The nonlinear membrane energy: Variational derivation under the constraint "det u 0"

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Abstract: Acerbi, Buttazzo and Percivale gave a variational definition of the nonlinear string energy under the constraint "det u>0" (see [E. Acerbi, G. Buttazzo, D. Percivale, A variational definition of the strain energy for an elastic string, J. Elasticity 25 (1991) 137–148]). In the same spirit, we obtain the nonlinear membrane energy under the simpler constraint "det u 0".

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THE NONLINEAR MEMBRANE ENERGY: VARIATIONAL DERIVATION UNDER THE CONSTRAINT "det$\nabla u > 0$"

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Abstract. In [4] we gave a variational definition of the nonlinear membrane energy under the constraint "det$\nabla u \neq 0$". In this paper we obtain the nonlinear membrane energy under the more realistic constraint "det$\nabla u > 0$".

1. Introduction

Consider an elastic material occupying in a reference configuration the bounded open set $\Sigma_\varepsilon \subset \mathbb{R}^3$ given by

$$\Sigma_\varepsilon := \Sigma \times \Big]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\Big[,$$

where $\varepsilon > 0$ is very small and $\Sigma \subset \mathbb{R}^2$ is Lipschitz, open and bounded. A point of Σ_ε is denoted by (x, x_3) with $x \in \Sigma$ and $x_3 \in]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$. Let $W : \mathbb{M}^{3\times 3} \to [0, +\infty]$ be the stored-energy function supposed to be *continuous* and *coercive*, i.e., $W(F) \geq C|F|^p$ for all $F \in \mathbb{M}^{3\times 3}$ and some $C > 0$. In order to take into account the important physical properties that the interpenetration of matter does not occur and that an infinite amount of energy is required to compress a finite volume into zero volume, i.e.,

$$W(F) \to +\infty \text{ as } \det F \to 0,$$

where $\det F$ denotes the determinant of the 3×3 matrix F, we assume that:

(C_1) $W(F) = +\infty$ *if and only if* $\det F \leq 0$;
(C_2) *for every* $\delta > 0$, *there exists* $c_\delta > 0$ *such that for all* $F \in \mathbb{M}^{3\times 3}$,

$$\text{if } \det F \geq \delta \text{ then } W(F) \leq c_\delta(1 + |F|^p).$$

Our goal is to show that as $\varepsilon \to 0$ the three-dimensional free energy functional $E_\varepsilon : W^{1,p}(\Sigma_\varepsilon; \mathbb{R}^3) \to [0, +\infty]$ (with $p > 1$) defined by

$$E_\varepsilon(u) := \frac{1}{\varepsilon}\int_{\Sigma_\varepsilon} W\big(\nabla u(x, x_3)\big)dxdx_3 \tag{1}$$

converges in a variational sense (see Definition 2.1) to the two-dimensional free energy functional $E_{\rm mem} : W^{1,p}(\Sigma; \mathbb{R}^3) \to [0, +\infty]$ given by

$$E_{\rm mem}(v) := \int_\Sigma W_{\rm mem}\big(\nabla v(x)\big)dx \tag{2}$$

with $W_{\rm mem} : \mathbb{M}^{3\times 2} \to [0, +\infty]$. Usually, $E_{\rm mem}$ is called the nonlinear membrane energy associated with the two-dimensional elastic material with respect to the reference configuration Σ. Furthermore we wish to give a representation formula for $W_{\rm mem}$.

To our knowledge, the problem of giving a variational definition of the nonlinear membrane energy was studied for the first time by Percivale in [18]. His paper deals with the constraint "det$\nabla u>0$" but seems to contain some mistakes. It never was published. Then, in [17] Le Dret and Raoult treated the simpler case where W is of p-polynomial growth, i.e., $W(F)\leq c(1+|F|^p)$ for all $F\in\mathbb{M}^{3\times 3}$ and some $c>0$. Later, in [8, Theorem 1] Ben Belgacem announced to have succeed to handle the constraint "det$\nabla u>0$". In [9], which is the paper corresponding to the note [8], the statement [8, Theorem 1] is partly proved, but a more complete proof can be found in his thesis [7]. Recently, in [4] we gave a variational definition of the nonlinear membrane energy under the constraint "det$\nabla u\neq 0$". In the present paper, by the same method as in [4], we obtain the nonlinear membrane energy under the more realistic constraint "det$\nabla u>0$".

An outline of the paper is as follows. The variational convergence of E_ε to $E_{\rm mem}$ as $\varepsilon\to 0$ as well as a representation formula for $W_{\rm mem}$ are given by Corollary 2.9 in Sect. 2.4. Corollary 2.9 is a consequence of Theorems 2.5, 2.6 and 2.8. Roughly, Theorems 2.5 and 2.6 establish the existence of the variational limit of E_ε as $\varepsilon\to 0$ (see Sect. 2.2), and Theorem 2.8 gives an integral representation for the corresponding variational limit, and so a representation formula for $W_{\rm mem}$ (see Sect. 2.3).

Theorem 2.5 is proved in Section 4. The principal ingredients are Theorem 2.6 and Theorem 3.4 whose proof (given in Section 3) uses an interchange theorem of infimum and integral that we obtained in [2]. (Note that the techniques used to prove Theorems 2.5 and 3.4 are the same as in [4, Sections 3 and 4].)

Theorem 2.6 is proved is Section 5. The main arguments are two approximation theorems developed by Ben Belgacem-Bennequin (see [7]) and Gromov-Eliashberg (see [14]). These theorems are stated in Appendix A.

Theorem 2.8 is proved in [4, Appendix A] (see also [3]).

2. Results

2.1. **Variational convergence.** To accomplish our asymptotic analysis, we use the notion of convergence introduced by Anzellotti, Baldo and Percivale in [5] in order to deal with dimension reduction problems in mechanics. Let $\pi=\{\pi_\varepsilon\}_\varepsilon$ be the family of maps $\pi_\varepsilon: W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)\to W^{1,p}(\Sigma;\mathbb{R}^3)$ defined by

$$\pi_\varepsilon(u):=\frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}}u(\cdot,x_3)dx_3.$$

Definition 2.1. We say that E_ε $\Gamma(\pi)$-converges to $E_{\rm mem}$ as $\varepsilon\to 0$, and we write $E_{\rm mem}=\Gamma(\pi)\text{-}\lim_{\varepsilon\to 0}E_\varepsilon$, if the following two assertions hold:

(i) for all $v\in W^{1,p}(\Sigma;\mathbb{R}^3)$ and all $\{u_\varepsilon\}_\varepsilon\subset W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)$,

$$\text{if } \pi_\varepsilon(u_\varepsilon)\to v \text{ in } L^p(\Sigma;\mathbb{R}^3) \text{ then } E_{\rm mem}(v)\leq\liminf_{\varepsilon\to 0}E_\varepsilon(u_\varepsilon);$$

(ii) for all $v\in W^{1,p}(\Sigma;\mathbb{R}^3)$, there exists $\{u_\varepsilon\}_\varepsilon\subset W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)$ such that:

$$\pi_\varepsilon(u_\varepsilon)\to v \text{ in } L^p(\Sigma;\mathbb{R}^3) \text{ and } E_{\rm mem}(v)\geq\limsup_{\varepsilon\to 0}E_\varepsilon(u_\varepsilon).$$

In fact, Definition 2.1 is a variant of De Giorgi's Γ-convergence. This is made clear by Lemma 2.3. Consider $\mathcal{E}_\varepsilon: W^{1,p}(\Sigma;\mathbb{R}^3)\to[0,+\infty]$ defined by

$$\mathcal{E}_\varepsilon(v):=\inf\Big\{E_\varepsilon(u):\pi_\varepsilon(u)=v\Big\}.$$

Definition 2.2. We say that $\mathcal{E}_\varepsilon$ Γ-converges to $E_{\rm mem}$ as $\varepsilon \to 0$, and we write $E_{\rm mem} = \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon$ if for every $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$,

$$\Big(\Gamma\text{-}\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon\Big)(v) = \Big(\Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon\Big)(v) = E_{\rm mem}(v),$$

where $(\Gamma\text{-}\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon)(v) := \inf\big\{\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(v_\varepsilon) : v_\varepsilon \to v \text{ in } L^p(\Sigma;\mathbb{R}^3)\big\}$ and $(\Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon)(v) := \inf\big\{\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(v_\varepsilon) : v_\varepsilon \to v \text{ in } L^p(\Sigma;\mathbb{R}^3)\big\}$.

For a deeper discussion of the Γ-convergence theory we refer to the book [11]. Clearly, Definition 2.2 is equivalent to assertions (i) and (ii) in definition 2.1 with "$\pi(u_\varepsilon) \to v$" replaced by "$v_\varepsilon \to v$". It is then obvious that

Lemma 2.3. $E_{\rm mem} = \Gamma(\pi)\text{-}\lim_{\varepsilon\to 0} E_\varepsilon$ *if and only if* $E_{\rm mem} = \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon$.

The $\Gamma(\pi)$-convergence of E_ε in (1) to $E_{\rm mem}$ in (2) as $\varepsilon \to 0$ as well as a representation formula for $W_{\rm mem}$ are given by Corollary 2.9. It is a consequence of Theorems 2.5, 2.6 and 2.8. Roughly, Theorems 2.5 and 2.6 establish the existence of the $\Gamma(\pi)$-limit of E_ε as $\varepsilon \to 0$ (see Sect. 2.2), and Theorem 2.8 gives an integral representation for the corresponding $\Gamma(\pi)$-limit, and so a representation formula for $W_{\rm mem}$ (see Sect. 2.3).

2.2. **Γ-convergence of $\mathcal{E}_\varepsilon$ as $\varepsilon \to 0$.** Denote by $C^1(\overline{\Sigma};\mathbb{R}^3)$ the space of all restrictions to $\overline{\Sigma}$ of C^1-differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}^3$, and set

$$C^1_*(\overline{\Sigma};\mathbb{R}^3) := \Big\{v \in C^1(\overline{\Sigma};\mathbb{R}^3) : \partial_1 v(x) \wedge \partial_2 v(x) \neq 0 \text{ for all } x \in \overline{\Sigma}\Big\},$$

where $\partial_1 v(x)$ (resp. $\partial_2 v(x)$) denotes the partial derivative of v at $x = (x_1, x_2)$ with respect to x_1 (resp. x_2). (In fact, $C^1_*(\overline{\Sigma};\mathbb{R}^3)$ is the set of all C^1-immersions from $\overline{\Sigma}$ to $\mathbb{R}^3$.) Let $\mathcal{E} : W^{1,p}(\Sigma;\mathbb{R}^3) \to [0,+\infty]$ be defined by

$$\mathcal{E}(v) := \begin{cases} \displaystyle\int_\Sigma W_0\big(\nabla v(x)\big)dx & \text{if } v \in C^1_*(\overline{\Sigma};\mathbb{R}^3) \\ \quad +\infty & \text{otherwise,} \end{cases}$$

where $W_0 : \mathbb{M}^{3\times 2} \to [0,+\infty]$ is given by

$$W_0(\xi) := \inf_{\zeta\in\mathbb{R}^3} W(\xi \mid \zeta)$$

with $(\xi \mid \zeta)$ denoting the element of $\mathbb{M}^{3\times 3}$ corresponding to $(\xi,\zeta) \in \mathbb{M}^{3\times 2} \times \mathbb{R}^3$. (As W is coercive, it is easy to see that W_0 *is coercive*, i.e., $W_0(\xi) \geq C|\xi|^p$ for all $\xi \in \mathbb{M}^{3\times 2}$ and some $C > 0$.) The following lemma gives three elementary properties of W_0 (the proof is left to the reader). Note that conditions (C_1) and (C_2) imply W_0 is not of p-polynomial growth.

Lemma 2.4. *Denote by $\xi_1 \wedge \xi_2$ the cross product of vectors $\xi_1, \xi_2 \in \mathbb{R}^3$.*

(i) W_0 *is continuous.*
(ii) *If* (C_1) *holds then*
   $(\overline{C}_1)$ $W_0(\xi_1 \mid \xi_2) = +\infty$ *if and only if* $\xi_1 \wedge \xi_2 = 0$.
(iii) *If* (C_2) *holds then*
   $(\overline{C}_2)$ *for all* $\delta > 0$, *there exists* $c_\delta > 0$ *such that for all* $\xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3\times 2}$,
$$\text{if } |\xi_1 \wedge \xi_2| \geq \delta \text{ then } W_0(\xi) \leq c_\delta(1 + |\xi|^p).$$

Taking Lemma 2.3 into account, we see that the existence of the $\Gamma(\pi)$-limit of E_ε as $\varepsilon \to 0$ follows from Theorem 2.5.

Theorem 2.5. *Let assumptions* (C_1) *and* (C_2) *hold. Then* $\Gamma\text{-}\lim_{\varepsilon\to0}\mathcal{E}_\varepsilon=\overline{\mathcal{E}}$ *with* $\overline{\mathcal{E}}:W^{1,p}(\Sigma;\mathbb{R}^3)\to[0,+\infty]$ *given by*

$$\overline{\mathcal{E}}(v):=\inf\left\{\liminf_{n\to+\infty}\mathcal{E}(v_n):W^{1,p}(\Sigma;\mathbb{R}^3)\ni v_n\to v\ \textit{in}\ L^p(\Sigma;\mathbb{R}^3)\right\}.$$

The proof of Theorem 2.5 is established in Section 4. It uses Theorem 3.4 (see Section 3) and Theorem 2.6.

Theorem 2.6. *If* $(\overline{\mathrm{C}}_2)$ *holds then* $\overline{\mathcal{E}}(v)=\mathcal{I}(v)$ *for all* $v\in W^{1,p}(\Sigma;\mathbb{R}^3)$*, where* $\mathcal{I}:W^{1,p}(\Sigma;\mathbb{R}^3)\to[0,+\infty]$ *is given by*

$$\mathcal{I}(v):=\inf\left\{\liminf_{n\to+\infty}\int_\Sigma W_0\big(\nabla v_n(x)\big)dx:W^{1,p}(\Sigma;\mathbb{R}^3)\ni v_n\to v\ \textit{in}\ L^p(\Sigma;\mathbb{R}^3)\right\}.$$

Theorem 2.6 is proved in Section 6 by using two approximation theorems developed by Ben Belgacem-Bennequin (see [7]) and Gromov-Eliashberg (see [14]). These theorems are stated in Appendix A.

2.3. **Integral representation of $\mathcal{I}$.** From now on, given a bounded open set $D\subset\mathbb{R}^2$ with $|\partial D|=0$, we denote by $\mathrm{Aff}(D;\mathbb{R}^3)$ the space of all continuous piecewise affine functions from D to $\mathbb{R}^3$, i.e., $v\in\mathrm{Aff}(D;\mathbb{R}^3)$ *if and only if v is continuous and there exists a finite family $(D_i)_{i\in I}$ of open disjoint subsets of D such that $|\partial D_i|=0$ for all $i\in I$, $|D\setminus\cup_{i\in I}D_i|=0$ and for every $i\in I$, $\nabla v(x)=\xi_i$ in D_i with $\xi_i\in\mathbb{M}^{3\times2}$* (where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^2$). Define $\mathcal{Z}W_0:\mathbb{M}^{3\times2}\to[0,+\infty]$ by

$$\mathcal{Z}W_0(\xi):=\inf\left\{\int_Y W_0\big(\xi+\nabla\phi(y)\big)dy:\phi\in\mathrm{Aff}_0(Y;\mathbb{R}^3)\right\}\tag{3}$$

where $Y:=]0,1[^2$ and $\mathrm{Aff}_0(Y;\mathbb{R}^3):=\{\phi\in\mathrm{Aff}(Y;\mathbb{R}^3):\phi=0\text{ on }\partial Y\}$. (As W_0 is coercive, it is easy to see that $\mathcal{Z}W_0$ *is coercive.*) Recall the definitions of quasiconvexity and quasiconvex envelope:

Definition 2.7. Let $f:\mathbb{M}^{3\times2}\to[0,+\infty]$ be a Borel measurable function.

(i) We say that f is quasiconvex if for every $\xi\in\mathbb{M}^{3\times2}$, every bounded open set $D\subset\mathbb{R}^2$ with $|\partial D|=0$ and every $\phi\in W_0^{1,\infty}(D;\mathbb{R}^3)$,
$$f(\xi)\le\frac{1}{|D|}\int_D f(\xi+\nabla\phi(x))dx.$$
(ii) By the quasiconvex envelope of f, we mean the unique function (when it exists) $\mathcal{Q}f:\mathbb{M}^{3\times2}\to[0,+\infty]$ such that:
- $\mathcal{Q}f$ is Borel measurable, quasiconvex and $\mathcal{Q}f\le f$;
- for all $g:\mathbb{M}^{3\times2}\to[0,+\infty]$, if g is Borel measurable, quasiconvex and $g\le f$, then $g\le\mathcal{Q}f$.

(Usually, for simplicity, we say that $\mathcal{Q}f$ is the greatest quasiconvex function which less than or equal to f.)

Under $(\overline{\mathrm{C}}_2)$, we proved that $\mathcal{Z}W_0$ is of p-polynomial growth and so continuous (see [4, Propositions A.3 and A.1(iii)]) and that $\mathcal{Z}W_0$ is the quasiconvex envelope of W_0, i.e., $\mathcal{Z}W_0=\mathcal{Q}W_0$ (see [4, Proposition A.5]). Taking Theorems 2.5 and 2.6 together with Lemmas 2.3 and 2.4(iii) into account, we see that Theorem 2.8 gives an integral representation for the $\Gamma(\pi)$-limit of E_ε as $\varepsilon\to0$ as well as a representation formula for W_{mem}.

Theorem 2.8. *If* $(\overline{\mathrm{C}}_2)$ *holds then for every* $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$,

$$\mathcal{I}(v) = \int_\Sigma \mathcal{Q}W_0\big(\nabla v(x)\big)dx.$$

Theorem 2.8 is proved in [4, Appendix A] (see also [3]).

2.4. **$\Gamma(\pi)$-convergence of E_ε to E_{mem} as $\varepsilon \to 0$.** According to Lemmas 2.3 and Lemma 2.4(iii), a direct consequence of Theorems 2.5, 2.6 and 2.8 is the following.

Corollary 2.9. *Let assumptions* (C_1) *and* (C_2) *hold. Then as* $\varepsilon \to 0$, E_ε *in* (1) $\Gamma(\pi)$*-converge to* E_{mem} *in* (2) *with* $W_{\mathrm{mem}} = \mathcal{Q}W_0$.

Remark 2.10. Corollary 2.9 can be applied when $W : \mathbb{M}^{3\times 3} \to [0, +\infty]$ is given by

$$W(F) := h(\mathrm{det}F) + |F|^p,$$

where $h : \mathbb{R} \to [0, +\infty]$ is a continuous function such that:

- $h(t) = +\infty$ if and only if $t \le 0$;
- for every $\delta > 0$, there exists $r_\delta > 0$ such that $h(t) \le r_\delta$ for all $t \ge \delta$.

3. Representation of $\mathcal{E}$

The goal of this section is to show Theorem 3.4. To this end, we begin by proving two lemmas.

For every $v \in C^1_*(\overline{\Sigma};\mathbb{R}^3)$ and $j \ge 1$, we define the multifunction $\Lambda^j_v : \overline{\Sigma} \rightrightarrows \mathbb{R}^3$ by

$$\Lambda^j_v(x) := \left\{\zeta \in \mathbb{R}^3 : \det(\nabla v(x) \mid \zeta) \ge \frac{1}{j}\right\}.$$

Lemma 3.1. *Let* $v \in C^1_*(\overline{\Sigma};\mathbb{R}^3)$. *Then:*

(i) *for every* $j \ge 1$, Λ^j_v *is a nonempty convex closed-valued lower semicontinuous*[1] *multifunction*;

(ii) *for every* $x \in \overline{\Sigma}$, $\Lambda^1_v(x) \subset \cdots \subset \Lambda^j_v(x) \subset \cdots \subset \cup_{j\ge 1}\Lambda^j_v(x) = \Lambda_v(x)$, *where* $\Lambda_v(x) := \{\zeta \in \mathbb{R}^3 : \det(\nabla v(x) \mid \zeta) > 0\}$.

Proof. (ii) is obvious. Prove then (i). Let $j \ge 1$. It is easy to see that for every $x \in \overline{\Sigma}$, $\Lambda^j_v(x)$ is nonempty, convex and closed. Let X be a closed subset of $\mathbb{R}^3$, let $x \in \overline{\Sigma}$, and let $\{x_n\}_{n\ge 1} \subset \overline{\Sigma}$ such that $|x_n - x| \to 0$ as $n \to +\infty$ and $\Lambda^j_v(x_n) \subset X$ for all $n \ge 1$. Let $\zeta \in \Lambda^j_v(x)$ and let $\{\zeta_m\}_{m\ge 1} \subset \mathbb{R}^3$ be given by $\zeta_m := \zeta + \frac{1}{m}\zeta$. Then, for every $m \ge 1$,

$$\det\big(\nabla v(x) \mid \zeta_m\big) = \det\big(\nabla v(x) \mid \zeta\big) + \frac{1}{m}\det\big(\nabla v(x) \mid \zeta\big) \ge \frac{1}{j} + \frac{1}{mj}. \tag{4}$$

Fix any $m \ge 1$. Since $\det(\nabla v(x_n) \mid \zeta_m) \to \det(\nabla v(x) \mid \zeta_m)$ as $n \to +\infty$, using (4) we see that $\det(\nabla v(x_{n_0}) \mid \zeta_m) > \frac{1}{j}$ for some $n_0 \ge 1$, so that $\zeta_m \in \Lambda^j_v(x_{n_0})$. Thus $\zeta_m \in X$ for all $m \ge 1$. As X is closed we have $\zeta = \lim_{m\to+\infty} \zeta_m \in X$. □

In the sequel, given $\Lambda : \overline{\Sigma} \rightrightarrows \mathbb{R}^3$ we set

$$C(\overline{\Sigma};\Lambda) := \Big\{\phi \in C\big(\overline{\Sigma};\mathbb{R}^3\big) : \phi(x) \in \Lambda(x) \text{ for all } x \in \overline{\Sigma}\Big\},$$

where $C(\overline{\Sigma};\mathbb{R}^3)$ denotes the space of all continuous functions from $\overline{\Sigma}$ to $\mathbb{R}^3$.

[1] A multifunction $\Lambda : \overline{\Sigma} \to \mathbb{R}^3$ is said to be lower semicontinuous if for every closed subset X of $\mathbb{R}^3$, every $x \in \overline{\Sigma}$ and every $\{x_n\}_{n\ge 1} \subset \overline{\Sigma}$ such that $|x_n - x| \to 0$ as $n \to +\infty$ and $\Lambda(x_n) \subset X$ for all $n \ge 1$, we have $\Lambda(x) \subset X$ (see [6] for more details).

Lemma 3.2. *Given* $v \in C^1_*(\Sigma;\mathbb{R}^3)$ *and* $j \geq 1$, *if* (C_2) *holds, then*

$$\inf_{\phi\in C(\overline{\Sigma};\Lambda^j_v)}\int_\Sigma W\big(\nabla v(x)\mid\phi(x)\big)dx=\int_\Sigma\inf_{\zeta\in\Lambda^j_v(x)}W\big(\nabla v(x)\mid\zeta\big)dx.$$

To prove Lemma 3.2 we need the following interchange theorem of infimum and integral (that we proved in [2, Corollary 5.4]).

Theorem 3.3. *Let* $\Gamma:\overline{\Sigma}\rightrightarrows\mathbb{R}^3$ *and let* $f:\overline{\Sigma}\times\mathbb{R}^3\to[0,+\infty]$. *Assume that*:

- (H_1) f *is a Carathéodory integrand*;
- (H_2) Γ *is a nonempty convex closed-valued lower semicontinuous multifunction*;
- (H_3) $C(\overline{\Sigma};\Gamma)\neq\emptyset$ *and for every* $\phi,\hat{\phi}\in C(\overline{\Sigma};\Gamma)$,

$$\int_\Sigma\max_{\alpha\in[0,1]}f\big(x,\alpha\phi(x)+(1-\alpha)\hat{\phi}(x)\big)dx<+\infty.$$

Then,

$$\inf_{\phi\in C(\overline{\Sigma};\Gamma)}\int_\Sigma f\big(x,\phi(x)\big)dx=\int_\Sigma\inf_{\zeta\in\Gamma(x)}f(x,\zeta)dx.$$

Proof of Lemma 3.2. Since W is continuous, (H_1) holds with $f(x,\zeta)=W(\nabla v(x)\mid\zeta)$. Lemma 3.1 shows that (H_2) is satisfied with $\Gamma=\Lambda^j_v$, and $C(\overline{\Sigma};\Lambda^j_v)\neq\emptyset$ (for example $\Phi:\overline{\Sigma}\to\mathbb{R}^3$ defined by (8) belongs to $C(\overline{\Sigma};\Lambda^j_v)$). Given $\phi,\hat{\phi}\in C(\overline{\Sigma};\Lambda^j_v)$, it is clear that $\det(\nabla v(x)\mid\alpha\phi(x)+(1-\alpha)\hat{\phi}(x))\geq 1/j$ for all $\alpha\in[0,1]$ and all $x\in\overline{\Sigma}$. From (C_2) it follows that there exists $c>0$ depending only on j, v, ϕ and $\hat{\phi}$ such that $W(\nabla v(x)\mid\alpha\phi(x)+(1-\alpha)\hat{\phi}(x))\leq c$ for all $x\in\overline{\Sigma}$. Thus (H_3) is verified with $f(x,\zeta)=W(\nabla v(x)\mid\zeta)$ and $\Gamma=\Lambda^j_v$, and Lemma 3.2 follows from Lemma 3.3. □

Here is our (non integral) representation theorem for $\mathcal{E}$.

Theorem 3.4. *If* (C_1) *and* (C_2) *hold, then for every* $v\in C^1_*(\overline{\Sigma};\mathbb{R}^3)$,

$$\mathcal{E}(v)=\inf_{j\geq1}\inf_{\phi\in C(\overline{\Sigma};\Lambda^j_v)}\int_\Sigma W\big(\nabla v(x)\mid\phi(x)\big)dx. \tag{5}$$

Proof. Fix $v\in C^1_*(\overline{\Sigma};\mathbb{R}^3)$ and denote by $\hat{\mathcal{E}}(v)$ the right-hand side of (5). It is easy to verify that $\mathcal{E}(v)\leq\hat{\mathcal{E}}(v)$. We are thus reduced to prove that

$$\hat{\mathcal{E}}(v)\leq\mathcal{E}(v). \tag{6}$$

Using Lemma 3.2, we obtain

$$\hat{\mathcal{E}}(v)\leq\inf_{j\geq1}\int_\Sigma\inf_{\zeta\in\Lambda^j_v(x)}W\big(\nabla v(x)\mid\zeta\big)dx. \tag{7}$$

Consider the continuous function $\Phi:\overline{\Sigma}\to\mathbb{R}^3$ defined by

$$\Phi(x):=\frac{\partial_1 v(x)\wedge\partial_2 v(x)}{|\partial_1 v(x)\wedge\partial_2 v(x)|^2}. \tag{8}$$

Then, $\det(\nabla v(x)\mid\Phi(x))=1$ for all $x\in\overline{\Sigma}$. Using (C_2) we deduce that there exists $c>0$ depending only on p such that

$$\int_\Sigma\inf_{\zeta\in\Lambda^1_v(x)}W(\nabla v(x)\mid\zeta)dx\leq c\big(|\Sigma|+\|\nabla v\|^p_{L^p(\Sigma;\mathbb{M}^{3\times2})}+\|\Phi\|^p_{L^p(\Sigma;\mathbb{R}^3)}\big).$$

It follows that $\inf_{\zeta\in\Lambda_v^1(\cdot)} W(\nabla v(\cdot)\mid\zeta)\in L^1(\Sigma)$. From Lemma 3.1(i) and (ii), we see that $\{\inf_{\zeta\in\Lambda_v^j(\cdot)} W(\nabla v(\cdot)\mid\zeta)\}_{j\geq 1}$ is non-increasing, and that for every $x\in\overline{\Sigma}$,

$$\inf_{j\geq 1}\inf_{\zeta\in\Lambda_v^j(x)} W\big(\nabla v(x)\mid\zeta\big)=W_0\big(\nabla v(x)\big), \tag{9}$$

and (19) follows from (7) and (9) by using Lebesgue's dominated convergence theorem. □

4. Proof of Theorem 2.5

In this section we prove Theorem 2.5. Since $\Gamma\text{-}\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon\leq\Gamma\text{-}\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon$, we only need to show that:

(a) $\overline{\mathcal{E}}\leq\Gamma\text{-}\liminf\limits_{\varepsilon\to 0}\mathcal{E}_\varepsilon$;

(b) $\Gamma\text{-}\limsup\limits_{\varepsilon\to 0}\mathcal{E}_\varepsilon\leq\overline{\mathcal{E}}$.

In the sequel, we follow the notation used in Section 3.

4.1. Proof of (a).

Let $v\in W^{1,p}(\Sigma;\mathbb{R}^3)$ and let $\{v_\varepsilon\}_\varepsilon\subset W^{1,p}(\Sigma;\mathbb{R}^3)$ be such that $v_\varepsilon\to v$ in $L^p(\Sigma;\mathbb{R}^3)$. We have to prove that

$$\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(v_\varepsilon)\geq\overline{\mathcal{E}}(v). \tag{10}$$

Without loss of generality we can assume that $\sup_{\varepsilon>0}\mathcal{E}_\varepsilon(v_\varepsilon)<+\infty$. To every $\varepsilon>0$ there corresponds $u_\varepsilon\in\pi_\varepsilon^{-1}(v_\varepsilon)$ such that

$$\mathcal{E}_\varepsilon(v_\varepsilon)\geq E_\varepsilon(u_\varepsilon)-\varepsilon. \tag{11}$$

Defining $\hat{u}_\varepsilon:\Sigma_1\to\mathbb{R}^3$ by $\hat{u}_\varepsilon(x,x_3):=u_\varepsilon(x,\varepsilon x_3)$ we have

$$E_\varepsilon\big(u_\varepsilon\big)=\int_{\Sigma_1}W\Big(\partial_1\hat{u}_\varepsilon(x,x_3)\mid\partial_2\hat{u}_\varepsilon(x,x_3)\mid\frac{1}{\varepsilon}\partial_3\hat{u}_\varepsilon(x,x_3)\Big)dxdx_3. \tag{12}$$

Using the coercivity of W, we deduce that $\|\partial_3\hat{u}_\varepsilon\|_{L^p(\Sigma_1;\mathbb{R}^3)}\leq c\varepsilon^p$ for all $\varepsilon>0$ and some $c>0$, and so $\|\hat{u}_\varepsilon-v_\varepsilon\|_{L^p(\Sigma_1;\mathbb{R}^3)}\leq c'\varepsilon^p$ by Poincaré-Wirtinger's inequality, where $c'>0$ is a constant which does not depend on ε. It follows that $\hat{u}_\varepsilon\to v$ in $L^p(\Sigma_1;\mathbb{R}^3)$. For $x_3\in]-\frac{1}{2},\frac{1}{2}[$, let $w_\varepsilon^{x_3}\in W^{1,p}(\Sigma;\mathbb{R}^3)$ given by $w_\varepsilon^{x_3}(x):=\hat{u}_\varepsilon(x,x_3)$. Then (up to a subsequence) $w_\varepsilon^{x_3}\to v$ in $L^p(\Sigma;\mathbb{R}^3)$ for a.e. $x_3\in]-\frac{1}{2},\frac{1}{2}[$. Taking (11) and (12) into account and using Fatou's lemma, we obtain

$$\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(v_\varepsilon)\geq\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\liminf_{\varepsilon\to 0}\int_\Sigma W_0\big(\nabla w_\varepsilon^{x_3}(x)\big)dx\right)dx_3,$$

and so $\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(v_\varepsilon)\geq\mathcal{I}(v)$, and (10) follows by using Theorem 2.6. □

4.2. Proof of (b).

As $\Gamma\text{-}\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon$ is lower semicontinuous with respect to the strong topology of $L^p(\Sigma;\mathbb{R}^3)$ (see [11, Proposition 6.8 p. 57]), it is sufficient to prove that for every $v\in C_*^1(\overline{\Sigma};\mathbb{R}^3)$,

$$\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(v)\leq\mathcal{E}(v). \tag{13}$$

Given $v\in C_*^1(\overline{\Sigma};\mathbb{R}^3)$, fix any $j\geq 1$, and any $n\geq 1$. Using Theorem 3.4 we obtain the existence of $\phi\in C(\overline{\Sigma};\Lambda_v^j)$ such that

$$\int_\Sigma W\big(\nabla v(x)\mid\phi(x)\big)dx\leq\mathcal{E}(v)+\frac{1}{n}. \tag{14}$$

By Stone-Weierstrass's approximation theorem, there exists $\{\phi_k\}_{k\geq 1}\subset C^\infty(\overline{\Sigma};\mathbb{R}^3)$ such that

$$\phi_k\to\phi \text{ uniformly as } k\to+\infty. \tag{15}$$

We claim that:

(c$_1$) $\det\big(\nabla v(x)\mid\phi_k(x)\big)\geq\dfrac{1}{2j}$ for all $x\in\overline{\Sigma}$, all $k\geq k_v$ and some $k_v\geq 1$;

(c$_2$) $\displaystyle\lim_{k\to+\infty}\int_\Sigma W\big(\nabla v(x)\mid\phi_k(x)\big)dx=\int_\Sigma W\big(\nabla v(x)\mid\phi(x)\big)dx.$

Indeed, setting $\mu_v:=\sup_{x\in\overline{\Sigma}}|\partial_1 v(x)\wedge\partial_2 v(x)|$ ($\mu_v>0$) and using (15), we deduce that there exists $k_v\geq 1$ such that for every $k\geq k_v$,

$$\sup_{x\in\overline{\Sigma}}\big|\phi_k(x)-\phi(x)\big|<\frac{1}{2j\mu_v}. \tag{16}$$

Let $x\in\overline{\Sigma}$, and let $k\geq k_v$. As $\phi\in C(\overline{\Sigma};\Lambda_v^j)$ we have

$$\det\big(\nabla v(x)\mid\phi_k(x)\big)\geq\frac{1}{j}-\det\big(\nabla v(x)\mid\phi_k(x)-\phi(x)\big). \tag{17}$$

Noticing that $\det(\nabla v(x)\mid\phi_k(x)-\phi(x))\leq|\partial_1 v(x)\wedge\partial_2 v(x)||\phi_k(x)-\phi(x)|$, from (16) and (17) we deduce that $\det\big(\nabla v(x)\mid\phi_k(x)\big)\geq\frac{1}{2j}$, and (c$_1$) is proved. Combining (c$_1$) with (C$_2$) we see that $\sup_{k\geq k_v}W(\nabla v(\cdot)\mid\phi_k(\cdot))\in L^1(\Sigma)$. As W is continuous we have $\lim_{k\to+\infty}W(\nabla v(x)\mid\phi_k(x))=W(\nabla v(x)\mid\phi(x))$ for all $x\in V$, and (c$_2$) follows by using Lebesgue's dominated convergence theorem, which completes the claim.

Fix any $k\geq k_v$ and define $\theta:]-\frac{1}{2},\frac{1}{2}[\to\mathbb{R}$ by $\theta(x_3):=\inf_{x\in\overline{\Sigma}}\det(\nabla v(x)+x_3\nabla\phi_k(x)\mid\phi_k(x))$. Clearly θ is continuous. By (c$_1$) we have $\theta(0)\geq\frac{1}{2j}$, and so there exists $\eta_v\in]0,\frac{1}{2}[$ such that $\theta(x_3)\geq\frac{1}{4j}$ for all $x_3\in]-\eta_v,\eta_v[$. Let $u_k:\Sigma_1\to\mathbb{R}$ be given by $u_k(x,x_3):=v(x)+x_3\phi_k(x)$. From the above it follows that

(c$_3$) $\det\nabla u_k(x,\varepsilon x_3)\geq\frac{1}{4j}$ for all $\varepsilon\in]0,\eta_v[$ and all $(x,x_3)\in\overline{\Sigma}\times]-\frac{1}{2},\frac{1}{2}[$.

As in the proof of (c$_1$), from (c$_3$) together with (C$_2$) and the continuity of W, we obtain

$$\lim_{\varepsilon\to 0}E_\varepsilon(u_k)=\lim_{\varepsilon\to 0}\int_{\Sigma_1}W\big(\nabla u_k(x,\varepsilon x_3)\big)dxdx_3=\int_\Sigma W\big(\nabla v(x)\mid\phi_k(x)\big)dx. \tag{18}$$

For every $\varepsilon>0$ and every $k\geq k_v$, since $\pi_\varepsilon(u_k)=v$ we have $\mathcal{E}_\varepsilon(v)\leq E_\varepsilon(u_k)$. Using (18), (c$_2$) and (14), we deduce that

$$\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(v)\leq\mathcal{E}(v)+\frac{1}{n},$$

and (13) follows by letting $n\to+\infty$. □

5. Proof of Theorem 2.6

In this section, we prove of Theorem 2.6. It is based upon two approximation theorems by Ben Belgacem-Bennequin (see Sect. A.1) and Gromov-Eliasberg (see Sect. A.2).

Recall the definition of rank one convexity and rank one convex envelope:

Definition 5.1. Let $f:\mathbb{M}^{3\times 2}\to[0,+\infty]$ be a Borel measurable function.

(i) We say that f is rank one convex if for every $\alpha \in]0,1[$ and every $\xi, \xi' \in \mathbb{M}^{3\times 2}$ with $\mathrm{rank}(\xi - \xi')$=1,

$$f(\alpha\xi + (1-\alpha)\xi') \leq \alpha f(\xi) + (1-\alpha) f(\xi').$$

(ii) By the rank one convex envelope of f, that we denote by $\mathcal{R}f$, we mean the greatest rank one convex function which less than or equal to f.

In [7, Proposition 7 p. 32 and Lemma 8 p. 34] (see also [9, Sect. 5.1], [19, Proposition 3.4.4 p. 112] and [20, Lemma 6.5]) Ben Belgacem proved the following lemma that we will use in the proof of Theorem 2.6. (As W_0 is coercive, it is easy to see that $\mathcal{R}W_0$ is coercive.)

Lemma 5.2. *If* $(\overline{\mathrm{C}}_2)$ *holds then*:

(i) $\mathcal{R}W_0(\xi) \leq c(1+|\xi|^p)$ *for all* $\xi \in \mathbb{M}^{3\times 2}$ *and some* $c > 0$;
(ii) $\mathcal{R}W_0$ *is continuous.*

Define $I : W^{1,p}(\Sigma;\mathbb{R}^3) \to [0,+\infty]$ by

$$I(v) := \inf\left\{\liminf_{n\to+\infty} \int_\Sigma W_0(\nabla v_n(x))dx : \mathrm{Aff}_{\mathrm{li}}(\Sigma;\mathbb{R}^3) \ni v_n \to v \text{ in } L^p(\Sigma;\mathbb{R}^3)\right\}$$

with $\mathrm{Aff}_{\mathrm{li}}(\Sigma;\mathbb{R}^3) := \{v \in \mathrm{Aff}(\Sigma;\mathbb{R}^3) : v \text{ is locally injective}\}$ ($\mathrm{Aff}(\Sigma;\mathbb{R}^3)$ is defined in Sect. 2.3). To prove Theorem 2.6 we will use Proposition 5.3.

Proposition 5.3. $I = J$ *with* $J : W^{1,p}(\Sigma;\mathbb{R}^3) \to [0,+\infty]$ *given by*

$$J(v) := \inf\left\{\liminf_{n\to+\infty} \int_\Sigma \mathcal{R}W_0(\nabla v_n(x))dx : \mathrm{Aff}_{\mathrm{li}}(\Sigma;\mathbb{R}^3) \ni v_n \to v \text{ in } L^p(\Sigma;\mathbb{R}^3)\right\}.$$

To prove Proposition 5.3 we need Lemma 5.4 whose proof is contained in the thesis of Ben Belgacem [7]. Since it is difficult to lay hands on this thesis (which is written in French), we give the proof of Lemma 5.4 in appendix B.

Lemma 5.4. $I(v) \leq \displaystyle\int_\Sigma \mathcal{R}W_0(\nabla v(x))dx$ *for all* $v \in \mathrm{Aff}_{\mathrm{li}}(\Sigma;\mathbb{R}^3)$.

Proof of Proposition 5.3. Clearly $J \leq I$. We are thus reduced to prove that

$$I \leq J. \tag{19}$$

Fix any $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$ and any sequence $v_n \to v$ in $L^p(\Sigma;\mathbb{R}^3)$ with $v_n \in \mathrm{Aff}_{\mathrm{li}}(\Sigma;\mathbb{R}^3)$. Using Lemma 5.4 we have $I(v_n) \leq \int_\Sigma \mathcal{R}W_0(\nabla v_n(x))dx$ for all $n \geq 1$. Thus,

$$I(v) \leq \liminf_{n\to+\infty} I(v_n) \leq \liminf_{n\to+\infty} \int_\Sigma \mathcal{R}W_0(\nabla v_n(x))dx,$$

and (19) follows. □

Proof of Theorem 2.6. We first prove that

$$\overline{\mathcal{E}} \leq I. \tag{20}$$

As in the proof of Proposition 5.3, it suffices to show that if $v \in \mathrm{Aff}_{\mathrm{li}}(\Sigma;\mathbb{R}^3)$ then

$$\overline{\mathcal{E}}(v) \leq \int_\Sigma W_0(\nabla v(x))dx. \tag{21}$$

Let $v \in \mathrm{Aff}_{\rm li}(\Sigma;\mathbb{R}^3)$. By Theorem A.1-bis (and Lemma A.2), there exists $\{v_n\}_{n\geq 1} \subset C^1_*(\overline{\Sigma};\mathbb{R}^3)$ such that (A_1) and (A_2) holds and $\nabla v_n(x) \to \nabla v(x)$ a.e. in Σ. As W_0 is continuous (see Lemma 2.4(i)), we have

$$\lim_{n\to+\infty} W_0\big(\nabla v_n(x)\big) = W_0\big(\nabla v(x)\big) \ \text{ a.e. in } \Sigma.$$

Using $(\overline{\mathrm{C}}_2)$ together with (A_2), we deduce that there exists $c > 0$ such that for every $n \geq 1$ and every measurable set $A \subset \Sigma$,

$$\int_A W_0\big(\nabla v_n(x)\big)dx \leq c\Big(|A| + \int_A |\nabla v_n(x) - \nabla v(x)|^p dx + \int_A |\nabla v(x)|^p dx\Big).$$

But $\nabla v_n \to \nabla v$ in $L^p(\Sigma;\mathbb{M}^{3\times 2})$ by (A_1), hence $\{W_0(\nabla v_n(\cdot))\}_{n\geq 1}$ is absolutely uniformly integrable. Using Vitali's theorem, we obtain

$$\lim_{n\to+\infty} \int_\Sigma W_0(\nabla v_n(x))dx = \int_\Sigma W_0(\nabla v(x))dx,$$

and (21) follows.

We now prove that

$$J \leq \overline{J}, \tag{22}$$

with $\overline{J} : W^{1,p}(\Sigma;\mathbb{R}^3) \to [0,+\infty]$ given by

$$\overline{J}(v) := \inf\left\{\liminf_{n\to+\infty} \int_\Sigma \mathcal{R}W_0(\nabla v_n(x))dx : W^{1,p}(\Sigma;\mathbb{R}^3) \ni v_n \to v \text{ in } L^p(\Sigma;\mathbb{R}^3)\right\}.$$

It is sufficient to show that

$$J(v) \leq \int_\Sigma \mathcal{R}W_0(\nabla v(x))dx. \tag{23}$$

Let $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$. By Corollary A.6, there exists $\{v_n\}_{n\geq 1} \subset \mathrm{Aff}_{\rm li}(\Sigma;\mathbb{R}^3)$ such that $\nabla v_n \to \nabla v$ in $L^p(\Sigma;\mathbb{R}^3)$ and $\nabla v_n(x) \to \nabla v(x)$ a.e. in Σ. Taking Lemma 5.2 into account, from Vitali's lemma, we see that

$$\lim_{n\to+\infty} \int_\Sigma \mathcal{R}W_0(\nabla v_n(x))dx = \int_\Sigma \mathcal{R}W_0(\nabla v(x))dx,$$

and (23) follows.

Noticing that $\mathcal{I} \leq \overline{\mathcal{E}}$ and $\overline{J} \leq \mathcal{I}$, and combining Proposition 5.3 with (20) and (22), we conclude that $\overline{\mathcal{E}} = \mathcal{I}$. □

Appendix A. Approximation theorems

A.1. **Ben Belgacem-Bennequin's theorem.** Denote by $\mathrm{Aff}^{ET}(\Sigma;\mathbb{R}^3)$ the space of Ekeland-Temam continuous piecewise affine functions from Σ to $\mathbb{R}^3$, i.e., $u \in \mathrm{Aff}^{ET}(\Sigma;\mathbb{R}^3)$ *if and only if* v *is continuous and there exists a finite family* $(V_i)_{i\in I}$ *of open disjoint subsets of* Σ *such that* $|\Sigma \setminus \cup_{i\in I} V_i| = 0$ *and for every* $i \in I$*, the restriction of* v *to* V_i *is affine.* Note that from Ekeland-Temam [12], we know that $\mathrm{Aff}^{ET}(\Sigma;\mathbb{R}^3)$ is strongly dense in $W^{1,p}(\Sigma;\mathbb{R}^3)$. Set

$$\mathrm{Aff}^{ET}_{\rm li}(\Sigma;\mathbb{R}^3) := \Big\{v \in \mathrm{Aff}^{ET}(\Sigma;\mathbb{R}^3) : v \text{ is locally injective}\Big\}.$$

In [7, Lemma 8 p. 114] (see also [19, Proposition C.0.4 p. 127] and [20, Lemma 1.3]) Ben Belgacem and Bennequin proved the following result.

Theorem A.1. *For every* $v \in \mathrm{Aff}^{ET}_{\rm li}(\Sigma;\mathbb{R}^3)$*, there exists* $\{v_n\}_{n\geq 1} \subset C^1_*(\overline{\Sigma};\mathbb{R}^3)$ *such that*:

(A_1) $v_n \to v$ in $W^{1,p}(\Sigma;\mathbb{R}^3)$;
(A_2) *there exists* $\delta > 0$ *such that* $|\partial_1 v_n(x) \wedge \partial_2 v_n(x)| \geq \delta$ *for all* $x \in \overline{\Sigma}$ *and all* $n \geq 1$.

Denote by $\mathrm{Aff}^V(\Sigma;\mathbb{R}^3)$ the space of Vitali continuous piecewise affine functions from Σ to $\mathbb{R}^3$ (introduced by Ben Belgacem in [7, 9]), i.e., $v \in \mathrm{Aff}^V(\Sigma;\mathbb{R}^3)$ *if and only if* v *is continuous and there exists a finite or countable family* $(O_i)_{i\in I}$ *of dsjoint open subsets of* Σ *such that* $|\partial O_i| = 0$ *for all* $i \in I$, $|\Sigma \setminus \cup_{i\in I} O_i| = 0$, *and* $v(x) = \xi_i \cdot x + a_i$ *if* $x \in O_i$, *where* $a_i \in \mathbb{R}^3$, $\xi_i \in \mathbb{M}^{3\times 2}$ *and* $\mathrm{Card}\{\xi_i : i \in I\}$ *is finite.* In [19, Lemma 3.1.5 p. 99] Trabelsi remarked that Theorem A.1 can be generalized replacing the space $\mathrm{Aff}^{ET}_{\mathrm{li}}(\Sigma;\mathbb{R}^3)$ by

$$\mathrm{Aff}^V_{\mathrm{li}}(\Sigma;\mathbb{R}^3) := \Big\{v \in \mathrm{Aff}^V(\Sigma;\mathbb{R}^3) : v \text{ is locally injective}\Big\}.$$

Theorem A.1-bis. *For every* $v \in \mathrm{Aff}^V_{\mathrm{li}}(\Sigma;\mathbb{R}^3)$, *there exists* $\{v_n\}_{n\geq 1} \subset C^1_*(\overline{\Sigma};\mathbb{R}^3)$ *satisfying* (A_1) and (A_2).

Here we consider the space $\mathrm{Aff}(\Sigma;\mathbb{R}^3)$ defined in Sect. 2.3. It is clear that $\mathrm{Aff}^{ET}(\Sigma;\mathbb{R}^3) \subset \mathrm{Aff}(\Sigma;\mathbb{R}^3)$, and so $\mathrm{Aff}(\Sigma;\mathbb{R}^3)$ is strongly dense in $W^{1,p}(\Sigma;\mathbb{R}^3)$. Moreover, we have

Lemma A.2. $\mathrm{Aff}(\Sigma;\mathbb{R}^3) = \mathrm{Aff}^V(\Sigma;\mathbb{R}^3)$.

Proof. Setting $D_i := \{x \in \cup_{i\in I} O_i : \nabla v(x) = \xi_i\}$ with $v \in \mathrm{Aff}^V(\Sigma;\mathbb{R}^3)$, we see that $\mathrm{Card}\{D_i : i \in I\}$ is finite. Thus $\mathrm{Aff}^V(\Sigma;\mathbb{R}^3) \subset \mathrm{Aff}(\Sigma;\mathbb{R}^3)$. Given $v \in \mathrm{Aff}(\Sigma;\mathbb{R}^3)$, let $(O_j)_{j\in J_i}$ be the connected components of D_i with $i \in I$ (where I is finite). Since D_i is open, O_j is open for all $j \in J_i$, hence J_i is finite or countable because $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$. Moreover, for each $j \in J_i$, the restriction of v to O_j is affine. Thus $\mathrm{Aff}(\Sigma;\mathbb{R}^3) \subset \mathrm{Aff}^V(\Sigma;\mathbb{R}^3)$. □

A.2. **Gromov-Eliashberg's theorem.** In [14, Theorem 1.3.4B] (see also [15, Theorem B_1' p. 20]) Gromov and Eliashberg proved the following result.

Theorem A.3. *Let* $1 \leq N < m$ *be two integers and let* M *be a compact* N*-dimensional manifold which can be immersed in* $\mathbb{R}^m$. *Then, for each* C^1*-differentiable function* v *from* M *to* $\mathbb{R}^m$ *there exists a sequence* $\{v_n\}_n$ *of* C^1*-immersions from* M *to* $\mathbb{R}^m$ *such that* $v_n \to v$ *in* $W^{1,p}(M;\mathbb{R}^m)$.

In our context, we have

Theorem A.4. *For every* $v \in C^1(\overline{\Sigma};\mathbb{R}^3)$ *there exists* $\{v_n\}_{n\geq 1} \subset C^1_*(\overline{\Sigma};\mathbb{R}^3)$ *such that* $v_n \to v$ *in* $W^{1,p}(\Sigma;\mathbb{R}^3)$.

Moreover, from [19, Proposition 3.1.7 p. 100], we have

Proposition A.5. *For every* $v \in C^1_*(\overline{\Sigma};\mathbb{R}^3)$ *there exists* $\{v_n\}_{n\geq 1} \subset \mathrm{Aff}^{ET}_{\mathrm{li}}(\Sigma;\mathbb{R}^3)$ *such that* $v_n \to v$ *in* $W^{1,p}(\Sigma;\mathbb{R}^3)$.

Thus, as a consequence of Theorem A.4 and Proposition A.5, we obtain

Corollary A.6. $\mathrm{Aff}^{ET}_{\mathrm{li}}(\Sigma;\mathbb{R}^3)$ *is strongly dense in* $W^{1,p}(\Sigma;\mathbb{R}^3)$.

Appendix B. Proof of Lemma 5.4

B.1. **Preliminaries.** Define the sequence $\{\mathcal{R}_iW_0\}_{i\geq 0}$ by $\mathcal{R}_0W_0 = W_0$ and for every $i \geq 1$ and every $\xi \in \mathbb{M}^{3\times 2}$,

$$\mathcal{R}_{i+1}W_0(\xi) := \inf_{\substack{a\in\mathbb{R}^2\\ b\in\mathbb{R}^3\\ t\in[0,1]}} \Big\{(1-t)\mathcal{R}_iW_0(\xi - ta\otimes b) + t\mathcal{R}_iW_0(\xi + (1-t)a\otimes b)\Big\}.$$

Recall that W_0 is coercive and continuous (see Lemma 2.4(i)). The following lemma is due to Kohn and Strang [16].

Lemma B.1. *$\mathcal{R}_{i+1}W_0 \leq \mathcal{R}_iW_0$ for all $i \geq 0$ and $\mathcal{R}W_0 = \inf_{i\geq 0}\mathcal{R}_iW_0$.*

Fix any $i \geq 0$ and any $v \in \mathrm{Aff}_{\rm li}(\Sigma;\mathbb{R}^3) := \{v \in \mathrm{Aff}(\Sigma;\mathbb{R}^3) : v \text{ is locally injective}\}$ (with $\mathrm{Aff}(\Sigma;\mathbb{R}^3)$ defined in Sect. 2.3). By definition, there exists a finite family $(V_j)_{j\in J}$ of open disjoint subsets of Σ such that $|\partial V_j| = 0$ for all $j \in J$, $|\Sigma\setminus\cup_{j\in J}V_j| = 0$ and, for every $j \in J$, $\nabla v(x) = \xi_j$ in V_j with $\xi_j \in \mathbb{M}^{3\times 2}$. (As v is locally injective we have $\mathrm{rank}(\xi_j) = 2$ for all $j \in J$.) Fix any $j \in J$. For a proof of Lemmas B.2 and B.3 we refer to [19, Proposition 3.1.2 p. 96].

Lemma B.2. *$\mathcal{R}_iW_0$ is continuous.*

Lemma B.3. *There exist $a \in \mathbb{R}^2$, $b \in \mathbb{R}^3$ and $t \in [0,1]$ such that*

$$\mathcal{R}_{i+1}W_0(\xi_j) = (1-t)\mathcal{R}_iW_0(\xi_j - ta\otimes b) + t\mathcal{R}_iW_0(\xi_j + (1-t)a\otimes b).$$

Without loss of generality we can assume that $a = (1,0)$. For each $n \geq 3$ and each $k \in \{0,\cdots,n-1\}$, consider $A^-_{k,n}, A^+_{k,n}, B_{k,n}, B^-_{k,n}, B^+_{k,n}, C_{k,n}, C^-_{k,n}, C^+_{k,n} \subset Y$ given by:

$$\begin{aligned}
A^-_{k,n} &:= \Big\{(x_1,x_2)\in Y : \tfrac{k}{n} \leq x_1 \leq \tfrac{k}{n} + \tfrac{1-t}{n} \text{ and } \tfrac{1}{n} \leq x_2 \leq 1 - \tfrac{1}{n}\Big\};\\
A^+_{k,n} &:= \Big\{(x_1,x_2)\in Y : \tfrac{k}{n} + \tfrac{1-t}{n} \leq x_1 \leq \tfrac{k+1}{n} \text{ and } \tfrac{1}{n} \leq x_2 \leq 1 - \tfrac{1}{n}\Big\};\\
B_{k,n} &:= \Big\{(x_1,x_2)\in Y : \tfrac{k}{n} \leq x_1 \leq \tfrac{k+1}{n} \text{ and } 0 \leq x_2 \leq -x_1 + \tfrac{k+1}{n}\Big\};\\
B^-_{k,n} &:= \Big\{(x_1,x_2)\in Y : -x_2 + \tfrac{k+1}{n} \leq x_1 \leq -tx_2 + \tfrac{k+1}{n} \text{ and } 0 \leq x_2 \leq \tfrac{1}{n}\Big\};\\
B^+_{k,n} &:= \Big\{(x_1,x_2)\in Y : -tx_2 + \tfrac{k+1}{n} \leq x_1 \leq \tfrac{k+1}{n} \text{ and } 0 \leq x_2 \leq \tfrac{1}{n}\Big\};\\
C_{k,n} &:= \Big\{(x_1,x_2)\in Y : \tfrac{k}{n} \leq x_1 \leq \tfrac{k+1}{n} \text{ and } x_1 + 1 - \tfrac{k+1}{n} \leq x_2 \leq 1\Big\};\\
C^-_{k,n} &:= \Big\{(x_1,x_2)\in Y : x_2 - 1 + \tfrac{k+1}{n} \leq x_1 \leq t(x_2-1) + \tfrac{k+1}{n} \text{ and } 0 \leq x_2 \leq \tfrac{1}{n}\Big\};\\
C^+_{k,n} &:= \Big\{(x_1,x_2)\in Y : t(x_2-1) + \tfrac{k+1}{n} \leq x_1 \leq \tfrac{k+1}{n} \text{ and } 0 \leq x_2 \leq \tfrac{1}{n}\Big\},
\end{aligned}$$

and define $\{\sigma_n\}_{n\geq 1} \subset \mathrm{Aff}_0(Y;\mathbb{R})$ by

$$\sigma_n(x_1,x_2) := \begin{cases} -t(x_1 - \frac{k}{n}) & \text{if } (x_1,x_2) \in A^-_{k,n}\\ (1-t)(x_1 - \frac{k+1}{n}) & \text{if } (x_1,x_2) \in A^+_{k,n} \cup B^+_{k,n} \cup C^+_{k,n}\\ -t(x_1 + x_2 - \frac{k+1}{n}) & \text{if } (x_1,x_2) \in B^-_{k,n}\\ -t(x_1 - x_2 + 1 - \frac{k+1}{n}) & \text{if } (x_1,x_2) \in C^-_{k,n}\\ 0 & \text{if } (x_1,x_2) \in B_{k,n} \cup C_{k,n} \end{cases}$$

(see Figure B.1).

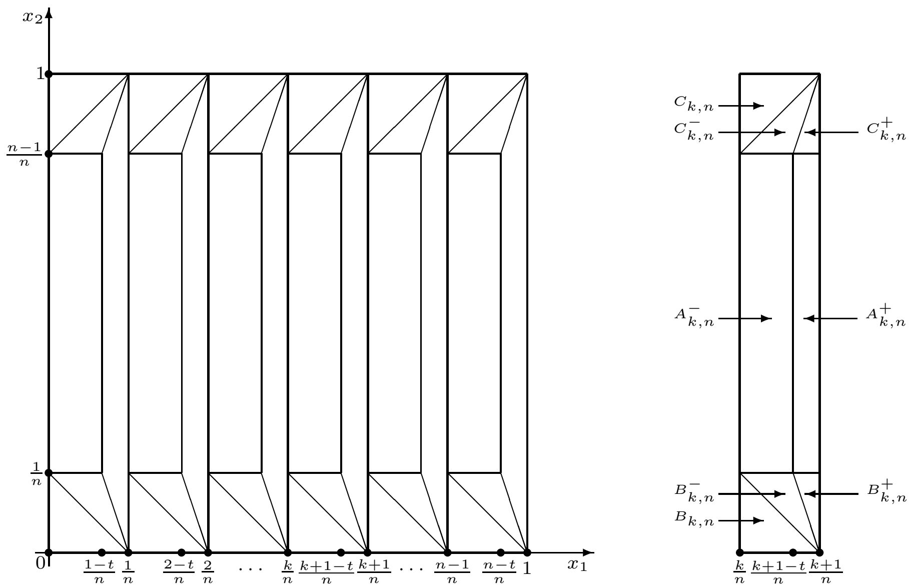

Figure B.1. The function σ_n and the sets $A^-_{k,n}, A^+_{k,n}, B_{k,n}, B^-_{k,n}, B^+_{k,n}, C_{k,n}, C^-_{k,n}, C^+_{k,n}$.

Set

$$b_\ell := \begin{cases} b & \text{if } b \notin \mathrm{Im}\xi_j \\ b + \frac{1}{\ell}\nu & \text{if } b \in \mathrm{Im}\xi_j \end{cases}$$

(with $\mathrm{Im}\xi_j := \{\xi_j \cdot x : x \in \mathbb{R}^2\}$) where $\ell \geq 1$ and $\nu \in \mathbb{R}^3$ is a normal vector to $\mathrm{Im}\xi_j$.

Lemma B.4. *Define* $\{\theta_{n,\ell}\}_{n,\ell\geq 1} \subset \mathrm{Aff}_0(Y;\mathbb{R}^3)$ *by*

$$\theta_{n,\ell}(x) := \sigma_n(x) b_\ell.$$

Then:

(i) *for every* $\ell \geq 1$, $\theta_{n,\ell} \to 0$ *in* $L^p(Y;\mathbb{R}^3)$;

(ii) $\displaystyle\lim_{\ell\to+\infty}\lim_{n\to+\infty}\int_Y \mathcal{R}_i W_0(\xi_j + \nabla\theta_{n,\ell}(x))dx = \mathcal{R}_{i+1}W_0(\xi_j)$.

Proof. (i) It suffices to prove that $\sigma_n \to 0$ in $L^p(Y;\mathbb{R})$. For every $k \in \{0, \cdots, n-1\}$, it is clear that $|\sigma_n(x)|^p \leq \frac{t^p(1-t)^p}{n^p}$ for all $x \in]\frac{k}{n}, \frac{k+1}{n}[\times]0,1[$, and so

$$\int_{]\frac{k}{n},\frac{k+1}{n}[\times]0,1[} |\sigma_n(x)|^p dx \leq \frac{t^p(1-t)^p}{n^{p+1}}.$$

As

$$\int_Y |\sigma_n(x)|^p dx = \sum_{k=0}^{n-1}\int_{]\frac{k}{n},\frac{k+1}{n}[\times]0,1[} |\sigma_n(x)|^p dx$$

it follows that

$$\int_Y |\sigma_n(x)|^p dx \leq \frac{t^p(1-t)^p}{n^p},$$

which gives the desired conclusion.

(ii) Recalling that $a=(1,0)$ we see that

$$\xi_j+\nabla\theta_{n,\ell}(x):=\begin{cases}\xi_j-ta\otimes b_\ell & \text{if } x\in\mathrm{int}(A^-_{k,n})\\ \xi_j+(1-t)a\otimes b_\ell & \text{if } x\in\mathrm{int}(A^+_{k,n}\cup B^+_{k,n}\cup C^+_{k,n})\\ \xi_j-t(a+a^\perp)\otimes b_\ell & \text{if } x\in\mathrm{int}(B^-_{k,n})\\ \xi_j-t(a-a^\perp)\otimes b_\ell & \text{if } x\in\mathrm{int}(C^-_{k,n})\\ \xi_j & \text{if } x\in\mathrm{int}(B_{k,n})\cup\mathrm{int}(C_{k,n})\end{cases}$$

with $a^\perp=(0,1)$ (and $\mathrm{int}(E)$ denotes the interior of the set E). Moreover, we have:

$$\int_{\cup_{k=0}^{n-1}A^-_{k,n}}\mathcal{R}_iW_0(\xi_j-ta\otimes b_\ell)dx=(1-t)(1-\tfrac{2}{n})\mathcal{R}_iW_0(\xi_j-ta\otimes b_\ell);$$
$$\int_{\cup_{k=0}^{n-1}A^+_{k,n}}\mathcal{R}_iW_0(\xi_j+(1-t)a\otimes b_\ell)dx=t(1-\tfrac{2}{n})\mathcal{R}_iW_0(\xi_j+(1-t)a\otimes b_\ell);$$
$$\int_{\cup_{k=0}^{n-1}(B^+_{k,n}\cup C^+_{k,n})}\mathcal{R}_iW_0(\xi_j+(1-t)a\otimes b_\ell)dx=\tfrac{t}{n}\mathcal{R}_iW_0(\xi_j+(1-t)a\otimes b_\ell);$$
$$\int_{\cup_{k=0}^{n-1}B^-_{k,n}}\mathcal{R}_iW_0(\xi_j-t(a+a^\perp)\otimes b_\ell)dx=\tfrac{1-t}{2n}\mathcal{R}_iW_0(\xi_j-t(a+a^\perp)\otimes b_\ell);$$
$$\int_{\cup_{k=0}^{n-1}C^-_{k,n}}\mathcal{R}_iW_0(\xi_j-t(a-a^\perp)\otimes b_\ell)dx=\tfrac{1-t}{2n}\mathcal{R}_iW_0(\xi_j-t(a-a^\perp)\otimes b_\ell);$$
$$\int_{\cup_{k=0}^{n-1}(B_{k,n}\cup C_{k,n})}\mathcal{R}_iW_0(\xi_j)dx=\tfrac{1}{n}\mathcal{R}_iW_0(\xi_j).$$

Hence

$$\begin{aligned}\int_Y\mathcal{R}_iW_0(\xi_j+\nabla\theta_{n,\ell}(x))dx &= \Big(1-\frac{2}{n}\Big)\Big[(1-t)\mathcal{R}_iW_0(\xi_j-ta\otimes b_\ell)+t\mathcal{R}_iW_0(\xi_j\\ &\quad+(1-t)a\otimes b_\ell)\Big]+\frac{1}{n}\Big[t\mathcal{R}_iW_0(\xi_j+(1-t)a\otimes b_\ell)\\ &\quad+\frac{1-t}{2}\big(\mathcal{R}_iW_0(\xi_j-t(a+a^\perp)\otimes b_\ell)+\mathcal{R}_iW_0(\xi_j-\\ &\quad t(a-a^\perp)\otimes b_\ell)\big)+\mathcal{R}_iW_0(\xi_j)\Big]\end{aligned}$$

for all $n,\ell\geq 1$. It follows that for every $\ell\geq 1$,

$$\begin{aligned}\lim_{n\to+\infty}\int_Y\mathcal{R}_iW_0(\xi_j+\nabla\theta_{n,\ell}(x))dx &= (1-t)\mathcal{R}_iW_0(\xi_j-ta\otimes b_\ell)\\ &\quad+t\mathcal{R}_iW_0(\xi_j+(1-t)a\otimes b_\ell).\end{aligned}$$

Taking Lemma B.2 into account and noticing that $b_\ell\to b$, we deduce that

$$\begin{aligned}\lim_{\ell\to+\infty}\lim_{n\to+\infty}\int_Y\mathcal{R}_iW_0(\xi_j+\nabla\theta_{n,\ell}(x))dx &= (1-t)\mathcal{R}_iW_0(\xi_j-ta\otimes b)\\ &\quad+t\mathcal{R}_iW_0(\xi_j+(1-t)a\otimes b),\end{aligned}$$

and (ii) follows by using Lemma B.3. □

Consider $V^j_q\subset V_j$ given by $V^j_q:=\{x\in V_j:\mathrm{dist}(x,\partial V_j)>\frac{1}{q}\}$ with $q\geq 1$ large enough. By Vitali's covering theorem, there exists a finite or countable family $(r_m+\rho_mY)_{m\in M}$ of disjoint subsets of V^j_q, with $r_m\in\mathbb{R}^2$ and $\rho_m\in]0,1[$, such that

$|V_q^j \setminus \cup_{m\in M}(r_m + \rho_m Y)| = 0$ (and so $\sum_{m\in M}\rho_m^2 = |V_q^j|$). Let $\{\phi_{n,\ell,q}\}_{n,\ell,q\geq 1} \subset \mathrm{Aff}_0(V_j;\mathbb{R}^3)$ be given by

$$\phi_{n,\ell,q}(x) := \begin{cases} \rho_m\theta_{n,\ell}\left(\dfrac{x-r_m}{\rho_m}\right) & \text{if } x \in r_m + \rho_m Y \subset V_q^j \\ 0 & \text{if } x \in V_j \setminus V_q^j. \end{cases}$$

Lemma B.5. *Define* $\{\Phi^j_{n,\ell,q}\}_{n,\ell,q\geq 1} \subset \mathrm{Aff}(V_j;\mathbb{R}^3)$ *by*

$$\Phi^j_{n,\ell,q}(x) := v(x) + \phi_{n,\ell,q}(x). \tag{24}$$

Then:

- (i) *for every* $n,\ell,q \geq 1$, $\Phi^j_{n,\ell,q}$ *is locally injective*;
- (ii) *for every* $\ell,q\geq 1$, $\Phi^j_{n,\ell,q} \to v$ *in* $L^p(V_j;\mathbb{R}^3)$;
- (iii) $\displaystyle\lim_{q\to+\infty}\lim_{\ell\to+\infty}\lim_{n\to+\infty}\int_{V_j}\mathcal{R}_iW_0(\nabla\Phi^j_{n,\ell,q}(x))dx = |V_j|\mathcal{R}_{i+1}W_0(\xi_j)$.

Proof. (i) Let $x \in V_j$ and let $W \subset V_j$ be the connected component of V_j such that $x \in W$ (as V_j is open, so is W). Since $\nabla v = \xi_j$ in W, there exists $c \in \mathbb{R}^3$ such that $v(x') = \xi_j\cdot x' + c$ for all $x' \in W$. We claim that $\Phi^j_{n,\ell,q}\lfloor_W$ is injective. Indeed, let $x' \in W$ be such that $\Phi^j_{n,\ell,q}(x) = \Phi^j_{n,\ell,q}(x')$. One the three possibilities holds:

- (a) $\Phi^j_{n,\ell,q}(x) = \xi_j\cdot x + c + \rho_m\sigma_n\big(\frac{x-r_m}{\rho_m}\big)b_\ell$ and $\Phi^j_{n,\ell,q}(x') = \xi_j\cdot x' + c + \rho_{m'}\sigma_n\big(\frac{x'-r_{m'}}{\rho_{m'}}\big)b_\ell$;
- (b) $\Phi^j_{n,\ell,q}(x) = \xi_j\cdot x + c + \rho_m\sigma_{n,\ell}\big(\frac{x-r_m}{\rho_m}\big)b_\ell$ and $\Phi^j_{n,\ell,q}(x') = \xi_j\cdot x' + c$;
- (c) $\Phi^j_{n,\ell,q}(x) = \xi_j\cdot x + c$ and $\Phi^j_{n,\ell,q}(x') = \xi_j\cdot x' + c$.

Setting $\alpha := \rho_m\sigma_n(\frac{x-r_m}{\rho_m}) - \rho_{m'}\sigma_n(\frac{x'-r_{m'}}{\rho_{m'}})$ and $\beta := \rho_m\sigma_n(\frac{x-r_m}{\rho_m})$ we have:

$$\begin{cases} \xi_j(x'-x) = 0 & \text{if } \alpha = 0 \\ b_\ell = \frac{1}{\alpha}\xi_j(x'-x) & \text{if } \alpha \neq 0 \end{cases} \quad \text{when (a) is satisfied;}$$

$$\begin{cases} \xi_j(x'-x) = 0 & \text{if } \beta = 0 \\ b_\ell = \frac{1}{\beta}\xi_j(x'-x) & \text{if } \beta \neq 0 \end{cases} \quad \text{when (b) is satisfied;}$$

$$\xi_j(x'-x) = 0 \text{ when (c) is satisfied.}$$

It follows that if $x \neq x'$ then either $\mathrm{rank}(\xi_j) < 2$ or $b_\ell \in \mathrm{Im}\xi_j$ which is impossible. Hence $x = x'$, and the claim is proved. Thus $\Phi^j_{n,\ell,q}$ is locally injective.

(ii) As $\rho_m \in]0,1[$ for all $m \in M$ and $\sum_{m\in M}\rho_m^2 = |V_q^j|$ we have

$$\int_{V_q^j}|\phi_{n,\ell,q}(x)|^p dx \leq |V_q^j|\int_Y |\theta_{n,\ell}(x)|^p dx.$$

Using Lemma B.4(i) we deduce that for every $\ell,q\geq 1$,

$$\lim_{n\to+\infty}\int_{V_q^j}|\phi_{n,\ell,q}(x)|^p dx = 0,$$

and (ii) follows.

(iii) Recalling that $\sum_{m\in M}\rho_m^2=|V_q^j|$ we see that

$$\begin{aligned}\int_{V_j}\mathcal{R}_iW_0(\nabla\Phi^j_{n,\ell,q}(x))dx &= \int_{V_j}\mathcal{R}_iW_0(\xi_j+\nabla\phi_{n,\ell,q}(x))dx\\ &= \int_{V_q^j}\mathcal{R}_iW_0(\xi_j+\nabla\phi_{n,\ell,q}(x))dx+|V_j\setminus V_q^j|\mathcal{R}_iW_0(\xi_j)\\ &= |V_q^j|\int_Y\mathcal{R}_iW_0(\xi_j+\nabla\theta_{n,\ell}(x))dx+|V_j\setminus V_q^j|\mathcal{R}_iW_0(\xi_j).\end{aligned}$$

Using Lemma B.4(ii) we deduce that for every $q\geq 1$,

$$\lim_{\ell\to+\infty}\lim_{n\to+\infty}\int_{V_j}\mathcal{R}_iW_0(\nabla\Phi^j_{n,\ell,q}(x))dx=|V_q^j|\mathcal{R}_{i+1}W_0(\xi_j)+|V_j\setminus V_q^j|\mathcal{R}_iW_0(\xi_j),$$

and (iii) follows by noticing that $|V_q^j|\to|V_j|$ and $|V_j\setminus V_q^j|\to 0$. □

B.2. **Proof of Lemma 5.4.** According to Lemma B.1, it is sufficient to show that for every $i\geq 0$,

$$(P_i)\qquad I(v)\leq\int_\Sigma\mathcal{R}_iW_0(\nabla v(x))dx\text{ for all }v\in\mathrm{Aff}_{\mathrm{li}}(\Sigma;\mathbb{R}^3).$$

The proof is by induction on i. As $R_0W_0=W_0$ it is clear that (P_0) is true. Assume that (P_i) is true, and prove that (P_{i+1}) is true. Let $v\in\mathrm{Aff}_{\mathrm{li}}(\Sigma;\mathbb{R}^3)$. By definition, there exists a finite family $(V_j)_{j\in J}$ of open disjoint subsets of Σ such that $|\partial V_j|=0$ for all $j\in J$, $|\Sigma\setminus\cup_{j\in J}V_j|=0$ and, for every $j\in J$, $\nabla v(x)=\xi_j$ in V_j with $\xi_j\in\mathbb{M}^{3\times 2}$. Define $\{\Psi_{n,\ell,q}\}_{n,\ell,q\geq 1}\subset\mathrm{Aff}(\Sigma;\mathbb{R}^3)$ by

$$\Psi_{n,\ell,q}(x):=\Phi^j_{n,\ell,q}(x)\text{ if }x\in V_j$$

with $\Phi^j_{n,\ell,q}$ given by (24). Taking Lemma B.5(i) into account (and recalling that v is locally injective), it is easy to see that $\Psi_{n,\ell,q}$ is locally injective. Using (P_i) we can assert that

$$I(\Psi_{n,\ell,q})\leq\int_\Sigma R_iW_0(\nabla\Psi_{n,\ell,q}(x))dx\text{ for all }n,\ell,q\geq 1.$$

By Lemma B.5(ii) it is clear that for every $\ell,q\geq 1$, $\Psi_{n,l,q}\to v$ in $L^p(\Sigma;\mathbb{R}^3)$. It follows that

$$I(v)\leq\varliminf_{n\to+\infty}I(\Psi_{n,\ell,q})\leq\varliminf_{n\to+\infty}\int_\Sigma R_iW_0(\nabla\Psi_{n,\ell,q}(x))dx\text{ for all }\ell,q\geq 1.$$

Moreover, from Lemma B.5(iii) we see that

$$\lim_{q\to+\infty}\lim_{\ell\to+\infty}\lim_{n\to+\infty}\int_\Sigma R_iW_0(\nabla\Psi_{n,\ell,q}(x))dx=\int_\Sigma\mathcal{R}_{i+1}W_0(\nabla v(x))dx.$$

Hence

$$I(v)\leq\int_\Sigma\mathcal{R}_{i+1}W_0(\nabla v(x))dx,$$

and the proof is complete. □

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