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ON THE GEOMETRY OF PREQUANTIZATION SPACES

MARCO ZAMBON AND CHENCHANG ZHU

ABSTRACT. Given a Poisson (or more generally Dirac) manifold P, there are two approaches to its geometric quantization: one involves a circle bundle Q over P endowed with a Jacobi (or Jacobi-Dirac) structure; the other one involves a circle bundle with a (pre)contact groupoid structure over the (pre)symplectic groupoid of P. We study the relation between these two prequantization spaces. We show that the circle bundle over the (pre)symplectic groupoid of P is obtained from the Lie groupoid of Q via an S^1 reduction that preserves both the Lie groupoid and the geometric structures.

CONTENTS

1. INTRODUCTION

The geometric quantization of symplectic manifolds is a classical problem that has been much studied over years. The first step is to find a prequantization. A symplectic manifold (P, ω) is prequantizable iff $[\omega]$ is an integer cohomology class. Finding a prequantization means finding a faithful representation of the Lie algebra of functions on (P, ω) (endowed with the Poisson bracket) mapping the function 1 to a multiple of the identity. Such a representation space consists usually of sections of a line bundle over P [14], or equivalently of S^1-antiequivariant complex functions on the total space Q of the corresponding circle bundle [18].

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For more general kinds of geometric structure on P, such as Poisson or even more generally Dirac [5] structures, there are two approaches to extend the geometric quantization of symplectic manifolds, at least as far as prequantization is concerned:

- To build a circle bundle Q over P compatible with the Possion (resp. Dirac) structure on P (see Souriau [18] for the symplectic case, [12][20][4] for the Poisson case, and [25] for the Dirac case)
- To build the symplectic (resp. presymplectic) groupoid of P first and construct a circle bundle over the groupoid [24], with the hope to quantize Poisson manifolds "all at once" as proposed by Weinstein [23].

We call Q as above a "prequantization space" for P because, when P is prequantizable, out of the hamiltonian vector fields on Q one can construct a representation of the admissible functions on P, which form a Poisson algebra, on the space of S^1 anti-equivariant functions on Q (see Prop. 5.1 of [25]). Usually however this representation is not faithful.

Since the (pre)symplectic groupoid $\Gamma_s(P)$ of P is the canonical global object associated to P, the prequantization circle bundle over $\Gamma_s(P)$ can be considered an "alternative prequantization space" for P. Furthermore, since there is a submersive Poisson (Dirac) map $\Gamma_s(P) \to P$, the admissible functions on P can be viewed as a Poisson subalgebra of the functions on $\Gamma_s(P)$, which can be prequantized whenever $\Gamma_s(P)$ is a prequantizable (pre)symplectic manifold. The resulting representation is faithful but the representation space is unsuitable because much too large.

In this paper we will not be interested in representations but only in the geometry that arises from the prequantization spaces associated to a given a Dirac manifold (P, L). Indeed our main aim is to study the relation between the two prequantization spaces above, which we will explain in Thm. 4.2, Thm. 4.9 and Thm. 4.11.

We start searching for a more transparent description of the geometric structures on the circle bundles Q, which are Jacobi-Dirac structures [25] $\bar{L}$. This will be done in Section 2, both in terms of subbundles and in terms of brackets of functions, paying particular attention to the Lie algebroid structure that $\bar{L}$ carries.

Secondly, in Section 3, we relate the Lie algebroid $\bar{L}$ associated to Q to the Lie algebroid of the prequantization of $\Gamma_s(P)$. We do this using S^1 precontact reduction, paralleling one of the motivating examples of symplectic reduction: $T^*M//_0G = T^*(M/G)$. This gives us evidence at the infinitesimal level for the relation between the Lie groupoid associated to Q and the prequantization of $\Gamma_s(P)$. The latter relation between Lie groupoids will be described in Section 4, again as an S^1 precontact reduction. We provide a direct proof in the Poisson case. In the general Dirac case, the proof is done by integrating the results of Section 3 to the level of Lie groupoids with the help of Lie algebroid path spaces. As a byproduct, we obtain the prequantization condition for $\Gamma_s(P)$ in terms of period groups on P. Then we show that this condition is automatically satisfied when the Dirac manifold P admits a prequantization circle bundle Q over it. This generalizes some of the results in [8] and [2].

This paper ends with three appendices. Appendix A provides a useful tool to perform computations on precontact groupoids, and Appendix B describes explicitly the Lie groupoid of a locally conformal symplectic manifold. In Appendix C we apply a construction of Vorobjev to the setting of Section 2.

Notation: Throughout the paper, unless otherwise specified, (P, L) will always denote a Dirac manifold, $\pi : Q \to P$ will be a circle bundle and $\bar{L}$ will be a Jacobi-Dirac structure on Q. By Γ_s and Γ_c we will denote presymplectic and precontact groupoids respectively, and we adopt the convention that the source map induces the (Dirac and Jacobi-Dirac respectively) structures on the bases of the groupoids. By "precontact structure" on a manifold we will just mean a 1-form on the manifold.

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2. Constructing the prequantization of P

The aim of this section is to describe in an intrinsic way the geometric structures (Jacobi-Dirac structures $\bar{L}$) on the circle bundles Q induced by prequantizable Dirac manifolds (P, L), paying particular attention to the associated Lie algebroid structures. In Subsection 2.1 we will recall the non-intrinsic construction of $\bar{L}$ given in [25]. In Subsection 2.2 we will describe $\bar{L}$ intrinsicly in terms of subbundles and in Subsection 2.3 by specifying the bracket on functions that it induces.

We first recall few definitions from [25].

Definition 2.1. A *Dirac structure* on a manifold P is a subbundle of $TP \oplus T^*P$ which is maximal isotropic w.r.t. the symmetric pairing $\langle X_1 \oplus \xi_1, X_2 \oplus \xi_2 \rangle_+ = \frac{1}{2}(i_{X_2}\xi_1 + i_{X_1}\xi_2)$ and whose sections are closed under the Courant bracket

$$[X_1 \oplus \xi_1, X_2 \oplus \xi_2]_{Cou} = \big([X_1, X_2] \ \oplus \ \mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1 + \frac{1}{2}d(i_{X_2}\xi_1 - i_{X_1}\xi_2)\big).$$

If ω is a 2-form on P then its graph $\{X \oplus \omega(X, \bullet) : X \in TP\}$ is a Dirac structure iff $d\omega = 0$. Given a bivector Λ on P, the graph $\{\Lambda(\bullet, \xi) \oplus \xi : \xi \in T^*P\}$ is a Dirac structure iff Λ is a Poisson bivector. A Dirac structure L on P gives rise to (and is encoded by) a singular foliation of P, whose leaves are endowed with presymplectic forms.

A function f on a Dirac manifold (Q, L) is *admissible* if there exists a smooth vector field X_f such that $X_f \oplus df$ is a section of L. A vector field X_f as above is called a *hamiltonian vector field* of f. The set of admissible functions, with the bracket $\{f, g\} = X_g \cdot f$, forms a Lie (indeed a Poisson) algebra. Given a map $\pi : Q \to P$ and a Dirac structure L on Q, for every $q \in Q$ one can define the subspace $(\pi_\star L)_{\pi(q)} := \{\pi_* X \oplus \mu : X \oplus \pi^*\mu \in L_q\}\}$ of $T_{\pi(q)}P \oplus T^*_{\pi(q)}P$. Whenever $\pi_\star L$ is a well-defined and smooth subbundle of $TP \oplus T^*P$ it is automatically a Dirac structure on P. In this case $\pi : (Q, L) \to (P, \pi_\star L)$ is said to be a *forward Dirac map*. Similarly, if P is endowed with some Dirac structure L, $(\pi^\star L)(q) := \{Y \oplus \pi^*\xi : \pi_* Y \oplus \xi \in L_{\pi(q)}\}$ (when a smooth subbundle) defines a Dirac structure on Q, and $\pi : (Q, \pi^\star L) \to (P, L)$ is said to be a *backward Dirac map*.

Definition 2.2. A *Jacobi-Dirac structure* on Q is defined as a subbundle of $\mathcal{E}^1(Q) := (TQ \times \mathbb{R}) \oplus (T^*Q \times \mathbb{R})$ which is maximal isotropic w.r.t. the symmetric pairing

$$\langle (X_1, f_1) \oplus (\xi_1, g_1) \ , \ (X_2, f_2) \oplus (\xi_2, g_2) \rangle_+ = \frac{1}{2}(i_{X_2}\xi_1 + i_{X_1}\xi_2 + g_2 f_1 + g_1 f_2)$$

and whose space of sections is closed under the extended Courant bracket on $\mathcal{E}^1(Q)$ given by

$$\begin{aligned}
[(X_1,f_1)\oplus(\xi_1,g_1)\ ,\ (X_2,f_2)\oplus(\xi_2,g_2)]_{\mathcal{E}^1(Q)} &= \big([X_1,X_2], X_1\cdot f_2 - X_2\cdot f_1\big)\\
&\oplus \big(\mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1 + \frac{1}{2}d(i_{X_2}\xi_1 - i_{X_1}\xi_2)\\
&+ f_1\xi_2 - f_2\xi_1 + \frac{1}{2}(g_2 df_1 - g_1 df_2 - f_1 dg_2 + f_2 dg_1),\\
&X_1\cdot g_2 - X_2\cdot g_1 + \frac{1}{2}(i_{X_2}\xi_1 - i_{X_1}\xi_2 - f_2 g_1 + f_1 g_2)\big).
\end{aligned} \tag{1}$$

We mention two examples. Given any 1-form (precontact structure) σ on Q, Graph $\left(\begin{smallmatrix} d\sigma & \sigma \\ -\sigma & 0\end{smallmatrix}\right) \subset \mathcal{E}^1(Q)$ is a Jacobi-Dirac structure. Given a bivector field Λ and a vector field E on Q and with the notation $\tilde{\Lambda}\xi := \Lambda(\bullet,\xi)$, Graph $\left(\begin{smallmatrix} \tilde{\Lambda} & -E \\ E & 0\end{smallmatrix}\right) \subset \mathcal{E}^1(Q)$ is a Jacobi-Dirac structure iff (Λ, E) is a Jacobi structure, i.e. by definition if it satisfies the Schouten bracket conditions $[E,\Lambda]=0$ and $[\Lambda,\Lambda]=2E\wedge\Lambda$. Further to a Dirac structure $L\subset TQ\oplus T^*Q$ there is an associated Jacobi-Dirac structure

$$L^c := \{(X,0)\oplus(\xi,g) : (X,\xi)\in L, g\in\mathbb{R}\}\subset\mathcal{E}^1(Q).$$

A function f on a Jacobi-Dirac manifold $(Q,\bar{L})$ is *admissible* if there exists a smooth vector field X_f and a smooth function φ_f such that $(X_f,\varphi_f)\oplus(df,f)$ is a section of $\bar{L}$, and X_f is called a *hamiltonian vector field* of f. The set of admissible functions, denoted by $C^\infty_{adm}(Q)$, together with the bracket $\{f,g\} = X_g\cdot f + f\varphi_g$ forms a Lie algebra. There is a notion of forward and backward Jacobi-Dirac maps analogous to the one for Dirac structures.

Definition 2.3. A *Lie algebroid* over a manifold P is a vector bundle A over P together with a Lie bracket $[\cdot,\cdot]$ on its space of sections and a bundle map $\rho: A\to TP$ (the *anchor*) such that the Leibniz rule $[s_1, fs_2] = \rho s_1(f)\cdot s_2 + f\cdot[s_1,s_2]$ is satisfied for all sections s_1, s_2 of A and functions f on P.

One can think of Lie algebroids as generalizations of tangent bundles. To every Lie algebroid A one associates cochains (the sections of the exterior algebra of A^*) and a certain differential d_A; the associated *Lie algebroid cohomology* $H^\bullet_A(P)$ can be thought of as a generalization of deRham cohomology. One also defines an *A-connection* on a vector bundle $K\to P$ as map $\Gamma(A)\times\Gamma(K)\to\Gamma(K)$ satisfying the usual properties of a contravariant connection.

A Dirac structure $L\subset TP\oplus T^*P$ is automatically a Lie algebroid over P, with bracket on sections of L given by the Courant bracket and anchor the projection $\rho_{TP}: L\to TP$. Similarly, a Jacobi-Dirac structure $\bar{L}\subset\mathcal{E}^1(Q)$, with the extended Courant bracket and projection onto the first factor as anchor, is a Lie algebroid.

2.1. A non-instrinsic description of $\bar{L}$.

We now recall the prequantization construction of [25], which associates to a Dirac manifold a circle bundle Q with a Jacobi-Dirac structure.

Let (P,L) be a Dirac structure. We saw above that L is a Lie algebroid with the restricted Courant bracket and anchor $\rho_{TP}: L\to TP$ (which is just the projection onto the tangent component). This anchor gives a Lie algebra homomorphism from $\Gamma(L)$ to $\Gamma(TP)$ endowed with the Lie bracket of vector fields. The pullback by the anchor therefore induces a map $\rho^*_{TP}: \Omega^\bullet_{dR}(P,\mathbb{R})\to\Omega^\bullet_L(P)$, the sections of the exterior algebra of L^*, which descends to a map from de Rham cohomology to the Lie algebroid cohomology $H^\bullet_L(P)$ of L. There is a distinguished class in $H^2_L(P)$: on $TP \oplus T^*P$ there is an anti-symmetric pairing given by

$$\langle X_1 \oplus \xi_1, X_2 \oplus \xi_2 \rangle_- = \frac{1}{2}(i_{X_2}\xi_1 - i_{X_1}\xi_2). \tag{2}$$

Its restriction Υ to L satisfies $d_L\Upsilon = 0$. The *prequantization condition* (which for Poisson manifolds was first formulated by Vaisman) is

$$[\Upsilon] = \rho^*_{TP}[\Omega] \tag{3}$$

for some integer deRham 2-class $[\Omega]$. (3) can be equivalently phrased as

$$\rho^*_{TP}\Omega = \Upsilon + d_L\beta, \tag{4}$$

where Ω is a closed integral 2-form and β a 1-cochain for the Lie algebroid L, i.e. a section of L^*. Let $\pi : Q \to P$ be an S^1-bundle with connection form σ having curvature Ω; denote by E the infinitesimal generator of the S^1-action. In Theorem 4.1 of [25] Q was endowed with the following geometric structure, described in terms of the triple (Q, σ, β):

Theorem 2.4. *The subbundle $\bar{L}$ of $\mathcal{E}^1(Q)$ given by the direct sum of*

$$\{(X^H + \langle X \oplus \xi, \beta\rangle E, 0) \oplus (\pi^*\xi, 0) : X \oplus \xi \in L\}$$

and the line bundles generated by $(-E, 0) \oplus (0, 1)$ and $(-A^H, 1) \oplus (\sigma - \pi^*\alpha, 0)$ is a Jacobi-Dirac structure on Q. Here, $A \oplus \alpha$ is an isotropic section of $TP \oplus T^*P$ satisfying $\beta = 2\langle A \oplus \alpha, \cdot\rangle_+|_L$. Such a section always exists, and the subbundle above is independent of the choice of $A \oplus \alpha$.

We call $(Q, \bar{L})$ a "prequantization space" for (P, L) because the assignment $g \mapsto \{\pi^*g, \bullet\} = -X_{\pi^*g}$ is a representation of $C^\infty_{adm}(P)$ on the space of S^1 anti-equivariant functions on Q [25].

Triples (Q, σ, β) as above define a hermitian L-connection with curvature $2\pi i\Upsilon$ on the line bundle K corresponding to Q, via the formula

$$D_\bullet = \nabla_{\rho_{TP}\bullet} - 2\pi i\langle \bullet, \beta\rangle \tag{5}$$

where ∇ is the covariant connection corresponding to σ (Lemma 6.2 in [25]). We have

Proposition 2.5. *For a prequantizable Dirac manifold (P, L), the Jacobi-Dirac structure $\bar{L}$ constructed in Thm. 2.4 on Q is determined by a choice of hermitian L-connection on K with curvature $2\pi i\Upsilon$.*

Proof. We described above how the triples (Q, σ, β) used to construct $\bar{L}$ give rise to hermitian L-connections with curvature $2\pi i\Upsilon$. Conversely, all hermitian L-connections with curvature $2\pi i\Upsilon$ arise from triples (Q, σ, β) as above (Proposition 6.1 in [25]). A short computation shows that the triples that define the same L-connection as (Q, σ, β) are exactly those of the form $(Q, \sigma + \pi^*\gamma, \beta + \rho^*_{TP}\gamma)$ for some 1-form γ on P, and that these triples all define the same Jacobi Dirac structure $\bar{L}$ (Lemma 4.1 in [25]; see also the last comment in Sect. 6.1 there). □

In the next two subsections we will construct $\bar{L}$ directly from the L-connection. We end this subsection by commenting on how the various Jacobi-Dirac structure $\bar{L}$ defined above are related.

Remark 2.6. Two L-connections on K are *gauge equivalent* if the differ by $d_L\phi$ for some function $\phi : P \to S^1$. Gauge-equivalent L-connections D on K with curvature $2\pi i\Upsilon$ give rise to isomorphic Jacobi-Dirac structures: denoting by Φ the bundle automorphism of Q given by $q \mapsto q \cdot \pi^*\phi$, using the proof of Proposition 4.1 in [25] one can show that if $D_2 = D_1 - 2\pi i d_L\phi$ then $(\Phi_*, Id) \oplus ((\Phi^{-1})^*, Id)$ is an isomorphism from the Jacobi-Dirac structure induced by D_1 to the one induced by D_2. (Alternatively one can check directly that for the bracket of functions, which by Remark 2.17 determine the Jacobi-Dirac structures, $\Phi^*\{\cdot,\cdot\}_{D_2} = \{\Phi^*\cdot, \Phi^*\cdot\}_{D_1}$. The gauge-equivalence classes of L-connections with curvature $2\pi i\Upsilon$ are a principal homogeneous space for $H^1_L(P, U(1))$ (see the proof of Prop. 6.1 in [25]).

Remark 2.7. It's easy to see that the prequantization space Q of a prequantizable Dirac manifold (P, L) can be endowed with various non-isomorphic Jacobi-Dirac structures $\bar{L}$. Even more is true: $(Q, \bar{L}_1)$ and $(Q, \bar{L}_2)$ will usually not even be Morita equivalent, for any reasonable notion of Morita equivalence of Jacobi-Dirac manifold (or of their respective precontact groupoids). Indeed for $P = \mathbb{R}$ with the zero Poisson structure, choosing $(Q, \sigma, \beta) = (S^1 \times \mathbb{R}, d\theta, x\partial_x)$ as in Example 4.13 one obtains a Jacobi structure on Q with three leaves, whereas choosing $(S^1 \times \mathbb{R}, d\theta, 0)$ one obtains a Jacobi structure with uncountably many leaves (namely all $S^1 \times \{q\}$). On the other hand, one of the general properties of Morita equivalence is to induce a bijection on the space of leaves.

2.2. **An intrinsic characterization of $\bar{L}$.** In this subsection we fix an L-connection D on the line bundle $K \to P$ with curvature $2\pi i\Upsilon$ and construct the Lie algebroid $\bar{L}$ from L and D directly. (In Prop. 3.4 we will perform the inverse construction, i.e. we will recover L from $\bar{L}$). An alternative approach that works in particular cases is presented in Appendix C.

We begin with a useful lemma concerning flat Lie algebroid connections (compare also to Lemma 6.1 in [25]).

Lemma 2.8. *Let E be any Lie algebroid over a manifold M, K a line bundle over M, and D a Hermitian E-connection on K. Consider the central extension $E \oplus_\eta \mathbb{R}$, where $2\pi i\eta$ equals the curvature of D; then $\tilde{D}_{(Y,g)} = D_Y + 2\pi i g$ defines an $E \oplus_\eta \mathbb{R}$-connection on K which is moreover flat.*

Proof. One checks easily that $\tilde{D}$ is indeed a Lie algebroid connection. Recall that the bracket on $E \oplus_\eta \mathbb{R}$ is defined as $[(a_1, f_1), (a_2, f_2)]_{E\oplus_\eta\mathbb{R}} = ([e_1, e_2]_E, \rho(a_1)f_2 - \rho(a_2)f_1 + \eta(a_1, a_2))$, where ρ is the anchor, and that the curvature of $\tilde{D}$ is

$$R_{\tilde{D}}(e_1, e_2)s = \tilde{D}_{e_1}\tilde{D}_{e_2}s - \tilde{D}_{e_2}\tilde{D}_{e_1}s - \tilde{D}_{[e_1,e_2]}s$$

for elements e_i of $E \oplus_\eta \mathbb{R}$ and s of K. The flatness of $\tilde{D}$ follows by a straightforward calculation. □

We will use of this construction, which is just a way to make explicit the structure of a transformation algebroid (see Remark 2.10 below).

Lemma 2.9. *Let A be any Lie algebroid over a manifold P, $\pi_Q : Q \to P$ a principle $SO(n)$-bundle, $\pi_K : K \to P$ the vector bundle associated to the standard representation of $SO(n)$ on $\mathbb{R}^n$, and $\tilde{D}$ a flat A-connection on K preserving its fiber-wise metric. The A-connection induces a bundle map $h_Q : \pi_Q^*A \to TQ$ (the "horizontal lift") that can be used to extend, by the Leibniz rule, the obvious bracket on $SO(n)$-invariant sections of π_Q^*A to all sections of π_Q^*A. The vector bundle π_Q^*A, with this bracket and h_Q as an anchor, is a Lie algebroid over Q.*

Proof. We first recall some facts from Section 2.5 in [11]. The A-connection $\tilde{D}$ on the vector bundle K defines a map (the "horizontal lift") $h_K : \pi_K^* A \to TK$ covering the anchor $A \to TP$ by taking parallel translations of elements of K along A-paths. See Section 4.2 for the definition of A-paths. Explicitly, fix an A-path $a(t)$ with base path $\gamma(t)$, a point $x \in \pi_K^{-1}(\gamma(0))$ and let $\tilde{\gamma}(t)$ the unique path in K (over $\gamma(t)$) starting at x with $\tilde{D}_{a(t)}\tilde{\gamma}(t) = 0$. We can always write $\tilde{D} = \nabla_{\rho\bullet} - \tilde{\beta}$ where ∇ is a metric TP-connection on A and $\tilde{\beta} \in \Gamma(A^*) \otimes \mathfrak{so}(K)$; then $\nabla_{\rho a(t)}\tilde{\gamma}(t) = \langle \tilde{\beta}, a(t)\rangle \tilde{\gamma}(t)$. Since the left hand side is the projection of the velocity of $\tilde{\gamma}(t)$ along the Ehresmann distribution H corresponding to ∇, we obtain $\frac{d}{dt}\tilde{\gamma}(t) = (\frac{d}{dt}\gamma(t))^H + \langle \tilde{\beta}, a(t)\rangle\tilde{\gamma}(t)$, so that

$$h_K(a(0), x) := \frac{d}{dt}|_{t=0}\tilde{\gamma}(t) = \rho(a(0))^H + \langle \tilde{\beta}, a(0)\rangle x. \tag{6}$$

Of course h_K does not depend on ∇ or $\tilde{\beta}$ directly, but just on $\tilde{D}$. By our assumptions h_K is induced by a "horizontal lift" for the principle bundle Q, i.e. by a $SO(n)$-equivariant map $h_Q : \pi_Q^* A \to TQ$ covering the anchor of A. Since our A-connection $\tilde{D}$ is flat, the map that associates to a section s of A the vector field $h_Q(\pi_Q^* s)$ on Q is a Lie algebra homomorphism.

On sections $\pi_Q^* s_1$, $\pi_Q^* s_2$ of $\pi_Q^* A$ which are pullbacks of sections of A we define the bracket to be $\pi_Q^*[s_1, s_2]$, and we extend it to all sections of $\pi_Q^* A$ by using h_Q as an anchor and forcing the Leibniz rule. We have to show that the resulting bracket satisfies the Jacobi identity. Given sections s_i of A and a function f on Q one can show that the Jacobiator $[[\pi_Q^* s_1, f \cdot \pi_Q^* s_2], \pi_Q^* s_3] + c.p. = 0$ by using the facts that the bracket on sections of A satisfies the Jacobi identity and that the correspondence $\pi_Q^* s_i \mapsto h_Q(\pi_Q^* s_i)$ is a Lie algebra homomorphism. Similarly, the Jacobiator of arbitrary sections of Q is also zero due to fact that h_Q actually induces a homomorphism on *all* sections of $\pi_Q^* A$. □

Remark 2.10. Using h_K instead of h_Q in the construction of the previous lemma leads to a Lie algebroid structure on $\pi_K^* A \to K$. As Kirill Mackenzie pointed out to us, $\pi_K^* A$ is just the transformation algebroid arising from the Lie algebroid action of A on K given by the flat connection $\tilde{D}$. Similarly, the Lie algebroid structure on $\pi_Q^* A$ we constructed in the lemma is the transformation algebroid structure coming from h_Q, which is viewed here as a Lie algebroid action of A on Q.

Now we come back to our original setting, where we consider the Lie algebroid L over P and a hermitian L-connection D on the line bundle K over P. Consider L^c, the Jacobi-Dirac structure on P naturally associated to L. There is a canonical isomorphism $L^c \to L \oplus_\Upsilon \mathbb{R}$, $(X, 0) \oplus (\xi, g) \mapsto (X, \xi, g)$ of Lie algebroids over P [8]. Lemma 2.8 provides us with a flat $L \oplus_\Upsilon \mathbb{R}$-connection $\tilde{D}$ on K , and by Lemma 2.9 the pullback of $L \oplus_\Upsilon \mathbb{R}$ to Q (the circle bundle associated to K) is endowed with a Lie algebroid structure. Using equation (6) one sees that its anchor $h_Q : \pi_Q^*(L \oplus_\Upsilon \mathbb{R}) \to TQ$, at any point of Q, is given by

$$h_Q(X, \xi, g) = X^H + (\langle X \oplus \xi, \beta\rangle - g)E \tag{7}$$

(here we make immaterial choices to write D as in equation (5) and denote by ^{H} the horizontal lift w.r.t. $\ker \sigma$). This formula for the anchor suggests how to identify $\pi_Q^*(L \oplus_\Upsilon \mathbb{R})$ with a subbundle of $\mathcal{E}^1(Q)$: we will show that the natural injection

$$I : \pi_Q^*(L \oplus_\Upsilon \mathbb{R}) \to \bar{L} \subset \mathcal{E}^1(Q), \quad I(X, \xi, g) = (h_Q(X, \xi, g), 0) \oplus (\pi^*\xi, g)$$

is a Lie algebroid morphism, whose image is a codimension one subalgebroid of $\bar{L}$ which we denote by $\bar{L}_0$. We regard $\bar{L}_0$ as a "lift" of L (or rather L^c) obtained using the hermitian

L-connection D. Now we can describe the Jacobi-Dirac structure $\bar{L}$ prequantizing L in invariant terms and characterize partially (see also Remark 2.14) its Lie algebroid structure:

Theorem 2.11. *Assume that the Dirac manifold (P, L) satisfies the prequantization condition (3). Fix the line bundle K over P associated with $[\Omega]$ and a Hermitian L-connection D on K with curvature $2\pi i\Upsilon$. Denote as above by $\bar{L}_0$ the lift of L^c by the connection D. Then $\bar{L}$, the subbundle defined in Thm. 2.4, is characterized as the unique Jacobi-Dirac structure on Q which contains $\bar{L}_0$ and which is different from $(\pi^\star L)^c$ (where $\pi^\star L$ denotes the pullback Dirac structure of L). Further $\bar{L}_0$ is canonically isomorphic to $\pi_Q^*(L \oplus_\Upsilon \mathbb{R})$ as a Lie algebroid.*

Proof. We first show that $I : \pi_Q^*(L \oplus_\Upsilon \mathbb{R}) \to \bar{L}$ is indeed a Lie algebroid morphism. We compute for S^1 invariant sections

$$\begin{aligned} &[I(X_1,\xi_1,0),\, I(X_2,\xi_2,0)]_{\mathcal{E}^1(Q)} \\ &\qquad =I([(X_1,\xi_1),(X_2,\xi_2)]_{Cou},0) + \langle (X_1,\xi_1),(X_2,\xi_2)\rangle_- \left((-E,0)\oplus(0,1)\right) \\ &\qquad =I([(X_1,\xi_1,0),(X_2,\xi_2,0)]_{\pi_Q^*(L\oplus_\Upsilon\mathbb{R})}) \end{aligned} \tag{8}$$

and $[I(X,\xi,0), I(0,0,1)]_{\mathcal{E}^1(Q)} = 0$; then one checks that I respects the anchor maps of $\pi_Q^*(L \oplus_\Upsilon \mathbb{R})$ and $\bar{L}$.

To prove the above characterization of $\bar{L}$ we show that there are exactly two maximally isotropic subbundles of $\mathcal{E}^1(Q)$ containing $\bar{L}_0$. Indeed, denoting by $(\bar{L}_0)^\perp$ the orthogonal of $\bar{L}_0$ w.r.t. the pairing $\langle \bullet, \bullet\rangle_+$, the quotient $(\bar{L}_0)^\perp/\bar{L}_0$ is a rank 2 vector bundle over Q which inherits a non-degenerate symmetric pairing on its fibers. Every fiber of such bundle is isomorphic to $\mathbb{R}^2$ with pairing $\langle (a,b),(a',b')\rangle = \frac{1}{2}(ab' + ba')$, which clearly contains exactly two isotropic subspaces of rank one (namely $\mathbb{R}(1,0)$ and $\mathbb{R}(0,1)$). So there are at most two maximally isotropic subbundles of $\mathcal{E}^1(Q)$ containing $\bar{L}_0$; indeed there are exactly two: $\bar{L}$ and $\bar{L}_0 \oplus \mathbb{R}((0,0)\oplus(0,1))$. The latter is $\pi^\star L = \{Y \oplus \pi^*\xi : \pi_*(Y) \oplus \xi \in L\}$ viewed as a Jacobi-Dirac structure on Q, hence we are done. □

Remark 2.12. Using the canonical identifications of Lie algebroids $L \oplus_\Upsilon \mathbb{R} \cong L^c$ and $\pi_Q^*(L \oplus_\Upsilon \mathbb{R}) \cong \bar{L}_0$ the natural Lie algebroid morphism $\pi_Q^*(L \oplus_\Upsilon \mathbb{R}) \to L \oplus_\Upsilon \mathbb{R}$ is

$$\Phi : \bar{L}_0 \to L^c, (X,0)\oplus(\pi^*\xi, g) \mapsto (\pi_* X, 0) \oplus (\xi, g). \tag{9}$$

Remark 2.13. The construction of Thm. 2.11 gives a quick way to see that the subbundle $\bar{L}$ of $\mathcal{E}^1(Q)$, as defined in Thm. 2.4, is indeed closed under the extended Courant bracket: $\bar{L}_0$ is closed since we realized it as a Lie algebroid, and the sum with the span of the section $(-A^H, 1)\oplus(\sigma - \pi^*\alpha, 0)$ is closed under the bracket because $\langle [s_1, s_2]_{\mathcal{E}^1(Q)}, s_3\rangle_+$ (for s_i sections of $\mathcal{E}^1(Q)$) is a totally skew-symmetric tensor [13].

Remark 2.14. The characterization of $\bar{L}_0$ as the transformation algebroid of some action of $L \oplus_\Upsilon \mathbb{R} \cong L^c$ on Q (Thm. 2.11) shows that if the Lie algebroid L^c is integrable then $\bar{L}_0$ is integrated by the corresponding transformation groupoid. Unfortunately using Thm. 2.11 we are not able to make the same conclusion for $\bar{L}$. Looking at the brackets on $\bar{L}$ is not very illuminating: it is determined by (8) and

$$\begin{aligned} &[I(X,\xi,0),\, (-A^H,1)\oplus(\sigma-\pi^*\alpha,0)]_{\mathcal{E}^1(Q)} = I(-[(X,\xi),(A,\alpha)]_{Cou},0) \\ &\qquad +I(0,\Omega(X)-\xi+\frac{1}{2}d\langle X\oplus\xi,\beta\rangle,0) - \langle A,\xi\rangle\left((-E,0)\oplus(0,1)\right). \end{aligned} \tag{10}$$

The remaining brackets between sections of the form $I(X, \xi, 0)$, $I(0, 0, 1)$ and $(-A^H, 1) \oplus (\sigma - \pi^*\alpha, 0)$ vanish, and by the Leibniz rule these brackets determine the bracket for arbitrary sections of $\bar{L}$.

Remark 2.15. Different choices of L-connection on the line bundle K with curvature $2\pi i\Upsilon$ usually lead to Lie algebroids $\bar{L}$ with different foliations (see Remark 2.7), which therefore can not be isomorphic. However the subalgebroids $\bar{L}_0$ are always isomorphic. Indeed any two connections with the same curvature are of the form D and $D' = D + 2\pi i\gamma$, where γ is a closed section of L^* (see Prop. 6.1 in [25]). A computation using $d_L\gamma = 0$ shows that $(X, \xi) \oplus g \mapsto (X, \xi) \oplus (g - \langle (X, \xi), \gamma\rangle)$ is a Lie algebroid automorphism of $L \oplus_\Upsilon \mathbb{R}$. Further this automorphism intertwines the Lie algebroid actions (7) of $L \oplus_\Upsilon \mathbb{R}$ on Q given by the "horizontal lifts" of the flat connections $\tilde{D}$ and $\tilde{D}'$. Hence the transformation algebroids of the two actions are isomorphic, as is clear from the description of Lemma 2.9.

We exemplify the fact that actions coming from different flat connections are intertwined by a Lie algebroid automorphism (something that can not occur if the anchor of the Lie algebroid is injective) in the simple case when the Dirac structure on P comes from a close 2-form ω: the Lie algebroid action of $TP \oplus_\omega \mathbb{R}$ on Q via a connection ∇ (with curvature $2\pi i\omega$) is intertwined to the obvious action of the Atiyah algebroid TQ/S^1 on Q (essentially given by the identity map) via $TP \oplus_\omega \mathbb{R} \cong TQ/S^1$ is $(X, g) \mapsto X^H - \pi^* gE$, where σ is the connection on the circle bundle Q corresponding to ∇.

2.3. **Describing $\bar{L}$ via the bracket on functions.** In this subsection we will describe the geometric structure $\bar{L}$ on the circle bundle Q in terms of the bracket on the admissible functions on Q; by Remark 2.17 below the bracket on functions uniquely determines $\bar{L}$.

We adopt the following notation. F_S denotes the function on Q associated to a section S of the line bundle K: F_S is just the restriction to the bundle of unit vectors Q of the fiberwise linear function on K given by $\langle \cdot, S\rangle$, where $\langle \cdot, \cdot\rangle$ is the S^1-invariant *real* inner product on K corresponding to the chosen Hermitian form on K. Alternatively F_S can be described as the real part of the S^1-antiequivariant function on Q that naturally corresponds to the section S. By iS we denote the image of the section S by the action of $i \in S^1$ (i.e. S rotated by $90°$), and f and g are functions on P.

Proposition 2.16. *Assume that the Dirac manifold (P, L) satisfies the prequantization condition (3). Fix the line bundle K over P associated with $[\Omega]$ and a Hermitian L-connection D on K with curvature $2\pi i\Upsilon$. Denote by $\tilde{D}$ the flat connection induced as in Lemma 2.8 and by $h_Q : \pi_Q^*(L \oplus_\Upsilon \mathbb{R}) \to TQ$ the horizontal lift associated to $\tilde{D}$ given by (7).*

Suppose a Jacobi-Dirac structure $\hat{L}$ on Q has the following two properties: first, nearby any $q \in Q$ such that $TP \cap L$ is regular near $\pi(q)$, the admissible functions for $\hat{L}$ are exactly those that are constant along the leaves of $\{h_Q(X, 0, 0) : X \in TP \cap L\}$. Second, the bracket on locally defined admissible functions is given by

- $\{\pi^* f, \pi^* g\}_Q = \pi^*\{f, g\}_P$
- $\{\pi^* f, F_S\}_Q = F_{-\tilde{D}_{X_f, df, f} S}$
- $\{\pi^* f, 1\}_Q = 0$
- $\{F_S, 1\}_Q = -2\pi F_{iS}$.

Then $\hat{L}$ must be the Jacobi-Dirac structure $\bar{L}$ given in Thm. 2.4.

Conversely, the Jacobi-Dirac structure $\bar{L}$ given in Thm. 2.4 has the two properties above.

Proof. We start by showing that the Jacobi-Dirac structure $\bar{L}$ constructed in Thm. 2.4 satisfies the above two properties. On the set of points where the "characteristic distribution"

$C := \bar{L} \cap (TQ \times \mathbb{R}) \oplus (0,0)$ of any Jacobi-Dirac structure has constant rank the admissible functions are exactly the functions f such that (df, f) annihilate C. In our case $C = \{X^H + \langle \alpha, X \rangle E : X \in L \cap TP\} = \{h_Q(X, 0, 0) : X \in TP \cap L\}$ is actually contained in TQ, so the admissible functions are those constant on the leaves of C as claimed.

Now we check that the four formulae for the bracket hold. The first equation follows from the fact that the pushforward of $\bar{L}$ is the Jacobi-Dirac structure associated to L (see Section 5 in [25]).

For the second equation we make use of the formulae

$$E(F_S) = -2\pi F_{iS} \qquad \text{and } X^H(F_S) = F_{\nabla_X S},$$

where we make some choice to express D as in equation (5) and X^H denotes to horizontal lift of $X \in TP$ using the connection on Q corresponding to the covariant derivative ∇ on K. Using these formulae we see

$$\begin{aligned} \{\pi^* f, F_S\}_Q &= -\langle dF_S, X_f^H + \langle (X_f, df), \beta \rangle E - fE \rangle \\ &= F_{-\nabla_{X_f} S + 2\pi i (\langle (X_f, df), \beta \rangle - f} \\ &= F_{-\tilde{D}_{X_f, df, f} S}. \end{aligned}$$

For the last two equations just notice that, since $(-E, 0) \oplus (0, 1)$ is a section of $\bar{L}$, the bracket of any admissible function with the constant function 1 amounts to applying $-E$ to that function.

Now we show that if a Jacobi-Dirac structure $\hat{L}$ satisfies the two properties in the statement of the proposition, then it must be $\bar{L}$. By Remark 2.17, the bracket of $dimQ - rkC + 1$ independent functions at regular points of $C := \hat{L} \cap (TQ \times \mathbb{R}) \oplus (0,0)$ determines $\hat{L}$, so we have to show that our two properties carry the information of the bracket of $dimQ - rkC + 1$ independent functions at regular points of C .

It will be enough to consider the open dense subset of the regular points of C where $C = \{h_Q(X, 0, 0) : X \in TP \cap L\}$ (This subset is dense because it includes the points q such that C is regular near q and $TP \cap L$ is regular near $\pi(q)$). Since there C is actually contained in TQ it is clear that 1 and $\pi^* f$ are admissible functions, for f any admissible function on P (this means that f is constant along the leaves of $L \cap TP$; there are $dimP - rkC$ such f which are linearly independent at $\pi(q)$). Further we can construct an admissible function F_S as follows: take a submanifold Y near $\pi(q)$ which is transverse to the foliation given by $L \cap TP$, and define the section $S|_Y$ so that it has norm one (i.e. its image lies in $Q \subset K$). Then extend S to a neighborhood of $\pi(q)$ by starting at a point y of Y and "following" the leaf of C through $S(y)$ (notice that C is a flat partial connection on $Q \to P$ covering the distribution $L \cap TP$ on P). Since C is S^1 invariant, the resulting function F_S is clearly constant along the leaves of C, hence admissible. Altogether we obtain $dimQ - rkC + 1$ admissible functions in a neighborhood of q for which we know the brackets, so we are done. □

Remark 2.17. On any Jacobi-Dirac manifold $(Q, \hat{L})$ the bracket on the sheaf of admissible functions $(C^\infty_{adm}(Q), \{\cdot, \cdot\})$ determines the subbundle $\hat{L}$ of $\mathcal{E}^1(Q)$. (This might seem a bit surprising at first, since the set of admissible functions is usually much smaller than $C^\infty(Q)$).

The set of points where $C := \hat{L} \cap (TQ \times \mathbb{R}) \oplus (0,0)$ (an analog of a "characteristic distribution") has locally constant rank is an open dense subset of Q, since C is an intersection of subbundles. Hence by continuity it is enough to reconstruct the subbundle $\bar{L}$ on each point q of this open dense set.

Since we assume that C has constant rank near q, given $C^\infty_{adm}(Q)$ in a neighborhood of q we can reconstruct C as the distribution annihilated by (df, f) where f ranges over $C^\infty_{adm}(Q)$. We can clearly find $dimQ - rkC + 1$ admissible functions f_i such that $\{(df_i, f_i)\}$ forms a basis of $\rho_{T^*Q\times\mathbb{R}}(\hat{L}) = C^\circ$ near q. The fact that each f_i is an admissible function means that there exist (X_i, ϕ_i) such that $(X_i, \phi_i) \oplus (df_i, f_i)$ is a smooth section of $\hat{L}$. Now knowing the bracket of any f_j with the other f_i's, i.e. the pairing of (X_j, ϕ_j) with all elements of $\rho_{T^*Q\times\mathbb{R}}(\hat{L})$, does not quite determine (X_j, ϕ_j). However it determines (X_j, ϕ_j) up to sections of C, hence the direct sum of the span of all $(X_i, \phi_i) \oplus (df_i, f_i)$ and of C is a well defined subbundle of $\mathcal{E}^1(Q)$. Moreover it has the same dimension as $\hat{L}$ and it is spanned by sections of $\hat{L}$, so it is $\hat{L}$.

3. Prequantization and reduction of Jacobi-Dirac structures

In the last section we considered a prequantizable Dirac manifold (P, L) and endowed Q (the total space of the circle bundle over P) with distinguished Jacobi-Dirac structures $\bar{L}$.

We are interested in the relation between the *Lie algebroid* structures on $\bar{L}$ and L^c (the Jacobi-Dirac structure canonically associated to L), because they will give an indication of the relation between the Lie groupoids integrating them. The map Φ of (9) is a natural surjective morphism of Lie algebroids from the codimension one subalgebroid $\bar{L}_0$ of $\bar{L}$ to L^c, so one may hope to extend Φ to a Lie algebroid morphism defined on $\bar{L}$. However in general there cannot be any Lie algebroid morphism from $\bar{L}$ to L^c or L with base map π: recall that a morphism of Lie algebroids maps each orbit of the source Lie algebroid into an orbit of the target Lie algebroid. If the map $\pi : Q \to P$ induced a morphism of Lie algebroids, then the orbits[1] of $\bar{L}$ would be mapped into the orbits of L^c (which coincide with those of L). However this happens exactly when (one and hence all choices of) the vector field A appearing in Thm. 2.4 is tangent to the foliation of L (see Section 4.1 of [25]). In the case of Example 4.13, i.e. $Q = S^1 \times \mathbb{R}$ and $P = \mathbb{R}$, the orbits of $T^*Q \times \mathbb{R}$ are exactly three (namely $S^1 \times \mathbb{R}_+, S^1 \times \{0\}$ and $S^1 \times \mathbb{R}_-$), and π does not map them into the orbits of T^*P, which are just points.

In this section we will take advantage of the fact that $\bar{L}$, in addition to the Lie algebroid structure, also carries a geometric structure, namely a precontact structure $\theta_{\bar{L}} \in \Omega^1(\bar{L})$ defined as follows:

$$\theta_{\bar{L}} := pr^*(\theta_c + dt), \tag{11}$$

where θ_c is the canonical 1-form on the cotangent bundle T^*Q, t is the coordinate on $\mathbb{R}$, and pr is the projection of $\bar{L} \subset \mathcal{E}^1(Q)$ onto $T^*Q \times \mathbb{R}$. We will use the the 1-form $\theta_{\bar{L}}$ to recover the Lie algebroid L^c from $\bar{L}$ via a precontact reduction procedure, which we will globalize to the corresponding Lie groupoids in the next Section.

3.1. **Reduction of Jacobi-Dirac structures as precontact reduction.** We recall a familiar fact: in symplectic geometry, we have the well-known motivating example of symplectic reduction $T^*M//_0G = T^*(M/G)$. In [9], it is extended to contact geometry by replacing T^*M by the cosphere bundle of M. Here we prove a similar result by replacing T^*M by $T^*M \times \mathbb{R}$–another natural contact manifold associated to any manifold M. Later on we will use this to reduce a G-invariant Jacobi-Dirac structure on M to a Jacobi-Dirac

[1]The orbits of a Lie algebroid are the leaves integrating the (singular) distribution given by the image of the anchor map.

structure on M/G.

Let a Lie group G act on a contact manifold (C, θ) preserving the contact form θ.

Then, a moment map is a map J from the manifold M to $\mathfrak{g}^*$ (the dual of the Lie algebra) such that for all v in the Lie algebra $\mathfrak{g}$:

$$\langle J, v\rangle = \theta_M(v_M), \tag{12}$$

where v_M is the infinitesimal generator of the action on M given by v. The moment map J is automatically equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$ given by $\xi \cdot g = L_g^* R_{g^{-1}}^* \xi$. A group action as above together with its moment map is called *Hamiltonian.* Notice that any group action preserving the contact form is Hamiltonian. In the above setting there are two ways to perform contact reduction, developed by Albert [1] and Willett [26] respectively, which agree when one performs reduction at $0 \in \mathfrak{g}^*$:

$$C//_0 G := J^{-1}(0)/G$$

is again a smooth contact manifold with induced contact form $\bar{\theta}$ such that $\pi^*(\bar{\theta}) = \theta|_{J^{-1}(0)}$.

Lemma 3.1. *Let the group G act on manifold M freely and properly. Then G has an induced action on the contact manifold $(C := T^*M \times \mathbb{R}, \theta := \theta_c + dt)$ where θ_c is the canonical 1-form on T^*M and t is the coordinate on $\mathbb{R}$. Then this action is Hamiltonian and the contact reduction at 0 is*

$$T^*M \times \mathbb{R}//_0 G = T^*(M/G) \times \mathbb{R}.$$

Proof. The induced G action on $T^*M \times \mathbb{R}$ is by $g \cdot (\xi, t) = ((g^{-1})^*\xi, t)$, and it preserves the 1-form $\theta_c + dt$. The projection of this action on M is the G action on M so it is also free and proper. Then the moment map J is determined by

$$\langle J(\xi, t), v\rangle = (\theta_c + dt)_{(\xi,t)}(v_C) = \theta_c(v_C) = \langle \xi, v_M\rangle,$$

where v_C (resp. v_M) denotes the vector filed corresponding to the infinitesimal action of G on the manifold C (resp. M). Since all infinitesimal generators v_C are nowhere proportional to the Reeb vector field $\frac{\partial}{\partial t}$, by Remark 3.2 in [26] all points of $T^*M \times \mathbb{R}$ are regular points of J. So $J^{-1}(0) = \{(\xi, t) : \langle \xi, v_M\rangle = 0 \ \forall v \in \mathfrak{g}\} = \{(\pi^*\mu, t) : \mu \in T^*(M/G)\}$ (with $\pi : M \to (M/G)$) is a smooth manifold. Therefore it is not hard to see that there is a well-defined

$$\Phi : J^{-1}(0)/G \to T^*(M/G) \times \mathbb{R}, \quad ([\xi], t) \mapsto (\mu, t),$$

where μ is uniquely determined by $\pi^*\mu = \xi$ and we used the notation $[\cdot]$ to denote the quotient of points (and later tangent vectors) of $J^{-1}(0)$ by the G action. It is not hard to see that Φ is an isomorphism since the two sides have the same dimension and Φ is obviously surjective. The contact form on $T^*(M/G) \times \mathbb{R}$ corresponding to the reduced contact form $\bar{\theta}$ via the isomorphism Φ is the canonical one: for a tangent vector $([v], \lambda\frac{\partial}{\partial t}) \in T_{[\xi],t}(J^{-1}(0)/G)$,

$$\bar{\theta}_{[\xi],t}([v], \lambda\frac{\partial}{\partial t}) = \theta_{\xi,t}(v, \lambda\frac{\partial}{\partial t}) = \xi(p_* v) + \lambda = \mu(\bar{p}_*\Phi_*[v]) + \lambda,$$

where $p : T^*M \to M$ and $\bar{p} : T^*(M/G) \to M/G$. Here we used $\bar{p}_*\Phi_*[v] = \pi_* p_* v$, which follows from the fact that Φ is a vector bundle map, and we abuse notation by denoting with the same symbol a restriction of Φ. □

This result extends to the precontact situation: instead of the contact manifold $T^*M \times \mathbb{R}$ we consider a Jacobi-Dirac subbundle $\bar{L} \subset \mathcal{E}^1(M)$, which together with the 1-form $\theta_{\bar{L}} \in \Omega^1(\bar{L})$ defined in (11) is a precontact manifold.

Proposition 3.2. *When $(Q, \bar{L})$ is a Jacobi-Dirac manifold, $\bar{L}$ is a precontact manifold as described above. If the group G acts freely and properly on Q preserving the Jacobi-Dirac structure, the action lifts to a free proper Hamiltonian action on $\bar{L}$ with moment map J,*

$$\langle J((X,f) \oplus (\xi, g)), v\rangle = \theta_{\bar{L}}|_{(X,f)\oplus(\xi,g)}(v_{\bar{L}}) = \xi(v_Q).$$

Write $\mathfrak{g}_Q$ as a short form for $\{v_Q : v \in \mathfrak{g}\} \subset TQ$, and let $\pi_\star \bar{L} \subset \mathcal{E}^1(P)$ be the pushforward of $\bar{L}$ via $\pi : Q \to P := Q/G$. Then

1. *$J^{-1}(0)$ is a subalgebroid of $\bar{L}$ iff $\bar{L} \cap (\mathfrak{g}_Q, 0) \oplus (0,0)$ has constant rank, and in that case $\bar{L}//_0 G := J^{-1}(0)/G$ has an induced Lie algebroid structure;*
2. *$J^{-1}(0)/G \cong \pi_\star \bar{L}$ both as Lie algebroids and precontact manifolds, iff $\bar{L} \cap (\mathfrak{g}_Q, 0) \oplus (0,0) = \{0\}$. Here the precontact forms are the reduced 1-form on $J^{-1}(0)/G$ and the one defined as in (11) on $\pi_\star \bar{L}$ respectively.*

Proof. The G action on Q lifts to $\bar{L}$ by $g \cdot (X,f) \oplus (\xi, g) = (g_* X, f) \oplus ((g^{-1})^* \xi, g)$, and the resulting moment map J is clearly as claimed in the statement.

To prove (1) we start with some linear algebra and fix $x \in Q$. We have a map $\pi_* : T_x Q \to T_{\pi(x)}(Q/G)$, hence we can push forward $\bar{L}_x$ to

$$(\pi_\star \bar{L})_{\pi(x)} = \{(\pi_* X, f) \oplus (\mu, g) : (X, f) \oplus (\pi^* \mu, g) \in \bar{L}_x\}$$

to obtain a linear Jacobi-Dirac subspace of $\mathcal{E}^1(Q/G)_{\pi(x)}$. Since $\bar{L}$ is G invariant, doing this at every $x \in Q$ we obtain a well defined subbundle of $\mathcal{E}^1(Q/G)$, which however might fail to be smooth[2]. We have a surjective map

$$\begin{aligned} \Phi : J^{-1}(0) = \{(X,f) \oplus (\xi, g) \in \bar{L} : \xi = \pi^* \mu \text{ for some } \mu \in T_{\pi(x)}(Q/G)\} \to \pi_\star \bar{L} \\ (X,f) \oplus (\xi, g) \mapsto (\pi_* X, f) \oplus (\mu, g) \end{aligned} \tag{13}$$

whose kernel is exactly $J^{-1}(0) \cap (\mathfrak{g}_Q, 0) \oplus (0,0)$ (Notice that the map is well defined for π is a submersion). So $J^{-1}(0)$ has constant rank iff $J^{-1}(0) \cap (\mathfrak{g}_Q, 0) \oplus (0,0) = \bar{L} \cap (\mathfrak{g}_Q, 0) \oplus (0,0)$ does. In this case it is easy to see that $J^{-1}(0)$ is closed under the Courant bracket: the Courant bracket of two sections of $J^{-1}(0)$ lie in $\bar{L}$ (because $\bar{L}$ is closed under the bracket), therefore one just has to show that its cotangent component is annihilated by $\mathfrak{g}_Q$. By a straight-forward computation this is true for G-invariant sections, and by the Leibniz rule it follows for all sections of $J^{-1}(0)$, i.e. $J^{-1}(0)$ is a subalgebroid. Clearly $J^{-1}(0)/G$ becomes a Lie algebroid with the bracket induced from the one on $J^{-1}(0)$ and anchor $([X], f) \oplus ([\xi], g) \mapsto \pi_* X$ (where $[\cdot]$ denotes the equivalence relation given by the G action).

To prove (2) consider the map Φ above. It induces an isomorphism of vector bundles over P between $J^{-1}(0)/G$ and $\pi_\star \bar{L}$ iff it is fiberwise injective, i.e. iff $\bar{L} \cap (\mathfrak{g}_Q, 0) \oplus (0,0) = \{0\}$. Since $J^{-1}(0)/G$ (being a precontact reduction) is a smooth manifold and $J^{-1}(0)/G \cong \pi_\star \bar{L}$ is point-wise a subbundle of $\mathcal{E}^1(P)$, it follows that $\pi_\star \bar{L}$ is a smooth vector bundle over P. We are left with showing that Φ induces an isomorphism of Lie algebroids and precontact manifolds. Using the fact that operations appearing in the definition of Courant bracket such as taking Lie derivatives commute with taking quotient of G (for example $\pi^*(L_{\pi_* X} \mu) = L_X \pi^* \mu$) we deduce that $\Phi : J^{-1}(0) \to \pi_\star \bar{L}$ is a surjective morphism of Lie algebroids, hence the induced map $\Phi : J^{-1}(0)/G \to \pi_\star \bar{L}$ an isomorphism of Lie algebroids.

[2]For example it is not smooth when $G = \mathbb{R}$, $Q = \mathbb{R}^2$, $v_Q = \frac{\partial}{\partial x}$ and $\bar{L}$ is the graph of the 1-form $\frac{y^2}{2} dx$.

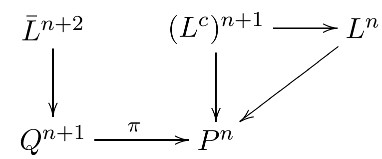
The isomorphism of precontact manifolds follows from an entirely similar argument as in Lemma 3.1. We consider a tangent vector $([w],\kappa\frac{\partial}{\partial s})\oplus([v],\lambda\frac{\partial}{\partial t})\in T_{([X],f)\oplus([\xi],g)}(J^{-1}(0)/G)$, then $\Phi(([X],f)\oplus([\xi],g))=(\pi_*X,f)\oplus(\mu,g)$, where $\pi^*\mu=\xi$. So the induced 1-form $\bar\theta$ on $J^{-1}(0)/G$ satisfies,

$$\bar\theta_{[X],f,[\xi],g}([w],\kappa\frac{\partial}{\partial s})\oplus([v],\lambda\frac{\partial}{\partial t})=\theta_{X,f,\xi,g}(w,\kappa\frac{\partial}{\partial t})\oplus(v,\lambda\frac{\partial}{\partial t})=\xi(p_*v)+\lambda=\mu(\bar p_*\Phi_*[v])+\lambda,$$

where $p:\bar L\to Q$ and $\bar p:\pi_\star\bar L\to P$ are projections. Therefore $\bar\theta=\Phi^*\theta_{\pi_\star\bar L}$ with $\theta_{\pi_\star\bar L}$ the canonical 1-form as in (11). □

Remark 3.3. A special case of Prop. 3.2 is the usual reduction of basic 1-forms: if the Jacobi-Dirac structure $\bar L$ of Prop. 3.2 comes from 1-form σ on Q such that $\mathfrak{g}_Q\subset ker\sigma$, then the pushforward $\pi_\star\bar L$ is given by the unique 1-form σ_{red} on $P=Q/G$ satisfying $\pi^*\sigma_{red}=\sigma$.

3.2. **Reduction of prequantizing Jacobi-Dirac structures.** Now we adapt the general theory of reduction of Jacobi-Dirac manifolds discussed in the previous subsection to our situation, namely we consider a prequantization Q of Dirac manifold (P,L). Then Q is Jacobi-Dirac with a free and proper S^1 action which preserves the Jacobi-Dirac structure $\bar L$. Let $L^c=\{(X,0)\oplus(\xi,g):(X,\xi)\in L,g\in\mathbb{R}\}$ denote the Jacobi-Dirac structure associated to the Dirac manifold (P,L). Then L^c naturally has a precontact form as described in (11). The algebroids $\bar L$, L^c and L fit into the following diagram (where we denote dimensions and ranks by superscripts):



The left two Lie algebroids in the diagram are related by the reduction described in the next proposition:

Proposition 3.4. *When $(Q,\bar L)$ is a prequantization of Dirac manifold (P,L) we have $J^{-1}(0)=\bar L_0$ (recall that $\bar L_0$ was defined at the end of Section 2.2) and the isomorphisms of precontact manifolds and Lie algebroids,*

$$\bar L//_0S^1\cong L^c.$$

Proof. The equality is clear from the characterization of $J^{-1}(0)$ in eq. (13) and from the definition of $\bar L_0$. For the isomorphism notice that $L^c=\pi_\star\bar L$ (this is equivalent to saying that π is a forward Jacobi-Dirac map) and apply Prop. 3.2 (which holds because the assumption $\bar L\cap(\mathfrak{g}_q,0)\oplus(0,0)=\{0\}$ is satisfied, as is clear from the definition of $\bar L$ in Theorem 2.4). □

In the rest of this subsection we want to see what Lemma 3.4 says about the objects that integrate the Lie algebroids $\bar L$ and L^c. We first recall few definitions.

Definition 3.5. A *Lie groupoid* over a manifold P is a manifold Γ endowed with surjective submersions $\mathbf{s}$,$\mathbf{t}$ (called source and target) to the base manifold P, a smooth associative multiplication m defined on elements $g,h\in\Gamma$ satisfying $\mathbf{s}(g)=\mathbf{t}(h)$, an embedding of P into Γ as the spaces of "identities" and a smooth inversion map $\Gamma\to\Gamma$ satisfying certain compatibility conditions (see for example [16])

Every Lie algebroid Γ has an associated Lie algebroid, whose total space is $ker(\mathbf{s}_*|_P)\subset T\Gamma|_P$, with a bracket on sections defined using right invariant vector fields on Γ and $\mathbf{t}_*|_P$

as anchor. A Lie algebroid A is said to be integrable if there exists a Lie groupoid whose associated Lie algebroid is isomorphic to A; in this case there is a unique (up to isomorphism) source simply connected (s.s.c.) Lie groupoid integrating A.

The following two definition are adapted from [3] and [17] respectively to match up the conventions of [8] and [27].

Definition 3.6. A *presymplectic groupoid* is a Lie groupoid Γ over a manifold P, with $\dim\Gamma = 2\dim P$, equipped with a closed 2-form Ω_Γ satisfying

$$m^*\Omega_\Gamma = pr_1^*\Omega_\Gamma + pr_2^*\Omega_\Gamma$$

and the non-degeneracy condition

$$\ker \mathbf{t}_* \cap \ker \mathbf{s}_* \cap \ker \Omega_\Gamma = \{0\}.$$

By [3] the Dirac structure on Γ given by the graph of Ω pushes down via $\mathbf{s}$ to a Dirac structure L on the base P which, as a Lie algebroid, is isomorphic to the Lie algebroid of Γ. Conversely, if (P, L) is any Dirac manifold, then L (if integrable) integrates to a s.s.c. presymplectic groupoid as above. The latter is unique (up to presymplectic groupoid automorphism), and will be denoted by $\Gamma_s(P)$ in this paper.

Hence presymplectic groupoids are the objects integrating Dirac structures. The objects integrating Jacobi-Dirac structures are the following:

Definition 3.7. A *precontact groupoid* is a Lie groupoid Γ over a manifold Q, $\dim\Gamma = 2\dim Q + 1$, equipped with a 1-form θ_Γ and a function f_Γ satisfying $f_\Gamma(gh) = f_\Gamma(g)f_\Gamma(h)$ and

$$m^*\theta_\Gamma = pr_1^*\theta_\Gamma pr_2^* f_\Gamma + pr_2^*\theta_\Gamma$$

and the non-degeneracy condition

$$\ker \mathbf{t}_* \cap \ker \mathbf{s}_* \cap \ker \theta_\Gamma \cap \ker d\theta_\Gamma = \{0\}.$$

The 1-form θ_Γ, viewed as a Jacobi-Dirac structure on Γ, pushes forward via the source map to a Jacobi-Dirac structure on M which is isomorphic to the Lie algebroid of Γ. (The formula for a canonical isomorphism is given in Appendix A). Conversely, if $(Q, \bar{L})$ is any Jacobi-Dirac manifold, then $\bar{L}$ (if integrable) integrates to a s.s.c. unique precontact groupoid as above, which will be denoted by $\Gamma_s(P)$ in this paper. Notice that a Dirac manifold (P, L), in addition to the presymplectic groupoid $\Gamma_s(P)$ associated as above, also has has an associated precontact groupoid $\Gamma_c(P)$ integrating the Jacobi-Dirac structure L^c corresponding to L.

When the presymplectic groupoid $\Gamma_s(P)$ is prequantizable its prequantization circle bundle can be view as an "alternative prequantization space" for (P, L), because $\Gamma_s(P)$ is the global object that corresponds to the Dirac manifold (P, L). We will see in items (4) and (5) of Thm. 4.11 that the prequantizability and integrability of (P, L) implies that $\Gamma_s(P)$ is prequantizable, and that the prequantization bundle $\tilde{\Gamma}_c(P)$ is a groupoid integrating L^c, so $A(\tilde{\Gamma}_c(P)) \cong L^c$ where "A" denote the functor that takes the Lie algebroid of a Lie groupoid. (In the Poisson case this follows from [8] and [2]).

There is a canonical Lie algebroid isomorphism between $\ker \mathbf{s}_*|_P \subset T\tilde{\Gamma}_c(P)|_P$ and L^c, given by Lemma A.1. It matches the restriction to $\ker \mathbf{s}_*|_P$ of the 1-form on $\tilde{\Gamma}_c(P)$ and the precontact form θ_{L^C} on L^c (see eq. (11)) at points of P (notice that at points of the zero section P the precontact form on L^c is just pr^*dt, i.e. the projection onto the last component). Similarly the canonical isomorphism between $\ker \mathbf{s}_*|_Q$ (where here $\mathbf{s}$ denotes the source map of $\Gamma_c(Q)$) and $\bar{L}$ matches the restriction of the 1-form on $\Gamma_c(Q)$ and $\theta_{\bar{L}}$. Hence the reduction of Prop. 3.4 matches the 1-forms on the groupoids $\Gamma_c(Q)$ and $\tilde{\Gamma}_c(P)$ at points of the identity sections.

As we will see in the next section, there is an S^1 action on the precontact groupoid $(\Gamma_c(Q), \theta_\Gamma, f_\Gamma)$ of $(Q, \bar{L})$ which is canonically induced by the S^1 action on Q and which hence makes the source map equivariant and which respects the 1-form and multiplicative function on the groupoid. The equivariance makes sure that taking derivatives along the identity one gets an S^1 action on $\ker \mathbf{s}_*|_Q$ by vector bundle isomorphism. Further, under the canonical isomorphism (see Lemma A.1) $\ker \mathbf{s}_*|_Q \cong \bar{L}$, the S^1 action is the natural one described at the beginning of the proof of Prop. 3.2, because the S^1 action on $\Gamma_c(Q)$ respects $\mathbf{t}$,r_Γ and θ_Γ. We conclude that the S^1 action we considered in this subsection is the infinitesimal version of the S^1 action on $(\Gamma_c(Q), \theta_\Gamma)$. We summarize:

Proposition 3.8. *The natural S^1 action on Q lifts to an action on $A(\Gamma_c(Q)) \cong \bar{L}$, whose precontact reduction is $L^c \cong A(\tilde{\Gamma}_c(P))$, endowed with the Lie algebroid and precontact structures given by the Lie groupoid $\tilde{\Gamma}_c(P)$.*

In the next section we will show that the precontact reduction of $\Gamma_c(Q)$ is isomorphic, both as precontact manifold and a groupoid, to the s.s.c. precontact groupoid of P, and that $\tilde{\Gamma}_c(Q)$ is a discrete quotient of it. This means that precontact reduction commutes with the Lie algebroid functor:

$$A(\Gamma_c(Q)//_0 S^1) = A(\Gamma_c(Q))//_0 S^1.$$

Further we also have a correspondence at the intermediate step of the reduction, namely for the zero level sets of the moment maps (see item (3) of Thm. 4.9).

4. Prequantization and reduction of precontact groupoids

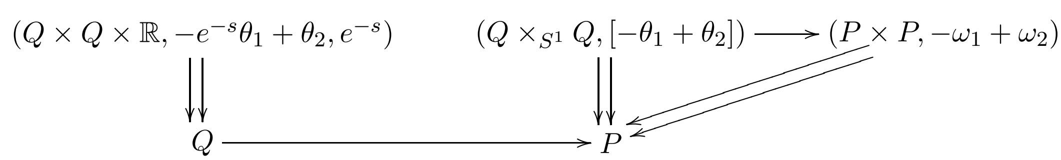
In this section we analyze the relation between the groupoids associated to (P, L) and $(Q, \bar{L})$, leading to an "integrated" version of Proposition 3.4 (i.e. to reduction of groupoids). In Subsection 4.1 we will perform the reduction using finite dimensional arguments, restricting ourselves for simplicity to the case when P is a Poisson manifold. If on one hand our finite dimensional proof might appeal more to geometric intuition, it will not allow to conclude whether the reduced groupoids we obtain are source simply connected. In Subsection 4.2, for the general case when P is a Dirac manifold, we will obtain a complete description of the reduction using path spaces. We will conclude with two examples.

4.1. The Poisson case.

In this subsection we show our results for Poisson manifold without using the infinite dimensional path spaces.

We start displaying a simple example, which was also a motivating example in [6].

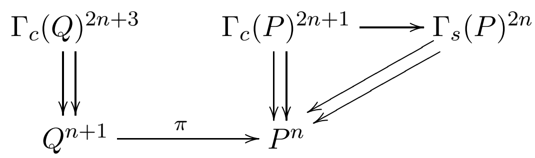
Example 4.1. Let (P, ω) be a simply connected integral symplectic manifold, and (Q, θ) a prequantization. We have the following diagram of groupoids:



The first groupoid is a (usually not s.s.c.) contact groupoid of (Q, θ), with coordinate s on the $\mathbb{R}$ factor. The second is a contact groupoid of (P, ω) which is a prequantization of the third groupoid (the s.s.c. symplectic groupoid of (P, ω)). The S^1 action on Q induces a circle action on its contact groupoid with moment map given by $\langle J, 1\rangle = -e^{-s} + 1$, so that its zero level set is obtained setting $s = 0$, and dividing by the circle action we obtain exactly the second groupoid above, i.e. the prequantization of the s.s.c. groupoid of (P, ω).

Let P be a Poisson manifold, consider the Dirac structure L given by the graph of the Poisson bivector, and assume that (P, L) is prequantizable and that it is integrable, in which case it integrates to a s.s.c symplectic[3] groupoid $\Gamma_s(P)$. The prequantizability of (P, L) implies that the period group of any source fiber of $\Gamma_s(P)$ is contained in $\mathbb{Z}$ (see Section 3.3 of [2], or Theorem 4.2 below for a straightforward generalization). This last condition is equivalent to saying that the symplectic groupoid $\Gamma_s(P)$ is prequantizable in the sense of [6] (see Prop. 2 in [2] or Thm. 3 in [8]). Its unique prequantization will be denoted by $\tilde{\Gamma}_c(P)$ and turns out to be a (usually not s.s.c.) contact[4] groupoid of P, i.e. it integrates the Lie algebroid L^c. Fix a prequantization $(Q, \bar{L})$ and assume that the Lie algebroid $\bar{L}$ is integrable; denote by $\Gamma_c(Q)$ the integrating s.s.c. contact groupoid. Now, "integrating" the reduction statements of the last section, we will clarify the relation between $\Gamma_c(Q)$ (the global object attached to the prequantization bundle Q) and the prequantization of $\Gamma_s(P)$ (which can be thought of as a different way to prequantize (P, L)).

The (smooth) groupoids we consider fit into the following diagram; we omitted $\tilde{\Gamma}_c(P)$, which is just a discrete quotient of the s.s.c. contact groupoid $\Gamma_c(P)$. This diagram corresponds to the diagram of Lie algebroids in Subsection 3.2, and again we denote dimensions by superscripts.



Theorem 4.2. *Let (P, L) be an integrable prequantizable Poisson manifold, and $(Q^{n+1}, \bar{L})$ one of its prequantizations as in Subsection 2.1, which we assume to be integrable. Then:*

a) *The s.s.c contact groupoid $\Gamma_c(P)$ of (P, L) is obtained from the s.s.c. contact groupoid $\Gamma_c(Q)$ of $(Q, \bar{L})$ by S^1 contact reduction.*
b) *The prequantization of the s.s.c. symplectic groupoid $\Gamma_s(P)$ is a discrete quotient of $\Gamma_c(P)$.*

Proof. S^1 acts on Q, and it acts also on $TQ \oplus T^*Q$ by the tangent and cotangent lifts. The S^1 action preserves the subbundle given by the Jacobi-Dirac structure $\bar{L}$, hence we obtain an S^1 action on the Lie algebroid $\bar{L} \to Q$. The source simply connected (s.s.c.) contact groupoid $(\Gamma_c(Q), \theta_\Gamma, f_\Gamma)$ of $(Q, \bar{L})$ is constructed canonically from the Lie algebroid $\bar{L}$ via the path-space construction [7], so it inherits an S^1 action that preserves its geometric and groupoid structures. In particular the source and target maps are S^1 equivariant, and similarly the multiplication map $\Gamma_c(Q)_{\mathbf{s}} \times_{\mathbf{t}} \Gamma_c(Q) \to \Gamma_c(Q)$. Also, the S^1 action preserves the contact form, so there is a moment map $J_\Gamma : \Gamma_c(Q) \to \mathbb{R}$ by $J_\Gamma(g) = \theta_\Gamma(v_\Gamma(g))$ where v_Γ denotes the infinitesimal generator of the S^1 action. We divide the proof in three steps.

[3]This means that the 2-form on the presymplectic groupoid intergrating L is non-degenerate.
[4]This means that the 1-form on the precontact groupoid satisfies $\theta_\Gamma \wedge (d\theta_\Gamma)^{dim(P)} \neq 0$.

Step 1: $J_\Gamma^{-1}(0)$ is a s.s.c. Lie subgroupoid of $\Gamma_c(Q)$.
We start by showing that $J_\Gamma = 1 - f_\Gamma$; this explicit[5] formula will turn out to be necessary in Step 2.

To do this we will use several properties of contact groupoids, for which to refer to Remark 2.2 in [27]. The identity $J_\Gamma + f_\Gamma = 1$ is clear along the identity section Q, since f_Γ is a multiplicative function and v_Γ is tangent to Q which is a Legendrian submanifold of $(\Gamma_c(Q), \theta_\Gamma)$. So to show that the statement holds at any point of $\Gamma_c(Q)$ it is enough to show that $\langle d(f_\Gamma + J_\Gamma), X_{f_\Gamma \mathbf{t}^* u}\rangle = 0$ for functions $u \in C^\infty(Q)$, since hamiltonian vector fields $X_{f_\Gamma \mathbf{t}^* u}$ span $\ker \mathbf{s}_*$. The statement follows by two computations: first

$$\begin{aligned}\langle df_\Gamma, X_{f_\Gamma \mathbf{t}^* u}\rangle &= \langle df_\Gamma, f_\Gamma \mathbf{t}^* u E_\Gamma + \Lambda_\Gamma d(f_\Gamma \mathbf{t}^* u)\rangle \\ &= f_\Gamma \cdot \langle df_\Gamma, \Lambda_\Gamma d(\mathbf{t}^* u)\rangle = -f_\Gamma \cdot d(\mathbf{t}^* u) X_{f_\Gamma} = f_\Gamma \cdot E(u),\end{aligned} \tag{14}$$

where we used twice $E_\Gamma(f_\Gamma) = 0$ and the fact that $\mathbf{t}$ is a $-f_\Gamma$-Jacobi map. Second,

$$\langle d(\theta_\Gamma(v_\Gamma)), X_{f_\Gamma \mathbf{t}^* u}\rangle = -d\theta_\Gamma(v_\Gamma, X_{f_\Gamma \mathbf{t}^* u}) = \langle -d(f_\Gamma \mathbf{t}^* u), (v_\Gamma - \theta_\Gamma(v_\Gamma) E_\Gamma)\rangle = -f_\Gamma \cdot E(u),$$

where we use the fact that $\mathcal{L}_{v_\Gamma}\theta_\Gamma = 0$ in the first equality, the formula $d\theta_\Gamma(X_\phi, w) = -\langle d\phi, w^H\rangle$ valid for any function ϕ on a contact groupoid (where w^H is the projection of the tangent vector w to $\ker \theta_\Gamma$ along the Reeb vector field E_Γ) in the second one, and in the last equality that $E_\Gamma(f_\Gamma)$,$v_\Gamma(f_\Gamma)$,$\mathbf{t}_* E_\Gamma$ all vanish and that the S^1 actions on $\Gamma_c(Q)$ and Q are intertwined by the target map $\mathbf{t}$.

Since f_Γ is multiplicative, it is clear that $J_\Gamma^{-1}(0) = f_\Gamma^{-1}(1)$ is a subgroupoid.

Further $J_\Gamma^{-1}(0)$ is a smooth submanifold of $\Gamma_c(Q)$: by Prop. 3.1.4 in [26] $g \in \Gamma_c(Q)$ is a singular point of J_Γ iff $v_\Gamma(g)$ is a non-zero multiple of $E_\Gamma(g)$. Since $\theta_\Gamma(E_\Gamma) = 1$ this is never the case if $g \in J_\Gamma^{-1}(0)$, so 0 is a regular value of J_Γ.

To show that $J_\Gamma^{-1}(0)$ is a Lie subgroupoid we still need to show that its source and target maps are submersions onto Q. We do so by showing explicitly that $(\ker \mathbf{s}_* \cap \ker df_\Gamma)$ (which along Q will be the Lie algebroid of $J_\Gamma^{-1}(0)$) has rank one less than $\ker \mathbf{t}_*$; this is clear since by the first equation of Step 1 it is just $\{X_{f_\Gamma \mathbf{t}^* \pi^* v} : v \in C^\infty(P)\}$.

For the proof of the source simply connectedness of the subgroupoid $J_1^{-1}(0)$ we refer to Thm. 4.9.

Step 2: The contact reduction $J_\Gamma^{-1}(0)/S^1$ is the s.s.c. contact groupoid $\Gamma_c(P)$ of P.
$J_\Gamma^{-1}(0)/S^1$ is smooth because the S^1 action is free and proper, and by contact reduction it is a contact manifold, so we just have to show that the Lie groupoid structure descends and is a compatible one.

The S^1 equivariance of the source and target maps of $\Gamma_c(Q)$ ensure that source and target descend to maps $J_\Gamma^{-1}(0)/S^1 \to P(= Q/S^1)$. Since the multiplication on $\Gamma_c(Q)$ is S^1 equivariant, the multiplication on $J_\Gamma^{-1}(0)$ induces a multiplication on $J_\Gamma^{-1}(0)/S^1$. It is routine to check this makes $J_\Gamma^{-1}(0)/S^1$ into a groupoid over P. Further, since the source map intertwines the S^1 action on $J^{-1}(0)$ and the free S^1 action on the base Q, the source fibers of $J_\Gamma^{-1}(0)/S^1$ will be diffeomorphic to the corresponding source fibers of $J_\Gamma^{-1}(0)$, hence

[5]The claim of Step 1 follows even without knowing the explicit formula for J_Γ. Indeed one can show that $J_\Gamma^{-1}(0)$ is a subgroupoid by means of the identity $J_\Gamma(gh) = f(h)J_\Gamma(g) + J_\Gamma(h)$, which is derived using the multiplicativity of θ_Γ and the fact that v_Γ is a multiplicative vector field (i.e. $v_\Gamma(g) \cdot v_\Gamma(h) = v_\Gamma(gh)$; this is just the infinitesimal version of the statement that the multiplication map is S^1 equivariant). Since $J_\Gamma^{-1}(0)$ is a smooth wide subgroupoid it is transverse to the $\mathbf{s}$ fibers nearby the identity, therefore its source and target maps are submersions and hence it is actually a Lie subgroupoid.

we obtain a s.s.c. Lie groupoid. Since $J_\Gamma^{-1}(0) \to J_\Gamma^{-1}(0)/S^1$ is a surjective submersion, the f_Γ-twisted multiplicativity of θ_Γ implies that the induced 1-form $\hat{\theta}_\Gamma$ is multiplicative, i.e. $(J_\Gamma^{-1}(0)/S^1, \hat{\theta}_\Gamma, \hat{f}_\Gamma)$ is a contact groupoid.

In order to prove that the above contact groupoid corresponds to the original Poisson structure Λ_P on P, we have to show that the source map $\hat{\mathbf{s}} : J_\Gamma^{-1}(0)/S^1 \to P$ is a Jacobi map (i.e. a forward Jacobi-Dirac map). Consider the diagram

$$\begin{array}{ccc} J_\Gamma^{-1}(0) & \xrightarrow{\pi_{J_\Gamma}} & J_\Gamma^{-1}(0)/S^1 \\ \mathbf{s}\big\downarrow & & \hat{\mathbf{s}}\big\downarrow \\ Q & \xrightarrow{\pi} & P. \end{array}$$

We adopt the following short-form notation: for a 1-form α, L_α will denote the Jacobi-Dirac structure associated to α [22]. Then for the pullback Jacobi-Dirac structure we have $i^* L_{\theta_\Gamma} = L_{i^*\theta_\Gamma}$, where i is the inclusion of $J_\Gamma^{-1}(0)$ into $\Gamma_c(Q)$, and the reduced 1-form is recovered as $\pi_{J_\Gamma *} i^* L_{\theta_\Gamma} = L_{\hat{\theta}_\Gamma}$. So by the functoriality of the pushforward, it is enough to show that $\pi_* \mathbf{s}_* L_{i^*\theta_\Gamma}$, which by definition is

$$\{((\pi \circ \mathbf{s})_* Y, f) \oplus (\xi, g) : (Y, f) \oplus ((\pi \circ \mathbf{s})^* \xi, g) \in L_{i^*\theta_\Gamma}\}, \tag{15}$$

equals the Jacobi-Dirac structure given by Λ_P. First we determine which tangent vectors Y to $J_\Gamma^{-1}(0)$ and $f \in \mathbb{R}$ have the property that $i^*(d\theta_\Gamma(Y) + f\theta_\Gamma)$ annihilates $\ker(\pi \circ \mathbf{s})_*$, which using equation (14) is equal to $\{X_{f_\Gamma \mathbf{t}^* \pi^* v} : v \in C^\infty(P)\} \oplus \mathbb{R} v_\Gamma$. A computation similar to those carried out in Step 1 and using the explicit formula $J = 1 - f_\Gamma$ shows that this is the case when $f = 0$ and $\pi_* \mathbf{t}_* Y = 0$, which by a computation similar to (14) amounts to $Y \in \{X_{\mathbf{s}^* \pi^* v} : v \in C^\infty(P)\} \oplus \mathbb{R} v_\Gamma$. These will be exactly the "Y" and "f" appearing in (15); a short computation using the facts that the source map of $\Gamma_c(Q)$ and π are Jacobi maps shows that (15) equals $\{(-\Lambda_P \xi, 0) \oplus (\xi, g) : \xi \in T^*P, g \in \mathbb{R}\}$, as was to be shown.

Step 3: $((J_\Gamma^{-1}(0)/S^1)/\mathbb{Z}, \hat{\theta}_\Gamma)$ is the prequantization of the s.s.c. symplectic groupoid $\Gamma_s(P)$ of P. Here $\mathbb{Z}$ acts as a subgroup of $\mathbb{R}$ by the flow of the Reeb vector field $\hat{E}_\Gamma$.
Consider the action on $J_\Gamma^{-1}(0)/S^1$ by its Reeb vector field $\hat{E}_\Gamma$, which by the contact reduction procedure is the projection of the Reeb vector field E_Γ of $\Gamma_c(Q)$ under $J_\Gamma^{-1}(0) \to J_\Gamma^{-1}(0)/S^1$.

The $\mathbf{t}$-image of a v_Γ orbit is an orbit of the S^1 action on Q, since the target map is S^1 equivariant. Hence each v_Γ orbit meets each $\mathbf{t}$-fiber at most once. Further each E_Γ-orbit is contained in a single $\mathbf{t}$-fiber (since $\mathbf{t}_* E_\Gamma = 0$), so an E_Γ orbit meets any orbit of the S^1 action on $\Gamma_c(Q)$ at most once. Therefore the period of an E_Γ orbit and of the corresponding $\bar{E}_\Gamma$ orbit are equal, and the first period is always an integer number (because $\mathbf{s}_* E_\Gamma = E_Q$, the generator of the circle action on Q).

Now the we know that the periods of $\bar{E}_\Gamma$ are integers, we can just apply Theorems 2 and 3 of [8] to prove our claim. □

4.2. **Path space constructions and the general Dirac case.** In this subsection we generalize Thm. 4.2 allowing P to be a general Dirac manifold, using the explicit description of Lie groupoids as quotients of path spaces as a powerful tool. The generalization will be presented in Thm. 4.9 and Thm. 4.11.

Definition 4.3. Let $\pi : A \to M$ be a Lie algebroid with anchor ρ. The *A-path space* $P_a(A)$ consists of all paths $a : [0,1] \to A$ satisfying $\frac{d}{dt}(\pi \circ a)(t) = \rho(a(t))$.

There is an equivalence relation in P_aA, called *A-homotopy* [7].

Definition 4.4. Let $a(t,s)$ be a family of A-paths which is C^2 in s. Assume that the base paths $\gamma(t,s) := \pi \circ a(t,s)$ have fixed end points. For a connection ∇ on A, consider the equation

$$\partial_t b - \partial_s a = T_\nabla(a,b), \quad b(0,s) = 0. \tag{16}$$

Here T_∇ is the torsion of the connection defined by $T_\nabla(\alpha,\beta) = \nabla_{\rho(\beta)}\alpha - \nabla_{\rho(\alpha)}\beta + [\alpha,\beta]$. Two paths $a_0 = a(0,\cdot)$ and $a_1 = a(1,\cdot)$ are homotopic if the solution $b(t,s)$ satisfies $b(1,s) = 0$.

More geometrically, for every Lie algebroid A, (notice that tangent bundles are Lie algebroids), we associate A a simplicial set $S(A) = [...S_2(A) \Rrightarrow S_1(A) \rightrightarrows S_0(A)]$ with,

$$S_i(A) = \hom_{algd}(T\Delta^i, A) := \{\text{Lie algebroid morphisms } T\Delta^i \xrightarrow{f} A\}, \tag{17}$$

and face and degeneracy maps $d_i^n : S_n(A) \to S_{n-1}(A)$ and $s_i^n : S_n(A) \to S_{n+1}(A)$ induced from the natual face and degeneracy maps $\Delta^n \to \Delta^{n-1}$ and $\Delta^n \to \Delta^{n+1}$. Here Δ^i is the i-dimensional standard simplex viewed as a smooth Riemannian manifold with boundary, hence it is isomorphic to the i-dimensional closed ball. Then as explained in [28, Section 2],

- it is easy to check that $S_0 = M$;
- S_1 is exactly the A-path space P_aA;
- bigons in S_2 are exactly the A-homotopies in P_aA since a bigon $f : T(d_2^2)^{-1}(Ts_0^1(T\Delta^0)) \to A$ can be written as $a(t,s)dt + b(t,s)ds$ over the base map $\gamma(t,s)$ after a suitable choice of parametrization[6] of the disk $(d_2^2)^{-1}(s_0^1(\Delta^0))$. Then we naturally have $b(0,s) = f(0,s)(\frac{\partial}{\partial s}) = 0$ and $b(1,s) = f(1,s)(\frac{\partial}{\partial s}) = 0$. Moreover the morphism is a Lie algebroid morphism if and only if $a(t,s)$ and $b(t,s)$ satisfy equation (16) which defines the A-homotopy.

The s.s.c. groupoid of any integrable Lie algebroid A can be constructed as the quotient of the A-path space by a foliation $\mathcal{F}$, whose leaves consists of the A-paths that are A-homotopic to each other [7]. In particular the precontact groupoid $(\Gamma_c(Q), \theta, f)$ of a Jacobi-Dirac manifold Q can be constructed via the A-path space $P_a(\bar{L})$, with θ and f coming from a corresponding 1-form and function on the path space. We refer to [8] [6] [17] and summarize the results in Thm. 4.5 below. The advantage of this method is that it can be used to generalize Theorem 4.2 to the setting of Dirac manifolds (see Theorems 4.9 and 4.11) and that it can be applied to a general group G action as in [10].

Theorem 4.5. *The s.s.c. precontact groupoid $(\Gamma_c(Q), \theta_\Gamma, f_\Gamma)$ of an integrable Jacobi-Dirac manifold $(Q, \bar{L})$ is the quotient space of the A-path space $P_a(\bar{L})$ by A-homotopies, and θ_Γ and f_Γ come from a 1-form $\tilde{\theta}$ and a function $\tilde{f}$ on $P_a(\bar{L})$. At the point $a = (a_4, a_3, a_1, a_0) \in P_a(\bar{L})$, where (a_4, a_3, a_1, a_0) are components in $TQ \oplus \mathbb{R} \oplus T^*Q \oplus \mathbb{R}$, $\tilde{\theta}$ and $\tilde{f}$ are*

$$\begin{aligned} \tilde{\theta}_a(X) &= -\int_0^1 \left\langle e(t)X(t), d\left(\int_0^1 a_0(t)dt\right)\right\rangle dt + \int_0^1 \langle e(t)X(t), pr^*\theta_c\rangle\, dt, \\ \tilde{f}(a) &= e(1), \quad \textit{with } e(t) := e^{\int_0^t -a_3} \end{aligned} \tag{18}$$

where X is a tangent vector to $P_a(\bar{L})$, hence a path itself (parameterized by t), and $pr^*\theta_c$ is the pull-back via $pr : \bar{L} \to T^*Q$ of the canonical 1-form on T^*Q.

---

[6] We need the one with $\gamma(0,s) = x$ and $\gamma(1,s) = y$ for all $s \in [0,1]$.

Proof. The equation for $\tilde{f}$ is taken from Prop. 3.5(i) of [8]. It is shown there that $\tilde{f}$ descends to the function f_Γ on $\Gamma_c(Q)$. To get the formula for $\tilde{\theta}$, we recall from Section 3.4 of [8] that the following map ϕ is an isomorphism preserving A-homotopies:

$$\phi : P_a(\bar{L}) \times \mathbb{R} \to P_a(\bar{L} \times_\psi \mathbb{R}),$$

mapping (a, s) with base path γ_1 to $\tilde{a} := e^{\gamma_0(t)} a$ with base path (γ_1, γ_0), where $\gamma_0 := s - \int_0^t a_3$. Here ψ is the 1-cocycle on $\bar{L}$ given by $(X, f) \oplus (\xi, g) \mapsto f$; $\bar{L} \times_\psi \mathbb{R}$ is the Lie algebroid on $Q \times \mathbb{R}$ obtained from the Lie algebroid $\bar{L}$ and the 1-cocycle ψ, and it is isomorphic to the Lie algebroid given by the Dirac structure on $Q \times \mathbb{R}$ obtained from the "Diracization" of $(Q, \bar{L})$ (see Section 2.3 in [17]).

The correspondence on the level of tangent spaces given by $T\phi$ maps $(\delta\gamma_1, \delta s, \delta a)$ to $(\delta\gamma_1, \delta\gamma_0, \delta\tilde{a})$ and satisfies

$$\begin{aligned}
\delta\gamma_0 &= \delta s - \int_0^t a_3, \\
\delta\tilde{a}_1 &= e^{\gamma_0}(\delta a_1 + (\delta s - \int_0^t \delta a_3) a_1), \\
\delta\tilde{a}_0 &= e^{\gamma_0}(\delta a_0 + (\delta s - \int_0^t \delta a_3) a_0).
\end{aligned}$$

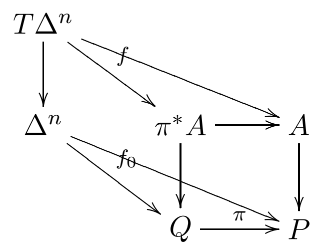
We identify $\bar{L} \times_\psi \mathbb{R}$ with the Dirac structure on $Q \times \mathbb{R}$ given by the Diracization of $(Q, \bar{L})$. Then on the whole space $P(\bar{L} \times_\psi \mathbb{R})$ of paths in $\bar{L} \times_\psi \mathbb{R}$ there is a symplectic form ω coming from integrating the pull-back of the canonical symplectic form on $T^*(Q \times \mathbb{R})$ (see Section 5 in [3]). This form restricted to the A-path space $P_a(\bar{L} \times_\psi \mathbb{R})$ is homogeneous w.r.t. the $\mathbb{R}$ component, i.e. $\varphi_s \omega = e^s \omega$, where φ_s is the flow of $\frac{\partial}{\partial s}$ with s the coordinate of $\mathbb{R}$. This is because φ_s acts on vector fields $\delta\tilde{a}_1$ and $\delta\tilde{a}_0$ by rescaling by an e^s factor as the formula of $T\phi$ and γ_0 show. This homogeneity survives the quotient to groupoids as shown in [8]. Therefore θ_Γ comes from the 1-form $\tilde{\theta}$ whose associated homogeneous symplectic form is ω, i.e. $\tilde{\theta} = -i_0^* i(\frac{\partial}{\partial s})\omega$. With a straightforward calculation and the formula of $T\phi$, we have the formula for $\tilde{\theta}$ in (18). □

Remark 4.6. The formula for $\tilde{\theta}$ is a generalization of Theorem 4.2 in [6] in the case $\bar{L}$ that comes from a Dirac structure. To get the formula of the 1-form there up to sign[7], one just has to put $e(t) = 1$ which corresponds to the case that $a_3 = 0$.

In Lemma 2.9 we constructed a Lie algebroid structure on $\pi^* A$, the pull back via $\pi : Q \to P$ of any Lie algebroid A on P, provided there is a flat A-connection $\tilde{D}$ on the vector bundle K corresponding to the principal bundle Q. ($\pi^* A$ turned out to be the transformation algebroid w.r.t. the action by the flat connection). Now we show some functorial property of algebroid paths in $\pi^* A$. Later in this section we will apply them to $A = L^c$, for $\pi^* L^c$ is identified with a Lie subalgebroid of $\bar{L}$ (Thm. 2.11), whose integrating groupoid we can describe in term of A-paths (Thm. 4.5).

[7] In [6] 1-forms on contact groupoids are so that the target map is a Jacobi map, whereas here we adopt the convention (as in [27]) that the source map be Jacobi.

Lemma 4.7. *An A-path a in A can be lifted to an A-path in π^*A. The same is true for A-homotopies. In other words, in the following diagram (for $n = 1, 2$),*



any Lie algebroid morphism $f : T\Delta^n \to A$ lifts to a Lie algebroid morphism from $T\Delta^n$ to π^*A.

Proof. Let γ be the base path of an A-path a, and let $\tilde{\gamma}$ be the parallel translation along a of some $\tilde{\gamma}(0) \in \pi^{-1}(\gamma(0))$ as in the proof of Lemma 2.9. Denoting by π^*a the lift of a to π^*A with base path $\tilde{\gamma}$, we have $\rho(\pi^*a) = h_Q(a(\gamma(t)), \tilde{\gamma}(t)) = d/dt(\tilde{\gamma})$, with ρ the anchor of π^*A (see equation (6)). That is, π^*a is an A-path in π^*A over $\tilde{\gamma}$. The lifting of a is not unique. In fact it is determined by the choice of a point in $\pi^{-1}(\gamma(0))$ as initial value.

Now we prove the same statement for A-homotopies. Suppose $a(\epsilon, t)$ is an A-homotopy over $\gamma(\epsilon, t)$, i.e. there exist A-paths (w.r.t. parameter ϵ) $b(\epsilon, t)$ also over γ satisfying

$$\partial_t b - \partial_\epsilon a = \nabla_{\rho(b)} a - \nabla_{\rho(a)} b + [a, b], \tag{19}$$

and the boundary condition $b(\epsilon, 0) = b(\epsilon, 1) = 0$, for any choice of connection ∇ on TP. As above, we can lift γ to $\tilde{\gamma}(\epsilon, t)$. In fact, once we choose $\tilde{\gamma}(0, 0)$, we can use $\tilde{\gamma}(0, 0)$ to obtain the lift $\tilde{\gamma}(\epsilon, 0)$ and then $\tilde{\gamma}(\epsilon, t)$. (The lift does not depend on whether we lift $\gamma(\epsilon, 0)$ or $\gamma(0, t)$ first, because the connection $\tilde{D}$ is flat). Then π^*a and π^*b are A-paths over $\tilde{\gamma}$ w.r.t. parameters t and ϵ respectively. Moreover, we choose a connection $\tilde{\nabla}$ on Q induced from the connection ∇ on P such that $\tilde{\nabla}_{X^H} Y^H = (\nabla_X Y)^H$, $\tilde{\nabla}_{X^H} E = 0$, $\tilde{\nabla}_E Y^H = 0$ and $\tilde{\nabla}_E E = 0$, where the superscript H denotes the horizontal lift with respect to some connection we fix on the circle bundle $\pi : Q \to P$. (Since $E(\pi^* f) = 0$ and $X^H(\pi^* f) = X(f)$ these requirements are consistent. In fact, the connection $\tilde{\nabla}$ on $TQ = \pi^*TP \oplus \mathbb{R}E$ is just the sum of the pullback connection on π^*TP and of the trivial connection). Now we will prove that π^*a and π^*b satisfy (19) w.r.t. $\tilde{\nabla}$. Notice that $\langle \pi^*\eta, \tilde{\nabla}_E X\rangle = 0$ for all vector fields X, so we have

$$\tilde{\nabla}_E \pi^*\eta = 0, \quad \tilde{\nabla}_{(\frac{\partial}{\partial\epsilon}\gamma)^H} \pi^*\eta = \pi^*(\nabla_{\frac{\partial}{\partial\epsilon}\gamma}\eta).$$

Therefore $\tilde{\nabla}_{\frac{\partial}{\partial\epsilon}\tilde{\gamma}} \pi^*\eta = \pi^*(\nabla_{\frac{\partial}{\partial\epsilon}\gamma}\eta)$. So $\partial_\epsilon \pi^*a = \pi^*(\partial_\epsilon a)$. The same is true for π^*b. Moreover, since $\rho(\pi^*a) = (\rho(a))^H + \langle\tilde{\beta}, a\rangle E$ (upon writing $\tilde{D}$ as in equation (5) and denoting by ^{H} the horizontal lift w.r.t. $\ker\sigma$), similarly we have $\tilde{\nabla}_{\rho(\pi^*a)}\pi^*b = \pi^*(\nabla_{\rho(a)} b)$ as well as the analog term obtained switching a and b. By the definition of Lie bracket on π^*A, we also have $[\pi^*a, \pi^*b] = \pi^*([a, b])$. Therefore a, b satisfying (19) implies that the same equation holds for π^*a and π^*b. The boundary condition $\pi^*b(\epsilon, 0) = \pi^*b(\epsilon, 1) = 0$ is obvious. Hence, π^*a is an A-homotopy in π^*A. □

Remark 4.8. We claim that all the A-paths and A-homotopies in π^*A are of the form π^*a. Indeed consider a π^*A path $\hat{a}$ over a base path $\hat{\gamma}$, i.e. $\rho(\hat{a}(t)) = \frac{d}{dt}\hat{\gamma}(t)$. Let $\gamma := \pi \circ \hat{\gamma}$ and let $a(t)$ be equal to $\hat{a}(t)$, seen as an element of $A_{\gamma(t)}$. The commutativity of

$$\begin{array}{ccc} \pi^*A & \xrightarrow{h_Q=\rho} & TQ \\ \downarrow & & \pi_* \downarrow \\ A & \xrightarrow{\rho_A} & TP \end{array}$$

implies that a is an A-path over γ. Further, the horizontal lift of a starting at $\hat{\gamma}(0)$ satisfies by definition $\frac{d}{dt}\tilde{\gamma}(t) = h_Q(a(\gamma(t)), \tilde{\gamma}(t))$, so it coincides with $\hat{\gamma}$. The same holds for A-homotopies.

The next theorem generalizes Thm. 4.2a).

Theorem 4.9. *Let (P, L) be an integrable prequantizable Dirac manifold and $(Q, \bar{L})$ one of its prequantization. We use the notation $[\cdot]_A$ to denote A-homotopy classes in the Lie algebroid A. Then we have the following results:*

(1) there is an S^1 action on the precontact groupoid $\Gamma_c(Q)$ with moment map $J_\Gamma = 1 - f_\Gamma$;

(2) $J_\Gamma^{-1}(0)$ is a source connected and simply connected subgroupoid of $\Gamma_c(Q)$ and is isomorphic to the action groupoid $\Gamma_c(P) \ltimes Q \rightrightarrows Q$.

(3) In terms of path spaces,

$$J_\Gamma^{-1}(0) = \{[\pi^* a]_{\bar{L}}\} = \{[\pi^* a]_{\bar{L}_0}\},$$

where a is an A-path in L^c and $\pi^* a$ is defined as in Lemma 4.7 (we identify $\pi^* L^c$ with $\bar{L}_0 \subset \bar{L}$ as in Thm. 2.11). Hence the Lie algebroid of $J_\Gamma^{-1}(0)$ is $\bar{L}_0 = J^{-1}(0)$ (see Prop. 3.4).

(4) the precontact reduction $\Gamma_c(Q)//_0 S^1$ is isomorphic to the s.s.c. contact groupoid $\Gamma_c(P)$ via the inverse of the following map

$$p : [a]_{L^c} \mapsto [\pi^* a]_{\bar{L}, S^1},$$

where $[\cdot]_{\bar{L}, S^1}$ denotes S^1 equivalence classes of $[\cdot]_{\bar{L}}$.

Remark 4.10. The isomorphism p gives the same contact groupoid structure on $\Gamma_c(Q)//_0 S^1$ as in Theorem 4.2 in the case when P is Poisson.

Proof. 1) The definition of the S^1 action is the same as in Theorem 4.2. J_Γ is defined by $J_\Gamma(g) = \theta_\Gamma(v_\Gamma(g))$, where v_Γ is induced by the S^1 action on Q hence on $\bar{L}$. More explicitly, $T(P_a(\bar{L}))$ is a subspace of the space of paths in $T\bar{L}$. If we take a connection ∇ on Q, then $T\bar{L}$ decomposes as $TQ \oplus \bar{L}$. At $(a_4, a_3, a_1, a_0) \in P_a(\bar{L})$ the infinitesimal S^1 action $\tilde{v}$ on the path space is $\tilde{v} = (E(\gamma(t)), *, *, *, 0)$. So

$$J_\Gamma([a]) = \tilde{\theta}_a(\tilde{v}) = \int_0^1 (\langle a_1(t), E\rangle e^{-\int_0^t \langle a_1, E\rangle dt}) dt = -\int_0^1 d(e^{-\int_0^t \langle a_1, E\rangle dt}) = 1 - f_\Gamma.$$

2) By 1) $J_\Gamma^{-1}(0) = f_\Gamma^{-1}(1)$. Since f_Γ is multiplicative, it is clear that $f_\Gamma^{-1}(1)$ is a subgroupoid. Moreover using Thm. 4.5 we see that $f_\Gamma^{-1}(1)$ is made up by paths $a = (a_4, a_3, a_1, a_0)$ such that

$$\int_0^1 \langle a_1(t), E\rangle dt = 0. \tag{20}$$

Notice that this are not exactly the same as A-paths in $\bar{L}_0$, which are the A-paths such that $\langle a_1(t), E\rangle \equiv 0$ for all $t \in [0, 1]$ (see Thm. 2.11).

Now we show that $J_\Gamma^{-1}(0)$ is source connected. Take $g \in \mathbf{s}^{-1}(x)$, and choose an A-path $a(t)$ representing g over a base path $\gamma(t) : I \to Q$. We will connect g to x within $J_\Gamma^{-1}(0) \cap \mathbf{s}^{-1}(x)$ in two steps: first we deform g to some other point h which can be represented by an A-path in $\bar{L}_0$; then we "linearly shrink" h to x.

Suppose the vector bundle $\bar{L}$ is trivial on a neighborhood U of the image of γ in Q. Choose a frame $Y_0, \dots, Y_{\dim Q}$ for $\bar{L}|_U$, with the property that $Y_0 = (-A^H, 1) \oplus (\sigma - \pi^*\alpha, 0)$ (with σ, A and α as in Thm. 2.4) and that all other Y_i satisfy $\langle a_1, E\rangle = 0$. In this frame, $a(t) = \sum_{i=0}^{\dim Q} p_i(t) Y_i|_{\gamma(t)}$ for some time-dependent coefficients $p_i(t)$. Define the following section of $\bar{L}|_U$: $Y_{t,\epsilon} = (1-\epsilon)p_0(t)Y_0 + \sum_{i=1}^{\dim Q} p_i(t)Y_i$. Define a deformation $\gamma(\epsilon, t)$ of $\gamma(t)$ by

$$\frac{d}{dt}\gamma(\epsilon, t) = \rho(Y_{t,\epsilon}), \quad \gamma(\epsilon, 0) = x,$$

where ρ is the anchor of $\bar{L}$ (one might have to extend U to make $\gamma(\epsilon, t) \in U$ for $t \in [0, 1]$). Let $a(\epsilon, t) := Y_{t,\epsilon}|_{\gamma(\epsilon,t)}$. For each ϵ it is an A-path by construction, and $a(0, t) = a(t)$. Using $g \in J_\Gamma^{-1}(0)$ (so that $\int_I p_0(t)dt = 0$) we have

$$\int_0^1 \langle a_1(\epsilon, t), E\rangle dt = \int_0^1 \langle (1-\epsilon)p_0(t)Y_0 + \sum_{i=1}^{\dim Q} p_i(t)Y_i, (E, 0, 0, 0)\rangle_- dt = (1-\epsilon)\int_I p_0(t)dt = 0,$$

so $[a(\epsilon, \cdot)]$ lies in $J_\Gamma^{-1}(0)$. Notice that $a(1, t)$ satisfies $\langle a_1(1, t), E\rangle \equiv 0$ for all t; hence an A-path in $\bar{L}_0$. We denote $h := [a(1, t)]$ and define a continuous map $pr : P_a(\bar{L}|_U) \to P_a(\bar{L}_0|_U)$ by $a(t) \mapsto a(1, t)$.

Then we can shrink linearly $a(1, t)$ to the zero path, via $a^\delta(1, t) := \delta a(1, \delta t)$ which is an A-path over $\gamma(1, \delta t)$. Taking equivalence classes we obtain a path from h to x, which moreover lies in $J_\Gamma^{-1}(0)$ because $\langle a_1(1, t), E\rangle \equiv 0$.

Now we show that $J_\Gamma^{-1}(0)$ is source simply connected. If there is a loop $g(s) = [a(1, s, t)]$ in a source fibre of $J_\Gamma^{-1}(0)$, then $g(s)$ can shrink to $x := \mathbf{s}(g(s))$ inside the big (s.s.c.!) groupoid $\Gamma_c(Q)$ via $g(\epsilon, s) = [a(\epsilon, s, t)]$. We can assume $a(\epsilon, s, t) = sa(\epsilon, 1, st)$. This is easy to realize since we can simply take $a(\epsilon, s, t) = g(\epsilon, st)^{-1}d/dt(g(\epsilon, st))$. Then the $a(i, 1, \cdot)$'s are A-paths in $\bar{L}_0$ for $i = 0, 1$. This is because both $g(s)$ and x are paths in $J_\Gamma^{-1}(0)$ which implies $\int_0^1 sa(i, 1, st) = 0$ for all $s \in [0, 1]$. Moreover the base paths $\gamma(\epsilon, s, t)$ form an embedded disk (one can assume that the deformation $g(\epsilon, s)$ has no self-intersections) in Q. So we can take a simply connected open set (for example a tubular neighborhood of this disk) $U \subset Q$ containing $\gamma(\epsilon, s, t)$. Then $L|_U$ is trivial. Therefore there is a continuous map pr such that $\bar{a}(\epsilon, 1, \cdot) = pr(a(\epsilon, 1, \cdot))$ is an A-path in $\bar{L}_0$ and $\bar{a}(1, 1, \cdot) = a(1, 1, \cdot)$. Then we can shrink $g(s) = \bar{g}(1, s)$ to $x = \bar{g}(0, s)$ via

$$\bar{g}(\epsilon, s) := [s\bar{a}(\epsilon, 1, st)],$$

which is inside of $J_\Gamma^{-1}(0)$ since $\langle \bar{a}_1(\epsilon, 1, t), E\rangle \equiv 0$.

3) To show that $J_\Gamma^{-1}(0) = \{[\pi^* a]_{\bar{L}}\}$, we just have to show that an A-path in $\bar{L}$ satisfying (20) is A-homotopic (equivalent) to an A-path lying contained in $\bar{L}_0$. Since $J_\Gamma^{-1}(0)$ has connected source fibres, given a point $g = [a]$ in $J_\Gamma^{-1}(0)$, there is a path $g(t)$ connecting g to $\mathbf{s}(g)$ lying in $J_\Gamma^{-1}(0)$. Differentiating $g(t)$ we get an A-path $b(t) = g(t)^{-1}\dot{g}(t)$ which is A-homotopic to a and $sb(st)$ represents the point $g(st) \in J^{-1}(0)$. Therefore $\int_0^1 \langle sb_1(st), E\rangle dt = 0$, for all $s \in [0, 1]$. Hence $\langle b_1(t), E\rangle \equiv 0$ for all $t \in [0, 1]$, i.e. b is a path in $\bar{L}_0$.

To further show that $J_\Gamma^{-1}(0) = \{[\pi^*a]_{\bar{L}_0}\}$, we only have to show that if two A-paths in $\bar{L}_0$ are A-homotopic in $\bar{L}$ then they are also A-homotopic in $\bar{L}_0$. Let $a(1,\cdot)$ and $a(0,\cdot)$ be two A-paths in $\bar{L}_0$, A-homotopic in $\bar{L}$ and representing an element $g \in J_\Gamma^{-1}(0)$. Integrate $sa(i,st)$ to get $g(i,t)$ for $i = 0,1$. Namely we have $sa(i,st) = g(i,s)^{-1}\frac{d}{dt}|_{t=s} g(i,t)$. Then $g(i,t)$ are two paths connecting g and $x := \mathbf{s}(g)$ lying in the subgroupoid $J_\Gamma^{-1}(0)$ since $a(i,t)$ are paths in $\bar{L}_0$. Since the source fibre of $J_\Gamma^{-1}(0)$ is simply connected, there is a homotopy $g(\epsilon,t) \in J_\Gamma^{-1}(0)$ linking $g(0,t)$ and $g(1,t)$. So $sa(\epsilon,st) := g(\epsilon,s)^{-1}\frac{d}{dt}|_{t=s} g(\epsilon,t)$ is an A-path in the variable t representing the element $g(\epsilon,s) \in J_\Gamma^{-1}(0)$ for every fixed s. Hence $sa(\epsilon,st)$ satisfies (20) for every $s \in [0,1]$. Therefore $\langle a_1(\epsilon,t), E\rangle \equiv 0$. Then $a(\epsilon,t) \subset \bar{L}_0$ is an A-homotopy between $a(0,t)$ and $a(1,t)$.

Therefore $J_\Gamma^{-1}(0)$ is the s.s.c. Lie groupoid integrating $J^{-1}(0) = \bar{L}_0$.

4) First of all, given an A-path a of L^c over the base path γ and a point $\tilde{\gamma}(0)$ over $\gamma(0)$ in Q, we lift it to an A-path π^*a of $\bar{L}$ as described in Lemma 4.7. By the same lemma, we see that (L^c) A-homotopic A-paths in L^c lift to ($\bar{L}_0$) A-homotopic A-paths in $\pi^*L^c \cong \bar{L}_0 \subset \bar{L}$, so the map p is well defined Different choices of $\tilde{\gamma}(0)$ give exactly the S^1 orbit of (some choice of) $[\pi^*a]_{\bar{L}}$. Surjectivity of the map p follows from the statement about A-paths in Remark 4.8. Injectivity follows from the fact that $\{[\pi^*a]_{\bar{L}}\} = \{[\pi^*a]_{\bar{L}_0}\}$ in 3) and the statement about A-homotopies in Remark 4.8. □

We saw in Subsection 3.2 that, given any integrable Dirac manifold (P,L), there are two groupoids attached to it. One is the presymplectic groupoid $\Gamma_s(P)$ integrating L; the other is the precontact groupoid $\Gamma_c(P)$ integrating L^c. In the non-integrable case, these two groupoids still exist as stacky groupoids carrying the same geometric structures (presymplectic and precontact) [19]. In this paper, to simplify the treatment, we view them as topological groupoids carrying the same name and when the topological groupoids are smooth manifolds they have additional presymplectic and precontact structures. Item (4) of the following theorem generalizes Thm. 4.2b). The other items generalize from the Poisson case to the Dirac case Theorem 2 and 3 in [8] and a result in [2].

Theorem 4.11. *For a Dirac manifold (P,L), there is a short exact sequence of topological groupoids*

$$1 \to \mathcal{G} \to \Gamma_c(P) \xrightarrow{\tau} \Gamma_s(P) \to 1,$$

where $\mathcal{G}$ is the quotient of the trivial groupoid $\mathbb{R}\times P$ by a group bundle $\mathcal{P}$ over P defined by

$$\mathcal{P}_x := \{\int_{[\gamma]} \omega_F : [\gamma] \in \pi_2(F,x) \text{ and } \gamma \text{ is the base of an } A\text{-homotopy between paths representing } 1_x \text{ in } L.\},$$

with F the presymplectic leaf passing through $x \in P$ and ω_F the presymplectic form on F. In the case that (P,L) is integrable as a Dirac manifold, then

(1) *the presymplectic form Ω on $\Gamma_s(P)$ is related to the precontact form θ on $\Gamma_c(P)$ by*

$$\tau^* d\theta = \Omega,$$

and the infinitesimal action R of $\mathbb{R}$ on $\Gamma_c(P)$ via $\mathbb{R}\times P \to \mathcal{G}$ satisfies

$$\mathcal{L}_R\theta = 0, \quad i(R)\theta = 1.$$

(2) *R is the left invariant vector field extending the section $(0,0)\oplus(0,-1)$ of $L^c \subset \mathcal{E}^1(P)$ as in Cor. A.2;*

(3) *the group $\mathcal{P}_x$ is generated by the periods of R;*

(4) $\Gamma_s(P)$ *is prequantizable iff* $\mathcal{P} \subset P \times \mathbb{Z}$*; in this case its prequantization is* $\Gamma_c(P)/\mathbb{Z}$*, where* $\mathbb{Z}$ *acts on* $\Gamma_c(P)$ *as a subgroup of* $\mathbb{R}$*.*

(5) If P *is prequantizable as a Dirac manifold, then* $\Gamma_s(P)$ *is prequantizable.*

Proof. The proof of (1) and (4) is the same as Section 4 of [8]. One only has to replace the Poisson bivector π by Υ and the leaf-wise symplectic form of π by ω_F. (3) is clear since R generates the $\mathbb{R}$ action and $\mathcal{G} = \mathbb{R}/\mathcal{P}$.

For (2), we identify $(0,0) \oplus (0,-1)$ with a section of $\ker \mathbf{t}_*$ using Lemma A.1 and then extend it to a left invariant vector field on $J^{-1}(0)/S^1$. Using Cor. A.2 we see that the resulting vector field is killed by $\mathbf{s}_*$, $\mathbf{t}_*$ and $d\theta_\Gamma$ and that it pairs to 1 with θ_Γ, so by the "non-degeneracy" condition in Def. 3.7 it must be equal to R.

For (5), if P is prequantizable as a Dirac manifold, then $\Upsilon = \rho^*\Omega + d_L\beta$ for some integral form Ω on P and $\beta \in \Gamma(L^*)$. Suppose $f = ad\epsilon + bdt$ is a Lie algebroid homomorphism from the tangent bundle $T\Box$ of a square $[0,1] \times [0,1]$ to L over the base map $\gamma : \Box \to P$, i.e. $a(\epsilon, t)$ is an A-homotopy over γ via $b(\epsilon, t)$ as in (19). Denoting by ω_F the presymplectic form of the leaf F in which $\gamma(\Box)$ lies, we have (see also Sect. 3.3 of [2]),

$$\int_\gamma \omega_F = \int_\Box \omega_F(\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial\epsilon}) = \int_\Box \langle ad\epsilon, bdt\rangle_- = \int_\Box f^*\Upsilon$$
$$= \int_\Box f^*(\rho^*\Omega + d_L\beta) = \int_\Box f^*(\rho^*\Omega) = \int_\Box \gamma^*\omega = \int_\gamma \omega \in \mathbb{Z}$$

where we used $\Upsilon = \rho^*\omega_F$ in the second equation and $f^*d_L\beta = d_{dR}(f^*\beta)$ in the fifth. □

4.3. **Two examples.** We present two explicit examples for Thm. 4.2, 4.9 and 4.11.

The first one generalizes Example 4.1.

Example 4.12. Let (P,ω) be an integral symplectic manifold (non necessarily simply connected), and (Q,θ) a prequantization. The s.s.c. contact groupoid of (Q,θ) is $(\bar{Q} \times_{\pi_1(Q)} \bar{Q} \times \mathbb{R}, -e^{-s}\theta_1 + \theta_2, e^{-s})$ where $\bar{Q}$ denotes the universal cover of Q. As in Example 4.1 the moment map is given by $J_\Gamma = -e^{-s} + 1$ and the reduced manifold at zero is $((\bar{Q} \times_{\pi_1(Q)} \bar{Q})/S^1, [-\theta_1 + \theta_2])$, where $\pi_1(Q)$ acts diagonally and the diagonal S^1 action is realized by following the Reeb vector field on $\bar{Q}$.

Notice that the Reeb vector field of $(\bar{Q} \times_{\pi_1(Q)} \bar{Q})/S^1$ is the Reeb vector field of the second copy of $\bar{Q}$. Dividing $\bar{Q}$ by $\mathbb{Z} \subset$ (Flow of Reeb v.f.) is the same as dividing by the $\pi_1(\tilde{Q})$ action on $\bar{Q}$, where $\tilde{Q}$ is the pullback of $Q \to P$ via the universal covering $\tilde{P} \to P$. To see this use that $\pi_1(\tilde{Q})$ is generated by any of its Reeb orbits (look at the long exact sequence corresponding to $S^1 \to \tilde{Q} \to \tilde{P}$), and that the Reeb vector field of $\bar{Q}$ is obtained lifting the one on $\tilde{Q}$. Also notice that $\pi_1(\tilde{Q})$ embeds into $\pi_1(Q)$ (as the subgroup generated by the Reeb orbits of Q) and that the quotient by the embedded image is isomorphic to $\pi_1(P)$, by the long exact sequence for $S^1 \to Q \to P$. So the quotient of $(\bar{Q} \times_{\pi_1(Q)} \bar{Q})/S^1$ by the $\pi_1(\tilde{Q})$ action on the second factor is $(\tilde{Q} \times_{\pi_1(P)} \tilde{Q})/S^1$ where we used $\bar{Q}/\pi_1(\tilde{Q}) = \tilde{Q}$ on each factor. This groupoid, together with the induced 1-form $[-\theta_1 + \theta_2]$, is clearly the prequantization of the s.s.c. symplectic groupoid $(\tilde{P} \times_{\pi_1(P)} \tilde{P}, -\omega_1 + \omega_2)$ of (P,ω).

In the second example we consider a Lie algebra $\mathfrak{g}$. Its dual $\mathfrak{g}^*$ is endowed with a linear Poisson structure Λ, called Lie-Poisson structure, and the Euler vector field A satisfies $\Lambda = -d_\Lambda A$ where d_Λ is the Poisson cohomology differential. So the prequantization condition (3) for $(\mathfrak{g}^*, \Lambda)$ is satisfied, with $\Omega = 0$ and $\beta = A$. We display the contact groupoid integrating the induced prequantization $(Q, \bar{L})$ for the simple case that $\mathfrak{g}$ be one dimensional; then we show that (a discrete quotient of) the S^1 contact reduction of this groupoid is the prequantization of the symplectic groupoid of $\mathfrak{g}^*$.

Example 4.13. Let $\mathfrak{g} = \mathbb{R}$ be the one-dimensional Lie algebra. We claim that the prequantization $Q = S^1 \times \mathfrak{g}^*$ of $\mathfrak{g}^*$ as above has as a s.s.c. contact groupoid $\Gamma_c(Q)$ the quotient of

$$(\mathbb{R}^5, x d\epsilon - e^t d\theta_1 + d\theta_2, e^t) \tag{21}$$

by the diagonal $\mathbb{Z}$ action on the variables (θ_1, θ_2). Here the coordinates on the five factors of $\mathbb{R}^5$ are $(\theta_1, t, \epsilon, \theta_2, x)$. The groupoid structure is the product of the following three groupoids: $\mathbb{R} \times \mathbb{R} = \{(\theta_1, \theta_2)\}$ the pair groupoid; $\mathbb{R} \times \mathbb{R} = \{(t, x)\}$ the action groupoid given by the flow of the vector field $-x\partial_x$ on $\mathbb{R}$, i.e. $(t', e^{-t}x) \cdot (t, x) = (t' + t, x)$; and $\mathbb{R} = \{\epsilon\}$ the group.

To see this, first determine the prequantization of $(\mathfrak{g}^*, \Lambda)$: it is $Q = S^1 \times \mathbb{R}$ with Jacobi structure $(E \wedge x\partial_x, E)$, where $E = \partial_\theta$ is the infinitesimal generator of the circle action and $x\partial_x$ is just the Euler vector field on $\mathfrak{g}^*$ (see [4]). This Jacobi manifold has two open leaves, and we first focus on one of them, say $Q_+ = S^1 \times \mathbb{R}_+$. This is a locally conformal symplectic leaf, with structure $(d\theta \wedge \frac{dx}{x}, \frac{dx}{x})$.

We determine the s.s.c contact groupoid $\Gamma_c(Q_+)$ of $(Q_+, d\theta \wedge \frac{dx}{x}, \frac{dx}{x})$ applying Lemma B.1 (choosing $\tilde{g} = \log x$, so that $e^{-\tilde{g}}\tilde{\Omega} = d(x^{-1} d\theta)$ there). We obtain the quotient of

$$(\tilde{Q}_+ \times \mathbb{R} \times \tilde{Q}_+, x_2 d\epsilon - \frac{x_2}{x_1} d\theta_1 + d\theta_2, \frac{x_2}{x_1})$$

by the diagonal $\mathbb{Z}$ action on the variables (θ_1, θ_2). Here (θ_i, x_i) are the coordinates on the two copies of the universal cover $\tilde{Q}_+ \cong \mathbb{R} \times \mathbb{R}_+$ and ϵ is the coordinate on the $\mathbb{R}$ factor. The groupoid structure is given by the product of the pair groupoid over $\tilde{Q}_+$ and group $\mathbb{R}$. This contact groupoid, and the one belonging to $Q_- = S^1 \times \mathbb{R}_-$, will sit as open contact subgroupoids in the contact groupoid of Q, and the question is how to "complete" the disjoint union of $\Gamma_c(Q_+)$ and $\Gamma_c(Q_-)$ to obtain the contact groupoid of Q. A clue comes from the simplest case of groupoid with two open orbits and a closed one to separate them, namely the transformation groupoid of a vector field on $\mathbb{R}$ with exactly one zero. The transformation groupoid associated to $-x\partial_x$ is $\mathbb{R} \times \mathbb{R} = \{(t, x)\}$ with source given by x, target given by $e^{-t}x$ and multiplication $(t', e^{-t}x) \cdot (t, x) = (t' + t, x)$. Notice that, on each of the two open orbits $\mathbb{R}_+$ and $\mathbb{R}_-$ the groupoid is isomorphic to a pair groupoid by the correspondence $(t, x) \in \mathbb{R} \times \mathbb{R}_\pm \mapsto (e^{-t}x, x) \in \mathbb{R}_\pm \times \mathbb{R}_\pm$, with inverse $(x_1, x_2) \mapsto (\log(\frac{x_2}{x_1}), x_2)$.

Now we embed $\Gamma_c(Q_+)$ into the groupoid $\Gamma_c(Q)$ described in (21) by the mapping

$$(\theta_1, x_1, \epsilon, \theta_2, x_2) \mapsto \left(\theta_1, t = \log(\frac{x_2}{x_1}), \epsilon, \theta_2, x = x_2\right),$$

and similarly for $\Gamma_c(Q_-)$. The contact forms and function translate to those indicated in (21), which as a consequence also satisfy the multiplicativity condition. One checks directly that the one form is a contact form also on the complement $\{x = 0\}$ of the two open subgroupoids. Therefore the one described in (21) is a contact groupoid, and since we know that the source map is a Jacobi map on the open dense set sitting over Q_+ and Q_-, it is the contact groupoid of $(Q, E \wedge x\partial_x, E)$.

Now we consider the S^1 contact reduction of the above s.s.c. groupoid $\Gamma_c(Q)$. As shown in the proof of Theorem 4.2 the moment map is $J_\Gamma = 1 - f_\Gamma = 1 - e^t$, so its zero level set is $\{t = 0\}$. The definition of moment map and the fact that the infinitesimal generator v_Γ of the S^1 action projects to E both via source and via target imply that on $\{t = 0\}$ we have $v_\Gamma = (\partial_{\theta_1}, 0, 0, \partial_{\theta_2}, 0)$. So $J^{-1}(0)/S^1$ is $\mathbb{R}^3$ with coordinates $(\theta := \theta_2 - \theta_1, \epsilon, x)$, 1-form $d\theta + xd\epsilon$, source and target both given by x and groupoid multiplication given by addition in the θ and ϵ factors. Upon division of the θ factor by $\mathbb{Z}$ (notice that the Reeb vector field of $\Gamma_c(Q)$ is ∂_{θ_2}) this is clearly just the prequantization of $T^*\mathbb{R}$, endowed with the canonical symplectic form $dx \wedge d\epsilon$ and fiber addition as groupoid multiplication, i.e. the prequantization of the symplectic groupoid of the Poisson manifold $(\mathbb{R}, 0)$.

Appendix A. Lie algebroids of precontact groupoids

Lemma A.1. *Let $(\Gamma, \theta_\Gamma, f_\Gamma)$ be a precontact groupoid (as in Definition 3.7) over the Jacobi-Dirac manifold $(Q, \bar{L})$, so that the source map be a Jacobi-Dirac map. Then a Lie algebroid isomorphism between $\ker \mathbf{s}_*|_Q$ and $\bar{L}$ is given by*

$$Y \mapsto (\mathbf{t}_* Y, -r_{\Gamma *} Y) \oplus (-d\theta_\Gamma(Y)|_{TQ}, \theta_\Gamma(Y)) \tag{22}$$

where $e^{-r_\Gamma} = f_\Gamma$. A Lie algebroid isomorphism between $\ker \mathbf{t}_*|_Q$ and $\bar{L}$ (obtained composing the above with i_* for i the inversion) is

$$Y \mapsto (\mathbf{s}_* Y, r_{\Gamma *} Y) \oplus (d\theta_\Gamma(Y)|_{TQ}, -\theta_\Gamma(Y)) \tag{23}$$

Proof. Consider the groupoid $\Gamma \times \mathbb{R}$ over $Q \times \mathbb{R}$ with target map $\tilde{\mathbf{t}}(g,t) = (\mathbf{t}(g), t - r_\Gamma(g))$ and the obvious source $\tilde{\mathbf{s}}$ and multiplication. $(\Gamma \times \mathbb{R}, d(e^t \theta_\Gamma))$ is then a presymplectic groupoid with the property that $\tilde{\mathbf{s}}$ is a forward Dirac map onto $(Q \times \mathbb{R}, \tilde{L})$, where

$$\tilde{L}_{(q,t)} = \{(X, f) \oplus e^t(\xi, g) : (X, f) \oplus (\xi, g) \in L_q\}$$

is the "Diracization" ([25][17]) of the Jacobi-Dirac structure $\bar{L}$ and t is the coordinate on $\mathbb{R}$. In the special case that $\bar{L}$ corresponds to a Jacobi structure this is just Prop. 2.7 of [8]; in the general case (but assuming different conventions for the multiplicativity of θ_Γ and for which of source and target is a Jacobi-Dirac map) this is Prop. 3.3 in [17]. We will prove only the first isomorphism above (the one for $\ker \mathbf{s}_*|_Q$); the other one follows by composing the first isomorphism with i_*. Now we consider the following diagram of spaces of sections (on the left column we have sections over Q, on the right column sections over $Q \times \mathbb{R}$):

$$\begin{array}{ccc}
\Gamma(\ker \mathbf{s}_*|_Q) & \xrightarrow{\Phi_{\mathbf{s}}} & \Gamma(\ker \tilde{\mathbf{s}}_*|_{Q\times\mathbb{R}}) \\
\downarrow & & \Phi \downarrow \\
\Gamma(\bar{L}) & \xrightarrow{\Phi_L} & \tilde{L}.
\end{array}$$

The first horizontal arrow $\Phi_{\mathbf{s}}$ is $Y \mapsto \tilde{Y}$, where the latter denotes the constant extension of Y along the $\mathbb{R}$ direction of the base $Q \times \mathbb{R}$. Notice that the projection $pr : \Gamma \times \mathbb{R} \to \Gamma$ is a groupoid morphism, so it induces a surjective Lie algebroid morphism $pr_* : \ker \tilde{\mathbf{s}}_*|_{Q\times\mathbb{R}} \to \ker \mathbf{s}_*|_Q$. Since sections $\tilde{Y}$ as above are projectable, by Prop. 4.3.8. in [15] we have $pr_*[\tilde{Y}_1, \tilde{Y}_2] = [Y_1, Y_2]$, and since pr_* is a fiberwise isomorphism we deduce that $\Phi_{\mathbf{s}}$ is a bracket-preserving map.

The vertical arrow Φ is induced from the following isomorphism of Lie algebroids (Cor. 4.8 iii of [3][8]) valid for any presymplectic manifold $(\tilde{\Gamma}, \Omega)$ over a Dirac manifold $(N, \tilde{L})$ for which the source map is Dirac:

$$\ker \tilde{\mathbf{s}}_*|_N \to \tilde{L}\ ,\ Z \mapsto (\tilde{\mathbf{t}}_* Z, -\Omega(Z)|_{TN}).$$

In our case, as mentioned above, the presymplectic form is $d(e^t \theta_\Gamma)$.

[8]In [3] the authors adopted the convention that the target map be a Dirac map. Here we use their result applied to the pre-symplectic form $-\Omega$.

The second horizontal arrow Φ_L is the natural map

$$(X,f)\oplus(\xi,g)\in L_q\mapsto(X,f)\oplus e^t(\xi,g)\in\bar{L}_{(q,t)}$$

which preserves the Lie algebroid bracket (see the remarks after Definition 3.2 of [25]).

One can check that $(\Phi\circ\Phi_{\mathbf{s}})(Y)=(\tilde{\mathbf{t}}_*\tilde{Y})\oplus(-d(e^t\theta_\Gamma)(\tilde{Y})|_{TQ\times\mathbb{R}})$ lies in the image of the injective map Φ_L. The resulting map from $\Gamma(\ker\mathbf{s}_*)$ to $\Gamma(\bar{L})$ is given by (22) and the arguments above show that this map preserves brackets. Further it is clear that this map of sections is induced by a vector bundle morphism given by the same formula, which clearly preserves not only the bracket of sections but also the anchor, so that the map $\ker\mathbf{s}_*|_Q\to\bar{L}$ given by (22) is a Lie algebroid morphism.

To show that it is an isomorphism one can argue noticing that $\ker\mathbf{s}_*$ and $\bar{L}$ have the same dimension and show that the vector bundle map is injective, by using the "non-degeneracy condition" in Def. 3.7 and the fact that the source and target fibers of $\Gamma\times\mathbb{R}$ are pre-symplectic orthogonal to each other. □

The vector bundle morphisms in the above lemma give a characterization of vectors tangent to the $\mathbf{s}$ or $\mathbf{t}$ fibers of a precontact groupoid as follows. Consider for instance a vector λ in $\bar{L}_x$, where $\bar{L}$ is the Jacobi-Dirac structure on the base Q. This vector corresponds to some $Y_x\in\ker\mathbf{t}_*$ by the isomorphism (23), and by left translation we obtain a vector field Y tangent to $\mathbf{t}^{-1}(x)$. Of course, every vector tangent to $\mathbf{t}^{-1}(x)$ arises in this way for a unique λ. The vector field Y satisfies the following equations at every point g of $\mathbf{t}^{-1}(x)$, which follow by simple computation from the multiplicativity of θ_Γ: $\theta_\Gamma(Y_g)=\theta_\Gamma(Y_x)$, $d\theta_\Gamma(Y_g,Z)=d\theta_\Gamma(Y_x,\mathbf{s}_*Z)-r_{\Gamma*}Y_x\cdot\theta_\Gamma(Z)$ for all $Z\in T_g\Gamma$, $r_{\Gamma*}Y_g=r_{\Gamma*}Y_x$ and $\mathbf{s}_*Y_g=\mathbf{s}_*Y_x$. Notice that the right hand sides of this properties can be expressed in terms of the four components of $\lambda\in\mathcal{E}^1(Q)$, and that by the "non-degeneracy" of θ_Γ these properties are enough to uniquely determine Y_g. We sum up this discussion into the following corollary, which can be used as a tool in computations on precontact groupoids in the same way that hamiltonian vector fields are used on contact or symplectic groupoids (such as the proof of Thm. 4.2):

Corollary A.2. *Let $(\Gamma,\theta_\Gamma,f_\Gamma)$ be a precontact groupoid (as in Definition 3.7) and denote by $\bar{L}$ the Jacobi-Dirac structure on the base Q so that source map is Jacobi-Dirac. Then there is bijection between sections of $\bar{L}$ and vector fields on Γ which are tangent to the $\mathbf{t}$-fibers and are left invariant. To a section $(X,f)\oplus(\xi,g)$ of $\bar{L}\subset\mathcal{E}^1(Q)$ corresponds the unique vector field Y tangent to the $\mathbf{t}$-fibers which satisfies*

- $\theta_\Gamma(Y)=-g$
- $d\theta_\Gamma(Y)=\mathbf{s}^*\xi-f\theta_\Gamma$
- $\mathbf{s}_*Y=X.$

Y furthermore satisfies $r_{\Gamma*}Y=f$.

Appendix B. Groupoids of locally conformal symplectic manifolds

A locally conformal symplectic (l.c.s.) manifold is a manifold (Q,Ω,ω) where Ω is a non-degenerate 2-form and ω is a closed 1-form satisfying $d\Omega=\omega\wedge\Omega$. Any Jacobi manifold is foliated by contact and l.c.s. leaves (see for example [27]); in particular a l.c.s. manifold is a Jacobi manifold, and hence, when it is integrable, it has an associated s.s.c. contact groupoid. In this appendix we will construct explicitly this groupoid; we make use of it in Example 4.13.

Lemma B.1. *Let (Q,Ω,ω) a locally conformal symplectic manifold. Consider the pullback structure on the universal cover $(\tilde{Q},\tilde{\Omega},\tilde{\omega})$, and write $\tilde{\omega}=d\tilde{g}$. Then Q is integrable as a Jacobi manifold iff the symplectic form $e^{-\tilde{g}}\tilde{\Omega}$ is a multiple of an integer form. In that case, choosing $\tilde{g}$ so that $e^{-\tilde{g}}\tilde{\Omega}$ is integer, the s.s.c. contact groupoid of (Q,Ω,ω) is the quotient of*

$$\left(\tilde{R}\times_{\mathbb{R}}\tilde{R}, e^{\tilde{\mathbf{s}}^*\tilde{g}}(-\tilde{\sigma}_1+\tilde{\sigma}_2), \frac{e^{\tilde{\mathbf{s}}^*\tilde{g}}}{e^{\tilde{\mathbf{t}}^*\tilde{g}}}\right), \tag{24}$$

a groupoid over $\tilde{Q}$, by a natural $\pi_1(Q)$ action. Here $(\tilde{R},\tilde{\sigma})$ is the universal cover (with the pullback 1-form) of a prequantization (R,σ) of $(\tilde{Q},e^{-\tilde{g}}\tilde{\Omega})$, and the group $\mathbb{R}$ acts by the diagonal lift of the S^1 action on R.

Proof. Using for example the Lie algebroid integrability criteria of [7], one sees that (Q,Ω,ω) is integrable as a Jacobi manifold iff $(\tilde{Q},\tilde{\Omega},\tilde{\omega})$ is. Lemma 1.5 in Appendix I of [27] states that, given a contact groupoid, multiplying the contact form by $\mathbf{s}^*u$ and the multiplicative function by $\frac{\mathbf{s}^*u}{\mathbf{t}^*u}$ gives another contact groupoid, for any non-vanishing function u on the base. Such an operation corresponds to twisting the groupoid, viewed just as a Jacobi manifold, by the function $\mathbf{s}^*u^{-1}$, hence the Jacobi structure induced on the base by the requirement that the source be a Jacobi map is the twist of the original one by u^{-1}. So $(\tilde{Q},\tilde{\Omega},\tilde{\omega})$ is integrable iff the symplectic manifold $(\tilde{Q},e^{-\tilde{g}}\tilde{\Omega})$ is Jacobi integrable, and by Section 7 of [8] this happens exactly when the class of $e^{-\tilde{g}}\tilde{\Omega}$ is a multiple of an integer one.

Choose $\tilde{g}$ so that this class is actually integer. A contact groupoid of $(\tilde{Q},e^{-\tilde{g}}\tilde{\Omega})$ is clearly $(R\times_{S^1}R,[-\sigma_1+\sigma_2],1)$, where the S^1 action on $R\times R$ is diagonal and "[ ]" denotes the form descending from $R\times R$. This groupoid is not s.s.c.; the s.s.c. one is $\tilde{R}\times_{\mathbb{R}}\tilde{R}$, where the $\mathbb{R}$ action on $\tilde{R}$ is the lift of the S^1 action on R. The source simply connectedness follows since $\mathbb{R}$ acts transitively (even though not necessarily freely) on each fiber of the map $\tilde{R}\to\tilde{Q}$, and this in turns holds because any S^1 orbit in R generates $\pi_1(R)$ and because the fundamental group of a space always acts (by lifting loops) transitively on the fibers of its universal cover.

By the above cited Lemma from [27] we conclude that (24) is the s.s.c. contact groupoid of $(\tilde{Q},\tilde{\Omega},\tilde{\omega})$. The fundamental group of Q acts on $\tilde{Q}$ respecting its geometric structure, so it acts on its Lie algebroid $T^*\tilde{Q}\times\mathbb{R}$. Since the path-space construction of the s.s.c. groupoid is canonical (see Subsection 4.2), $\pi_1(Q)$ acts on the s.s.c. groupoid (24) preserving the groupoid and geometric structure. Hence the quotient is a s.s.c. contact groupoid over (Q,Ω,ω), and its source map is a Jacobi map, so it is the s.s.c. contact groupoid of (Q,Ω,ω). □

Appendix C. On a construction of Vorobjev

In Section 2 we derived the geometric structure on the circle bundles Q from a prequantizable Dirac manifold (P,L) and a suitable choice of connection D. In this appendix we describe an alternative attempt; even though we can make our construction work only if we start with a symplectic manifold, we believe the construction is interesting on its own right.

First we recall Vorobjev's construction in Section 4 of [21], which the author there uses to study the linearization problem of Poisson manifolds near a symplectic leaf. Consider a transitive algebroid A over a base P with anchor ρ; the kernel $\ker\rho$ is a bundle of Lie algebras. Choose a splitting $\gamma: TP\to A$ of the anchor. Its curvature R_γ is a 2-form on P with values in $\Gamma(\ker\rho)$ (given by $R_\gamma(v,w)=[\gamma v,\gamma w]_A-\gamma[v,w]$). The splitting γ also induces a (TP-)covariant derivative ∇ on $\ker\rho$ by $\nabla_v s=[\gamma v,s]_A$. Now, if P is endowed with a symplectic form ω, a neighborhood of the zero section in $(\ker\rho)^*$ inherits a Poisson structure $\Lambda_{vert}+\Lambda_{hor}$ as follows (Theorem 4.1 in [21]): denoting by F_s the fiberwise linear function on $(\ker\rho)^*$ obtained by contraction with the section s of $\ker\rho$, the Poisson bivector has a vertical component determined by $\Lambda_{vert}(dF_{s_1}, dF_{s_2}) = F_{[s_1,s_2]}$. It also has a component Λ_{hor} which is tangent to the Ehresmann connection Hor given by the dual connection[9] to ∇ on the bundle $(\ker\rho)^*$; Λ_{hor} at $e \in (\ker\rho)^*$ is obtained by restricting the non-degenerate form $\omega - \langle R_\gamma, e\rangle$ to Hor_e and inverting it. (Here we are identifying Hor_e and the corresponding tangent space to P.)

To apply Vorobjev's construction in our setting, let (P,ω) be a prequantizatible symplectic manifold and (K,∇_K) its prequantization line bundle with Hermitian connection of curvature $2\pi i\omega$. By Lemma 2.8 we obtain a flat $TP\oplus_\omega\mathbb{R}$-connection $\tilde{D}_{(X,f)} = \nabla_X + 2\pi i f$ on K. Now we make use of the following well know fact about extensions, which can be proven by direct computation:

Lemma C.1. *Let A be a Lie algebroid over M, V a vector bundle over M, and $\tilde{D}$ a flat A-connection on V. Then $A\oplus V$ becomes a Lie algebroid with the anchor of A as anchor and bracket*

$$[(Y_1,S_1),(Y_2,S_2)] = ([Y_1,Y_2]_A, \tilde{D}_{Y_1}S_2 - \tilde{D}_{Y_2}S_1).$$

Therefore $A := TP\oplus_\omega\mathbb{R}\oplus K$ is a transitive Lie algebroid over P, with isotropy bundle $\ker\rho = \mathbb{R}\oplus K$ and bracket $[(f_1,S_1),(f_2,S_2)] = [(0, 2\pi i(f_1S_2 - f_2S_1)]$ there. Now choosing the canonical splitting γ of the anchor $TM\oplus_\omega\mathbb{R}\oplus K\to TM$ we see that its curvature is $R_\gamma(X_1,X_2) = (0,\omega(X_1,X_2),0)$. The horizontal distribution on the dual of the isotropy bundle is the product of the trivial one on $\mathbb{R}$ and of the one corresponding to ∇_K on K (upon identification of K and K^* by the metric). By the above, there is a Poisson structure on $\mathbb{R}\oplus K$, at least near the zero section: the Poisson bivector at (t,q) has a horizontal component given by lifting the inverse of $(1-t)\omega$ and a vertical component which turns out to be $2\pi(iq\partial_q)\wedge\partial t$, where "$iq\partial_q$" denotes the vector field tangent to the circle bundles in K obtained by turning by 90° the Euler vector field $q\partial_q$. A symplectic leaf is clearly given by $\{t<1\}\times Q$ (where $Q = \{|q| = 1\}$). On this leaf the symplectic structure is seen to be given by $(1-t)\omega + \theta\wedge dt = d((1-t)\theta)$, where θ is the connection 1-form on Q corresponding to the connection ∇_K on K (which by definition satisfies $d\theta = \pi^*\omega$). This means that the leaf is just the symplectification $(\mathbb{R}_+\times Q, d(r\theta))$ of (Q,θ) (here $r = 1-t$), which is a "prequantization space" for our symplectic manifold (P,ω). Unfortunately we are not able to modify Vorobjev's construction appropriately when P is a Poisson or Dirac manifold.

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