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ON THE GEOMETRY OF PREQUANTIZATION SPACES

MARCO ZAMBON AND CHENCHANG ZHU

ABSTRACT. Given a Poisson (or more generally Dirac) manifold P , there are two approaches to its geometric quantization: one involves a circle bundle Q over P endowed with a Jacobi (or Jacobi-Dirac) structure; the other one involves a circle bundle with a (pre-) contact groupoid structure over the (pre-) symplectic groupoid of P . We study the relation between these two prequantization spaces. We show that the circle bundle over the (pre-) symplectic groupoid of P is obtained from the groupoid of Q via an S^1 reduction that preserves both the groupoid and the geometric structure.

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1. INTRODUCTION

The geometric quantization of symplectic manifolds is a classical problem that has been much studied over years. The first step is to find a prequantization. A symplectic manifold (P, ω) is prequantizable iff $[\omega]$ is an integer cohomology class. Finding a prequantization means finding a faithful representation of the Lie algebra of functions on (P, ω) (endowed with the Poisson bracket) mapping the function 1 to a multiple of the identity. Such a representation space consists usually of sections of a line bundle over P [14], or equivalently of S^1 -antiequivariant complex functions on the total space Q of the corresponding circle bundle [17]. In the latter case, endowing Q with a suitable contact structure, functions on P act by the hamiltonian vector fields of their pullbacks to Q .

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For more general kinds of geometric structure on P , such as Poisson or even more generally Dirac [6] structures, there are two approaches to extend the geometric quantization of symplectic manifolds, at least as far as prequantization is concerned: to build a circle bundle on P compatible with the Poisson (resp. Dirac) structure over P (see Souriau [17] for the symplectic case, [13][19][5] for the Poisson case, and [24] for the Dirac case); to build a symplectic (resp. presymplectic) groupoid first and construct a circle bundle on the groupoid [23], with the hope to quantize Poisson manifolds “all at once” as proposed by Weinstein [22].

The aim of the paper is to study the relation between these two prequantization spaces given a Dirac manifold (P, L) . First of all, we search for a more transparent description of the geometric structures (the Jacobi-Dirac structures \bar{L} of [24]) on the circle bundles Q over (P, L) . This will be done in Section 2, both in terms of subbundles and in terms of brackets of functions.

Secondly, in Section 3, we relate the algebroid associated to Q to the algebroid of the prequantization space in the sense of Weinstein. We do this using precontact reduction, paralleling one of the motivating examples of symplectic reduction: $T^*M/_0G = T^*(M/G)$. This gives us evidence at the infinitesimal level for the relation between the groupoid of Q and the prequantization of the symplectic groupoid of P , which we describe in Section 4 as an S^1 reduction. We provide a direct proof in the Poisson case. In the general Dirac case, the proof is done by integrating the results of Section 3 to the level of groupoids with the help of algebroid path spaces. As a byproduct, we obtain the prequantization condition for the presymplectic groupoid of a Dirac manifold P in terms of period groups on P . Then we show that this condition is implied by the prequantization condition (à la Vaisman) for Dirac manifolds. This generalizes some of the results in [9] and [2]. We also argue why this seems the only way to describe the relation and display two examples.

This paper ends with three appendices. Appendix 1 provides a useful tool to perform computations on precontact groupoids, and Appendix 2 describes explicitly the groupoid of a locally conformal symplectic manifold. Appendix 3 is an attempt to apply a construction of Vorobjev to the setting of Section 2.

Notation Throughout the paper, unless otherwise specified, (P, L) will always denote a Dirac manifold, $\pi : Q \rightarrow P$ will be a circle bundle and \bar{L} will be a Jacobi-Dirac structure on Q . By Γ_s and Γ_c we will denote presymplectic and precontact groupoids respectively, and we adopt the convention that the source map induces the (Dirac and Jacobi-Dirac respectively) structures on the bases of the groupoids. By “precontact structure” on a manifold we will just mean a 1-form on the manifold.

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2. CONSTRUCTING THE PREQUANTIZATION OF P

The aim of this section is to describe in an intrinsic way the geometric structures (Jacobi-Dirac structures \bar{L}) on the circle bundles Q induced by prequantizable Dirac manifolds

(P, L) . In Subsection 2.1 we will attempt to do so in terms of subbundles, but we will succeed only in reconstructing a codimension one subalgebroid of \bar{L} . In Subsection 2.2 we will determine \bar{L} by specifying the bracket on functions that it induces.

A description of \bar{L} in non-intrinsic terms is given by the prequantization construction of [24] (to which we refer for the main definitions), which we now recall. Let (P, L) be a Dirac structure. This means that L is a maximal isotropic subbundle of $TP \oplus T^*P$ whose sections are closed under the Courant bracket. L is a Lie algebroid with the restricted Courant bracket and anchor $\rho_{TP} : L \rightarrow TP$ (which is just the projection onto the tangent component). This anchor gives a Lie algebra homomorphism from $\Gamma(L)$ to $\Gamma(TP)$ endowed with the Lie bracket of vector fields. The pullback by the anchor therefore induces a map $\rho_{TP}^* : \Omega_{dR}^\bullet(P, \mathbb{R}) \rightarrow \Omega_L^\bullet(P)$, the sections of the exterior algebra of L^* , which descends to a map from de Rham cohomology to the Lie algebroid cohomology $H_L^\bullet(P)$ of L . There is a distinguished class in $H_L^2(P)$: on $TP \oplus T^*P$ there is an anti-symmetric pairing given by

$$(1) \quad \langle X_1 \oplus \xi_1, X_2 \oplus \xi_2 \rangle_- = \frac{1}{2}(i_{X_2}\xi_1 - i_{X_1}\xi_2).$$

Its restriction Υ to L satisfies $d_L\Upsilon = 0$. The prequantization condition (which for Poisson manifolds was first formulated by Vaisman) is

$$(2) \quad [\Upsilon] = \rho_{TP}^*[\Omega]$$

for some integer deRham 2-class $[\Omega]$. When this is satisfied there exists a Hermitian L -connection with curvature $2\pi i\Upsilon$ on the line bundle K associated to $[\Omega]$. Any such connection determines on the corresponding circle bundle Q a Jacobi-Dirac structure¹ \bar{L} , whose hamiltonian vector fields provide a prequantization representation for the Lie algebra $C_{adm}^\infty(P)$ of admissible functions on P .

More explicitly the construction of \bar{L} goes as follows. (2) can be equivalently phrased as

$$(3) \quad \rho_{TP}^*\Omega = \Upsilon + d_L\beta,$$

where Ω is a closed integral 2-form and β a 1-cochain for the Lie algebroid L , i.e. a section of L^* . Let $\pi : Q \rightarrow P$ be an S^1 -bundle with connection form σ having curvature Ω ; denote by E the infinitesimal generator of the S^1 -action. In Theorem 4.1 of [24] Q was endowed with the following geometric structure:

Theorem 2.1. *The subbundle \bar{L} of $\mathcal{E}^1(Q)$ given by the direct sum of*

$$\{(X^H + \langle X \oplus \xi, \beta \rangle E, 0) \oplus (\pi^*\xi, 0) : X \oplus \xi \in L\}$$

and the line bundles generated by $(-E, 0) \oplus (0, 1)$ and $(-A^H, 1) \oplus (\sigma - \pi^\alpha, 0)$ is a Jacobi-Dirac structure on Q . Here, $A \oplus \alpha$ is an isotropic section of $TP \oplus T^*P$ satisfying $\beta = 2\langle A \oplus \alpha, \cdot \rangle_+|_L$. Such a section always exists, and the subbundle above is independent of the choice of $A \oplus \alpha$.*

It turns out that the geometric structure (\bar{L}, Q) depends on less data than (Q, σ, β) . Triples (Q, σ, β) as above define a hermitian L connection with curvature $2\pi i\Upsilon$ on the line bundle corresponding to Q (Lemma 6.2 in [24]), via the formula

$$(4) \quad D_\bullet = \nabla_{\rho_{TP}\bullet} - 2\pi i\langle \bullet, \beta \rangle$$

¹A Jacobi-Dirac structure on Q is a maximally isotropic subbundle of $\mathcal{E}^1(Q) := (TQ \times \mathbb{R}) \oplus (T^*Q \times \mathbb{R})$ whose sections are closed under the extended Courant bracket. Examples of Jacobi-Dirac manifolds include precontact structures (i.e. 1-forms), Jacobi manifolds and Dirac manifolds.

where ∇ is the covariant connection corresponding to σ . Further all hermitian L connections with curvature $2\pi i\Upsilon$ arise from triples (Q, σ, β) as above (Proposition 6.1 in [24]). A short computation shows that the triples that define the same L connection as (Q, σ, β) are exactly those of the form $(Q, \sigma + \pi^*\gamma, \beta + \rho_{TP}^*\gamma)$ for some 1-form γ on P , and that these triples all define the same Jacobi Dirac structure \bar{L} (Lemma 4.1 in [24]; see also the last comment in Sect. 6.1 there).

Hence, given a prequantizable Dirac manifold (P, L) , the Jacobi-Dirac structure \bar{L} constructed in Thm. 2.1 on Q depends only on a choice of hermitian L -connection with curvature $2\pi i\Upsilon$. We call (Q, \bar{L}) a “prequantization space” for (P, L) because the assignment $g \mapsto \{\pi^*g, \bullet\} = -X_{\pi^*g}$ is a representation of $C_{adm}^\infty(P)$ on the space of functions on Q . We refer to Remark 2.10 for a comment about how many such geometric structures there are.

2.1. The codimension one subalgebroid \bar{L}_0 of \bar{L} . In this subsection we fix an L connection D on K with curvature $2\pi i\Upsilon$ and attempt to construct the algebroid \bar{L} from L and D directly. (In Prop. 3.3 we will perform the inverse construction, i.e. we will recover L from \bar{L}). We will only succeed in reconstructing a codimension one subalgebroid \bar{L}_0 , which however turns out to be an important object in Subsection 3.3. Somehow unexpectedly, it turns out that the isomorphism class of the Lie algebroid \bar{L}_0 is independent of the choice of connection D .

We begin with a useful lemma concerning flat algebroid connections (compare also to Lemma 6.1 in [24]).

Lemma 2.2. *Let E be any algebroid over a manifold M , K a line bundle over M , and D a Hermitian E -connection on K . Consider the central extension $E \oplus_\eta \mathbb{R}$, where $2\pi i\eta$ equals the curvature of D ; then $\tilde{D}_{(Y,g)} = D_Y + 2\pi ig$ defines an $E \oplus_\eta \mathbb{R}$ -connection on K which is moreover flat.*

Proof. One checks easily that \tilde{D} is indeed an algebroid connection. With the definition of central extension of Lie algebroids and curvature [24],

$$R_{\tilde{D}}(e_1, e_2)s = \tilde{D}_{e_1}\tilde{D}_{e_2}s - \tilde{D}_{e_2}\tilde{D}_{e_1}s - \tilde{D}_{[e_1, e_2]}s$$

for elements e_i of $E \oplus_\eta \mathbb{R}$ and s of K , the result follows by a straightforward calculation. \square

We will use of this construction, which is just a way to make explicit the structure of a transformation algebroid (see Remark 2.4 below).

Lemma 2.3. *Let A be any algebroid over a manifold P , $\pi_Q : Q \rightarrow P$ a principle $SO(n)$ -bundle, $\pi_K : K \rightarrow P$ the vector bundle associated to the standard representation of $SO(n)$ on \mathbb{R}^n , and \tilde{D} a flat A -connection on K preserving its fiber-wise metric. The A -connection induces a bundle map $h_Q : \pi_Q^*A \rightarrow TQ$ that can be used to extend, by the Leibniz rule, the obvious bracket on $SO(n)$ -invariant sections of π_Q^*A to all sections of π_Q^*A . The vector bundle π_Q^*A , with this bracket and h_Q as an anchor, is a Lie algebroid.*

Proof. We first recall some facts from Section 2.5 in [12]. The A -connection \tilde{D} on the vector bundle K defines a map (the “horizontal lift”) $h_K : \pi_K^*A \rightarrow TK$ covering the anchor $A \rightarrow TP$ by taking parallel translations of elements of K along A -paths. Explicitly, fix an A -path $a(t)$ with base path $\gamma(t)$, a point $x \in \pi_K^{-1}(\gamma(0))$ and let $\tilde{\gamma}(t)$ the unique path in K (over $\gamma(t)$) starting at x with $\tilde{D}_{a(t)}\tilde{\gamma}(t) = 0$. We can always write $\tilde{D} = \nabla_{\rho_\bullet} - \tilde{\beta}$ where ∇ is a metric TP -connection on A and $\tilde{\beta} \in \Gamma(A^*) \otimes \mathfrak{so}(K)$; then $\nabla_{\rho a(t)}\tilde{\gamma}(t) = \langle \tilde{\beta}, a(t) \rangle \tilde{\gamma}(t)$. Since

the left hand side is the projection of the velocity of $\tilde{\gamma}(t)$ along the Ehresmann distribution H corresponding to ∇ , we obtain $\frac{d}{dt}\tilde{\gamma}(t) = (\frac{d}{dt}\gamma(t))^H + \langle \tilde{\beta}, a(t) \rangle \tilde{\gamma}(t)$, so that

$$(5) \quad h_K(a(0), x) := \frac{d}{dt}|_{t=0}\tilde{\gamma}(t) = \rho(a(0))^H + \langle \tilde{\beta}, a(0) \rangle x.$$

Of course h_K does not depend on ∇ or $\tilde{\beta}$ directly, but just on \tilde{D} . By our assumptions h_K is induced by a ‘‘horizontal lift’’ for the principle bundle Q , i.e. by a $SO(n)$ -equivariant map $h_Q : \pi_Q^*A \rightarrow TQ$ covering the anchor of A . Since our A -connection \tilde{D} is flat, the map that associates to a section s of A the vector field $h_Q(\pi_Q^*s)$ on Q is a Lie algebra homomorphism.

On sections $\pi_Q^*s_1, \pi_Q^*s_2$ of π_Q^*A which are pullbacks of sections of A we define the bracket to be $\pi_Q^*[s_1, s_2]$, and we extend it to all sections of π_Q^*A by using h_Q as an anchor and forcing the Leibniz rule. We have to show that the resulting bracket satisfies the Jacobi identity. Given sections s_i of A and a function f on Q one can show that the Jacobiator $[[\pi_Q^*s_1, f \cdot \pi_Q^*s_2], \pi_Q^*s_3] + c.p. = 0$ by using the facts that the bracket on sections of A satisfies the Jacobi identity and that the correspondence $\pi_Q^*s_i \mapsto h_Q(\pi_Q^*s_i)$ is a Lie algebra homomorphism. Similarly, the Jacobiator of arbitrary sections of Q is also zero due to fact that h_Q actually induces a homomorphism on *all* sections of π_Q^*A . \square

Remark 2.4. Using h_K instead of h_Q , the construction of the previous lemma leads to an algebroid structure on $\pi_K^*A \rightarrow K$. As Kirill Mackenzie pointed out to us, π_K^*A is just the transformation algebroid arising from the algebroid action of A on K given by the flat connection \tilde{D} . Similarly, the algebroid structure on π_Q^*A we constructed in the lemma is the transformation algebroid structure coming from h_Q seen as an algebroid action of A on Q .

Now we come back to our original setting, where we consider the algebroid L over P and a hermitian L -connection on the line bundle K over P . Lemma 2.2 provides us with a flat $L \oplus_{\Upsilon} \mathbb{R}$ -connection \tilde{D} on K , and by Lemma 2.3 the pullback of $L \oplus_{\Upsilon} \mathbb{R}$ to Q is endowed with a Lie algebroid structure. Using equation (5) one sees that its anchor $h_Q : \pi_Q^*(L \oplus_{\Upsilon} \mathbb{R}) \rightarrow TQ$, at any point of Q , is given by

$$(6) \quad h_Q(X, \xi, g) = X^H + (\langle X \oplus \xi, \beta \rangle - g)E$$

(here we make choices to write D as in equation (4) and denote by H the horizontal lift w.r.t. $\ker \sigma$).

We claim that the natural injection $I : \pi_Q^*(L \oplus_{\Upsilon} \mathbb{R}) \rightarrow \bar{L} \subset \mathcal{E}^1(Q)$, given by $I(X, \xi, g) = (h_Q(X, \xi, g), 0) \oplus (\pi^*\xi, g)$ is a Lie algebroid morphism. Its image is the codimension one subalgebroid \bar{L}_0 we are after. To show that I is an algebroid morphism we compute for S^1 invariant sections

$$(7) \quad \begin{aligned} & [I(X_1, \xi_1, 0), I(X_2, \xi_2, 0)]_{\mathcal{E}^1(Q)} \\ &= I([(X_1, \xi_1), (X_2, \xi_2)]_c, 0) + \langle (X_1, \xi_1), (X_2, \xi_2) \rangle - ((-E, 0) \oplus (0, 1)) \\ &= I([(X_1, \xi_1, 0), (X_2, \xi_2, 0)]_{\pi_Q^*(L \oplus_{\Upsilon} \mathbb{R})}) \end{aligned}$$

and $[I(X, \xi, 0), I(0, 0, 1)]_{\mathcal{E}^1(Q)} = 0$, where $[\cdot, \cdot]_c$ denotes the usual Courant bracket on $TP \oplus T^*P$, and that I respects the anchor maps of $\pi_Q^*(L \oplus_{\Upsilon} \mathbb{R})$ and \bar{L} .

We summarize the above:

Proposition 2.5. *Assume that the Dirac manifold (P, L) satisfies the prequantization condition (2). Fix the line bundle K over P associated with $[\Omega]$ and a Hermitian L -connection*

D on K with curvature $2\pi i\Upsilon$. The transformation algebroid $\pi_Q^*(L \oplus_{\Upsilon} \mathbb{R})$ is canonically isomorphic to a codimension one subalgebroid \bar{L}_0 of \bar{L} .

Notice that there is a natural Lie algebroid morphism from $\pi_Q^*(L \oplus_{\Upsilon} \mathbb{R})$ (and hence \bar{L}_0) to $L \oplus_{\Upsilon} \mathbb{R}$.

Unfortunately we are not able to give a similar intrinsic description of \bar{L} (see Remark 2.7); from such a description one could presumably infer the integrability of the Lie algebroid \bar{L} from the integrability and prequantizability of (P, L) , as it happens for \bar{L}_0 .

Remark 2.6. Different choices of L -connection on the line bundle K with curvature $2\pi i\Upsilon$ usually lead to Lie algebroids \bar{L} with different foliations (see Remark 2.11), which therefore can not be isomorphic. However the subalgebroids \bar{L}_0 are always isomorphic. Indeed any two connections with the same curvature are of the form D and $D' = D + 2\pi i\gamma$, where γ is a closed section of L^* (see Prop. 6.1 in [24]). A computation using $d_L\gamma = 0$ shows that $(X, \xi) \oplus g \mapsto (X, \xi) \oplus (g - \langle (X, \xi), \gamma \rangle)$ is a Lie algebroid automorphism of $L \oplus_{\Upsilon} \mathbb{R}$. Further this automorphism intertwines the Lie algebroid actions (6) of $L \oplus_{\Upsilon} \mathbb{R}$ on Q given by the ‘‘horizontal lifts’’ of the flat connections \tilde{D} and \tilde{D}' . Hence the transformation algebroids of the two actions are isomorphic, as is clear from the description of Lemma 2.3.

We exemplify the fact that actions coming from different flat connections are intertwined by a Lie algebroid automorphism (something that can not occur if the anchor of the Lie algebroid is injective) in the simple case when the Dirac structure on P comes from a close 2-form ω : the Lie algebroid action of $TP \oplus_{\omega} \mathbb{R}$ on Q via a connection ∇ (with curvature $2\pi i\omega$) is intertwined to the obvious action of the Atiyah algebroid TQ/S^1 on Q (essentially given by the identity map) via $TP \oplus_{\omega} \mathbb{R} \cong TQ/S^1$ is $(X, g) \mapsto X^H - \pi^*gE$, where σ is the connection on the circle bundle Q corresponding to ∇ .

Remark 2.7. The bracket on \bar{L} is determined by (7) and

$$(8) \quad \begin{aligned} [I(X, \xi, 0), (-A^H, 1) \oplus (\sigma - \pi^*\alpha, 0)] &= I(-[(X, \xi), (A, \alpha)]_c) \\ &+ I(0, \Omega(X) - \xi + \frac{1}{2}d\langle X \oplus \xi, \beta \rangle) - \langle A, \xi \rangle ((-E, 0) \oplus (0, 1)). \end{aligned}$$

The remaining brackets between sections of the form $I(X, \xi, 0)$, $I(0, 0, 1)$ and $(-A^H, 1) \oplus (\sigma - \pi^*\alpha, 0)$ vanish, and by the Leibniz rule these brackets determine the bracket for arbitrary sections of \bar{L} . Our difficulty in understanding the structure of the algebroid \bar{L} is due to fact that the section $(-A^H, 1) \oplus (\sigma - \pi^*\alpha, 0)$ depends on more choices than just the L -connection D that determines \bar{L} .

2.2. Describing \bar{L} via the bracket on functions. In this subsection we will succeed in describing the geometric structure on the circle bundle Q in terms of the bracket on the admissible functions on Q (see [24] for the definition).

We adopt the following notation. F_S denotes the function on Q associated to a section S of the line bundle K : F_S is just the restriction to the bundle of unit vectors Q of the fiberwise linear function on K given by $\langle \cdot, S \rangle$, where $\langle \cdot, \cdot \rangle$ is the S^1 -invariant *real* inner product on K corresponding to the chosen Hermitian form on K . Alternatively F_S can be described as the real part of the S^1 -antiequivariant function on Q that naturally corresponds to the section S . By iS we denote the image of the section S by the action of $i \in S^1$ (i.e. S rotated by 90°), and f and g are functions on P .

Proposition 2.8. *Assume that the Dirac manifold (P, L) satisfies the prequantization condition (2). Fix the line bundle K over P associated with $[\Omega]$ and a Hermitian L -connection*

D on K with curvature $2\pi i\Upsilon$. Denote by \tilde{D} the flat connection induced as in Lemma 2.2 and by $h_Q : \pi_Q^*(L \oplus_{\Upsilon} \mathbb{R}) \rightarrow TQ$ the horizontal lift associated to \tilde{D} given by (6).

Suppose a Jacobi-Dirac structure \hat{L} on Q has the following two properties: first, nearby any $q \in Q$ such that $TP \cap L$ is regular near $\pi(q)$, the admissible functions for \hat{L} are exactly those that are constant along the leaves of $\{h_Q(X, 0, 0) : X \in TP \cap L\}$. Second, the bracket on locally defined admissible functions is given by

- $\{\pi^*f, \pi^*g\}_Q = \pi^*\{f, g\}_P$
- $\{\pi^*f, F_S\}_Q = F_{-\tilde{D}_{X_f, df, f}S}$
- $\{\pi^*f, 1\}_Q = 0$
- $\{F_S, 1\}_Q = -2\pi F_{iS}$.

Then \hat{L} must be the Jacobi-Dirac structure \bar{L} given in Thm. 2.1.

Conversely, the Jacobi-Dirac structure \bar{L} given in Thm. 2.1 has the two properties above.

Proof. We start by showing that the Jacobi-Dirac structure \bar{L} constructed in Thm. 2.1 satisfies the above two properties. On the set of points where the ‘‘characteristic distribution’’ $C := \bar{L} \cap (TQ \times \mathbb{R}) \oplus (0, 0)$ of any Jacobi-Dirac structure has constant rank the admissible functions are exactly the functions f such that (df, f) annihilate C . In our case $C = \{X^H + \langle \alpha, X \rangle E : X \in L \cap TP\} = \{h_Q(X, 0, 0) : X \in TP \cap L\}$ is actually contained in TQ , so the admissible functions are those constant on the leaves of C as claimed.

Now we check that the four formulae for the bracket hold. The first equation follows from the fact that the pushforward of \bar{L} is the Jacobi-Dirac structure associated to L (see Section 5 in [24]).

For the second equation we make use of the formulae

$$E(F_S) = -2\pi F_{iS} \quad \text{and} \quad X^H(F_S) = F_{\nabla_X S},$$

where we make some choice to express D as in equation (4) and X^H denotes to horizontal lift of $X \in TP$ using the connection on Q corresponding to the covariant derivative ∇ on K . Using these formulae we see

$$\begin{aligned} \{\pi^*f, F_S\}_Q &= -\langle dF_S, X_f^H + \langle (X_f, df), \beta \rangle E - fE \rangle \\ &= F_{-\nabla_{X_f} S + 2\pi i(\langle (X_f, df), \beta \rangle - f)} \\ &= F_{-\tilde{D}_{X_f, df, f}S}. \end{aligned}$$

For the last two equations just notice that, since $(-E, 0) \oplus (0, 1)$ is a section of \bar{L} , the bracket of any admissible function with the constant function 1 amounts to applying $-E$ to that function.

Now we show that if a Jacobi-Dirac structure \hat{L} satisfies the two properties in the statement of the proposition, then it must be \bar{L} . By Remark 2.9, the bracket of $\dim Q - rkC + 1$ independent functions at regular points of $C := \hat{L} \cap (TQ \times \mathbb{R}) \oplus (0, 0)$ determines \hat{L} , so we have to show that our two properties carry the information of the bracket of $\dim Q - rkC + 1$ independent functions at regular points of C .

It will be enough to consider the open dense subset of the regular points of C where $C = \{h_Q(X, 0, 0) : X \in TP \cap L\}$ (This subset is dense because it includes the points q such that C is regular near q and $TP \cap L$ is regular near $\pi(q)$). Since there C is actually contained in TQ it is clear that 1 and $\{\pi^*f\}$ are admissible functions, for f any admissible function on P (this means that f is constant along the leaves of $L \cap TP$; there are $\dim P - rkC$ such f which are linearly independent at $\pi(q)$). Further we can construct an admissible function

F_S as follows: take a submanifold Y near $\pi(q)$ which is transverse to the foliation given by $L \cap TP$, and define the section $S|_Y$ so that it has norm one (i.e. its image lies in $Q \subset K$). Then extend S to a neighborhood of $\pi(q)$ by starting at a point y of Y and “following” the leaf of C through $S(y)$ (notice that C is a flat partial connection on $Q \rightarrow P$ covering the distribution $L \cap TP$ on P). Since C is S^1 invariant, the resulting function F_S is clearly constant along the leaves of C , hence admissible. Altogether we obtain $\dim Q - rkC + 1$ admissible functions in a neighborhood of q for which we know the brackets, so we are done. \square

Remark 2.9. On any Jacobi-Dirac manifold (Q, \hat{L}) the bracket on the sheaf of admissible functions $(C_{adm}^\infty(Q), \{\cdot, \cdot\})$ determines the subbundle \hat{L} of $\mathcal{E}^1(Q)$. (This might seem a bit surprising at first, since the set of admissible functions is usually much smaller than $C^\infty(Q)$).

The set of points where $C := \hat{L} \cap (TQ \times \mathbb{R}) \oplus (0, 0)$ (an analog of a “characteristic distribution”) has locally constant rank is an open dense subset of Q , since C is an intersection of subbundles. Hence by continuity it is enough to reconstruct the subbundle \bar{L} on each point q of this open dense set.

Since we assume that C has constant rank near q , given $C_{adm}^\infty(Q)$ in a neighborhood of q we can reconstruct C as the distribution annihilated by (df, f) where f ranges over $C_{adm}^\infty(Q)$. We can clearly find $\dim Q - rkC + 1$ admissible functions f_i such that $\{(df_i, f_i)\}$ forms a basis of $\rho_{T^*Q \times \mathbb{R}}(\hat{L}) = C^\circ$ near q . The fact that each f_i is an admissible function means that there exist (X_i, ϕ_i) such that $(X_i, \phi_i) \oplus (df_i, f_i)$ is a smooth section of \hat{L} . Now knowing the bracket of any f_j with the other f_i 's, i.e. the pairing of (X_j, ϕ_j) with all elements of $\rho_{T^*Q \times \mathbb{R}}(\hat{L})$, does not quite determine (X_j, ϕ_j) . However it determines (X_j, ϕ_j) up to sections of C , hence the direct sum of the span of all $(X_i, \phi_i) \oplus (df_i, f_i)$ and of C is a well defined subbundle of $\mathcal{E}^1(Q)$. Moreover it has the same dimension as \hat{L} and it is spanned by sections of \hat{L} , so it is \hat{L} .

We end this section by commenting on “how many prequantization spaces” there are and on Morita equivalence.

Remark 2.10. Two L -connections on K are *gauge equivalent* if they differ by $d_L\phi$ for some function $\phi : P \rightarrow S^1$. Gauge-equivalent L -connections D on K with curvature $2\pi i\Upsilon$ give rise to isomorphic Jacobi-Dirac structures: denoting by Φ the bundle automorphism of Q given by $q \mapsto q \cdot \pi^*\phi$, using the proof of Proposition 4.1 in [24] one can show that if $D_2 = D_1 - 2\pi i d_L\phi$ then $(\Phi_*, Id) \oplus ((\Phi^{-1})^*, Id)$ is an isomorphism from the Jacobi-Dirac structure induced by D_1 to the one induced by D_2 . Alternatively one can check directly for the bracket of functions that $\Phi^*\{\cdot, \cdot\}_{D_2} = \{\Phi^*\cdot, \Phi^*\cdot\}_{D_1}$. The gauge-equivalence classes of L -connections with curvature $2\pi i\Upsilon$ are a principal homogeneous space for $H_L^1(P, U(1))$ (see the proof of Prop. 6.1 in [24]).

Remark 2.11. We have seen that the prequantization space Q of a prequantizable Dirac manifold (P, L) can be endowed with various non-isomorphic Jacobi-Dirac structures \bar{L} . It is easy to see that (Q, \bar{L}_1) and (Q, \bar{L}_2) will usually not even be Morita equivalent, for any reasonable notion of Morita equivalence of Jacobi-Dirac manifold (or of their respective precontact groupoids). Indeed for $P = \mathbb{R}$ with the zero Poisson structure, choosing $(Q, \sigma, \beta) = (S^1 \times \mathbb{R}, d\theta, x\partial_x)$ as in Example 4.12 one obtains a Jacobi structure on Q with three leaves, whereas choosing $(S^1 \times \mathbb{R}, d\theta, 0)$ one obtains a Jacobi structure with uncountably many leaves (namely all $S^1 \times \{q\}$). On the other hand, one of the general properties of Morita equivalence is to induce a bijection on the space of leaves.

3. PREQUANTIZATION AND REDUCTION OF JACOBI-DIRAC STRUCTURES

In the last section we considered a prequantizable Dirac manifold (P, L) and endowed Q (the total space of the circle bundle over P) with distinguished Jacobi-Dirac structures \bar{L} . Even though L^c , the Jacobi-Dirac structure canonically associated to L , is just the pushforward of \bar{L} , there is no Lie algebroid morphism $\bar{L} \rightarrow L$ in general (we will elaborate more on this in Subsection 3.3). In this section we will recover the Lie algebroid L from \bar{L} via a reduction procedure, which we will globalize to the corresponding Lie groupoids in the next Section.

3.1. Reduction of Jacobi-Dirac structures as precontact reduction. We recall a familiar fact: in symplectic geometry, we have the well-known motivating example of symplectic reduction $T^*M//_0G = T^*(M/G)$. In [10], it is extended to contact geometry by replacing T^*M by the cosphere bundle of M . Here we prove a similar result by replacing T^*M by $T^*M \times \mathbb{R}$ —another natural contact manifold associated to any manifold M . Let a Lie group G acts on a contact manifold (C, θ) preserving the contact form θ . Then, a moment map is a map J from the manifold M to \mathfrak{g}^* (the dual of the Lie algebra) such that for all v in the Lie algebra \mathfrak{g} :

$$(9) \quad \langle J, v \rangle = \theta_M(v_M),$$

where v_M is the infinitesimal generator of the action on M given by v . The moment map J is automatically equivariant with respect to the coadjoint action of G on \mathfrak{g}^* given by $\xi \cdot g = L_g^* R_{g^{-1}}^* \xi$. A group action as above together with its moment map is called *Hamiltonian*. Notice that any group action preserving the contact form is Hamiltonian. There are two sorts of contact reduction by Albert and Willett respectively [1] [25]. But they are the same if reduced at 0, which is used here, namely,

$$C//_0G := J^{-1}(0)/G,$$

is again a smooth contact manifold with the induced contact form $\bar{\theta}$ such that $\pi^*(\bar{\theta}) = \theta|_{J^{-1}(0)}$.

Lemma 3.1. *Let group G act on manifold M freely and properly. Then G has an induced action on the contact manifold $(C := T^*M \times \mathbb{R}, \theta := \theta_c + dt)$ where θ_c is the canonical 1-form on T^*M and t is the coordinate on \mathbb{R} . Then this action is Hamiltonian and the contact reduction at 0*

$$T^*M \times \mathbb{R} //_0 G = T^*(M/G) \times \mathbb{R}.$$

Proof. The induced G action on $T^*M \times \mathbb{R}$ is by $g \cdot (\xi, t) = ((g^{-1})^* \xi, t)$, and it preserves the 1-form $\theta_c + dt$. The projection of this action on M is the G action on M so it is also free and proper. Then the moment map J is determined by

$$\langle J(\xi, t), v \rangle = (\theta_c + dt)_{(\xi, t)}(v_C) = \theta_c(v_C) = \langle \xi, v_M \rangle,$$

where v_C (resp. v_M) denotes the vector field corresponding to the infinitesimal action of G on the manifold C (resp. M). Since all infinitesimal generators v_C are nowhere proportional to the Reeb vector field $\frac{\partial}{\partial t}$, by Remark 3.2 in [25] all points of $T^*M \times \mathbb{R}$ are regular points of J . So $J^{-1}(0) = \{(\xi, t) : \langle \xi, v_M \rangle = 0 \ \forall v \in \mathfrak{g}\} = \{(\pi^* \mu, t) : \mu \in T^*(M/G)\}$ (with $\pi : M \rightarrow (M/G)$) is a smooth manifold. Therefore it is not hard to see that there is a well-defined

$$\Phi : J^{-1}(0)/G \rightarrow T^*(M/G) \times \mathbb{R}, \quad \text{by } ([\xi], t) \mapsto (\mu, t),$$

where μ is uniquely determined by $\pi^*\mu = \xi$ and we used the notation $[\cdot]$ to denote the quotient of points (and later tangent vectors) of $J^{-1}(0)$ by the G action. It is not hard to see that Φ is an isomorphism since the two sides have the same dimension and Φ is obviously surjective. The contact form on $T^*(M/G) \times \mathbb{R}$ corresponding to the reduced contact form $\bar{\theta}$ via the isomorphism Φ is the canonical one: for a tangent vector $([v], \lambda \frac{\partial}{\partial t}) \in T_{[\xi], t}(J^{-1}(0)/G)$,

$$\bar{\theta}_{[\xi], t}([v], \lambda \frac{\partial}{\partial t}) = \theta_{\xi, t}(v, \lambda \frac{\partial}{\partial t}) = \xi(p_*v) + \lambda = \mu(\bar{p}_*\Phi_*[v]) + \lambda,$$

where $p : T^*M \rightarrow M$ and $\bar{p} : T^*(M/G) \rightarrow M/G$. Here we used $\bar{p}_*\Phi_*[v] = \pi_*p_*v$, which follows from the fact that Φ is a vector bundle map, and we abuse notation by denoting with the same symbol a restriction of Φ . \square

This result extends to the precontact situation. Let \bar{L} be a subbundle of $\mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$. It is a precontact manifold with a precontact 1-form

$$(10) \quad \theta_{\bar{L}} = pr^*(\theta_c + dt),$$

where pr is the projection $\bar{L} \rightarrow T^*M \times \mathbb{R}$.

Proposition 3.2. *When (Q, \bar{L}) is a Jacobi-Dirac manifold, \bar{L} is a precontact manifold as described above. If the group G acts freely and properly on Q preserving the Jacobi-Dirac structure, the action lifts to a free proper Hamiltonian action on \bar{L} with moment map J ,*

$$\langle J((X, f) \oplus (\xi, g)), v \rangle = \theta_{\bar{L}}|_{(X, f) \oplus (\xi, g)}(v_{\bar{L}}) = \xi(v_Q).$$

Write \mathfrak{g}_Q as a short form for $\{v_Q : v \in \mathfrak{g}\} \subset TQ$, and let $L_P \subset \mathcal{E}^1(P)$ be the pushforward of \bar{L} via $\pi : Q \rightarrow P := Q/G$. Then

- (1) $J^{-1}(0)$ is a subalgebroid of \bar{L} iff $\bar{L} \cap (\mathfrak{g}_Q, 0) \oplus (0, 0)$ has constant rank, and in that case $\bar{L}/_0G := J^{-1}(0)/G$ has an induced Lie algebroid structure;
- (2) $J^{-1}(0)/G \cong L_P$ both as Lie algebroids and precontact manifolds, iff $\bar{L} \cap (\mathfrak{g}_Q, 0) \oplus (0, 0) = \{0\}$. Here the precontact forms are the reduced 1-form on $J^{-1}(0)/G$ and the one defined as in (10) on L_P respectively.

Proof. The G action on Q lifts to \bar{L} by $g \cdot (X, f) \oplus (\xi, g) = (g_*X, f) \oplus ((g^{-1})^*\xi, g)$, and the resulting moment map J is clearly as claimed in the statement.

To prove (1) we start with some linear algebra and fix $x \in Q$. We have a map $\pi_* : T_xQ \rightarrow T_{\pi(x)}(Q/G)$, hence we can push forward $\bar{L}|_x$ to

$$L_P|_{\pi(x)} := \{(\pi_*X, f) \oplus (\mu, g) : (X, f) \oplus (\pi^*\mu, g) \in \bar{L}|_x\}$$

to obtain a linear Jacobi-Dirac subspace of $\mathcal{E}^1(Q/G)|_{\pi(x)}$. Since \bar{L} is G invariant, doing this at every $x \in Q$ we obtain a well defined subbundle of $\mathcal{E}^1(Q/G)$, which however might fail to be smooth². We have a surjective map

$$(11) \quad \begin{aligned} \Phi : J^{-1}(0) &= \{(X, f) \oplus (\xi, g) \in \bar{L} : \xi = \pi^*\mu \text{ for some } \mu \in T_{\pi(x)}(Q/G)\} \rightarrow L_P \\ &(X, f) \oplus (\xi, g) \mapsto (\pi_*X, f) \oplus (\mu, g) \end{aligned}$$

whose kernel is exactly $J^{-1}(0) \cap (\mathfrak{g}_Q, 0) \oplus (0, 0)$ (Notice that the map is well defined for π is a submersion). So $J^{-1}(0)$ has constant rank iff $J^{-1}(0) \cap (\mathfrak{g}_Q, 0) \oplus (0, 0) = \bar{L} \cap (\mathfrak{g}_Q, 0) \oplus (0, 0)$ does. In this case it is easy to see that $J^{-1}(0)$ is closed under the Courant bracket: the Courant bracket of two sections of $J^{-1}(0)$ lie in \bar{L} (because \bar{L} is closed under the bracket),

²For example it is not smooth when $G = \mathbb{R}$, $Q = \mathbb{R}^2$, $v_Q = \frac{\partial}{\partial x}$ and \bar{L} is the graph of the 1-form $\frac{y^2}{2}dx$.

therefore one just has to show that its cotangent component is annihilated by \mathfrak{g}_Q . By a straight-forward computation this is true for G -invariant sections, and by the Leibniz rule it follows for all sections of $J^{-1}(0)$, i.e. $J^{-1}(0)$ is a subalgebroid. Clearly $J^{-1}(0)/G$ becomes an algebroid with the bracket induced from the one on $J^{-1}(0)$ and anchor $([X], f) \oplus ([\xi], g) \mapsto \pi_* X$ (where $[\cdot]$ denotes the equivalence relation given by the G action).

To prove (2) consider the map Φ above. It induces an isomorphism of vector bundles over P between $J^{-1}(0)/G$ and L_P iff it is fiberwise injective, i.e. iff $\bar{L} \cap (\mathfrak{g}_Q, 0) \oplus (0, 0) = \{0\}$. Since $J^{-1}(0)/G$ (being a precontact reduction) is a smooth manifold and $J^{-1}(0)/G \cong L_P$ is point-wise a subbundle of $\mathcal{E}^1(P)$, it follows that L_P is a smooth vector bundle over P . We are left with showing that Φ induces an isomorphism of Lie algebroids and precontact manifolds. Using the fact that operations appearing in the definition of Courant bracket such as taking Lie derivatives commute with taking quotient of G (for example $\pi^*(L_{\pi_* X} \mu) = L_X \pi^* \mu$) we deduce that $\Phi : J^{-1}(0) \rightarrow L_P$ is a surjective morphism of Lie algebroids, hence the induced map $\Phi : J^{-1}(0)/G \rightarrow L_P$ an isomorphism of Lie algebroids.

The isomorphism of precontact manifolds follows from an entirely similar argument as in Lemma 3.1. We consider a tangent vector $([w], \kappa \frac{\partial}{\partial s}) \oplus ([v], \lambda \frac{\partial}{\partial t}) \in T_{([X], f) \oplus ([\xi], g)}(J^{-1}(0)/G)$, then $\Phi(([X], f) \oplus ([\xi], g)) = (\pi_* X, f) \oplus (\mu, g)$, where $\pi^* \mu = \xi$. So the induced 1-form $\bar{\theta}$ on $J^{-1}(0)/G$ satisfies,

$$\bar{\theta}_{[X], f, [\xi], g}([w], \kappa \frac{\partial}{\partial s}) \oplus ([v], \lambda \frac{\partial}{\partial t}) = \theta_{X, f, \xi, g}(w, \kappa \frac{\partial}{\partial t}) \oplus (v, \lambda \frac{\partial}{\partial t}) = \xi(p_* v) + \lambda = \mu(\bar{p}_* \Phi_* [v]) + \lambda,$$

where $p : \bar{L} \rightarrow Q$ and $\bar{p} : L_P \rightarrow P$ are projections. Therefore $\bar{\theta} = \Phi^* \theta_{L_P}$ with θ_{L_P} the canonical 1-form as in (10). □

3.2. Reduction of prequantizing Jacobi-Dirac structures. Now we adapt the general theory of reduction of Jacobi-Dirac manifolds as we discussed above to our situation, namely we consider a prequantization Q of Dirac manifold (P, L) . Then Q is Jacobi-Dirac with a free and proper S^1 action which preserves the Jacobi-Dirac structure \bar{L} . Let $L^c := \{(X, 0) \oplus (\xi, g) : (X, \xi) \in L, g \in \mathbb{R}\}$ denote the Jacobi-Dirac structure associated to the Dirac manifold (P, L) . Then L^c naturally has a precontact form as described in (10). The algebroids \bar{L} , L^c and L fit into the following diagram (where we denote dimensions and ranks by superscripts):

$$\begin{array}{ccc} \bar{L}^{n+2} & & (L^c)^{n+1} \longrightarrow L^n \\ \downarrow & & \downarrow \swarrow \\ Q^{n+1} & \xrightarrow{\pi} & P^n \end{array}$$

The left two algebroids in the diagram are related by the reduction described in the next proposition:

Proposition 3.3. *When (Q, \bar{L}) is a prequantization of Dirac manifold (P, L) we have $J^{-1}(0) = \bar{L}_0$ (recall that \bar{L}_0 was defined at the end of Section 2.1) and the isomorphisms of precontact manifolds and algebroids,*

$$\bar{L}/_0 S^1 \cong L^c.$$

Proof. The equality is clear from the characterization of $J^{-1}(0)$ in Prop. 3.2 and from the definition of \bar{L}_0 . For the isomorphism notice that $L^c = L_P$ (this is equivalent to saying that

π is a forward Jacobi-Dirac map) and apply Prop. 3.2 (which holds because the assumption $\bar{L} \cap (\mathfrak{g}_q, 0) \oplus (0, 0) = \{0\}$ is satisfied, as is clear from the definition of \bar{L} in Theorem 2.1). \square

Now we make use of the last lemma to establish a relation to the relevant Lie groupoids. See the remark below for an interpretation in terms of infinitesimal counterpart to a Lie groupoid reduction.

Lemma 3.4. *If the presymplectic groupoid $\Gamma_s(P)$ exists, then*

$$\bar{L} //_0 S^1 \cong A(\tilde{\Gamma}_c(P)),$$

as algebroids, where $A(\tilde{\Gamma}_c(P))$ is the algebroid of the prequantization $\tilde{\Gamma}_c(P)$ of the s.s.c. symplectic groupoid of P . Further, along points of P , the reduced 1-form on $\bar{L} //_0 S^1$ coincides with the 1-form on $\tilde{\Gamma}_c(P)$.

Proof. We will see in items (4) and (5) of Thm. 4.10 that the prequantizability and integrability of (P, L) implies that $\Gamma_s(P)$ is prequantizable, and that the prequantization bundle $\tilde{\Gamma}_c(P)$ is a groupoid integrating L^c . (In the Poisson case this follows from [9] and [2]). Hence the algebroid isomorphism follows from Prop. 3.3. Equation (18) in Lemma 5.1 gives a Lie algebroid isomorphism between $\ker \mathbf{s}_*|_P$ and L^c , under which the restriction of the 1-form on $\tilde{\Gamma}_c(P)$ and the precontact form (10) on L^c at points of P correspond (notice that at points of the zero section P the precontact form on L^c is just pr^*dt , i.e. the projection onto the last component). \square

Remark 3.5. As we will see in the next section, there is an S^1 action on the precontact groupoid $(\Gamma_c(Q), \theta_\Gamma, f_\Gamma)$ of (Q, \bar{L}) (see Definition 4.3), which is canonically induced by the S^1 action on Q and which hence makes the source map equivariant and which respects the 1-form and multiplicative function on the groupoid. The equivariance makes sure that taking derivatives along the identity one gets an S^1 action on $\ker \mathbf{s}_*|_Q$ by vector bundle isomorphism. Further, as we will show in Lemma 5.1 in the appendix, one can show that $\ker \mathbf{s}_*|_Q \cong \bar{L}$ via

$$Y \mapsto (\mathbf{t}_*Y, -r_{\Gamma_*}Y) \oplus (-d\theta_\Gamma(Y)|_{TQ}, \theta_\Gamma(Y))$$

is an isomorphism of Lie algebroids where $e^{-r_\Gamma} = f_\Gamma$. Under this identification, the S^1 action is the natural one described at the beginning of the proof of Prop. 3.2, because the S^1 action on $\Gamma_c(Q)$ respects \mathbf{t}, r_Γ and θ_Γ . Further, under the above isomorphism, $\theta_\Gamma|_Q$ and (the restriction to the zero section of) the 1-form (10) on \bar{L} clearly coincide. We conclude that the S^1 action we considered in this subsection is the infinitesimal version of the S^1 action on $(\Gamma_c(Q), \theta_\Gamma)$.

Therefore the question arises of the relation between the two S^1 -contact reductions at 0. The answer is contained in Lemma 3.4 and the next section, where we will show that the contact reduction of $\Gamma_c(Q)$ is isomorphic, both as contact manifold and a groupoid, to the s.s.c. contact groupoid of P , and that $\tilde{\Gamma}_c(Q)$ is a discrete quotient of it. This means that contact reduction commutes with taking Lie algebroid, that is

$$A(\Gamma_c(Q) //_0 S^1) = A(\Gamma_c(Q)) //_0 S^1.$$

Further we also have a correspondence at the intermediate step of the reduction, namely for the zero level sets of the moment maps (see item (3) of Thm. 4.8).

3.3. Alternative approaches. We explain why our construction seems the only way to describe the relation between the algebroids (and hence the groupoids) appearing in the picture. We just need to assume that (P, L) be a prequantizable Dirac manifold and we fix a connection D with curvature $2\pi i\Upsilon$ as in Subsection 2.1, i.e. we fix a prequantization (Q, \bar{L}) . As shown in [24] the projection $\pi : Q \rightarrow P$ is a forward Jacobi-Dirac map, so L can be recovered as the pushforward by π of \bar{L} ; however this is unsatisfactory because it does not imply anything about the relation between the corresponding groupoids.

For the codimension one subalgebroid \bar{L}_0 of \bar{L} the natural map

$$\Phi : \bar{L}_0 \rightarrow L^c, (X, 0) \oplus (\pi^*\xi, g) \mapsto (\pi_*X, 0) \oplus (\xi, g)$$

is a (surjective) morphism of Lie algebroids. (See the proofs of Prop. 3.2 and Prop. 3.3. This is the same algebroid morphism mentioned after Prop. 2.5, upon usage of the Lie algebroid isomorphism $L^c \rightarrow L \oplus_{\Upsilon} \mathbb{R}$ from [9] or Section 5.2 of [16]).

However this map can not be usually extended to an algebroid morphism defined on \bar{L} . Indeed usually there cannot be any Lie algebroid morphism from \bar{L} to L^c or L with base map π : recall that a morphism of algebroids maps each orbit of the source algebroid into an orbit of the target algebroid. If the map $\pi : Q \rightarrow P$ induced a morphism of algebroids, then the orbits of \bar{L} would be mapped into the orbits of L^c (which coincide with those of L). However this happens exactly when (one and hence all choices of) the vector field A appearing in Thm. 2.1 is tangent to the foliation of L (see Section 4.1 of [24]). In the case of Example 4.12, i.e. $Q = S^1 \times \mathbb{R}$ and $P = \mathbb{R}$, the orbits of $T^*Q \times \mathbb{R}$ are exactly three (namely $S^1 \times \mathbb{R}_+$, $S^1 \times \{0\}$ and $S^1 \times \mathbb{R}_-$), and π does not map them into the orbits of T^*P , which are just points.

Hence \bar{L} and L^c are usually not related by an algebroid morphism, but bringing into the picture the 1-form $\theta_{\bar{L}}$ allows to relate them by S^1 -reduction: the zero level set of the moment map is \bar{L}_0 , and the natural algebroid morphism $\Phi : \bar{L}_0 \rightarrow L^c$ shows that the reduced space is isomorphic to L^c .

4. PREQUANTIZATION AND REDUCTION OF PRECONTACT GROUPOIDS

In this section we analyze the relation between the groupoids associated to (P, L) and (Q, \bar{L}) , leading to an “integrated” version of Proposition 3.3 (i.e. to reduction of groupoids). In Subsection 4.1 we will perform the reduction using finite dimensional arguments, restricting ourselves for simplicity to the case when P is a Poisson manifold. If on one hand our finite dimensional proof might appeal more to geometric intuition, it will not allow to conclude whether the reduced groupoids we obtain are source simply connected. In Subsection 4.2, for the general case when P is a Dirac manifold, we will obtain a complete description of the reduction using path spaces. We will conclude with two examples.

4.1. The Poisson case. In this subsection we show our results for Poisson manifold without using the infinite dimensional path spaces.

We start displaying a simple example, which was also a motivating example in [7].

Example 4.1. Let (P, ω) be a simply connected integral symplectic manifold, and (Q, θ) a prequantization. We have the following diagram of groupoids:

$$\begin{array}{ccc} (Q \times Q \times \mathbb{R}, -e^{-s}\theta_1 + \theta_2, e^{-s}) & (Q \times_{S^1} Q, [-\theta_1 + \theta_2]) & \longrightarrow (P \times P, -\omega_1 + \omega_2) \\ \Downarrow & \Downarrow & \swarrow \\ Q & \longrightarrow & P \end{array}$$

The first groupoid is a (usually not s.s.c.) contact groupoid of (Q, θ) , with coordinate s on the \mathbb{R} factor. The second is a contact groupoid of (P, ω) which is a prequantization of the third groupoid (the s.s.c. symplectic groupoid of (P, ω)). The S^1 action on Q induces a circle action on its contact groupoid with moment map given by $\langle J, 1 \rangle = -e^{-s} + 1$, so that its zero level set is obtained setting $s = 0$, and dividing by the circle action we obtain exactly the second groupoid above, i.e. the prequantization of the s.s.c. groupoid of (P, ω) .

Let (P, L) be a Poisson manifold, and assume that it is prequantizable and integrable to a s.s.c symplectic groupoid $\Gamma_s(P)$. Here we look at P as a Dirac manifold, i.e. L is the graph of the Poisson bivector of P . As shown in Section 3.3 of [2] (see Theorem 4.2 for a straightforward generalization), the prequantizability of (P, L) implies that the period group of any source fiber of $\Gamma_s(P)$ is contained in \mathbb{Z} . By Prop. 2 in [2] or Thm. 3 in [9] this last condition is equivalent to saying that the symplectic groupoid $\Gamma_s(P)$ is prequantizable in the sense of [7]. Its unique prequantization will be denoted by $\tilde{\Gamma}_c(P)$ and turns out to be a (usually not s.s.c.) contact groupoid of P , i.e. it integrates the Lie algebroid L^c . Hence “integrating” the reduction statements of the last section we will clarify the relation between the “global object” associated to the prequantization Q (i.e. the s.s.c. contact groupoid $\Gamma_c(Q)$) and the prequantization of $\Gamma_s(P)$ (which in a way can be thought of as a different way to prequantize (P, L) , for example because it allows to construct a prequantization representation of the Lie algebra of functions on P).

Assuming that (P, L) be prequantizable, integrable (as a Poisson manifold) and that (Q, \bar{L}) be integrable, we obtain (smooth) groupoids that fit into the following diagram; we omitted $\tilde{\Gamma}_c(P)$, which is just a discrete quotient of the s.s.c. contact groupoid $\Gamma_c(P)$. See the previous section for the diagram of the corresponding algebroids; again we denote dimensions by superscripts.

$$\begin{array}{ccccc}
 \Gamma_c(Q)^{2n+3} & & \Gamma_c(P)^{2n+1} & \longrightarrow & \Gamma_s(P)^{2n} \\
 \Downarrow & & \Downarrow & \swarrow & \swarrow \\
 Q^{n+1} & \xrightarrow{\pi} & P^n & &
 \end{array}$$

Theorem 4.2. *Let (P, L) be an integrable prequantizable Poisson manifold, and (Q^{n+1}, \bar{L}) one of its prequantizations, which we assume to be integrable. Then:*

- *The s.s.c contact groupoid $\Gamma_c(P)$ of (P, L) is obtained from the s.s.c. contact groupoid $\Gamma_c(Q)$ of (Q, \bar{L}) by S^1 contact reduction.*
- *The prequantization of the s.s.c. symplectic groupoid $\Gamma_s(P)$ is a discrete quotient of $\Gamma_c(P)$.*

Proof. S^1 acts on Q , and it acts also on $TQ \oplus T^*Q$ by the tangent and cotangent lifts. The S^1 action preserves the subbundle given by the Jacobi-Dirac structure \bar{L} , hence we obtain an S^1 action on the algebroid $\bar{L} \rightarrow Q$. The source simply connected (s.s.c.) contact groupoid $(\Gamma_c(Q), \theta_\Gamma, f_\Gamma)$ of (Q, \bar{L}) is constructed canonically from the algebroid \bar{L} via the path-space construction, so it inherits an S^1 action that preserves its geometric and groupoid structures. In particular the source and target maps are S^1 equivariant, and similarly the multiplication map $\Gamma_c(Q)_s \times_t \Gamma_c(Q) \rightarrow \Gamma_c(Q)$. Also, the S^1 action preserves the contact form, so there is a moment map $J_1 : \Gamma_c(Q) \rightarrow \mathbb{R}$ by $J_1(g) = \theta_\Gamma(v_\Gamma(g))$ where v_Γ denotes the infinitesimal generator of the S^1 action. We divide the proof in three steps.

Step 1: $J_1^{-1}(0)$ is a s.s.c. Lie subgroupoid of $\Gamma_c(Q)$.

We start by showing that $J_1 = 1 - f_\Gamma$; this explicit³ formula will turn out to be necessary in Step 2.

To do this we will use several properties of contact groupoids, for which to refer to Remark 2.2 in [26]. The identity $J_1 + f_\Gamma = 1$ is clear along the identity section Q , since f_Γ is a multiplicative function and v_Γ is tangent to Q which is a Legendrian submanifold of $(\Gamma_c(Q), \theta_\Gamma)$. So to show that the statement holds at any point of $\Gamma_c(Q)$ it is enough to show that $\langle d(f_\Gamma + J_1), X_{f_\Gamma \mathbf{t}^* u} \rangle = 0$ for functions $u \in C^\infty(Q)$, since hamiltonian vector fields $X_{f_\Gamma \mathbf{t}^* u}$ span $\ker \mathbf{s}_*$. The statement follows by two computations: first

$$(12) \quad \begin{aligned} \langle df_\Gamma, X_{f_\Gamma \mathbf{t}^* u} \rangle &= \langle df_\Gamma, f_\Gamma \mathbf{t}^* u E_\Gamma + \Lambda_\Gamma d(f_\Gamma \mathbf{t}^* u) \rangle \\ &= f_\Gamma \cdot \langle df_\Gamma, \Lambda_\Gamma d(\mathbf{t}^* u) \rangle = -f_\Gamma \cdot d(\mathbf{t}^* u) X_{f_\Gamma} = f_\Gamma \cdot E(u), \end{aligned}$$

where we used twice $E_\Gamma(f_\Gamma) = 0$ and the fact that \mathbf{t} is a $-f_\Gamma$ -Jacobi map. Second,

$$\langle d(\theta_\Gamma(v_\Gamma)), X_{f_\Gamma \mathbf{t}^* u} \rangle = -d\theta_\Gamma(v_\Gamma, X_{f_\Gamma \mathbf{t}^* u}) = \langle -d(f_\Gamma \mathbf{t}^* u), (v_\Gamma - \theta_\Gamma(v_\Gamma) E_\Gamma) \rangle = -f_\Gamma \cdot E(u),$$

where we use the fact that $\mathcal{L}_{v_\Gamma} \theta_\Gamma = 0$ in the first equality, the formula $d\theta_\Gamma(X_\phi, w) = -\langle d\phi, w^H \rangle$ valid for any function ϕ on a contact groupoid (where w^H is the projection of the tangent vector w to $\ker \theta_\Gamma$ along the Reeb vector field E_Γ) in the second one, and in the last equality that $E_\Gamma(f_\Gamma), v_\Gamma(f_\Gamma), \mathbf{t}_* E_\Gamma$ all vanish and that the S^1 actions on $\Gamma_c(Q)$ and Q are intertwined by the target map \mathbf{t} .

Since f_Γ is multiplicative, it is clear that $J_1^{-1}(0) = f_\Gamma^{-1}(1)$ is a subgroupoid.

Further $J_1^{-1}(0)$ is a smooth submanifold of $\Gamma_c(Q)$: by Prop. 3.1.4 in [25] $g \in \Gamma_c(Q)$ is a singular point of J_1 iff $v_\Gamma(g)$ is a non-zero multiple of $E_\Gamma(g)$. Since $\theta_\Gamma(E_\Gamma) = 1$ this is never the case if $g \in J_1^{-1}(0)$, so 0 is a regular value of J_1 .

To show that $J_1^{-1}(0)$ is a Lie subgroupoid we still need to show that its source and target maps are submersions onto Q . We do so by showing explicitly that $(\ker \mathbf{s}_* \cap \ker df_\Gamma)$ (which along Q will be the algebroid of $J_1^{-1}(0)$) has rank one less than $\ker \mathbf{t}_*$; this is clear since by the first equation of Step 1 it is just $\{X_{f_\Gamma \mathbf{t}^* \pi^* v} : v \in C^\infty(P)\}$.

For the proof of the source simply connectedness of the subgroupoid $J_1^{-1}(0)$ we refer to Thm. 4.8.

Step 2: The contact reduction $J_1^{-1}(0)/S^1$ is the s.s.c. contact groupoid $\Gamma_c(P)$ of P .

$J_1^{-1}(0)/S^1$ is smooth because the S^1 action is free and proper, and by contact reduction it is a contact manifold, so we just have to show that the Lie groupoid structure descends and is a compatible one.

The S^1 equivariance of the source and target maps of $\Gamma_c(Q)$ ensure that source and target descend to maps $J_1^{-1}(0)/S^1 \rightarrow P (= Q/S^1)$. Since the multiplication on $\Gamma_c(Q)$ is S^1 equivariant, the multiplication on $J_1^{-1}(0)$ induces a multiplication on $J_1^{-1}(0)/S^1$. It is routine to check this makes $J_1^{-1}(0)/S^1$ into a groupoid over P . Further, since the source map intertwines the S^1 action on $J_1^{-1}(0)$ and the free S^1 action on the base Q , the source fibers of $J_1^{-1}(0)/S^1$ will be diffeomorphic to the corresponding source fibers of $J_1^{-1}(0)$, hence we obtain a s.s.c. Lie groupoid. Since $J_1^{-1}(0) \rightarrow J_1^{-1}(0)/S^1$ is a surjective submersion, the

³The claim of Step 1 follows even without knowing the explicit formula for J_1 . Indeed one can show that $J_1^{-1}(0)$ is a subgroupoid by means of the identity $J_1(gh) = f(h)J_1(g) + J_1(h)$, which is derived using the multiplicativity of θ_Γ and the fact that v_Γ is a multiplicative vector field (i.e. $v_\Gamma(g) \cdot v_\Gamma(h) = v_\Gamma(gh)$); this is just the infinitesimal version of the statement that the multiplication map is S^1 equivariant). Since $J_1^{-1}(0)$ is a smooth wide subgroupoid it is transverse to the \mathbf{s} fibers nearby the identity, therefore its source and target maps are submersions and hence it is actually a Lie subgroupoid.

f_Γ -twisted multiplicativity of θ_Γ implies that the induced 1-form $\hat{\theta}_\Gamma$ is multiplicative, i.e. $(J_1^{-1}(0)/S^1, \hat{\theta}_\Gamma, \hat{f}_\Gamma)$ is a contact groupoid.

In order to prove that the above contact groupoid corresponds to the original Poisson structure Λ_P on P , we have to show that the source map $\hat{\mathbf{s}} : J_1^{-1}(0)/S^1 \rightarrow P$ is a Jacobi map (i.e. a forward Jacobi-Dirac map). Consider the diagram

$$\begin{array}{ccc} J_1^{-1}(0) & \xrightarrow{\pi_{J_1}} & J_1^{-1}(0)/S^1 \\ \mathbf{s} \downarrow & & \hat{\mathbf{s}} \downarrow \\ Q & \xrightarrow{\pi} & P. \end{array}$$

We adopt the following short-form notation: for a 1-form α , L_α will denote the Jacobi-Dirac structure associated to α [21]. Then for the pullback Jacobi-Dirac structure we have $i^*L_{\theta_\Gamma} = L_{i^*\theta_\Gamma}$, where i is the inclusion of $J_1^{-1}(0)$ into $\Gamma_c(Q)$, and the reduced 1-form is recovered as $\pi_{J_1*}i^*L_{\theta_\Gamma} = L_{\hat{\theta}_\Gamma}$. So by the functoriality of the pushforward, it is enough to show that $\pi_*\mathbf{s}_*L_{i^*\theta_\Gamma}$, which by definition is

$$(13) \quad \{((\pi \circ \mathbf{s})_*Y, f) \oplus (\xi, g) : (Y, f) \oplus ((\pi \circ \mathbf{s})^*\xi, g) \in L_{i^*\theta_\Gamma}\},$$

equals the Jacobi-Dirac structure given by Λ_P . First we determine which tangent vectors Y to $J_1^{-1}(0)$ and $f \in \mathbb{R}$ have the property that $i^*(d\theta_\Gamma(Y) + f\theta_\Gamma)$ annihilates $\ker(\pi \circ \mathbf{s})_*$, which using equation (12) is equal to $\{X_{f_\Gamma + \mathbf{t}^*\pi^*v} : v \in C^\infty(P)\} \oplus \mathbb{R}v_\Gamma$. A computation similar to those carried out in Step 1 and using the explicit formula $J = 1 - f_\Gamma$ shows that this is the case when $f = 0$ and $\pi_*\mathbf{t}_*Y = 0$, which by a computation similar to (12) amounts to $Y \in \{X_{\mathbf{s}^*\pi^*v} : v \in C^\infty(P)\} \oplus \mathbb{R}v_\Gamma$. These will be exactly the “ Y ” and “ f ” appearing in (13); a short computation using the facts that the source map of $\Gamma_c(Q)$ and π are Jacobi maps shows that (13) equals $\{(-\Lambda_P\xi, 0) \oplus (\xi, g) : \xi \in T^*P, g \in \mathbb{R}\}$, as was to be shown.

Step 3: $((J_1^{-1}(0)/S^1)/\mathbb{Z}, \hat{\theta}_\Gamma)$ is the prequantization of the s.s.c. symplectic groupoid $\Gamma_s(P)$ of P . Here \mathbb{Z} acts as a subgroup of \mathbb{R} by the flow of the Reeb vector field \hat{E}_Γ .

Consider the action on $J_1^{-1}(0)/S^1$ by its Reeb vector field \hat{E}_Γ , which by the contact reduction procedure is the projection of the Reeb vector field E_Γ of $\Gamma_c(Q)$ under $J_1^{-1}(0) \rightarrow J_1^{-1}(0)/S^1$.

The \mathbf{t} -image of a v_Γ orbit is an orbit of the S^1 action on Q , since the target map is S^1 equivariant. Hence each v_Γ orbit meets each \mathbf{t} -fiber at most once. Further each E_Γ -orbit is contained in a single \mathbf{t} -fiber (since $\mathbf{t}_*E_\Gamma = 0$), so an E_Γ orbit meets any orbit of the S^1 action on $\Gamma_c(Q)$ at most once. Therefore the period of an E_Γ orbit and of the corresponding \bar{E}_Γ orbit are equal, and the first period is always an integer number (because $\mathbf{s}_*E_\Gamma = E_Q$, the generator of the circle action on Q).

Now we know that the periods of \bar{E}_Γ are integers, we can just apply Theorems 2 and 3 of [9] to prove our claim. \square

4.2. Path space constructions and the general case. In this subsection we generalize Thm. 4.2 allowing P to be a general Dirac manifold, using the explicit description of Lie groupoids as quotients of path spaces as a powerful tool. The s.s.c. groupoid of any integrable algebroid A can be constructed as the quotient of the A -path space by a foliation \mathcal{F} [8]. Specifically, the precontact groupoid $(\Gamma_c(Q), \theta, f)$ of a Jacobi-Dirac manifold Q can be constructed via the A -path space $P_a(\bar{L})$ with θ and f coming from a corresponding 1-form and function on the path space. We refer to [9] [7] [16] and summarize the results in Thm. 4.4 below. The advantage of this method is that we can use it in Theorems 4.8 and

4.10 to generalize Theorem 4.2 to the setting of Dirac manifolds and one can apply it for a general group G action as in [11].

Precontact groupoids are defined in [16]; we will adapt the definition there to match up the conventions of [9] and [26].

Definition 4.3. A precontact groupoid is a Lie groupoid Γ over M equipped with a 1-form θ_Γ and a function f_Γ satisfying $f_\Gamma(gh) = f_\Gamma(g)f_\Gamma(h)$ and

$$m^*\theta_\Gamma = pr_1^*\theta_\Gamma pr_2^*f_\Gamma + pr_2^*\theta_\Gamma$$

and the non-degeneracy condition

$$\ker \mathbf{t}_* \cap \ker \mathbf{s}_* \cap \ker \theta_\Gamma \cap \ker d\theta_\Gamma = \{0\}.$$

The 1-form θ_Γ gives rise to a Jacobi-Dirac structure on Γ which can be pushed-forward via the source map to obtain a Jacobi-Dirac structure on M .

Theorem 4.4. *The s.s.c. precontact groupoid $(\Gamma_c(Q), \theta_\Gamma, f_\Gamma)$ of an integrable Jacobi-Dirac manifold (Q, \bar{L}) is the quotient space of the A -path space $P_a(\bar{L})$ by A -homotopies, and θ_Γ and f_Γ come from a 1-form $\tilde{\theta}$ and a function \tilde{f} on $P_a(\bar{L})$. At the point $a = (a_4, a_3, a_1, a_0) \in P_a(\bar{L})$, where (a_4, a_3, a_1, a_0) are components in $TQ \oplus \mathbb{R} \oplus T^*Q \oplus \mathbb{R}$, $\tilde{\theta}$ and \tilde{f} are*

$$(14) \quad \begin{aligned} \tilde{\theta}_a(X) &= - \int_0^1 \left\langle e(t)X(t), d \left(\int_0^1 a_0(t)dt \right) \right\rangle dt + \int_0^1 \langle e(t)X(t), pr^*\theta_c \rangle dt, \\ \tilde{f}(a) &= e(1), \quad \text{with } e(t) := e^{\int_0^t -a_3} \end{aligned}$$

where X is a tangent vector to $P_a(\bar{L})$, hence a path itself (parameterized by t), and $pr^*\theta_c$ is the pull-back via $pr : \bar{L} \rightarrow T^*Q$ of the canonical 1-form on T^*Q .

Proof. The equation for \tilde{f} is taken from Prop. 3.5(i) of [9]. It is shown there that \tilde{f} descends to the function f_Γ on $\Gamma_c(Q)$. To get the formula for $\tilde{\theta}$, we recall from Section 3.4 of [9] that the following map ϕ is an isomorphism preserving A -homotopy:

$$\phi : P_a(\bar{L}) \times \mathbb{R} \rightarrow P_a(\bar{L} \times_\psi \mathbb{R}),$$

mapping (a, s) with base path γ_1 to $\tilde{a} := e^{\gamma_0(t)}a$ with base path (γ_1, γ_0) , where $\gamma_0 := s - \int_0^t a_3$. Here ψ is the 1-cocycle on \bar{L} given by $(X, f) \oplus (\xi, g) \mapsto f$; $\bar{L} \times_\psi \mathbb{R}$ is the algebroid on $Q \times \mathbb{R}$ obtained from the algebroid \bar{L} and the 1-cocycle ψ , and it is isomorphic to the algebroid given by the Dirac structure on $Q \times \mathbb{R}$ obtained from the ‘‘Diracization’’ of (Q, \bar{L}) (see Section 2.3 in [16]).

The correspondence on the level of tangent spaces given by $T\phi$ maps $(\delta\gamma_1, \delta s, \delta a)$ to $(\delta\gamma_1, \delta\gamma_0, \delta\tilde{a})$ and satisfies

$$\begin{aligned} \delta\gamma_0 &= \delta s - \int_0^t a_3, \\ \delta\tilde{a}_1 &= e^{\gamma_0}(\delta a_1 + (\delta s - \int_0^t \delta a_3)a_1), \\ \delta\tilde{a}_0 &= e^{\gamma_0}(\delta a_0 + (\delta s - \int_0^t \delta a_3)a_0). \end{aligned}$$

We identify $\bar{L} \times_\psi \mathbb{R}$ with the Dirac structure on $Q \times \mathbb{R}$ induced via Diracization. Then on the whole space $P(\bar{L} \times_\psi \mathbb{R})$ of paths in $\bar{L} \times_\psi \mathbb{R}$ there is a symplectic form ω coming from integrating the pull-back of the canonical symplectic form on $T^*(Q \times \mathbb{R})$ (see Section 5 in [3]). This form restricted to the A -path space $P_a(\bar{L} \times_\psi \mathbb{R})$ is homogeneous w.r.t. the \mathbb{R}

component, i.e. $\varphi_s \omega = e^s \omega$, where φ_s is the flow of $\frac{\partial}{\partial s}$ with s the coordinate of \mathbb{R} . This is because φ_s acts on vector fields $\delta \tilde{a}_1$ and $\delta \tilde{a}_0$ by rescaling by an e^s factor as the formula of $T\phi$ and γ_0 show. This homogeneity survives the quotient to groupoids as shown in [9]. Therefore θ_Γ comes from the 1-form $\tilde{\theta}$ whose associated homogeneous symplectic form is ω , namely $\tilde{\theta} = -i_0^* i(\frac{\partial}{\partial s}) \omega$. With a straightforward calculation and the formula of $T\phi$, we have the formula for $\tilde{\theta}$ in (14). \square

Remark 4.5. The formula for $\tilde{\theta}$ is a generalization of Theorem 4.2 in [7] in the case \bar{L} that comes from a Dirac structure. To get the formula of the 1-form there up to sign⁴, one just has to put $e(t) = 1$ which corresponds to the case that $a_3 = 0$.

In Lemma 2.3, we constructed a Lie algebroid structure on π^*A the pull back via $\pi : Q \rightarrow P$ of any Lie algebroid A on P , provided there is a flat A -connection \tilde{D} on the line bundle K corresponding to Q . (π^*A turns out to be the transformation algebroid w.r.t. the action by the flat connection). Now we show some functorial property of this algebroid π^*A .

Lemma 4.6. *An A -path a in A can be lifted to an A -path in π^*A . The same is true for A -homotopies. In other words, in the following diagram (for $\square^n := [0, 1]^n$, $n = 1, 2$),*

$$\begin{array}{ccccc}
 T\square^n & & & & \\
 \downarrow & \searrow f & & & \\
 \square^n & & \pi^*A & \longrightarrow & A \\
 & \searrow f_0 & \downarrow & & \downarrow \\
 & & Q & \xrightarrow{\pi} & P
 \end{array}$$

any Lie algebroid morphism $f : T\square^n \rightarrow A$ satisfying a suitable boundary condition [4] lifts to a Lie algebroid morphism from $T\square^n$ to π^*A satisfying corresponding boundary conditions.

Proof. Let γ be the base path of an A -path a , and let $\tilde{\gamma}$ be the parallel translation along a of some $\tilde{\gamma}(0) \in \pi^{-1}(\gamma(0))$ as in the proof of Lemma 2.3. Denoting by π^*a the lift of a to π^*A with base path $\tilde{\gamma}$, we have $\rho(\pi^*a) = h_Q(a(\gamma(t)), \tilde{\gamma}(t)) = d/dt(\tilde{\gamma})$, with ρ the anchor of π^*A (see equation (5)). That is, π^*a is an A -path in π^*A over $\tilde{\gamma}$. The lifting of a is not unique. In fact it is decided by the choice of a point in $\pi^{-1}(\gamma(0))$ as initial value.

Now we prove the same statement for A -homotopies. Suppose $a(\epsilon, t)$ is an A -homotopy over $\gamma(\epsilon, t)$, i.e. there exist A -paths (w.r.t. parameter ϵ) $b(\epsilon, t)$ also over γ satisfying⁵,

$$(15) \quad \partial_t b - \partial_\epsilon a = \nabla_{\rho_A b} a - \nabla_{\rho_A a} b + [a, b],$$

and the boundary condition $b(\epsilon, 0) = b(\epsilon, 1) = 0$, for any choice of connection ∇ on TP . As above, we can lift γ to $\tilde{\gamma}(\epsilon, t)$. In fact, once we choose $\tilde{\gamma}(0, 0)$, we can use $\tilde{\gamma}(0, 0)$ to obtain the lift $\tilde{\gamma}(\epsilon, 0)$ and then $\tilde{\gamma}(\epsilon, t)$. (The lift does not depend on whether we lift $\gamma(\epsilon, 0)$ or $\gamma(0, t)$ first, because the connection \tilde{D} is flat). Then π^*a and π^*b are A -paths over $\tilde{\gamma}$ w.r.t. parameters t and ϵ respectively. Moreover, we choose a connection $\tilde{\nabla}$ on Q induced from the connection ∇ on P such that $\tilde{\nabla}_{X^H} Y^H = (\nabla_X Y)^H$, $\tilde{\nabla}_{X^H} E = 0$, $\tilde{\nabla}_E Y^H = 0$

⁴In [7] 1-forms on contact groupoids are so that the target map is a Jacobi map, whereas here we adopt the convention (as in [26]) that the source map be Jacobi.

⁵Strictly speaking, to make sense of the following equation, one needs to extend a and b to time-dependent local sections. For example, $\partial_\epsilon a := \nabla_{\frac{\partial}{\partial \epsilon} \gamma} \eta(\epsilon, x) + \frac{d}{d\epsilon} \eta(\epsilon, x)$, where $\eta(\epsilon, x)$ extending $a(\epsilon, t)$ is a time-dependent local section. The same holds for b . The result is independent of the choice of extension; we refer the reader to [8] for details.

and $\tilde{\nabla}_E E = 0$, where the superscript H denotes the horizontal lift with respect to some connection we fix on the circle bundle $\pi : Q \rightarrow P$. (Since $E(\pi^*f) = 0$ and $X^H(\pi^*f) = X(f)$ these requirements are consistent. In fact, the connection $\tilde{\nabla}$ on $TQ = \pi^*TP \oplus \mathbb{R}E$ is just the sum of the pullback connection on π^*TP and of the trivial connection). Now we will prove that π^*a and π^*b satisfy (15) w.r.t. $\tilde{\nabla}$. Notice that $\langle \pi^*\eta, \tilde{\nabla}_E X \rangle = 0$ for all vector fields X , so we have

$$\tilde{\nabla}_E \pi^* \eta = 0, \quad \tilde{\nabla}_{(\frac{\partial}{\partial \epsilon} \gamma)^H} \pi^* \eta = \pi^* (\nabla_{\frac{\partial}{\partial \epsilon} \gamma} \eta).$$

Therefore $\tilde{\nabla}_{\frac{\partial}{\partial \epsilon} \tilde{\gamma}} \pi^* \eta = \pi^* (\nabla_{\frac{\partial}{\partial \epsilon} \gamma} \eta)$. So $\partial_\epsilon \pi^* a = \pi^* (\partial_\epsilon a)$. The same is true for $\pi^* b$. Moreover, since $\rho(\pi^* a) = (\rho(a))^H + \langle \tilde{\beta}, a \rangle E$ (upon writing \tilde{D} as in equation (4) and denoting by H the horizontal lift w.r.t. $\ker \sigma$), similarly we have $\tilde{\nabla}_{\rho(\pi^* a)} \pi^* b = \pi^* (\nabla_{\rho(a)} b)$ as well as the analog term obtained switching a and b . By the definition of Lie bracket on $\pi^* A$, we also have $[\pi^* a, \pi^* b] = \pi^* ([a, b])$. Therefore a, b satisfying (15) implies that the same equation holds for $\pi^* a$ and $\pi^* b$. The boundary condition $\pi^* b(\epsilon, 0) = \pi^* b(\epsilon, 1) = 0$ is obvious. Hence, $\pi^* a$ is an A -homotopy in $\pi^* A$. \square

Remark 4.7. We claim that all the A -paths and A -homotopies in $\pi^* A$ are of the form $\pi^* a$. Indeed consider a $\pi^* A$ path \hat{a} over a base path $\hat{\gamma}$, i.e. $\rho(\hat{a}(t)) = \frac{d}{dt} \hat{\gamma}(t)$. Let $\gamma := \pi \circ \hat{\gamma}$ and let $a(t)$ be equal to $\hat{a}(t)$, seen as an element of $A_{\gamma(t)}$. The commutativity of

$$\begin{array}{ccc} \pi^* A & \xrightarrow{h_Q = \rho} & TQ \\ \downarrow & & \downarrow \pi_* \\ A & \xrightarrow{\rho_A} & TP \end{array}$$

implies that a is an A -path over γ . Further, the horizontal lift of a starting at $\hat{\gamma}(0)$ satisfies by definition $\frac{d}{dt} \tilde{\gamma}(t) = h_Q(a(\gamma(t)), \tilde{\gamma}(t))$, so it coincides with $\hat{\gamma}$. The same holds for A -homotopies.

The next theorem generalizes the second item of Thm. 4.2.

Theorem 4.8. *Let (P, L) be an integrable prequantizable Dirac manifold and (Q, \bar{L}) one of its prequantization. We denote $[\cdot]_A$ as the A -homotopy class in the Lie algebroid A . Then we have the following results:*

- (1) *there is an S^1 action on the precontact groupoid $\Gamma_c(Q)$ with moment map $J_1 = 1 - f_\Gamma$;*
- (2) *$J_1^{-1}(0)$ is a source connected and simply connected subgroupoid of $\Gamma_c(Q)$ and is isomorphic to the action groupoid $\Gamma_c(P) \times Q \rightrightarrows Q$.*
- (3) *In terms of path spaces,*

$$J_1^{-1}(0) = \{[\pi^* a]_{\bar{L}}\} = \{[\pi^* a]_{\bar{L}_0}\},$$

where a is an A -path in L^c and $\pi^ a$ is defined as in Lemma 4.6 (we identify $\pi^* L^c$ with $\bar{L}_0 \subset \bar{L}$ as in Prop. 2.5). Hence we see that the Lie algebroid of $J_1^{-1}(0)$ is \bar{L}_0 , which by Prop. 3.3 is equal to $J^{-1}(0)$.*

- (4) *the precontact reduction $\Gamma_c(Q) //_0 S^1$ is isomorphic to the s.s.c. contact groupoid $\Gamma_c(P)$ via the inverse of the following map*

$$p : [a]_{L^c} \mapsto [\pi^* a]_{\bar{L}, S^1},$$

where $[\cdot]_{\bar{L}, S^1}$ denotes S^1 equivalence classes of $[\cdot]_{\bar{L}}$.

Remark 4.9. The isomorphism p gives the same contact groupoid structure on $\Gamma_c(Q) //_0 S^1$ as in Theorem 4.2 in the case when P is Poisson.

Proof. 1) The definition of the S^1 action is the same as in Theorem 4.2. J_1 is defined by $J_1(g) = \theta_\Gamma(v_\Gamma(g))$, where v_Γ is induced by the S^1 action on Q hence on \bar{L} . More explicitly, $T(P_a(\bar{L}))$ is a subspace of the space of paths in $T\bar{L}$. If we take a connection ∇ on Q , then $T\bar{L}$ decomposes as $TQ \oplus \bar{L}$. At $(a_4, a_3, a_1, a_0) \in P_a(\bar{L})$ the infinitesimal S^1 action \tilde{v} on the path space is $\tilde{v} = (E(\gamma(t)), *, *, *, 0)$. So

$$J_1([a]) = \tilde{\theta}_a(\tilde{v}) = \int_0^1 \langle a_1(t), E \rangle e^{-\int_0^t \langle a_1, E \rangle dt} dt = - \int_0^1 d(e^{-\int_0^t \langle a_1, E \rangle dt}) = 1 - f_\Gamma.$$

2) By 1) $J_1^{-1}(0) = f_\Gamma^{-1}(1)$. Since f_Γ is multiplicative, it is clear that $f_\Gamma^{-1}(1)$ is a subgroupoid. Moreover using Thm. 4.4 we see that $f_\Gamma^{-1}(1)$ is made up by paths $a = (a_4, a_3, a_1, a_0)$ such that

$$(16) \quad \int_0^1 \langle a_1(t), E \rangle dt = 0.$$

Notice that this are not exactly the same as A -paths in \bar{L}_0 , which are the A -paths such that $\langle a_1(t), E \rangle \equiv 0$ for all $t \in [0, 1]$ (see Prop. 2.5).

Now we show that $J_1^{-1}(0)$ is source connected. Take $g \in \mathfrak{s}^{-1}(x)$, and choose an A -path $a(t)$ representing g over a base path $\gamma(t) : I \rightarrow Q$. We will connect g to x within $J_1^{-1}(0) \cap \mathfrak{s}^{-1}(x)$ in two steps: first we deform g to some other point h which can be represented by an A -path in \bar{L}_0 ; then we “linearly shrink” h to x .

Suppose the vector bundle \bar{L} is trivial on a neighborhood U of the image of γ in Q . Choose a frame $Y_0, \dots, Y_{\dim Q}$ for $\bar{L}|_U$, with the property that $Y_0 = (-A^H, 1) \oplus (\sigma - \pi^*\alpha, 0)$ (with σ, A and α as in Thm. 2.1) and that all other Y_i satisfy $\langle a_1, E \rangle = 0$. In this frame, $a(t) = \sum_{i=0}^{\dim Q} p_i(t) Y_i|_{\gamma(t)}$ for some time-dependent coefficients $p_i(t)$. Define the following section of $\bar{L}|_U$: $Y_{t,\epsilon} = (1 - \epsilon)p_0(t)Y_0 + \sum_{i=1}^{\dim Q} p_i(t)Y_i$. Define a deformation $\gamma(\epsilon, t)$ of $\gamma(t)$ by

$$\frac{d}{dt}\gamma(\epsilon, t) = \rho(Y_{t,\epsilon}), \quad \gamma(\epsilon, 0) = x,$$

where ρ is the anchor of \bar{L} (one might have to extend U to make $\gamma(\epsilon, t) \in U$ for $t \in [0, 1]$). Let $a(\epsilon, t) := Y_{t,\epsilon}|_{\gamma(\epsilon,t)}$. For each ϵ it is an A -path by construction, and $a(0, t) = a(t)$. Using $g \in J_1^{-1}(0)$ (so that $\int_I p_0(t)dt = 0$) we have

$$\int_0^1 \langle a_1(\epsilon, t), E \rangle dt = \int_0^1 \langle (1 - \epsilon)p_0(t)Y_0 + \sum_{i=1}^{\dim Q} p_i(t)Y_i, (E, 0, 0, 0) \rangle_- dt = (1 - \epsilon) \int_I p_0(t)dt = 0,$$

so $[a(\epsilon, \cdot)]$ lies in $J_1^{-1}(0)$. Notice that $a(1, t)$ satisfies $\langle a_1(1, t), E \rangle \equiv 0$ for all t ; hence an A -path in \bar{L}_0 . We denote $h := [a(1, t)]$ and define a continuous map $pr : P_a(\bar{L}|_U) \rightarrow P_a(\bar{L}_0|_U)$ by $a(t) \mapsto a(1, t)$.

Then we can shrink linearly $a(1, t)$ to the zero path, via $a^\delta(1, t) := \delta a(1, \delta t)$ which is an A -path over $\gamma(1, \delta t)$. Taking equivalence classes we obtain a path from h to x , which moreover lies in $J_1^{-1}(0)$ because $\langle a_1(1, t), E \rangle \equiv 0$.

Now we show that $J_1^{-1}(0)$ is source simply connected. If there is a loop $g(s) = [a(1, s, t)]$ in a source fibre of $J_1^{-1}(0)$, then $g(s)$ can shrink to $x := \mathfrak{s}(g(s))$ inside the big (s.s.c.!) groupoid $\Gamma_c(Q)$ via $g(\epsilon, s) = [a(\epsilon, s, t)]$. We can assume $a(\epsilon, s, t) = sa(\epsilon, 1, st)$. This is easy to realize since we can simply take $a(\epsilon, s, t) = g(\epsilon, st)^{-1}d/dt(g(\epsilon, st))$. Then the $a(i, 1, \cdot)$'s are A -paths in \bar{L}_0 for $i = 0, 1$. This is because both $g(s)$ and x are paths in $J_1^{-1}(0)$ which

implies $\int_0^1 sa(i, 1, st) = 0$ for all $s \in [0, 1]$. Moreover the base paths $\gamma(\epsilon, s, t)$ form an embedded disk (one can assume that the deformation $g(\epsilon, s)$ has no self-intersections) in Q . So we can take a simply connected open set (for example a tubular neighborhood of this disk) $U \subset Q$ containing $\gamma(\epsilon, s, t)$. Then $L|_U$ is trivial. Therefore there is a continuous map pr such that $\bar{a}(\epsilon, 1, \cdot) = pr(a(\epsilon, 1, \cdot))$ is an A -path in \bar{L}_0 and $\bar{a}(1, 1, \cdot) = a(1, 1, \cdot)$. Then we can shrink $g(s) = \bar{g}(1, s)$ to $x = \bar{g}(0, s)$ via

$$\bar{g}(\epsilon, s) := [s\bar{a}(\epsilon, 1, st)],$$

which is inside of $J_1^{-1}(0)$ since $\langle \bar{a}_1(\epsilon, 1, t), E \rangle \equiv 0$.

3) To show that $J_1^{-1}(0) = \{[\pi^*a]_{\bar{L}}\}$, we just have to show that an A -path in \bar{L} satisfying (16) is A -homotopic (equivalent) to an A -path lying contained in \bar{L}_0 . Since $J_1^{-1}(0)$ has connected source fibres, given a point $g = [a]$ in $J_1^{-1}(0)$, there is a path $g(t)$ connecting g to $\mathbf{s}(g)$ lying in $J_1^{-1}(0)$. Differentiating $g(t)$ we get an A -path $b(t) = g(t)^{-1}\dot{g}(t)$ which is A -homotopic to a and $sb(st)$ represents the point $g(st) \in J^{-1}(0)$. Therefore $\int_0^1 \langle sb_1(st), E \rangle dt = 0$, for all $s \in [0, 1]$. Hence $\langle b_1(t), E \rangle \equiv 0$ for all $t \in [0, 1]$, i.e. b is a path in \bar{L}_0 .

To further show that $J_1^{-1}(0) = \{[\pi^*a]_{\bar{L}_0}\}$, we only have to show that if two A -paths in \bar{L}_0 are A -homotopic in \bar{L} then they are also A -homotopic in \bar{L}_0 . Let $a(1, \cdot)$ and $a(0, \cdot)$ be two A -paths in \bar{L}_0 , A -homotopic in \bar{L} and representing an element $g \in J_1^{-1}(0)$. Integrate $sa(i, st)$ to get $g(i, t)$ for $i = 0, 1$. Namely we have $sa(i, st) = g(i, s)^{-1} \frac{d}{dt}|_{t=s} g(i, t)$. Then $g(i, t)$ are two paths connecting g and $x := \mathbf{s}(g)$ lying in the subgroupoid $J_1^{-1}(0)$ since $a(i, t)$ are paths in \bar{L}_0 . Since the source fibre of $J_1^{-1}(0)$ is simply connected, there is a homotopy $g(\epsilon, t) \in J_1^{-1}(0)$ linking $g(0, t)$ and $g(1, t)$. So $sa(\epsilon, st) := g(\epsilon, s)^{-1} \frac{d}{dt}|_{t=s} g(\epsilon, t)$ is an A -path in the variable t representing the element $g(\epsilon, s) \in J_1^{-1}(0)$ for every fixed s . Hence $sa(\epsilon, st)$ satisfies (16) for every $s \in [0, 1]$. Therefore $\langle a_1(\epsilon, t), E \rangle \equiv 0$. Then $a(\epsilon, t) \subset \bar{L}_0$ is an A -homotopy between $a(0, t)$ and $a(1, t)$.

Therefore $J_1^{-1}(0)$ is the s.s.c. Lie groupoid integrating $J^{-1}(0) = \bar{L}_0$.

4) First of all, given an A -path a of L^c over the base path γ and a point $\tilde{\gamma}(0)$ over $\gamma(0)$ in Q , we lift it to an A -path π^*a of \bar{L} as described in Lemma 4.6. By the same lemma, we see that (L^c) A -homotopic A -paths in L^c lift to (\bar{L}_0) A -homotopic A -paths in $\pi^*L^c \cong \bar{L}_0 \subset \bar{L}$, so the map p is well defined. Different choices of $\tilde{\gamma}(0)$ give exactly the S^1 orbit of (some choice of) $[\pi^*a]_{\bar{L}}$. Surjectivity of the map p follows from the statement about A -paths in Remark 4.7. Injectivity follows from the fact that $\{[\pi^*a]_{\bar{L}}\} = \{[\pi^*a]_{\bar{L}_0}\}$ in 3) and the statement about A -homotopies in Remark 4.7. \square

Given any integrable Dirac manifold (P, L) , there are two groupoids attached to it. One is the presymplectic groupoid $\Gamma_s(P)$ integrating L ; the other is the precontact groupoid $\Gamma_c(P)$ integrating L^c . In the non-integrable case, these two groupoids still exist as stacky groupoids carrying the same geometric structures (presymplectic and precontact) [18]. In this paper, to simplify the treatment, we view them as topological groupoids carrying the same name and when the topological groupoids are smooth manifolds they have additional presymplectic and precontact structures. We state a theorem generalizing Theorem 2 and 3 in [9] and the result in [2] from the Poisson case to the Dirac case and sketch the proof.

Theorem 4.10. *For a Dirac manifold (P, L) , there is a short exact sequence of topological groupoids*

$$1 \rightarrow \mathcal{G} \rightarrow \Gamma_c(P) \xrightarrow{\tau} \Gamma_s(P) \rightarrow 1,$$

where \mathcal{G} is the quotient of the trivial groupoid $\mathbb{R} \times P$ by a group bundle \mathcal{P} over P defined by

$$\mathcal{P}_x := \left\{ \int_{[\gamma]} \omega_F : [\gamma] \in \pi_2(F, x) \text{ and } \gamma \text{ is the base of an } A\text{-homotopy between paths representing } 1_x \text{ in } L. \right\},$$

with F the presymplectic leaf passing through $x \in P$ and ω_F the presymplectic form on F . In the case that (P, L) is integrable as a Dirac manifold, then

(1) the presymplectic form Ω on $\Gamma_s(P)$ is related to the precontact form θ on $\Gamma_c(P)$ by

$$\tau^* d\theta = \Omega,$$

and the infinitesimal action R of \mathbb{R} on $\Gamma_c(P)$ via $\mathbb{R} \times P \rightarrow \mathcal{G}$ satisfies

$$\mathcal{L}_R \theta = 0, \quad i(R)\theta = 1.$$

- (2) R is the left invariant vector field extending the section $(0, 0) \oplus (0, -1)$ of $L^c \subset \mathcal{E}^1(P)$ as in Cor. 5.2;
(3) the group \mathcal{P}_x is generated by the period of R ;
(4) $\Gamma_s(P)$ is prequantizable iff $\mathcal{P} \subset P \times \mathbb{Z}$; in this case, the prequantization is $\Gamma_c(P)/\mathbb{Z}$, where the \mathbb{Z} viewed as a subgroup of \mathbb{R} acts on $\Gamma_c(P)$.
(5) If P is prequantizable as a Dirac manifold, then $\Gamma_s(P)$ is prequantizable.

Proof. The proof of (1) and (4) is the same as Section 4 of [9]. One only has to replace the Poisson bivector π by Υ and the leaf-wise symplectic form of π by ω_F . (3) is clear since R generates the \mathbb{R} action and $\mathcal{G} = \mathbb{R}/\mathcal{P}$.

For (2), we identify $(0, 0) \oplus (0, -1)$ with a section of $\ker \mathbf{t}_*$ using Lemma 5.1 and then extend it to a left invariant vector field on $J^{-1}(0)/S^1$. Using Cor. 5.2 we see that the resulting vector field is killed by \mathbf{s}_* , \mathbf{t}_* and $d\theta_\Gamma$ and that it pairs to 1 with θ_Γ , so by the “non-degeneracy” condition in Def. 4.3 it must be equal to R .

For (5), if P is prequantizable as a Dirac manifold, then $\Upsilon = \rho^* \Omega + d_L \beta$ for some integral form Ω on P and $\beta \in \Gamma(L^*)$. Suppose $f = ade + bdt$ is an algebroid homomorphism from the tangent bundle $T\Box$ of a square $[0, 1] \times [0, 1]$ to L over the base map $\gamma : \Box \rightarrow P$, i.e. $a(\epsilon, t)$ is an A -homotopy over γ via $b(\epsilon, t)$ as in (15). Denoting by ω_F the presymplectic form of the leaf F in which $\gamma(\Box)$ lies, we have (see also Sect. 3.3 of [2]),

$$\begin{aligned} \int_\gamma \omega_F &= \int_\Box \omega_F \left(\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial \epsilon} \right) = \int_\Box \langle ade\epsilon, bdt \rangle_- = \int_\Box f^* \Upsilon \\ &= \int_\Box f^* (\rho^* \Omega + d_L \beta) = \int_\Box f^* (\rho^* \Omega) = \int_\Box \gamma^* \omega = \int_\gamma \omega \in \mathbb{Z} \end{aligned}$$

where we used $\Upsilon = \rho^* \omega_F$ in the second equation and $f^* d_L \beta = d_{dR}(f^* \beta)$ in the fifth. \square

4.3. Two examples. We present two explicit examples for Thm. 4.2 and 4.8.

The first one generalizes Example 4.1.

Example 4.11. Let (P, ω) be an integral symplectic manifold (non necessarily simply connected), and (Q, θ) a prequantization. The s.s.c. contact groupoid of (Q, θ) is $(\bar{Q} \times_{\pi_1(Q)} \bar{Q} \times \mathbb{R}, -e^{-s}\theta_1 + \theta_2, e^{-s})$ where \bar{Q} denotes the universal cover of Q . As in Example 4.1 the moment map is given by $J_1 = -e^{-s} + 1$ and the reduced manifold at zero is $((\bar{Q} \times_{\pi_1(Q)} \bar{Q})/S^1, [-\theta_1 + \theta_2])$, where $\pi_1(Q)$ acts diagonally and the diagonal S^1 action is realized by following the Reeb vector field on \bar{Q} .

Notice that the Reeb vector field of $(\bar{Q} \times_{\pi_1(Q)} \bar{Q})/S^1$ is the Reeb vector field of the second copy of \bar{Q} . Dividing \bar{Q} by $\mathbb{Z} \subset (\text{Flow of Reeb v.f.})$ is the same as dividing by the $\pi_1(\tilde{Q})$ action on \bar{Q} , where \tilde{Q} is the pullback of $Q \rightarrow P$ via the universal covering $\tilde{P} \rightarrow P$. To see this use that $\pi_1(\tilde{Q})$ is generated by any of its Reeb orbits (look at the long exact sequence corresponding to $S^1 \rightarrow \tilde{Q} \rightarrow \tilde{P}$), and that the Reeb vector field of \bar{Q} is obtained lifting the one on \tilde{Q} . Also notice that $\pi_1(\tilde{Q})$ embeds into $\pi_1(Q)$ (as the subgroup generated by the Reeb orbits of Q) and that the quotient by the embedded image is isomorphic to $\pi_1(P)$, by the long exact sequence for $S^1 \rightarrow Q \rightarrow P$. So the quotient of $(\bar{Q} \times_{\pi_1(Q)} \bar{Q})/S^1$ by the $\pi_1(\tilde{Q})$ action on the second factor is $(\tilde{Q} \times_{\pi_1(P)} \tilde{Q})/S^1$ where we used $\bar{Q}/\pi_1(\tilde{Q}) = \tilde{Q}$ on each factor. This groupoid, together with the induced 1-form $[-\theta_1 + \theta_2]$, is clearly the prequantization of the s.s.c. symplectic groupoid $(\tilde{P} \times_{\pi_1(P)} \tilde{P}, -\omega_1 + \omega_2)$ of (P, ω) .

In the second example we consider a Lie algebra \mathfrak{g} . Its dual \mathfrak{g}^* is endowed with a linear Poisson structure Λ , called Lie-Poisson structure, and the Euler vector field A satisfies $\Lambda = -d_\Lambda A$. So the prequantization condition (2) for $(\mathfrak{g}^*, \Lambda)$ is satisfied, with $\Omega = 0$ and $\beta = A$. We display the contact groupoid integrating the induced prequantization (Q, \bar{L}) for the simple case that \mathfrak{g} be one dimensional; then we show that (a discrete quotient of) the S^1 contact reduction of this groupoid is the prequantization of the symplectic groupoid of \mathfrak{g}^* .

Example 4.12. Let $\mathfrak{g} = \mathbb{R}$ be the one-dimensional Lie algebra. We claim that the prequantization $Q = S^1 \times \mathfrak{g}^*$ of \mathfrak{g}^* as above has as a s.s.c. contact groupoid $\Gamma_c(Q)$ the quotient of

$$(17) \quad (\mathbb{R}^5, x d\epsilon - e^t d\theta_1 + d\theta_2, e^t)$$

by the diagonal \mathbb{Z} action on the variables (θ_1, θ_2) . Here the coordinates on the five factors of \mathbb{R}^5 are $(\theta_1, t, \epsilon, \theta_2, x)$. The groupoid structure is the product of the following three groupoids: $\mathbb{R} \times \mathbb{R} = \{(\theta_1, \theta_2)\}$ the pair groupoid; $\mathbb{R} \times \mathbb{R} = \{(t, x)\}$ the action groupoid given by the flow of the vector field $-x\partial_x$ on \mathbb{R} , i.e. $(t', e^{-t}x) \cdot (t, x) = (t' + t, x)$; and $\mathbb{R} = \{\epsilon\}$ the group.

To see this, first determine the prequantization of $(\mathfrak{g}^*, \Lambda)$: it is $Q = S^1 \times \mathbb{R}$ with Jacobi structure $(E \wedge x\partial_x, E)$, where $E = \partial_\theta$ is the infinitesimal generator of the circle action and $x\partial_x$ is just the Euler vector field on \mathfrak{g}^* (see [5]). This Jacobi manifold has two open leaves, and we first focus on one of them, say $Q_+ = S^1 \times \mathbb{R}_+$. This is a locally conformal symplectic leaf, with structure $(d\theta \wedge \frac{dx}{x}, \frac{dx}{x})$.

We determine the s.s.c contact groupoid $\Gamma_c(Q_+)$ of $(Q_+, d\theta \wedge \frac{dx}{x}, \frac{dx}{x})$ applying Lemma 6.1 (choosing $\tilde{g} = \log x$, so that $e^{-\tilde{g}}\tilde{\Omega} = d(x^{-1}d\theta)$ there). We obtain the quotient of

$$(\tilde{Q}_+ \times \mathbb{R} \times \tilde{Q}_+, x_2 d\epsilon - \frac{x_2}{x_1} d\theta_1 + d\theta_2, \frac{x_2}{x_1})$$

by the diagonal \mathbb{Z} action on the variables (θ_1, θ_2) . Here (θ_i, x_i) are the coordinates on the two copies of the universal cover $\tilde{Q}_+ \cong \mathbb{R} \times \mathbb{R}_+$ and ϵ is the coordinate on the \mathbb{R} factor. The groupoid structure is given by the product of the pair groupoid over \tilde{Q}_+ and group \mathbb{R} . This contact groupoid, and the one belonging to $Q_- = S^1 \times \mathbb{R}_-$, will sit as open contact subgroupoids in the contact groupoid of Q , and the question is how to “complete” the disjoint union of $\Gamma_c(Q_+)$ and $\Gamma_c(Q_-)$ to obtain the contact groupoid of Q . A clue comes from the simplest case of groupoid with two open orbits and a closed one to separate them, namely the transformation groupoid of a vector field on \mathbb{R} with exactly one zero. The transformation groupoid associated to $-x\partial_x$ is $\mathbb{R} \times \mathbb{R} = \{(t, x)\}$ with source given by x , target given by $e^{-t}x$ and multiplication $(t', e^{-t}x) \cdot (t, x) = (t' + t, x)$. Notice that, on each of the two open

orbits \mathbb{R}_+ and \mathbb{R}_- the groupoid is isomorphic to a pair groupoid by the correspondence $(t, x) \in \mathbb{R} \times \mathbb{R}_\pm \mapsto (e^{-t}x, x) \in \mathbb{R}_\pm \times \mathbb{R}_\pm$, with inverse $(x_1, x_2) \mapsto (\log(\frac{x_2}{x_1}), x_2)$.

Now we embed $\Gamma_c(Q_+)$ into the groupoid $\Gamma_c(Q)$ described in (17) by the mapping

$$(\theta_1, x_1, \epsilon, \theta_2, x_2) \mapsto \left(\theta_1, t = \log\left(\frac{x_2}{x_1}\right), \epsilon, \theta_2, x = x_2 \right),$$

and similarly for $\Gamma_c(Q_-)$. The contact forms and function translate to those indicated in (17), which as a consequence also satisfy the multiplicativity condition. One checks directly that the one form is a contact form also on the complement $\{x = 0\}$ of the two open subgroupoids. Therefore the one described in (17) is a contact groupoid, and since we know that the source map is a Jacobi map on the open dense set sitting over Q_+ and Q_- , it is the contact groupoid of $(Q, E \wedge x\partial_x, E)$.

Now we consider the S^1 contact reduction of the above s.s.c. groupoid $\Gamma_c(Q)$. As shown in the proof of Theorem 4.2 the moment map is $J_1 = 1 - f_\Gamma = 1 - e^t$, so its zero level set is $\{t = 0\}$. The definition of moment map and the fact that the infinitesimal generator v_Γ of the S^1 action projects to E both via source and via target imply that on $\{t = 0\}$ we have $v_\Gamma = (\partial_{\theta_1}, 0, 0, \partial_{\theta_2}, 0)$. So $J^{-1}(0)/S^1$ is \mathbb{R}^3 with coordinates $(\theta := \theta_2 - \theta_1, \epsilon, x)$, 1-form $d\theta + x d\epsilon$, source and target both given by x and groupoid multiplication given by addition in the θ and ϵ factors. Upon division of the θ factor by \mathbb{Z} (notice that the Reeb vector field of $\Gamma_c(Q)$ is ∂_{θ_2}) this is clearly just the prequantization of $T^*\mathbb{R}$, endowed with the canonical symplectic form $dx \wedge d\epsilon$ and fiber addition as groupoid multiplication, i.e. the prequantization of the symplectic groupoid of the Poisson manifold $(\mathbb{R}, 0)$.

5. APPENDIX I

Lemma 5.1. *Let $(\Gamma, \theta_\Gamma, f_\Gamma)$ be a precontact groupoid (as in Definition 4.3) over the Jacobi-Dirac manifold (Q, \bar{L}) , so that the source map be a Jacobi-Dirac map. Then a Lie algebroid isomorphism between $\ker \mathbf{s}_*|_Q$ and \bar{L} is given by*

$$(18) \quad Y \mapsto (\mathbf{t}_*Y, -r_{\Gamma_*}Y) \oplus (-d\theta_\Gamma(Y)|_{TQ}, \theta_\Gamma(Y))$$

where $e^{-r_\Gamma} = f_\Gamma$. An algebroid isomorphism between $\ker \mathbf{t}_*|_Q$ and \bar{L} (obtained composing the above with i_* for i the inversion) is

$$(19) \quad Y \mapsto (\mathbf{s}_*Y, r_{\Gamma_*}Y) \oplus (d\theta_\Gamma(Y)|_{TQ}, -\theta_\Gamma(Y))$$

Proof. Consider the groupoid $\Gamma \times \mathbb{R}$ over $Q \times \mathbb{R}$ with target map $\tilde{\mathbf{t}}(g, t) = (\mathbf{t}(g), t - r_\Gamma(g))$ and the obvious source $\tilde{\mathbf{s}}$ and multiplication. $(\Gamma \times \mathbb{R}, d(e^t\theta_\Gamma))$ is then a presymplectic groupoid with the property that $\tilde{\mathbf{s}}$ is a forward Dirac map onto $(Q \times \mathbb{R}, \tilde{L})$, where

$$\tilde{L}_{(q,t)} = \{(X, f) \oplus e^t(\xi, g) : (X, f) \oplus (\xi, g) \in L_q\}$$

is the ‘‘Diracization’’ ([24][16]) of the Jacobi-Dirac structure \bar{L} and t is the coordinate on \mathbb{R} . In the special case that \bar{L} corresponds to a Jacobi structure this is just Prop. 2.7 of [9]; in the general case (but assuming different conventions for the multiplicativity of θ_Γ and for which of source and target is a Jacobi-Dirac map) this is Prop. 3.3 in [16]. We will prove only the first isomorphism above (the one for $\ker \mathbf{s}_*|_Q$); the other one follows by composing the first isomorphism with i_* . Now we consider the following diagram of spaces of sections

(on the left column we have sections over Q , on the right column sections over $Q \times \mathbb{R}$):

$$\begin{array}{ccc} \Gamma(\ker \mathbf{s}_*|_Q) & \xrightarrow{\Phi_{\mathbf{s}}} & \Gamma(\ker \tilde{\mathbf{s}}_*|_{Q \times \mathbb{R}}) \\ \downarrow & & \Phi \downarrow \\ \Gamma(\bar{L}) & \xrightarrow{\Phi_L} & \tilde{L}. \end{array}$$

The first horizontal arrow $\Phi_{\mathbf{s}}$ is $Y \mapsto \tilde{Y}$, where the latter denotes the constant extension of Y along the \mathbb{R} direction of the base $Q \times \mathbb{R}$. Notice that the projection $pr : \Gamma \times \mathbb{R} \rightarrow \Gamma$ is a groupoid morphism, so it induces a surjective algebroid morphism $pr_* : \ker \tilde{\mathbf{s}}_*|_{Q \times \mathbb{R}} \rightarrow \ker \mathbf{s}_*|_Q$. Since sections \tilde{Y} as above are projectable, by Prop. 4.3.8. in [15] we have $pr_*[\tilde{Y}_1, \tilde{Y}_2] = [Y_1, Y_2]$, and since pr_* is a fiberwise isomorphism we deduce that $\Phi_{\mathbf{s}}$ is a bracket-preserving map.

The vertical arrow Φ is induced from the following isomorphism of Lie algebroids (Cor. 4.8 iii of [3]⁶) valid for any presymplectic manifold $(\tilde{\Gamma}, \Omega)$ over a Dirac manifold (N, \bar{L}) for which the source map is Dirac:

$$\ker \tilde{\mathbf{s}}_*|_N \rightarrow \tilde{L}, \quad Z \mapsto (\tilde{\mathbf{t}}_*Z, -\Omega(Z)|_{TN}).$$

In our case, as mentioned above, the presymplectic form is $d(e^t\theta_{\Gamma})$.

The second horizontal arrow Φ_L is the natural map

$$(X, f) \oplus (\xi, g) \in L_q \mapsto (X, f) \oplus e^t(\xi, g) \in \bar{L}_{(q,t)}$$

which preserves the Lie algebroid bracket (see the remarks after Definition 3.2 of [24]).

One can check that $(\Phi \circ \Phi_{\mathbf{s}})(Y) = (\tilde{\mathbf{t}}_*\tilde{Y}) \oplus (-d(e^t\theta_{\Gamma})(\tilde{Y})|_{TQ \times \mathbb{R}})$ lies in the image of the injective map Φ_L . The resulting map from $\Gamma(\ker \mathbf{s}_*)$ to $\Gamma(\bar{L})$ is given by (18) and the arguments above show that this map preserves brackets. Further it is clear that this map of sections is induced by a vector bundle morphism given by the same formula, which clearly preserves not only the bracket of sections but also the anchor, so that the map $\ker \mathbf{s}_*|_Q \rightarrow \bar{L}$ given by (18) is a Lie algebroid morphism.

To show that it is an isomorphism one can argue noticing that $\ker \mathbf{s}_*$ and \bar{L} have the same dimension and show that the vector bundle map is injective, by using the “non-degeneracy condition” in Def. 4.3 and the fact that the source and target fibers of $\Gamma \times \mathbb{R}$ are pre-symplectic orthogonal to each other. \square

The vector bundle morphisms in the above lemma give a characterization of vectors tangent to the \mathbf{s} or \mathbf{t} fibers of a precontact groupoid as follows. Consider for instance a vector λ in \bar{L}_x , where \bar{L} is the Jacobi-Dirac structure on the base Q . This vector corresponds to some $Y_x \in \ker \mathbf{t}_*$ by the isomorphism (19), and by left translation we obtain a vector field Y tangent to $\mathbf{t}^{-1}(x)$. Of course, every vector tangent to $\mathbf{t}^{-1}(x)$ arises in this way for a unique λ . The vector field Y satisfies the following equations at every point g of $\mathbf{t}^{-1}(x)$, which follow by simple computation from the multiplicativity of θ_{Γ} : $\theta_{\Gamma}(Y_g) = \theta_{\Gamma}(Y_x)$, $d\theta_{\Gamma}(Y_g, Z) = d\theta_{\Gamma}(Y_x, \mathbf{s}_*Z) - r_{\Gamma_*}Y_x \cdot \theta_{\Gamma}(Z)$ for all $Z \in T_g\Gamma$, $r_{\Gamma_*}Y_g = r_{\Gamma_*}Y_x$ and $\mathbf{s}_*Y_g = \mathbf{s}_*Y_x$. Notice that the right hand sides of this properties can be expressed in terms of the four components of $\lambda \in \mathcal{E}^1(Q)$, and that by the “non-degeneracy” of θ_{Γ} these properties are enough to uniquely determine Y_g . We sum up this discussion into the following corollary, which can be used as a tool in computations on precontact groupoids in the same way that

⁶In [3] the authors adopted the convention that the target map be a Dirac map. Here we use their result applied to the pre-symplectic form $-\Omega$.

hamiltonian vector fields are used on contact or symplectic groupoids (such as the proof of Thm. 4.2):

Corollary 5.2. *Let $(\Gamma, \theta_\Gamma, f_\Gamma)$ be a precontact groupoid (as in Definition 4.3) and denote by \bar{L} the Jacobi-Dirac structure on the base Q so that source map is Jacobi-Dirac. Then there is bijection between sections of \bar{L} and vector fields on Γ which are tangent to the \mathfrak{t} -fibers and are left invariant. To a section $(X, f) \oplus (\xi, g)$ of $\bar{L} \subset \mathcal{E}^1(Q)$ corresponds the unique vector field Y tangent to the \mathfrak{t} -fibers which satisfies*

- $\theta_\Gamma(Y) = -g$
- $d\theta_\Gamma(Y) = \mathfrak{s}^*\xi - f\theta_\Gamma$
- $\mathfrak{s}_*Y = X$.

Y furthermore satisfies $r_{\Gamma_*}Y = f$.

6. APPENDIX II

A locally conformal symplectic (l.c.s.) manifold is a manifold (Q, Ω, ω) where Ω is a non-degenerate 2-form and ω is a closed 1-form satisfying $d\Omega = \omega \wedge \Omega$. Any Jacobi manifold is foliated by contact and l.c.s. leaves (see for example [26]); in particular a l.c.s. manifold is a Jacobi manifold, and hence, when it is integrable, it has an associated s.s.c. contact groupoid. In this appendix we will construct explicitly this groupoid; we make use of it in Example 4.12.

Lemma 6.1. *Let (Q, Ω, ω) a locally conformal symplectic manifold. Consider the pullback structure on the universal cover $(\tilde{Q}, \tilde{\Omega}, \tilde{\omega})$, and write $\tilde{\omega} = d\tilde{g}$. Then Q is integrable as a Jacobi manifold iff the symplectic form $e^{-\tilde{g}}\tilde{\Omega}$ is a multiple of an integer form. In that case, choosing \tilde{g} so that $e^{-\tilde{g}}\tilde{\Omega}$ is integer, the s.s.c. contact groupoid of (Q, Ω, ω) is the quotient of*

$$(20) \quad \left(\tilde{R} \times_{\mathbb{R}} \tilde{R}, e^{\tilde{\mathfrak{s}}^*\tilde{g}}(-\tilde{\sigma}_1 + \tilde{\sigma}_2), \frac{e^{\tilde{\mathfrak{s}}^*\tilde{g}}}{e^{\tilde{\mathfrak{t}}^*\tilde{g}}} \right),$$

a groupoid over \tilde{Q} , by a natural $\pi_1(Q)$ action. Here $(\tilde{R}, \tilde{\sigma})$ is the universal cover (with the pullback 1-form) of a prequantization (R, σ) of $(\tilde{Q}, e^{-\tilde{g}}\tilde{\Omega})$, and the group \mathbb{R} acts by the diagonal lift of the S^1 action on R .

Proof. Using for example the algebroid integrability criteria of [8], one sees that (Q, Ω, ω) is integrable as a Jacobi manifold iff $(\tilde{Q}, \tilde{\Omega}, \tilde{\omega})$ is. Lemma 1.5 in Appendix I of [26] states that, given a contact groupoid, multiplying the contact form by \mathfrak{s}^*u and the multiplicative function by $\frac{\mathfrak{s}^*u}{\mathfrak{t}^*u}$ gives another contact groupoid, for any non-vanishing function u on the base. Such an operation corresponds to twisting the groupoid, viewed just as a Jacobi manifold, by the function \mathfrak{s}^*u^{-1} , hence the Jacobi structure induced on the base by the requirement that the source be a Jacobi map is the twist of the original one by u^{-1} . So $(\tilde{Q}, \tilde{\Omega}, \tilde{\omega})$ is integrable iff the symplectic manifold $(\tilde{Q}, e^{-\tilde{g}}\tilde{\Omega})$ is Jacobi integrable, and by Section 7 of [9] this happens exactly when the class of $e^{-\tilde{g}}\tilde{\Omega}$ is a multiple of an integer one.

Choose \tilde{g} so that this class is actually integer. A contact groupoid of $(\tilde{Q}, e^{-\tilde{g}}\tilde{\Omega})$ is clearly $(\tilde{R} \times_{S^1} \tilde{R}, [-\sigma_1 + \sigma_2], 1)$, where the S^1 action on $\tilde{R} \times \tilde{R}$ is diagonal and “[]” denotes the form descending from $\tilde{R} \times \tilde{R}$. This groupoid is not s.s.c.; the s.s.c. one is $\tilde{R} \times_{\mathbb{R}} \tilde{R}$, where the \mathbb{R} action on \tilde{R} is the lift of the S^1 action on \tilde{R} . The source simply connectedness follows since \mathbb{R} acts transitively (even though not necessarily freely) on each fiber of the map $\tilde{R} \rightarrow \tilde{Q}$, and

this in turns holds because any S^1 orbit in R generates $\pi_1(R)$ and because the fundamental group of a space always acts (by lifting loops) transitively on the fibers of its universal cover.

By the above cited Lemma from [26] we conclude that (20) is the s.s.c. contact groupoid of $(\tilde{Q}, \tilde{\Omega}, \tilde{\omega})$. The fundamental group of Q acts on \tilde{Q} respecting its geometric structure, so it acts on its algebroid $T^*\tilde{Q} \times \mathbb{R}$. Since the path-space construction of the s.s.c. groupoid is canonical (see Subsection 4.2), $\pi_1(Q)$ acts on the s.s.c. groupoid (20) preserving the groupoid and geometric structure. Hence the quotient is a s.s.c. contact groupoid over (Q, Ω, ω) , and its source map is a Jacobi map, so it is the s.s.c. contact groupoid of (Q, Ω, ω) . \square

7. APPENDIX III

In this Appendix, we describe an alternative attempt to derive the geometric structure on the circle bundles Q from a prequantizable Dirac manifold (P, L) and a suitable choice of connection D . Even though we can make our construction work only if we start with a symplectic manifold, we believe the construction is interesting on its own right.

First we recall Vorobjev's construction in Section 4 of [20], which the author there uses to study the linearization problem of Poisson manifolds near a symplectic leaf. Consider a transitive algebroid A over a base P with anchor ρ ; the kernel $\ker \rho$ is a bundle of Lie algebras. Choose a splitting $\gamma : TP \rightarrow A$ of the anchor. Its curvature R_γ is a 2-form on P with values in $\Gamma(\ker \rho)$ (given by $R_\gamma(v, w) = [\gamma v, \gamma w]_A - \gamma[v, w]$). The splitting γ also induces a (TP-)covariant derivative ∇ on $\ker \rho$ by $\nabla_v s = [\gamma v, s]_A$. Now, if P is endowed with a symplectic form ω , a neighborhood of the zero section in $(\ker \rho)^*$ inherits a Poisson structure $\Lambda_{vert} + \Lambda_{hor}$ as follows (Theorem 4.1 in [20]): denoting by F_s the fiberwise linear function on $(\ker \rho)^*$ obtained by contraction with the section s of $\ker \rho$, the Poisson bivector has a vertical component determined by $\Lambda_{vert}(dF_{s_1}, dF_{s_2}) = F_{[s_1, s_2]}$. It also has a component Λ_{hor} which is tangent to the Ehresmann connection Hor given by the dual connection⁷ to ∇ on the bundle $(\ker \rho)^*$; Λ_{hor} at $e \in (\ker \rho)^*$ is obtained by restricting the non-degenerate form $\omega - \langle R_\gamma, e \rangle$ to Hor_e and inverting it. (Here we are identifying Hor_e and the corresponding tangent space to P .)

To apply Vorobjev's construction in our setting, let (P, ω) be a prequantizable symplectic manifold and (K, ∇_K) its prequantization line bundle with Hermitian connection of curvature $2\pi i\omega$. By Lemma 2.2 we obtain a flat $TP \oplus_\omega \mathbb{R}$ -connection $\tilde{D}_{(X, f)} = \nabla_X + 2\pi i f$ on K . Now we make use of the following well know fact about extensions, which can be proven by direct computation:

Lemma 7.1. *Let A be a Lie algebroid over M , V a vector bundle over M , and \tilde{D} a flat A -connection on V . Then $A \oplus V$ becomes a Lie algebroid with the anchor of A as anchor and bracket*

$$[(X_1, s_1), (X_2, s_2)] = ([X_1, X_2]_A, \tilde{D}_{X_1} s_2 \tilde{D}_{X_2} s_1).$$

Therefore $A := TP \oplus_\omega \mathbb{R} \oplus K$ is a transitive Lie algebroid over P , with isotropy bundle $\ker \rho = \mathbb{R} \oplus K$ and bracket $[(f_1, S_1), (f_2, S_2)] = [(0, 2\pi i(f_1 S_2 - f_2 S_1))]$ there. Now choosing the canonical splitting γ of the anchor $TM \oplus_\omega \mathbb{R} \oplus K \rightarrow TM$ we see that its curvature is $R_\gamma(X_1, X_2) = (0, \omega(X_1, X_2), 0)$. The horizontal distribution on the dual of the isotropy bundle is the product of the trivial one on \mathbb{R} and of the one corresponding to ∇_K on K (upon identification of K and K^* by the metric). By the above, there is a Poisson structure on $\mathbb{R} \oplus K$, at least near the zero section: the Poisson bivector at (t, q) has a horizontal component given by lifting the inverse of $(1-t)\omega$ and a vertical component which turns out

⁷In [20] the author phrases this condition as $\mathcal{L}_{hor(X)} F_s = F_{\nabla_X s}$.

to be $2\pi(iq\partial_q) \wedge dt$, where “ $iq\partial_q$ ” denotes the vector field tangent to the circle bundles in K obtained by turning by 90° the Euler vector field $q\partial_q$. A symplectic leaf is clearly given by $\{t < 1\} \times Q$ (where $Q = \{|q| = 1\}$). On this leaf the symplectic structure is seen to be given by $(1-t)\omega + \theta \wedge dt = d((1-t)\theta)$, where θ is the connection 1-form on Q corresponding to the connection ∇_K on K (which by definition satisfies $d\theta = \pi^*\omega$). This means that the leaf is just the symplectification $(\mathbb{R}_+ \times Q, d(r\theta))$ of (Q, θ) (here $r = 1-t$), which is a “prequantization space” for our symplectic manifold (P, ω) . Unfortunately we are not able to modify Vorobjev’s construction appropriately when P is a Poisson or Dirac manifold.

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