheta_+'}{\lambda\vartheta_-\vartheta_+\vartheta}(\vartheta_+ - \vartheta_-)\end{aligned}$$

or, *equivalently*,

$$(\phi^t_{\mathbb{X}_H} m_0) \circ \mathscr{L}^t_H = \left(\frac{\vartheta_-\vartheta_+}{\vartheta^2}\right)^2 m_0.$$

It is remarkable that *all* updated expressions arising in the study of individual flows are given in terms of the theta functions (4). In fact, McKean had previously integrated the CH equation on the line by means of a triple of "theta-like" Fredholm determinants [McKean 2003]. The nomenclature is prompted from the fact that these determinants as well as their individual theta functions counterparts (4) satisfy a number of properties which are reminiscent of those met by Riemann's theta function together with its translates. Notably, there is only one theta-like (determinant) function, the others being produced from it by infinitesimal addition [McKean 2001]. Moreover, these (determinants) functions satisfy curious algebraic identities among themselves [McKean 2003]; the most significant one being

$$\vartheta^2 = \vartheta_-\vartheta_+ + \vartheta_-'\vartheta_+ - \vartheta_-\vartheta_+'.$$

In this paper, we will show that these underlying algebraic structures prevail when considering composite flows of a particular but sufficiently general class (see Theorem 2 below). Our aim is to offer a detailed account of the integration of the aforementioned pair flows associated with the CH hierarchy and discuss their large-time asymptotics. It will become clear to the reader that exactly the same method can be applied to integrate composite flows arising from Hamiltonians involving an arbitrary number of eigenvalues up to the full CH Hamiltonian which, when m is summable, satisfies

$$-H_{\mathrm{CH}} = \tfrac{1}{2}\int m\mathrm{G} * m = \frac{1}{4}\sum \frac{1}{\lambda_n^2}$$

where $\mathrm{G} = e^{-|\cdot|}/2$ is the Green's function $(1 - D^2)\mathrm{G} = \delta$, i.e., $\mathrm{G} * m = v$; the sum accounting for all spectrum. Henceforth, we will focus in leading the reader along the hints and observations embodying the concatenation of lucky occurrences that culminate in the final expressions that substantiate the following main result.

THEOREM 2. *The Hamiltonian flow of the CH hierarchy arising from* $H = (1/\lambda_+^2 + 1/\lambda_-^2)/4$ (*in which* $\lambda_\pm$ *denote an arbitrary pair of eigenvalues of the acoustic equation* $(f_\pm^0)'' = (1/4-\lambda_\pm m_0) f_\pm^0$ *which is associated to the summable initial data* $m_0 = v_0 - v_0''$) *is integrated explicitly in terms of the latter with the help of its private "Lagrangian" scale*, *specified by*

$$\partial\mathscr{L}_H^t/\partial t = -(\phi^t_{\mathbb{X}_H *}\partial H/\partial m_0)\circ\mathscr{L}_H^t, \qquad \mathscr{L}_H^0 = \mathrm{id}$$

and three "theta" determinants (*each of which depends on t and a spatial variable denoted by* $\cdot$ *if unspecified*, *e.g.*, $\Theta = \Theta(t, \cdot; \lambda m_0)$ *and so on*), *namely*

$$\begin{matrix}\Theta_- \\ \Theta \\ \Theta_+\end{matrix} \;:= \det\left[\mathrm{Id} + \mathscr{E}(t,\Lambda)\left\{\begin{matrix}\int_{-\infty}^{\cdot} \Upsilon_{-,0}\otimes\Upsilon_{-,0} \\ \Lambda\Phi_0 \\ \int_{-\infty}^{\cdot} \Upsilon_{+,0}\otimes\Upsilon_{+,0}\end{matrix}\right.\right]. \tag{5}$$

where Id= *the* 2×2 *identity matrix*,

$$\Lambda\colon = \begin{pmatrix}\lambda_- & 0\\ 0 & \lambda_+\end{pmatrix}, \qquad \mathscr{E}(t,\Lambda)\colon = (e^{t/(2\Lambda)} - \mathrm{Id}),$$

$$\Upsilon_{-,0}\colon = \begin{pmatrix}(f_-^0)' - \frac{1}{2}f_-^0\\ (f_+^0)' - \frac{1}{2}f_+^0\end{pmatrix}, \qquad \Upsilon_{+,0}\colon = \begin{pmatrix}(f_-^0)' + \frac{1}{2}f_-^0\\ (f_+^0)' + \frac{1}{2}f_+^0\end{pmatrix},$$

and

$$\Phi_0 := \int_{-\infty}^{\cdot} m_0\boldsymbol{f}_0\otimes\boldsymbol{f}_0 = \begin{pmatrix}\varphi_-^0/\lambda_- & \varphi_0\\ \varphi_0 & \varphi_+^0/\lambda_+\end{pmatrix}$$

where $\boldsymbol{f}_0 := (f_-^0, f_+^0)^\dagger$ *and*

$$\varphi_0 := \int_{-\infty}^{\cdot} m_0 f_-^0 f_+^0; \quad \varphi_\pm^0 := \int_{-\infty}^{\cdot}\lambda_\pm m_0 (f_\pm^0)^2.$$

To wit,

$$(\phi^t_{\mathbb{X}_H} v_0) \circ \mathscr{L}^t_H = \frac{\Theta_-}{\Theta_+}\left(\frac{v_0 + v_0'}{2} + \tfrac{1}{2}\Upsilon^\dagger_{+,0}(\mathcal{M}\Lambda)^{-1}\mathscr{E}\Upsilon_{+,0}\right) + \frac{\Theta_+}{\Theta_-}\left(\frac{v_0 - v_0'}{2} - \tfrac{1}{2}\Upsilon^\dagger_{-,0}(\mathcal{M}\Lambda)^{-1}\mathscr{E}\Upsilon_{-,0}\right)$$

$$\frac{\partial(\phi^t_{\mathbb{X}_H} v_0) \circ \mathscr{L}^t_H}{\partial \mathscr{L}^t_H} = \frac{\Theta_-}{\Theta_+}\left(\frac{v_0 + v_0'}{2} + \tfrac{1}{2}\Upsilon^\dagger_{+,0}(\mathcal{M}\Lambda)^{-1}\mathscr{E}\Upsilon_{+,0}\right) - \frac{\Theta_+}{\Theta_-}\left(\frac{v_0 - v_0'}{2} - \tfrac{1}{2}\Upsilon^\dagger_{-,0}(\mathcal{M}\Lambda)^{-1}\mathscr{E}\Upsilon_{-,0}\right)$$

where $\mathcal{M}(t,\cdot) := \mathrm{Id} + \mathscr{E}(t,\Lambda)\Lambda\Phi_0$ (*so that* $\Theta = \det \mathcal{M}$) *or*, *equivalently*,

$$(\phi^t_{\mathbb{X}_H} m_0) \circ \mathscr{L}^t_H = \left(\frac{\Theta_-\Theta_+}{\Theta^2}\right)^2 m_0\,.$$

The algebraic similitude of the formulas in Theorem 1 and Theorem 2 might, in part, be at the core of why most interesting features pertaining to the full flow are already reflected at the level of its components. Furthermore, we interpret this fact as stronger evidence substantiating the nature, interplay, and relevance of individual flows to the understanding of the underlying mechanisms that are involved in soliton formation and soliton interactions.

Indeed, the explicit formulas of Theorem 1 were shown to be valuable while conducting the large-time asymptotic analysis in that they afforded a mathematical treatment to establish the eventual emergence (provided we waited long enough) of a soliton escaping to infinity at a speed commensurable to the eigenvalue characterizing the individual flow at play. See [Loubet 2006].

THEOREM 3. *Assume* $\lambda > 0$, *and let the real summable initial data* $m_0 = v_0 - v_0''$ *be such that* $m_0 = o(1)$ *for* $x \ll 0$ *and disposed as in* $\operatorname{sign}\{\lambda m_0(x)\} = \operatorname{sign}\{x\}$. *Then*, *provided that we wait long enough*, *we see that*, *to leading order*, *the velocity profile* (*run by the individual flow arising from the Hamiltonian* $H = 1/\lambda$) *shapes itself like the escaping soliton. In symbols*:

$$\mathscr{L}^t_H(x) \in [\mathscr{L}^t_H(\mathrm{R}_-(t,\kappa)), \mathscr{L}^t_H(\mathrm{R}_+(t,\kappa))] \quad \textit{for all } x,$$
$$\left|[(\phi^t_{\mathbb{X}_H} v_0) \circ \mathscr{L}^t_H](x) - \frac{1}{2\lambda} e^{-|\mathscr{L}^t_H(x) - \mathscr{L}^t_H(\mathrm{R}_0(t))|}\right| = o(1) \quad \textit{as } t \uparrow +\infty,$$

where

$$\begin{aligned} \mathscr{L}^t_H(\mathrm{R}_-(t,\kappa)) + t &= -\log\left(\frac{1+\kappa}{\kappa}\right) + o(1) \\ \mathscr{L}^t_H(\mathrm{R}_0(t)) + t &= o(1) \qquad\qquad \textit{as} \quad t \uparrow +\infty, \\ \mathscr{L}^t_H(\mathrm{R}_+(t,\kappa)) + t &= +\log\left(\frac{1+\kappa}{\kappa}\right) + o(1) \end{aligned}$$

and R_0, $\mathrm{R}_\pm$ *are, for sufficiently large times, defined respectively by*

$$\vartheta(t, \mathrm{R}_0(t)) := 0, \quad \textit{and} \quad \vartheta_\pm(t, \mathrm{R}_\mp(t,\kappa)) := 1 + \kappa^{\pm 1}$$

κ *being a nonnegative parameter*.

Note that the signature disposition $\mathrm{sign}\{\lambda m_0(x)\} = \mathrm{sign}\{x\}$ on initial data m_0 guarantees breakdown (i.e., $v' \downarrow -\infty$ for some $0 < t < +\infty$; see [McKean 1998; McKean 2004]) or, what is the same thing, the vanishing of $\vartheta(t, \cdot)$ for a sufficiently large time

$$t > T := \log\left[1 + \left(\int_{-\infty}^0 -\lambda m_0 f_0^2\right)^{-1}\right]$$

at a unique site $\mathrm{R}_0(t) < 0$. Actually, the fact that the soliton of Theorem 3 (escaping to $-\infty$) has its peak (for $t > T$) precisely at the root $\mathrm{R}_0(t)$ of $\vartheta(t, \cdot) = 0$ is merely accidental. Indeed, even in the case where no breakdown occurs (i.e., where ϑ no longer vanishes), one can adapt the analysis as in [Loubet 2006] to conclude about the genesis a soliton moving at the right of the origin (see concluding remarks in that reference). As the direction and speed of soliton propagation are given, respectively, by the signature and magnitude of the underlying eigenvalue, we see that similar large-time asymptotic behavior (as that following the results of Theorem 3 describing events way ahead into the future) would take place when going far back into the past.

What is more, the algebraic robustness of our formulas as $t \to \pm\infty$ offered further quantitative confirmation of the qualitative description of soliton genesis [Loubet $\geq$ 2007a; $\geq$ 2007b]. Indeed, as $t \to \pm\infty$, the soliton that emerged escapes to infinity leaving behind a stationary profile $\lim_{t\to\pm\infty}[(\phi^t_{\mathbb{X}_H} v_0) \circ \mathscr{L}^t_H]$. We have corroborated these facts from the energetic and spectral standpoints with the help of the exact limiting formulas describing the latter, and we have shown that the energy of the residual profile is less than the energy of the initial profile by an amount that corresponds exactly to the energy that is embodied (at any time $|t| < +\infty$) by the escaping soliton. On the other hand, the isospectrality of individual flows, $\mathrm{spec}\,[(\phi^t_{\mathbb{X}_H} m_0) \circ \mathscr{L}^t_H] = \mathrm{spec}\, m_0$, though true for any given $|t| < +\infty$, ceases to hold in the limits $t \to \pm\infty$. Indeed, the maps

$$m_0 \mapsto \lim_{t\to\pm\infty} [(\phi^t_{\mathbb{X}_H} m_0) \circ \mathscr{L}^t_H]$$

from initial to residual profiles have a Darboux-type character in that precisely the eigenvalue λ of (3) associated to m_0 from which the underlying individual flow was based upon is excised, i.e., it is no longer part of

$$\operatorname{spec}\Big(\lim_{t\to\pm\infty}[(\phi^t_{\mathbb{X}_H} m_0)\circ \mathscr{L}^t_H]\Big),$$

the discrete spectrum associated to the stationary profiles. In short, the discrete train of solitons generated by a suitable superposition of finitely many individual actions should be regarded as a caricature of the infinite soliton train describing the large-time asymptotics of the full CH model. Indeed, under current evidence [Loubet 2006], it is hard to disbelieve that the aforementioned pair flows (with $\lambda_- < 0 < \lambda_+$) will eventually give rise to symmetrically disposed pairs of solitons escaping from the origin, each with a fixed speed (which must be) regulated by their corresponding eigenvalue. We also elaborate on some of these themes in the present paper. Nonetheless, a rigorous mathematical verification of this intuitive picture may not be, a priori, as simple to establish as in the case of individual flows. On the one hand, the formulas pertaining to the pair flows involve theta-like determinants which in principle are harder to manipulate. More significantly, the waiting times before solitons occur need now to be distinguished quantitatively and not merely qualitatively ("long enough") as before. Indeed, the asymptotic analysis that is related to a Hamiltonian depending on a couple of distinct spectral values would require a precise estimate of the patience one must bear — which should depend somehow on the ratio of the intervening eigenvalues — before it is possible to detect a slower developing soliton trailing behind a faster sibling. In any case, even if such an attempt is proved to be successful — which in our opinion would constitute an instructive exercise — once we move up to the next stage in complexity, say, when considering "quadruple" flows and beyond, there is little hope that we would be able to discern the large-time asymptotics directly from the corresponding theta-like determinants of matrices of higher rank with components depending (rationally) on the eigenvalues. In short, we believe that a new approach is required to reach a concise mathematical understanding of soliton train formation associated to the CH equation. Be that as it may, our formulas might spur useful potential numerical experiments that might shed light into aspects of the genesis of solitons and their interactions (prior to their escape at infinity) that may encourage new promising strategies. Indeed, the algebraic similarity of the recipes that arise in each of the cases that we have considered so far, with the Fredholm analogues which McKean [2003] employed to give account of CH on the line, cannot simply be accidental.

2. Preparations

2.1. Identification of the pair flow. Let $H = (1/\lambda_+^2 + 1/\lambda_-^2)/4$ be the Hamiltonian corresponding (up to the constant factor 1/4) to the reciprocal of the squares of *any* pair of eigenvalues $\lambda_\pm$ of the spectral problem (3 with real summable m_0), its associated H^1 eigenfunctions $f_\pm^0$ normalized as in

$$|f_\pm^0|_1^2 = \int \lambda_\pm m_0 (f_\pm^0)^2 = \int [((f_\pm^0)')^2 + \tfrac{1}{4}(f_\pm^0)^2] = 1. \tag{6}$$

(Here we have used the notation established on page 272.) A routine computation establishes that the H^0-functional gradient of the reciprocal of any eigenvalue (spectral invariant) is given by the square of the associated (normalized) eigenfunction, so that

$$\frac{\partial H}{\partial m_0} = \frac{1}{2\lambda_-}(f_-^0)^2 + \frac{1}{2\lambda_+}(f_+^0)^2 .$$

Hence, the Hamiltonian pair flow is regulated by

$$m^\bullet = (mD + Dm)\left(\frac{1}{2\lambda_-} f_-^2 + \frac{1}{2\lambda_+} f_+^2\right), \quad m(0,\cdot) = m_0 \tag{7}$$

where $f_\pm := (\phi^t_{\mathbb{X}_H *} f_\pm^0)$ denote the normalized time t updates of $f_\pm^0$.

2.2. Induced flow on eigenfunctions. For summable m_0, the spectrum of (3) is discrete and simple [Loubet 2006]. Hence, as eigenfunctions are well-defined functionals of m_0, their variation is to be inferred from that of the potential, e.g., the evolution of the updates $f_\pm$ of the normalized eigenfunctions $f_\pm^0$ is dictated from that of m_0 and the normalization constraint. More precisely, it is prescribed by the solution of the inhomogeneous acoustic problem — which arises after taking the Lie derivative of the original acoustic problem along the vector field $\mathbb{X}_H$ i.e., after differentiating with respect to t the (time t) updated acoustic problem associated to $f_\pm$ — which conforms to the preservation of normalization. Let

$$\mathcal{A}_\pm := [1/4 - D^2 - \lambda_\pm m].$$

Then, by (7), the motion $\mathbb{X}_H[m](f_\pm)$ of the updates $f_\pm$ satisfies

$$\mathcal{A}_\pm(\mathbb{X}_H[m](f_\pm)) = \lambda_\pm f_\pm(\mathbb{X}_H[m]) = \lambda_\pm f_\pm \mathcal{J}\left(\frac{1}{2\lambda_-} f_-^2 + \frac{1}{2\lambda_+} f_+^2\right),$$

where $\mathcal{J} \equiv \mathcal{J}(m)$. Now, from the results in [Loubet 2006] pertaining to individual flows, we know that

$$\Omega_\pm := \tfrac{1}{2} f_\pm - f_\pm \int_{-\infty}^{\cdot} \lambda_\pm m f_\pm^2 = f_\pm(\tfrac{1}{2} - \varphi_\pm),$$
$$\Pi_\pm := -f_\mp \int_{-\infty}^{\cdot} \lambda_\pm m f_- f_+ = -\lambda_\pm f_\mp \varphi$$

satisfy $\mathcal{A}_\pm(\Omega_\pm) = \lambda_\pm f_\pm \mathcal{J}(f_\pm^2)$ and $\mathcal{A}_\pm(\Pi_\pm) = \lambda_\pm f_\pm \mathcal{J}(f_\mp^2)$. Then, as $\mathcal{A}_\pm$ and $\mathcal{J}$ are linear, we have

$$\mathcal{A}_\pm\left(\frac{1}{2\lambda_\pm}\Omega_\pm + \frac{1}{2\lambda_\mp}\Pi_\pm\right) = \lambda_\pm f_\pm \mathcal{J}\left(\frac{1}{2\lambda_-} f_-^2 + \frac{1}{2\lambda_+} f_+^2\right).$$

In these expressions

$$\varphi_\pm := (\phi^t_{\mathbb{X}_H *} \varphi_\pm^0) \qquad \text{and} \qquad \varphi := (\phi^t_{\mathbb{X}_H *} \varphi_0)$$

denote the time t updates of, respectively,

$$\varphi_\pm^0 := \int_{-\infty}^{\cdot} \lambda_\pm m_0 (f_\pm^0)^2 \quad \text{and} \quad \varphi_0 := \int_{-\infty}^{\cdot} m_0 f_-^0 f_+^0, \tag{8}$$

as the action of the (induced) flow commutes with integration,

$$(\phi^t_{\mathbb{X}_H *} \lambda_\pm m_0 (f_\pm^0)^2) = \lambda_\pm (\phi^t_{\mathbb{X}_H} m_0)(\phi^t_{\mathbb{X}_H *} f_\pm^0)^2 \equiv \lambda_\pm m f_\pm^2$$

and so on. Hence, we are tempted to declare that

$$\overset{\bullet}{f}_\pm = \mathbb{X}_H[m](f_\pm) := \frac{1}{2\lambda_\pm}\Omega_\pm + \frac{1}{2\lambda_\mp}\Pi_\pm = \frac{f_\pm}{2\lambda_\pm}(\tfrac{1}{2} - \varphi_\pm) - \frac{\lambda_\pm f_\mp}{2\lambda_\mp}\varphi. \tag{9}$$

To convince ourselves that this is the correct recipe, we need to check whether or not, under such evolution, the norm is preserved. The verification is simple: Let $\mathcal{N}_\pm(t) := \int \lambda_\pm m f_\pm^2$. Then, according to the "tentative" prescription (9),

$$\overset{\bullet}{\mathcal{N}}_\pm = \int \left[\lambda_\pm f_\pm^2 \mathcal{J}\Big(\frac{f_-^2}{2\lambda_-} + \frac{f_+^2}{2\lambda_+}\Big) + 2\lambda_\pm m f_\pm \Big(\frac{f_\pm}{2\lambda_\pm}(\tfrac{1}{2} - \varphi_\pm) - \frac{\lambda_\pm f_\mp}{2\lambda_\mp}\varphi\Big)\right].$$

As $\mathcal{J}$ is skew-symmetric and $f_\pm$ vanish at infinity, the integral of $f_\pm^2 \mathcal{J}(f_\pm^2)$ vanishes. On the other hand, $f_\pm^2 \mathcal{J}(f_\mp^2) = \big(m f_-^2 f_+^2 + (\lambda_\pm - \lambda_\mp)\varphi^2\big)'$, and since eigenfunctions associated to different eigenvalues are orthogonal ($\int m f_- f_+ = 0$) the last display reduces to

$$\overset{\bullet}{\mathcal{N}}_\pm = \frac{1}{2\lambda_\pm}\mathcal{N}_\pm(1 - \mathcal{N}_\pm)$$

from where it is plain that $\mathcal{N}_\pm(t) \equiv 1$ for every $|t| < +\infty$ since $\mathcal{N}_\pm(0) = 1$. In other words, the right-hand side of (9) dictates the evolution of $f_\pm$ that conforms to normalization.

2.3. Constants of motion and private Lagrangian scale. In addition to the infinite number of independent functionals in involution which are preserved by any flow of the CH hierarchy as follows from Magri's observation [1978] that *compatibility of brackets is equivalent to saying that the class of raisable functions is one and the same as the class of lowerable ones* and the Lenard scheme (starting from the lower/upper pair H_{CH} and H^{+}_{CH}) alluded to in the introduction, the CH equation has another infinite collection of fundamental invariants. They are defined as follows. Every nice functional H defined on phase space gives rise to a flow $\phi^t_{\mathbb{X}_H}$ i.e., a one-parameter group of diffeomorphisms of a domain of phase space into itself characterized by the solution curves $m := (\phi^t_{\mathbb{X}_H} m_0)$ starting at m_0 of the dynamical system — *in an underlying "original" spatial scale* x — associated to the (locally Lipschitz) Hamiltonian vector field: $m^{\bullet} := \mathbb{X}_H[m] = \mathcal{J}(\partial H/\partial m)$. It also gives rise to a *new* "Lagrangian" scale $\mathcal{L}^t_H(x) =: \bar{x}(t,x)$ characterized by $\bar{x}^{\bullet} = -(\partial H/\partial m)\circ\bar{x}$, $\bar{x}(0,x) = x$. That is, at any given time t up to breakdown [McKean 1998; 2004], the map $\bar{x}(t,\cdot) = \mathcal{L}^t_H$ is a diffeomorphism of the real line issuing from the identity. The upshot being that, at any time t before (possible) breakdown and for every x in $\mathbb{R}$, $[(\phi^t_{\mathbb{X}_H} m_0)\circ\mathcal{L}^t_H](x)\times[\partial\mathcal{L}^t_H(x)/\partial x]^2$ is a constant of motion $\equiv m_0(x)$. The verification is straightforward (see Remark 4 below). In particular, for the pair flow in question the associated Lagrangian scale obeys

$$\bar{x}^{\bullet} = -\left(\frac{1}{2\lambda_-} f_-^2 + \frac{1}{2\lambda_+} f_+^2\right)\circ\bar{x}, \quad \bar{x}(0,\cdot) = \mathrm{id}. \tag{10}$$

REMARK 4. The fundamental invariant

$$[(\mathcal{L}^t_H)']^2(\phi^t_{\mathbb{X}_H} m_0)\circ\mathcal{L}^t_H = m_0 \tag{11}$$

— or $(\bar{x}')^2 m\circ\bar{x} = m_0$ for short — in combination with the explicit form of the Green's function $\mathrm{G} = e^{-|\cdot|}/2$ through which $v = \mathrm{G} * m$ is connected to m show that the integration of *any* flow of the CH hierarchy boils down to the determination of the associated Lagrangian scale. Indeed, the formulas of Theorem 2 are obtained essentially following the explicit characterization of $\mathcal{L}^t_H$. The invariants (11) originate from intrinsic symmetries [Khesin and Misiołek 2003]. Indeed, CH (1) satisfies the least-action principle as it is a reexpression of geodesic flow on the group of smooth orientation preserving compressible diffeomorphisms on the line with respect to the right-invariant H^1-metric assimilated as the energy [Misiołek 1998]. On the other hand, Noether's theorem guarantees the existence of a first integral from each one-parameter subgroup that leaves the energy functional unchanged. By right-invariance, the elements of every orbit emanating from the identity constitute such a subgroup, and since these are plenty (one such for each initial direction in the tangent space at the identity alias the Lie algebra associated to the group); the corresponding infinite collection of associated invariants turns out to be embodied in a (one-parameter) identity (11). In other words, the CH equation (1) is nothing but a reexpression of the (time) invariance of the right-hand side of (11). More precisely, the Euler-Lagrange equation describing the critical points of the right-invariant H^1-energy functional reads

$$[(\mathscr{L}^t_{H_{\mathrm{CH}}})']^2 \cdot (1) \circ \mathscr{L}^t_{H_{\mathrm{CH}}} = (11)^{\bullet}_{H=H_{\mathrm{CH}}} = 0.$$

in which $\mathscr{L}^t_{H_{\mathrm{CH}}}$ is the "true" Lagrangian scale, that is,

$$(\mathscr{L}^t_{H_{\mathrm{CH}}})^{\bullet} = v \circ \mathscr{L}^t_{H_{\mathrm{CH}}} \quad \text{and} \quad \mathscr{L}^0_{H_{\mathrm{CH}}} = \mathrm{id}.$$

McKean [2003] (see also [Loubet 2006]) made the crucial observation that the first integral (11) remains in force for *all* other flows (i.e., flows originating from *any* Hamiltonian H) of the CH hierarchy provided that in each case a suitable "Lagrangian" scale is employed.

NOTATION. From now on, $m(t,x)$ will be short for the more cumbersome expression $(\phi^t_{\mathbb{X}_H} m_0)(x)$ describing the evaluation at x of the time t update of the solution curve starting at m_0, of the dynamical system (in phase space) defined by the Hamiltonian vector field $\mathbb{X}_H$, as explained in Section 2.3. Similarly, we will denote by

$$f_{\pm}(t,x) := (\phi^t_{\mathbb{X}_H *} f^0_{\pm})(x)$$

the evaluations at x of the (normalized time t) updates $(\phi^t_{\mathbb{X}_H *} f^0_{\pm})$ of the normalized eigenfunctions $f^0_{\pm}$ associated, respectively, to the spectral parameters $\lambda_{\pm}$, and so on. Moreover, whenever not confusing, we will occasionally omit the explicit dependence and write m and $f_{\pm}$ plain for brevity. In other words, all expressions with an upper/lower index "0" refer to (and hence involve purely) initial data whereas their counterparts where the label "0" is dropped account implicitly for their (time t) updates when acted upon by the (induced pair) flow. We emphasize that all initial and updated expressions are *functions* of an underlying independent spatial variable (denoted by $\cdot$ if left out), unspecified unless explicitly stated otherwise. In addition, in an effort to avoid unnecessary details which should be clear from the context, we omit writing explicitly the dummy variables and the differentials intervening in the integrands of integral expressions (for example, $\int_{-\infty}^{\cdot} e^y m_0$ is short for $\int_{-\infty}^{\cdot} e^y m_0(y)dy$ and so on.) Also, sometimes the same expressions will be used to denote both the functions and their evaluations at x. For example, depending on the context, when we write $\int^{\overline{x}}$ we might mean the function $\int^{\overline{x}(t,\cdot)}$, as in (13) below, or its evaluation at x, namely, $\int^{\overline{x}(t,x)}$, as in (19a). Finally, identities employing subscript $\pm$ are short for *two* such expressions, one with subscript $+$ and one with $-$.

3. The road to integration: lucky facts

3.1. Building up integrable expressions. As pointed out in Remark 4, the integration of the pair flows succumbs to the computation of their associated Lagrangian scale. This suggests that the key to integration is to play around with "sensible" objects involving the latter. The idea is to look for expressions incorporating the Lagrangian scale and functions of interest, whose evolution (under the pair flow) leads to integrable formulas from which to infer subsequently the integration of the items we actually care about. As in the case pertaining to individual flows [Loubet 2006] we start by analyzing whether

$$(\phi^t_{\mathbb{X}_H *}\varphi^0_\pm)\circ\mathscr{L}^t_H := \left(\phi^t_{\mathbb{X}_H *}\int_{-\infty}^{\cdot}\lambda_\pm m_0(f^0_\pm)^2\right)\circ\mathscr{L}^t_H$$

i.e., the composition of the time t update $\varphi_\pm$ of $\varphi^0_\pm$ with the diffeomorphism on the line given by the Lagrangian scale at t (i.e., $\mathscr{L}^t_H$) can be expressed in an alternative closed form. To this end, we compute the time derivative of $\varphi_\pm\circ\bar{x} = \int_{-\infty}^{\bar{x}}\lambda_\pm m f^2_\pm$ and explore to what extent the resulting equation is integrable. Direct computation using (7) and (9) yields

$$\begin{aligned}&[\varphi_\pm\circ\bar{x}]^\bullet = \lambda_\pm(mf^2_\pm)\circ\bar{x}\cdot\bar{x}^\bullet\\ &+\int_{-\infty}^{\bar{x}}\left[\lambda_\pm f^2_\pm\mathscr{I}\Big(\frac{1}{\lambda_-}f^2_- + \frac{1}{\lambda_+}f^2_+\Big) + 2\lambda_\pm m f_\pm\Big(\frac{f_\pm}{2\lambda_\pm}(\tfrac{1}{2}-\varphi_\pm) - \frac{\lambda_\pm f_\mp}{2\lambda_\mp}\varphi\Big)\right].\end{aligned}$$

As $f^2_\pm\mathscr{I}(f^2_\pm) = \big(mf^4_\pm\big)'$ and $f^2_\pm\mathscr{I}(f^2_\mp) = \big(mf^2_- f^2_+ + (\lambda_\pm-\lambda_\mp)\varphi^2\big)'$ it follows by (10) after cancellations and appropriate identifications that

$$[\varphi_\pm\circ\bar{x}]^\bullet = \left(\frac{1}{2\lambda_\pm}\varphi_\pm(1-\varphi_\pm) - \frac{\lambda_\pm}{2}\varphi^2\right)\circ\bar{x}. \tag{12a}$$

In sharp contrast with the analogue equation arising in the study of individual flows [Loubet 2006], equation (12a) *is not* an ODE, as the presence of the term $\lambda_\pm\varphi^2/2$ shows; see (8). It accounts for the mutual interaction of the underlying individual flows comprising the pair flow. Hence, as it stands, equation (12a) can only be useful if we manage somehow to determine, a priori, $\varphi\circ\bar{x}$. Now, analogous manipulations to the ones leading to (12a) show that

$$[\varphi\circ\bar{x}]^\bullet = \left(\left[\frac{1}{2\lambda_-}(\tfrac{1}{2}-\varphi_-) + \frac{1}{2\lambda_+}(\tfrac{1}{2}-\varphi_+)\right]\varphi\right)\circ\bar{x}. \tag{12b}$$

Neither of the *coupled* equations (12a) and (12b) pertaining to the evolution under the pair flow of the Lagrangian-scaled valued updates of $\varphi^0_\pm$, respectively

φ_0, are seen to be integrable when looked at individually, but if we combine these expressions suitably in a (symmetric) matrix as in

$$\Phi_0 := \begin{pmatrix} \frac{1}{\lambda_-}\varphi_-^0 & \varphi_0 \\ \varphi_0 & \frac{1}{\lambda_+}\varphi_+^0 \end{pmatrix} = \int_{-\infty}^{\cdot} m_0 \boldsymbol{f}_0 \otimes \boldsymbol{f}_0, \tag{12c}$$

where $\boldsymbol{f}_0 := (f_-^0, f_+^0)^\dagger$, we learn that (12a) and (12b) translate into

$$[\Phi \circ \bar{x}]^\bullet = \left(-\tfrac{1}{2}\Phi^2 + \tfrac{1}{4}[\Lambda^{-1}\Phi + \Phi\Lambda^{-1}]\right) \circ \bar{x}, \tag{12d}$$

where $\Phi := (\phi^t_{\mathbb{X}_H *}\Phi_0)$ and $\Lambda := \operatorname{diag}(\lambda_-, \lambda_+)$. This is the kind of luck that we were after: all we have to do now is solve an ODE! The last term on the right-hand side of (12d) suggests an ansatz of the form $\Phi \circ \bar{x} := e^{t/(4\Lambda)}\mathcal{S}(t, \cdot)e^{t/(4\Lambda)}$. By direct computation, the latter is seen to satisfy

$$[\Phi \circ \bar{x}]^\bullet = \frac{1}{4}[\Lambda^{-1}\Phi + \Phi\Lambda^{-1}] \circ \bar{x} + e^{t/(4\Lambda)} \overset{\bullet}{\mathcal{S}}(t, \cdot)e^{t/(4\Lambda)}.$$

We would be done if we could find $\mathcal{S}(t, x)$ such that $\mathcal{S}(0, x) = \Phi_0(x)$ and $e^{t/(4\Lambda)}\mathcal{S}^\bullet(t, \cdot)e^{t/(4\Lambda)} = -\frac{1}{2}[\Phi \circ \bar{x}]^2$. Now, the derivative of the inverse of a matrix is *quadratic*, i.e., $(\mathbb{O}^{-1})^\bullet = -\mathbb{O}^{-1}\mathbb{O}^\bullet\mathbb{O}^{-1}$, so that if $\mathcal{S}(t, x) := \mathcal{P}(x)\mathbb{O}^{-1}(t, x)$ then $e^{t/(4\Lambda)}\mathcal{S}^\bullet(t, \cdot)e^{t/(4\Lambda)} = -[\Phi \circ \bar{x}]e^{-t/(4\Lambda)}\mathbb{O}^\bullet\mathbb{O}^{-1}e^{t/(4\Lambda)}$, and thus, it would suffice to find $\mathbb{O}$ such that $e^{-t/(4\Lambda)}\mathbb{O}^\bullet\mathbb{O}^{-1}e^{t/(4\Lambda)} = \frac{1}{2}\Phi \circ \bar{x}$ or, what is the same thing, $\mathbb{O}^\bullet = \frac{1}{2}e^{t/(2\Lambda)}\mathcal{P} = (\Lambda e^{t/(2\Lambda)}\mathcal{P})^\bullet$. This implies that $\mathbb{O}(t, x) = \Lambda e^{t/(2\Lambda)}\mathcal{P}(x) + \mathcal{D}(x, \Lambda)$ for some 2×2 matrix $\mathcal{D}$. Finally, we observe that the initial constraint, $\Phi_0 = \mathcal{P}(\Lambda\mathcal{P} + \mathcal{D})^{-1}$, can be met by setting $\mathcal{D}(\cdot, \Lambda) := \mathrm{Id} - \Lambda\mathcal{P}$ and $\mathcal{P} := \Phi_0$. In short, the solution of (12d) is given by

$$\Phi \circ \bar{x} = e^{t/(4\Lambda)}\Phi_0[\mathrm{Id} + (e^{t/(2\Lambda)} - \mathrm{Id})\Lambda\Phi_0]^{-1}e^{t/(4\Lambda)}. \tag{13}$$

It will be helpful to introduce a shorthand and record the latter (or its evaluation at x: see Notation on page 272) as

$$[\Phi \circ \bar{x}](x) = \mathcal{T}(t)\Phi_0(x)[\mathcal{M}(t, x)]^{-1}\mathcal{T}(t),$$

where

$$\mathcal{M}(t, x) := \mathrm{Id} + \mathcal{E}(t, \Lambda)\Lambda\Phi_0(x) = \mathrm{Id} + \mathcal{C}(t, \Lambda)\Phi_0(x) \tag{14a}$$

in which

$$\mathcal{C}(t, \Lambda) := \mathcal{E}(t, \Lambda)\Lambda, \quad \mathcal{E}(t, \Lambda) := (\mathcal{T}^2(t) - \mathrm{Id}), \quad \mathcal{T}(t) := e^{t/(4\Lambda)}. \tag{14b}$$

For reasons that will be clear in a moment, we also need to investigate whether the evolution of other (Lagrangian-scaled valued updates of) integral expressions involving the eigenfunctions $f_\pm^0$, which are associated to the pair of eigenvalues $\lambda_\pm$ that define the flow and the so called "improper" eigenfunctions $e^{\pm \cdot/2}$ of the acoustic equation ($\lambda = 0$ is not in the spectrum of (3 with summable m_0)) admit alternative spellings. In other words, we play the same game as before with the exception that this time we look at the evolution of the truncated integrals

$$\begin{aligned}\mathscr{F}^{\downarrow}_{0,\pm} &:= \int_{-\infty}^{\cdot} \lambda_\pm m_0 f^0_\pm e^{y/2} = -e^{\cdot/2}((f^0_\pm)' - \tfrac{1}{2} f^0_\pm) = -[f^0_\pm, e^{\cdot/2}],\\ \mathscr{F}^{\uparrow}_{0,\pm} &:= \int_{\cdot}^{+\infty} \lambda_\pm m_0 f^0_\pm e^{-y/2} = e^{-\cdot/2}((f^0_\pm)' + \tfrac{1}{2} f^0_\pm) = [f^0_\pm, e^{-\cdot/2}],\end{aligned} \tag{15a}$$

where the bracket $[f, g]$ is short for the Wronskian $f'g - fg'$. These expressions can also be written in integrated form in terms of Wronskians. It develops after some work that $\mathscr{F}^{\downarrow}_\pm := (\phi^t_{\mathbb{X}_H *}\mathscr{F}^{\downarrow}_{0,\pm})$ and $\mathscr{F}^{\uparrow}_\pm := (\phi^t_{\mathbb{X}_H *}\mathscr{F}^{\uparrow}_{0,\pm})$ satisfy

$$\begin{aligned}[\mathscr{F}^{\downarrow}_\pm \circ \bar{x}]^\bullet &= \Big(\frac{1}{2\lambda_\pm}(\tfrac{1}{2} - \varphi_\pm)\mathscr{F}^{\downarrow}_\pm - \frac{\lambda_\pm}{2\lambda_\mp}\varphi \mathscr{F}^{\downarrow}_\mp\Big) \circ \bar{x},\\ [\mathscr{F}^{\uparrow}_\pm \circ \bar{x}]^\bullet &= \Big(\frac{1}{2\lambda_\pm}(\tfrac{1}{2} - \varphi_\pm)\mathscr{F}^{\uparrow}_\pm - \frac{\lambda_\pm}{2\lambda_\mp}\varphi \mathscr{F}^{\uparrow}_\mp\Big) \circ \bar{x}.\end{aligned}$$

To compute the solutions of these coupled systems of equations (and hence produce the desired tentative new spellings), we are led to pack the expressions (15a) into the vectors

$$\begin{aligned}\mathscr{F}^{\downarrow}_0 &:= \begin{pmatrix} \mathscr{F}^{\downarrow}_{0,-}/\lambda_- \\ \mathscr{F}^{\downarrow}_{0,+}/\lambda_+ \end{pmatrix} = -\Lambda^{-1}\begin{pmatrix} [f^0_-, e^{\cdot/2}] \\ [f^0_+, e^{\cdot/2}] \end{pmatrix} = -e^{\cdot/2}\Lambda^{-1}\Upsilon_{-,0},\\ \mathscr{F}^{\uparrow}_0 &:= \begin{pmatrix} \mathscr{F}^{\uparrow}_{0,-}/\lambda_- \\ \mathscr{F}^{\uparrow}_{0,+}/\lambda_+ \end{pmatrix} = \Lambda^{-1}\begin{pmatrix} [f^0_-, e^{-\cdot/2}] \\ [f^0_+, e^{-\cdot/2}] \end{pmatrix} = e^{-\cdot/2}\Lambda^{-1}\Upsilon_{+,0},\end{aligned} \tag{15b}$$

where $\Upsilon_{\pm,0} := \pm(\frac{1}{2} \pm D)\boldsymbol{f}_0$, i.e.,

$$\Upsilon_{-,0} := \begin{pmatrix} (f^0_-)' - \frac{1}{2} f^0_- \\ (f^0_+)' - \frac{1}{2} f^0_+ \end{pmatrix} \quad \text{and} \quad \Upsilon_{+,0} := \begin{pmatrix} (f^0_-)' + \frac{1}{2} f^0_- \\ (f^0_+)' + \frac{1}{2} f^0_+ \end{pmatrix}, \tag{15c}$$

so that $\mp(\frac{1}{2} \pm D)\Upsilon_{\mp,0} = \Lambda m_0 \boldsymbol{f}_0$. Indeed, the evolution of their respective components (as displayed lines above) is gathered nicely in the system of *uncoupled* ODE's

$$\begin{aligned}[\mathscr{F}^{\downarrow} \circ \bar{x}]^\bullet &= \{\tfrac{1}{2}(\tfrac{1}{2}\Lambda^{-1} - \Phi)\mathscr{F}^{\downarrow}\} \circ \bar{x},\\ [\mathscr{F}^{\uparrow} \circ \bar{x}]^\bullet &= \{\tfrac{1}{2}(\tfrac{1}{2}\Lambda^{-1} - \Phi)\mathscr{F}^{\uparrow}\} \circ \bar{x},\end{aligned}$$

$\Phi \circ \bar{x}$ being already known from (13). Since $\mathscr{F}^{\downarrow} := (\phi^t_{\mathbb{X}_H *}\mathscr{F}^{\downarrow}_0)$ and $\mathscr{F}^{\uparrow} := (\phi^t_{\mathbb{X}_H *}\mathscr{F}^{\uparrow}_0)$ differ only in their initial values ($\mathscr{F}^{\downarrow}_0$ and $\mathscr{F}^{\uparrow}_0$), we only need to deal further with either of them, say $\mathscr{F}^{\downarrow}$. Direct computation shows that the educated guesses

$$\begin{aligned}\mathcal{F}^{\downarrow}\circ\bar{x} &= e^{t/(4\Lambda)}[(\mathcal{M}(t,\cdot))^{\dagger}]^{-1}\mathcal{F}_0^{\downarrow},\\ \mathcal{F}^{\uparrow}\circ\bar{x} &= e^{t/(4\Lambda)}[(\mathcal{M}(t,\cdot))^{\dagger}]^{-1}\mathcal{F}_0^{\uparrow},\end{aligned}\tag{16}$$

with $\mathcal{M}$ as in (14a), provide the answer. Indeed, the time derivative of the right-hand side of the second line yields

$$\mathcal{T}^{\bullet}(\mathcal{M}^{\dagger})^{-1}\mathcal{F}_0^{\downarrow}-\mathcal{T}(\mathcal{M}^{\dagger})^{-1}(\mathcal{M}^{\dagger})^{\bullet}(\mathcal{M}^{\dagger})^{-1}\mathcal{F}_0^{\downarrow}.$$

As $\mathcal{T}^{\bullet}=\frac{1}{4}\Lambda^{-1}\mathcal{T}$ and $(\mathcal{M}^{\dagger})^{\bullet}=2\Phi_0\Lambda\mathcal{T}\mathcal{T}^{\bullet}=\frac{1}{2}\Phi_0\mathcal{T}^2$ — see (14a) and (14b) — the latter reduces to

$$\big(\tfrac{1}{4}\Lambda^{-1}-\tfrac{1}{2}\mathcal{T}(\mathcal{M}^{\dagger})^{-1}\Phi_0\mathcal{T}\big)[\mathcal{F}^{\downarrow}\circ\bar{x}].$$

The verification is completed by appealing to the identity

$$(\mathcal{M}^{\dagger})^{-1}\Phi_0=\Phi_0\mathcal{M}^{-1}\tag{17}$$

and the preliminary integration result (13) so that

$$\mathcal{T}(\mathcal{M}^{\dagger})^{-1}\Phi_0\mathcal{T}=\mathcal{T}\Phi_0\mathcal{M}^{-1}\mathcal{T}\equiv\Phi\circ\bar{x}.$$

4. Determination of the Lagrangian scale

The trick to get an explicit formula for the Lagrangian scale $\bar{x}=\bar{x}(t,x)$ of section 2.3 in terms of time t, the original spatial scale x, and initial data, is to "peal off," in an orderly fashion, the integrated expressions of section 3. More precisely, we start by differentiating with respect to the underlying variable x both sides of the identity (see (13), (12c), (8), and the notation clarifications on page 272)

$$\int_{-\infty}^{\bar{x}} m\boldsymbol{f}\otimes\boldsymbol{f}=[\Phi\circ\bar{x}](x)=e^{t/(4\Lambda)}\Phi_0(x)[\mathcal{M}(t,x)]^{-1}e^{t/(4\Lambda)}$$

where $\boldsymbol{f}:=(f_-,f_+)^{\dagger}=(\phi^t_{\mathbb{X}_H *}\boldsymbol{f}_0)$ is the update of $\boldsymbol{f}_0:=(f_-^0,f_+^0)^{\dagger}$. Writing $\mathcal{M}\equiv\mathrm{Id}+\mathcal{C}\Phi_0$ for simplicity as in (14a), we get

$$(m\boldsymbol{f}\otimes\boldsymbol{f})\circ\bar{x}\cdot\bar{x}'=e^{t/(4\Lambda)}\big(\Phi_0\mathcal{M}^{-1}\big)'e^{t/(4\Lambda)}.$$

From (17) and the equality $\mathcal{M}^{\dagger}=\mathrm{Id}+\Phi_0\mathcal{C}$ ($\mathcal{C}$ and Φ_0 being symmetric) we see that $(\Phi_0\mathcal{M}^{-1})'$ is equal to

$$(\mathrm{Id}-\Phi_0\mathcal{M}^{-1}\mathcal{C})\Phi_0'\mathcal{M}^{-1}=(\mathrm{Id}-(\mathcal{M}^{\dagger})^{-1}\Phi_0\mathcal{C})\Phi_0'\mathcal{M}^{-1}=(\mathcal{M}^{\dagger})^{-1}\Phi_0'\mathcal{M}^{-1}.$$

As $\Phi_0'=m_0\boldsymbol{f}_0\otimes\boldsymbol{f}_0$, the latter in combination with the fundamental invariant (11), reduce (by associativity and linearity) the previous to last display to $m_0/\bar{x}'$ times

$$(\boldsymbol{f}\otimes\boldsymbol{f})\circ\bar{x}=\bar{x}'(e^{t/(4\Lambda)}(\mathcal{M}^{\dagger})^{-1}\boldsymbol{f}_0)\otimes(e^{t/(4\Lambda)}(\mathcal{M}^{\dagger})^{-1}\boldsymbol{f}_0).$$

As m_0 is not identically zero and since $\bar{x}' > 0$ (at least for small times; see (10)), we use linearity once more to recognize that

$$\boldsymbol{f} \circ \bar{x} = \sqrt{\bar{x}'} e^{t/(4\Lambda)} [(\mathcal{M}(t,\cdot))^\dagger]^{-1} \boldsymbol{f}_0. \tag{18}$$

It is not surprising that the determination of the Lagrangian-scaled update of $\boldsymbol{f}_0$ given by the left-hand side of (18) would follow once we have an explicit formula for $\bar{x}$. In fact (18) should be interpreted the other way around, namely, as a step towards the determination of $\bar{x}$: by equation (10), the trace identity

$$\mathrm{Tr}\big(\tfrac{1}{2}\Lambda^{-1}(\boldsymbol{f} \otimes \boldsymbol{f}) \circ \bar{x}\big) = \Big(\frac{1}{2\lambda_-} f_-^2 + \frac{1}{2\lambda_+} f_+^2\Big) \circ \bar{x} \equiv -\bar{x}^{\bullet},$$

in combination with the *partial* result (18) leads to

$$\bar{x}^{\bullet} + \Gamma(t,x)\bar{x}' = 0,$$

in which $\Gamma(t,x)$ is short for the trace of

$$\tfrac{1}{2}\Lambda^{-1}(e^{t/(4\Lambda)}[(\mathcal{M}(t,x))^\dagger]^{-1}\boldsymbol{f}_0(x)) \otimes (e^{t/(4\Lambda)}[(\mathcal{M}(t,x))^\dagger]^{-1}\boldsymbol{f}_0(x)).$$

This is a first order linear evolution equation from which, in principle, the Lagrangian scale can be computed, $\bar{x}(0,x) = x$ being known. But to find an explicit expression for the solution of this seemingly trivial equation (say, by the method of characteristics) is not an easy matter. We actually take a different route that bumps into yet another piece of grace. The "problem" with (18) is that we do not have sufficient information about the shape of $\boldsymbol{f} \circ \bar{x}$ to infer that of $\bar{x}$. To fix this, we apply the previous method to identities that incorporate the improper eigenfunctions $e^{\pm\cdot/2}$ ($\lambda = 0$ is not in spec(m_0) of (3) for summable m_0) whose shape is explicit and, more importantly, *fixed* for all times, improper eigenfunctions being insensitive to the potential. As a matter of fact, the computation of the scale $\bar{x}$ is not particularly sensitive to the actual shape of improper eigenfunctions but rather, and this is the key, to the fact that they are inverses of one another as we now show. Focus on the (evaluations at x) of the Lagrangian-scaled updates of (15a) for which we have found explicit alternative expressions (16), namely,

$$\int_{-\infty}^{\bar{x}} e^{y/2} m \boldsymbol{f} = [\mathscr{F}^{\downarrow} \circ \bar{x}](x) = e^{t/(4\Lambda)}[(\mathcal{M}(t,x))^\dagger]^{-1}\mathscr{F}_0^{\downarrow}(x), \tag{19a}$$

$$\int_{\bar{x}}^{+\infty} e^{-y/2} m \boldsymbol{f} = [\mathscr{F}^{\uparrow} \circ \bar{x}](x) = e^{t/(4\Lambda)}[(\mathcal{M}(t,x))^\dagger]^{-1}\mathscr{F}_0^{\uparrow}(x). \tag{19b}$$

Now, keeping in mind identity (11) i.e., $(\bar{x})'^2 m \circ \bar{x} = m_0$ for short, definitions (14a) and (12c), the fact that $\boldsymbol{f}_0 \otimes \boldsymbol{f}_0 = \boldsymbol{f}_0 \boldsymbol{f}_0{}^\dagger$ (for column vectors $\boldsymbol{f}_0$) and the partial result (18), the x-derivative of both ends of (19a) yields (omitting here and there the explicit x-dependence)

$$\frac{m_0}{\bar{x}'}[\boldsymbol{f}\circ\bar{x}]e^{\bar{x}/2}=\frac{m_0}{\bar{x}'}[\boldsymbol{f}\circ\bar{x}]\sqrt{\bar{x}'}e^{x/2}\big(1-e^{-x/2}(\mathscr{C}\boldsymbol{f}_0)^\dagger((\mathcal{M}^\dagger)^{-1}\mathscr{F}_0^\downarrow)\big).$$

A similar expression follows from differentiating (19a) instead (the next display embodies both of them). Hence, dropping the common factor $m_0[\boldsymbol{f}\circ\bar{x}]/\bar{x}'$, which is assumed to be different from zero (as is the case at least for small times), by linearity and appealing to the characterizations (15b) and (15c) involving the initial data, we have

$$e^{\pm\bar{x}/2}=\sqrt{\bar{x}'}e^{\pm x/2}\big(1+(\mathscr{C}\boldsymbol{f}_0)^\dagger((\Lambda\mathcal{M}^\dagger)^{-1}\Upsilon_{\mp,0})\big).$$

It is clear that the desired expressions for $\bar{x}'$ and in fact $\bar{x}$ can be obtained simply by *multiplying*, respectively, *dividing* the above identities. But before we actually do that, we wish to find more palatable expressions for the intervening factors. The coefficients on the right-hand side of the last display are of the form $1+\boldsymbol{a}^\dagger\boldsymbol{b}$ (for column vectors $\boldsymbol{a}$ and $\boldsymbol{b}$), so we can invoke the identity

$$1+\boldsymbol{a}^\dagger\boldsymbol{b}\equiv 1+\boldsymbol{a}\cdot\boldsymbol{b}=\det[\mathrm{Id}+\boldsymbol{a}\otimes\boldsymbol{b}]$$

to express them in terms of *determinants*. To wit,

$$1+(\mathscr{C}\boldsymbol{f}_0)^\dagger((\Lambda\mathcal{M}^\dagger)^{-1}\Upsilon_{\mp,0})=\det[\mathrm{Id}+\mathscr{E}(\Phi_0\Lambda+\boldsymbol{f}_0\otimes\Upsilon_{\mp,0})]/\det\mathcal{M},\quad(20)$$

where in the last step we used the formula for the determinant of a product of matrices and the fact that $\mathrm{Id}+\boldsymbol{a}\otimes(\Lambda\mathcal{M}^\dagger)^{-1}\boldsymbol{b}$ equals

$$\mathrm{Id}+\boldsymbol{a}[(\Lambda\mathcal{M}^\dagger)^{-1}\boldsymbol{b}]^\dagger=\mathrm{Id}+\boldsymbol{a}\otimes\boldsymbol{b}(\mathcal{M}\Lambda)^{-1}=[\mathcal{M}\Lambda+\boldsymbol{a}\otimes\boldsymbol{b}]\Lambda^{-1}\mathcal{M}^{-1},$$

so that by (14a)

$$\mathrm{Id}+(\mathscr{C}\boldsymbol{f}_0)\otimes((\Lambda\mathcal{M}^\dagger)^{-1}\Upsilon_{\mp,0})=\Lambda(\mathrm{Id}+\mathscr{E}(\Phi_0\Lambda+\boldsymbol{f}_0\otimes\Upsilon_{\mp,0}))\Lambda^{-1}\mathcal{M}^{-1}.$$

Moreover, recalling (12c) we have by linearity, the acoustic equation (3), and definitions (15c) that $\Phi_0\Lambda$ is equal to

$$\int_{-\infty}^{\cdot}\boldsymbol{f}_0\otimes(m_0\Lambda\boldsymbol{f}_0)=\int_{-\infty}^{\cdot}\boldsymbol{f}_0\otimes(1/4-D^2)\boldsymbol{f}_0=\mp\int_{-\infty}^{\cdot}\boldsymbol{f}_0\otimes(\tfrac{1}{2}\pm D)\Upsilon_{\mp,0}.$$

Hence, upon integrating by parts, we see that

$$\Phi_0\Lambda+\boldsymbol{f}_0\otimes\Upsilon_{\mp,0}=\int_{-\infty}^{\cdot}\Upsilon_{\mp,0}\otimes\Upsilon_{\mp,0}.$$

Now, substituting this expression into (20) and recalling the definitions of the (evaluations at x of the) theta determinants (5) (bear in mind that all expressions following (19a) are assumed to be implicitly evaluated at x), we learn that

$$e^{\pm\bar{x}/2} = \sqrt{\bar{x}'}\, e^{\pm x/2}\frac{\Theta_\mp}{\Theta}.$$

Finally, as advertised, looking at the ratio and the product of the latter identities we get

$$e^{\bar{x}} = e^{x}\frac{\Theta_-}{\Theta_+}, \quad \text{respectively} \quad \bar{x}' = \frac{\Theta^2}{\Theta_-\Theta_+}. \tag{21}$$

REMARK 5. Actually, these identities are equivalent. Indeed, $\bar{x}(t,\pm\infty) = \pm\infty$ since $\Theta_\pm(t,-\infty) = 1$ and $\Theta_\pm(t,+\infty) = e^{t(\lambda_-^{-1}+\lambda_+^{-1})/2}$ by inspection of (5) and the normalizations of $f_\pm^0$; see (6). Also, note that $\Theta_\pm$ vanish nowhere, being the determinants of matrices

$$\mathcal{M}_\pm(t,x) := \mathrm{Id} + \mathscr{E}(t,\Lambda)\int_{-\infty}^{x} \Upsilon_{\pm,0}\otimes\Upsilon_{\pm,0} \tag{22}$$

(see (14b) and (15c)) with positive definite associated quadratic forms.

REMARK 6. Differentiating with respect to x the first expression in (21) and using the substitution afforded by the second we get the curious quadratic identity

$$\Theta^2 = \Theta_-\Theta_+ + \Theta_-'\Theta_+ - \Theta_-\Theta_+'$$

relating the theta determinants; compare with [McKean 2003; Loubet 2006].

REMARK 7. The integration of the pair flow is now more or less completed. Indeed, substituting the second identity of (21) into identity (11) yields

$$(\phi^t_{\mathbb{X}_H} m_0)\circ\mathscr{L}^t_H = \left(\frac{\Theta_-\Theta_+}{\Theta^2}\right)^2 m_0.$$

Hence, all that remains is to compute $(\phi^t_{\mathbb{X}_H} v_0)\circ\mathscr{L}^t_H = (\mathrm{G} * (\phi^t_{\mathbb{X}_H} m_0))\circ\mathscr{L}^t_H$ as is done in Appendix B of [Loubet 2006]. Nonetheless, in the next section we will present a more direct route to the formulas in Theorem 2.

REMARK 8. The logarithmic (time) derivative of the first identity in (21) shows that $\bar{x}^\bullet = (\log\det(\mathcal{M}_-\mathcal{M}_+^{-1}))^\bullet$. Hence, upon invoking the identity $(\log\det\mathcal{Q})^\bullet = \mathrm{Tr}(\mathcal{Q}^\bullet\mathcal{Q}^{-1})$ valid for any differentiable square matrix $\mathcal{Q}$, we obtain an alternative description of the right-hand side of (10). To wit,

$$\mathrm{Tr}\big(\tfrac{1}{2}\Lambda^{-1}(\boldsymbol{f}\otimes\boldsymbol{f})\circ\bar{x}\big) = \left(\frac{1}{2\lambda_-}f_-^2 + \frac{1}{2\lambda_+}f_+^2\right)\circ\bar{x} = \mathrm{Tr}(\mathcal{M}_+^\bullet\mathcal{M}_+^{-1} - \mathcal{M}_-^\bullet\mathcal{M}_-^{-1}).$$

We invite the reader to check this, starting directly from (22) with the help of (21) and the identity preceding (18). More significantly, upon substituting (21) into (18), we obtain an exact formula for the Lagrangian-valued time t update of $\boldsymbol{f}_0$. To wit,

$$\boldsymbol{f}\circ\bar{x}=\frac{\Theta}{\sqrt{\Theta_-\Theta_+}}e^{t/(4\Lambda)}[(\mathcal{M}(t,\cdot))^{\dagger}]^{-1}\boldsymbol{f}_0, \tag{23a}$$

or, componentwise,

$$f_{\pm}\circ\bar{x}=\frac{1}{\sqrt{\Theta_-\Theta_+}}e^{t/(4\lambda_{\pm})}\big(f^0_{\pm}+(e^{t/(2\lambda_{\mp})}-1)[f^0_{\pm}\varphi^0_{\mp}-\lambda_{\mp}f^0_{\mp}\varphi_0]\big). \tag{23b}$$

Similarly, one can compute the Lagrangian-valued (time t) updates of eigenfunctions of (3) other than $f_{\pm}$ but that is not our purpose here.

5. Integration of the pair flow

In this section we finally show how to exploit the integrated expressions of section 3 and section 4 — notably the characterization of the Lagrangian scale (21) — to obtain the explicit formulas of Theorem 2. The trick is to stick to the successful algorithm that was used systematically in the bulk of the previous sections and apply it to

$$\mathcal{V}^{\downarrow}_0:=\int_{-\infty}^{\cdot}e^{y}m_0=e^{\cdot}(v_0-v_0'),\quad \mathcal{V}^{\uparrow}_0:=\int_{\cdot}^{+\infty}e^{-y}m_0=e^{-\cdot}(v_0+v_0').$$

It develops (using (15a) and (16)) that

$$[\mathcal{V}^{\downarrow}\circ\bar{x}]^{\bullet}=-\tfrac{1}{2}[(\mathcal{F}^{\downarrow}_-/\lambda_-)^2+(\mathcal{F}^{\downarrow}_+/\lambda_+)^2]\circ\bar{x}\equiv-\tfrac{1}{2}|\mathcal{F}^{\downarrow}\circ\bar{x}|^2,$$
$$[\mathcal{V}^{\uparrow}\circ\bar{x}]^{\bullet}=\tfrac{1}{2}[(\mathcal{F}^{\uparrow}_-/\lambda_-)^2+(\mathcal{F}^{\uparrow}_+/\lambda_+)^2]\circ\bar{x}\equiv\tfrac{1}{2}|\mathcal{F}^{\uparrow}\circ\bar{x}|^2,$$

where $\mathcal{V}^{\downarrow}:=(\phi^t_{\mathbb{X}_H *}\mathcal{V}^{\downarrow}_0)$, and $\mathcal{V}^{\uparrow}:=(\phi^t_{\mathbb{X}_H *}\mathcal{V}^{\uparrow}_0)$. Closer inspection of the squares of the Euclidean norms in the right-hand side of the equations above via the preliminary integrations (i.e., alternative spellings) of $\mathcal{F}^{\downarrow}\circ\bar{x}$ and $\mathcal{F}^{\uparrow}\circ\bar{x}$ reveals the last piece of the puzzle. Here is how. By identities (16), keeping in mind the definitions (14a), and omitting writing explicitly the dependence on independent variables, we have

$$|\mathcal{F}^{\downarrow}\circ\bar{x}|^2=(\mathcal{T}(\mathcal{M}^{\dagger})^{-1}\mathcal{F}^{\downarrow}_0)^{\dagger}(\mathcal{T}(\mathcal{M}^{\dagger})^{-1}\mathcal{F}^{\downarrow}_0)=(\mathcal{F}^{\downarrow}_0)^{\dagger}\mathcal{M}^{-1}\mathcal{T}^2(\mathcal{M}^{\dagger})^{-1}\mathcal{F}^{\downarrow}_0$$
$$|\mathcal{F}^{\uparrow}\circ\bar{x}|^2=(\mathcal{T}(\mathcal{M}^{\dagger})^{-1}\mathcal{F}^{\uparrow}_0)^{\dagger}(\mathcal{T}(\mathcal{M}^{\dagger})^{-1}\mathcal{F}^{\uparrow}_0)=(\mathcal{F}^{\uparrow}_0)^{\dagger}\mathcal{M}^{-1}\mathcal{T}^2(\mathcal{M}^{\dagger})^{-1}\mathcal{F}^{\uparrow}_0.$$

Now, since $\frac{1}{2}\mathcal{T}^2\Phi_0=\mathcal{M}^{\bullet}$ by (14a), and because $(\mathcal{M}^{-1})^{\bullet}=-\mathcal{M}^{-1}\mathcal{M}^{\bullet}\mathcal{M}^{-1}$,

$$-\tfrac{1}{2}\mathcal{M}^{-1}\mathcal{T}^2(\mathcal{M}^{\dagger})^{-1}=-\mathcal{M}^{-1}\mathcal{M}^{\bullet}\Phi_0^{-1}(\mathcal{M}^{\dagger})^{-1}=(\mathcal{M}^{-1})^{\bullet}\mathcal{M}\Phi_0^{-1}(\mathcal{M}^{\dagger})^{-1}.$$

Moreover, by identity (17)

$$\mathcal{M}\Phi_0^{-1}(\mathcal{M}^{\dagger})^{-1}=\mathcal{M}(\mathcal{M}^{\dagger}\Phi_0)^{-1}=\mathcal{M}(\Phi_0\mathcal{M})^{-1}=\Phi_0^{-1}.$$

Altogether, from the last two displays and the fact that the functions Φ_0, $\mathscr{F}_0^{\downarrow}$ and $\mathscr{F}_0^{\uparrow}$ are independent of t, we learn that $-\frac{1}{2}|\mathscr{F}^{\downarrow}\circ\bar{x}|^2$ and $\frac{1}{2}|\mathscr{F}^{\uparrow}\circ\bar{x}|^2$ can be written as *time derivatives* i.e.,

$$[\mathscr{V}^{\downarrow}\circ\bar{x}]^{\bullet} = [(\mathscr{F}_0^{\downarrow})^{\dagger}(\Phi_0\mathscr{M})^{-1}\mathscr{F}_0^{\downarrow}]^{\bullet},$$
$$[\mathscr{V}^{\uparrow}\circ\bar{x}]^{\bullet} = [-(\mathscr{F}_0^{\uparrow})^{\dagger}(\Phi_0\mathscr{M})^{-1}\mathscr{F}_0^{\uparrow}]^{\bullet}.$$

Integrating the latter with respect to the time variable from 0 to t yields

$$e^{\bar{x}}(v-v')\circ\bar{x} = \mathscr{V}^{\downarrow}\circ\bar{x} = e^{\cdot}(v_0-v_0') + (\mathscr{F}_0^{\downarrow})^{\dagger}\{(\Phi_0\mathscr{M})^{-1}-\Phi_0^{-1}\}\mathscr{F}_0^{\downarrow},$$
$$e^{-\bar{x}}(v+v')\circ\bar{x} = \mathscr{V}^{\uparrow}\circ\bar{x} = e^{-\cdot}(v_0+v_0') - (\mathscr{F}_0^{\uparrow})^{\dagger}\{(\Phi_0\mathscr{M})^{-1}-\Phi_0^{-1}\}\mathscr{F}_0^{\uparrow},$$

where, as before, the underlying spatial variable (denoted by $\cdot$) is left unspecified. By associativity, definitions (14a) and linearity, it is immediate to see that

$$(\Phi_0\mathscr{M})^{-1}-\Phi_0^{-1} = \mathscr{M}^{-1}(\mathrm{Id}-\mathscr{M})\Phi_0^{-1} = \mathscr{M}^{-1}(-\mathscr{C}\Phi_0)\Phi_0^{-1} = -\mathscr{M}^{-1}\mathscr{C}.$$

Now we use in the last-but-one pair of identities the connection between the scales (21) (with the understanding that in the latter the scale x is now to be left unspecified i.e., $e^{\bar{x}} = e^{\cdot}\,\Theta_-/\Theta_+$). Together with linearity and the identifications $e^{-\cdot/2}\mathscr{F}_0^{\downarrow} = -\Lambda^{-1}\Upsilon_{-,0}$ and $e^{\cdot/2}\mathscr{F}_0^{\uparrow} = \Lambda^{-1}\Upsilon_{+,0}$ of (15b), in combination with the preceding identity and the equality $\mathscr{C}\Lambda^{-1} = \mathscr{E}$, from (14b), this yields

$$(v-v')\circ\bar{x} = \frac{\Theta_+}{\Theta_-}\big(v_0-v_0'-\Upsilon_{-,0}^{\dagger}(\mathscr{M}\Lambda)^{-1}\mathscr{E}\Upsilon_{-,0}\big),$$
$$(v+v')\circ\bar{x} = \frac{\Theta_-}{\Theta_+}\big(v_0+v_0'+\Upsilon_{+,0}^{\dagger}(\mathscr{M}\Lambda)^{-1}\mathscr{E}\Upsilon_{+,0}\big).$$

Finally, taking half of the sum, respectively, of the difference of these identities reproduce the punch line formulas of Theorem 2, and we are done.

6. Large-time asymptotics and limiting behavior

If $\lambda_- < 0 < \lambda_+$, the Lagrangian-scaled updates of eigenfunctions associated to the eigenvalues $\lambda_\pm$ (cf. (23a) and (23b)) vanish as $t\to\pm\infty$. This prompts that $\lambda_\pm$ are *excised* from the spectrum associated with the *residual* profiles. Indeed, $e^{t/(2\lambda_\mp)} = o(1)$ as $t\to\pm\infty$, and thus by (22) and (15c),

$$\Theta_\pm := \det\mathscr{M}_\pm(t,\cdot) = \begin{cases} e^{t/(2\lambda_+)}(\theta_\pm^{+\infty}+o(1)) & \text{as} \quad t\uparrow+\infty,\\ e^{t/(2\lambda_-)}(\theta_\pm^{-\infty}+o(1)) & \text{as} \quad t\downarrow-\infty,\end{cases}$$

where

$$\begin{matrix}\theta_\pm^{+\infty}\\ \theta_\pm^{-\infty}\end{matrix} := \Phi_\pm - \begin{cases}[a_\pm(\varphi_-^0-1)+b_\pm\varphi_+^0]\\ [a_\pm\varphi_-^0+b_\pm(\varphi_+^0-1)]\end{cases} \tag{24}$$

are the corresponding limiting/stationary theta counterparts, $\varphi^0_\pm$ being defined in (8), and

$$\begin{aligned}\Phi_\pm &:= \lambda_+ f^0_+ \varphi^0_+ ((f^0_-)' \pm \tfrac{1}{2} f^0_-) + \lambda_- f^0_- \varphi^0_- ((f^0_+)' \pm \tfrac{1}{2} f^0_+) + (1+\lambda_-\lambda_+)\varphi^0_-\varphi^0_+ ,\\ a_\pm &:= \varphi^0_+ + f^0_+((f^0_+)' \pm \tfrac{1}{2} f^0_+) = \int_{-\infty}^{\cdot} ((f^0_+)' \pm \tfrac{1}{2} f^0_+)^2 ,\\ b_\pm &:= \varphi^0_- + f^0_-((f^0_-)' \pm \tfrac{1}{2} f^0_-) = \int_{-\infty}^{\cdot} ((f^0_-)' \pm \tfrac{1}{2} f^0_-)^2 .\end{aligned}$$

Hence, by (23b) we have

$$\lim_{t\to\pm\infty} [(\phi^t_{\mathbb{X}_H *} f^0_+) \circ \mathscr{L}^t_H] = 0, \quad \text{and} \quad \lim_{t\to\pm\infty} [(\phi^t_{\mathbb{X}_H *} f^0_-) \circ \mathscr{L}^t_H] = 0.$$

On the other hand, using (14a) and (12c) we see that

$$\Theta := \det \mathcal{M}(t,\cdot) = e^{t/(2\lambda_\pm)}(\theta^{\pm\infty} + o(1)) \quad \text{as } t \to \pm\infty, \tag{25}$$

where

$$\theta^{\pm\infty} := \varphi^0_\pm (1 - \varphi^0_\mp) + \lambda_-\lambda_+ \varphi^2_0 . \tag{26}$$

In other words, either limits (in the remote past or future) give rise to stationary Lagrangian scales (cf. (21))

$$(\mathscr{L}^{\pm\infty}_H)' := \lim_{t\to\pm\infty} \frac{\Theta^2}{\Theta_-\Theta_+} = \frac{(\theta^{\pm\infty})^2}{\theta^{\pm\infty}_-\theta^{\pm\infty}_+},$$

or, what is the same,

$$e^{\mathscr{L}^{\pm\infty}_H} = e^{\cdot}\, \frac{\theta^{\pm\infty}_-}{\theta^{\pm\infty}_+}$$

and residual potentials

$$\lim_{t\to\pm\infty} [(\phi^t_{\mathbb{X}_H} m_0) \circ \mathscr{L}^t_H] = (\phi^{\pm\infty}_{\mathbb{X}_H} m_0) \circ \mathscr{L}^{\pm\infty}_H = \left(\frac{\theta^{\pm\infty}_-\theta^{\pm\infty}_+}{(\theta^{\pm\infty})^2} \right)^2 m_0 .$$

NOTE. It is amusing to check directly from the definitions (24), (26), and (8) that (dropping the upper indexes $\pm\infty$)

$$\theta^2 = \theta_-\theta_+ + \theta'_-\theta_+ - \theta_-\theta'_+ .$$

i.e., that the algebraic structure of the identity in Remark 6 relating the theta-determinants remains valid (in the limits $t \to \pm\infty$) for either of their stationary analogues.

Either of the residual profiles can be computed from the corresponding residual potentials via the Green's function as in (cf. Remark 7),

$$\lim_{t\to\pm\infty}[(\phi^t_{\mathbb{X}_H} v_0)\circ \mathcal{L}^t_H] = (\phi^{\pm\infty}_{\mathbb{X}_H} v_0)\circ \mathcal{L}^{\pm\infty}_H = (\mathrm{G} * \phi^{\pm\infty}_{\mathbb{X}_H} m_0)\circ \mathcal{L}^{\pm\infty}_H\,,$$

but it is more efficient to infer them directly by taking (respectively) the limits as $t \to \pm\infty$ of the formulas of Theorem 2. Indeed, for $\lambda_- < 0 < \lambda_+$, we have[1]

$$\lim_{t\to\pm\infty}\frac{\Theta_\pm}{\Theta_\mp} = \frac{\theta^{\pm\infty}_\pm}{\theta^{\pm\infty}_\mp}$$

where $\theta^{\pm\infty}_\pm$ are given by (24) (see also (8)). On the other hand, we can check that, in the notation of (14a), (12c) and (8),

$$\Psi_\pm(\cdot,\Lambda) := \lim_{t\to\pm\infty}[(\mathcal{M}(t,\cdot)\Lambda)^{-1}\mathcal{E}(t,\Lambda)] = \begin{cases} \dfrac{1}{\theta^{+\infty}}\begin{pmatrix} -\varphi^0_+/\lambda_- & \varphi_0 \\ \varphi_0 & (1-\varphi^0_-)/\lambda_+ \end{pmatrix} \\[2ex] \dfrac{1}{\theta^{-\infty}}\begin{pmatrix} (1-\varphi^0_+)/\lambda_- & \varphi_0 \\ \varphi_0 & -\varphi^0_-/\lambda_+ \end{pmatrix} \end{cases}$$

where $\theta^{\pm\infty}$ are given by (26) (see also (8)). Altogether, it follows directly from the formulas of Theorem 2 that

$$\begin{aligned} \lim_{t\to\pm\infty}[(\phi^t_{\mathbb{X}_H} v_0)\circ \mathcal{L}^t_H] &= \frac{\theta^{\pm\infty}_-}{\theta^{\pm\infty}_+}\left(\frac{v_0+v_0'}{2} + \tfrac{1}{2}\Upsilon^\dagger_{+,0}\Psi_\pm\Upsilon_{+,0}\right) \\ &\quad + \frac{\theta^{\pm\infty}_+}{\theta^{\pm\infty}_-}\left(\frac{v_0-v_0'}{2} - \tfrac{1}{2}\Upsilon^\dagger_{-,0}\Psi_\pm\Upsilon_{-,0}\right), \\ \lim_{t\to\pm\infty}\left[\frac{\partial(\phi^t_{\mathbb{X}_H} v_0)\circ \mathcal{L}^t_H}{\partial \mathcal{L}^t_H}\right] &= \frac{\theta^{\pm\infty}_-}{\theta^{\pm\infty}_+}\left(\frac{v_0+v_0'}{2} + \tfrac{1}{2}\Upsilon^\dagger_{+,0}\Psi_\pm\Upsilon_{+,0}\right) \\ &\quad - \frac{\theta^{\pm\infty}_+}{\theta^{\pm\infty}_-}\left(\frac{v_0-v_0'}{2} - \tfrac{1}{2}\Upsilon^\dagger_{-,0}\Psi_\pm\Upsilon_{-,0}\right). \end{aligned} \tag{27}$$

The reader is invited to check from these equations that the H^1-energy associated to either of the stationary profiles (27) falls short of the one associated to the initial profile v_0 by *exactly* an amount that is equal to the sum of the energies that, at any given time $|t| < +\infty$, each of the solitons escaping (respectively) at speeds $1/(2\lambda_\pm)$ embody. But there is more, one can verify, adapting the general method of sections 3 and 4, that the limits (as $t \to \pm\infty$) of the Lagrangian-scaled updates of the remaining eigenfunctions that we refer to at the end of Remark 8 do not vanish. In fact, one can check that the latter constitute a basis in H^1. In

[1] In $\theta^{\pm\infty}_\pm$, the signature of the upper index $\pm\infty$ indicates which of the limits $t \to \pm\infty$ is meant, while the lower indexes merely distinguish which of the theta functions, θ_- or θ_+, is being referred to; cf. Notation on page 272.

short, as in the large-time asymptotics pertaining to the individual flows where it is shown that the eigenvalue defining the flow at play is excised [Loubet ≥ 2007a; ≥ 2007b], in the case of pair flows, we have evidence that the maps from initial to residual profiles, as described in the introduction, are also of Darboux-type with the difference that two eigenvalues (the ones involved in the Hamiltonian defining the flow) are excised instead of just one.

7. Conclusion

Closer inspection to the bulk of sections 3 and 4 shows that the method therein employed, can be adapted to produce analogous explicit exact formulas for the updates of profiles when run by flows of the CH hierarchy associated to Hamiltonians of the form $\sum_{|j|\leq N} 1/(4\lambda_j^2)$ with arbitrary N in $\mathbb{Z}^+$, all the way up — with due technical precautions in order to guarantee convergence, etc. — to (the limiting case where $N \uparrow +\infty$ corresponding to) the full CH flow [McKean 2003]. Moreover, the present analysis suggests that all these expressions will be sufficiently robust to afford (at least) a quantitative description of soliton train development. Nonetheless, it remains to explore in more detail how manageable all these expressions really are in helping reveal any more qualitative and quantitative phenomena pertaining to soliton emergence and soliton interaction.

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