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REDUCTION OF BRANES IN GENERALIZED COMPLEX GEOMETRY

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ABSTRACT. We show that certain submanifolds of generalized complex manifolds ("weak branes") admit a natural quotient which inherits a generalized complex structure. This is analog to quotienting coisotropic submanifolds of symplectic manifolds. In particular Gualtieri's generalized complex submanifolds ("branes") quotient to space-filling branes. Along the way we perform reductions by foliations (i.e. no group action is involved) for exact Courant algebroids - interpreting the reduced Ševera class - and for Dirac structures.

CONTENTS

1. INTRODUCTION

Consider the following setup in ordinary geometry: a manifold M and a submanifold C endowed with some integrable distribution $\mathcal{F}$ so that $\underline{C} := C/\mathcal{F}$ be smooth. Then we have a projection $pr : C \to \underline{C}$ which induces a vector bundle morphism $pr_* : TC \to T\underline{C}$. If M is endowed with some geometric structure, such as a symplectic 2-form ω, one can ask when ω induces a symplectic form on $\underline{C}$.

This happens for example when C is a coisotropic submanifold[1]. Indeed in this case the pullback $i^*\omega$ of ω to C has a kernel $\mathcal{F}$ which is of constant rank and integrable, and the closeness of ω ensures that if p and q lie in the same $\mathcal{F}$-leaf then $(i^*\omega)_p$ and $(i^*\omega)_q$ project to the same linear symplectic form at $pr(p) = pr(q)$, so that one obtains a well-defined symplectic form on $\underline{C}$. An instance of the above is when there is a Lie group G acting hamiltonianly on M with moment map $\nu : M \to \mathfrak{g}^*$ and C is the zero level set of ν (Marsden-Weinstein

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[1]This means that the symplectic orthogonal of TC is contained in TC.

reduction [17]).

In this paper we consider the geometry that arises when one replaces the tangent bundle TM with an exact Courant algebroid E over M (any such E is non-canonically isomorphic to $TM \oplus T^*M$). In this context reduction by the action of a Lie group has been considered by several authors (Burstzyn-Cavalcanti-Gualtieri [3], Hu [11, 12], Stienon-Xu [19], Tolman-Lin [15, 16]); in this paper we do not assume any group action. Unlike the tangent bundle case, knowing $\underline{C}$ does not automatically determine the exact Courant algebroid over it. We have to replace the foliation $\mathcal{F}$ by more data, namely a suitable subbundle K of $E|_C$ (projecting to $\mathcal{F}$ under the anchor map $\pi : E \to TM$); we determine conditions on K that allow to construct by a quotienting procedure a Courant algebroid $\underline{E}$ on $\underline{C}$ (Theorem 3.7) endowed with a morphism from E to $\underline{E}$ (Remark 3.9). Our construction follows closely the one of Bursztyn-Cavalcanti-Gualtieri [3], in which a suitable group action on E is assumed. In [3] the group action provides an identification between fibers of E at different points; in our case we make up for this asking that there exist enough "basic sections" (Def. 3.3). We also describe how a submanifold C with a foliation $\mathcal{F}$, once equipped with a suitable maximal isotropic subbundle L of $E|_C$, naturally has a reduced Courant algebroid over its leaf-space $\underline{C}$ (see Prop. 3.14). We describe in a simple way (see Def. 3.11) which splittings of E induce 3-forms on M (representing the Ševera class of E) which descend to 3-forms on $\underline{C}$ (representing the Ševera class of $\underline{E}$). Finally, in the case when the exact Courant algebroid E is split, we give an explicit and simple description of the reduction procedure of Thm. 3.7 in terms of differential forms (Prop. 3.18).

Once we know how to reduce an exact Courant algebroid, we can ask when geometric structures defined on them descend to the reduced exact Courant algebroid. We consider Dirac structures (suitable subbundles of E) and generalized complex structures (suitable endomorphisms of E). We give sufficient conditions for these structures to descend in Prop. 4.1 and Prop. 5.1 respectively. The ideas and techniques are borrowed the literature cited above, in particular from [3] and [19] (however our proof differs from these two references in that we reduce generalized complex structures directly and not viewing them as Dirac structures in the complexification of E).

The heart of this paper is Section 6, where we identify the objects that automatically satisfy the assumptions needed to perform generalized complex reduction. When M is a generalized complex manifold we consider pairs consisting of a submanifold C of M and suitable maximal isotropic subbundle L of $E|_C$ (we call them "weak branes" in Def. 6.9). We show in Prop. 6.10 that weak branes admit a canonical quotient $\underline{C}$ which is endowed with an exact Courant algebroid and a generalized complex structure; this construction is inspired by Thm. 2.1 of Vaisman's work [21] in the setting of the standard Courant algebroid.

Particular cases of weak branes are generalized complex submanifolds (C, L) (also known as "branes", see Def. 6.3), which were first introduced by Gualtieri [9] and are relevant to physics [14]. Using our reduction of Dirac structures we show in Thm. 6.4 that the quotients $\underline{C}$ of branes, which by the above are generalized complex manifolds, are also endowed with the structure of a space-filling brane (i.e. $\underline{C}$ together with the reduction of L forms a brane). This is interesting also because space filling branes induce an honest complex structure on the underlying manifold [8].

The reduction statements we had to develop in order to prove the results of Section 6 are versions "without group action" of statements that already appeared in the literature

[3][11, 12] [2, 19] [15, 16] [21]. Consequently many ideas and techniques are borrowed from the existing literature; we make appropriate references in the text whenever possible. In particular we followed closely [3] (also as far as notation and conventions are concerned).

Plan of the paper: in Section 2 we review exact Courant algebroids, mainly following [3]. In Section 3 we perform the reduction of exact Courant algebroids, determine objects that naturally satisfy the assumptions needed for the reduction, and comment on the reduced Ševera class. In Section 4 we perform the reduction of Dirac structures, and present as an example the coisotropic reduction in Poisson manifolds. In Section 5 we reduce generalized complex structures and comment briefly on generalized Kähler reduction. The main section of this paper is Section 6: we reduce branes and weak branes, providing few examples. We also give a criteria that allows to obtain weak branes by restricting to cosymplectic submanifolds.

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2. Review of Courant algebroids

We review the notion of exact Courant algebroid; see [3] and [11] for more details.

Definition 2.1. A *Courant algebroid* over a manifold M is a vector bundle $E \to M$ equipped with a fibrewise non-degenerate symmetric bilinear form $\langle \cdot,\cdot \rangle$, a bilinear bracket $[\cdot,\cdot]$ on the smooth sections $\Gamma(E)$, and a bundle map $\pi : E \to TM$ called the *anchor*, which satisfy the following conditions for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$:

C1) $[e_1,[e_2,e_3]] = [[e_1,e_2],e_3] + [e_2,[e_1,e_3]]$,
C2) $\pi([e_1,e_2]) = [\pi(e_1),\pi(e_2)]$,
C3) $[e_1, fe_2] = f[e_1,e_2] + (\pi(e_1)f)e_2$,
C4) $\pi(e_1)\langle e_2,e_3\rangle = \langle [e_1,e_2],e_3\rangle + \langle e_2,[e_1,e_3]\rangle$,
C5) $[e_1,e_1] = \mathcal{D}\langle e_1,e_1\rangle$,

where $\mathcal{D} = \frac{1}{2}\pi^* \circ d : C^\infty(M) \to \Gamma(E)$ (using $\langle \cdot,\cdot \rangle$ to identify E with E^*).

We see from axiom $C5$) that the bracket is not skew-symmetric:

$$[e_1,e_2] = -[e_2,e_1] + 2\mathcal{D}\langle e_1,e_2\rangle.$$

Hence we have the following "Leibniz rule for the first entry": $[fe_1,e_2] = f[e_1,e_2] - (\pi(e_2)f)e_1 + 2\langle e_1,e_2\rangle \mathcal{D}f$.

Definition 2.2. A Courant algebroid is *exact* if the following sequence is exact:

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0 \tag{1}$$

To simplify the notation, in the sequel we will often omit the map $T^*M \xrightarrow{\pi^*} E^* \cong E$ and think of T^*M as being a subbundle of E. Given an exact Courant algebroid, we may always choose a right splitting $\sigma : TM \to E$ whose image in E is isotropic with respect to $\langle \cdot, \cdot \rangle$. Such a splitting induces the closed 3-form on M given by

$$H(X, Y, Z) = 2\langle [\sigma X, \sigma Y], \sigma Z \rangle.$$

Using the bundle isomorphism $\nabla + \frac{1}{2}\pi^* : TM \oplus T^*M \to E$, one can transport the Courant algebroid structure onto $TM \oplus T^*M$. The resulting structure is as follows (where $X_i + \xi_i \in \Gamma(TM \oplus TM^*)$): the bilinear pairing is

$$\langle X_1 + \xi_1, X_2 + \xi_2 \rangle = \frac{1}{2}(\xi_2(X_1) + \xi_1(X_2)), \tag{2}$$

and the bracket is

$$[X_1 + \xi_1, X_2 + \xi_2]_H = [X_1, X_2] + \mathcal{L}_{X_1}\xi_2 - i_{X_2}d\xi_1 + i_{X_2}i_{X_1}H, \tag{3}$$

which is the H-twisted Courant bracket on $TM \oplus T^*M$ [18]. Isotropic splittings of (1) differ by 2-forms $b \in \Omega^2(M)$, and a change of splitting modifies the curvature H by the exact form db. Hence there is a well-defined cohomology class $[H] \in H^3(M, \mathbb{R})$ attached to the exact Courant algebroid structure on E; $[H]$ is called the *Ševera class* of E.

We refer to [3] and [11] for information on the group of automorphisms $Aut(E)$ and its Lie algebra $Der(E)$. Here we just mention few facts, the first of which underlies many of our constructions: for any $e \in \Gamma(E)$, $[e, \cdot]$ is an element of $Der(E)$ and hence integrates to an automorphism of the Courant algebroid E. Notice that for closed 1-forms ξ (seen as sections of $T^*M \subset E$) we have $[\xi, \cdot] = 0$ by (3). Further, any 2-form B on M determines a vector bundle map $TM \oplus TM^* \to TM \oplus TM^*$ by $e^B : X + \xi \mapsto X + \xi + i_X B$ [9] and these "gauge transformations" satisfy

$$[e^B \cdot, e^B \cdot]_H = e^B[\cdot, \cdot]_{H+dB}. \tag{4}$$

3. The case of exact Courant algebroids

In this section we reduce exact Courant algebroids (see Thm. 3.7 and Prop. 3.18), display objects whose quotient is naturally endowed with a reduced exact Courant algebroid, and then comment on the relation between a Courant algebroid and its reduction, as well as on the relation between the Ševera classes.

3.1. **Reducing exact Courant algebroids.** Let M be a manifold and E an exact Courant algebroid over M. We fix a submanifold C.

Lemma 3.1. *Let $D \to C$ be a subbundle of E such that $\pi(D^\perp) \subset TC$ (where $D^\perp$ denotes the orthogonal to D w.r.t. the symmetric pairing), and e_1, e_2 sections of $D^\perp$. Then the expression $[\tilde{e}_1, \tilde{e}_2]|_C$, where $\tilde{e}_i$ are extensions of e_i to sections of $E \to M$, depends on the extensions only up to sections of D.*

Proof. Fix extensions $\tilde{e}_i$ of e_i $(i = 1, 2)$. We have to show that for functions f_i vanishing on C and sections $\hat{e}_i$ of E we have $[\tilde{e}_1 + f_1\hat{e}_1, \tilde{e}_2 + f_2\hat{e}_2]|_C = [\tilde{e}_1, \tilde{e}_2]|_C$ up to sections of D. By the Leibniz rule $C3$) and since $\pi(e_1) \subset TC$ we have $[\tilde{e}_1, f_2\hat{e}_2]|_C = 0$. Also $[f_1\hat{e}_1, \tilde{e}_2]|_C = 2\langle \hat{e}_1, \tilde{e}_2 \rangle (\mathcal{D}f_1)|_C \subset N^*C \subset (\pi(D^\perp))^\circ = D \cap T^*M$ [2]. The term $[f_1\hat{e}_1, f_2\hat{e}_2]|_C$ vanishes by the above since $(f_1\hat{e}_1)|_C$ is a section of D. $\square$

[2] Indeed for any subspace D of a vector space $T \oplus T^*$, denoting by π the projection onto T, we have $D \cap T^* = (\pi(D^\perp))^\circ$. This follows from $(D \cap T^*)^\perp = D^\perp + T^* = \pi^{-1}(\pi(D^\perp))$.

Remark 3.2. If $D \to C$ is a subbundle of E such that $\pi(D^\perp) \subset TC$ we can make sense of a statement like "$[e_1, e_2] \in \Gamma(D)$" for $e_1, e_2 \in \Gamma(D^\perp)$: it means that $[\tilde{e}_1, \tilde{e}_2]|_C \in \Gamma(D)$ for one (or equivalently, by Lemma 3.1, for all) extensions $\tilde{e}_i$ to sections of $E \to M$. Similarly, we take $[\Gamma(D^\perp), \Gamma(D^\perp)] \subset \Gamma(D)$ to mean $[e_1, e_2] \in \Gamma(D)$ for all $e_1, e_2 \in \Gamma(D^\perp)$.

Now fix an *isotropic* subbundle $K \to C$ of E, i.e. $K \subset K^\perp$, such that $\pi(K_x^\perp) = T_x C$ at each $x \in C$.

Definition 3.3. We define the space of sections of $K^\perp$ which are *basic w.r.t.* K as

$$\Gamma_{bas}(K^\perp) := \{e \in \Gamma(K^\perp) : [\Gamma(K), e] \subset \Gamma(K)\}. \tag{5}$$

Remark 3.4. To ensure that a section e of $K^\perp$ be basic it suffices to consider locally defined sections of K that span K point-wise. That is, it suffices to show that for every point of C there is a neighborhood $U \subset C$ and a subset $S \subset \Gamma(K|_U)$ with $span\{k_p : k \in S\} = K_p$ (for every $p \in U$) so that $[S, e|_U] \subset \Gamma(K|_U)$. Indeed from the "Leibniz rule in the first entry" it follows that $[\Gamma(K), e] \subset \Gamma(K)$.

Lemma 3.5. *Assume that the sections of $\Gamma_{bas}(K^\perp)$ span $K^\perp$ at every point, i.e. that $span\{e_p : e \in \Gamma_{bas}(K^\perp)\} = K_p^\perp$ for every $p \in C$. Then*

1) $[\Gamma(K), \Gamma(K^\perp)] \subset \Gamma(K^\perp)$
2) $[\Gamma(K), \Gamma(K)] \subset \Gamma(K)$.

Proof. Fix a subset of sections $\{e_i\} \subset \Gamma_{bas}(K^\perp)$ that spans point-wise $K^\perp$. For any section k of K and functions f_i (so that the sum $\sum f_i e_i$ is locally finite) by the Leibniz rule we have $[k, \sum f_i e_i] \subset K^\perp$, proving 1). Now 1) is equivalent to 2), as can be seen using axiom C4) in the definition of Courant algebroid: let k_1, k_2 be sections of K and e a section of $K^\perp$. Then $\langle [k_1, e], k_2 \rangle + \langle e, [k_1, k_2] \rangle = \pi(k_1)\langle e, k_2 \rangle = 0$ because $\pi(K) \subset \pi(K^\perp) = TC$. $\square$

Remark 3.6. A converse to Lemma 3.5 for local sections is given in [4].

The proof of the following theorem is modeled on Thm. 3.3 of [3].

Theorem 3.7 (Exact Courant algebroid reduction)**.** *Let E be an exact Courant algebroid over M, C a submanifold of M, and K an isotropic subbundle of E over C such that $\pi(K^\perp) = TC$. Assume that the space of (global) sections $\Gamma_{bas}(K^\perp)$ spans point-wise $K^\perp$ (i.e. that $span\{e_p : e \in \Gamma_{bas}(K^\perp)\} = K_p^\perp$ for every $p \in C$) and that the quotient $\underline{C}$ of C by the foliation integrating $\pi(K)$ be a smooth manifold. Then there is an exact Courant algebroid $\underline{E}$ over $\underline{C}$ that fits in the following pullback diagram of vector bundles:*

$$\begin{array}{ccc} K^\perp/K & \longrightarrow & \underline{E} \\ \downarrow & & \downarrow \\ C & \longrightarrow & \underline{C} \end{array} .$$

Proof. Notice that since $\pi(K)$ has constant rank iff $\pi(D^\perp)$ does (use the previous footnote or eq. (2.17) of [20]) it follows that $\pi(K)$ is a regular distribution on C. Further, by the assumption on basic sections and item 2) of Lemma 3.5, $\pi(K)$ is an integrable distribution, so there exists a regular foliation integrating $\pi(K)$. We divide the proof in 3 steps.

Step 1 To describe the vector bundle $\underline{E}$ we have to explain how we identify fibers of $K^\perp/K$ over two points p, q lying in the same leaf F of $\pi(K)$. We do this as follows: we identify two elements $\hat{e}(p) \in (K^\perp/K)_p$ and $\hat{e}(q) \in (K^\perp/K)_q$ iff there is a section $e \in$

$\Gamma_{bas}(K^{\perp})$ which under the projection $K^{\perp} \to K^{\perp}/K$ maps[3] to $\hat{e}(p)$ at p and $\hat{e}(q)$ at q. To show that this procedure gives a well-defined identification of $(K^{\perp}/K)_p$ and $(K^{\perp}/K)_q$, we need to show that if e_1 and e_2 are sections of $\Gamma_{bas}(K^{\perp})$ such that $e_1(p)$ and $e_2(p)$ map to $\hat{e}(p)$, then $e_1(q)$ and $e_2(q)$ map to the same element of $(K^{\perp}/K)_q$.

Pick a finite sequence of local sections $k_1, \dots, k_n$ of K that join p to q, i.e. such that following successively the vector fields $\pi(k_i)$ for times t_i the point p is mapped to q. Extend each k_i to a section $\tilde{k}_i$ of E. Denote by $e^{ad_{\tilde{k}_i}}$ the Courant algebroid automorphism of E obtained integrating $ad_{\tilde{k}_i} = [\tilde{k}_i, \cdot]$, and by Φ the composition $e^{ad_{t_n \tilde{k}_n}} \circ \dots \circ e^{ad_{t_1 \tilde{k}_1}}$. Since e_1 is a basic section we have $[k_i, e_1] \subset K$ for all i. So $\Phi(e_1(p)) - e_1(q) \in K_q$, and similarly for e_2. Now $e_1(p) - e_2(p) \in K_p$ by assumption, so because of item 2) of Lemma 3.5 we have $\Phi(e_1(p) - e_2(p)) \in K_q$. We deduce that $e_1(q) - e_2(q)$ also belong to K_q and therefore project to the zero vector in $(K^{\perp}/K)_q$.

It is clear that $\underline{E}$, obtained from $K^{\perp}/K$ by identifying the fibers over each leaf of $\pi(K)$ as above, is endowed with a projection $\underline{pr}$ onto $\underline{C}$ (induced from the projection $pr : K^{\perp}/K \to C$). $\underline{E}$ is indeed a smooth vector bundle: given any point $\underline{p}$ of $\underline{C}$ choose a preimage $p \in C$ and a submanifold $S \subset C$ through p transverse to the leaves of $\pi(K)$. S provides a chart around $\underline{p}$ for the manifold $\underline{C}$, and $pr^{-1}(S)$ is a vector subbundle of $K^{\perp}/K$ proving a chart for $\underline{E}$ around $\underline{p}$.

Notice that pulling back by the vector bundle epimorphism $K^{\perp}/K \to \underline{E}$ we can embed the space of sections of $\underline{E}$ into the space of sections of $K^{\perp}/K$, the image being the image of $\Gamma_{bas}(K^{\perp})$ under the map $K^{\perp} \to K^{\perp}/K$. In other words, we have a canonical identification $\Gamma(\underline{E}) \cong \Gamma_{bas}(K^{\perp})/\Gamma(K)$.

Step 2 The pairing $\langle \cdot, \cdot \rangle$ on the fibers of E induces a symmetric bilinear form on each fiber of $K^{\perp}/K$, which is moreover non-degenerate because it is obtained by "odd linear symplectic reduction". This pairing descends to $\underline{E}$, because for any two given sections $e_1, e_2 \in \Gamma_{bas}(K^{\perp})$ the expression $\langle e_1, e_2 \rangle$ is constant along each leaf of $\pi(K)$: for any section k of K we have $\pi(k)\langle e_1, e_2 \rangle = 0$ using C4).

For the bracket of sections of $\underline{E}$, first notice that $\Gamma_{bas}(K^{\perp})$ is closed (in the sense of Remark 3.2) under the bracket $[\cdot, \cdot]$ of E: if $e_1, e_2 \in \Gamma_{bas}(K^{\perp})$, then for any section k of K we have by C4) $\langle [e_1, e_2], k \rangle = -\langle e_2, [e_1, k] \rangle + \pi(e_1)\langle e_2, k \rangle$. This vanishes since e_1 is basic and $\pi(e_1)$ is tangent to C, so $[e_1, e_2]$ is a section of $K^{\perp}$. Further it is basic again by the "Jacobi identity" C1): for any section k of K we have $[k, [e_1, e_2]] = [[k, e_1], e_2] + [e_1, [k, e_2]]$. Now by definition of basic section each $[k, e_i]$ lies in K, and applying once more the definition of basic section[4] we see that $[k, [e_1, e_2]] \subset K$, i.e. that $[e_1, e_2]$ is basic. In the light of Lemma 3.1, what we really have a well-defined bilinear form $\Gamma_{bas}(K^{\perp}) \times \Gamma_{bas}(K^{\perp}) \to \Gamma_{bas}(K^{\perp})/\Gamma(K)$. Using the definition of basic section we then have an induced bracket on $\Gamma_{bas}(K^{\perp})/\Gamma(K)$, which as we saw is canonically isomorphic to $\Gamma(\underline{E})$.

[3]In other words, we give a canonical trivialization of $(K^{\perp}/K)|_F$ by projecting into it a frame for $K^{\perp}|_F$ consisting of basic sections; by assumptions we have enough basic sections to really get a frame for $(K^{\perp}/K)|_F$.

[4]Together with the fact that for any section $\hat{k}$ of K we have $[e_1, \hat{k}] = -[\hat{k}, e_1] + 2\mathcal{D}\langle e_1, \hat{k} \rangle$ and $\mathcal{D}\langle e_1, \hat{k} \rangle \subset N^*C = K \cap T^*M$.

We define the anchor $\underline{\pi}:\underline{E}\to T\underline{C}$ to make the following diagram of vector bundle morphisms commute:

$$\begin{array}{ccccc} K^{\perp} & \longrightarrow & K^{\perp}/K & \longrightarrow & \underline{E} \\ \downarrow{\scriptstyle \pi} & & & & \downarrow{\scriptstyle \underline{\pi}} \\ TC & & \longrightarrow & & T\underline{C} \end{array} .$$

To show that $\underline{\pi}$ is well-defined we choose an element v of $\underline{E}_{\underline{p}}$ and view it as a section $\tilde{e}$ of $(K^{\perp}/K)|_F$, where F is the leaf of $\pi(K)$ corresponding to $\underline{p}$. We define $\underline{\pi}(v)$ as $\pi(\tilde{e}_p)\in T_pC/\pi(K_p)\cong T_{\underline{p}}\underline{C}$, for $p\in F$ and abusing notation by calling π the map $(K^{\perp}/K)_p\to T_pC/\pi(K_p)$. We have to show that the above definition is independent of the point $p\in F$: take any basic section $e\in\Gamma_{bas}(K^{\perp})$ defined near F and mapping to $\tilde{e}$ under $K^{\perp}\to K^{\perp}/K$. We have to show that $\pi(e)$ is a projectable vector field; this is the case since for any vector field Y on C tangent to the leaves of $\pi(K)$ we can write $Y=\pi(k)$ for a smooth section of K, and by C2) and the definition of basic section $[Y,\pi(e)]=\pi([k,e])\subset\pi(K)$.

Step 3 Up to now we have defined the vector bundle $\underline{E}\to\underline{C}$ and endowed it with a fiber-wise non-degenerate symmetric pairing, with a bilinear bracket on $\Gamma(\underline{E})$ and an anchor $\underline{\pi}$. It is straightforward to check that the axioms C1)-C5) in the definition of Courant algebroid (Def. 2.1) are fulfilled.

We are left with showing that $\underline{E}$ is an exact Courant algebroid. To this aim it suffices to show that $rk(\underline{E})=2dim(\underline{C})$ and that the kernel of the anchor $\underline{\pi}$ is isotropic in $\underline{E}$[5]. The dimension of $\underline{C}$ is equal to the rank of $\pi(K^{\perp})/\pi(K)$, which is $dim(M)-rk(K)$ as can be seen using $K\cap T^*M=(\pi(K^{\perp}))^{\circ}$. The rank of $\underline{E}$ is the rank of $K^{\perp}/K$, which is $2(dim(M)-rk(K))$. The kernel of $\underline{\pi}$ is the image under $K^{\perp}\to K^{\perp}/K\to\underline{E}$ of $(\pi|_{K^{\perp}})^{-1}(\pi(K))$ by the commutativity of the above diagram, and it's isotropic iff the latter is. Now $(\pi|_{K^{\perp}})^{-1}(\pi(K))=K+(K^{\perp}\cap ker(\pi))$, which is isotropic because both K and $ker(\pi)$ are. □

Remark 3.8. We give an alternative way to describe the identification (see Step 1 of the above proof) between fibers of $K^{\perp}/K$ over two points p,q lying in the same leaf of $\pi(K)$: they are identified by the action of any sequence of sections of K joining p to q. More precisely, pick a finite sequence of local sections $k_1,\dots,k_n$ of K that join p to q, pick extensions $\tilde{k}_i\in\Gamma(E)$ and denote again by Φ the induced Courant algebroid automorphism of E. By item 1) of Lemma 3.5 Φ preserves the subbundle $K^{\perp}$ of E. By item 2) of the same lemma Φ preserves K, hence it induces a linear map $(K^{\perp}/K)_p\to(K^{\perp}/K)_q$. For any $e\in\Gamma_{bas}(K^{\perp})$ we have $\Phi(e(p))-e(q)\in K_q$. So, when $\Gamma_{bas}(K^{\perp})$ spans point-wise $K^{\perp}$, the map $(K^{\perp}/K)_p\to(K^{\perp}/K)_q$ gives the same identification as in the proof of Thm. 3.7 (hence it is independent of the choices of $\tilde{k}_i$'s and their extensions).

[5]Any Courant algebroid satisfying these two conditions is exact, as we show now (sticking to our previous notation for $\underline{E}$). By dimension counting it follows that $\underline{\pi}$ is surjective and that $ker(\underline{\pi})$ is maximal isotropic. Fix a covector $\xi\in T^*\underline{C}$. Then for all $e\in ker(\underline{\pi})$ we have

$$0=\langle\underline{\pi}(e),\xi\rangle_{T\underline{C},T^*\underline{C}}=\langle e,\underline{\pi}^*(\xi)\rangle_{\underline{E},\underline{E}^*}=\langle e,\Psi(\underline{\pi}^*(\xi))\rangle_{\underline{E}}.$$

Here the subscripts indicate that the first two are pairings of a vector space with its dual and the third one the non-degenerate symmetric bilinear form on $\underline{E}$; $\Psi:\underline{E}^*\to\underline{E}$ is the induced isomorphism. Now since $ker(\underline{\pi})$ is maximal isotropic it follows that $\Psi(\underline{\pi}^*(\xi))\in ker(\underline{\pi})$. Since this holds for all covectors ξ we obtain $\underline{\pi}^*(T^*\underline{C})\subset\ker(\underline{\pi})$, and since $\underline{\pi}^*$ is injective because $\underline{\pi}$ is surjective, we deduce that $\underline{E}$ is exact.

Remark 3.9. The Courant algebroids E and $\underline{E}$ in Thm. 3.7 give rise to two pieces of data: the submanifold $S := \{(p, \underline{p}) | p \in C\}$ of $M \times \underline{C}$ and a subbundle $F := \{(e, \underline{e}) | e \in K^\perp\}$ of $(E \times \underline{E})|_S$. The subbundle F is maximal isotropic in $E \times \underline{E}^-$ (where the superscript "$^-$" denotes that we invert the sign of the symmetric pairing on $\underline{E}$), we have $(\pi \times \underline{\pi})(F) = TS$, and F is closed under the Courant bracket on the product Courant algebroid $E \times \underline{E}^-$ (in the sense of Remark 3.2). These three statements are easily checked using the Courant algebroid structure on $\underline{E}$ as defined in the proof of Thm. 3.7. Hence the subbundle $F \to S$ provides a morphism of Courant algebroids from E to $\underline{E}$ as defined [6] in Def. 6.12 of [1] or in Def. 3.5.1 of [13].

We present a simple example:

Example 3.10 (Quotients of submanifolds). Take E to be $TM \oplus T^*M$ with the untwisted bracket, i.e. the one given by setting $H = 0$ in (3). Let C be a submanifold endowed with a regular distribution $\mathcal{F}$, and assume that the quotient $\underline{C} = C/\mathcal{F}$ be smooth. Take $K := \mathcal{F} \oplus N^*C$ (so $K^\perp = TC \oplus \mathcal{F}^\circ$). We want to check that the basic sections of $K^\perp$ span $K^\perp$. $\Gamma(K)$ is spanned by vector fields on C lying in $\mathcal{F}$ and differentials of functions vanishing on C. Since the latter (as all closed 1-forms) act trivially, it is enough to consider the action of a vector field $X \subset \mathcal{F}$. Let $Y \oplus df|_C$ be a section of $K^\perp$, where Y is a projectable vector field and f is the extension to M of the pullback of a function on $\underline{C}$. The action of X on this sections is just $[X, Y] \oplus (\mathcal{L}_X df)|_C$, which lies again in K. Since such $Y \oplus df|_C$ span $K^\perp$ we can apply Thm. 3.7 and obtain a reduced Courant algebroid on $\underline{C}$. Of course this is just $T\underline{C} \oplus T^*\underline{C}$ with the untwisted bracket.

The above example can be also easily recovered from Prop. 3.14 below (choosing $L = TC \oplus N^*C$ there) or from Prop. 3.18 (choosing $F \in \Omega^2(C)$ to be zero).

3.2. **Adapted splittings.** In this subsection we consider "good" splittings of an exact Courant algebroid $E \to M$, and using their existence we determine simple data on a foliated submanifold that induce an exact Courant algebroid on the leaf space.

Let E be an exact Courant algebroid over M and let C be a submanifold endowed with a coisotropic subbundle $K^\perp$ of E satisfying $\pi(K^\perp) = TC$. Assume that $\pi(K)$ is integrable and $\underline{C} := C/\pi(K)$ smooth.

Definition 3.11. We call a splitting $\sigma : TM \to E$ of the sequence (1) *adapted to* K if
a) The image of σ is isotropic
b) $\sigma(TC) \subset K^\perp$
c) for any vector field X on C which is projectable to $\underline{C}$ we have $\sigma(X) \in \Gamma_{bas}(K^\perp)$.

Remark 3.12. For such a splitting it follows automatically that $\sigma(\pi(K)) \subset K$. Indeed by $\pi(K^\perp) = TC$, b) in the definition above and $K^\perp \cap T^*M = (\pi(K))^\circ$ we have $K^\perp = \sigma(TC) + (\pi(K))^\circ$. Now $\langle \sigma(\pi(K)), \sigma(TC) \rangle = 0$ by a) in the definition above and $\langle \sigma(\pi(K)), (\pi(K))^\circ \rangle$ is equal to one-half the pairing of $\pi(K)$ and $(\pi(K))^\circ$, which is zero. Hence $\sigma(\pi(K))$ has zero symmetric pairing with $K^\perp$, so $\sigma(\pi(K)) \subset K$

The following proposition says that splittings adapted to K exist if and only if the reduced exact Courant algebroid $\underline{E}$ as in Thm. 3.7 exists.

[6] We actually use a slight modification of the definitions of [1] and [13], for in these two references S is required to be the graph of an honest map $M \to \underline{C}$. Further in [1] Courant algebroids are endowed with the skew-symmetric Courant bracket.

Proposition 3.13. *Let $K \to C$ be an isotropic subbundle of E with $\pi(K^\perp) = TC$ and assume that $\pi(K)$ is integrable and $\underline{C} := C/\pi(K)$ smooth. Then splittings adapted to K exist if and only if $\Gamma_{bas}(K^\perp)$ spans $K^\perp$ at every point of C.*

Proof. Assume first that a splitting σ adapted to K exists. Let X be a projectable vector field on C. By c) of Def. 3.11 $\sigma(X)$ we will lie in $\Gamma_{bas}(K^\perp)$. Take a function on $\underline{C}$, pull it back to a function on C and extend it to a function f on M. Then $df|_C$ is a section of $(\pi(K))^\circ = T^*M \cap K^\perp$. Further it lies in $\Gamma_{bas}(K^\perp)$: for any $k \in \Gamma(K)$ we have

$$[k, df|_C] = -[df|_C, k] + d\langle k, df|_C\rangle \subset N^*C \subset K$$

because df as a closed 1-form acts trivially and it annihilates $\pi(K)$. Since $K^\perp = \sigma(TC) + (T^*M \cap K^\perp)$, taking all projectable vector fields X and functions f as above we see that $\Gamma_{bas}(K^\perp)$ spans $K^\perp$ at every point of M.

Conversely, assume now that $\Gamma_{bas}(K^\perp)$ spans $K^\perp$ at every point of M. Then by Thm. 3.7 the Courant algebroid $\underline{E}$ over $\underline{C}$ exists. We show that any isotropic splitting $\underline{\sigma} : T\underline{C} \to \underline{E}$ can be "lifted" to a splitting of E adapted to K. For any point $p \in C$ (and its image $\underline{p} \in \underline{C}$) consider the commutative diagram

$$\begin{tikzcd}
K_p^\perp \arrow[r, "\pi"] \arrow[d] & T_pC \ . \arrow[d] \\
(K^\perp/K)_p \cong \underline{E}_{\underline{p}} \arrow[r, "\pi"] & T_{\underline{p}}\underline{C}
\end{tikzcd}$$

To simplify the notation we leave out the footpoints until the end of this paragraph. Choose any subspace S such that $S \oplus K = K^\perp$, and use the isomorphism $\pi|_S : S \to K^\perp/K$ to obtain from $\underline{\sigma}(T\underline{C}) \subset \underline{E} \cong K^\perp/K$ a subspace $A \subset S$. By construction A maps isomorphically onto $T\underline{C}$ by the left and bottom map of the diagram, hence the same is true also using the top and right map. In particular $\pi(A)$ is a complement to $\pi(K)$ in TC, allowing us to define $\sigma : \pi(A) \to A \subset K^\perp$. We summarize the situation in the following commutative diagram:

$$\begin{tikzcd}
K_p^\perp & \supset & A_p \arrow[r, "\pi"] \arrow[d] & \pi(A_p) \arrow[d] & \subset & T_pC \ . \\
\underline{E}_{\underline{p}} \cong (K^\perp/K)_p & \supset & \underline{\sigma}(T_{\underline{p}}\underline{C}) \arrow[r, "\pi"] & T_{\underline{p}}\underline{C} & &
\end{tikzcd}$$

Notice that A is isotropic in E because $\underline{\sigma}$ was an isotropic splitting of $\underline{E}$. Extend $\sigma : \pi(A) \to A$ to TC so that the resulting $\sigma : TC \to K^\perp$ maps $\pi(K)$ to K. Then b) of Def. 3.11 is satisfied, and since $\sigma(TC)$ is isotropic condition a) is also satisfied.

We are left with showing that c) of Def. 3.11 holds, i.e. that if X is a basic vector field on C, then $\sigma(X) \in \Gamma_{bas}(K^\perp)$. Writing $X = X_{\pi(A)} + X_{\pi(K)}$ we see that $\sigma(X)_p$ is the sum of a section of K_p and the lift to A_p of $\underline{\sigma}(\underline{X})_{\underline{p}} \in \underline{E}_{\underline{p}} \cong (K^\perp/K)_p$. The projection of $\sigma(X)_p$ to $(K^\perp/K)_p$ is just $\underline{\sigma}(\underline{X})_{\underline{p}}$, i.e. it does not depend on p but just on its image $\underline{p} \in \underline{C}$. This shows that $\sigma(X)$ induces a well-defined section of $\underline{E}$ and hence lies in $\Gamma_{bas}(K^\perp)$. Now one can extend[7] $\sigma : TC \to K^\perp$ to the whole of M and obtain an isotropic splitting $TC \to E$. $\square$

[7] At any point p of C first extend σ from T_pC to T_pM as follows. Again we suppress the index "p". Since $\sigma(TC) \cap T^*M = \{0\}$ it follows that $(\sigma(TC))^\perp$ maps surjectively onto TM under π; choose a subspace W with $\sigma(TC) \subset W \subset (\sigma(TC))^\perp$ which maps isomorphically onto TM. W is a complement in E to the (maximal) isotropic subspace $ker(\pi)$, hence we can deform it canonically to a (maximal) isotropic subspace

Now we are able to determine a class of objects[8] that admit a reduced exact Courant algebroid on their quotient:

Proposition 3.14. *Let E be an exact Courant algebroid over a manifold M, C a submanifold endowed with a regular integrable foliation $\mathcal{F}$ so that $C/\mathcal{F}$ be smooth, and L a maximal isotropic subbundle $L \subset E|_C$ with $\pi(L) = TC$ such that $[\Gamma(K), \Gamma(L)] \subset \Gamma(L)$ where $K := L \cap \pi^{-1}(\mathcal{F})$. Then the assumptions of Thm. 3.7 are satisfied, hence E descends to an exact Courant algebroid on $C/\mathcal{F}$.*

Proof. Notice that K is isotropic and has constant rank, because $ker(\pi|_K) = K \cap T^*M = L \cap T^*M = N^*C$ has constant rank and $\pi(K) = \mathcal{F}$ has constant rank by assumption. Also $K^\perp = L + \mathcal{F}^\circ$, so $\pi(K^\perp) = TC$. Let $\sigma : TM \to E$ be an isotropic splitting such that $\sigma(TC) \subset L$. Since $\pi(L) = TC$ such a splitting always exists. We claim that σ is automatically a splitting adapted to K (Def. 3.11). Since $L \subset K^\perp$ we just have to check that if $k \in \Gamma(K)$ and X is a projectable vector field on C then $[k, \sigma(X)] \in \Gamma(K)$. By assumption this bracket is a section of L, and $\pi([k, \sigma(X)]) = [\pi(k), X] \subset \mathcal{F}$ since $\pi(k)$ lies in $\mathcal{F}$ and X is projectable, so altogether it follows that σ is a splitting adapted to K. By Prop. 3.13 the existence of a splitting adapted to K implies that $\Gamma_{bas}(K^\perp)$ spans pointwise $K^\perp$, and we are done. $\square$

3.3. **The Ševera class of the reduced Courant algebroid.** As in the previous subsection let E be an exact Courant algebroid over M and let C be a submanifold endowed with a coisotropic subbundle $K^\perp$ of E satisfying $\pi(K^\perp) = TC$. In Theorem 3.7 we showed that, when certain assumptions are met, one obtains an exact Courant algebroid $\underline{E}$ over the quotient $\underline{C}$ of C by the distribution $\pi(K)$. In this subsection we will discuss how to obtain the Ševera class of $\underline{E}$ from the one of E.

Assume that $\pi(K)$ is integrable, $\underline{C} := C/\pi(K)$ smooth, and σ a splitting adapted to K. We start observing that j^*H_σ descends to a 3-form on $\underline{C}$, where j is the inclusion of C in M. We need to check that $i_X(j^*H_\sigma) = 0$ and $\mathcal{L}_X(j^*H_\sigma) = 0$ for any vector field X on C tangent to $\pi(K)$. Since H_σ is closed by Cartan's formula for the Lie derivative we just need to check the first condition: take a vector $X \in \pi(K_p)$ and extend it to a vector field tangent to $\pi(K)$; take vectors $Y, Z \in T_pC$ and extend them locally to projectable vector fields of C. Since σ is an splitting adapted to K we know that $\sigma(Y) \in \Gamma_{bas}(K^\perp)$, and since $\sigma(X) \subset K$ (by Remark 3.12) we have $[\sigma(X), \sigma(Y)] \subset K$. Therefore

$$H_\sigma(X, Y, Z) \quad = \quad 2\langle[\sigma(X), \sigma(Y)], \sigma(Z)\rangle = 0, \tag{6}$$

which is what we needed to prove. Even more is true by the following, which is an analog of Prop. 3.6 of [3] (but unlike that proposition does not involve equivariant cohomology; see also [16, 15]).

Proposition 3.15. *Assume that $\underline{C}$ is a smooth manifold. If σ is a splitting adapted to K then $j^*(H_\sigma)$ descends to a closed 3-form on $\underline{C}$ which represents the Ševera class of $\underline{E}$.*

$\hat{W}$ of E as one does in symplectic linear algebra (see Chapter 8 of [5]; here we think of $\langle \cdot, \cdot \rangle$ as an odd linear symplectic form). Explicitly, we define a map $\phi : W \to ker(\pi)$ by $\langle \phi w, \cdot \rangle|_W = -\frac{1}{2}\langle w, \cdot \rangle|_W$ and define $\hat{W}$ as the graph of ϕ. Since ϕ maps $\sigma(TC)$ to zero we have $\sigma(TC) \subset \hat{W}$, and $\hat{W}$ is still transverse to the kernel of π, allowing us to define $\sigma : TM \to \hat{W} \subset E$. Now we just extend $\hat{W} \subset E|_M$ in any way to a subbundle of $E \to M$ and apply the same construction as above to deform it into an isotropic subbundle.

[8]Compare also with Def. 6.1 and Def. 6.9.

Proof. We first describe an isotropic splitting $\underline{\sigma}$ of $\underline{E}$ induced by σ. Fix a distribution B on C such that $\pi(K) \oplus B = TC$. Fix a point $p \in C$ and define the subspace D_p as the image of $\sigma(B_p)$ under $K_p^\perp \to (K^\perp/K)_p$ (here we use $\sigma(B_p) \subset K_p^\perp$ by b) in Def. 3.11). Notice that since $\sigma(B_p) \cap K_p = \{0\}$ all four arrows of this commutative diagram are isomorphisms:

$$\begin{tikzcd} K_p^\perp & \supset & \sigma(B_p) \arrow[r, "\pi"] \arrow[d] & B_p \arrow[d] & \subset & T_pC\ . \\ \underline{E}_{\underline{p}} \cong (K^\perp/K)_p & \supset & D_p \arrow[r, "\pi"] & T_{\underline{p}}\underline{C} & & \end{tikzcd}$$

It is clear that D_p is isotropic because $\sigma(B_p)$ is. Reversing the bottom isomorphism we obtain a linear map $\underline{\sigma}_p : T_{\underline{p}}\underline{C} \to D_p \subset \underline{E}_{\underline{p}}$. We want to show that this map depends only on $\underline{p}$ and not on the choice of point p in the $\pi(K)$-leaf F sitting over $\underline{p}$. To this aim take $\underline{X} \in T_{\underline{p}}\underline{C}$, lift it (using the Ehresmann connection B for the submersion $C \to \underline{C}$) to $X \in \Gamma(B|_F)$. $\underline{\sigma}_p(\underline{X})$ is by definition the image of $\sigma(X_p)$ under the left vertical isomorphism, and it depends only on $\underline{p}$ because by c) in Def. 3.11 $\sigma(X)$ is a section of $\Gamma_{bas}(K^\perp)$ (defined over F). So we obtain a well-defined splitting $\underline{\sigma}$ of the Courant algebroid $\underline{E}$ over $\underline{C}$.

To compute the 3-form on $\underline{C}$ induced by j^*H_σ pick three tangent vectors on $\underline{C}$ at some point $\underline{p}$, which by abuse by notation we denote by $\underline{X}, \underline{Y}, \underline{Z}$. Extend them to vector fields on $\underline{C}$ and lift them to obtain vector fields X, Y, Z which are projectable. $\sigma(Z)$ lies in $\Gamma_{bas}(K^\perp)$, and using the commutativity of the above diagram we see that it is a lift of $\underline{\sigma}(\underline{Z}) \in \Gamma(\underline{E})$. The same holds for X and Y, therefore, by the definition of Courant bracket on $\underline{E}$, we know that $[\sigma(X), \sigma(Y)] \in \Gamma_{bas}(K^\perp)$ is a lift of $[\underline{\sigma}(\underline{X}), \underline{\sigma}(\underline{Y})] \in \Gamma(\underline{E})$. Hence

$$H_\sigma(X,Y,Z) = 2\langle[\sigma(X), \sigma(Y)], \sigma(Z)\rangle \tag{7}$$
$$= 2\langle[\underline{\sigma}(\underline{X}), \underline{\sigma}(\underline{Y})], \underline{\sigma}(\underline{Z})\rangle. \tag{8}$$

This shows that H_σ descends to the curvature 3-form of $\underline{E}$ induced by the isotropic splitting $\underline{\sigma}$. □

Remark 3.16. If σ and $\hat{\sigma}$ are any two isotropic splittings for $E \to TM$ then there is a 2-form $b \in \Omega^2(M)$ for which $\sigma(X) - \hat{\sigma}(X) = b(X, \cdot) \in T^*M$ for all $X \in TM$. It is also known that H_σ and $H_{\hat{\sigma}}$ differ by db. Now let σ and $\hat{\sigma}$ be adapted to K (Def. 3.11). Then the interior product of a vector X tangent to $\pi(K)$ with $d(j^*b)$ vanishes, because $d(j^*b)$ is the difference of 3-forms which descend to $\underline{C}$. Also, $b(X, \cdot) = \sigma(X) - \hat{\sigma}(X) \in K \cap T^*M = N^*C$. So the interior product of X with j^*b vanishes too and j^*b descends to a 2-form on $\underline{C}$. This is consistent with the fact that by Prop. 3.15 H_σ and $H_{\hat{\sigma}}$ descend to 3-forms that represent the same element of $H^3(\underline{C}, \mathbb{R})$ (namely the Ševera class of $\underline{E}$).

As an instance of how a splitting adapted to K is used to compute the Ševera class of the reduced Courant algebroid we revisit Example 3.12 of [3], because it is simple and displays how a reduced Courant algebroid can have non-trivial Ševera class even though the original one has trivial Ševera class. We will reconsider this example in Ex. 3.20 below.

Example 3.17. Let $M = C = S^3 \times S^1$, denote by ∂_t the infinitesimal generator of the action of the circle on S^3 giving rise to the Hopf bundle $p : S^3 \to S^2$, and by s the coordinate on the second factor S^1. Let $E = TM \oplus T^*M$ the untwisted (i.e. $H = 0$) Courant algebroid on M. We choose the rank-one subbundle K to be spanned by $\partial_t + ds$. Choose a connection one form α for the circle bundle $S^3 \to S^2$, and denote by $X^H \in TS^3$ the horizontal lift of a vector X on S^2. $K^\perp$ is spanned by $\{\partial_t, \partial_s - \alpha, X^H, p^*\xi, ds\}$ where X (resp. ξ) runs over all vectors (resp. covectors) on S^2. Since ds is closed the adjoint action of $\partial_t + ds$ is just the Lie derivative w.r.t. ∂_t, which kills any of $\partial_t, \alpha, X^H, p^*\xi, \partial_s, ds$. In particular $\Gamma_{bas}(K^\perp)$ spans $K^\perp$. Hence the assumptions of Thm. 3.7 are satisfied, and on $S^2 \times S^1$ we have a reduced exact Courant algebroid. Now we choose the splitting $\sigma : TM \to K^\perp$ as follows:

$$\sigma(\partial_t) = \partial_t + ds, \quad \sigma(X^H) = X^H + 0 \text{ for all } X \in TS^2, \quad \sigma(\partial_s) = \partial_s - \alpha.$$

This splitting is isotropic, its image lies in $K^\perp$ and it maps projectable vector fields to elements of $\Gamma_{bas}(K^\perp)$ as one checks directly using $[\partial_t + ds, \cdot] = \mathcal{L}_{\partial_t}$. Hence σ satisfies the conditions of Def. 3.11, i.e. it is a splitting adapted to K.

Now we compute H_σ. If X, Y are vector fields on S^2 we have $[\sigma(X^H), \sigma(Y^H)] = [X^H, Y^H] + 0 = ([X,Y]^H - F(X,Y)\partial_t) + 0$ where $F \in \Omega^2(S^2)$ is the curvature of α. Also $[\sigma(\partial_s), \sigma(X^H)] = 0 + p^*(i_X F)$, and the analog computation for other other combinations of pairs of $\sigma(\partial_t), \sigma(X^H), \sigma(\partial_s)$ is zero. From this we deduce that $H_\sigma = p^*F \wedge ds$. This form descends to the 3-form $F \wedge ds$ on $S^2 \times S^1$, and by Prop. 3.15 it represents the Ševera class of the reduced Courant algebroid $\underline{E}$.

As pointed out in [3] $F \wedge ds$ defines a non-trivial cohomology class. An "explanation" for this fact is that by Prop. 3.15 to obtain a 3-form on C that descends to a representative of the Ševera class of $\underline{E}$ we need to choose a splitting adapted to K; the trivial splitting $\hat{\sigma} : TM \to TM \oplus T^*M$, which delivers $H_{\hat{\sigma}} = 0$, fails to be one because it does not map into $K^\perp$.

3.4. **Explicit formulae in the split case.** In this subsection we consider a split exact Courant algebroid and write down in explicit terms our reduction procedure for exact Courant algebroids (Thm. 3.7).

Let E be an exact Courant algebroid over M and let C be a submanifold endowed with a coisotropic subbundle $K^\perp$ of E satisfying $\pi(K^\perp) = TC$. Assume that $\mathcal{F} := \pi(K)$ is integrable and $C/\mathcal{F}$ smooth. Now consider the case that E is equal to $(TM \oplus T^*M, [\cdot,\cdot]_H)$, where H is some closed 3-form on M. Then there is a unique bilinear form $\hat{F} : TC \times \mathcal{F} \to \mathbb{R}$ with

$$K^\perp = \{(X, \xi) : X \in TC, \xi|_{\mathcal{F}} = \hat{F}(X, \cdot)\},$$

and the restriction of $\hat{F}$ to $\mathcal{F} \times \mathcal{F}$ is skew-symmetric (Prop. 2.2 of [20]). Since the subbundle $K^\perp$ and the bilinear form $\hat{F}$ determine each other, in the following we will use interchangeably the one or the other.

Proposition 3.18. *Consider the Courant algebroid $(TM \oplus T^*M, [\cdot,\cdot]_H)$ where H is a closed 3-form on M, and let $j : C \to M$ be a submanifold endowed with a regular integrable foliation $\mathcal{F}$ so that $\underline{C} := C/\mathcal{F}$ be smooth. Let $\hat{F} : TC \times \mathcal{F} \to \mathbb{R}$ be a bilinear form which is skew-symmetric on $\mathcal{F} \times \mathcal{F}$.*

Then $\hat{F}$ induces an exact Courant algebroid on $\underline{C}$ as in Thm. 3.7 iff there exists an extension $F \in \Omega^2(C)$ of $\hat{F}$ so that $dF + j^*H$ descends to $\underline{C}$. For any $F \in \Omega^2(C)$ as above, $dF + j^*H$ descends to a 3-form representing the Ševera class of the reduced Courant algebroid.

Remark 3.19. In the course of the proof and later on we will use the following fact which holds for any 2-form F on C and follows by a straight-forward computation using eq. (3): if $X_i + \xi_i$ are sections of the maximal isotropic subbundle $\tau_C^F := \{(X, \xi) \in TC \oplus T^*M|_C : \xi|_{TC} = i_X F\}$ then

$$2\langle [X_1 + \xi_1, X_2 + \xi_2]_H, X_3 + \xi_3 \rangle = (j^*H + dF)(X_1, X_2, X_3). \tag{9}$$

Proof. Suppose that there exists an extension $F \in \Omega^2(C)$ of $\hat{F}$ so that $dF + j^*H$ descends to $\underline{C}$. Let $B \in \Omega^2(M)$ any extension of F, and σ the induced splitting of $TM \oplus T^*M$ (so $\sigma(Y) = (Y, i_Y B)$ for $Y \in TM$). We show now that σ is a splitting adapted to K; then by Prop. 3.13 we can conclude that $\hat{F}$ induces an exact Courant algebroid over $\underline{C}$. To check that σ is an adapted splitting, notice first that $\sigma(TC) \subset K^\perp$ because B extends $\hat{F}$. Pick a projectable vector field Y on C; we want to show that $\langle [k, (Y, i_Y B)], e\rangle = 0$ for all $k \in \Gamma(K), e \in \Gamma(K^\perp)$. Since $K = \sigma(\mathcal{F}) + N^*C$ by Remark 3.12 and N^*C is spanned by closed 1-forms (which act trivially via the bracket), we may assume that $k = (X, i_X B)$ for some vector field $X \subset \mathcal{F}$. Since $K^\perp = \sigma(TC) + \mathcal{F}^\circ$ we can write $e = (Z, i_Z B) + \xi$ where $Z \in TC$ and $\xi \in \mathcal{F}^\circ$. Since the tangent component of $[(X, i_X B), (Y, i_Y B)]$ lies in $\mathcal{F}$ (because Y is a projectable vector field), we are left with showing that $\langle [(X, i_X B), (Y, i_Y B)], (Z, i_Z B)\rangle$ vanishes. Since $B \in \Omega^2(M)$ is an extension of $F \in \Omega^2(C)$, by (9) this expression is equal to $\frac{1}{2}(dF + j^*H)(X, Y, Z)$, which vanishes since we assumed that $dF + j^*H$ descends to $\underline{C}$. So we showed that σ is a splitting adapted to K.

Conversely, let us assume that $\hat{F}$ induces an exact Courant algebroid $\underline{E}$ on $\underline{C}$ as in Thm. 3.7. By Prop. 3.13 there exists a splitting $\sigma : TM \to TM \oplus T^*M$ adapted to K, which is necessarily of the form $\sigma(Y) = (Y, i_Y B)$ for some $B \in \Omega^2(M)$. As above, the fact that $\sigma(TC) \subset K^\perp$ means that B extends $\hat{F}$. Since σ is an adapted splitting, by Prop. 3.15 $j^*(H_\sigma) \in \Omega^3(C)$ descends (to a representative of the Ševera class of $\underline{E}$). By definition of H_σ we have

$$j^*(H_\sigma)(X, Y, Z) = 2\langle [(X, i_X B), (Y, i_Y B)], (Z, i_Z B)\rangle,$$

which together with eq. (9) shows that $j^*(H_\sigma)$ is equal to $dF + j^*H$, where $F = j^*B$; hence $dF + j^*H$ descends.

To conclude the proof of the theorem notice that, as we showed in the first half of the proof, any extension $F \in \Omega^2(C)$ of $\hat{F}$ so that $dF + j^*H$ descends to $\underline{C}$ is the restriction of a $B \in \Omega^2(M)$ corresponding to a splitting adapted to K. □

Example 3.20. Consider again Example 3.17: $M = C = S^3 \times S^1$, $H = 0$, and K is spanned by $\partial_t + ds$ where ∂_t is the infinitesimal generator of the action of the circle on S^3 (giving rise to the Hopf bundle $S^3 \to S^2$) and s the coordinate on the second factor S^1. $K^\perp$ corresponds to $\hat{F} : TM \times \mathbb{R}\partial_t \to \mathbb{R}$ given by $-ds \otimes (\alpha|_{\mathbb{R}\partial_t})$, where α is a connection one form for the circle bundle $S^3 \to S^2$. $\hat{F}$ extends to $F = \alpha \wedge ds \in \Omega^2(M)$, and dF descends to $F_\alpha \wedge ds$ on $S^2 \times S^1$ (where $F_\alpha \in \Omega^2(S^2)$ is the curvature of α), which by Prop. 3.18 represents the Ševera class of the the Courant algebroid obtained reducing $(TM \oplus T^*M, [\cdot,\cdot]_0)$ via the subbundle K.

In the above example one sees easily that any exact Courant algebroid on $S^2 \times S^1$ can be obtained from $(TM \oplus T^*M, [\cdot,\cdot]_0)$ via reduction, where $M = S^3 \times S^1$. Indeed (adopting the notation of the example above) any class $[\tilde{H}]$ in $H^3(S^2 \times S^1, \mathbb{R})$ has a representative of the form $\lambda F_\alpha \wedge ds$ for some $\lambda \in \mathbb{R}$, and restricting $F := \lambda\alpha \wedge ds \in \Omega^2(M)$ to $TM \times \mathbb{R}\partial_t$ gives rise to a subbundle $K^\perp \subset TM \oplus T^*M$ which by Prop. 3.18 produces by reduction the desired $[\tilde{H}]$-twisted Courant algebroid.

This is an instance of the following

Proposition 3.21. *Let M be a manifold endowed with an integrable distribution $\mathcal{F}$ so that $\underline{M} := M/\mathcal{F}$ is smooth, denote by p the projection, and let $H \in \Omega^3(M)$. The Ševera classes of the Courant algebroids on $\underline{M}$ obtained from $(TM \oplus T^*M, [\cdot,\cdot]_H)$ by reduction (as in Thm. 3.7) are exactly the preimages of $[H]$ under $p^* : H^3(\underline{M}, \mathbb{R}) \to H^3(M, \mathbb{R})$.*

Proof. Given any isotropic subbundle K of $(TM \oplus T^*M, [\cdot,\cdot]_H)$ with $\pi(K) = \mathcal{F}$ and satisfying the assumption of Thm. 3.7, choose an adapted splitting σ. By Prop. 3.15 the curvature H_σ of the splitting descends to a 3-form $\underline{H_\sigma}$ representing the Ševera class of the reduced Courant algebroid of $\underline{M}$, and $p^*[\underline{H_\sigma}] = [\overline{H_\sigma}] = [H]$.

Conversely let $\tilde{H}$ be a 3-form on $\underline{M}$ so that $p^*[\tilde{H}] = [H]$. This means that there exists a 2-form F on M so that $dF + H = p^*\tilde{H}$. In particular $dF + H$ descends, and by Prop. 3.18 $\hat{F} := F|_{TC\times\mathcal{F}}$ corresponds to a coisotropic subbundle $K^\perp$ of $(TM \oplus T^*M, [\cdot,\cdot]_H)$ which by reduction produces an exact Courant algebroid on $\underline{M}$ with Ševera class $[\tilde{H}]$. □

4. The case of Dirac structures

Let E be an exact Courant algebroid over M. Recall [7] that a *Dirac structure* is a maximal isotropic subbundle of E which is closed under the Courant bracket. Now we let C be a submanifold of M and consider a maximal isotropic subbundle $L \subset E$ defined over C (not necessarily satisfying $\pi(L) \subset TC$).

The following is analog to Thm. 4.2 of [3].

Proposition 4.1 (Dirac reduction)**.** *Let $E \to M$ and $K \to C$ satisfy the assumptions of Thm. 3.7, so that we have an exact Courant algebroid $\underline{E} \to \underline{C}$. Let L be a maximal isotropic subbundle of $E|_C$ such that $L \cap K^\perp$ has constant rank, and assume that*

$$[\Gamma(K), \Gamma(L \cap K^\perp)] \subset \Gamma(L + K). \tag{10}$$

Then L descends to a maximal isotropic subbundle $\underline{L}$ of $\underline{E} \to \underline{C}$. If furthermore

$$[\Gamma_{bas}(L \cap K^\perp), \Gamma_{bas}(L \cap K^\perp)] \subset \Gamma(L + K). \tag{11}$$

then $\underline{L}$ is an (integrable) Dirac structure. Here $\Gamma_{bas}(L \cap K^\perp) := \Gamma(L) \cap \Gamma_{bas}(K^\perp)$

Proof. At every $p \in C$ we have a Lagrangian relation[9] between E_p and $(K^\perp/K)_p$ given by $\{(e, e + K_p) : e \in K_p^\perp\}$. The image of L_p under this relation, which we denote by $\underline{L}(p)$, is maximal isotropic because L_p is. Doing this at every point of C we obtain a maximal isotropic subbundle of $K^\perp/K$, which is furthermore smooth because $\underline{L}(p)$ is the image of $(L \cap K^\perp)_p$, which has constant rank by assumption, under the projection $K_p^\perp \to (K^\perp/K)_p$.

Recall that in Thm. 3.7 we identified $(K^\perp/K)_p$ and $(K^\perp/K)_q$ when p and q lie in the same leaf of $\pi(K)$, and that the identification was induced by the Courant algebroid automorphism Φ of E obtained integrating any sequence of locally defined sections $k_1, \dots, k_n$ of K that join p to q (see Remark 3.8). Assumption (10) (together with Lemma 3.5 1)) is exactly what is needed to ensure that Φ maps $L \cap K^\perp$ into $(L + K) \cap K^\perp = (L \cap K^\perp) + K$, so that $\underline{L}(p)$ gets identified with $\underline{L}(q)$. As a consequence we obtain a well-defined smooth maximal isotropic subbundle $\underline{L}$ of the reduced Courant algebroid $\underline{E}$, i.e. an almost Dirac structure for $\underline{E}$. Now assume that (11) holds, and take two sections of $\underline{L}$, which by abuse of notation we denote $\underline{e}_1, \underline{e}_2$. Since $L \cap K^\perp$ has constant rank we can lift them to sections e_1, e_2 of $\Gamma_{bas}(L \cap K^\perp)$. As for all elements of $\Gamma_{bas}(K^\perp)$ their bracket lies in $\Gamma_{bas}(K^\perp)$, and by assumption it also lies in $L+K$, so $[e_1, e_2]$ is a basic section of $(L+K) \cap K^\perp = (L \cap K^\perp) + K$. Its projection under $K^\perp/K \to \underline{E}$, which is by definition the bracket of $\underline{e}_1$ and $\underline{e}_2$, lies then in $\underline{L}$. □

[9] A lagrangian (or canonical) relation between two vector spaces V, W endowed with (ever or odd) symplectic forms σ_V, σ_W is a maximal isotropic subspace of $(V, \sigma_V) \times (W, -\sigma_W)$.

Example 4.2 (Coisotropic reduction). Let (M, Π) be a Poisson manifold and C a coisotropic submanifold[10]. It is known [7] that the characteristic distribution $\mathcal{F} := \sharp N^*C$ is a singular integrable distribution; assume that it is regular and the quotient $\underline{C} = C/\mathcal{F}$ be smooth. It is known that $D = \{(\sharp\xi, \xi) : \xi \in T^*P\}$ is a Dirac structure for the standard (i.e. $H = 0$) Courant algebroid $TM \oplus T^*M$. By Example 3.10, choosing $K = \mathcal{F} \oplus N^*C$, we know that we can reduce this Courant algebroid and obtain the standard Courant algebroid on $\underline{C}$.

Using Prop. 4.1 now we show that $L := D|_C$ also descends. $L \cap K^\perp$ has constant rank since it's isomorphic to $\mathcal{F}^\circ$. To check (10) we use the fact that K is spanned by closed 1-forms and hamiltonian vector fields of functions vanishing on C. The former act trivially, the latter (acting by Lie derivative) map $\Gamma(L)$ to itself because hamiltonian vector fields preserve the Poisson structure. An arbitrary section of K maps $\Gamma(L \cap K^\perp)]$ to $\Gamma(L+K)$ by the "Leibniz rule in the first entry" (see Section 2), so (10) is satisfied. Further it's known [7] that the integrability of Π is equivalent to $\Gamma(D)$ being closed under the Courant bracket, so (11) holds. Hence Prop. 4.1 tells us that Π descends to a Dirac structure on $\underline{C}$. This of course is the well-known Poisson structure on $\underline{C}$ determined by $pr^*\{\underline{f}_1, \underline{f}_2\} = \{f_1, f_2\}|_C$, where $pr : C \to \underline{C}$ and f_i is any extension on $pr^*\underline{f}_i$ to M.

Prop. 4.1 allows us to interpret some results of Section 3 in a more conceptual way.

Remark 4.3. Let E be an exact Courant algebroid over M, and $K \to C$ a subbundle satisfying the assumptions of Thm. 3.7. Then, by the prescription $\sigma \mapsto L := \sigma(TM)$, splittings σ adapted to K correspond exactly to subbundles $L \subset E$ with $\pi(L) = TM$ satisfying (compare with Def. 3.11)

a) L is maximal isotropic
b) $\pi(L \cap K^\perp) = TC$
c) $[\Gamma(K), \Gamma(L \cap K^\perp)] \subset \Gamma(L+K)$, which is just eq. (10).

In particular L satisfies the assumptions of Prop. 4.1 and therefore descends to a maximal isotropic subbundle $\underline{L}$ of $\underline{E}$. Because of $\pi(L \cap K^\perp) = TC$ it is clear that the anchor maps $\underline{L}$ onto $T\underline{C}$, hence $\underline{L}$ corresponds to an isotropic splitting of the reduced Courant algebroid $\underline{E}$. This splitting is just the splitting $\underline{\sigma}$ constructed n Prop. 3.15. Remark 4.4 below will make clear that the induced splitting $\underline{\sigma}$ doesn't depend on the whole of L but actually depends only on j^*L, the pullback of L to C. This explains also why Prop. 3.18 involves only a 2-form F on C (which encodes j^*L).

Now we comment on why we chose to perform our reductions (Thm. 3.7 and Prop. 4.1) directly and not by first pulling back our subbundles to the submanifold C.

Remark 4.4. Let E be an exact Courant algebroid over M and C a submanifold of M. Then with $\hat{K} = N^*C$ (and $\hat{K}^\perp = \pi^{-1}(TC)$) the assumptions of Thm. 3.7 are satisfied; indeed all the sections of $\hat{K}^\perp$ are basic. Hence we recover Lemma 3.7 of [3], which says that $E_C := \hat{K}^\perp/\hat{K}$ is an exact Courant algebroid over C.

Now let K be an isotropic subbundle of E over C such that $\pi(K^\perp) = TC$. For any $p \in C$ we have the inclusion of coisotropic subspaces $K_p^\perp \subset \hat{K}_p^\perp$. Hence, applying (the odd version of) symplectic reduction in stages we know that $K_p^\perp/K_p \cong i^*K_p^\perp/i^*K_p$ as vector spaces with non-degenerate symmetric pairing, where i^*K_p denotes the (isotropic) subspace of $(E_C)_p$ given by the image of K_p under $\hat{K}_p^\perp \to \hat{K}_p^\perp/\hat{K}_p$. Now assume that the quotient $\underline{C}$ of C by the foliation integrating $\pi(K)$ be a smooth manifold. One can check that the assumptions of Thm. 3.7 for the coisotropic subbundle $K^\perp \subset E$ and for the coisotropic

[10] This means that $\sharp N^*C \subset TC$, where $\sharp : T^*M \to TM$ is the contraction with Π.

subbundle $\hat{K}^{\perp} \subset E_C$ are equivalent, and that when they are satisfied the two reduced exact Courant algebroids over $\underline{C}$ obtained via Thm. 3.7 coincide. Hence, reducing directly $K^{\perp}$ or first restricting to C and then reducing amounts exactly to the same thing.

Now we introduce a new piece of data, namely a maximal isotropic subbundle $L \subset E|_C$ such that $L \cap K^{\perp}$ has constant rank. The restriction i^*L of L to C generally is not a smooth subbundle. If we *assume* that $L \cap N^*C$ has constant rank then i^*L is smooth. In this case using $i^*(L \cap K^{\perp}) = i^*L \cap i^*K^{\perp}$ one can show that $L \cap K^{\perp}$ has constant rank iff $i^*L \cap i^*K^{\perp}$ does, that the remaining assumptions of Prop. 4.1 (i.e. (10) and (11)) for $L \subset E$ and $i^*L \subset E_C$ are equivalent, and that the reduced Dirac structures on $\underline{C}$ coincide. Since restricting to C forces an extra assumption on L, altogether it is preferable to reduce L directly than first restricting to C.

In the next section we will consider a generalized complex structure on M, which is in particular an endomorphism $\mathcal{J}$ of E which leaves invariant $\langle \cdot, \cdot \rangle$, and ask when it descends to the Courant algebroid $\underline{E}$ induced by $K^{\perp} \subset E$. The endomorphism $\mathcal{J}$ can not generally be pulled back to C: as the composition of three Lagrangian relations[11] $E_C \sim E \sim E \sim E_C$ the endomorphism $\mathcal{J}$ will induce a Lagrangian relation from E_C to itself, but this will usually not be the graph of an honest endomorphism[12]. As it is easier to induce an endomorphism of $\underline{E}$ from one on E rather than from a Lagrangian relation on E_C , we made our constructions so to reduce directly rather than first restrict to C.

An alternative description of generalized complex structures on M is given in terms of a Dirac structure $L_{\mathbb{C}}$ in the complexification of E; however even from this perspective it is preferable to reduce directly $L_{\mathbb{C}}$ rather than than first restrict to C, in order to avoid extra assumptions on $L_{\mathbb{C}}$.

5. The case of generalized complex structures

Let E be an exact Courant algebroid over M. Recall that a generalized complex structure is a vector bundle endomorphism $\mathcal{J}$ of E which preserves $\langle \cdot, \cdot \rangle$, squares to $-Id_E$ and for which the Nijenhuis tensor

$$N_{\mathcal{J}}(e_1, e_2) := [\mathcal{J}e_1, \mathcal{J}e_2] - [e_1, e_2] - \mathcal{J}([e_1, \mathcal{J}e_2] + [\mathcal{J}e_1, e_2]). \tag{12}$$

vanishes[13].

The analog of the following proposition when a group action is present is Thm. 4.8 of [19]; we borrow the first part of our proof from them, but use different arguments to prove the integrability of the reduced generalized complex structure.

Proposition 5.1 (Generalized complex reduction)**.** *Let $E \to M$ and $K \to C$ satisfy the assumptions of Thm. 3.7, so that we have an exact Courant algebroid $\underline{E} \to \underline{C}$. Let $\mathcal{J}$ be a generalized complex structure on M such that $\mathcal{J}K \cap K^{\perp}$ has constant rank and is contained in K. Assume further that $[\Gamma(K), \mathcal{J}(\Gamma_{bas}(K^{\perp} \cap \mathcal{J}K^{\perp}))] \subset \Gamma(K)$ (i.e. that $\mathcal{J}$ applied to any basic section of $\mathcal{J}K^{\perp} \cap K^{\perp}$ is again a basic section). Then $\mathcal{J}$ descends to a generalized complex structure $\underline{\mathcal{J}}$ on $\underline{E} \to \underline{C}$.*

[11] The second relation is the graph of $\mathcal{J}$, the third one is given by $\{(e, e + \hat{K}) : e \in \hat{K}^{\perp}\}$, and similarly the first one.

[12] It is exactly when $\mathcal{J}\hat{K} \cap \hat{K}^{\perp} \subset \hat{K}$.

[13] $N_{\mathcal{J}}$ coincides with the Nijenhuis tensor of $\mathcal{J}$ written with the skew-symmetrized Courant bracket, more commonly found in the literature, as can be seen using that $\mathcal{J}$ preserves the symmetric pairing and squares to -1.

Remark 5.2. The linear algebra conditions on $\mathcal{J}K \cap K^{\perp}$ are in particular satisfied when $\mathcal{J}$ preserves K (in which case the proof below simplifies quite a bit as well), for in that case $\mathcal{J}K \cap K^{\perp} = K$. The opposite extreme case is when $\mathcal{J}K \cap K^{\perp} = \{0\}$.

Proof. First we show that $\underline{J}$ induces a smooth[14] endomorphism of the vector bundle $K^{\perp}/K$ over C. Indeed $\mathcal{J}K \cap K^{\perp} \subset K$ is equivalent to $\mathcal{J}K^{\perp} + K \supset K^{\perp}$, so that $K^{\perp} = K^{\perp} \cap (\mathcal{J}K^{\perp} + K) = (K^{\perp} \cap \mathcal{J}K^{\perp}) + K$. From this it is clear that $K^{\perp} \cap \mathcal{J}K^{\perp}$ maps surjectively under $\Pi : K^{\perp} \to K^{\perp}/K$. Since $ker(\Pi|_{K^{\perp} \cap \mathcal{J}K^{\perp}}) = (K^{\perp} \cap \mathcal{J}K^{\perp}) \cap K = K \cap \mathcal{J}K^{\perp}$, by our constant rank assumption we obtain a smooth vector bundle $K^{\perp} \cap \mathcal{J}K^{\perp}/ker(\Pi|_{K^{\perp} \cap \mathcal{J}K^{\perp}})$ canonically isomorphic to $K^{\perp}/K$.

We use again the assumption $\mathcal{J}K \cap K^{\perp} \subset K$, interpreting it as follows: if e lies in the kernel of $\Pi : K^{\perp} \to K^{\perp}/K$ and $\mathcal{J}e \in K^{\perp}$ then $\mathcal{J}e$ is still in the kernel. This applies in particular to all $e \in ker(\Pi|_{K^{\perp} \cap \mathcal{J}K^{\perp}})$ (since $K^{\perp} \cap \mathcal{J}K^{\perp}$ is $\mathcal{J}$-invariant), so we deduce that $\mathcal{J}$ leaves $ker(\Pi_{K^{\perp} \cap \mathcal{J}K^{\perp}})$ invariant, i.e. $\mathcal{J}$ induces a well-defined endomorphism on $K^{\perp} \cap \mathcal{J}K^{\perp}/ker(\Pi|_{K^{\perp} \cap \mathcal{J}K^{\perp}}) \cong K^{\perp}/K$. Further it is clear that it squares to -1 and preserves the induced symmetric pairing on $K^{\perp}/K$.

Now take a section $\underline{e}$ of $\underline{E}$, lift it to a (automatically basic) section e of $K^{\perp} \cap \mathcal{J}K^{\perp}$. Then by assumption $\mathcal{J}e$ is again a basic section; this shows that the endomorphism on $K^{\perp} \cap \mathcal{J}K^{\perp}/ker(\Pi|_{K^{\perp} \cap \mathcal{J}K^{\perp}})$ descends to an endomorphism $\underline{\mathcal{J}}$ of $\underline{E}$.

We are left with showing that $\underline{\mathcal{J}}$ is integrable, i.e. with showing that the Nijenhuis tensor $N_{\underline{\mathcal{J}}}$ vanishes. Let $\underline{e}_1, \underline{e}_2$ be elements of $\underline{E}$, extend them to local sections and pull them back to basic sections e_1, e_2 of $K^{\perp} \cap \mathcal{J}K^{\perp}$. We claim that $N_{\mathcal{J}}(e_1, e_2)$ is a lift of $N_{\underline{\mathcal{J}}}(\underline{e}_1, \underline{e}_2)$; since the former vanishes, the latter vanishes too and we are done.

To prove our claim we reason as follows. By the definition of $\underline{\mathcal{J}}$ we known that $\mathcal{J}e_i \in \Gamma_{bas}(K^{\perp} \cap \mathcal{J}K^{\perp})$ is a lift of $\underline{\mathcal{J}}(\underline{e}_i)$, hence the four Courant brackets of sections appearing on the r.h.s. of (12) are lifts of the analogous brackets in $\underline{E}$. Since $\Gamma_{bas}(K^{\perp})$ is closed under the Courant bracket we know that the term

$$([e_1, \mathcal{J}e_2] + [\mathcal{J}e_1, e_2]) \tag{13}$$

of (12) lies[15] in $\Gamma_{bas}(K^{\perp})$. However, to conclude that applying $\underline{\mathcal{J}}$ to (13) we obtain a lift of the analogous term in $\underline{E}$ (and hence that $N_{\mathcal{J}}(e_1, e_2)$ is a lift of $N_{\underline{\mathcal{J}}}(\underline{e}_1, \underline{e}_2)$), we still need to show that (13) is a section of $\mathcal{J}K^{\perp}$, because then it will lie in $\mathcal{J}K^{\perp} \cap K^{\perp}$ which is where we let $\mathcal{J}$ act to define $\underline{\mathcal{J}}$. To this aim pick a section k of K, and apply the Leibniz rule C4) to $\pi(e_1)\langle \mathcal{J}e_2, \mathcal{J}k\rangle$ and to $\pi(\mathcal{J}e_1)\langle e_2, \mathcal{J}k\rangle$, both of which vanish because $e_2, \mathcal{J}e_2 \subset \mathcal{J}K^{\perp} \cap K^{\perp} = (K + \mathcal{J}K)^{\perp}$ and $\pi(e_1), \pi(\mathcal{J}e_1) \subset \pi(K^{\perp}) = TC$. Taking the sum of the two equations we obtain

$$0 = \langle [e_1, \mathcal{J}e_2] + [\mathcal{J}e_1, e_2], \mathcal{J}k\rangle + \langle e_2, -\mathcal{J}[e_1, \mathcal{J}k] + [\mathcal{J}e_1, \mathcal{J}k]\rangle. \tag{14}$$

Now the vanishing of $N_{\mathcal{J}}(e_1, k)$ means that $-\mathcal{J}[e_1, \mathcal{J}k] + [\mathcal{J}e_1, \mathcal{J}k] = [e_1, k] + \mathcal{J}[\mathcal{J}e_1, k]$, and the latter lies in $K + \mathcal{J}K$ because e_1 and $\mathcal{J}e_1$ are in particular basic sections of $K^{\perp}$. Hence the last term in (14) vanishes, and we deduce that $[e_1, \mathcal{J}e_2] + [\mathcal{J}e_1, e_2]$ has zero symmetric pairing with $\mathcal{J}K$, i.e. that it lies in $\mathcal{J}K^{\perp}$. ∎

In Prop. 5.1 the condition $[\Gamma(K), \mathcal{J}(\Gamma_{bas}(K^{\perp} \cap \mathcal{J}K^{\perp}))] \subset \Gamma(K)$ does not follow from the integrability of $\mathcal{J}$ (see Ex. 5.3 below for an explicit example); this is not surprising. In Section 6 we will consider submanifolds C for which the integrability of $\mathcal{J}$ does imply

[14]This is clear when $\mathcal{J}$ preserves $K^{\perp}$.

[15]This concludes the proof in the case $\mathcal{J}K^{\perp} = K^{\perp}$.

all the assumptions of Prop. 5.1, in analogy to the case of coisotropic submanifolds in the Poisson setting (see also Example 4.2).

Example 5.3 (Complex foliations). Take E to be the standard Courant algebroid and $\mathcal{J}$ be given by a complex structure J on M. Take $\mathcal{F}$ to be a real integrable distribution on M preserved by J (so J induces the structure of a complex manifold on each leaf of $\mathcal{F}$) and $K = \mathcal{F} \oplus 0$, so that $\underline{M} := M/\pi(K) = M/\mathcal{F}$ be smooth. The generalized complex structure $\mathcal{J}$ preserves K. If $\mathcal{J}$ mapped $\Gamma_{bas}(K^\perp)$ into itself[16] then by Prop. 5.1 it would follow that $\underline{M}$ would have an induced generalized complex structure. Further, it would necessarily correspond to an honest complex structure on $\underline{M}$ that makes $M \to \underline{M}$ into a holomorphic map. However there are examples for which such a complex structure on $\underline{M}$ does not exist; in [22] Winkelmann quotes an example where M is a twistor space of real dimension 6 and $\underline{M}$ is the 4-dimensional torus.

Example 5.4 (Symplectic foliations). Take again E to be the standard Courant algebroid and $\mathcal{J}$ be given by a symplectic form ω on M. Take $K = \mathcal{F}$ to be a real integrable distribution on M. One checks that $\mathcal{J}K \cap K^\perp$ is contained in K only if it is trivial, which is equivalent to saying that the leaves of $\mathcal{F}$ are symplectic submanifolds. $\mathcal{J}$ maps basic sections of $\mathcal{J}K^\perp \cap K^\perp = \mathcal{F}^\omega \oplus \mathcal{F}^\circ$ into basic sections iff the hamiltonian vector field $X_{\pi^* f}$ is a projectable vector field for any function f, where $pr : M \to \underline{M} := M/\mathcal{F}$. When this is the case the induced generalized complex structure on $\underline{M}$ is the symplectic structure given by the isomorphism of vector spaces $\mathcal{F}^\omega_x \cong T_{pr(x)}\underline{M}$ (where $x \in M$).

Remark 5.5. We recall that a submanifold C of a Poisson manifold (M, Π) is called *coisotropic* if $\sharp N^*C \subset TC$, where $\sharp : T^*M \to TM$ is given by contraction with Π. In this case $\sharp N^*C$ is a singular integrable distribution on C, called *characteristic distribution*, and it is well-known that when it is regular and the quotient $C/\sharp N^*C$ is a smooth manifold then it has an induced Poisson structure.

It is known that a generalized complex manifold $(M, \mathcal{J})$ comes with a canonical Poisson structure Π, whose sharp map $\sharp$ is given by the composition $T^*M \hookrightarrow E \xrightarrow{\mathcal{J}} E \xrightarrow{\pi} TM$. If in Prop. 5.1 we assume that $\mathcal{J}$ preserves K, then C is a necessarily a coisotropic submanifold, because from $N^*C = (\pi(K^\perp))^\circ = K \cap ker(\pi) \subset K$ we have $\sharp(N^*C) = \pi(\mathcal{J}N^*C) \subset \pi(K) \subset \pi(K^\perp) = TC$. So $C/\sharp N^*C$ (if smooth) has an induced Poisson structure. We know that also $\underline{C} := C/\pi(K)$ has a Poisson structure, induced from the reduced generalized complex structure. In general $\pi(K)$ is *not* the characteristic distribution of C; we just have an inclusion $\sharp N^*C \subset \pi(K)$[17]. The Poisson structure on $C/\pi(K)$ is induced from the one on M in a natural way, namely $pr^*\{\underline{f}, \underline{g}\}_{\underline{C}} = \{f, g\}_M|_C$ where $pr : C \to \underline{C}$ and f, g are any extensions to M of $pr^*(\underline{f})$ and $pr^*(\underline{g})$. Indeed $df \in \Gamma(K^\perp)$, the commutativity of diagram in the proof of Thm. 3.7, and the non-degeneracy of the symmetric pairing on $\underline{E}$ imply that df is basic and indeed a lift of $d\underline{f} \in \Gamma(\underline{E})$. Hence $\mathcal{J}df$ is a lift of $\underline{\mathcal{J}}(d\underline{f})$ and

$$pr^*\{\underline{f}, \underline{g}\}_{\underline{C}} = pr^*\langle \underline{\mathcal{J}}(d\underline{f}), d\underline{g}\rangle = \langle \mathcal{J}df, dg\rangle|_C = \{f, g\}_M|_C.$$

Given an exact Courant algebroid E on M, recall that a *generalized Kähler structure* consists of two commuting generalized complex structures $\mathcal{J}_1, \mathcal{J}_2$ such that the symmetric bilinear form on E given by $\langle \mathcal{J}_1\mathcal{J}_2 \cdot, \cdot\rangle$ be positive definite. The following result borrows the

[16]This is equivalent to saying that for any vector field X on M which is projectable the vector field $J(X)$ is also projectable.

[17]A case in which this inclusion is strict is when $\mathcal{J}$ corresponds to the standard complex structure on $M = \mathbb{C}^n$ (with complex coordinates $z_k = x_k + iy_k$) and $K = span\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\}$.

proof of Thm. 6.1 of [3], except for the fact that the integrability of the reduced generalized complex structures is automatic by Prop. 5.1.

Proposition 5.6 (Generalized Kähler reduction). *Let $E \to M$ and $K \to C$ satisfy the assumptions of Prop. 3.7, so that we have an exact Courant algebroid $\underline{E} \to \underline{C}$. Let $\mathcal{J}_1, \mathcal{J}_2$ be a generalized Kähler structure on M such that $\mathcal{J}_1 K = K$. Assume further that $\mathcal{J}_1$ maps $\Gamma_{bas}(K^\perp)$ into itself and that $\mathcal{J}_2$ maps $\Gamma_{bas}(\mathcal{J}_2 K^\perp \cap K^\perp)$ into itself. Then $\mathcal{J}_1, \mathcal{J}_2$ descend to a generalized Kähler structure on $\underline{E} \to \underline{C}$.*

Proof. By Thm. 5.1 $\mathcal{J}_1$ induces a generalized complex structure $\underline{\mathcal{J}_1}$ on $\underline{E}$. The orthogonal $K^{\mathcal{G}}$ of K w.r.t. $\langle \mathcal{J}_1 \mathcal{J}_2 \cdot, \cdot \rangle$ is $(\mathcal{J}_2 \mathcal{J}_1 K)^\perp = \mathcal{J}_2 K^\perp$. Because of the identity $K^\perp = K \oplus (K^{\mathcal{G}} \cap K^\perp)$ the restriction to $\mathcal{J}_2 K^\perp \cap K^\perp$ of the projection $K^\perp \to K^\perp / K$ is an isomorphism. So we can apply Prop. 5.1 to $\mathcal{J}_2$ and obtain a generalized complex structure $\underline{\mathcal{J}_2}$ on $\underline{E}$. Notice that both $\mathcal{J}_1$ and $\mathcal{J}_2$ preserve $\mathcal{J}_2 K^\perp \cap K^\perp$; pulling back sections of $\underline{E}$ to basic sections of $\mathcal{J}_2 K^\perp \cap K^\perp$ one sees that $\underline{\mathcal{J}_1}, \underline{\mathcal{J}_2}$ form a generalized Kähler structure on $\underline{E}$. ☐

6. The case of (weak) branes

In this section we define branes and show that they admit a natural quotient which is a generalized complex manifold endowed with a space-filling brane. Then we notice that quotients of more general objects, which we call "weak branes", also inherit a generalized complex structure; examples of weak branes are coisotropic submanifolds in symplectic manifolds. Finally we show how weak branes can be obtained by passing from a generalized complex manifold to a suitable submanifold.

6.1. **Reducing branes.**

Definition 6.1. Let E be an exact Courant algebroid over a manifold M. A *generalized submanifold* is a pair (C, L) consisting of a submanifold $C \subset M$ and a maximal isotropic subbundle $L \subset E$ over C with $\pi(L) = TC$ which is closed under the Courant bracket (i.e. $[\Gamma(L), \Gamma(L)] \subset \Gamma(L)$ with the conventions of Remark 3.2).

We show that this definition, which already appeared in the literature[18], is just a splitting-independent rephrasing[19] of Gualtieri's original definition (Def. 7.4 of [9]). See also Lemma 3.2.3 of [13].

Lemma 6.2. *Let E be an exact Courant algebroid over M. Choose an isotropic splitting σ for E, giving rise to an isomorphism of Courant algebroids $(E, [\cdot,\cdot]) \cong (TM \oplus T^*M, [\cdot,\cdot]_{H_\sigma})$ where H_σ is the curvature 3-form of the splitting (see Section 2). Then pairs (C, L) as in Def. 6.1 correspond bijectively to pairs (C, F), where $F \in \Omega^2(C)$ satisfies $-i^* H_\sigma = dF$ (for i the inclusion of C in M).*

Proof. The fact that $L \subset E$ is maximal isotropic and $\pi(L) = TC$ means that under the isomorphism it maps to

$$\tau_C^F := \{(X, \xi) \in TC \oplus T^*M|_C : \xi|_{TC} = i_X F\}$$

for some 2-form F on C. The correspondence $L \leftrightarrow F$ is clearly bijective. Equation (9) shows that the integrability conditions also correspond.

[18]It appeared in Def. 3.2.2 of [13] with the name "maximally isotropic extended submanifold". Also, a subbundle L as above but for which we just ask $\pi(L) \subset TC$ is called generalized Dirac structure in Def. 6.8 of [1] (in the setting of the skew-symmetric Courant bracket).

[19]Up to a sign, since Def. 7.4 of [9] requires $i^* H_\sigma = dF$ (in the notation of this lemma).

□

By Lemma 6.2 the following definition is equivalent to Gualtieri's original one (i.e. to Def. 7.6 of [9], again up to a sign):

Definition 6.3. Let E be an exact Courant algebroid over a manifold M and $\mathcal{J}$ be a generalized complex structure on E. A *generalized complex submanifold* or *brane* is a generalized submanifold (C, L) satisfying $\mathcal{J}(L) = L$.

Now we state the main theorem of this paper. Recall that we gave the definition of coisotropic submanifold in Remark 5.5.

Theorem 6.4 (Brane reduction). *Let E be an exact Courant algebroid over a manifold M, $\mathcal{J}$ a generalized complex structure on E, and (C, L) a brane. Then C is coisotropic w.r.t. the Poisson structure induced by $\mathcal{J}$ on M. If the quotient $\underline{C}$ of C by its characteristic foliation is smooth, then*

a) *E induces an exact Courant algebroid $\underline{E}$ over $\underline{C}$*
b) *$\mathcal{J}$ induces a generalized complex structure $\underline{\mathcal{J}}$ on $\underline{E} \to \underline{C}$*
c) *L induces the structures of a space-filling brane on $\underline{C}$ and the Ševera class of $\underline{E}$ is trivial.*

Proof. Recall that the Poisson structure Π induced by $\mathcal{J}$ on M (or rather its sharp map $\sharp$) is given by the composition $T^*M \hookrightarrow E \xrightarrow{\mathcal{J}} E \xrightarrow{\pi} TM$. Since $N^*C = (\pi(L))^\circ = L \cap ker(\pi) \subset L$ we have $\sharp(N^*C) = \pi(\mathcal{J}N^*C) \subset \pi(L) = TC$, so C is a coisotropic submanifold. As above we let $\mathcal{F} := \sharp N^*C$, assume that it be a regular distribution and that $\underline{C} := C/\mathcal{F}$ be a smooth manifold.

a) C, L and $\mathcal{F}$ satisfy the assumptions of Prop. 3.14. Hence we can apply Thm. 3.7 with $K := L \cap \pi^{-1}(\mathcal{F})$ and obtain an exact Courant algebroid $\underline{E}$ over $\underline{C}$. Notice that we have not made use of the integrability of $\mathcal{J}$ here, if not for the fact that the induced bivector Π is integrable and hence the distribution $\mathcal{F}$ is involutive.

b) Now we check that the assumptions of Prop. 5.1 are satisfied. From $L \cap T^*M = N^*C$, the fact that $\mathcal{J}N^*C$ is contained in L and that it projects onto $\mathcal{F}$ we deduce that $K = N^*C + \mathcal{J}N^*C$, which is clearly preserved by $\mathcal{J}$. So we just need to check that, for any basic section e of $K^\perp$, $\mathcal{J}e$ is again basic. Locally we can write $K = span\{(dg_i)|_C, \mathcal{J}(dg_i)|_C\}$ where $g_1, \dots, g_{codim(C)}$ are local functions on M vanishing on C. Since each dg_i is a closed one form, $[(dg_i)|_C, \mathcal{J}e] \subset K$. Using the fact that the Nijenhuis tensor $N_{\mathcal{J}}$ vanishes (12) we have

$$[\mathcal{J}(dg_i)|_C, \mathcal{J}e] = \mathcal{J}[\mathcal{J}(dg_i)|_C, e] + \mathcal{J}[(dg_i)|_C, \mathcal{J}e] + [(dg_i)|_C, e].$$

The first term on the r.h.s. lies in K because e is a basic section, and the last two because dg_i is a closed 1-form. So $[\mathcal{J}(dg_i)|_C, \mathcal{J}e] \subset K$, hence e is again a basic section. Hence the assumptions of Prop. 5.1 are satisfied, concluding the proof of b).

c) We want to apply Prop. 4.1 to obtain a brane on $\underline{C}$. Since $L \subset K^\perp$ the assumption (10) needed for L to descend reads $[\Gamma(K), \Gamma(L)] \subset \Gamma(L)$, and the integrability assumption (11) reads $[\Gamma_{bas}(L), \Gamma_{bas}(L)] \subset \Gamma(L)$. As L is closed under the bracket both assumptions hold, and we obtain an (integrable) Dirac structure $\underline{L}$ on $\underline{C}$. Furthermore from the fact that $\mathcal{J}$ preserves L we see that $\underline{\mathcal{J}}$ preserves $\underline{L}$. Hence $(\underline{C}, \underline{L})$ is a brane for the generalized complex structure $\underline{\mathcal{J}}$ on $\underline{E}$.

If we chose any isotropic splitting for $\underline{E}$, as discussed in Lemma 6.2, then $\underline{L}$ gives rise to a 2-form $\hat{F}$ on $\underline{C}$ such that $-d\hat{F}$ equals the curvature of the splitting, which hence is an exact 3-form. This concludes the proof of c) and of the theorem.

□

Remark 6.5. We saw in Thm. 6.4 that branes C are coisotropic and their quotient by the characteristic foliation is endowed with a generalized complex structure. As pointed out in Remark 5.5, if one starts with a $\mathcal{J}$-invariant coisotropic subbundle $K^{\perp}$ of $E|_C$ (instead of constructing one from the brane (C, L) as in Thm. 6.4) in general it is a different quotient of C that is endowed with a generalized complex structure (via Prop. 5.1). If one picks just any arbitrary coisotropic submanifold C, its quotient by the characteristic foliation inherits a Poisson structure, but in general it does not inherit a generalized complex structure: take for example any odd dimensional submanifold of a complex manifold.

Remark 6.6. When the characteristic foliation of a brane $(C, L) \subset M$ is regular, using coordinates adapted to the foliation one sees that the quotient of small enough open sets U of C by the characteristic foliation is smooth, and Thm. 6.4 gives a local statement. However in general the characteristic foliation is singular, as the following example shows. Take $M = \mathbb{C}^2$, the untwisted exact Courant algebroid as E, and as $\mathcal{J}$ take $\begin{pmatrix} I & \Pi \\ 0 & -I^* \end{pmatrix}$. Here $I(\partial_{x_i}) = \partial_{y_i}$ is the canonical complex structure on $\mathbb{C}^2$ and $\Pi = y_1(\partial_{x_1} \wedge \partial_{x_2} - \partial_{y_1} \wedge \partial_{y_2}) - x_1(\partial_{y_1} \wedge \partial_{x_2} + \partial_{x_1} \wedge \partial_{y_2})$ is the imaginary part of the holomorphic Poisson bivector (see [8][10]) $z_1 \partial_{z_1} \wedge \partial_{z_2}$. It is easy to check that $C = \{z_2 = 0\}$ with $F = 0$ define a brane for $\mathcal{J}$, and that the characteristic distribution of C has rank zero at the origin and rank 2 elsewhere.

Example 6.7 (Branes in symplectic manifolds [9]). Consider a symplectic manifold (M, ω) and view it as a generalized complex structure on the standard Courant algebroid. Example 7.8 of [9] states that if a generalized submanifold (C, F) (so F is a closed 2-form on C) is a brane then F descends to the quotient $\underline{C}$ (which we assume to be smooth), and $\underline{F} + i\underline{\omega}$ is a holomorphic symplectic form on $\underline{C}$.

Remark 6.8. Suppose that in the setting of Thm. 6.4 E is additionally endowed with some $\mathcal{J}_2$ so that $\mathcal{J}_1, \mathcal{J}_2$ form a generalized Kähler structure. Then using Prop. 5.6 we see that *if* $\mathcal{J}_2$ descends to $\underline{E}$ then $\underline{E}$ is endowed with a generalized Kähler structure too.

6.2. **Reducing weak branes.** We weaken the conditions in the definition of brane; at least for the time being, we refer to resulting object as "weak branes".

Definition 6.9. Let E be an exact Courant algebroid over a manifold M, $\mathcal{J}$ a generalized complex structure on E. We will call *weak brane* a pair (C, L) consisting of a submanifold C and a maximal isotropic subbundle $L \subset E|_C$ with $\pi(L) = TC$ such that

$$\mathcal{J}(N^*C) \subset L, \qquad [\Gamma(K), \Gamma(L)] \subset \Gamma(L) \tag{15}$$

(where $K := L \cap \pi^{-1}(\mathcal{F})$ and $\mathcal{F} := \sharp N^*C$, or equivalently $K = N^*C + \mathcal{J}N^*C$.)

Notice that weak branes for which $\mathcal{F}$ has constant rank automatically satisfy the assumptions of Prop. 3.14. Also notice that in the proof of Thm. 6.4 (except for c)) we just used properties of weak branes, hence we obtain

Proposition 6.10. *If in Thm. 6.4 we let (C, L) be a weak brane then C is a coisotropic submanifold and a) and b) of Thm. 6.4 still hold, i.e. there is a reduced Courant algebroid and a reduced generalized complex structure on $\underline{C}$ (when it is a smooth manifold).*

We describe how weak branes look like in the split case, i.e. when $E = (TM \oplus T^*M, [\cdot,\cdot]_H)$. We write $\mathcal{J}$ in matrix form as $\begin{pmatrix} A & \Pi \\ \omega & -A^* \end{pmatrix}$ where A is an endomorphism of TM, Π the Poisson bivector canonically associated to $\mathcal{J}$, and ω a 2-form on M.

Corollary 6.11. *Let C be a submanifold of M and $F \in \Omega^2(C)$. Fix an extension $B \in \Omega^2(M)$ of F. Then (C, τ_C^F) is a weak brane (with smooth quotient $\underline{C}$) iff C is coisotropic (with smooth quotient $\underline{C}$), $A + \Pi B : TM \to TM$ preserves TC , and the 3-form $dF + i^*H$ on C descends to $\underline{C}$.*

In this case the Ševera class of the reduced Courant algebroid $\underline{E}$ is represented by the pushforward of $dF + i^*H$. Further there is a splitting of $\underline{E}$ in which the reduced generalized complex structure is

$$\underline{\tilde{\mathcal{J}}} = \begin{pmatrix} \underline{\tilde{A}} & \underline{\tilde{\Pi}} \\ \underline{\tilde{\omega}} & -\underline{\tilde{A}}^* \end{pmatrix},$$

where the endomorphism $\underline{\tilde{A}}$ is the pushforward of $(A + \Pi B)|_{TC}$, the Poisson bivector $\underline{\tilde{\Pi}}$ is induced by Π, and the 2-form $\underline{\tilde{\omega}}$ is the pushforward of $i^*(\omega - B\Pi B - BA - A^*B)$.

Proof. Since K is $\tau_C^F \cap \pi^{-1}(\mathcal{F})$ equation (9) shows that $[\Gamma(K), \Gamma(\tau_C^F)] \subset \Gamma(\tau_C^F))$ is equivalent to the fact that the closed 3-form $i^*H + dF$ descend to $\underline{C}$. Now perform a $-B$-transformation; the transformed objects are $\tilde{L} = TC \oplus N^*C$ and $\tilde{\mathcal{J}} = \begin{pmatrix} \tilde{A} & \tilde{\Pi} \\ \tilde{\omega} & -\tilde{A}^* \end{pmatrix}$, with components $\tilde{A} = A + \Pi B$, $\tilde{\Pi} = \Pi$ and $\tilde{\omega} = \omega - B\Pi B - BA - A^*B$ (see for example [21]). Hence we see that the first condition in (15) is equivalent to C begin coisotropic and $A + \Pi B$ preserving TC (a condition independent of the extension B). Further, since by the proof of Thm. 6.4 $\mathcal{J}$ preserves $TC \oplus \mathcal{F}^\circ$ and $\mathcal{F} \oplus N^*C$, it is clear that in the induced splitting of $\underline{E}$ the components of $\underline{\tilde{\mathcal{J}}}$ are induced from those of $\tilde{\mathcal{J}}$.

Since we saw that $dF + i^*H$ descend to $\underline{C}$, by Prop. 3.18 the Ševera class of the reduced Courant algebroid $\underline{E}$ is represented by the pushforward of $dF + i^*H$. □

We use the characterization of Cor. 6.11 in the following examples.

Example 6.12 (Coisotropic reduction). If $\mathcal{J}$ corresponds to a symplectic structure on M, then any coisotropic submanifold C endowed with $F = 0$ is a weak brane. The generalized complex structure on $\underline{C}$ (assumed to be a smooth manifold) corresponds to the reduced symplectic form.

If $\mathcal{J}$ corresponds to a complex structure, then any weak brane is necessarily a complex submanifold. If $\mathcal{J}$ is obtained deforming a complex structure in direction of a holomorphic Poisson structure [8][10] this is no longer the case, as in the following two examples. In both cases however the reduced generalized complex structures we obtain are quite trivial.

Example 6.13. Similarly to Remark 6.6 take M to be the open halfspace $\{(x_1, y_1, x_2, y_2) : y_1 > 0\} \subset \mathbb{C}^2$, the untwisted exact Courant algebroid as E, and as $\mathcal{J}$ take $\begin{pmatrix} I & \Pi \\ 0 & -I^* \end{pmatrix}$ where $I(\partial_{x_i}) = \partial_{y_i}$ is the canonical complex structure on $\mathbb{C}^2$ and $\Pi = y_1(\partial_{x_1} \wedge \partial_{x_2} - \partial_{y_1} \wedge \partial_{y_2}) - x_1(\partial_{y_1} \wedge \partial_{x_2} + \partial_{x_1} \wedge \partial_{y_2})$ is the imaginary part of the holomorphic Poisson bivector $z_1 \partial_{z_1} \wedge \partial_{z_2}$. We now take $C = \{(x_1, y_1, x_2, 0) : y_1 > 0\}$ and on C the closed 2-form $F := -\frac{1}{y_1} dy_1 \wedge dx_2$. We show that the pair (C, F) forms a weak brane. By dimension reasons C is coisotropic (the characteristic distribution is regular and spanned by $x_1 \partial_{x_1} + y_1 \partial_{y_1}$), so we just have to check that $I+\Pi B$ preserves TC, where B the 2-form on M given by the same formula as F. This is true as one computes $I+\Pi B: \partial_{x_1} \mapsto \partial_{y_1},\ \ \partial_{y_1} \mapsto -\frac{x_1}{y_1}\partial_{y_1},\ \ \partial_{x_2} \mapsto -\frac{x_1}{y_1}\partial_{x_2}$.

Now we want to compute the generalized complex structure on $\underline{C}$ given by Prop. 6.10, We do so by first applying the gauge transformation by $-B$ to obtain a generalized complex structure $\tilde{\mathcal{J}}$ and then using the diffeomorphism $\underline{C} \cong (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}$ induced by $C \to (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}, (x_1,y_1,x_2) \mapsto (\theta := arctg(\frac{x_1}{y_1}), x_2)$. The Poisson bracket of the coordinate functions θ and x_2 on $\underline{C}$ is computed by pulling back the two functions to C, extending them to the whole of M and taking their Poisson bracket there. This gives the constant function 1. Next the coordinate vector field ∂_θ on $\underline{C}$ is lifted by the vector field $\frac{x_1^2+y_1^2}{y_1}\partial_{x_1}$ on C, and of course ∂_{x_2} on $\underline{C}$ is lifted by ∂_{x_2} on C. Applying the endomorphism $I+\Pi B$ of TC we see the induced endomorphism on $T\underline{C}$ is just multiplication by $-tg(\theta)$. Finally, the component $\tilde{\omega}$ of $\tilde{\mathcal{J}}$ is given by $-BI-B\Pi B-I^*B$, which on C restricts to the 2-form $\frac{1}{y_1^2}(y_1dx_1-x_1dy_1)\wedge dx_2$, which in turn is the pullback of the 2-form $(1+tg^2(\theta))d\theta\wedge dx_2$ on $\underline{C}$. Hence the induced generalized complex structure on $\underline{C}$ is

$$\begin{pmatrix} -tg(\theta)\cdot Id & \partial_\theta\wedge\partial_{x_2} \\ (1+tg^2(\theta))d\theta\wedge dx_2 & tg(\theta)\cdot Id \end{pmatrix}.$$

This is just the gauge transformation by the closed 2-form $tg(\theta)d\theta\wedge dx_2$ of the generalized complex structure on $(-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}$ that corresponds to the symplectic form $d\theta\wedge dx_2$.

Example 6.14. Similarly to the previous example we take $M=\mathbb{C}^2$, the untwisted exact Courant algebroid as E, and as $\mathcal{J}$ we take $\begin{pmatrix} I & \Pi \\ 0 & -I^* \end{pmatrix}$ where $I(\partial_{x_i})=\partial_{y_i}$ is the canonical complex structure on $\mathbb{C}^2$ and $\Pi = y_1(\partial_{x_1}\wedge\partial_{x_2}-\partial_{y_1}\wedge\partial_{y_2})-x_1(\partial_{y_1}\wedge\partial_{x_2}+\partial_{x_1}\wedge\partial_{y_2})$. Now we let C be the hypersurface $\{x_1^2+y_1^2=1\}$. The characteristic distribution is generated by ∂_{y_2}, so the quotient $\underline{C}$ is a cylinder. Let $a,b,c\in C^\infty(C)$ so that, denoting by $F_{(a,b,c)}$ the pullback to C of

$$B_{(a,b,c)} := a\cdot dx_1\wedge dy_1 + b\cdot dx_1\wedge dx_2 + c\cdot dy_1\wedge dx_2 - y_1\cdot dx_1\wedge dy_2 + +x_1\cdot dy_1\wedge dy_2,$$

$dF_{(a,b,c)}$ descends[20] to $\underline{C}$. One checks that $I^*+B_{(a,b,c)}\Pi$ preserves N^*C, so that $(C,F_{(a,b,c)})$ is a weak brane. A computation analog to the one of the previous example shows that the reduced generalized complex structure on $\underline{C}=S^1\times\mathbb{R}$ with coordinates θ and x_2 is given by

$$\begin{pmatrix} \lambda_{(a,b)}\cdot Id & \partial_\theta\wedge\partial_{x_2} \\ (1+\lambda_{(a,b)}^2)d\theta\wedge dx_2 & -\lambda_{(a,b)}\cdot Id \end{pmatrix}$$

where $\lambda_{(a,b)}\in C^\infty(\underline{C})$ is the function that lifts to $-by_1+cx_1\in C^\infty(C)$ via $C\to\underline{C}$. Again this is a gauge transformation of the standard symplectic structure on $S^1\times\mathbb{R}$.

A consequence is that for no choice of a,b,c as above the weak brane $(C,F_{(a,b,c)})$ is actually a brane. Indeed if this was the case by Thm. 6.4 we would obtain a space-filling brane for a symplectic structure on $S^1\times\mathbb{R}$; applying again Thm. 6.4, by Example 7.8 of [9], we would obtain the structure of a holomorphic symplectic manifold on $S^1\times\mathbb{R}$, which can not exist because holomorphic symplectic manifolds have real dimension $4k$ for some integer k.

[20]This happens exactly when $F_{(a,b,c)}$ is closed.

6.3. **Cosymplectic submanifolds.** Recall that a submanifold $\tilde{M}$ of a Poisson manifold (M,Π) is *cosymplectic* if $\sharp N^*\tilde{M} \oplus T\tilde{M} = TM|_{\tilde{M}}$. It is known (see for example [23]) that a cosymplectic submanifold inherits canonically a Poisson structure. The following lemma, which follows also from more general results of [2], says that generalized complex structures are also inherited by cosymplectic submanifolds:

Lemma 6.15. *Let E be an exact Courant algebroid over a manifold M, $\mathcal{J}$ a generalized complex structure on E and $\tilde{M}$ a cosymplectic submanifold of M (w.r.t. the natural Poisson structure on M induced by $\mathcal{J}$). Then $\tilde{M}$ is naturally endowed with a generalized complex structure.*

Proof. We want apply Prop. 5.1 with $K = N^*\tilde{M}$ (so $K^\perp = \pi^{-1}(T\tilde{M})$). The intersection $\mathcal{J}K \cap K^\perp$ is trivial. Indeed if $\xi \in N^*\tilde{M}$ and $\pi(\mathcal{J}\xi) \in T\tilde{M}$ then by the definition of cosymplectic submanifold $\pi(\mathcal{J}\xi) = 0$ (recall that $\sharp = \pi\mathcal{J}|_{T^*M}$) and the restriction $\sharp$ to $N^*\tilde{M}$ is injective, so that $\xi = 0$. Further, as seen in Remark 4.4, all sections of $K^\perp$ are basic, so $\mathcal{J}$ maps the set of basic sections of $\mathcal{J}K^\perp \cap K^\perp$ into itself. Hence the assumptions of Prop. 5.1 are satisfied and we obtain a generalized complex structure on $\tilde{M}$. □

Now we describe how a pair (C,L) which doesn't quite satisfy the conditions of Def. 6.9 can be regarded as a weak brane by passing to a cosymplectic submanifold.

Proposition 6.16. *Let E be an exact Courant algebroid over a manifold M, $\mathcal{J}$ a generalized complex structure on E, C a submanifold and L a maximal isotropic subbundle of $E|_C$ with $\pi(L) = TC$. Suppose that $\mathcal{J}(N^*C) \cap \pi^{-1}(TC)$ is contained in L and has constant rank. Then there exists a submanifold $\tilde{M}$ (containing C) which inherits a generalized complex structure $\tilde{\mathcal{J}}$ from M, and so that $\tilde{L}$ satisfies $\tilde{\mathcal{J}}(\tilde{N}^*C) \subset \tilde{L}$. Here $\tilde{L}$ is the pullback of L to $\tilde{M}$ and $\tilde{N}^*C$ the conormal bundle of C in $\tilde{M}$.*

Further assume that $[\Gamma(L \cap \pi^{-1}(\mathcal{F})), \Gamma(L)] \subset \Gamma(L)$ where $\mathcal{F} := \sharp N^*C \cap TC$ is the characteristic distribution of C. Then $[\Gamma(\tilde{L} \cap \tilde{\pi}^{-1}(\mathcal{F})), \Gamma(\tilde{L})] \subset \Gamma(\tilde{L})$. Hence $(C, \tilde{L})$ is a weak brane in $(\tilde{M}, \tilde{\mathcal{J}})$.

Proof. Since the intersection of $\mathcal{J}(N^*C)$ and $\pi^{-1}(TC)$ has constant rank the same holds for their sum and for $\pi(\mathcal{J}(N^*C) + \pi^{-1}(TC)) = \sharp N^*C + TC$. Hence C is a *pre-Poisson* submanifold [6] of (M,Π). Fix any complement R of $\sharp N^*C + TC$ in $TM|_C$; by Theorem 3.3 of [6], "extending" C in direction of R we obtain a submanifold $\tilde{M}$ of M which is cosymplectic. By Lemma 6.15 we know that $\tilde{M}$ is endowed with a generalized complex structure $\tilde{\mathcal{J}}$. Further by the same lemma $\mathcal{J}K \cap K^\perp$ is trivial. The projection $K^\perp \to K^\perp/K$ (for $K = N^*\tilde{M}$) maps $\mathcal{J}K^\perp \cap K^\perp$ isomorphically onto $K^\perp/K$, and $\tilde{\mathcal{J}}$ is induced by the action of $\mathcal{J}$ on $\mathcal{J}K^\perp \cap K^\perp$. Therefore, denoting by $\tilde{L} := L/K$ the pullback of L to $\tilde{M}$, requiring $\tilde{\mathcal{J}}(\tilde{N}^*C) \subset \tilde{L}$ is equivalent to requiring that $\mathcal{J}(N^*C \cap (\mathcal{J}K^\perp \cap K^\perp))$ maps into $\tilde{L}$ under $K^\perp \to K^\perp/K$, which in turn means $\mathcal{J}(N^*C) \cap K^\perp \subset L$. Now using $K^\perp = \pi^{-1}(T\tilde{M})$, $T\tilde{M}|_C = R \oplus TC$ and recalling that R was chosen so that $R \oplus (\sharp N^*C + TC) = TM|_C$, it follows that $\mathcal{J}(N^*C) \cap K^\perp = \mathcal{J}(N^*C) \cap \pi^{-1}(TC)$. So our assumption ensures that $\tilde{\mathcal{J}}(\tilde{N}^*C) \subset \tilde{L}$.

Finally notice that the projection $K^\perp \to K^\perp/K$ maps L onto $\tilde{L}$. Since $\pi^{-1}(\mathcal{F})$ is mapped onto $\tilde{\pi}^{-1}(\mathcal{F})$ we also have that $L \cap \pi^{-1}(\mathcal{F})$ is mapped onto $\tilde{L} \cap \tilde{\pi}^{-1}(\mathcal{F})$. Hence our assumption $[\Gamma(L \cap \pi^{-1}(\mathcal{F})), \Gamma(L)] \subset \Gamma(L)$ implies $[\Gamma(\tilde{L} \cap \tilde{\pi}^{-1}(\mathcal{F})), \Gamma(\tilde{L})] \subset \Gamma(\tilde{L})$. □

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