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THE G-BILIAISON CLASS OF SYMMETRIC DETERMINANTAL SCHEMES

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Abstract: We consider a family of schemes, that are defined by minors of a homogeneous symmetric matrix with polynomial entries. We assume that they have maximal possible codimension, given the size of the matrix and of the minors that define them. We show that these schemes are G-bilinked to a linear variety of the same dimension. In particular, they can be obtained from a linear variety by a finite sequence of ascending G-biliaisons on some determinantal schemes. We describe the biliaisons explicitely in the proof of Theorem 2.3. In particular, it follows that these schemes are glicci.

Introduction

A main open question in liaison theory consists of deciding whether every arithmetically Cohen-Macaulay scheme is in the same G-liaison class of a complete intersection (glicci). The Theorem of Gaeta says that every arithmetically Cohen-Macaulay scheme of codimension 2 belongs to the CI-liaison class of a complete intersection. This result was generalized by Kleppe, Migliore, Miró-Roig, Nagel, and Peterson in [13], where they proved that every standard determinantal scheme is in the G-liaison class of a complete intersection of the same codimension. Hartshorne strenghtened their result in [11], proving that every standard determinantal scheme is in the G-biliaison class of a complete intersection. In [4], Casanellas and Miró-Roig proved that any arithmetically Cohen-Macaulay divisor on a rational normal scroll surface is glicci. In her Ph.D. thesis [3], M. Casanellas generalized this result to arithmetically Cohen-Macaulay divisors on a rational normal scroll. Moreover, we know that any arithmetically Gorenstein scheme of codimension 3 is licci (i.e. it belongs to the CI-liaison class of a complete intersection). Therefore, any arithmetically Gorenstein scheme of codimension 3 is glicci. This result easily follows from the main theorem in the paper of Watanabe [18]. We are still far from being able to answer in full generality to the question of whether any arithmetically Cohen-Macaulay scheme is glicci. As far as we know, all the work in this direction deals with specific families of arithmetically-Cohen Macaulay schemes. In this note we consider a family of schemes whose saturated ideal is generated by minors of a fixed size of a symmetric matrix with polynomial entries. We prove that these schemes are G-bilinked to a complete intersection. In particular, they are glicci.

In the first section, we introduce the family of schemes that will be the object of our study. Their defining ideals are generated by the minors of a fixed size of a symmetric matrix with polynomial entries. We assume that the schemes have the highest possible codimension, for a fixed size of the matrix and of the minors that define them. We call a scheme of this kind **symmetric determinantal**. We observe that symmetric determinantal schemes do not exist for any given codimension, in fact they can only have codimension $\binom{b}{2}$, for some $b \geq 2$. Examples of symmetric determinantal schemes are complete intersections of admissible codimension, the Veronese surface in $\mathbb{P}^5$, and some standard and good determinantal schemes (see Examples 1.6, 1.7, and 1.8). In the first section we also introduce the concept of **almost-symmetric determinantal scheme**. Symmetric and almost-symmetric determinantal schemes are arithmetically Cohen-Macaulay. This follows from a result of Kutz [16]. In Theorem 1.18 we discuss when an almost-symmetric determinantal scheme is a generic complete intersection, by giving an equivalent condition and a sufficient condition. In Theorem 1.21 we state the analogous result for symmetric determinantal schemes. In Theorem 1.22 we give an upper bound on the height of the ideal of minors of size $t \times t$ of a symmetric $m \times m$ matrix modulo the ideal of minors of size $t \times t$ of the same matrix that do not involve the last row. From this, in Corollary 1.23 (resp. Corollary 1.26) we derive an upper bound on the height of the ideal of minors of size $t \times t$ modulo the ideal of minors of size $(t+1)\times(t+1)$ of a symmetric (resp. almost-symmetric) matrix. In Corollary 1.24 we show that if the ideal of $t \times t$ minors of a symmetric matrix M defines a symmetric determinantal scheme, then the ideal of minors of M of any bigger size also defines a symmetric determinantal scheme. In Corollary 1.27 we show that if the ideal of $t \times t$ minors of an almost-symmetric matrix M defines an almost-symmetric determinantal scheme, then the ideal of minors of M of any bigger size also defines an almost-symmetric determinantal scheme.

Proposition 2.1 in Section 2 clarifies the connection between symmetric and almost-symmetric determinantal schemes. For each symmetric determinantal scheme X we produce an almost-symmetric determinantal scheme Y such that X is a generalized divisor on Y, Y is arithmetically Cohen-Macaulay and a generic complete intersection. We also construct another symmetric determinantal scheme X' that is a generalized divisor on Y. Theorem 2.3 is the main result of this paper: any symmetric determinantal scheme belongs to the G-biliaison class of a linear variety. All the divisors involved in the G-biliaisons are symmetric determinantal, and the G-biliaisons are performed on almost-symmetric determinantal schemes.

We wish to emphasize the analogy from the point of view of liaison theory between the family of symmetric determinantal schemes and the family of standard determinantal schemes. See [15] for the definition of standard and good determinantal schemes. See also [13] for some of their properties, mainly in relation with liaison theory. In [13] Kleppe, Migliore, Mirò-Roig, Nagel and Peterson prove that standard determinantal schemes are glicci. Their argument is constructive, meaning that following the proof of Theorem 3.6 of [13] one can write down explicitely all the links. In [11], Hartshorne proves that standard determinantal schemes are in the same G-biliaison class of a linear variety. He shows that the G-bilinks constructed in [13] can indeed be regarded as elementary G-biliaisons. Our main theorem is analogous to the main theorem in [11], and the G-bilinks that one obtains following our proof are in the spirit of [13].

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1. Symmetric and almost-symmetric determinantal schemes

Let X be a scheme in $\mathbb{P}^n = \mathbb{P}^n_K$, where K is an algebraically closed field. We make no assumption on the characteristic of K. Let I_X be the saturated homogeneous ideal corresponding to X in the polynomial ring $R = K[x_0, x_1, \dots, x_n]$. We denote by $\mathfrak{m}$ the homogeneous irrelevant maximal ideal of R, $\mathfrak{m} = (x_0, x_1, \dots, x_n)$. For an ideal $I \subset R$, we denote by $H^0_*(I)$ the saturation of I with respect to the maximal ideal $\mathfrak{m}$.

Let $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^n}$ be the ideal sheaf of X. Let Y be a scheme that contains X. We denote by $\mathcal{I}_{X|Y}$ the ideal sheaf of X restricted to Y, i.e. the quotient sheaf $\mathcal{I}_X/\mathcal{I}_Y$. For $i \geq 0$, we let $H^i_*(\mathbb{P}^n, \mathcal{I}) = \oplus_{t\in\mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{I}(t))$ denote the i-th cohomology of the sheaf $\mathcal{I}$ on $\mathbb{P}^n$. We often write just $H^0_*(\mathcal{I})$, when the ambient space $\mathbb{P}^n$ is clearly defined.

Notation 1.1. Let $I \subseteq R$ be a homogeneous ideal. We let $\mu(I)$ denote the cardinality of a set of minimal generators of I.

In this paper we deal with schemes whose saturated ideals are defined by minors of matrices with polynomial entries.

Definition 1.2. Let M be a matrix of size $m \times m$ with entries in R. We say that M is *t-homogeneous* if the minors of M of size $s \times s$ are homogeneous polynomials for all $s \leq t$. We say that M is *homogeneous* if it is m-homogeneous.

We will always consider t-homogeneous matrices. Moreover, we will regard symmetric matrices up to invertible linear transformations that preserve their symmetry, and almost symmetric matrices up to invertible linear transformations that preserve the property of being almost-symmetric. We regard all matrices up to changes of coordinates. See Definition 1.9 for the definition of almost-symmetric matrix.

Definition 1.3. Let $X \subset \mathbb{P}^n$ be a scheme. We say that X is *symmetric determinantal* if:

(1) there exists a symmetric t-homogeneous matrix M of size $m \times m$ with entries in R, such that the saturated ideal of X is generated by the minors of size $t \times t$ of M, $I_X = I_t(M)$, and
(2) X has codimension $\binom{m-t+2}{2}$.

Remark 1.4. For any scheme X satisfying requirement (1) of Definition 1.3, we have

$$codim(X) \leq \binom{m-t+2}{2}.$$

See Theorem 2.1 of [12] for a proof of this fact.

The remark shows that symmetric determinantal schemes have highest possible codimension among the schemes defined by minors of a symmetric matrix, for a given size of the matrix and of the minors. Notice also that by requiring that $I_t(M)$ is the defining ideal of a scheme X, we are requiring that the ideal of $t \times t$ minors of M is saturated.

It is worth emphasizing that symmetric determinantal schemes do not occur for every codimension. However, complete intersections are a special case of symmetric determinantal schemes for the codimensions for which symmetric determinantal schemes do exist.

Remark 1.5. Symmetric determinantal schemes do not exist for every codimension. In fact, we have symmetric determinantal schemes of codimension c if and only if c is of the form $\binom{b}{2}$, for some integer $b \geq 2$.

Complete intersections are an easy example of symmetric determinantal schemes, for each admissible codimension.

Example 1.6. Let X be a complete intersection of codimension $\binom{b}{2}$. Then

$$I_X = (F_{ij} \mid 1 \leq i \leq j \leq b-1) = I_1(M)$$

where $M = (F'_{ij})$, $F'_{ij} = F_{ij}$ if $i \leq j$ and $F'_{ij} = F_{ji}$ if $i \geq j$. The matrix M is symmetric of size $(b-1) \times (b-1)$. Its entries, i.e. its minors of size 1 define X, and the codimension of X is $\binom{b}{2}$. Hence X is symmetric determinantal.

The Veronese surface $V \subset \mathbb{P}^5$ is an example of a symmetric determinantal scheme that is not a complete intersection.

Example 1.7. Let $V \subset \mathbb{P}^5$ be the Veronese surface. The saturated ideal of V is minimally generated by the (distinct) minors of size two by two of a symmetric matrix of indeterminates of size three by three:

$$I_V = I_2 \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_5 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix}.$$

The Veronese surface has codimension $3 = \binom{3-2+2}{2}$.

Theorem 2.3 of [12] provides an example of a symmetric determinantal scheme in $\mathbb{P}^n$ for $n = \binom{m+1}{2}$ and for each $t \leq m$.

Example 1.8. For any fixed $m \geq 1$, and for any choice of t such that $1 \leq t \leq m$, let $n = \binom{m+1}{2}$. Let $X \subset \mathbb{P}^n$ be the symmetric determinantal scheme whose saturated ideal is generated by the minors of size $t \times t$ of the symmetric matrix of indeterminates of size $m \times m$:

$$I_X = I_t \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{1,2} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & & \vdots \\ x_{1,m} & x_{2,m} & \cdots & x_{m,m} \end{bmatrix}.$$

From Theorem 2.3 in [12] we have that X has

$$codim(X) = depth(I_X) = \binom{m-t+2}{2},$$

therefore X is arithmetically Cohen-Macaulay and symmetric determinantal.

In his dissertation [5], A. Conca studied the ideals of Example 1.8, and in fact a larger family of ideals generated by minors of matrices of indeterminates. He computed the Gröbner basis of the schemes of Example 1.8 with respect to a diagonal monomial order. As a consequence, he was able to compute some of the invariants related to the Poincaré series of these rings. He also showed that the schemes corresponding to these ideals are reduced, irreducible and normal, and he characterized the arithmetically Gorenstein ones among them. See section 4 of [5] for more details.

Complete intersection schemes of codimension $\binom{b}{2}$ for some b are good determinantal schemes that are also symmetric determinantal (as observed in Example 1.6). Notice however that symmetric determinantal schemes are not a subfamily of standard or good determinantal schemes. For example, the Veronese surface in $\mathbb{P}^5$ is a symmetric determinantal scheme, but it is not standard determinantal (see Proposition 6.7 in [8]). Moreover, in Proposition 6.7 and Proposition 6.17 of [8] we provide a large class of examples of symmetric determinantal schemes that are not standard determinantal. They include the schemes of Example 1.8. Other examples are presented in [7].

Definition 1.9. Let O be a matrix of size $(m-1)\times m$. We say that O is *almost symmetric* if the submatrix of O consisting of the first $m-1$ columns is symmetric.

Definition 1.10. Let $Y \subset \mathbb{P}^n$ be a scheme. We say that Y is *almost-symmetric determinantal* if:

(1) there exists an almost-symmetric t-homogeneous matrix O of size $(m-1)\times m$ with entries in R, such that the saturated ideal of Y is generated by the minors of size $t \times t$ of O, $I_Y = I_t(O)$
(2) Y has codimension $\binom{m-t+2}{2} - 1$.

Notice that in analogy with the case of symmetric determinantal schemes, the ideal of $t \times t$ minors of O is saturated since it is the ideal associated to a projective scheme. Moreover, we require almost-symmetric determinantal schemes to have highest possible codimension among the schemes defined by minors of an almost-symmetric matrix, for a given size of the matrix and of the minors.

Remark 1.11. For any scheme Y satisfying requirement (1) of Definition 1.10, we have

$$codim(Y) \leq \binom{m-t+2}{2} - 1.$$

See for example the paper of Kutz [16]. A. Conca showed in his Ph.D. dissertation that if O is an almost-symmetric matrix of indeterminates, then

$$codim(Y) = \binom{m-t+2}{2} - 1$$

(see Proposition 4.6.2 of [5]).

The previous remark provides us with an example of almost-symmetric determinantal schemes.

Example 1.12. Let O be an almost-symmetric matrix of indeterminates of size $(m-1)\times m$. Let $n=\binom{m+1}{2}$. For any choice of $1\leq t\leq m-1$, let $Y_t\subseteq \mathbb{P}^n$ be the scheme whose saturated ideal is generated by the minors of size $t\times t$ of M. Then it follows from Proposition 4.6.2 of [5] that

$$codim(Y_t)=\binom{m-t+2}{2}-1.$$

Hence Y_t is an almost-symmetric determinantal scheme.

Similarly to the case of symmetric determinantal schemes, almost-symmetric determinantal schemes do not exist for any codimension. This is clear from part (2) of Definition 1.10. Notice also that complete intersections of codimension $\binom{b}{2}-1$ for some $b\geq 3$ are almost-symmetric determinantal.

Example 1.13. Every complete intersection of codimension $\binom{b}{2}-1$ for some $b\geq 3$ is an almost-symmetric determinantal scheme. In fact, let $Y\subset\mathbb{P}^n$ be a complete intersection of codimension $\binom{b}{2}-1$. Notice that $\binom{b}{2}-1=\binom{b-1}{2}+b-2$. Let

$$I_Y=(F_{i,j},G_k\mid 1\leq i\leq j\leq b-2,\ 1\leq k\leq b-2)$$

be a minimal system of generators of the ideal of Y. Then I_Y is generated by the entries of the almost-symmetric matrix

$$O=\left[\begin{array}{ccccc} F_{1,1} & F_{1,2} & \cdots & F_{1,b-2} & G_1\\ F_{1,2} & F_{2,2} & \cdots & F_{2,b-2} & G_2\\ \vdots & \vdots & & \vdots & \vdots\\ F_{1,b-2} & F_{2,b-2} & \cdots & F_{b-2,b-2} & G_{b-2}\end{array}\right],$$

hence Y is almost-symmetric determinantal.

Remark 1.14. The family of almost-symmetric determinantal schemes does not coincide with the family of symmetric determinantal schemes, since by Remark 1.4 and Remark 1.11 it follows that in general they have different codimensions.

Almost-symmetric determinantal schemes are not a subfamily of standard or good determinantal schemes. The schemes of Example 1.12 are a family of almost-symmetric determinantal schemes that are not standard determinantal whenever $1<t<m$.

Cohen-Macaulayness of symmetric and almost-symmetric determinantal schemes was proved by R. Kutz in Theorem 1 of [16]. We state here a special case of this result, as we will need it in this section.

Theorem 1.15 (Kutz)**.** *Symmetric and almost-symmetric determinantal schemes are arithmetically Cohen-Macaulay.*

We now estabilish some further properties of almost-symmetric determinantal schemes that will be needed in this paper. We use the notation of Definition 1.10.

We start by showing that a scheme defined by the $t\times t$ minors of a t-homogeneous matrix can be a complete intersection only when it is generated by the entries of the matrix, or by its determinant (in the case of a square matrix). In [14], Kotzev has shown that the ideal of $t \times t$ minors of a symmetric matrix of indeterminates of size $m \times m$ is Gorenstein if and only if $t = 1$ or $t = m$. However, here we need a result that applies to almost-symmetric matrices, and in general to matrices with polynomial entries.

Lemma 1.16. *Let M be a t-homogeneous matrix of size $m \times n$ with entries in R or in R_P for some prime P. Assume that M has no invertible entries. If $I_t(M)$ is a complete intersection, then $t = 1$ or $t = m = n$.*

Proof. Let $F_{l,m}$ denote the entry of M in row l and column m. Let $m_{(i,j)} = m_{(i_1,\ldots,i_t;j_1,\ldots,j_t)}$ be the minor of M corresponding to rows $i_1, \ldots, i_t$ and columns $j_1, \ldots, j_t$. Let $e_{(i,j)}$ be the basis vector corresponding to $m_{(i,j)}$. We assume that we are not in the situation $t = m = n$, i.e. that the ideal $I_t(M)$ has more than one generator. One can easily check that the following are syzygies of the $t \times t$ minors of M

$$\sum_{h=1}^{t+1} (-1)^h F_{i_h,j_k} e_{(i_1,\ldots,i_{h-1},i_{h+1},\ldots,i_{t+1};j_1,\ldots,j_t)} \tag{1}$$

for any choice of $1 \leq i_1 < \ldots < i_{t+1} \leq m$, $1 \leq j_1 < \ldots < j_t \leq m$, and $1 \leq k \leq t$. Similarly, the following are syzygies of the $t \times t$ minors of M

$$\sum_{k=1}^{t+1} (-1)^h F_{i_h,j_k} e_{(i_1,\ldots,i_t;j_1,\ldots,j_{k-1},j_{k+1},\ldots,j_{t+1})} \tag{2}$$

for any choice of $1 \leq i_1 < \ldots < i_t \leq m$, $1 \leq j_1 < \ldots < j_{t+1} \leq m$, and $1 \leq h \leq t$. The syzygies (1) are obtained choosing a $(t+1) \times t$ submatrix N of M and making it into a $(t+1) \times (t+1)$ matrix by choosing one of the columns of N and adding it as a first column. We then obtain a matrix O whose determinant is zero, since it has a repeated column. The syzygy among the $t \times t$ minors of N is obtained by developing the determinant of the matrix O with respect to the first column. The syzygies (2) are obtained in a similar fashion: we choose a $t \times (t+1)$ submatrix N of M and we make it into a $(t+1) \times (t+1)$ matrix by choosing one of the rows of N and adding it as a first row. We then obtain a matrix O whose determinant is zero, since it has a repeated row. The syzygy among the $t \times t$ minors of N is obtained by developing the determinant of the matrix O with respect to the first row.

We can obtain another family of syzygies of the $t \times t$ minors of M as follows. We choose a $(t+1) \times (t+1)$ submatrix of M and we develop its determinant with respect to one of its rows. Then we develop the same determinant with respect to one of the columns. The difference between these two expressions for the determinant gives a syzygy among the $t \times t$ minors of M. The syzygies that we obtain in this manner have the form

$$\sum_{h=1}^{t+1} (-1)^h F_{i_h,j_k} e_{(i_1,\ldots,i_{h-1},i_{h+1},\ldots,i_{t+1};j_1,\ldots,j_{k-1},j_{k+1},\ldots,j_{t+1})} - \tag{3}$$

$$\sum_{m=1}^{t+1} (-1)^m F_{i_n,j_m} e_{(i_1,\ldots,i_{n-1},i_{n+1},\ldots,i_{t+1};j_1,\ldots,j_{m-1},j_{m+1},\ldots,j_{t+1})}$$

for any choice of $1 \leq i_1 < \ldots < i_{t+1} \leq m$, $1 \leq j_1 < \ldots < j_{t+1} \leq m$, $1 \leq k \leq t+1$, and $1 \leq n \leq t+1$. Here $e_{(i,j)}$ indicates the basis vector of $R^{\binom{m}{t}^2}$ corresponding to $m_{(i,j)}$.

By contradiction, assume that $t \geq 2$ and that c of the $t \times t$ minors of M form a regular sequence. Denote these minors by $m_{(i^1,j^1)}, \ldots, m_{(i^c,j^c)}$. Here i^a and j^a denote vectors of integers, as for i and j above. A set of minimal generators for the syzygies of $m_{(i^1,j^1)}, \ldots, m_{(i^c,j^c)}$ is of the form

$$m_{(i^a,j^a)}e_{(i^b,j^b)} - m_{(i^b,j^b)}e_{(i^a,j^a)}. \tag{4}$$

Therefore each syzygy of type (1), (2) or (3) can be written as a combination with coefficients in R of syzygies of type (4). In particular, this implies that $F_{i,j}$ can be written as a combination with coefficients in R of some of the $t \times t$ minors of M. To fix ideas, let $F_{1,1} \neq 0$ be the entry of M of minimum degree. $F_{1,1}$ cannot be written as a combination of minors of M, since the degree of the minors of size $t \times t$ of M is strictly bigger than the minimum of the positive degrees of the entries that are involved in the minors. In particular, they all have strictly larger degree than $F_{1,1}$. This is a contradiction, therefore it must be $t = 1$. □

Definition 1.17. Let $X \subset \mathbb{P}^n$ be a scheme. X is a *generic complete intersection* if the localization $(I_X)_P$ is a complete intersection for every P minimal associated prime of I_X.

X is *generically Gorenstein*, abbreviated G_0, if the localization $(I_X)_P$ is a Gorenstein ideal for every P minimal associated prime of I_X.

The next theorem will be used in the proof of Theorem 2.3.

Theorem 1.18. *Let Y be an almost-symmetric determinantal scheme with defining matrix O, $I_Y = I_t(O)$. Let $c = \binom{m-t+2}{2} - 1$ be the codimension of X. The following are equivalent:*

1. *Y is a generic complete intersection.*
2. *ht $I_{t-1}(O) \geq c+1$.*

Let N be the symmetric matrix obtained from O by deleting the last column. The two equivalent conditions are verified if ht $I_{t-1}(N) = c+1$.

Proof. (1) $\Longrightarrow$ (2): since $I_t(O) \subseteq I_{t-1}(O)$, then ht $I_{t-1}(O) \geq c$. Therefore it suffices to show that ht $I_{t-1}(O) \neq c$. By contradiction, assume that there exists a minimal associated prime P of $I_{t-1}(O)$ of height c. Then P is also a minimal associated prime of $I_t(O)$. Let

$$\begin{array}{rcl} \varphi : \mathbb{F} & \longrightarrow & \mathbb{G} \\ v & \mapsto & Ov \end{array}$$

be the map induced by O. Here v is a column vector whose entries are polynomials, $\mathbb{F}$ and $\mathbb{G}$ are free R-modules of ranks m and $m-1$ respectively. By Proposition 16.3 in [2] we have that the map φ_P that we obtain from φ after localizing at the prime ideal P is an isomorphism on a direct summand R_P^s of $\mathbb{F}_P$, $\mathbb{G}_P$ for some $s \leq t-1$.

We let s be maximal with this property. The localization O_P of O at P can be reduced after invertible row and column operations to the form

$$O_P = \begin{bmatrix} I_s & 0 \\ 0 & B \end{bmatrix},$$

where I_s is an identity matrix of size $s \times s$, 0 represents a matrix of zeroes, and B is a matrix of size $(m-s) \times (m-1-s)$ that has no invertible entries. By assumption, $I_t(O)_P \subsetneq R_P$ is a complete intersection ideal. Since

$$I_t(O)_P = I_t(O_P) = I_{t-s}(B)$$

and B has no invertible entries, it follows by Lemma 1.16 that $t-s=1$, that is $s = t-1$. But then

$$I_{t-1}(O)_P = I_{t-1}(O_P) = R_P,$$

that contradicts the assumption that $P \supseteq I_{t-1}(O)$.

(2) $\Longrightarrow$ (1): let P be a minimal associated prime of $I_t(O)$. Then

$$\text{ht } P = c < c+1 \leq \text{ht } I_{t-1}(O)$$

so $P \not\supseteq I_{t-1}(O)$. Let

$$\begin{array}{rcl} \varphi : \mathbb{F} & \longrightarrow & \mathbb{G} \\ v & \mapsto & Ov \end{array}$$

be the map induced by O. Here v is a column vector whose entries are polynomials, $\mathbb{F}$ and $\mathbb{G}$ are free R-modules of ranks m and $m-1$ respectively. By Proposition 16.3 in [2] we have that the map φ_P that we obtain from φ after localizing at the prime ideal P is an isomorphism on a direct summand R_P^{t-1} of $\mathbb{F}_P$, $\mathbb{G}_P$. Hence, the localization O_P of O at P can be reduced, after elementary row and column operations, to the form

$$O_P = \begin{bmatrix} I_{t-1} & 0 \\ 0 & B \end{bmatrix},$$

where I_{t-1} is an identity matrix of size $(t-1) \times (t-1)$, 0 represents a matrix of zeroes, and B is a matrix of size $(m+1-t) \times (m-t)$. We claim that B is an almost-symmetric matrix. This is clear if the symmetric part of O_P contains an invertible minor of size $(t-1) \times (t-1)$. If instead we have an invertible $(t-1) \times (t-1)$ minor of O that contains the last column, we can write

$$O_P = \begin{bmatrix} I_{t-2} & 0 \\ 0 & B \end{bmatrix} \quad \text{where} \quad B = \begin{bmatrix} & 1 \\ b_{ij} & 0 \\ & \vdots \\ & 0 \end{bmatrix}, \quad b_{ij} = b_{ji}, \quad 1 \leq i,j \leq m+1-t.$$

In fact, up to invertible row and column operations that preserve the almost-symmetry of B, we can assume that B has a symmetric block of maximal size and an invertible entry in the last column. Then

$$B = \begin{bmatrix} 0\dots0 & 1 \\ & 0 \\ b_{ij} & \vdots \\ & 0 \end{bmatrix}, \quad 2 \le i \le m+1-t, \ 1 \le j \le m+1-t,$$

so it contains an almost-symmetric block of size $(m+1-t)\times(m-t)$. Summarizing, if we let $B' = (b_{ij})_{2\le i\le m+1-t, 1\le j\le m+1-t}$ we have that

$$O_P = \begin{bmatrix} I_{t-1} & 0 \\ 0 & B' \end{bmatrix}.$$

This completes the proof of our claim.

The localization of $I_t(O)$ at the prime ideal P is

$$I_t(O)_P = I_t(O_P) = I_1(B)$$

thus it is generated by the entries of B. Since B is an almost-symmetric matrix, we have

$$\mu(I_t(O)_P) \le \binom{m+1-t}{2} + (m-t) = \binom{m+2-t}{2} - 1 = c = \text{ht } I_t(O)_P.$$

Then $I_t(O)$ is locally generated by a regular sequence at all the minimal associated primes, i.e. Y is a generic complete intersection.

Assume now that ht $I_{t-1}(N) = c+1$. Since $I_{t-1}(N) \subseteq I_{t-1}(O)$, then

$$c+1 = \text{ht } I_{t-1}(N) \le \text{ht } I_{t-1}(O).$$

Then the two equivalent conditions hold. □

Remark 1.19. The condition that $I_{t-1}(N) = c+1$ means that Y contains a symmetric determinantal subscheme X' of codimension 1, whose defining ideal is $I_{X'} = I_{t-1}(N)$. Notice that whenever this is the case, Y is a generic complete intersection, hence it is G_0. Under this assumption we have a concept of generalized divisor on Y (see [9], [10] and [11] about generalized divisors). Then X' is a generalized divisor on Y. Theorem 1.18 proves that the existence of such a subscheme X' of codimension 1 guarantees that Y is locally a complete intersection. Notice the analogy with standard determinantal schemes: a standard determinantal scheme Y is good determinantal if and only if it is locally a complete intersection, if and only if it contains a standard determinantal subscheme of codimension 1, whose defining matrix is obtained by deleting a column from the defining matrix of Y.

The next example shows that in general the condition that ht $I_{t-1}(N) = c+1$ is stronger than the two equivalent conditions of Theorem 1.18.

Example 1.20. Let $R = k[x_o, x_1, x_2, x_3]$, and let

$$O = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_0 & x_3 \end{bmatrix}.$$

$I_2(O)$ is a Cohen-Macaulay ideal of height 2, $I_1(O) = (x_0, x_1, x_2, x_3)$ has height $4 \geq$ ht $I_2(O) + 1$. Deleting the last column of O yields the matrix

$$N = \begin{bmatrix} x_0 & x_1 \\ x_1 & x_0 \end{bmatrix}.$$

$I_1(N) = (x_0, x_1)$ is an ideal of height $2 < 2+1$. Notice that $I_1(N)$ does not change, even if we perform invertible row and column operations on O that preserve its almost symmetric structure, before deleting the last column.

Now assume that O is obtained from a homogeneous symmetric matrix M by deleting the last row. Then M is of the form

$$M = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_0 & x_3 \\ x_2 & x_3 & L \end{bmatrix}$$

for some linear form L (possibly $L = 0$). Notice that if we apply chosen invertible row and column operations that preserve the symmetry of M, we obtain a matrix

$$M = \begin{bmatrix} 2x_0 + 2x_1 + 2ax_2 + 2ax_3 + a^2L & x_1 + x_0 + ax_3 & x_2 + x_3 + aL \\ x_1 + x_0 + ax_3 & x_0 & x_3 \\ x_2 + x_3 + aL & x_3 & L \end{bmatrix}$$

for any choice of $a \neq 0$. If we now delete the last row we obtain a new matrix O'

$$O' = \begin{bmatrix} 2x_0 + 2x_1 + 2ax_2 + 2ax_3 + a^2L & x_1 + x_0 + ax_3 & x_2 + x_3 + aL \\ x_1 + x_0 + ax_3 & x_0 & x_3 \end{bmatrix}.$$

Deleting the last column of O' yields the matrix

$$N' = \begin{bmatrix} 2x_0 + 2x_1 + 2ax_2 + 2ax_3 + a^2L & x_1 + x_0 + ax_3 \\ x_1 + x_0 + ax_3 & x_0 \end{bmatrix}$$

and $I_1(N') = (x_0, x_1 + ax_3, a(2x_2 + aL))$. Then $I_1(N')$ has height 3 unless $2x_2 + aL$ is a linear combination of $x_0, x_1 + ax_3$. This is the case for a generic choice of a.

Summarizing the example, $I_2(O)$ has maximal height, ht $I_2(O) <$ ht $I_1(O)$, $I_1(M)$ has maximal height, and $I_1(N)$ does not have maximal height. However, after applying to M a general transformation that preserves its symmetry, we obtain O' and N' with the property that both $I_2(O')$ and $I_1(N')$ have maximal height.

The following is the analogous of Theorem 1.18 for symmetric determinantal schemes. We will not need it in the sequel, but we wish to emphasize that a result of this kind holds. The proof is very similar to that of the previous theorem, so we omit it.

Theorem 1.21. *Let X be a symmetric determinantal scheme with defining matrix M, $I_X = I_t(M)$. Let $c + 1 = \binom{m-t+2}{2}$ be the codimension of X. The following are equivalent:*

(1) *X is a generic complete intersection.*
(2) *ht $I_{t-1}(M) \geq c + 2$.*

Let O be the almost-symmetric matrix obtained from M by deleting the last row. If ht $I_{t-1}(O) \geq c + 2$, then the two equivalent conditions are verified.

Finally, we prove a result in the lines of the Eisenbud-Evans generalized principal ideal theorem (see [6]) and of its generalization by Bruns (see [1]). The result is not new for an arbitrary matrix M, but the estimate on the height can be sharpened in the case of symmetric matrices. We essentially follow the proof of Theorem 2 in [1].

Theorem 1.22. *Let M be a t-homogeneous symmetric matrix of size $m \times m$ with entries in R. Assume that M has no invertible entries. Let O be the matrix obtained from M by deleting the last row. Then*

$$ht\, I_t(M)/I_t(O) \leq 1.$$

Proof. Let

$$M = \begin{bmatrix} F_{1,1} & \dots & F_{1,m-1} & F_{1,m} \\ \vdots & & \vdots & \vdots \\ F_{1,m-1} & \dots & F_{m-1,m-1} & F_{m-1,m} \\ F_{1,m} & \dots & F_{m-1,m} & F_{m,m} \end{bmatrix}$$

and

$$O = \begin{bmatrix} F_{1,1} & \dots & F_{1,m-1} & F_{1,m} \\ \vdots & & \vdots & \vdots \\ F_{1,m-1} & \dots & F_{m-1,m-1} & F_{m-1,m} \end{bmatrix}.$$

Consider the matrix

$$L = \begin{bmatrix} F_{1,1} & \dots & F_{1,m-1} & F_{1,m} & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ F_{1,m-1} & \dots & F_{m-1,m-1} & F_{m-1,m} & 0 \\ F_{1,m} & \dots & F_{m-1,m} & F_{m,m} & -1 \end{bmatrix}.$$

Notice that L and M are related in the same way as φ and φ' in the proof of Theorem 2 of [1]. We regard the matrices over the ring $S = R/I_t(O)$. L defines a morphism $\psi : S^m \longrightarrow S^{m+1}$ where the images of a basis of S^m are given by the rows of L. Similarly, M defines a morphism $\varphi : S^m \longrightarrow S^m$. Let $\mathcal{M} := Coker\psi$ and $\mathcal{M}' := Coker\varphi$. Then $\mathcal{M}' \cong \mathcal{M}/e_{m+1}$, where $e_1, \dots, e_{m+1}$ denote the elements of the standard basis of S^{m+1}. For an $x \in \mathcal{M}$, we define

$$\mathcal{M}^*(x) := \{f(x) \; : \; f \in Hom_S(\mathcal{M}, S)\}.$$

As shown in Theorem 2 of [1], we have that

$$I_t(M)/I_t(O) \subseteq \mathcal{M}^*(e_{m+1}).$$

Following the proof of Theorem 2 of [1], one can show that ht $\mathcal{M}^*(e_{m+1}) \leq m - t + 1$. However we claim that in the special case of a symmetric matrix M, one has the sharper bound ht $\mathcal{M}^*(e_{m+1}) \leq 1$.

Since $e_{m+1} = F_{1,m}e_1 + \ldots + F_{m,m}e_m$, we have

$$\mathcal{M}^*(e_{m+1}) = \{f(F_{1,m}e_1 + \ldots + F_{m,m}e_m) \; : \; f \in Hom_S(\mathcal{M}, S)\}.$$

For each $f \in Hom_S(\mathcal{M}, S)$, we have

$$f(F_{1,m}e_1 + \ldots + F_{m,m}e_m) = F_{1,m}f(e_1) + \ldots + F_{m,m}f(e_m).$$

Let $f_i := f(e_i) \in S$. Assume by contradiction that ht $\mathcal{M}^*(e_{m+1}) \geq 2$. Then we can find $f, g \in Hom_S(\mathcal{M}, S)$ such that $f(e_{m+1})$ and $g(e_{m+1})$ form a regular sequence in S. For all $i = 1, \ldots, m-1$ we have

$$f(\sum_{j=1}^{m} F_{i,j}e_j) = \sum_{j=1}^{m} F_{i,j}f_j = 0$$

therefore

$$f_m F_{i,m} = -\sum_{j=1}^{m-1} F_{i,j}f_j.$$

Analogously,

$$g_m F_{i,m} = -\sum_{j=1}^{m-1} F_{i,j}g_j.$$

Hence

$$g_m f(e_{m+1}) = \sum_{i=1}^{m} F_{i,m}f_i g_m = f_m g_m F_{m,m} - \sum_{\substack{i=1,\ldots,m-1 \\ j=1,\ldots,m-1}} f_i g_j F_{i,j}$$

and since M is symmetric

$$g_m f(e_{m+1}) = f_m g(e_{m+1}).$$

By assumption $f(e_{m+1})$ and $g(e_{m+1})$ form a regular sequence in S, so $g_m = hg(e_{m+1})$ and $f_m = hf(e_{m+1})$ for some $h \in S$. Assume that $f_m, g_m \neq 0$. So $h \neq 0$ and

$$f_m = hf(e_{m+1}) = \sum_{i=1}^{m} hF_{i,m}f_i = \sum_{i \in I} hF_{i,m}f_i \tag{5}$$

where $I \subseteq \{1, \ldots, m\}$ is the set of indexes of the summands that effectively contribute to the sum. In other words, if $f = (f_1, \ldots, f_m)$ and $\sum_{i \notin I} F_{i,m}f_i = 0$ then $f(e_{m+1}) = \phi(e_{m+1})$ where $\phi = (\phi_1, \ldots, \phi_m)$ with $\phi_i = f_i$ if $i \in I$ and $\phi_i = 0$ if $i \notin I$. So we can replace f with ϕ. This proves that we can assume without loss of generality that since $f_m \neq 0$, then $m \in I$. Hence the term $F_{m,m}f_m$ appears in the sum (5). Comparing degrees in (5), we get

$$deg(f_m) \geq deg(h) + deg(F_{m,m}) + deg(f_m) \geq deg(F_{m,m}) + deg(f_m),$$

hence $deg(F_{m,m}) \leq 0$. By the assumption that M has no invertible entries, $F_{m,m} = 0$. But this is a contradiction: since we are allowing generic invertible row and column operations that preserve the symmetry of M we can always assume that $F_{m,m} \neq 0$ unless all the entries in the last row and column of M are zero. However, in that case $I_t(O) = I_t(M)$ and the thesis is trivially verified.

We still need to analyse the case when $f_m = g_m = 0$. We have

$$f(\sum_{j=1}^{m} F_{i,j}e_j) = \sum_{j=1}^{m-1} F_{i,j}f_j = 0,$$

therefore

$$f_{m-1}F_{i,m-1} = -\sum_{j=1}^{m-2} F_{i,j}f_j.$$

Analogously,

$$g_{m-1}F_{i,m-1} = -\sum_{j=1}^{m-2} F_{i,j}g_j.$$

Hence, proceeding as in the previous case,

$$f_{m-1}g(e_{m+1}) = g_{m-1}f(e_{m+1})$$

and either $f_{m-1}, g_{m-1} \neq 0$ or $f_{m-1} = g_{m-1} = 0$. In the first case, we can conclude as above, in the second case we obtain

$$f_{m-2}g(e_{m+1}) = g_{m-2}f(e_{m+1}).$$

We can keep iterating this reasoning until either $f_i, g_i \neq 0$ for some i, or $f = g = 0$. In both cases we get a contradiction. □

The following is an easy consequence of Theorem 1.22. It improves on the bound Theorem 2 of [1], in the special case of a symmetric matrix.

Corollary 1.23. *Let M be a $m \times m$ symmetric matrix with entries in R. Assume that M has no invertible entries. Then*

$$ht\ I_t(M)/I_{t+1}(M) \leq m+1-t.$$

Proof. Let O be the matrix obtained from M by deleting the last row. From Theorem 1.22

$$\text{ht } I_t(M)/I_t(O) \leq 1.$$

Letting $O = \varphi'$ and $M = \varphi$ in the proof of Theorem 2 of [1], we obtain that

$$\text{ht } I_t(O)/I_{t+1}(M) \leq m-t.$$

Therefore it follows that

$$\text{ht } I_t(M)/I_{t+1}(M) \leq m+1-t.$$

□

The following is another easy consequence of Theorem 1.22 and of Corollary 1.23.

Corollary 1.24. *Let M be a homogeneous $m \times m$ symmetric matrix with entries in R. Assume that M has no invertible entries. If ht $I_t(M) = \binom{m-t+2}{2}$, then ht $I_s(M) = \binom{m-s+2}{2}$ for any $s \geq t$. In other words, if the ideal of $t \times t$ minors of M defines a symmetric determinantal scheme, then the ideal of minors of M of any bigger size defines a symmetric determinantal scheme.*

Proof. It suffices to prove the thesis for $s = t+1$. From Corollary 1.23 we have

$$\text{ht } I_{t+1}(M) \geq \binom{m-t+2}{2} - (m+1-t) = \binom{m-t+1}{2}.$$

Therefore the equality holds. □

Remark 1.25. In particular, we recover the result that the ideal generated by the $t \times t$ minors of a symmetric matrix of indeterminates has maximal height (according to Remark 1.4). More generally, the ideal generated by the $t \times t$ minors of a symmetric matrix whose entries form a regular sequence has maximal height for all $t \geq 1$.

The same kind of statements holds for an almost-symmetric matrix.

Corollary 1.26. *Let O be a $(m-1) \times m$ almost-symmetric matrix with entries in R. Assume that O has no invertible entries. Then*

$$ht\, I_t(O)/I_{t+1}(O) \leq m + 1 - t.$$

Proof. Let M be a symmetric matrix obtained from O by adding a row, such that ht $I_{t+1}(M) = \binom{m-t+1}{2}$. Clearly a matrix M such that ht $I_t(M) = \binom{m-t+2}{2} =$ ht $I_t(O) + 1$ can be constructed, by choosing as F_{mm} a generic form of the appropriate degree. But then ht $I_{t+1}(M) = \binom{m-t+1}{2}$ by Corollary 1.23. From Theorem 1.22

$$\text{ht } I_{t+1}(M)/I_{t+1}(O) \leq 1.$$

Letting $O = \varphi'$ and $M = \varphi$ in the proof of Theorem 2 of [1], we obtain that

$$\text{ht } I_t(O)/I_{t+1}(M) \leq m - t.$$

Therefore it follows that

$$\text{ht } I_t(O)/I_{t+1}(O) \leq m + 1 - t.$$

□

Corollary 1.27. *Let O be a homogeneous $(m-1) \times m$ almost-symmetric matrix with entries in R. Assume that O has no invertible entries. If ht $I_t(O) = \binom{m-t+2}{2} - 1$, then ht $I_s(O) = \binom{m-s+2}{2} - 1$ for any $s \geq t$. In other words, if the ideal of $t \times t$ minors of O defines an almost-symmetric determinantal scheme, then the ideal of minors of O of any bigger size defines an almost-symmetric determinantal scheme.*

Proof. It suffices to prove the thesis for $s = t + 1$. From Corollary 1.26 we have

$$\text{ht } I_{t+1}(O) \geq \binom{m-t+2}{2} - 1 - (m + 1 - t) = \binom{m-t+1}{2} - 1.$$

Therefore the equality holds. □

2. Biliaison of symmetric determinantal schemes

In this section we prove that symmetric determinantal schemes are in the same G-bilaison class of a complete intersection of the same codimension. We start by proving that any symmetric determinantal scheme is a divisor on an almost-symmetric determinantal scheme. This result will be used in the proof of Theorem 2.3.

Proposition 2.1. *Let X be a symmetric determinantal scheme of codimension $c + 1$. Assume that the ideal of X is generated by the $t \times t$ minors of a t-homogeneous matrix M, $I_X = I_t(M)$. Let O be the matrix obtained from M by deleting the last row (after* *performing generic invertible row and column operations that preserve the symmetry of M). Let N be the matrix obtained from O by deleting the last column. Then:*

- *N is a t-homogeneous symmetric matrix. It defines a symmetric determinantal scheme X' of codimension $c+1$, with $I_{X'} = I_{t-1}(N)$.*
- *O is a t-homogeneous almost-symmetric matrix. It defines an almost-symmetric determinantal scheme Y of codimension c, with $I_Y = I_t(O)$. Y is an arithmetically Cohen-Macaulay, generic complete intersection scheme.*

Proof. Let N be the matrix obtained from M by deleting a row and the corresponding column, after performing generic invertible row and column operations on M that preserve its symmetry. Then N is symmetric, and by Remark 1.4 ht $I_{t-1}(N) \leq \binom{m-1-(t-1)+2}{2} = c+1$. Let P be a prime ideal of height $h = \text{ht } P \leq c$. We claim that $P \not\supseteq I_{t-1}(N)$. Clearly $P \not\supseteq I_t(M)$, since ht $I_t(M) = c+1 > h$. Then the localization M_P of M at P has the form (up to a change of coordinates)

$$M_P = \begin{bmatrix} I_t & 0 \\ 0 & C \end{bmatrix},$$

where I_t is an identity matrix of size $t \times t$, 0 represents a matrix of zeroes, and C is a symmetric matrix of size $(m-t) \times (m-t)$. N is obtained from M by deleting a row and a column, possibly after a generic invertible linear transformation. Then the localization N_P of N at P has the form (up to a change of coordinates)

$$N_P = \begin{bmatrix} I_{t-1} & 0 \\ 0 & C \end{bmatrix} \quad \text{or} \quad N_P = \begin{bmatrix} I_t & 0 \\ 0 & C' \end{bmatrix}.$$

In fact, the morphism φ_P defined by M_P is an isomorphism on a free direct summand of rank at least t, hence the morphism φ'_P defined by N_P is an isomorphism on a free direct summand of rank at least $t-1$. But then

$$I_{t-1}(N_P) = I_{t-1}(N)_P = R_P,$$

so $P \not\supseteq I_{t-1}(N)$. Since this holds for any prime P of R of height $h \leq c$, ht $I_{t-1}(N) = c+1$. Then $I_{t-1}(N)$ defines an arithmetically Cohen-Macaulay, symmetric determinantal scheme X' of codimension $c+1$, and $I_{t-1}(N) = I_{X'}$.

By Theorem 1.22,

$$\text{ht } I_t(M)/I_t(O) \leq 1,$$

then

$$\text{ht } I_t(O) \geq \text{ht } I_t(M) - 1 = c.$$

It follows that ht $I_t(O) = c$ and $I_t(O)$ is Cohen-Macaulay by Theorem 1.15. Therefore $I_Y = H^0_*(I_t(O)) = I_t(O)$, and the scheme Y is arithmetically Cohen-Macaulay of codimension c. Moreover,

$$\text{ht } I_{t-1}(O) \geq \text{ht } I_t(M) = c+1 > c = \text{ht } I_t(O),$$

so Y is a generic complete intersection by Theorem 1.18. □

Remark 2.2. If X is a symmetric determinantal scheme of codimension $c+1$ with defining matrix M, $I_X = I_t(M)$, we prove that:

- ht $I_{t-1}(O) \geq c+1$, where O is almost-symmetric and obtained from M by deleting the last row.
- ht $I_{t-1}(N) = c+1$, where N is symmetric and obtained from M by deleting the last row and column.

Notice that by Theorem 1.18 this implies that the scheme Y defined by $I_t(O)$ is a generic complete intersection. However, it does not imply the same result for X, nor for the scheme X' defined by $I_{t-1}(N)$.

We are now ready to prove the main result of this paper.

Theorem 2.3. *Any symmetric determinantal scheme in $\mathbb{P}^n$ can be obtained from a linear variety by a finite sequence of ascending elementary G-biliaisons.*

Proof. Let $X \subset \mathbb{P}^n$ be a symmetric determinantal scheme. We follow the notation of Definition 1.3. Let $M = (F_{ij})$ be the matrix whose minors of size $t \times t$ define X. From the definition $F_{ij} = F_{ji}$ for all i, j, and the matrix is t-homogeneous. Let $c+1$ be the codimension of X, $c := \binom{m-t+2}{2} - 1$. If $t = 1$ or $t = m$, then X is a complete intersection, therefore we can perform a finite sequence of descending elementary CI-biliaisons to a linear variety. Therefore, we concentrate on the case when $2 \leq t < m$.

Let O be the matrix obtained from M by deleting the last row, after performing generic invertible row and column operations that preserve the symmetry of M. O is a t-homogeneous matrix of size $(m-1) \times m$. Let Y be the scheme whose saturated ideal is generated by the $t \times t$ minors of O. By Proposition 2.1, Y is an arithmetically Cohen-Macaulay scheme of codimension c. Notice that Y is standard determinantal exactly when $t = m-1$. For our purpose, it is important to observe that Y is a generic complete intersection. In particular, it satisfies the property G_0. Therefore, a biliaison on Y is a G-biliaison, hence also an even G-liaison. This was proved in [13] for Y satisfying property G_1 and extended in [11] to Y satisfying property G_0.

Let N be the matrix obtained from M by deleting the last row and column. N is a homogeneous symmetric matrix of size $(m-1) \times (m-1)$. Let X' be the scheme cut out by the $(t-1) \times (t-1)$ minors of N. Both X and X' are contained in Y. We denote by H a hyperplane section divisor on Y. We are going to show that

$$X \sim X' + aH \quad \text{for some } a > 0,$$

where $\sim$ denotes linear equivalence of divisors. This will prove that X is obtained by an elementary biliaison from X'. Continuing in this manner, after $t-1$ biliaisons we reduce to the case $t = 1$, when the scheme X is a complete intersection. Then we can perform descending CI-biliaisons to a linear variety.

Let $\mathcal{I}_{X|Y}$, $\mathcal{I}_{X'|Y}$ be the ideal sheafs on Y of X and X'. We then need to show that

$$\mathcal{I}_{X|Y} \cong \mathcal{I}_{X'|Y}(-a) \quad \text{for some } a > 0. \tag{6}$$

A system of generators of $I_{X|Y} = H^0_*(\mathcal{I}_{X|Y}) = I_t(M)/I_Y$ is given by the images in the coordinate ring of Y of the $t \times t$ minors of M

$$I_{X|Y} = (M_{i_1,\ldots,i_t;j_1,\ldots,j_t} \mid 1 \leq i_1 < i_2 < \ldots < i_t \leq m, 1 \leq j_1 < j_2 < \ldots < j_t \leq m).$$

Here $M_{i_1,\dots,i_t;j_1,\dots,j_t}$ denotes the image of the determinant of the submatrix of M consisting of rows $i_1,\dots,i_t$ and columns $j_1,\dots,j_t$ in the coordinate ring of Y. The saturated ideal of Y is minimally generated by the minors of size $t\times t$ of M that do not involve the last row. Notice that this minimal system of generators of I_Y can be completed to a minimal system of generators of I_X by adding all the minors of size $t\times t$ of M that involve both the last row and the last column. Therefore, a minimal system of generators of $I_{X|Y}$ is given by

$$I_{X|Y}=(M_{i_1,\dots,i_{t-1},m;j_1,\dots,j_{t-1},m}\mid 1\leq i_1<\dots<i_{t-1}\leq m-1, 1\leq j_1<\dots<j_{t-1}\leq m-1).$$

A minimal system of generators of $I_{X'|Y}=H^0_*(\mathcal{I}_{X'|Y})=I_{t-1}(N)/I_Y$ is given by the images in the coordinate ring of Y of the $(t-1)\times(t-1)$ minors of N

$$I_{X'|Y}=(M_{i_1,\dots,i_{t-1};j_1,\dots,j_{t-1}}\mid 1\leq i_1<\dots<i_{t-1}\leq m-1, 1\leq j_1<\dots<j_{t-1}\leq m-1).$$

Here $M_{i_1,\dots,i_{t-1};j_1,\dots,j_{t-1}}$ denotes the image of the determinant of the submatrix of M consisting of rows $i_1,\dots,i_{t-1}$ and columns $j_1,\dots,j_{t-1}$ in the coordinate ring of Y.

In order to prove the isomorphism (6), it suffices to check that the quotients

$$\frac{M_{i_1,\dots,i_{t-1},m;j_1,\dots,j_{t-1},m}}{M_{i_1,\dots,i_{t-1};j_1,\dots,j_{t-1}}} \tag{7}$$

are all equal as elements of $H^0(\mathcal{K}_Y(a))$, where $\mathcal{K}_Y$ is the sheaf of total quotient rings of Y. This also gives us an easy way to compute the value of a as the difference $deg(M_{i_1,\dots,i_{t-1},m;j_1,\dots,j_{t-1},m})-deg(M_{i_1,\dots,i_{t-1};j_1,\dots,j_{t-1}})=deg(F_{m,m})$.

Equality (7) is readily verified, once we show that

$$M_{i_1,\dots,i_{t-1},m;j_1,\dots,j_{t-1},m}\cdot M_{k_1,\dots,k_{t-1};l_1,\dots,l_{t-1}}-M_{k_1,\dots,k_{t-1},m;l_1,\dots,l_{t-1},m}\cdot M_{i_1,\dots,i_{t-1};j_1,\dots,j_{t-1}}\in I_Y.$$

The proof is then completed by the following lemmas. □

Since we could not find an adequate reference in the literature, we need to prove the following two lemmas about the minors of a matrix.

Lemma 2.4. *Let M be a matrix of size $m\times m$ and let $M_{i_1,\dots,i_a;j_1,\dots,j_a}$ denote the minor of the submatrix of M consisting of rows $i_1,\dots,i_a$ and columns $j_1,\dots,j_a$. Let I be the ideal generated by the minors of M of size $(a+1)\times(a+1)$. Then*

$$M_{i_1,\dots,i_a;j_1,\dots,j_a}\cdot M_{k_1,\dots,k_a;l_1,\dots,l_a}-M_{k_1,\dots,k_a;j_1,\dots,j_a}\cdot M_{i_1,\dots,i_a;l_1,\dots,l_a}\in I.$$

Proof. We start by proving the thesis when $i_b=k_b$ for $b=1,\dots a-1$. So we want to show that

$$M_{i_1,\dots,i_a;j_1,\dots,j_a}\cdot M_{i_1,\dots,i_{a-1},k_a;l_1,\dots,l_a}-M_{i_1,\dots,i_{a-1},k_a;j_1,\dots,j_a}\cdot M_{i_1,\dots,i_a;l_1,\dots,l_a}\in I. \tag{8}$$

This is essentially an application of Sylvester's identity:

$$M_{i_1,\dots,i_a;j_1,\dots,j_a}\cdot M_{i_1,\dots,i_{a-1},k_a;j_1,\dots,j_{a-1},l_a}-M_{i_1,\dots,i_{a-1},k_a;j_1,\dots,j_a}\cdot M_{i_1,\dots,i_a;j_1,\dots,j_{a-1},l_a}=$$

$$M_{i_1,\dots,i_{a-1};j_1,\dots,j_{a-1}}\cdot M_{i_1,\dots,i_a,k_a;j_1,\dots,j_a,l_a}$$

For our purpouse, we only need that the differemce belongs to I. See [17], pg. 33, for a general statement and a proof of Sylvester's identity.

$$M_{i_1,\ldots,i_a;j_1,\ldots,j_a}\cdot M_{i_1,\ldots,i_{a-1},k_a;l_1,\ldots,l_a}-M_{i_1,\ldots,i_{a-1},k_a;j_1,\ldots,j_a}\cdot M_{i_1,\ldots,i_a;l_1,\ldots,l_a}=$$

$$M_{i_1,\ldots,i_a;j_1,\ldots,j_a}\cdot M_{i_1,\ldots,i_{a-1},k_a;l_1,\ldots,l_a}-\sum_{b=1}^{a-1}M_{i_1,\ldots,i_a;j_1,\ldots,j_b,l_{b+1},\ldots,l_a}\cdot M_{i_1,\ldots,i_{a-1},k_a;l_1,\ldots,l_b,j_{b+1},\ldots,j_a}+$$

$$\sum_{b=1}^{a-1}M_{i_1,\ldots,i_a;j_1,\ldots,j_b,l_{b+1},\ldots,l_a}\cdot M_{i_1,\ldots,i_{a-1},k_a;l_1,\ldots,l_b,j_{b+1},\ldots,j_a}-M_{i_1,\ldots,i_{a-1},k_a;j_1,\ldots,j_a}\cdot M_{i_1,\ldots,i_a;l_1,\ldots,l_a}=$$

$$=\sum_{b=1}^{a}(M_{i_1,\ldots,i_a;j_1,\ldots,j_b,l_{b+1},\ldots,l_a}\cdot M_{i_1,\ldots,i_{a-1},k_a;l_1,\ldots,l_b,j_{b+1},\ldots,j_a}-$$
$$M_{i_1,\ldots,i_a;j_1,\ldots,j_{b-1},l_b,\ldots,l_a}\cdot M_{i_1,\ldots,i_{a-1},k_a;l_1,\ldots,l_{b-1},j_b,\ldots,j_a}).$$

Each summand is of the form

$$M_{i_1,\ldots,i_a;j_1,\ldots,j_b,l_{b+1},\ldots,l_a}\cdot M_{i_1,\ldots,i_{a-1},k_a;l_1,\ldots,l_b,j_{b+1},\ldots,j_a}-$$
$$M_{i_1,\ldots,i_a;j_1,\ldots,j_{b-1},l_b,\ldots,l_a}\cdot M_{i_1,\ldots,i_{a-1},k_a;l_1,\ldots,l_{b-1},j_b,\ldots,j_a},$$

for $b=1,\ldots,a$. In particular, all the minors in the expression have all the rows and the columns in common, except possibly for one. So Sylvester's identity applies, and the thesis follows.

Let us now prove the thesis in full generality. We are going to use (8), and we will proceed in an analogous manner to the proof above. We want to show that

$$M_{i_1,\ldots,i_a;j_1,\ldots,j_a}\cdot M_{k_1,\ldots,k_a;l_1,\ldots,l_a}-M_{k_1,\ldots,k_a;j_1,\ldots,j_a}\cdot M_{i_1,\ldots,i_a;l_1,\ldots,l_a}\in I.$$

Rewrite the difference as

$$M_{i_1,\ldots,i_a;j_1,\ldots,j_a}\cdot M_{k_1,\ldots,k_a;l_1,\ldots,l_a}-M_{k_1,\ldots,k_a;j_1,\ldots,j_a}\cdot M_{i_1,\ldots,i_a;l_1,\ldots,l_a}=$$

$$M_{i_1,\ldots,i_a;j_1,\ldots,j_a}\cdot M_{k_1,\ldots,k_a;l_1,\ldots,l_a}-\sum_{a-1}^{b=1}M_{i_1,\ldots,i_b,k_{b+1},\ldots,k_a;j_1,\ldots,j_a}\cdot M_{k_1,\ldots,k_b,i_{b+1},\ldots,i_a;l_1,\ldots,l_a}+$$

$$\sum_{b=1}^{a-1}M_{i_1,\ldots,i_b,k_{b+1},\ldots,k_a;j_1,\ldots,j_a}\cdot M_{k_1,\ldots,k_b,i_{b+1},\ldots,i_a;l_1,\ldots,l_a}-M_{k_1,\ldots,k_a;j_1,\ldots,j_a}\cdot M_{i_1,\ldots,i_a;l_1,\ldots,l_a}=$$

$$\sum_{b=1}^{a}M_{i_1,\ldots,i_b,k_{b+1},\ldots,k_a;j_1,\ldots,j_a}\cdot M_{k_1,\ldots,k_b,i_{b+1},\ldots,i_a;l_1,\ldots,l_a}-$$
$$M_{i_1,\ldots,i_{b-1},k_b,\ldots,k_a;j_1,\ldots,j_a}\cdot M_{k_1,\ldots,k_{b-1},i_b,\ldots,i_a;l_1,\ldots,l_a}.$$

Each summand is of the form

$$M_{i_1,\ldots,i_b,k_{b+1},\ldots,k_a;j_1,\ldots,j_a}\cdot M_{k_1,\ldots,k_b,i_{b+1},\ldots,i_a;l_1,\ldots,l_a}-M_{i_1,\ldots,i_{b-1},k_b,\ldots,k_a;j_1,\ldots,j_a}\cdot M_{k_1,\ldots,k_{b-1},i_b,\ldots,i_a;l_1,\ldots,l_a},$$

hence by (8) it belongs to I. This concludes the proof. □

Remark 2.5. Notice that following the steps of the proof, one can write down explicitely $M_{i_1,\ldots,i_a;j_1,\ldots,j_a}\cdot M_{k_1,\ldots,k_a;l_1,\ldots,l_a}-M_{k_1,\ldots,k_a;j_1,\ldots,j_a}\cdot M_{i_1,\ldots,i_a;l_1,\ldots,l_a}$ as a combination of the minors of size $(a+1)\times(a+1)$ of the matrix M. This is not relevant for our purpouses.

The following lemma concludes the proof of Theorem 2.3.

Lemma 2.6. *Let M be a matrix of size $m \times m$ and let $M_{i_1,\dots,i_t;j_1,\dots,j_t}$ denote the minor of the submatrix of M consisting of rows $i_1,\dots,i_t$ and columns $j_1,\dots,j_t$. Let I_Y be the ideal generated by the determinants of the submatrices of M of size $t\times t$ of M, that do not contain the last row. Then*

$$M_{i_1,\dots,i_{t-1},m;j_1,\dots,j_{t-1},m}\cdot M_{k_1,\dots,k_{t-1};l_1,\dots,l_{t-1}} - M_{k_1,\dots,k_{t-1},m;l_1,\dots,l_{t-1},m}\cdot M_{i_1,\dots,i_{t-1};j_1,\dots,j_{t-1}} \in I_Y.$$

Proof. It is enough to prove that the statement holds for a matrix of indeterminates $M=(x_{ij})$, $1\leq i\leq j\leq m$. From Lemma 2.4, we have that

(9)

$$M_{i_1,\dots,i_{t-1},m;j_1,\dots,j_{t-1},m}\cdot M_{k_1,\dots,k_{t-1},m;l_1,\dots,l_{t-1},m} - M_{k_1,\dots,k_{t-1},m;j_1,\dots,j_{t-1},m}\cdot M_{i_1,\dots,i_{t-1},m;l_1,\dots,l_{t-1},m} \in I\subseteq I_Y.$$

Here I is the ideal generated by the minors of M of size $t+1$. Clearly, $I\subseteq I_Y$.

Expanding the determinant $M_{i_1,\dots,i_{t-1},m;j_1,\dots,j_{t-1},m}$ about column m we obtain

$$M_{i_1,\dots,i_{t-1},m;j_1,\dots,j_{t-1},m} = \sum_{h=1}^{t}(-1)^h x_{i_h,m} M_{i_1,\dots,i_{h-1},i_{h+1},\dots,i_{t-1},m;j_1,\dots,j_{t-1}}.$$

We are adopt the convention that $i_t=j_t=k_t=l_t=m$.

Substituting this expression in the equation (9), we get

$$M_{i_1,\dots,i_{t-1},m;j_1,\dots,j_{t-1},m}\cdot M_{k_1,\dots,k_{t-1},m;l_1,\dots,l_{t-1},m} - M_{k_1,\dots,k_{t-1},m;l_1,\dots,l_{t-1},m}\cdot M_{i_1,\dots,i_{t-1},m;j_1,\dots,j_{t-1},m} =$$

$$\sum_{h=1}^{t}(-1)^h x_{i_h,m} M_{i_1,\dots,i_{h-1},i_{h+1},\dots,i_{t-1},m;j_1,\dots,j_{t-1}} M_{k_1,\dots,k_{t-1},m;l_1,\dots,l_{t-1},m} -$$

$$\sum_{h=1}^{t}(-1)^h x_{k_h,m} M_{k_1,\dots,k_{h-1},k_{h+1},\dots,k_{t-1},m;l_1,\dots,l_{t-1}} M_{i_1,\dots,i_{t-1},m;j_1,\dots,j_{t-1},m} \in I_S$$

The coefficient of $x_{m,m}$ in (9) is then

$$(-1)^t M_{i_1,\dots,i_{t-1};j_1,\dots,j_{t-1}} M_{k_1,\dots,k_{t-1},m;l_1,\dots,l_{t-1},m} - (-1)^t M_{k_1,\dots,k_{t-1};l_1,\dots,l_{t-1}} M_{i_1,\dots,i_{t-1},m;j_1,\dots,j_{t-1},m}.$$

Since (9) is an equation of the form $\alpha x_{m,m}+\beta=0$ mod. I_Y, and since this equality has to hold for any $x_{m,m}$, we deduce that $\alpha=0$ mod. I_Y. Equivalently,

$$M_{i_1,\dots,i_{t-1};j_1,\dots,j_{t-1}} M_{k_1,\dots,k_{t-1},m;l_1,\dots,l_{t-1},m} - M_{k_1,\dots,k_{t-1};l_1,\dots,l_{t-1}} M_{i_1,\dots,i_{t-1},m;j_1,\dots,j_{t-1},m} \in I_Y,$$

that is what we wanted to prove. □

We want to emphasize a consequence of the proof of Theorem 2.3.

Corollary 2.7. *Every symmetric determinantal scheme X can be G-bilinked in m steps to a complete intersection, where m is the size of the matrix defining X. In particular, every symmetric determinantal scheme is glicci.*

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