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Quadri-tilings of the plane

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Abstract

We introduce *quadri-tilings* and show that they are in bijection with dimer models on a *family* of graphs $\{R^*\}$ arising from rhombus tilings. Using two height functions, we interpret a sub-family of all quadri-tilings, called *triangular quadri-tilings*, as an interface model in dimension 2+2. Assigning "critical" weights to edges of R^*, we prove an explicit expression, only depending on the local geometry of the graph R^*, for the minimal free energy per fundamental domain Gibbs measure; this solves a conjecture of [8]. We also show that when edges of R^* are asymptotically far apart, the probability of their occurrence only depends on this set of edges. Finally, we give an expression for a Gibbs measure on the set of *all* triangular quadri-tilings whose marginals are the above Gibbs measures, and conjecture it to be that of minimal free energy per fundamental domain.

1 Introduction

In this paper, we introduce **quadri-tilings** and a sub-family of all quadri-tilings called **triangular quadri-tilings**. In order to explain the originality of this model, let us go back to the yet classical domino and lozenge tilings, see for example [3, 5, 7, 18]. Both models are dimer models (see below) on a *fixed* graph, the square lattice $\mathbb{Z}^2$, and the equilateral triangular lattice $\mathbb{T}$, respectively. By the means of a *height function*, they can be interpreted as random discrete interfaces in dimension $2+1$, that is as random discrete surfaces of dimension 2 in a space of dimension 3 that have been projected to the plane [18]. Keeping this in mind, one can now explain the interesting feature of quadri-tilings: they correspond to dimer models on a *family* of graphs, instead of a fixed graph as was the case up to now. Moreover, by the means of *two* height functions, triangular quadri-tilings can be interpreted as random interfaces in dimension $2+2$, that is random surfaces of dimension 2 in a space of dimension 4, that have been projected to the plane. It is the first such model arising from dimer models. Using tools of the dimer model, we are able to give an explicit expression for a Gibbs measure on triangular quadri-tilings, as well as a surprising property of the asymptotics of this measure. In the course of doing so, we prove a conjecture of [8].

Quadri-tilings are defined as follows. Consider the set of right triangles whose hypotenuses have length two. Color the vertex at the right angle black, and the other two vertices white. A **quadri-tile** is a quadrilateral obtained from two such triangles in two different ways: either glue them along the hypotenuse, or supposing they have a leg of the same length, glue them along this edge matching the black (white) vertex to the black (white) one, see Figure 1. Note that both types of quadri-tiles have four vertices. A **quadri-tiling** of the plane is an edge-to-edge tiling of the plane by quadri-tiles that respects the coloring of the vertices, that is black (resp. white) vertices are

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matched to black (resp. white) ones. An example of quadri-tiling is given in Figure 1. In all that follows, we consider quadri-tilings of the plane that use finitely many different quadri-tiles up to isometry.

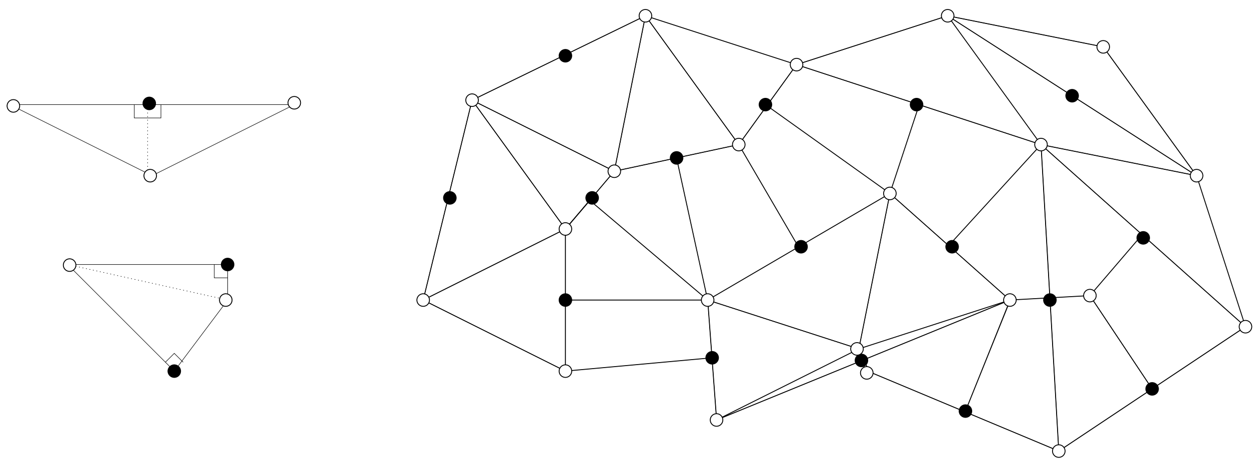

Figure 1: Two types of quadri-tiles (left), and a quadri-tiling (right).

The goal of Section 2 is to precisely describe the features of quadri-tilings. In order to give some insight, let us define **2-tiling models** or equivalently **dimer models**. A **2-tile** of an infinite graph G is a polygon made of two adjacent inner faces of G, and a **2-tiling** of G is a covering of G with 2-tiles, such that there are no holes and no overlaps. The **dual graph** G^* of G is the graph whose vertices correspond to faces of G, two vertices of G^* being joined by an edge if the corresponding faces are adjacent. A **dimer configuration** of G^*, also called **perfect matching**, is a subset of edges of G^* which covers each vertex exactly once. Then 2-tilings of the graph G are in bijection with dimer configurations of the dual graph G^*, as explained by the following correspondence. Denote by f^* the dual vertex of a face G, and consider an edge f^*g^* of G^*. We say that the 2-tile of G made of the adjacent faces f and g is the 2-tile **corresponding** to the edge f^*g^*. Then, 2-tiles corresponding to edges of a dimer configuration of G^* form a 2-tiling of G. Let us denote by $\mathcal{M}(G^*)$ the set of perfect matching of the graph G^*.

Prior to describing Section 2, we need one more definition. If R is a rhombus tiling of the plane, then the corresponding **rhombus-with-diagonals tiling**, denoted by R, is the graph obtained from R by adding the diagonals of the rhombi. In the whole of this paper, we suppose that rhombus tilings of the plane have only finitely many rhombus angles.

In Section 2.1, we prove the main feature of quadri-tilings, i.e. that they correspond to 2-tilings on a family of graphs, which consists of rhombus-with-diagonals tilings. More precisely, if T is a quadri-tiling, then by a geometric construction, we associate to T a rhombus-with-diagonals tiling $R(T)$, such that T is a 2-tiling of $R(T)$. The corresponding rhombus tiling $\mathsf{R}(T)$ is called the **underlying rhombus tiling** of T.

Triangular quadri-tilings consist in the sub-family of all quadri-tilings whose underlying tiling is a lozenge tiling, where **lozenges** are defined to be 60°-rhombi; refer to Figure 2 for an example, and to Section 2.1 for the construction of the underlying tiling. In order to distinguish general rhombus tilings, denoted R, from lozenge tilings, we denote the latter by L. The set of all triangular quadri-tilings up to isometry is denoted by $\mathcal{Q}$. Note that $\mathcal{Q}$ corresponds to two superposed dimer models.

Indeed, let T be a triangular quadri-tiling, then T is a 2-tiling of its underlying lozenge-with-diagonals tiling $L(T)$, moreover the corresponding lozenge tiling $\mathsf{L}(T)$ is a 2-tiling of the equilateral triangular lattice $\mathbb{T}$.

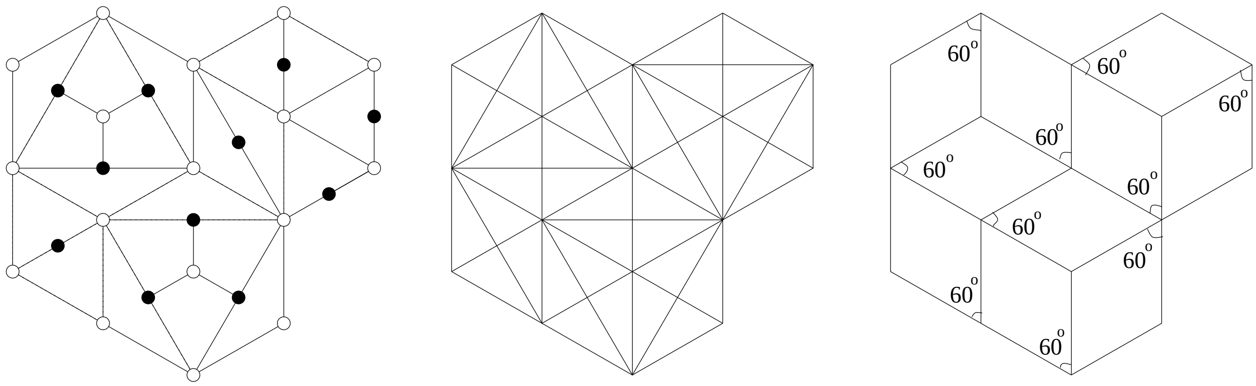

Figure 2: Triangular quadri-tiling T (left), underlying lozenge-with-diagonals tiling $L(T)$ (middle), corresponding lozenge tiling $\mathsf{L}(T)$ (right).

Section 2.2 consists in the geometric interpretation of triangular quadri-tilings using height functions. On the vertices of every triangular quadri-tiling T, we define a $\mathbb{Z}$-valued function h_1, called the **first height function**, corresponding to the "height" of T interpreted as a 2-tiling of its underlying lozenge-with-diagonals tiling $L(T)$. Then, we assign a **second height function** h_2 (Thurston's height function on lozenges [18]) corresponding to the height of $\mathsf{L}(T)$ interpreted as a 2-tiling of $\mathbb{T}$, see Figure 4. Hence triangular quadri-tilings are characterized by two height functions, and so can be interpreted as discrete interfaces in dimension $2+2$. In Section 2.3, we give elementary operations that allow to transform any triangular quadri-tiling of a simply connected region into any other.

The dimer model belongs to the field of statistical mechanics, hence there are natural measures to consider, Boltzmann and Gibbs measures, which are defined as follows. Let G be an infinite graph, and let ν be a positive weight function on the edges of G^*. Consider a finite sub-graph G^1 of G, then the **Boltzmann measure** on the set of dimer configurations $\mathcal{M}(G^{1*})$ of G^{1*}, corresponding to the weight function ν, is defined by

$$\mu^1(M) = \frac{\prod_{e\in M}\nu(e)}{Z(G^{1*},\nu)},$$

where $Z(G^{1*},\nu) = \sum_{M\in\mathcal{M}(G^{1*})}\prod_{e\in M}\nu(e)$ is the **dimer partition function**. A **Gibbs measure** is a probability measure on $\mathcal{M}(G^*)$ with the following property. If the matching in an annular region is fixed, the matchings inside and outside of the annulus are independent of each other, and the probability of any interior matching M is proportional to $\prod_{e\in M}\nu(e)$.

An important question in solving a dimer model is the study of local statistics, i.e. to obtain an explicit expression for the set of Gibbs measures. Kenyon, Okounkov and Sheffield [9] give such an expression for the two-parameter family of Gibbs measures on dimer configurations of doubly periodic bipartite graphs. The expression they obtain involves the limiting inverse Kasteleyn matrix which is hard to evaluate in general, often implying elliptic integrals. In another paper [8], for graphs G which have bipartite duals and satisfy a geometric condition called **isoradiality**, Kenyon defines a specific weight function on the edges of G^* called the **critical** weight function. He also defines the Dirac operator K indexed by the vertices of G^*, and gives an explicit expression for its inverse K^{-1} (see also Sections 3.1 and 3.2). The expression for K^{-1} has the interesting property of only depending on the *local geometry* of the graph. Kenyon conjectures that K^{-1} is, in some sense to be determined, the limiting inverse Kasteleyn matrix. In Section 3, we consider a general rhombus-with-diagonals tiling of the plane R. It has the property of being an isoradial graph, so that we assign the critical weight function to edges of R^*. Theorem 3.2 of Section 3.3 (see also Theorem 1.1 below) proves an explicit expression for a Gibbs measure μ^R on $\mathcal{M}(R^*)$, as a function of K and K^{-1}.

For every subset of edges $e_1 = w_1 b_1, \ldots, e_k = w_k b_k$ of R^*, the **cylinder** $\{e_1, \ldots, e_k\}$ is defined to be the set of dimer configurations of R^* that contain these edges. Then

Theorem 1.1 *There is a probability measure μ^R on $\mathcal{M}(R^*)$ such that, for every cylinders $\{e_1, \ldots, e_k\}$ of R^*,*

$$\mu^R(e_1, \ldots, e_k) = \left(\prod_{i=1}^{k} K(w_i, b_i) \right) \det_{1 \le i,\, j \le k} \left(K^{-1}(b_i, w_j) \right). \qquad (1)$$

Moreover μ^R is a Gibbs measure on $\mathcal{M}(R^*)$. When R^* is doubly periodic, μ^R is the unique Gibbs measure which has minimal free energy per fundamental domain among the two-parameter family of ergodic Gibbs measures of [9].

- Note that we do not ask the graph R^* to be periodic. The proof of Theorem 1.1 is the subject of Section 4. The argument in the case where R^* is not periodic relies on the argument in the case where R^* is doubly periodic, combined with a non-trivial geometric property of rhombus tilings proved in Proposition 4.1: "every finite simply connected sub-graph of a rhombus tiling can be embedded in a periodic rhombus tiling of the plane."

- From the proof of Theorem 1.1, it appears that the statement is true for all doubly periodic isoradial graphs with bipartite duals. Hence, Theorem 1.1 solves the conjecture of [8] of interpreting the inverse Dirac operator as the limiting inverse Kasteleyn matrix. Moreover, the fact that the measure μ^R is of minimal free energy per fundamental domain makes it of special interest among the two-parameter family of ergodic Gibbs measures of [9].

- Using the locality property of K^{-1} mentioned above, we deduce that the expression (1) only depends on the local geometry of the graph, hence it yields an easy way of computing local statistics explicitly. This is very surprising in regards of the expression obtained in [9], and we believe this locality property to be true only in the isoradial case with critical weights.

In Section 5, we extend the notion of Gibbs measure to the set of all triangular quadri-tilings $\mathcal{Q}$. Then, as a corollary to Theorem 1.1 we deduce an explicit expression for such a Gibbs measure μ, and conjecture it to be that of minimal free energy per fundamental domain among a four parameter family of Gibbs measures.

In Section 6, we consider a general rhombus-with-diagonals tiling of the plane R. We assign the critical weight function to edges of R^*, and let K be the Dirac operator indexed by vertices of R^*. Theorem 6.1 of Section 6.1 (see also Theorem 1.2 below) establishes that asymptotically (as $|b - w| \to \infty$) and up to the second order term, $K^{-1}(b, w)$ only depends on the rhombi to which the vertices b and w belong, and else is independent of the structure of the graph R. For a general isoradial graph, Kenyon [8] gives an asymptotic formula for $K^{-1}(b, w)$ which depends on the angles of an edge-path from w to b. Hence, it is an interesting and surprising fact that the dependence on the edges along the path should asymptotically disappear in the case of rhombus-with-diagonals tilings.

Theorem 1.2 *As $|b-w|\to\infty$, $K^{-1}(b,w)$ is equal to*

$$\frac{1}{2\pi}\left(\frac{1}{b-w}+\frac{e^{-i(\theta_1+\theta_2)}}{\bar{b}-\bar{w}}\right)+\frac{1}{2\pi}\left(\frac{e^{2i\theta_1}+e^{2i\theta_2}}{(b-w)^3}+\frac{e^{-i(3\theta_1+\theta_2)}+e^{-i(\theta_1+3\theta_2)}}{(\bar{b}-\bar{w})^3}\right)+O\left(\frac{1}{|b-w|^3}\right).$$

As a consequence of Theorem 1.2, we deduce that when edges $e_1,\dots,e_k$ of R^* are asymptotically far apart, $\mu^R(e_1,\dots,e_k)$ only depends on the rhombi to which the edges $e_1,\dots,e_k$ belong, and else is independent of the structure of the graph R (Corollary 6.2). We conclude by giving a consequence of Corollary 6.2 for the measure μ on triangular quadri-tilings (Corollary 6.4).

Acknowledgments: We thank Richard Kenyon for proposing the quadri-tiling model and asking the questions related to it; we are grateful to him for the many enlightening discussions. We also thank the referee for the many pertinent remarks which have helped to increase the quality of this paper.

2 Features of quadri-tilings

2.1 Underlying rhombus-with-diagonals tilings

Lemma 2.1 *Quadri-tilings are in one-to-one correspondence with 2-tilings of graphs which are rhombus-with-diagonals tilings of the plane.*

Proof: Consider a quadri-tiling of the plane T. Denote by R the tiling of the plane obtained from T by drawing, for each quadri-tile, the edge separating the two right triangles. Let b be a black vertex of R, denote by $w_1,\dots,w_k$ the neighbors of b in cclw (counterclockwise) order. In each right triangle, the black vertex is adjacent to two white vertices, and since the gluing respects the coloring of the vertices, $w_1,\dots,w_k$ are white vertices. Moreover, b is at the right angle, so $k=4$ and the edges w_1w_2, w_2w_3, w_3w_4, w_4w_1 are hypotenuses of right triangles. Therefore $w_1,\dots,w_4$ form a side-length-2 rhombus, and b stands at the crossing of its diagonals. This is true for any black vertex b of R, so R is a rhombus-with-diagonals tiling of the plane, and T is a 2-tiling of R. □

As a consequence of Lemma 2.1, a quadri-tiling T is a 2-tiling of a unique rhombus-with-diagonals tiling, which we call the **underlying rhombus-with-diagonals tiling**, and denote by $R(T)$, see Figure 2.

2.2 Height functions

We define a **first height function** h_1 on vertices of every quadri-tiling T. Moreover, when T is a triangular quadri-tiling, we define a **second height function** h_2 on vertices of T. Using h_1 and h_2, we interpret triangular quadri-tilings as discrete 2-dimensional surfaces in a 4-dimensional space projected to the plane.

2.2.1 First height function

Consider a quadri-tiling of the plane T, then T is a 2-tiling of its underlying rhombus-with-diagonals tiling $R(T)$. In order to define the first height function h_1, we need a bipartite coloring of the faces of $R(T)$, which is given by the following.

Lemma 2.2 *Let* R *be a rhombus tiling of the plane, and* R *be the corresponding rhombus-with-diagonals tiling. Then* R *has a bipartite coloring of its faces which is also a bipartite coloring of the vertices of* R^**.*

Proof: Cycles corresponding to the faces of the graph R have length four, thus R has a bipartite coloring of its vertices, say black and white. Consider a face of R and orient its boundary edges cclw. If the white vertex of the hypotenuse-edge comes before the black one, assign color black to the face, else assign color white. This defines a bipartite coloring of the faces of R, which is also a bipartite coloring of the vertices of R^* (see Figure 3).

□

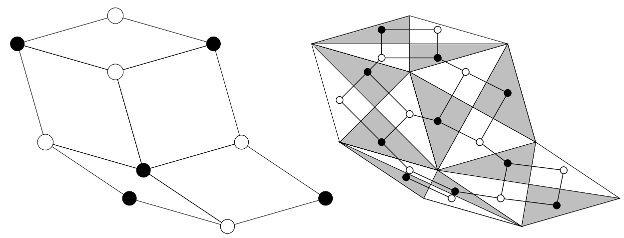

Figure 3: Bipartite coloring of the vertices of R (left), and corresponding bipartite coloring of the faces of R and of the vertices of R^* (right).

Consider the bipartite coloring of the faces of $R(T)$. Orient the edges around the black faces cclw, edges around the white faces are then oriented cw, and define h_1 on the vertices of T as follows. Fix a vertex v_1 on a boundary edge of a rhombus of $R(T)$, and set $h_1(v_1) = 0$. For every other vertex v of T, take an edge-path γ_1 from v_1 to v which follows the boundaries of the quadri-tiles of T. The first height function h_1 changes by ± 1 along each edge of γ_1: if an edge is oriented in the direction of the path, then h_1 increases by 1, if it is oriented in the opposite direction, then h_1 decreases by 1. The value $h_1(v)$ is independent of the path γ_1 because the plane is simply connected, and the height change around any quadri-tile is zero. An example of computation of h_1 is given in Figure 4.

The following lemma gives a bijection between 2-tilings of a rhombus-with-diagonals tiling R and first height functions defined on vertices of R.

Lemma 2.3 *Fix a vertex* v_1 *on a boundary edge of a rhombus of* R*. Let* $\tilde{h_1}$ *be a* $\mathbb{Z}$*-valued function on the vertices of* R *satisfying the following two conditions:*

- $\tilde{h_1}(v_1) = 0$,
- $\tilde{h_1}(v) = \tilde{h_1}(u) + 1$, *or* $\tilde{h_1}(v) = \tilde{h_1}(u) - 2$, *for any edge* uv *oriented from* u *to* v.

Then, there is a bijection between functions $\tilde{h_1}$ *satisfying these two conditions and 2-tilings of* R*.*

Proof: The idea of the proof closely follows [3]. If T is a 2-tiling of R, then the first height function defined above satisfies the two conditions of the lemma: if an edge uv, oriented from u to v, belongs to the boundary of a quadri-tile, it satisfies $h_1(v) = h_1(u) + 1$, else if it lies across a quadri-tile, it satisfies $h_1(v) = h_1(u) - 2$.
Conversely, consider a $\mathbb{Z}$-valued function $\tilde{h_1}$ as in the lemma. Then, anytime there is an edge uv

satisfying $|\tilde{h_1}(v) - \tilde{h_1}(u)| = 2$, put a quadri-tile made of the two right triangles adjacent to this edge. This defines a 2-tiling of R. $\square$

2.2.2 Second height function

Consider a triangular quadri-tiling T. Let $L(T)$ be its underlying lozenge-with-diagonals tiling, and h_1 be the first height function on vertices of T. The lozenge tiling $\mathsf{L}(T)$ corresponding to $L(T)$ is a 2-tiling of the equilateral triangular lattice $\mathbb{T}$. Moreover $\mathbb{T}$ has a bipartite coloring of its faces, say black and white. Orient the edges around the black faces cclw, edges around the white faces are then oriented cw. Thurston [18] defines the second height function h_2 as follows: choose a vertex v_2 of $\mathsf{L}(T)$, and set $h_2(v_2) = 0$. For every other vertex v of $\mathsf{L}(T)$, take an edge-path γ_2 from v_2 to v which follows the boundaries of the lozenges of $\mathsf{L}(T)$. The second height function h_2 changes by ± 1 along each edge of γ_2: if an edge is oriented in the direction of the path, then h_2 increases by 1, if it is oriented in the opposite direction, then h_2 decreases by 1. The value $h_2(v)$ is independent of the path γ_2. For convenience, we choose v_2 to be the same vertex as v_1, and denote this common vertex by v_0, so that $h_1(v_0) = h_2(v_0) = 0$. An analog to Lemma 2.3 gives a bijection between second height functions and lozenge tilings of the plane, hence we deduce that triangular quadri-tilings are characterized by h_1 and h_2.

Let us define a natural value for the second height function at the vertex in the center of the lozenges of $\mathsf{L}(T)$. When going cclw around the vertices of a lozenge ℓ of $\mathsf{L}(T)$, starting from the smallest value of h_2 say h, vertices take on successive values $h, h+1, h+2, h+1$, so that we assign value $h+1$ to the vertex in the center of the lozenge ℓ. An example of computation of h_2 is given in Figure 4.

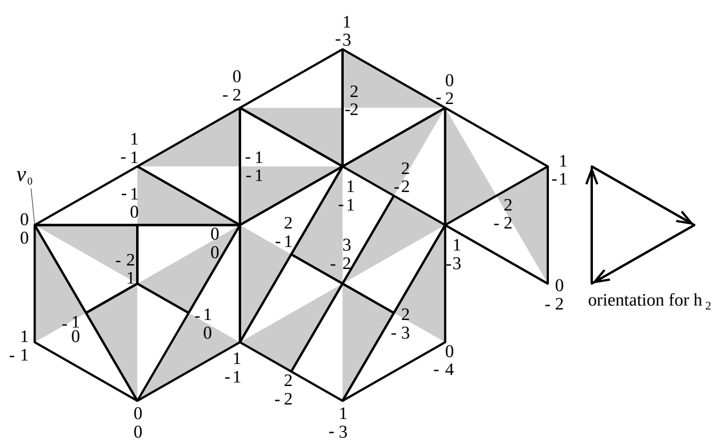

Figure 4: Triangular quadri-tiling with height functions h_1 (above) and h_2 (below).

In Thurston's geometric interpretation [18], a lozenge tiling is seen as a surface S in $\mathbb{Z}^3$ (where the diagonals of the cubes are orthogonal to the plane) that has been projected orthogonally to the plane. The surface S is determined by the height function h_2. In a similar way, a triangular quadri-tiling of the plane T can be seen as a surface S_1 in a 4-dimensional space that has been projected to the plane; S_1 can also be projected to $\widetilde{\mathbb{Z}}^3$ ($\widetilde{\mathbb{Z}}^3$ is the space $\mathbb{Z}^3$ where cubes are drawn with diagonals on their faces), and one obtains a surface S_2. When projected to the plane, S_2 is the underlying lozenge-with-diagonals tiling $L(T)$.

2.3 Elementary operations

Consider a finite simply connected sub-graph G of the equilateral triangular lattice $\mathbb{T}$, and let ∂G be the cycle of G consisting of its boundary edges. Denote by $\mathcal{Q}(\partial G)$ the set of triangular quadri-tilings whose underlying tilings are lozenge tilings of G. Let L^1 be a lozenge tiling of G, and L^1 be the corresponding lozenge-with-diagonals tiling. Then using the bijection between the first height function and 2-tilings of L^1 we obtain, in exactly the same way as Elkies, Kuperberg, Larsen, Propp [3] have for domino tilings, the following lemma:

Lemma 2.4 *Every 2-tiling of L^1 can be transformed into any other by a finite sequence of the following operations, (in brackets is the number of possible orientations for the graph corresponding to the operation):*

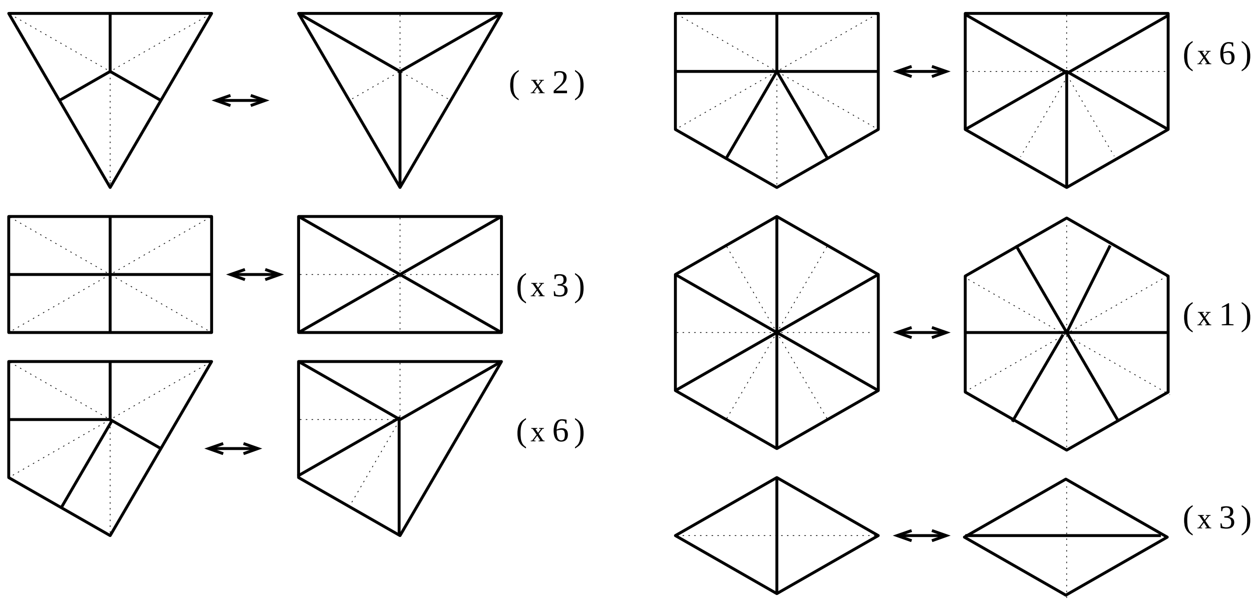

Figure 5: Quadri-tile operations.

Let us call **quadri-tile operations** the 21 operations described in Lemma 2.4. Moreover, every lozenge tiling of G can be transformed into any other by a finite sequence of **lozenge operations**:

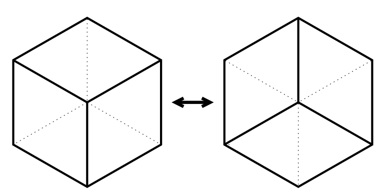

Figure 6: Lozenge operations.

Note that if L is any lozenge tiling of G, then L is quadri-tilable with quadri-tiles obtained by cutting in two every lozenge along one of its diagonals. Moreover, when one performs a lozenge operation on such a quadri-tiling, one still obtains a quadri-tiling of $\mathcal{Q}(\partial G)$. Let us call **elementary operations** the quadri-tile operations and the lozenge operations performed on quadri-tilings as described above. Then we have:

Lemma 2.5 *Every quadri-tiling of $\mathcal{Q}(\partial G)$ can be transformed into any other by a finite sequence of elementary operations.*

Proof: This results from Lemma 2.4, and the above observation. □

3 Gibbs measure on quadri-tilings

This section aims at giving a precise statement of Theorem 1.1 of the introduction (see Theorem 3.2 below). Sections 3.1 and 3.2 are taken form [8] and give a precise definition of an isoradial graph, the critical weight function, the Dirac and inverse Dirac operator. Section 3.3 consists in the statement of Theorem 3.2.

3.1 Isoradial graphs and critical weight function

The definition of the critical weight function follows [8]. It is defined on edges of graphs satisfying a geometric condition called **isoradiality**: all faces of an isoradial graph are inscribable in a circle, and all circumcircles have the same radius.

Note that if R is a rhombus tiling of the plane, then the corresponding rhombus-with-diagonals tiling R is an isoradial graph. Let us consider the embedding of the dual graph R^* (the same notation is used for the one-skeleton of a graph and its embedding) where the dual vertices are the circumcenters of the corresponding faces. Then R^* is also an isoradial graph and the circumcenters of the faces are the vertices of R.

To each edge e of R^*, we associate a unit side-length rhombus $R(e)$ whose vertices are the vertices of e and the vertices of its dual edge. Let $\widetilde{R} = \cup_{e \in R^*} R(e)$. Note that the dual edges corresponding to the boundary edges of the rhombi of R have length zero, and that the rhombi associated to these edges are degenerated.

For each edge e of R^*, define $\nu(e) = 2\sin\theta$, where 2θ is the angle of the rhombus $R(e)$ at the vertex it has in common with e; θ is called the **rhombus angle** of the edge e. Note that $\nu(e)$ is the length of e^*, the dual edge of e. The function ν is called the **critical weight function**.

3.2 Dirac and inverse Dirac operator

Results and definitions of this section are due to Kenyon [8], see also Mercat [14]. Note that this section (as the previous one) is true for general isoradial graphs with bipartite dual graphs.

Let R be a rhombus tiling of the plane, then by Lemma 2.2, R^* is a bipartite graph. Let B (resp. W) be the set of black (resp. white) vertices of R^*, and denote by ν the critical weight function on the edges of R^*. The Hermitian matrix K indexed by the vertices of R^* is defined as follows. If v_1 and v_2 are not adjacent $K(v_1, v_2) = 0$. If $w \in W$ and $b \in B$ are adjacent vertices, then $K(w, b) = \overline{K(b, w)}$ is the complex number of modulus $\nu(wb)$ and direction pointing from w to b. If w and b have the same image in the plane, then $|K(w, b)| = 2$, and the direction of $K(w, b)$ is that which is perpendicular to the corresponding dual edge, and has sign determined by the local orientation. The infinite matrix K defines the **Dirac operator** $K : \mathbb{C}^{V(R^*)} \to \mathbb{C}^{V(R^*)}$, by

$$(Kf)(v) = \sum_{u \in R^*} K(v,u) f(u).$$

where $V(R^*)$ denotes the set of vertices of the graph R^*.

The **inverse Dirac operator** K^{-1} is defined to be the operator satisfying:

1. $KK^{-1} = \mathrm{Id}$,
2. $K^{-1}(b,w) \to 0$, when $|b - w| \to \infty$.

Kenyon [8] obtains an explicit expression for K^{-1}. Before stating his theorem, we need to define the rational functions $f_{wv}(z)$. Let w be a white vertex of R^*. For every other vertex v, define $f_{wv}(z)$ as follows. Let $w = v_0, v_1, v_2, \ldots, v_k = v$ be an edge-path of $\widetilde{R}$, from w to v. Each edge $v_j v_{j+1}$ has exactly one vertex of R^* (the other is a vertex of R). Direct the edge away from this vertex if it is white, and towards this vertex if it is black. Let $e^{i\alpha_j}$ be the corresponding vector in $\widetilde{R}$ (which may point contrary to the direction of the path), then f_{wv} is defined inductively along the path, starting from

$$f_{ww}(z) = 1.$$

If the edge leads away from a white vertex, or towards a black vertex, then

$$f_{wv_{j+1}}(z) = \frac{f_{wv_j}(z)}{z - e^{i\alpha_j}},$$

else, if it leads towards a white vertex, or away from a black vertex, then

$$f_{wv_{j+1}}(z) = f_{wv_j}(z).(z - e^{i\alpha_j}).$$

The function $f_{wv}(z)$ is well defined (i.e. independent of the edge-path of $\widetilde{R}$ from w to v), because the multipliers for a path around a rhombus of $\widetilde{R}$ come out to 1. For a black vertex b the value $K^{-1}(b,w)$ will be the sum over the poles of $f_{wb}(z)$ of the residue of f_{wb} times the angle of z at the pole. However, there is an ambiguity in the choice of angle, which is only defined up to a multiple of 2π. To make this definition precise, angles are assigned to the poles of $f_{wb}(z)$. Working on the branched cover of the plane, branched over w, so that for each black vertex b in this cover, a real angle θ_0 is assigned to the complex vector $b - w$, which increases by 2π when b winds once around w. In the branched cover of the plane, a real angle in $[\theta_0 - \pi + \Delta, \theta_0 + \pi - \Delta]$ can be assigned to each pole of f_{wb}, for some small $\Delta > 0$.

Theorem 3.1 [8] *There exists a unique K^{-1} satisfying the above two properties, and K^{-1} is given by:*

$$K^{-1}(b,w) = \frac{1}{4\pi^2 i} \int_C f_{wb}(z) \log z \, dz,$$

where C is a closed contour surrounding cclw the part of the circle $\{e^{i\theta} | \theta \in [\theta_0 - \pi + \Delta, \theta_0 + \pi - \Delta]\}$, which contains all the poles of f_{wb}, and with the origin in its exterior, see Figure 7.

The remarkable property of $K^{-1}(b,w)$ is that it only depends on the local geometry of the graph, i.e. on an edge-path from w to b.

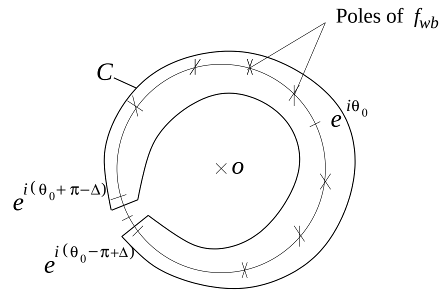

Figure 7: An example of contour C.

3.3 Statement of result

Let R be a rhombus tiling of the plane, and R be the corresponding rhombus-with-diagonals tiling. Suppose that the critical weight function ν is assigned to edges of R^*, and let K be the Dirac operator indexed by the vertices of R^*. Moreover, recall that if $e_1 = w_1b_1, \ldots, e_k = w_kb_k$ is a subset of edges of R^*, then the **cylinder** $\{e_1, \ldots, e_k\}$ is defined to be the set of dimer configurations of R^* which contain these edges. Let $\mathcal{A}$ be the field consisting of the empty set and of the finite disjoint unions of cylinders. Denote by $\sigma(\mathcal{A})$ the σ-field generated by $\mathcal{A}$.

Theorem 3.2 *There is a probability measure μ^R on $(\mathcal{M}(R^*), \sigma(\mathcal{A}))$ such that for every cylinder $\{e_1, \ldots, e_k\}$ of R^*,*

$$\mu^R(e_1, \ldots, e_k) = \left(\prod_{i=1}^{k} K(w_i, b_i)\right) \det_{1\leq i,\, j\leq k} \left(K^{-1}(b_i, w_j)\right). \tag{2}$$

Moreover μ^R is a Gibbs measure on $\mathcal{M}(R^*)$. When R^* is doubly periodic, μ^R is the unique Gibbs measure which has minimal free energy per fundamental domain among the two-parameter family of ergodic Gibbs measures of [9].

From the proof, it appears that Theorem 3.2 is in fact true for all doubly periodic isoradial graphs with bipartite dual graphs. We refer to the introduction for comments on Theorem 3.2.

4 Proof of Theorem 3.2

The proof of Theorem 3.2 uses Propositions 4.1 and 4.9 below. Proposition 4.1 is a geometric property of rhombus tilings and is the subject of Section 4.1. Proposition 4.9 concerns the convergence of the Boltzmann measure on some appropriate toroidal graphs, it is the subject of Section 4.3. In Section 4.2, we introduce the *real* Dirac operator and its inverse. This operator is related to the Dirac operator and is needed for the proof of Proposition 4.9. Theorem 3.2 is then proved in Section 4.4.

4.1 Geometric property of rhombus tilings

Proposition 4.1 *Let* R *be a rhombus tiling of the plane, then any finite simply connected sub-graph* P *of* R *can be embedded in a periodic rhombus tiling* S *of the plane.*

Proof: This proposition is a direct consequence of Lemmas 4.2, 4.3, and Theorem 4.4 below. □

The notion of **train-track** has been introduced by Mercat [13], see also Kenyon and Schlenker [8, 11]. A train-track of a rhombus tiling is a path of rhombi (each rhombus being adjacent along an edge to the previous rhombus) which does not turn: on entering a rhombus, it exits across the opposite edge. Train-tracks are assumed to be maximal in the sense that they extend in both directions as far as possible. Thus train-tracks of rhombus tilings of the plane are bi-infinite. Each rhombus in a train-track has an edge parallel to a fixed unit vector e, called the **transversal direction** of the train-track. Let us denote by t_e the train-track of transversal direction e. In an oriented train-track (i.e. the edges of the two parallel boundary paths of the train-track have the same given orientation), we choose the direction of e so that when the train-track runs in the direction given by the orientation, e points from the right to the left. The vector e is called the **oriented transversal direction** of the oriented train-track. A train-track cannot cross itself, and two different train-tracks can cross at most once. A finite simply connected sub-graph P of a rhombus tiling of the plane R is **train-track-convex**, if every train-track of R that intersects P crosses the boundary of P twice exactly.

Lemma 4.2 *Let* R *be a rhombus tiling of the plane, then any finite simply connected sub-graph* P *of* R *can be completed by a finite number of rhombi of* R *in order to become train-track-convex.*

Proof: Let $e_1, \dots, e_m$ be the boundary edges of P. Every rhombus of P belongs to two train-tracks of R, each of which can be continued in both directions up to the boundary of P. In both directions the intersection of each of the train-tracks and the boundary of P is an edge parallel to the transversal direction of the train-track. Thus, to take into account all train-tracks of R that intersect P, it suffices to consider for every i the train-tracks t_{e_i} associated to the boundary edges of P. Consider the following algorithm (see Figure 8).

Set $\mathsf{Q}_1 = \mathsf{P}$.

For $i = 1, \dots, m$, do the following:

Consider the train-track t_{e_i}, and let $2n_i$ be the number of times t_{e_i} intersects the boundary of Q_i.

- If $n_i > 1$: there are $n_i - 1$ portions of t_{e_i} that are outside of Q_i, denote them by ${t_{e_i}}^1, \dots, {t_{e_i}}^{n_i-1}$. Then, since Q_i is simply connected, for every j, $\mathsf{R} \setminus (\mathsf{Q}_i \cup {t_{e_i}}^j)$ is made of two disjoint sub-graphs of R, one of which is finite (it might be empty in the case where one of the two parallel boundary paths of ${t_{e_i}}^j$ is part of the boundary path of Q_i). Denote by ${g_{e_i}}^j$ the simply connected sub-graph of R made of the finite sub-graph of $\mathsf{R} \setminus (\mathsf{Q}_i \cup {t_{e_i}}^j)$ and of ${t_{e_i}}^j$. Denote by ${b_{e_i}}^j$ the portion of the boundary of Q_i which bounds ${g_{e_i}}^j$. Replace Q_i by $\mathsf{Q}_{i+1} = \mathsf{Q}_i \cup (\cup_{j=1}^{n_i-1} {g_{e_i}}^j)$. By this construction t_{e_i} intersects the boundary of Q_{i+1} exactly twice, and Q_{i+1} is simply connected.

- If $n_i = 1$: set $\mathsf{Q}_{i+1} = \mathsf{Q}_i$.

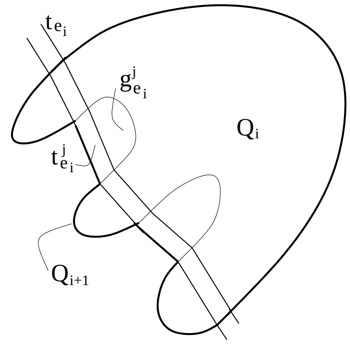

Figure 8: One step of the algorithm.

Let us show that at every step the train-tracks of R that intersect Q_i and Q_{i+1} are the same. By construction, boundary edges of Q_{i+1} are boundary edges of Q_i and of ${t_{e_i}}^j$, for every j. Let f be an edge on the boundary of Q_{i+1}, but not of Q_i, that is f is on the boundary of ${t_{e_i}}^j$ for some j, thus t_f crosses ${g_{e_i}}^j$. Since two train-tracks cross at most once, t_f has to intersect ${b_{e_i}}^j$, which means t_f also crosses Q_i. From this we also conclude that if a train-track intersects the boundary of Q_i twice, then it also intersects the boundary of Q_{i+1} twice.
Thus all train-tracks that intersect Q_{m+1} cross its boundary exactly twice, and Q_{m+1} contains P. □

Lemma 4.3 *Let* R *be a rhombus tiling of the plane. Then any finite simply connected train-track-convex sub-graph* P *of* R *can be completed by a finite number of rhombi in order to become a convex polygon* Q*, whose opposite boundary edges are parallel.*

Proof: Let $e_1, \dots, e_m$ be the boundary edges of P oriented cclw. Since P is train-track-convex, the train-tracks $t_{e_1}, \dots, t_{e_m}$ intersect the boundary of P twice, so that there are pairs of parallel boundary edges. Let us assume that the transversal directions of the train-tracks are all distinct (if this is not the case, one can always perturb the graph a little so that it happens). Let us also denote by $t_{e_1}, \dots, t_{e_m}$ the portions of the bi-infinite train-tracks of R in P. In what follows, indices will be denoted cyclically, that is $e_j = e_{m+j}$. Write x_j (resp. y_j) for the initial (resp. end) vertex of an edge e_j.
Let e_i, e_{i+1} be two adjacent boundary edges of P. Consider the translate e_{i+1}^t of e_{i+1} so that the initial vertex of e_{i+1}^t is adjacent to the initial vertex of e_i. Then we define the **turning angle from** e_i **to** e_{i+1} (also called exterior angle) to be the angle $\widehat{e_i e_{i+1}^t}$, and we denote it by $\theta_{e_i,e_{i+1}}$. If e_i, e_j are two boundary edges, then the **turning angle from** e_i **to** e_j is defined by $\sum_{\alpha=i}^{j-1} \theta_{e_\alpha,e_{\alpha+1}}$, and is denoted by θ_{e_i,e_j}.

Properties

1. $\sum_{\alpha=1}^{m} \theta_{e_\alpha,e_{\alpha+1}} = 2\pi$.

2. If e_i, e_j are two boundary edges, and if $\gamma = \{f_1, \ldots, f_n\}$ is an oriented edge-path in P from y_i to x_j, then $\theta_{e_i,e_j} = \theta_{e_i,f_1} + \sum_{\alpha=1}^{n-1} \theta_{f_\alpha,f_{\alpha+1}} + \theta_{f_n,e_j}$.

3. If e_i is a boundary edge of P, and e_k is the second boundary edge at which t_{e_i} intersects the boundary of P, then $\theta_{e_i,e_k} = \pi$. Thus e_k and e_i are oriented in the opposite direction, and we denote e_k by ${e_i}^{-1}$.

4. P is convex, if and only if every train-track of P crosses every other train-track of P.

We first end the proof of Lemma 4.3, and then prove Properties 1 to 4.
Note that Properties 1 and 2 are true for any finite simply connected sub-graph of R.
The number of train-tracks intersecting P is $n = m/2$. So that if every train-track crosses every other train-track, the total number of crossings is $n(n-1)/2$. Consider the following algorithm (see Figure 9 for an example).

Set $\mathsf{Q}_1 = \mathsf{P}$, $n_1 =$ the number of train-tracks that cross in Q_1.
For $i = 1, 2, \ldots$ do the following:

- If $n_i = n(n-1)/2$: then by Property 4, Q_i is convex.
- If $n_i < n(n-1)/2$: then by Property 4, $\theta_{e_{j_i},e_{j_i+1}} < 0$ for some $j_i \in \{1, \ldots, m\}$. Add the rhombus ℓ_{j_i} of parallel directions e_{j_i}, e_{j_i+1} along the boundary of Q_i. Set $\mathsf{Q}_{i+1} = \mathsf{Q}_i \cup \ell_{j_i}$, and rename the boundary edges $e_1, \ldots, e_m$ in cclw order. Then the number of train-tracks that cross in Q_{i+1} is $n_i + 1$, set $n_{i+1} = n_i + 1$. Note that if Property 4 is true for Q_i, it stays true for Q_{i+1}, and note that the same train-tracks intersect Q_i and Q_{i+1}.

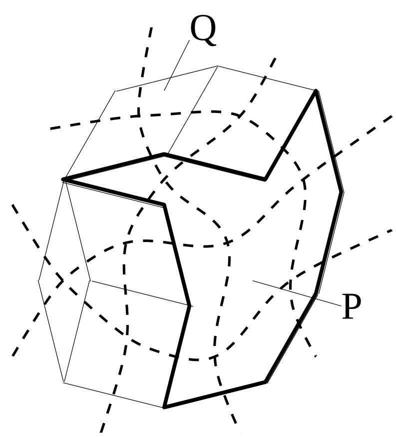

Figure 9: Example of application of the algorithm.

For the algorithm to be able to add the rhombus ℓ_{j_i} at every step, we need to check that:

$$\forall\, i,\ \theta_{e_{j_i-1},e_{j_i+1}} > -\pi, \text{ and } \theta_{e_{j_i},e_{j_i+2}} > -\pi. \tag{3}$$

Assume we have proved that for any finite simply connected train-track-convex sub-graph P of R we have:

$$\forall\, i, j,\ \theta_{e_i,e_j} > -\pi. \tag{4}$$

Then Properties 1 and 2 imply that if (4) is true for Q_i it stays true for Q_{i+1}, moreover (4) implies (3). So let us prove (4) by induction on the number of rhombi contained in P. If P is a rhombus, then (4) is clear. Now assume P is made of k rhombi. Consider the train-tracks in P adjacent to the boundary (every boundary edge e of P belongs to a rhombus of P which has parallel directions e and f; for every boundary edge e, the train-track of transversal direction f is the train-track adjacent to the boundary). Denote the train-tracks adjacent to the boundary by $t_1, \dots, t_p$ in cclw order, and write f_β for the oriented transversal direction of t_β (when the boundary edge-path of P is oriented cclw). Consider two adjacent boundary edges e_i, e_{i+1} of P that don't belong to the same boundary train-track. That is e_i belongs to t_β, and e_{i+1} to $t_{\beta+1}$. Then either $\widehat{f_\beta f_{\beta+1}} < 0$ or $\widehat{f_\beta f_{\beta+1}} > 0$, in the second case t_β and $t_{\beta+1}$ cross and their intersection is a rhombus ℓ_β of P. The rhombus ℓ_β has boundary edges e_i, e_{i+1}, and $f_{\beta+1} = {e_i}^{-1}, {f_\beta}^{-1} = {e_{i+1}}^{-1}$. Now Property 1 implies that $\sum_{\beta=1}^{p-1} \widehat{f_\beta f_{\beta+1}} = 2\pi$, so that there always exists β_0 such that $\widehat{f_{\beta_0} f_{\beta_0+1}} > 0$. Removing ℓ_{β_0} from P and using the assumption that P is train-track-convex, we obtain a graph P' made of $k-1$ rhombi which is train-track-convex. By induction, $\theta_{e,f} > -\pi$ for every boundary edge of P', and using Property 2, we conclude that this stays true for P.

Denote by Q the convex polygon obtained from P by the algorithm, and assume that opposite boundary edges are not parallel. Then there are indices i and j such that e_i comes before e_j, and e_j^{-1} comes before e_i^{-1}. This implies that $\theta_{e_i,e_j} = -\theta_{{e_j}^{-1},{e_i}^{-1}}$, so that one of the two angles is negative, which means Q can not be convex. Thus we have a contradiction, and we conclude that opposite boundary edges of Q are parallel.

Proof of Properties 1 to 4.

1. and 2. are straightforward.

3. When computing θ_{e_i,e_k} along the boundary edge-path of t_{e_i} we obtain π, so by Property 2 we deduce that $\theta_{e_i,e_k} = \pi$ in P.

4. P is convex if and only if, for every i, $\theta_{e_i,e_{i+1}} > 0$, which is equivalent to saying that, for every $i \neq j$, $\theta_{e_i,e_j} > 0$. Therefore Property 4 is equivalent to proving that $\theta_{e_i,e_j} > 0$, for every $i \neq j$, if and only if every train-track of P crosses every other train-track of P.
   Assume there are two distinct train-tracks t_{e_ℓ} and t_{e_k} that don't cross in P. Then, in cclw order around the boundary of P, we have either $e_\ell, {e_k}^{-1}, e_k, {e_\ell}^{-1}$, or $e_\ell, e_k, {e_k}^{-1}, {e_\ell}^{-1}$. It suffices to solve the second case, the first case being similar. By Property 1, $\theta_{e_\ell,e_k} + \theta_{e_k,{e_k}^{-1}} + \theta_{{e_k}^{-1},{e_\ell}^{-1}} + \theta_{{e_\ell}^{-1},e_\ell} = 2\pi$. Moreover by Property 3, $\theta_{e_k,{e_k}^{-1}} = \theta_{{e_\ell}^{-1},e_\ell} = \pi$, which implies $\theta_{e_\ell,e_k} = -\theta_{{e_k}^{-1},{e_\ell}^{-1}}$. Since all train-tracks have different transversal directions, either θ_{e_ℓ,e_k} or $\theta_{{e_k}^{-1},{e_\ell}^{-1}}$ is negative.
   Now take two boundary edges e_i, e_j of P (with $i \neq j$, and $e_j \neq {e_i}^{-1}$), and assume the train-tracks t_{e_i}, t_{e_j} cross inside P. Then in cclw order around the boundary of P, we have either $e_i, {e_j}^{-1}, {e_i}^{-1}, e_j$, or $e_i, e_j, {e_i}^{-1}, {e_j}^{-1}$. It suffices to solve the second case since the first case can be deduced from the second one. The intersection of t_{e_i} and t_{e_j} is a rhombus ℓ. Let $\tilde{e}_j^{-1}$ (resp. $\tilde{e}_i^{-1}$) be the boundary edge of ℓ parallel and closest to e_j (resp. e_i), oriented in the opposite direction, then $\theta_{\tilde{e}_j^{-1},\tilde{e}_i^{-1}} < 0$. Let γ_j (resp. γ_i) be the boundary edge-path of t_{e_j} (resp. t_{e_i}) from y_j to $\tilde{x}_j$ (resp. from $\tilde{y}_i$ to x_i), and let Q be the sub-graph of R whose boundary is $e_i, e_{i+1}, \dots, e_j, \gamma_j, \tilde{e}_j^{-1}, \tilde{e}_i^{-1}, \gamma_i$. Since t_{e_i} and t_{e_j} intersect the boundary of P twice, they also intersect the boundary of Q twice. Moreover t_{e_i} and t_{e_j} don't cross in Q, so that $\theta_{e_i,e_j} = -\theta_{\tilde{e}_j^{-1},\tilde{e}_i^{-1}} > 0$. □

□

Theorem 4.4 *Any convex 2n-gon* Q *whose opposite boundary edges are parallel and of the same length can be embedded in a periodic tiling of the plane by* Q *and rhombi.*

Proof: Let $e_1, \ldots, e_n, {e_1}^{-1}, \ldots, {e_n}^{-1}$ be the boundary edges of the polygon Q oriented cclw. If $n \leq 3$, then Q is either a rhombus or a hexagon, and it is straightforward that the plane can be tiled periodically with Q.
If $n > 4$, for $k = 1, \ldots, n-3$, do the following (see Figure 10): along e_{n-k} add the finite train-track $\tilde{t}_{e_{n-k}}$ of transversal direction e_{n-k}, going away from Q, whose boundary edges starting from the boundary of Q are:

$$e_1, \underbrace{e_2, e_1}, \underbrace{e_3, e_2, e_1}, \ldots, \underbrace{e_{n-k-2}, \ldots, e_1}.$$

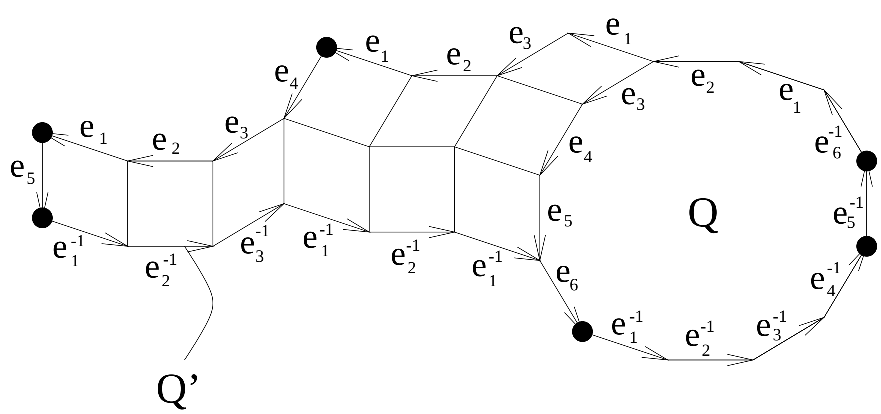

Figure 10: Fundamental domain of a periodic tiling of the plane by dodecagons and rhombi.

Since the polygon Q is convex, the rhombi that are added are well defined, moreover the intersection of $\tilde{t}_{e_i}$ and the boundary of Q is the edge e_i, and $\tilde{t}_{e_i}$ doesn't cross $\tilde{t}_{e_j}$ when $i \neq j$. So we obtain a new polygon Q' made of Q and rhombi, whose boundary edge-path is $\gamma_1, \ldots, \gamma_6$, (when starting from the edge ${e_n}^{-1}$ of Q), where:

$$\gamma_1 = {e_n}^{-1}, e_1, \underbrace{e_2, e_1}, \underbrace{e_3, e_2, e_1}, \ldots, \underbrace{e_{n-3}, \ldots, e_1},$$

$$\gamma_2 = e_{n-2}, \ldots, e_1,$$

$$\gamma_3 = e_{n-1},$$

$$\gamma_4 = \underbrace{{e_1}^{-1}, \ldots, {e_{n-3}}^{-1}}, \ldots, \underbrace{{e_1}^{-1}, {e_2}^{-1}}, {e_1}^{-1}, e_n,$$

$$\gamma_5 = {e_1}^{-1}, \ldots, {e_{n-2}}^{-1},$$

$$\gamma_6 = {e_{n-1}}^{-1}.$$

Noting that $\gamma_4 = {\gamma_1}^{-1}, \gamma_5 = {\gamma_2}^{-1}, \gamma_6 = {\gamma_3}^{-1}$, and using the fact that the plane can be tiled periodically with hexagons which have parallel opposite boundary edges, we deduce that the plane can be tiled with Q', that is it can be tiled periodically by Q and rhombi. $\square$

Remark 4.5 *It was pointed out by the referee that Theorem 4.4 might be known already. After a second look at the extensive literature on tilings, we were not able to find a reference, but any information is of course welcome.*

4.2 Real Dirac and inverse real Dirac operator

In the whole of this section we let R be a rhombus tiling of the plane, and R be the corresponding rhombus-with-diagonals tiling. Assume that the critical weight function ν is assigned to edges of R^*, and denote by K the Dirac operator indexed by the vertices of R^*.
The proof of Theorem 3.2 requires to take the limit of Boltzmann measures on some appropriate toroidal graphs (see Section 4.3). In order to do this, we need to introduce the **real Dirac operator** denoted by K. Both the Dirac operator K and the real Dirac operator K are represented by infinite weighted adjacency matrices indexed by the vertices of R^*. For K, the edges of R^* are un-oriented and weighted by their critical weight times a complex number of modulus 1. For K, edges of R^* are oriented with a *clockwise odd* orientation, and are weighted by their critical weight. Both weight functions yield the same probability measure on finite simply connected sub-graphs of R^*, but, and this is the reason why we introduce the real Dirac operator, these weights *do not* yield the same probability distribution on toroidal sub-graphs of R^*.
The structure of this section is close to that of Section 3.2. We first define the real Dirac operator K and its inverse K^{-1} and then, using results of [8], we prove the existence and uniqueness of K^{-1} by giving an explicit expression for K^{-1}. Note that this section is actually true for general isoradial graphs with bipartite dual graphs.

4.2.1 Real Dirac operator

Let us define an orientation of the edges of R^*. An elementary cycle C of R^* is said to be **clockwise odd** if, when traveling cw around the edges of C, the number of co-oriented edges is odd. Note that since R^* is bipartite, the number of contra-oriented edges is also odd. Kasteleyn [6] defines the orientation of the graph R^* to be **clockwise odd** if all elementary cycles are clockwise odd. He also proves that, for planar simply connected graphs, such an orientation always exists.
Consider a clockwise odd orientation of the edges of R^*. Define K to be the infinite adjacency matrix of the graph R^*, weighted by the critical weight function ν. That is, if v_1 and v_2 are not adjacent, $\mathsf{K}(v_1, v_2) = 0$. If $w \in W$ and $b \in B$ are adjacent vertices, then $\mathsf{K}(w, b) = -\mathsf{K}(b, w) = (-1)^{\mathbb{I}_{(w,b)}} \nu(wb)$, where $\mathbb{I}_{(w,b)} = 0$ if the edge wb is oriented from w to b, and 1 if it is oriented from b to w. The infinite matrix K defines the **real Dirac operator** K: $\mathbb{C}^{V(R^*)} \to \mathbb{C}^{V(R^*)}$, by

$$(\mathsf{K}f)(v) = \sum_{u \in R^*} \mathsf{K}(v, u) f(u),$$

The matrix K is also called a **Kasteleyn matrix** for the underlying dimer model.

4.2.2 Inverse real Dirac operator

The **inverse real Dirac operator** K^{-1} is defined to be the unique operator satisfying

1. $\mathsf{K}\mathsf{K}^{-1} = \mathrm{Id}$,

2. $\mathsf{K}^{-1}(b,w) \to 0$, when $|b-w| \to \infty$.

Let us define the rational functions $\mathsf{f}_{wx}(z)$. They are the analogous of the rational functions $f_{wv}(z)$, but are defined for vertices $x \in R^*$ (whereas the functions $f_{wv}(z)$ were defined for vertices $v \in \widetilde{R}$). Let $w \in W$, and let $x \in B$ (resp. $x \in W$); consider the edge-path $w = w_1, b_1, \ldots, w_k, b_k = x$ (resp. $w = w_1, b_1, \ldots, w_k, b_k, w_{k+1} = x$) of R^* from w to x. Let $R(w_j b_j)$ be the rhombus associated to the edge $w_j b_j$, and denote by w_j, x_j, b_j, y_j its vertices in cclw order; $e^{i\alpha_j}$ is the complex vector $y_j - w_j$, and $e^{i\beta_j}$ is the complex vector $x_j - w_j$. In a similar way, denote by w_{j+1}, x_j', b_j, y_j' the vertices of the rhombus $R(w_{j+1}b_j)$ in cclw order, then $e^{i\alpha_j'}$ is the complex vector $y_j' - w_{j+1}$, and $e^{i\beta_j'}$ is the complex vector $x_j' - w_{j+1}$. Then $\mathsf{f}_{wx}(z)$ is defined inductively along the path,

$$\begin{aligned}
\mathsf{f}_{ww}(z) &= 1, \\
\mathsf{f}_{wb_j}(z) &= \mathsf{f}_{ww_j}(z) \frac{(-1)^{\mathbb{I}_{(w_j,b_j)}} e^{i\frac{\alpha_j+\beta_j}{2}}}{(z-e^{i\alpha_j})(z-e^{i\beta_j})}, \\
\mathsf{f}_{ww_{j+1}}(z) &= \mathsf{f}_{wb_j}(z)(-1)^{\mathbb{I}_{(w_{j+1},b_j)}} e^{-i\frac{\alpha_j'+\beta_j'}{2}} (z-e^{i\alpha_j'})(z-e^{i\beta_j'}).
\end{aligned}$$

Remark 4.6

We have the following relations between the real and the complex case.

1. *$\forall\, w \in W$, $\forall\, x \in B \cup W$, $\mathsf{f}_{wx}(z) = \overline{\mathsf{f}_{wx}(0)} f_{wx}(z)$.*
2. *$\forall\, w \in W$, $\forall\, b \in B$, such that w is adjacent to b, $\mathsf{K}(w,b) = \mathsf{f}_{wb}(0) K(w,b)$.*

Proof:

1. This is a direct consequence of the definitions of the functions f_{wx} and f_{wx}.

2. Let $R(wb)$ be the rhombus associated to the edge wb, and let w, x, b, y be its vertices in cclw order. Denote by $e^{i\alpha}$ the complex vector $y - w$, and by $e^{i\beta}$ the complex vector $x - w$. Let θ be the rhombus angle of the edge wb. By definition we have,

$$\begin{aligned}
\mathsf{K}(w,b) &= (-1)^{\mathbb{I}_{(w,b)}} 2\sin\theta = (-1)^{\mathbb{I}_{(w,b)}} \frac{e^{i\frac{\alpha-\beta}{2}} - e^{-i\frac{\alpha-\beta}{2}}}{i}, \\
K(w,b) &= i(e^{i\beta} - e^{i\alpha}), \\
\mathsf{f}_{wb}(0) &= (-1)^{\mathbb{I}_{(w,b)}} e^{-i\frac{\alpha+\beta}{2}}.
\end{aligned}$$

Combining the above three equations yields 2. □

Lemma 4.7 *The function f_{wx} is well defined.*

Proof: Showing that the function f_{wx} is well defined amounts to proving that f_{wx} is independent of the edge-path of R^* from w to x. This is equivalent to proving the following: let $w_1, b_1, \ldots, w_k, b_k, w_{k+1} = w_1$ be the vertices of an elementary cycle C of R^*, where vertices are enumerated in cclw order; if $\mathsf{f}_{w_1w_1}(z) = 1$ then $\mathsf{f}_{w_1w_{k+1}}(z) = 1$. Let us use the notations introduced in the definition of f_{wx}, and denote indices cyclically, that is $k+1 \equiv 1$. By Remark 4.6, we have

$$\mathsf{f}_{w_1w_{k+1}}(z) = \overline{\mathsf{f}_{w_1w_{k+1}}(0)} f_{w_1w_{k+1}}(z).$$

Since the function f_{wx} is well defined, $f_{w_1 w_{k+1}}(z) = 1$. Hence, it remains to prove that $\overline{\mathsf{f}_{w_1 w_{k+1}}(0)} = 1$. By definition of f_{wx}, we have

$$\overline{\mathsf{f}_{w_1 w_{k+1}}(0)} = \prod_{j=1}^{k} \left((-1)^{(\mathbb{I}_{(w_j,b_j)} + \mathbb{I}_{(w_{j+1},b_j)})} e^{i\frac{\alpha_j+\beta_j}{2}} e^{-i\frac{\alpha'_j+\beta'_j}{2}} \right).$$

Moreover for every j, $\alpha'_j = \beta_j$, so that

$$\overline{\mathsf{f}_{w_1 w_{k+1}}(0)} = \prod_{j=1}^{k} \left((-1)^{(\mathbb{I}_{(w_j,b_j)} + \mathbb{I}_{(w_{j+1},b_j)})} e^{i\frac{\alpha_j-\beta_j}{2}} e^{i\frac{\alpha'_j-\beta'_j}{2}} \right).$$

Let θ_j (resp. θ'_j) be the rhombus angle of the edge $w_j b_j$ (resp. $w_{j+1} b_j$), then

$$\overline{\mathsf{f}_{w_1 w_{k+1}}(0)} = (-1)^{\sum_{j=1}^{k} (\mathbb{I}_{(w_j,b_j)} + \mathbb{I}_{(w_{j+1},b_j)})} e^{i \sum_{j=1}^{k} (\theta_j + \theta'_j)}. \tag{5}$$

The cycle C corresponds to a face of the graph R^*. Let c be the circumcenter of this face, and let τ_j (resp. τ'_j) be the angle of the rhombus $R(w_j b_j)$ (resp. $R(w_{j+1} b_j)$) at the vertex c. Then $\tau_j = \pi - 2\theta_j$, and $\tau'_j = \pi - 2\theta'_j$. Since $\sum_{j=1}^{k} (\tau_j + \tau'_j) = 2\pi$, we deduce $\sum_{j=1}^{k} (\theta_j + \theta'_j) = \pi(k-1)$. Hence,

$$e^{i \sum_{j=1}^{k} (\theta_j + \theta'_j)} = -(-1)^k. \tag{6}$$

Moreover $\mathbb{I}_{(w_{j+1},b_j)} = 1 - \mathbb{I}_{(b_j,w_{j+1})}$, and $(-1)^{1-\mathbb{I}_{(b_j,w_{j+1})}} = (-1)^{\mathbb{I}_{(b_j,w_{j+1})}-1}$, so

$$(-1)^{\sum_{j=1}^{k} (\mathbb{I}_{(w_j,b_j)} + \mathbb{I}_{(w_{j+1},b_j)})} = (-1)^{\sum_{j=1}^{k} (\mathbb{I}_{(w_j,b_j)} + \mathbb{I}_{(b_j,w_{j+1})}) - k}.$$

Note that $\sum_{j=1}^{k} (\mathbb{I}_{(w_j,b_j)} + \mathbb{I}_{(b_j,w_{j+1})})$ is the number of co-oriented edges encountered when traveling cclw around the cycle C. Since the orientation of the edges of R^* is clockwise odd, it is also counterclockwise odd, and so this number is odd. This implies

$$(-1)^{\sum_{j=1}^{k} (\mathbb{I}_{(w_j,b_j)} + \mathbb{I}_{(w_{j+1},b_j)})} = -(-1)^{-k}. \tag{7}$$

The proof is completed by combining equations (5), (6) and (7). □

As in the complex case, a real angle in $[\theta_0 - \pi + \Delta, \theta_0 + \pi - \Delta]$ can be assigned to each pole of f_{wb}, for some small $\Delta > 0$; where θ_0 is the real angle assigned to the vector $b - w$.

Lemma 4.8 *There exists a unique K^{-1} satisfying the above two properties, and K^{-1} is given by:*

$$\mathsf{K}^{-1}(b,w) = \frac{1}{4\pi^2 i} \int_C \mathsf{f}_{wb}(z) \log z \, dz, \tag{8}$$

where C is a closed contour surrounding cclw the part of the circle $\{e^{i\theta} \mid \theta \in [\theta_0 - \pi + \Delta, \theta_0 + \pi - \Delta]\}$, which contains all the poles of f_{wb}, and with the origin in its exterior.

Proof: Let $\mathsf{F}(b,w)$ be the right hand side of (8). Fix a vertex $w_0 \in W$, and let us prove that $\sum_{b \in B} \mathsf{K}(w_0,b)\mathsf{F}(b,w) = \delta_{w_0}(w)$. Denote by $b_1, \ldots, b_k$ the black neighbors of w_0. Using Remark 4.6, we obtain for every j,

$$\mathsf{K}(w_0,b_j) = \mathsf{f}_{w_0 b_j}(0) K(w_0,b_j),$$
$$\mathsf{f}_{wb_j}(z) = \overline{\mathsf{f}_{wb_j}(0)} f_{wb_j}(z).$$

Moreover $\forall w \in W, \forall b \in B$, we have $\overline{\mathsf{f}_{wb}(0)} = \mathsf{f}_{wb}(0)^{-1} = \mathsf{f}_{bw}(0)$, so that $\mathsf{f}_{w_0 b_j}(0)\overline{\mathsf{f}_{wb_j}(0)} = \mathsf{f}_{w_0 b_j}(0)\mathsf{f}_{b_j w}(0) = \mathsf{f}_{w_0 w}(0)$. Hence, using Theorem 3.1, we obtain for every j,

$$\mathsf{K}(w_0, b_j)\mathsf{F}(b_j, w) = \mathsf{f}_{w_0 w}(0) K(w_0, b_j) K^{-1}(b_j, w).$$

Since $\mathsf{K}(w_0, b) = 0$ when w_0 and b are not adjacent, and since K^{-1} is the inverse Dirac operator, we obtain

$$\begin{aligned} \sum_{b\in B} \mathsf{K}(w_0, b)\mathsf{F}(b, w) &= \sum_{j=1}^{k} \mathsf{K}(w_0, b_j)\mathsf{F}(b_j, w), \\ &= \mathsf{f}_{w_0 w}(0) \sum_{j=1}^{k} K(w_0, b_j) K^{-1}(b_j, w) = \mathsf{f}_{w_0 w}(0)\delta_{w_0}(w) = \delta_{w_0}(w). \end{aligned}$$

Uniqueness of K^{-1} follows from the uniqueness of K^{-1}. □

4.3 Convergence of the Boltzmann measure on the torus

Let R be a rhombus tiling of the plane, and R be the corresponding rhombus-with-diagonals tiling. Suppose that the critical weight function is assigned to edges of R^*, and denote by K the Dirac operator indexed by the vertices of R^*.
Consider a subset of edges $e_1 = w_1 b_1, \dots, e_k = w_k b_k$ of R^*, and let P be a finite simply connected sub-graph of R such that P^* contains these edges. By Proposition 4.1, there exists a periodic rhombus tiling of the plane S that contains P. Let S be the corresponding rhombus-with-diagonals tiling, and assign the critical weight function to edges of S^*. Denote by Λ the lattice which acts periodically on S, and suppose that the dual graph $\bar{S}_n^*$ of the toroidal graph $\bar{S}_n = S/n\Lambda$ is bipartite (this is possible by eventually replacing Λ by 2Λ). Denote by μ_n^S be the Boltzmann measure on dimer configurations $\mathcal{M}(\bar{S}_n^*)$ of $\bar{S}_n^*$. Then we have,

Proposition 4.9

$$\lim_{n\to\infty} \mu_n^S(e_1, \dots, e_k) = \left(\prod_{i=1}^{k} K(w_i, b_i)\right) \det_{1\le i,j\le k} K^{-1}(b_i, w_j).$$

Proof: Let us first define an orientation of the edges of S^*, and the four Kasteleyn matrices $\mathsf{K}_1^n, \dots, \mathsf{K}_4^n$ of the graph $\bar{S}_n^*$. Consider the graph $\bar{S}_1^*$, then it is a bipartite graph on the torus. Fix a reference matching M_0 of $\bar{S}_1^*$. For every other perfect matching M of $\bar{S}_1^*$, consider the superposition $M \cup M_0$ of M and M_0, then $M \cup M_0$ consists of doubled edges and cycles. Let us define four parity classes for perfect matchings M of $\bar{S}_1^*$: (e,e) consists of perfect matchings M for which cycles of $M \cup M_0$ circle the torus an even number of times horizontally and vertically; (e,o) consists of perfect matchings M, for which cycles of $M \cup M_0$ circle the torus an even number of times horizontally, and an odd number of times vertically; (o,e) and (o,o) are defined in a similar way. By Tesler [17], one can construct an orientation of the edges of $\bar{S}_1^*$, so that the corresponding weighted adjacency matrix K_1^1 has the following property: perfect matchings which belong to the same parity class have the same sign in the expansion of the determinant of K_1^1. By an appropriate choice of sign, we can make the (e,e) class have the plus sign in $\det \mathsf{K}_1^1$, and the other three have minus sign. Consider a horizontal and a vertical cycle of $\bar{S}_1$. Then define K_2^1 (resp. K_3^1) to be the matrix K_1^1 where the sign of the coefficients corresponding to edges crossing the horizontal (resp. vertical) cycle is reversed; and define K_4^1 to be the matrix K_1^1 where the sign of the coefficients corresponding to edges crossing both cycles are reversed.

The orientation of the edges of $\bar{S}_1^*$ defines a periodic orientation of the graph S^*. For every n, consider the graph $\bar{S}_n^*$ and the four matrices $\mathsf{K}_1^n, \mathsf{K}_2^n, \mathsf{K}_3^n, \mathsf{K}_4^n$ defined as above. These matrices are called the **Kasteleyn matrices** of the graph $\bar{S}_n^*$.

The orientation defined on the edges of the graph S^* is a clockwise odd orientation. Let K_S be the real Dirac operator indexed by the vertices of S^* corresponding to this clockwise orientation, and let K_S be the Dirac operator indexed by the vertices of S^*. Then Proposition 4.9 is a direct consequence of Lemmas 4.10, 4.14, 4.15 below.

Lemma 4.10

$$\lim_{n\to\infty} \mu_n^S(e_1,\ldots,e_k) = \left(\prod_{i=1}^k \mathsf{K}_S(w_i,b_i)\right) \det_{1\le i,j\le k} \mathsf{K}_S^{-1}(b_i,w_j). \tag{9}$$

Proof: The **toroidal partition function** $Z(\bar{S}_n^*,\nu)$ is defined to be the weighted sum (weighted by the function ν) of dimer configurations of the graph $\bar{S}_n^*$. Then, by Tesler [17] (it is a generalization of a theorem of Kasteleyn [5]), we have

Theorem 4.11 [5, 17]

$$Z(\bar{S}_n^*,\nu) = \frac{1}{2}(-\det \mathsf{K}_1^n + \det \mathsf{K}_2^n + \det \mathsf{K}_3^n + \det \mathsf{K}_4^n).$$

Kenyon gives the following theorem for the Boltzmann measure μ_n^S.

Theorem 4.12 [7] *$\mu_n^S(e_1,\ldots,e_k)$ is equal to $\left(\prod_{i=1}^k \mathsf{K}_S(w_i,b_i)\right)$ times*

$$\left(-\frac{\det \mathsf{K}_1^n}{2Z(\bar{S}_n^*,\nu)} \det_{1\le i,j\le k}\left((\mathsf{K}_1^n)^{-1}(b_i,w_j)\right) + \sum_{\ell=2}^4 \frac{\det \mathsf{K}_\ell^n}{2Z(\bar{S}_n^*,\nu)} \det_{1\le i,j\le k}\left((\mathsf{K}_\ell^n)^{-1}(b_i,w_j)\right)\right). \tag{10}$$

This part of the argument can be found in [7]. Equation (10) is a weighted average of the four quantities $\det_{1\le i,j\le k}\left((\mathsf{K}_\ell^n)^{-1}(b_i,w_j)\right)$, with weights $\frac{1}{2}\det \mathsf{K}_\ell^n / Z(\bar{S}_n^*,\nu)$. These weights are all in the interval $(-1,1)$ since, for every $\ell = 1,\ldots,4$, $2Z(\bar{S}_n^*,\nu) > |\det \mathsf{K}_\ell^n|$. Indeed, $Z(\bar{S}_n^*,\nu)$ counts the weighted sum of dimer configurations of $\bar{S}_n^*$, whereas $|\det \mathsf{K}_\ell^n|$ counts some configurations with negative sign. Moreover, by Theorem 4.11 these weights sum to 1, so that the weighted average converges to the same value as each $\det_{1\le i,j\le k}\left((\mathsf{K}_\ell^n)^{-1}(b_i,w_j)\right)$.

Denote by B_S (resp. W_S) the set of black (resp. white) vertices of S^*. Let us prove that for every $\ell = 1,\ldots,4$, and for every $w \in W_S$, $b \in B_S$, $(\mathsf{K}_\ell^n)^{-1}(b,w)$ converges to $\mathsf{K}_S^{-1}(b,w)$ on a subsequence of n's. The following theorem of [9] gives the convergence on a subsequence of n's of the inverse Kasteleyn matrices of the graph $\bar{S}_n^*$.

Theorem 4.13 [9] *For every $w \in W_S, b \in B_S$, $\ell = 1,\ldots,4$,*

$$\lim{}'_{n\to\infty} (\mathsf{K}_\ell^n)^{-1}(b,w) = \frac{1}{(2\pi)^2}\int_{S^1\times S^1} \frac{Q_{b,w}(z,u)u^x z^y}{P(z,u)}\frac{dz}{z}\frac{du}{u}, \tag{11}$$

where $Q_{b,w}$ and P are polynomials ($Q_{b,w}$ only depends on the equivalence class of w and b), and x (resp. y) is the horizontal (resp. vertical) translation from the fundamental domain of b to the fundamental domain of w.

Denote by $\mathsf{F}(b,w)$ the right hand side of (11). In [9], it is proved that $\mathsf{F}(b,w)$ converges to 0 as $|b-w|\to\infty$, as long as the dimer model is not in its frozen phase. Moreover, it is proved in [10] that dimer models on isoradial graphs are never in their frozen phase when the weight function is the critical one. Hence, we deduce that $\mathsf{F}(b,w)$ converges to 0 as $|b-w|\to\infty$.
Let us prove that for every $b\in B_S, w\in W_S$, $\mathsf{F}(b,w)=\mathsf{K}_S^{-1}(b,w)$. Consider $w_1,w_2\in W_S$, and denote by $b_1,\dots,b_k$ the neighbors of w_1. Assume n is large enough so that the graph $\bar{S}_n^*$ contains $w_1,w_2,b_1,\dots,b_k$, and so that the edges w_1b_j do not cross the horizontal and vertical cycle of $\bar{S}_n$. Then, for every $\ell=1,\dots,4$,

$$\sum_{b\in B_S}\mathsf{K}_\ell^n(w_1,b)(\mathsf{K}_\ell^n)^{-1}(b,w_2)=\sum_{j=1}^{k}\mathsf{K}_\ell^n(w_1,b_j)(\mathsf{K}_\ell^n)^{-1}(b_j,w_2)=\delta_{w_1w_2}.$$

Moreover, $\mathsf{K}_\ell^n(w_1,b_j)=\mathsf{K}_S(w_1,b_j)$, so that taking the limit on a subsequence of n's, and using Theorem 4.13, we obtain

$$\sum_{j=1}^{k}\mathsf{K}_S(w_1,b_j)\mathsf{F}(b_j,w_2)=\delta_{w_1w_2}.$$

This is true for all $w_1,w_2\in W_S$. Moreover, $\lim_{|b-w|\to\infty}\mathsf{F}(b,w)=0$, so that by definition of the inverse real Dirac operator, and by the existence and uniqueness Lemma 4.8, we deduce that for all $b\in B_S$, $w\in W_S$, $\mathsf{F}(b,w)=\mathsf{K}_S^{-1}(b,w)$.

Hence, $\mu_n^S(e_1,\dots,e_k)$ converges to the right hand side of (9) on a subsequence of n's. By Sheffield's Theorem [16], this is the unique limit of the Boltzmann measures μ_n^S, so that we have convergence for every n. □

Lemma 4.14

$$\left(\prod_{i=1}^{k}\mathsf{K}_S(w_i,b_i)\right)\det_{1\le i,\,j\le k}\left(\mathsf{K}_S^{-1}(b_i,w_j)\right)=\left(\prod_{i=1}^{k}K_S(w_i,b_i)\right)\det_{1\le i,\,j\le k}\left(K_S^{-1}(b_i,w_j)\right).\tag{12}$$

Proof: By definition of the determinant, the left hand side of (12) is equal to

$$\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}\sigma\left(\prod_{i=1}^{k}\mathsf{K}_S(w_i,b_i)\right)\mathsf{K}_S^{-1}(b_1,w_{\sigma(1)})\dots\mathsf{K}_S^{-1}(b_k,w_{\sigma(k)}),$$

where $\mathcal{S}_n$ is the set of permutations of n elements. A permutation $\sigma\in\mathcal{S}_n$ can be written as a product of disjoint cycles, so let us treat the case of each cycle separately. Refer to Section 4.2.2 for the definition of the function f_{wx}.

• Suppose that in the product there is a 1-cycle, that is a point j such that $\sigma(j)=j$. Then, using Remark 4.6 and Lemma 4.8, we obtain

$$\begin{aligned}\mathsf{K}_S(w_j,b_j)&=\mathsf{f}_{w_jb_j}(0)K_S(w_j,b_j),\\ \mathsf{K}_S^{-1}(b_j,w_j)&=\overline{\mathsf{f}_{w_jb_j}(0)}K_S^{-1}(b_j,w_j).\end{aligned}$$

Moreover, $\overline{\mathsf{f}_{w_jb_j}(0)}=\mathsf{f}_{w_jb_j}(0)^{-1}$, hence

$$\mathsf{K}_S(w_j,b_j)\mathsf{K}_S^{-1}(b_j,w_j)=K_S(w_j,b_j)K_S^{-1}(b_j,w_j).\tag{13}$$

• Suppose that in the product there is an ℓ-cycle, with $\ell \neq 1$. To simplify notations, let us assume $\sigma(1)=2,\dots,\sigma(\ell)=1$, and let us prove the following (indices are written cyclically, i.e. $\ell+1\equiv 1$),

$$\prod_{j=1}^{\ell} \mathsf{K}_S(w_j,b_j)\mathsf{K}_S^{-1}(b_j,w_{j+1}) = \prod_{j=1}^{\ell} K_S(w_j,b_j)K_S^{-1}(b_j,w_{j+1}). \tag{14}$$

Again, using Remark 4.6 and Lemma 4.8, we obtain

$$\prod_{j=1}^{\ell} \mathsf{K}_S(w_j,b_j)\mathsf{K}_S^{-1}(b_j,w_{j+1}) = \prod_{j=1}^{\ell} K_S(w_j,b_j)K_S^{-1}(b_j,w_{j+1})\mathsf{f}_{w_j b_j}(0)\mathsf{f}_{w_{j+1} b_j}(0)^{-1}.$$

Using the definition of the function f_{wx}, and the fact that it is well defined yields

$$\prod_{j=1}^{\ell} \mathsf{f}_{w_j b_j}(0)\mathsf{f}_{w_{j+1} b_j}(0)^{-1} = \prod_{j=1}^{\ell} \mathsf{f}_{w_j b_j}(0)\mathsf{f}_{b_j w_{j+1}}(0) = \prod_{j=1}^{\ell} \mathsf{f}_{w_j w_{j+1}}(0) = \mathsf{f}_{w_1 w_1}(0) = 1.$$

This proves equation (14). Combining equations (13), (14), and the fact that every permutation is a product of cycles, we obtain Lemma 4.14. □

Lemma 4.15

$$\left(\prod_{i=1}^{k} K_S(w_i,b_i)\right) \det_{1\leq i,\, j\leq k}\left(K_S^{-1}(b_i,w_j)\right) = \left(\prod_{i=1}^{k} K(w_i,b_i)\right) \det_{1\leq i,\, j\leq k}\left(K^{-1}(b_i,w_j)\right).$$

Proof: Since P is simply connected, for every $i,j=1,\dots,k$, it contains a path of $\widetilde{R}$ from w_j to b_i. Moreover by Theorem 3.1, the coefficient of the inverse Dirac operator corresponding to b_i,w_j only depends on such a path. Hence $K_S^{-1}(b_i,w_j)=K^{-1}(b_i,w_j)$. We also have $\forall\, i=1,\dots,k$, $K_S(w_i,b_i)=K(w_i,b_i)$, so that we deduce Lemma 4.15. □

□

4.4 Proof of Theorem 3.2

The edges of the graph R^* form a countable set. For every $i\in\mathbb{N}$, define $f_i:\mathcal{M}(R^*)\to\{0,1\}$ by

$$f_i(M) = \begin{cases} 1 & \text{if the edge } e_i \text{ belongs to } M, \\ 0 & \text{else.} \end{cases}$$

Fix $k\in\mathbb{N}$, and a k-tuple $(s_1,\dots,s_k)$ of distinct elements of $\mathbb{N}$. Let $H\in\mathcal{B}\{0,1\}^k$, where $\mathcal{B}\{0,1\}^k$ denotes the Borel σ-field of $\{0,1\}^k$, and define a **cylinder of rank** k by

$$A_{(s_1,\dots,s_k)}(H) = \{M\in\mathcal{M}(R^*) | (f_{s_1}(M),\dots,f_{s_k}(M))\in H\}.$$

Then $A_{(s_1,\dots,s_k)}(H)$ can be written as a disjoint union of cylinder sets,

$$A_{(s_1,\dots,s_k)}(H) = \bigcup_{i=1}^{m}\{e_{t_{i1}},\dots,e_{t_{i\ell_i}}\},$$

(recall that for every i, $\{e_{t_{i1}}, \ldots, e_{t_{i\ell_i}}\}$ denotes the set of dimer configurations of R^* containing the edges $e_{t_{i1}}, \ldots, e_{t_{i\ell_i}}$). Define

$$\mu_{(s_1,\ldots,s_k)}(H) = \sum_{i=1}^{m} \left(\left(\prod_{j=1}^{\ell_i} K(w_{t_{ij}}, b_{t_{ij}}) \right) \det_{1\leq j,k\leq \ell_i} (K^{-1}(b_{t_{ij}}, w_{t_{ik}})) \right).$$

Let P be a finite simply connected sub-graph of R such that, for every $i = 1, \ldots, m$, P^* contains the edges $e_{t_{i1}}, \ldots, e_{t_{i\ell_i}}$. Let S be a periodic rhombus tiling of the plane that contains P (given by Proposition 4.1). Then, by Proposition 4.9, $\mu_{(s_1,\ldots,s_k)}(H) = \lim_{n\to\infty} \mu_n^S(A_{(s_1,\ldots,s_k)}(H))$. From this we deduce that for every k, and for every k-tuple $(s_1, \ldots, s_k)$, $\mu_{(s_1,\ldots,s_k)}$ is a probability measure on $\mathcal{B}\{0,1\}^k$. Moreover, we deduce that the system of measures $\{\mu_{(s_1,\ldots,s_k)} : (s_1, \ldots, s_k)$ is a k-tuple of distinct elements of $\mathbb{N}\}$ satisfy Kolmogorov's two consistency conditions. Applying Kolmogorov's extension theorem, we obtain the existence of a unique measure μ^R, which satisfies (2).

Using the fact that the measure μ^R of a cylinder set is the limit of Boltzmann measures, we deduce that the measure μ^R is a Gibbs measure in the sense given in the introduction.

Assume moreover that the graph R^* is doubly periodic. Then, for every cylinder set $\{e_1, \ldots, e_k\}$ of R^*, $\mu^R(e_1, \ldots, e_k) = \lim_{n\to\infty} \mu_n^R(e_1, \ldots, e_k)$. Moreover by [16] (see also [9]), the Boltzmann measures μ_n^R converge to the minimal free energy per fundamental domain Gibbs measure which is unique, so that this proves Theorem 3.2. □

5 Gibbs measure on the set of all triangular quadri-tilings

Recall that $\mathcal{Q}$ is the set of all triangular quadri-tilings up to isometry, i.e. the set of all quadri-tilings whose underlying tiling is a lozenge tiling of the equilateral triangular lattice $\mathbb{T}$. Denote by $\mathcal{M}$ the set of dimer configurations corresponding to quadri-tilings of $\mathcal{Q}$. In this section, we first define the notion of Gibbs measure on $\mathcal{M}$; then we define a σ-algebra $\sigma(\mathcal{B})$ on $\mathcal{M}$, and give an explicit expression for a Gibbs measure μ on $(\mathcal{M}, \sigma(\mathcal{B}))$. We conjecture μ to be of minimal total free energy per fundamental domain among a four parameter family of ergodic Gibbs measures.

The notion of Gibbs measure on $\mathcal{Q}$ is a natural extension of the notion of Gibbs measure on dimer configurations of a fixed graph. Assume a weight function ν is assigned to quadri-tiles of triangular quadri-tilings of $\mathcal{Q}$, then a **Gibbs measure on $\mathcal{Q}$** is a probability measure with the following property. If a triangular quadri-tiling is fixed in an annular region, then quadri-tilings inside and outside of the annulus are independent. Moreover, the probability of any interior triangular quadri-tiling is proportional to the product of the weights of the quadri-tiles. Using the bijection between $\mathcal{Q}$ and $\mathcal{M}$, this yields the definition of a Gibbs measure on $\mathcal{M}$.

Define $\mathcal{L}$ to be the set of lozenge-with-diagonals tilings of the plane, up to isometry. Define $\mathcal{L}^*$ to be the graph (which is not planar) obtained by superposing the dual graphs L^* of lozenge-with-diagonals tilings $L \in \mathcal{L}$. Although some edges of $\mathcal{L}^*$ have length 0, we think of them as edges of the one skeleton of the graph, so that to every edge of $\mathcal{L}^*$ there corresponds a unique quadri-tile in a lozenge-with-diagonals tiling of $\mathcal{L}$. Let e be an edge of $\mathcal{L}^*$, and let q_e be the corresponding quadri-tile, then q_e is made of two adjacent right triangles. If the two triangles share the hypotenuse edge, they belong to two adjacent lozenges; else if they share a leg, they belong to the same lozenge. Let us call these lozenge(s) the **lozenge(s) associated to the edge** e, and denote it/them by l_e (that is l_e consists of either one or two lozenges). Let k_e be the edge(s) of $\mathbb{T}^*$ corresponding to the lozenge(s) l_e.

Let $e_1, \dots, e_k$ be a subset of edges of $\mathcal{L}^*$, and define the **cylinder set** $\{e_1, \dots, e_k\}$ of $\mathcal{L}^*$ to be the set of dimer configurations of $\mathcal{M}$ which contain these edges. Let us call **connected cylinder** any cylinder of $\mathcal{L}^*$ which has the property that the lozenge(s) associated to its edges form a connected path. Then every cylinder of $\mathcal{L}^*$ can be expressed as a disjoint union of connected cylinders. Consider $\mathcal{B}$ the field consisting of the empty set and of the finite disjoint unions of connected cylinders. Denote by $\sigma(\mathcal{B})$ the σ-field generated by $\mathcal{B}$.

Let $\mu^{\mathbb{T}}$ be the Gibbs measure on dimer configurations of the honeycomb lattice $\mathcal{M}(\mathbb{T}^*)$ given in [7]. As we have noted before, Theorem 3.2 is true for all doubly periodic isoradial graphs with critical weights on their edges. Then, the measure $\mu^{\mathbb{T}}$ coincides with the minimal free energy per fundamental domain Gibbs measure given by Theorem 3.2, when the doubly periodic isoradial graph is the honeycomb lattice. As a corollary to Theorem 3.2, we obtain:

Corollary 5.1 *There is a probability measure μ on $(\mathcal{M}, \sigma(B))$ such that for every connected cylinder $\{e_1, \dots, e_k\}$ of $\mathcal{L}^*$,*

$$\mu(e_1, \dots, e_k) = \mu^L(e_1, \dots, e_k)\mu^{\mathbb{T}}(\mathsf{k}_{e_1}, \dots, \mathsf{k}_{e_k}), \tag{15}$$

where L is the lozenge-with-diagonals tiling corresponding to any lozenge tiling L which contains the lozenges $\mathsf{l}_{e_1}, \dots, \mathsf{l}_{e_k}$. Moreover μ is a Gibbs measure on $\mathcal{M}$, where the critical weight function is assigned to quadri-tiles.

Proof: Expression (15) is well defined, i.e. independent of the lozenge tiling L which contains the lozenges $\mathsf{l}_{e_1}, \dots, \mathsf{l}_{e_k}$. Indeed, by definition of a connected cylinder set, the lozenges associated to the edges $e_1, \dots, e_k$ form a connected path of lozenges, say γ. Let L be a lozenge tiling that contains γ, and denote by K_L the complex Dirac operator indexed by the vertices of the graph L^*. Then $K_L^{-1}(b_i, w_j)$ is independent of the lozenge tiling L which contains γ, indeed $K_L^{-1}(b_i, w_j)$ only depends on an edge-path of $\widetilde{R}$ from w_j to b_i, and since L contains γ which is connected, we can choose the edge-path to be the same for all such lozenge-with-diagonals tilings L. We then use the fact that $\mu^{\mathbb{T}}$ and μ^L are probability measures to prove the two conditions of Kolmogorov's extension theorem.

The measure μ is a Gibbs measure on $\mathcal{M}$ as a consequence of the fact that μ^L and $\mu^{\mathbb{T}}$ are Gibbs measure on $\mathcal{M}(L^*)$ and $\mathcal{M}(\mathbb{T}^*)$ respectively. □

Assume that a weight function is assigned to quadri-tiles of quadri-tilings of $\mathcal{Q}$. Denote by Λ the lattice which acts periodically on $\mathbb{T}$, and by $\mathbb{T}_n = \mathbb{T}/n\Lambda$, moreover suppose that $\mathbb{T}_n^*$ is bipartite (this is possible by eventually replacing Λ by 2Λ). Define $\mathcal{Q}_n$ to be the set of triangular quadri-tilings whose underlying tiling is a lozenge tiling of $\mathbb{T}_n$. Denote by μ_n the Boltzmann measure on $\mathcal{Q}_n$; that is the probability of having a given subset of quadri-tiles in a quadri-tilings of $\mathcal{Q}_n$ chosen with respect to μ_n is proportional to the product of the weights of the quadri-tiles. We make the first following conjecture.

Conjecture 1 *Suppose the critical weight function is assigned to quadri-tiles, then the Gibbs measure of Corollary 5.1 is the limit of the Boltzmann measures μ_n.*

Now fix $(s,t,p,q) \in \mathbb{R}^4$, and let $\mathcal{Q}_n^{(s,t,p,q)}$ be the subset of quadri-tilings of $\mathcal{Q}_n$ whose first height change is $(\lfloor ns \rfloor, \lfloor nt \rfloor)$ and second height change is $(\lfloor np \rfloor, \lfloor nq \rfloor)$. Assuming that $\mathcal{Q}_n^{(s,t,p,q)}$ is non empty, let $\mu_n^{(s,t,p,q)}$ be the conditional measure induced by μ_n on $\mathcal{Q}_n^{(s,t,p,q)}$. Denote by $Z_n^{(s,t,p,q)}$ the weighted sum of quadri-tilings of $\mathcal{Q}_n^{(s,t,p,q)}$, and define the **total free energy per fundamental domain** $\sigma(s,t,p,q)$ by:

$$\sigma(s,t,p,q) = -\lim_{n\to\infty} \frac{1}{n^2} \log Z_n^{(s,t,p,q)}.$$

Then the following conjecture is inspired by a result of [16], see also [9].

Conjecture 2 *For each (s,t,p,q) for which $\mathcal{Q}_n^{(s,t,p,q)}$ is non empty for n sufficiently large, $\mu_n^{(s,t,p,q)}$ converges as $n \to \infty$ to an ergodic Gibbs measure $\mu^{(s,t,p,q)}$ of slope (s,t,p,q). Furthermore μ_n itself converges to $\mu^{(s_0,t_0,p_0,q_0)}$ where (s_0,t_0,p_0,q_0) is the limit of the slopes of μ_n. If (s_0,t_0,p_0,q_0) lies in the interior of the set (s,t,p,q) for which $\mathcal{Q}_n^{(s,t,p,q)}$ is non-empty for n sufficiently large, then every ergodic Gibbs measure of slope (s,t,p,q) is of the form $\mu^{(s,t,p,q)}$ for some (s,t,p,q) as above; that is $\mu^{(s,t,p,q)}$ is the unique ergodic Gibbs measure of slope (s,t,p,q). Moreover, the measure $\mu^{(s_0,t_0,p_0,q_0)}$ is the unique one which has minimal total free energy per fundamental domain.*

As a consequence, when the critical weight function is assigned to quadri-tiles, we conjecture the minimal total free energy per fundamental domain Gibbs measure to be given by the explicit expression of Corollary 5.1.

6 Asymptotics in the case of quadri-tilings

Section 6.1 aims at giving a precise statement of Theorem 1.2 of the introduction (see Theorem 6.1 below), and Section 6.2 gives consequences of Theorem 6.1 for the Gibbs measure μ^R of Theorem 3.2 and μ of Corollary 5.1.

6.1 Asymptotics of the inverse Dirac operator

Refer to Figure 11 for the following notations. Let ℓ_1', ℓ_2' be two disjoint side-length two rhombi in the plane, and let ℓ_1, ℓ_2 be the corresponding rhombi with-diagonals. Assume ℓ_1 and ℓ_2 have a fixed black and white bipartite coloring of their faces. Let r_1 and r_2 be the dual graphs of ℓ_1 and ℓ_2 (r_1 and r_2 are rectangles), with the corresponding bipartite coloring of the vertices. Let w be a white vertex of r_1, and b a black vertex of r_2, then w (resp. b) belongs to a boundary edge e_1 of ℓ_1 (resp. e_2 of ℓ_2). By Lemma 2.2, to the bipartite coloring of the faces of ℓ_1 and ℓ_2, there corresponds a bipartite coloring of the vertices of ℓ_1' and ℓ_2'. Let x_1 (resp. x_2) be the black vertex of the edge e_1 (resp. e_2). Orient the edge wx_1 from w to x_1, and let $e^{i\theta_1}$ be the corresponding vector. Orient the edge x_2b from x_2 to b, and let $e^{i\theta_2}$ be the corresponding vector. Assume ℓ_1' and ℓ_2' belong to a rhombus tiling of the plane R. Moreover, suppose that the bipartite coloring of the vertices of R^* is compatible with the bipartite coloring of the vertices of r_1 and r_2.

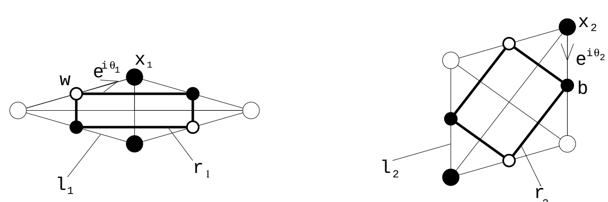

Figure 11: Rhombi with diagonals ℓ_1, ℓ_2 and their dual graphs r_1, r_2.

Then we have the following asymptotics for the inverse Dirac operator K^{-1} indexed by the vertices of R^*. Refer to the introduction for comments about Theorem 6.1.

Theorem 6.1 *As* $|b-w| \to \infty$, $K^{-1}(b,w)$ *is equal to*

$$\frac{1}{2\pi}\left(\frac{1}{b-w} + \frac{e^{-i(\theta_1+\theta_2)}}{\bar{b}-\bar{w}}\right) + \frac{1}{2\pi}\left(\frac{e^{2i\theta_1}+e^{2i\theta_2}}{(b-w)^3} + \frac{e^{-i(3\theta_1+\theta_2)}+e^{-i(\theta_1+3\theta_2)}}{(\bar{b}-\bar{w})^3}\right) + O\left(\frac{1}{|b-w|^3}\right),$$

where θ_1 *and* θ_2 *are defined above.*

Proof: Let us define an edge-path from w to b in $\widetilde{R}$ (the set of rhombi associated to the edges of R^*). Consider the bipartite coloring of the vertices of R (given by Lemma 2.2) associated to the bipartite coloring of the vertices of R^*. We define the graph N as follows. Vertices of N are black vertices of R, and two vertices of N are connected by an edge if they belong to the same rhombus in R. The graph N is connected because R is. Each face of N is inscribable in a circle of radius two. The circumcenter of a face of N is the intersection of the rhombi in R, to which the edges on the boundary cycle of the face belong. Thus the circumcenter is in the closure of the face, and so faces of N are convex. Note that the vertices x_1 and x_2 are vertices of the graph N.

Denote by (x,y) the line segment from a vertex x to a vertex y of N. An edge uv of N is called a **forward-edge** for the segment (x,y) if $< v-u, y-x > \geq 0$. An edge-path $v_1,\ldots,v_k$ of N is called a **forward-path** for the segment (x,y), if all the edges v_iv_{i+1} are forward-edges for (x,y). Similarly to what has been done in [8], let us define a forward-path of N for the segment (x_1,x_2), from x_1 to x_2 (see Figure 12). Let $F_1,\ldots,F_\ell$ be the faces of N whose interior intersect (x_1,x_2) (if some edge of N lies exactly on (x_1,x_2), perturb the segment (x_1,x_2) slightly, using instead a segment $(x_1+\varepsilon_1, x_2+\varepsilon_2)$ for two generic infinitesimal translations $(\varepsilon_1,\varepsilon_2)$). Note that the number of such faces is finite because the rhombus tiling of the plane R has only finitely many different rhombi. Then for $j=1,\ldots,\ell-1$, $F_j \cap F_{j+1}$ is an edge e_{j+1} of N crossing (x_1,x_2). Set $v_1 = x_1$, $v_\ell = x_2$, and for $j=1,\ldots,\ell-2$, define v_{j+1} to be the vertex of e_{j+1} such that the edge e_{j+1} oriented towards v_{j+1} is a forward-edge for (x_1,x_2). Then, for $j=1,\ldots,\ell-1$, the vertices v_j and v_{j+1} belong to the face F_j. Take an edge-path from v_j to v_{j+1} on the boundary cycle of F_j, such that it is a forward-path for (x_1,x_2). Such a path exists because faces of N are convex. Thus, we have built a forward-path of N for (x_1,x_2), from x_1 to x_2. Denote by $u_1 = x_1, u_2, \ldots, u_{k-1}, u_k = x_2$ the vertices of this path.

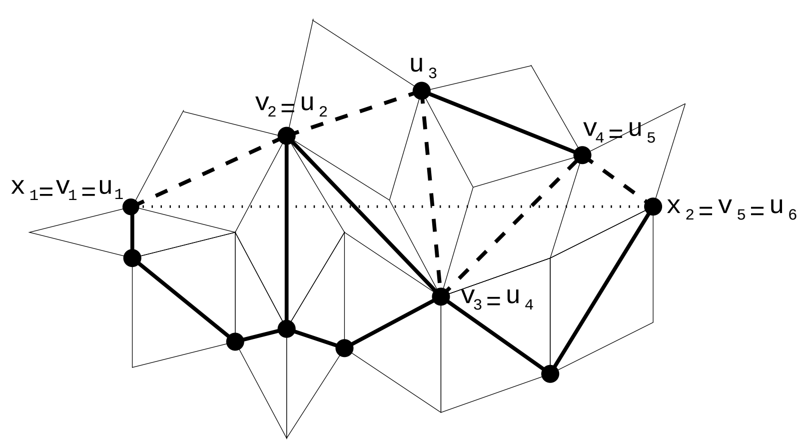

Figure 12: forward-path from x_1 to x_2 for the segment (x_1,x_2).

Let us now define an edge-path of $\widetilde{R}$ from w to b. Note that the edges wx_1 and x_2b are edges of $\widetilde{R}$. For $j = 1, \dots, k-1$, define the following edge-path of $\widetilde{R}$ from u_j to u_{j+1} (see Figure 13). Remember that u_ju_{j+1} is the diagonal of a rhombus of R, say $\tilde{\ell}'_j$. Let $\tilde{r_j}$ be the dual graph of $\tilde{\ell}_j$. Let u_j^1 be the black vertex in $\tilde{r_j}$ adjacent to u_j, let u_j^2 be the crossing of the diagonals of $\tilde{\ell}_j$, and let u_j^3 be the white vertex in $\tilde{r_j}$ adjacent to u_{j+1}. Then the path $u_j, u_j^1, u_j^2, u_j^3, u_{j+1}$ is an edge-path of $\widetilde{R}$. Thus $w, x_1 = u_1, u_1^1, u_1^2, u_1^3, u_2, \dots, u_{k-1}, u_{k-1}^1, u_{k-1}^2, u_{k-1}^3, u_k = x_2, b$ is an edge-path of $\widetilde{R}$, from w to b. Orient the edges in the path towards the black vertices of R^*, and away from the white vertices of R^*.

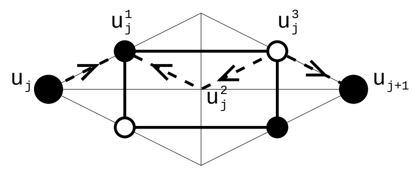

Figure 13: Edge-path of $\widetilde{R}$ from u_j to u_{j+1}.

Let $e^{i\beta_j^1}$, $e^{i\beta_j^2}$, $e^{i\alpha_j^1}$, $e^{i\alpha_j^2}$ be the vectors corresponding respectively to the edges $u_ju_j^1$, $u_j^3u_{j+1}$, $u_j^2u_j^1$, $u_j^3u_j^2$. Without loss of generality, suppose that $x_2 - x_1$ is real and positive. Then for $j = 1, \dots, k-1$, and $\ell = 1, 2$, we have:

$$\cos\beta_j^\ell - \cos\alpha_j^\ell = \frac{< u_{j+1} - u_j, x_2 - x_1 >}{2|x_2 - x_1|}.$$

Since $u_1, \dots, u_k$ is a forward-path for (x_1, x_2), this quantity is positive, thus $\cos\beta_j^\ell \geq \cos\alpha_j^\ell$. Moreover, since there is only a finite number of different rhombi in R, $k = O(|b-w|)$. For the same reason, there is a finite number of angles β_j^ℓ, and they are all in $[-\pi + \Delta, \pi - \Delta]$, for some small $\Delta > 0$ (in the general case where the angle of the vector $x_2 - x_1$ is θ_0, the angles β_j^ℓ would be in the interval $[\theta_0 - \pi + \Delta, \theta_0 + \pi - \Delta]$). Thus by Theorem 4.3 of [8], we have that $K^{-1}(b, w)$ is equal to:

$$\frac{1}{2\pi}\left(\frac{1}{b-w} + \frac{\gamma}{\bar{b}-\bar{w}}\right) + \frac{1}{2\pi}\left(\frac{\xi_2}{(b-w)^3} + \frac{\gamma\bar{\xi}_2}{(\bar{b}-\bar{w})^3}\right) + O\left(\frac{1}{|b-w|^3}\right), \tag{16}$$

where $\gamma = e^{-i(\theta_1+\theta_2)} \prod_{j=1}^{k-1}\prod_{\ell=1}^{2} e^{i(-\beta_j^\ell + \alpha_j^\ell)}$, and $\xi_2 = e^{2i\theta_1} + e^{2i\theta_2} + \sum_{j=1}^{k-1}\sum_{\ell=1}^{2} e^{2i\beta_j^\ell} - e^{2i\alpha_j^\ell}$.

Note that for $j = 1, \dots, k-1$, we have $\alpha_j^2 \equiv (\beta_j^1 + \pi) \bmod[2\pi]$, and $\beta_j^2 \equiv (\alpha_j^1 + \pi) \bmod[2\pi]$, thus:

$$\begin{aligned}
\prod_{\ell=1}^{2} e^{i(-\beta_j^\ell+\alpha_j^\ell)} &= e^{i(-\beta_j^1+\alpha_j^1)} e^{i(-\alpha_j^1-\pi+\beta_j^1+\pi)} = 1, \\
\sum_{\ell=1}^{2} e^{2i\beta_j^\ell} - e^{2i\alpha_j^\ell} &= e^{2i\beta_j^1} - e^{2i\alpha_j^1} + e^{2i(\alpha_j^1+\pi)} - e^{2i(\beta_j^1+\pi)} = 0.
\end{aligned}$$

Therefore $\gamma = e^{-i(\theta_1+\theta_2)}$, $\xi_2 = e^{2i\theta_1} + e^{2i\theta_2}$, which proves the theorem. □

6.2 Asymptotics of the Gibbs measures on quadri-tilings

Let R be a rhombus tiling of the plane, and R be the corresponding rhombus-with-diagonals tiling. Consider a subset of edges $e_1 = w_1b_1, \ldots, e_k = w_kb_k$ of R^*, and recall that μ^R is the Gibbs measure on $\mathcal{M}(R^*)$ given by Theorem 3.2.

Corollary 6.2 *When $\forall\, j \neq i$, $|w_j - b_i| \to \infty$, then up to the second order term, $\mu^R(e_1, \ldots, e_k)$ only depends on the rhombi of R to which the vertices b_i and w_j belong, and else is independent of the structure of the graph R.*

Proof: This is a consequence of the explicit formula for $\mu^R(e_1, \ldots, e_k)$ of Theorem 3.2, and of the asymptotic formula for the inverse Dirac operator of Theorem 6.1. □

Recall that $\mathcal{L}^*$ is the non-planar graph obtained by superposing duals of lozenge-with-diagonals tilings of $\mathcal{L}$. Let $e_1 = w_1b_1, \ldots, e_k = w_kb_k$ be a subset of edges of $\mathcal{L}^*$. Define $\mathcal{L}^E$ to be the set of lozenge-with-diagonals tilings of the plane that contain the lozenges associated to the edges $e_1, \ldots, e_k$.

Corollary 6.3 *When $\forall\, j \neq i$, $|w_j - b_i| \to \infty$, then up the second order term, $\mu^L(e_1, \ldots, e_k)$ is independent of $L \in \mathcal{L}^E$.*

Proof: As in Section 5, we choose an embedding of $\mathcal{L}^*$ so that every edge of $\mathcal{L}^*$ uniquely determines the lozenge(s) it belongs to. Corollary 6.3 is then a restatement of Corollary 6.2. □

Recall that μ is the Gibbs on triangular quadri-tilings given by Corollary 5.1. Let $\mathsf{l}_{e_1}, \ldots, \mathsf{l}_{e_k}$ be the lozenges associated to the edges $e_1, \ldots, e_k$ of $\mathcal{L}^*$, and let $\mathsf{k}_{e_1}, \ldots, \mathsf{k}_{e_k}$ be the corresponding edges of $\mathbb{T}^*$.

Corollary 6.4 *When $\forall\, j \neq i$, $|w_j - b_i| \to \infty$, and for every $L \in \mathcal{L}^E$, we have*

$$\mu(e_1, \ldots, e_k) = \mu^L(e_1, \ldots, e_k)\mu^{\mathbb{T}}(\mathsf{l}_{e_1}, \ldots, \mathsf{l}_{e_k}) + O\left(\frac{1}{(\bar{b} - \bar{w})^3}\right).$$

Proof: This is a consequence of the explicit formula for $\mu(e_1, \ldots, e_k)$ of Corollary 5.1, and of Corollary 6.3. □

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