Zurich Open Repository and Archive
University of Zurich
Main Library
Strickhofstrasse 39
CH-8057 Zurich
www.zora.uzh.ch

Year: 2006

The nonlinear membrane energy: Variational derivation under the constraint "det u 0"

Anza Hafsa, O ; Mandallena, J P

Abstract: Acerbi, Buttazzo and Percivale gave a variational definition of the nonlinear string energy under the constraint "det u>0" (see [E. Acerbi, G. Buttazzo, D. Percivale, A variational definition of the strain energy for an elastic string, J. Elasticity 25 (1991) 137–148]). In the same spirit, we obtain the nonlinear membrane energy under the simpler constraint "det u 0".

DOI: https://doi.org/10.1016/j.matpur.2006.01.004

Other titles: The nonlinear membrane energy: variational derivation under the constraint "det$\nabla u > 0$"

Posted at the Zurich Open Repository and Archive, University of Zurich
ZORA URL: https://doi.org/10.5167/uzh-21588
Journal Article
Accepted Version

Originally published at:
Anza Hafsa, O; Mandallena, J P (2006). The nonlinear membrane energy: Variational derivation under the constraint "det u 0". Journal de Mathematiques Pures et Appliquees, 86(2):100-115.
DOI: https://doi.org/10.1016/j.matpur.2006.01.004

arXiv:math/0507310v1 [math.CA] 15 Jul 2005

THE NONLINEAR MEMBRANE ENERGY: VARIATIONAL DERIVATION UNDER THE CONSTRAINT "det$\nabla u > 0$"

OMAR ANZA HAFSA AND JEAN-PHILIPPE MANDALLENA

Abstract. In [4] we gave a variational definition of the nonlinear membrane energy under the constraint "det$\nabla u \neq 0$". In this paper we obtain the nonlinear membrane energy under the more realistic constraint "det$\nabla u > 0$".

1. Introduction

Consider an elastic material occupying in a reference configuration the bounded open set $\Sigma_\varepsilon \subset \mathbb{R}^3$ given by

$$\Sigma_\varepsilon := \Sigma \times \left]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right[,$$

where $\varepsilon > 0$ is very small and $\Sigma \subset \mathbb{R}^2$ is Lipschitz, open and bounded. A point of Σ_ε is denoted by (x, x_3) with $x \in \Sigma$ and $x_3 \in]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$. Denote by $W : \mathbb{M}^{3\times3} \to [0, +\infty]$ the stored-energy function supposed to be *continuous*. In order to take into account the important physical properties that the interpenetration of matter does not occur and that an infinite amount of energy is required to compress a finite volume into zero volume, i.e.,

$$W(F) \to +\infty \text{ as } \det F \to 0^+,$$

where $\det F$ denotes the determinant of the 3×3 matrix F, we assume that:

(C_1) $W(F) = +\infty$ *if and only if* $\det F \leq 0$;
(C_2) *for every* $\delta > 0$*, there exists* $c_\delta > 0$ *such that for all* $F \in \mathbb{M}^{3\times3}$,

$$\textit{if } \det F \geq \delta \textit{ then } W(F) \leq c_\delta(1 + |F|^p).$$

Our goal is to show that as $\varepsilon \to 0$ the three-dimensional free energy functional $E_\varepsilon : W^{1,p}(\Sigma_\varepsilon; \mathbb{R}^3) \to [0, +\infty]$ (with $p > 1$) defined by

$$E_\varepsilon(u) := \frac{1}{\varepsilon}\int_{\Sigma_\varepsilon} W\big(\nabla u(x, x_3)\big)dxdx_3 \tag{1}$$

converge in a variational sense (cf. Definition 2.1) to the two-dimensional free energy functional $E_{\rm mem} : W^{1,p}(\Sigma; \mathbb{R}^3) \to [0, +\infty]$ given by

$$E_{\rm mem}(v) := \int_\Sigma W_{\rm mem}\big(\nabla v(x)\big)dx \tag{2}$$

with $W_{\rm mem} : \mathbb{M}^{3\times2} \to [0, +\infty]$. Usually, $E_{\rm mem}$ is called the nonlinear membrane energy associated with the two-dimensional elastic material with respect to the reference configuration Σ. Furthermore we wish to give a representation formula for $W_{\rm mem}$.

To our knowledge, the problem of giving a variational definition of the nonlinear membrane energy was studied for the first time by Percivale in [13]. His paper deals with the constraint "det$\nabla u > 0$" but seems to contain some mistakes. It never was published. Then, in [11] Le Dret and Raoult treated the simpler case where W is of p-polynomial growth, i.e., $W(F) \leq c(1+|F|^p)$ for all $F \in \mathbb{M}^{3\times 3}$ and some $c > 0$. Later, in [7] Ben Belgacem announced to have succeed to handle the constraint "$\det\nabla u > 0$". Although a detailed discussion appears in his thesis [6], a proof of his statement [7, Theorem 1] never was completely established and published. Recently, in [4] we gave a variational definition of the nonlinear membrane energy under the constraint "$\det\nabla u \neq 0$". In the present paper, by the same method as in [4], we obtain the nonlinear membrane energy under the more realistic constraint "$\det\nabla u > 0$".

An outline of the paper is as follows. The variational convergence of E_ε to $E_{\rm mem}$ as $\varepsilon \to 0$ as well as a representation formula for $W_{\rm mem}$ are given by Corollary 2.12 (see also Proposition 2.4). Corollary 2.12 is a consequence of Theorems 2.6, 2.7 and 2.11. Theorem 2.6 is proved in Section 4: the principal ingredients are Theorem 2.7 and Proposition 3.3 whose proof (given in Section 3) uses an interchange theorem of infimum and integral that we obtained in [2, Corollary 5.4]. Theorem 2.7 is proved is Section 6: the main arguments are Proposition 2.8 (whose proof is given in Sect. 5.2) and an approximation theorem by Ben Belgacem and Bennequin [6, Lemme 8 p. 114] (see also [14, Proposition C.0.4 p. 127]). Finally, Theorem 2.11 is proved in Section 7, (Theorem 2.11 is in fact contained in our paper [3], but for the convenience of the reader we give the proof).

2. Results

2.1. **Variational convergence.** To accomplish our asymptotic analysis, we use the notion of convergence introduced by Anzellotti, Baldo and Percivale in [5] in order to deal with dimension reduction problems in mechanics. Let $\pi = \{\pi_\varepsilon\}_\varepsilon$ be the family of maps $\pi_\varepsilon : W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3) \to W^{1,p}(\Sigma;\mathbb{R}^3)$ defined by

$$\pi_\varepsilon(u) := \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} u(\cdot, x_3)dx_3.$$

Definition 2.1. We say that E_ε $\Gamma(\pi)$-converges to $E_{\rm mem}$ as $\varepsilon \to 0$, and we write $E_{\rm mem} = \Gamma(\pi)\text{-}\lim_{\varepsilon\to 0} E_\varepsilon$, if the following two assertions hold:

(i) for all $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$ and all $\{u_\varepsilon\}_\varepsilon \subset W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)$,

$$\text{if } \pi_\varepsilon(u_\varepsilon) \rightharpoonup v \text{ in } W^{1,p}(\Sigma;\mathbb{R}^3) \text{ then } E_{\rm mem}(v) \leq \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon);$$

(ii) for all $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$, there exists $\{u_\varepsilon\}_\varepsilon \subset W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)$ such that:

$$\pi_\varepsilon(u_\varepsilon) \rightharpoonup v \text{ in } W^{1,p}(\Sigma;\mathbb{R}^3) \text{ and } E_{\rm mem}(v) \geq \limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon).$$

In fact, Definition 2.1 is a variant of De Giorgi's Γ-convergence. This is made clear by Lemma 2.3 below. Consider $\mathcal{E}_\varepsilon : W^{1,p}(\Sigma;\mathbb{R}^3) \to [0,+\infty]$ defined by

$$\mathcal{E}_\varepsilon(v) := \inf\Big\{E_\varepsilon(u) : \pi_\varepsilon(u) = v\Big\}.$$

Definition 2.2. We say that $\mathcal{E}_\varepsilon$ Γ-converges to $E_{\rm mem}$ as $\varepsilon \to 0$, and we write $E_{\rm mem} = \Gamma\text{-}\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon$ if for every $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$,

$$\Big(\Gamma\text{-}\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon\Big)(v) = \Big(\Gamma\text{-}\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon\Big)(v) = E_{\rm mem}(v)$$

with

$$\left(\Gamma\text{-}\liminf_{\varepsilon\to0}\mathcal{E}_\varepsilon\right)(v):=\inf\left\{\liminf_{\varepsilon\to0}\mathcal{E}_\varepsilon(v_\varepsilon):v_\varepsilon\rightharpoonup v\text{ in }W^{1,p}(\Sigma;\mathbb{R}^3)\right\}$$

and

$$\left(\Gamma\text{-}\limsup_{\varepsilon\to0}\mathcal{E}_\varepsilon\right)(v):=\inf\left\{\limsup_{\varepsilon\to0}\mathcal{E}_\varepsilon(v_\varepsilon):v_\varepsilon\rightharpoonup v\text{ in }W^{1,p}(\Sigma;\mathbb{R}^3)\right\}.$$

Clearly, Definition 2.2 is equivalent to assertions (i) and (ii) in definition 2.1 with "$\pi(u_\varepsilon)\rightharpoonup v$" replaced by "$v_\varepsilon\rightharpoonup v$". (For a deeper discussion of the Γ-convergence theory we refer to the book [9]). It is then obvious that

Lemma 2.3. *$E_{\rm mem}=\Gamma(\pi)\text{-}\lim_{\varepsilon\to0}E_\varepsilon$ if and only if $E_{\rm mem}=\Gamma\text{-}\lim_{\varepsilon\to0}\mathcal{E}_\varepsilon$.*

Suppose that the exterior loads derive from a potential $\Psi:\overline{\Sigma}_1\times\mathbb{R}^3\to\mathbb{R}$ given by $\Psi((x,x_3),\zeta):=\langle\psi(x,x_3),\zeta\rangle+|\zeta|^p$, where $\psi:\overline{\Sigma}_1\to\mathbb{R}^3$ is continuous and $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^3$, and define $L_\varepsilon:W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)\to\mathbb{R}$ and $L_{\rm mem}:W^{1,p}(\Sigma;\mathbb{R}^3)\to\mathbb{R}$ by

$$L_\varepsilon(u):=\frac{1}{\varepsilon}\int_{\Sigma_\varepsilon}\Psi\big((x,x_3),u(x,x_3)\big)dxdx_3\text{ and }L_{\rm mem}(v):=\int_\Sigma\Psi\big((x,0),v(x)\big)dx.$$

Consider also the following coercivity condition:

(C_3) *there exists $C>0$ such that $W(F)\geq C|F|^p$ for all $F\in\mathbb{M}^{3\times3}$.*

Then, using similar arguments to those in [1, proof of Proposition 3.1 p. 141 and proof of Theorem 2.1 p. 145], we obtain

Proposition 2.4. *Assume that (C_3) holds and E_ε in (1) $\Gamma(\pi)$-converge to $E_{\rm mem}$ in (2) as $\varepsilon\to0$, and consider $\{u_\varepsilon\}_\varepsilon\subset W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)$ such that*

$$E_\varepsilon(u_\varepsilon)+L_\varepsilon(u_\varepsilon)-\inf\Big\{E_\varepsilon(u)+L_\varepsilon(u):u\in W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)\Big\}\to0\text{ as }\varepsilon\to0.$$

Then, $\{\pi_\varepsilon(u_\varepsilon)\}_\varepsilon$ is weakly relatively compact and each of its cluster point $\bar{v}$ satisfies

$$E_{\rm mem}(\bar v)+L_{\rm mem}(\bar v)=\min\Big\{E_{\rm mem}(v)+L_{\rm mem}(v):v\in W^{1,p}(\Sigma;\mathbb{R}^3)\Big\}.$$

The $\Gamma(\pi)$-convergence of E_ε in (1) to $E_{\rm mem}$ in (2) as $\varepsilon\to0$ as well as a representation formula for $W_{\rm mem}$ are given by Corollary 2.12. It is a consequence of Theorems 2.6, 2.7 and 2.11. Roughly speaking, Theorems 2.6 and 2.7 establish the existence of the $\Gamma(\pi)$-limit of E_ε as $\varepsilon\to0$ (cf. Sect. 2.2), and Theorem 2.11 gives an integral representation for the corresponding $\Gamma(\pi)$-limit, and so a representation formula for $W_{\rm mem}$ (cf. Sect. 2.3).

2.2. **Γ-convergence of $\mathcal{E}_\varepsilon$ as $\varepsilon\to0$.** Denote by $C^1(\overline{\Sigma};\mathbb{R}^3)$ the space of all restrictions to $\overline{\Sigma}$ of C^1-differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}^3$, and set

$$C^1_*(\overline{\Sigma};\mathbb{R}^3):=\Big\{v\in C^1(\overline{\Sigma};\mathbb{R}^3):\partial_1v(x)\wedge\partial_2v(x)\neq0\text{ for all }x\in\overline{\Sigma}\Big\},$$

where $\partial_1v(x)$ (resp. $\partial_2v(x)$) denotes the partial derivative of v at $x=(x_1,x_2)$ with respect to x_1 (resp. x_2). Let $\mathcal{E}:W^{1,p}(\Sigma;\mathbb{R}^3)\to[0,+\infty]$ be defined by

$$\mathcal{E}(v):=\begin{cases}\displaystyle\int_\Sigma W_0\big(\nabla v(x)\big)dx & \text{if }v\in C^1_*(\overline{\Sigma};\mathbb{R}^3)\\ +\infty & \text{otherwise,}\end{cases}$$

where $W_0 : \mathbb{M}^{3\times 2} \to [0, +\infty]$ is given by

$$W_0(\xi) := \inf_{\zeta\in\mathbb{R}^3} W(\xi \mid \zeta)$$

with $(\xi \mid \zeta)$ denoting the element of $\mathbb{M}^{3\times 3}$ corresponding to $(\xi, \zeta) \in \mathbb{M}^{3\times 2} \times \mathbb{R}^3$. The following lemma gives three elementary properties of W_0 (its proof is left to the reader). Note that conditions (C_1) and (C_2) make that W_0 is not of p-polynomial growth.

Lemma 2.5. *Denote by $\xi_1 \wedge \xi_2$ the cross product of vectors $\xi_1, \xi_2 \in \mathbb{R}^3$.*

(i) *W_0 is continuous.*
(ii) *If (C_1) holds then*
$(\overline{\mathrm{C}}_1)$ *$W_0(\xi_1 \mid \xi_2) = +\infty$ if and only if $\xi_1 \wedge \xi_2 = 0$.*
(iii) *If (C_2) holds then*
$(\overline{\mathrm{C}}_2)$ *for all $\delta > 0$, there exists $c_\delta > 0$ such that for all $\xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3\times 2}$,*

$$\text{if } |\xi_1 \wedge \xi_2| \geq \delta \text{ then } W_0(\xi) \leq c_\delta(1 + |\xi|^p).$$

Taking Lemma 2.3 into account, we see that the existence of the $\Gamma(\pi)$-limit of E_ε as $\varepsilon \to 0$ follows from the following theorem.

Theorem 2.6. *Let assumptions (C_1), (C_2) and (C_3) hold. Then $\Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon = \overline{\mathcal{E}}$ with $\overline{\mathcal{E}} : W^{1,p}(\Sigma; \mathbb{R}^3) \to [0, +\infty]$ given by*

$$\overline{\mathcal{E}}(v) := \inf\left\{\liminf_{n\to+\infty} \mathcal{E}(v_n) : v_n \rightharpoonup v \text{ in } W^{1,p}(\Sigma; \mathbb{R}^3)\right\}.$$

The proof of Theorem 2.6 is established in Section 4. It uses Proposition 3.3 (cf. Section 3) and Theorem 2.7 below.

Theorem 2.7. *If $(\overline{\mathrm{C}}_2)$ holds then $\overline{\mathcal{E}}(v) = \mathcal{I}(v)$ for all $v \in W^{1,p}(\Sigma; \mathbb{R}^3)$, where $\mathcal{I} : W^{1,p}(\Sigma; \mathbb{R}^3) \to [0, +\infty]$ is given by*

$$\mathcal{I}(v) = \inf\left\{\liminf_{n\to+\infty} \int_\Sigma W_0\big(\nabla v_n(x)\big)dx : W^{1,p}(\Sigma; \mathbb{R}^3) \ni v_n \rightharpoonup v \text{ in } W^{1,p}(\Sigma; \mathbb{R}^3)\right\}.$$

Theorem 2.7 is proved in Section 6 by using Proposition 2.8 below (whose proof is given in Sect. 5.2) and an approximation theorem by Ben Belgacem and Bennequin [6, Lemme 8 p. 114] (see also [14, Proposition C.0.4 p. 127]). Denote by $\mathrm{Aff}(\Sigma; \mathbb{R}^3)$ the space of all continuous piecewise affine functions from Σ to $\mathbb{R}^3$.

Proposition 2.8. *If $(\overline{\mathrm{C}}_2)$ holds then $\mathcal{I}(v) = \mathcal{J}(v)$ for all $v \in W^{1,p}(\Sigma; \mathbb{R}^3)$, where $\mathcal{J} : W^{1,p}(\Sigma; \mathbb{R}^3) \to [0, +\infty]$ is given by*

$$\mathcal{J}(v) = \inf\left\{\liminf_{n\to+\infty} \int_\Sigma W_0\big(\nabla v_n(x)\big)dx : \mathrm{Aff}(\Sigma; \mathbb{R}^3) \ni v_n \rightharpoonup v \text{ in } W^{1,p}(\Sigma; \mathbb{R}^3)\right\}.$$

Remark 2.9. Theorem 2.6 can be applied when $W : \mathbb{M}^{3\times 3} \to [0, +\infty]$ is given by

$$W(F) := h(\det F) + |F|^p,$$

where $h : [0, +\infty[\to [0, +\infty]$ is a continuous function such that:

- $h(t) = +\infty$ if and only if $t \leq 0$;
- for every $\delta > 0$, there exists $r_\delta > 0$ such that $h(t) \leq r_\delta$ for all $t \geq \delta$.

2.3. **Integral representation of $\mathcal{I}$.** Define $\mathcal{Z}_1 W_0 : \mathbb{M}^{3\times 2} \to [0, +\infty]$ by

$$\mathcal{Z}_1 W_0(\xi) := \inf\left\{\int_Y W_0\big(\xi + \nabla\phi(y)\big)dy : \phi \in \mathrm{Aff}_0(Y;\mathbb{R}^3)\right\}$$

with $\mathrm{Aff}_0(Y;\mathbb{R}^3) := \{u \in \mathrm{Aff}(Y;\mathbb{R}^3) : u = 0 \text{ on } \partial Y\}$, and set $\mathcal{Z}_k W_0 := \mathcal{Z}_1[\mathcal{Z}_{k-1}W_0]$ for all integers $k \geq 2$ and denote by $\mathcal{Q}W_0$ the quasiconvex envelope of W_0. The following proposition is proved in Sect. 5.3.

Proposition 2.10. *If* $(\overline{\mathrm{C}}_2)$ *holds then* $\mathcal{Q}W_0 = \mathcal{Q}[Z_2 W_0] = \mathcal{Z}_3 W_0$, $\mathcal{Q}W_0$ *is continuous and there exists* $c > 0$ *such that* $\mathcal{Q}W_0(\xi) \leq c(1 + |\xi|^p)$ *for all* $\xi \in \mathbb{M}^{3\times 2}$.

Taking Theorems 2.6 and 2.7 together with Lemmas 2.3 and 2.5(iii) into account, we see that Theorem 2.11 below (proved in Section 7) gives an integral representation for the $\Gamma(\pi)$-limit of E_ε as $\varepsilon \to 0$ as well as a representation formula for W_{mem}.

Theorem 2.11. *If* $(\overline{\mathrm{C}}_2)$ *holds then for every* $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$,

$$\mathcal{I}(v) = \int_\Sigma \mathcal{Q}W_0\big(\nabla v(x)\big)dx.$$

2.4. **$\Gamma(\pi)$-convergence of E_ε to E_{mem} as $\varepsilon \to 0$.** According to Lemmas 2.3 and Lemma 2.5(iii), a direct consequence of Theorems 2.6, 2.7 and 2.11 is the following.

Corollary 2.12. *Let assumptions* (C_1), (C_2) *and* (C_3) *hold. Then as* $\varepsilon \to 0$, E_ε *in* (1) $\Gamma(\pi)$*-converge to* E_{mem} *in* (2) *with* $W_{\mathrm{mem}} = \mathcal{Q}W_0$.

3. Representation of $\mathcal{E}$

The goal of this section is to show Proposition 3.3 below. To this end, we begin by proving two lemmas.

For every $v \in C^1_*(\overline{\Sigma};\mathbb{R}^3)$ and $j \geq 1$, we define the multifunction $\Lambda^j_v : \overline{\Sigma} \rightrightarrows \mathbb{R}^3$ by

$$\Lambda^j_v(x) := \left\{\zeta \in \mathbb{R}^3 : \det(\nabla v(x) \mid \zeta) \geq \frac{1}{j}\right\}.$$

Lemma 3.1. *Let* $v \in C^1_*(\overline{\Sigma};\mathbb{R}^3)$. *Then*:

(i) *for every* $j \geq 1$, Λ^j_v *is a nonempty convex closed-valued lower semicontinuous*[1] *multifunction*;
(ii) *for every* $x \in \overline{\Sigma}$, $\Lambda^1_v(x) \subset \cdots \subset \Lambda^j_v(x) \subset \cdots \subset \cup_{j\geq 1}\Lambda^j_v(x) = \Lambda_v(x)$, *where* $\Lambda_v(x) := \{\zeta \in \mathbb{R}^3 : \det(\nabla v(x) \mid \zeta) > 0\}$.

Proof. (i) Let $j \geq 1$. It is easy to see that for every $x \in \overline{\Sigma}$, $\Lambda^j_v(x)$ is nonempty, convex and closed. Let X be a closed subset of $\mathbb{R}^3$, let $x \in \overline{\Sigma}$, and let $\{x_n\}_{n\geq 1} \subset \overline{\Sigma}$ such that $|x_n - x| \to 0$ as $n \to +\infty$ and $\Lambda^j_v(x_n) \subset X$ for all $n \geq 1$. Let $\zeta \in \Lambda^j_v(x)$ and let $\{\zeta_m\}_{m\geq 1} \subset \mathbb{R}^3$ be given by $\zeta_m := \zeta + \frac{1}{m}\zeta$. Then, for every $m \geq 1$,

$$\det\big(\nabla v(x) \mid \zeta_m\big) = \det\big(\nabla v(x) \mid \zeta\big) + \frac{1}{m}\det\big(\nabla v(x) \mid \zeta\big) \geq \frac{1}{j} + \frac{1}{mj}. \tag{3}$$

Fix any $m \geq 1$. Since $\det(\nabla v(x_n) \mid \zeta_m) \to \det(\nabla v(x) \mid \zeta_m)$ as $n \to +\infty$, using (3) we see that $\det(\nabla v(x_{n_0}) \mid \zeta_m) > \frac{1}{j}$ for some $n_0 \geq 1$, so that $\zeta_m \in \Lambda^j_v(x_{n_0})$. Thus $\zeta_m \in X$ for all $m \geq 1$. As X is closed we have $\zeta = \lim_{m\to+\infty}\zeta_m \in X$.

[1] A multifunction $\Lambda : \overline{\Sigma} \to \mathbb{R}^3$ is said to be lower semicontinuous if for every closed subset X of $\mathbb{R}^3$, every $x \in \overline{\Sigma}$ and every $\{x_n\}_{n\geq 1} \subset \overline{\Sigma}$ such that $|x_n - x| \to 0$ as $n \to +\infty$ and $\Lambda(x_n) \subset X$ for all $n \geq 1$, we have $\Lambda(x) \subset X$.

(ii) Let $x \in \overline{\Sigma}$. It is clear that $\Lambda_v^1(x) \subset \cdots \subset \Lambda_v^j(x) \subset \cdots \subset \cup_{j\geq 1}\Lambda_v^j(x) \subset \Lambda_v(x)$. If there exists $\zeta \in \Lambda_v(x) \setminus \cup_{j\geq 1}\Lambda_v^j(x)$, then $0 < \det(\nabla v(x) \mid \zeta) < \frac{1}{j}$ for all $j \geq 1$, and letting $j \to +\infty$ we obtain $0 < \det(\nabla v(x) \mid \zeta) \leq 0$, which is impossible. Hence $\cup_{j\geq 1}\Lambda_v^j(x) = \Lambda_v(x)$. □

In the sequel, given $\Lambda : \overline{\Sigma} \rightrightarrows \mathbb{R}^3$ we set

$$C(\overline{\Sigma};\Lambda) := \Big\{\phi \in C\big(\overline{\Sigma};\mathbb{R}^3\big) : \phi(x) \in \Lambda(x) \text{ for all } x \in \overline{\Sigma}\Big\},$$

where $C(\overline{\Sigma};\mathbb{R}^3)$ denotes the space of all continuous functions from $\overline{\Sigma}$ to $\mathbb{R}^3$.

Lemma 3.2. *Given $v \in C^1_*(\Sigma;\mathbb{R}^3)$ and $j \geq 1$, if* (C_2) *holds, then*

$$\inf_{\phi\in C(\overline{\Sigma};\Lambda_v^j)} \int_\Sigma W\big(\nabla v(x) \mid \phi(x)\big)dx = \int_\Sigma \inf_{\zeta\in\Lambda_v^j(x)} W\big(\nabla v(x) \mid \zeta\big)dx.$$

Proof. The lemma follows from the following interchange result of infimum and integral (that we proved in [2]).

Interchange of infimum and integral ([2, Corollary 5.4]). *Given $\Gamma : \overline{\Sigma} \rightrightarrows \mathbb{R}^3$ and $f : \overline{\Sigma} \times \mathbb{R}^3 \to [0,+\infty]$, suppose that*:

(H_1) *f is a Carathéodory integrand*;

(H_2) *Γ is a nonempty convex closed-valued lower semicontinuous multifunction*;

(H_3) *$C(\overline{\Sigma};\Gamma) \neq \emptyset$ and for every $\phi, \hat{\phi} \in C(\overline{\Sigma};\Gamma)$,*

$$\int_\Sigma \max_{\alpha\in[0,1]} f\big(x, \alpha\phi(x) + (1-\alpha)\hat{\phi}(x)\big)dx < +\infty.$$

Then,

$$\inf_{\phi\in C(\overline{\Sigma};\Gamma)} \int_\Sigma f\big(x,\phi(x)\big)dx = \int_\Sigma \inf_{\zeta\in\Gamma(x)} f(x,\zeta)dx.$$

Since W is continuous, (H_1) holds with $f(x,\zeta) = W(\nabla v(x) \mid \zeta)$. Lemma 3.1 shows that ($H_2$) is satisfied with $\Gamma = \Lambda_v^j$, and from Michael's selection continuous theorem [12], we deduce that $C(\overline{\Sigma};\Lambda_v^j) \neq \emptyset$. Given $\phi, \hat{\phi} \in C(\overline{\Sigma};\Lambda_v^j)$, it is clear that for every $\alpha \in [0,1]$ and every $x \in \overline{\Sigma}$, $\det(\nabla v(x) \mid \alpha\phi(x) + (1-\alpha)\hat{\phi}(x)) \geq \frac{1}{j}$. From ($C_2$) it follows that there exists $c > 0$ depending only on j, v, ϕ and $\hat{\phi}$ such that $W(\nabla v(x) \mid \alpha\phi(x) + (1-\alpha)\hat{\phi}(x)) \leq c$ for all $x \in \overline{\Sigma}$. Thus (H_3) is verified with $f(x,\zeta) = W(\nabla v(x) \mid \zeta)$ and $\Gamma = \Lambda_v^j$, and the proof of Lemma 3.2 is complete. □

Here is our (non integral) representation theorem for $\mathcal{E}$.

Proposition 3.3. *If* (C_1), (C_2) *and* (C_3) *hold, then for every $v \in C^1_*(\overline{\Sigma};\mathbb{R}^3)$,*

$$\mathcal{E}(v) = \inf_{j\geq 1} \inf_{\phi\in C(\overline{\Sigma};\Lambda_v^j)} \int_\Sigma W\big(\nabla v(x) \mid \phi(x)\big)dx. \tag{4}$$

Proof. Fix $v \in C^1_*(\overline{\Sigma};\mathbb{R}^3)$ and denote by $\hat{\mathcal{E}}(v)$ the right-hand side of (4). It is easy to verify that $\mathcal{E}(v) \leq \hat{\mathcal{E}}(v)$. We are thus reduced to prove that

$$\hat{\mathcal{E}}(v) \leq \mathcal{E}(v). \tag{5}$$

Using Lemma 3.2, we obtain

$$\hat{\mathcal{E}}(v) \leq \inf_{j\geq 1} \int_\Sigma \inf_{\zeta\in\Lambda_v^j(x)} W\big(\nabla v(x) \mid \zeta\big)dx. \tag{6}$$

Consider the continuous function $\Phi : \overline{\Sigma} \to \mathbb{R}^3$ defined by

$$\Phi(x) := \frac{\partial_1 v(x) \wedge \partial_2 v(x)}{|\partial_1 v(x) \wedge \partial_2 v(x)|^2}.$$

Then, $\det(\nabla v(x) \mid \Phi(x)) = 1$ for all $x \in \overline{\Sigma}$. Using (C$_2$) we deduce that there exists $c > 0$ depending only on p such that

$$\int_\Sigma \inf_{\zeta \in \Lambda_v^1(x)} W(\nabla v(x) \mid \zeta) dx \leq c\big(|\Sigma| + \|\nabla v\|^p_{L^p(\Sigma;\mathbb{M}^{3\times 2})} + \|\Phi\|^p_{L^p(\Sigma;\mathbb{R}^3)}\big).$$

It follows that $\inf_{\zeta \in \Lambda_v^1(\cdot)} W(\nabla v(\cdot) \mid \zeta) \in L^1(\Sigma)$. From Lemma 3.1(i) and (ii), we see that $\{\inf_{\zeta \in \Lambda_v^j(\cdot)} W(\nabla v(\cdot) \mid \zeta)\}_{j\geq 1}$ is non-increasing, and that for every $x \in \overline{\Sigma}$,

$$\inf_{j\geq 1} \inf_{\zeta \in \Lambda_v^j(x)} W\big(\nabla v(x) \mid \zeta\big) = W_0\big(\nabla v(x)\big), \tag{7}$$

and (21) follows from (6) and (7) by using Lebesgue's dominated convergence theorem. □

4. Proof of Theorem 2.6

In this section we prove Theorem 2.6. Since $\Gamma\text{-}\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon \leq \Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon$, we only need to show that:

(a) $\overline{\mathcal{E}} \leq \Gamma\text{-}\liminf\limits_{\varepsilon\to 0} \mathcal{E}_\varepsilon$;

(b) $\Gamma\text{-}\limsup\limits_{\varepsilon\to 0} \mathcal{E}_\varepsilon \leq \overline{\mathcal{E}}$.

In the sequel, we follow the notation used in Section 3.

4.1. Proof of (a).

Let $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$ and let $\{v_\varepsilon\}_\varepsilon \subset W^{1,p}(\Sigma;\mathbb{R}^3)$ be such that $v_\varepsilon \rightharpoonup v$ in $W^{1,p}(\Sigma;\mathbb{R}^3)$. We have to prove that

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(v_\varepsilon) \geq \overline{\mathcal{E}}(v). \tag{8}$$

Without loss of generality we can assume that $\sup_{\varepsilon>0} \mathcal{E}_\varepsilon(v_\varepsilon) < +\infty$ and $v_\varepsilon \to v$ in $L^p(\Sigma;\mathbb{R}^3)$. To every $\varepsilon > 0$ there corresponds $u_\varepsilon \in \pi_\varepsilon^{-1}(v_\varepsilon)$ such that

$$\mathcal{E}_\varepsilon(v_\varepsilon) \geq E_\varepsilon(u_\varepsilon) - \varepsilon. \tag{9}$$

Defining $\hat{u}_\varepsilon : \Sigma_1 \to \mathbb{R}^3$ by $\hat{u}_\varepsilon(x, x_3) := u_\varepsilon(x, \varepsilon x_3)$ we have

$$E_\varepsilon\big(u_\varepsilon\big) = \int_{\Sigma_1} W\Big(\partial_1 \hat{u}_\varepsilon(x,x_3) \mid \partial_2 \hat{u}_\varepsilon(x,x_3) \mid \frac{1}{\varepsilon}\partial_3 \hat{u}_\varepsilon(x,x_3)\Big) dx dx_3. \tag{10}$$

From (C$_3$) we deduce that there exists $c > 0$ such that $\|\partial_3 \hat{u}_\varepsilon\|_{L^p(\Sigma_1;\mathbb{R}^3)} \leq c\varepsilon^p$ for all $\varepsilon > 0$, and so $\|\hat{u}_\varepsilon - v_\varepsilon\|_{L^p(\Sigma_1;\mathbb{R}^3)} \leq c'\varepsilon^p$ by Poincaré-Wirtinger's inequality, where $c' > 0$ is a constant which does not depend on ε. It follows that $\hat{u}_\varepsilon \to v$ in $L^p(\Sigma_1;\mathbb{R}^3)$. For $x_3 \in]-\frac{1}{2},\frac{1}{2}[$, let $w_\varepsilon^{x_3} \in W^{1,p}(\Sigma;\mathbb{R}^3)$ given by $w_\varepsilon^{x_3}(x) := \hat{u}_\varepsilon(x,x_3)$. Then (up to a subsequence) $w_\varepsilon^{x_3} \to v$ in $L^p(\Sigma;\mathbb{R}^3)$ for a.e. $x_3 \in]-\frac{1}{2},\frac{1}{2}[$. Taking (9) and (10) into account and using Fatou's lemma, we obtain

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(v_\varepsilon) \geq \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\liminf_{\varepsilon\to 0} \int_\Sigma W_0\big(\nabla w_\varepsilon^{x_3}(x)\big) dx\right) dx_3,$$

and so $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(v_\varepsilon) \geq \hat{\mathcal{I}}(v)$ with $\hat{\mathcal{I}} : W^{1,p}(\Sigma;\mathbb{R}^3) \to [0,+\infty]$ given by

$$\hat{\mathcal{I}}(z) := \inf\left\{\liminf_{n\to+\infty} \int_\Sigma W_0\big(\nabla z_n(x)\big) dx : W^{1,p}(\Sigma;\mathbb{R}^3) \ni z_n \to z \text{ in } L^p(\Sigma;\mathbb{R}^3)\right\}.$$

Using (C_3) we see that $\hat{\mathcal{I}} = \mathcal{I}$, and (8) follows by using Theorem 2.7. □

4.2. **Proof of (b).** As $\Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Sigma;\mathbb{R}^3)$, it is sufficient to prove that for every $v \in C^1_*(\overline{\Sigma};\mathbb{R}^3)$,

$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(v) \le \mathcal{E}(v). \tag{11}$$

Given $v \in C^1_*(\overline{\Sigma};\mathbb{R}^3)$, fix any $j \ge 1$, and any $n \ge 1$. Using Proposition 3.3 we obtain the existence of $\phi \in C(\overline{\Sigma};\Lambda^j_v)$ such that

$$\int_\Sigma W\big(\nabla v(x) \mid \phi(x)\big)dx \le \mathcal{E}(v) + \frac{1}{n}. \tag{12}$$

By Stone-Weierstrass's approximation theorem, there exists $\{\phi_k\}_{k\ge 1} \subset C^\infty(\overline{\Sigma};\mathbb{R}^3)$ such that

$$\phi_k \to \phi \text{ uniformly as } k \to +\infty. \tag{13}$$

We claim that:

(c_1) $\det\big(\nabla v(x) \mid \phi_k(x)\big) \ge \dfrac{1}{2j}$ for all $x \in \overline{\Sigma}$, all $k \ge k_v$ and some $k_v \ge 1$;

(c_2) $\displaystyle\lim_{k\to+\infty} \int_\Sigma W\big(\nabla v(x) \mid \phi_k(x)\big)dx = \int_\Sigma W\big(\nabla v(x) \mid \phi(x)\big)dx.$

Indeed, setting $\mu_v := \sup_{x\in\overline{\Sigma}} |\partial_1 v(x) \wedge \partial_2(x)|$ ($\mu_v > 0$) and using (13), we deduce that there exists $k_v \ge 1$ such that for every $k \ge k_v$,

$$\sup_{x\in\overline{\Sigma}} \big|\phi_k(x) - \phi(x)\big| < \frac{1}{2j\mu_v}. \tag{14}$$

Let $x \in \overline{\Sigma}$, and let $k \ge k_v$. As $\phi \in C(\overline{\Sigma};\Lambda^j_v)$ we have

$$\det\big(\nabla v(x) \mid \phi_k(x)\big) \ge \frac{1}{j} - \det\big(\nabla v(x) \mid \phi_k(x) - \phi(x)\big). \tag{15}$$

Noticing that $\det(\nabla v(x) \mid \phi_k(x) - \phi(x)) \le |\partial_1 v(x) \wedge \partial_2 v(x)||\phi_k(x) - \phi(x)|$, from (14) and (15) we deduce that $\det\big(\nabla v(x) \mid \phi_k(x)\big) \ge \frac{1}{2j}$, and ($c_1$) is proved. Combining ($c_1$) with ($C_2$) we see that $\sup_{k\ge k_v} W(\nabla v(\cdot) \mid \phi_k(\cdot)) \in L^1(\Sigma)$. As W is continuous we have $\lim_{k\to+\infty} W(\nabla v(x) \mid \phi_k(x)) = W(\nabla v(x) \mid \phi(x))$ for all $x \in V$, and (c_2) follows by using Lebesgue's dominated convergence theorem, which completes the claim.

Fix any $k \ge k_v$ and define $\theta :]-\frac{1}{2},\frac{1}{2}[\to \mathbb{R}$ by $\theta(x_3) := \inf_{x\in\overline{\Sigma}} \det(\nabla v(x) + x_3\nabla\phi_k(x) \mid \phi_k(x))$. Clearly θ is continuous. By (c_1) we have $\theta(0) \ge \frac{1}{2j}$, and so there exists $\eta_v \in]0,\frac{1}{2}[$ such that $\theta(x_3) \ge \frac{1}{4j}$ for all $x_3 \in]-\eta_v,\eta_v[$. Let $u_k : \Sigma_1 \to \mathbb{R}$ be given by $u_k(x,x_3) := v(x) + x_3\phi_k(x)$. From the above it follows that

(c_3) $\det\nabla u_k(x,\varepsilon x_3) \ge \frac{1}{4j}$ for all $\varepsilon \in]0,\eta_v[$ and all $(x,x_3) \in \overline{\Sigma}\times]-\frac{1}{2},\frac{1}{2}[$.

As in the proof of (c_1), from (c_3) together with (C_2) and the continuity of W, we obtain

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_k) = \lim_{\varepsilon\to 0} \int_{\Sigma_1} W\big(\nabla u_k(x,\varepsilon x_3)\big)dxdx_3 = \int_\Sigma W\big(\nabla v(x) \mid \phi_k(x)\big)dx. \tag{16}$$

For every $\varepsilon > 0$ and every $k \ge k_v$, since $\pi_\varepsilon(u_k) = v$ we have $\mathcal{E}_\varepsilon(v) \le E_\varepsilon(u_k)$. Using (16), ($c_2$) and (12), we deduce that

$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(v) \le \mathcal{E}(v) + \frac{1}{n},$$

and (11) follows by letting $n \to +\infty$. □

5. Proof of Propositions 2.8 and 2.10

In fact, Proposition 2.8 and 2.10 are contained in our paper [4]. In order to be complete, we gives the proofs.

5.1. **Preliminary results.** The following lemma gives two interesting properties of functions $\mathcal{Z}_k W_0$ defined in Sect. 2.3.

Lemma 5.1. *Every function $\mathcal{Z}_k W_0$ with $k \geq 1$ satisfies the following two properties*:

(i) *for every bounded open set $D \subset \mathbb{R}^2$ with $|\partial D| = 0$ and every $\xi \in \mathbb{M}^{3\times 2}$,*

$$\mathcal{Z}_k W_0(\xi) = \inf\left\{\frac{1}{|D|}\int_D \mathcal{Z}_{k-1}W_0\big(\xi + \nabla\phi(y)\big)dy : \phi \in \mathrm{Aff}_0(D;\mathbb{R}^3)\right\}.$$

(ii) *$\mathcal{Z}_k W_0$ is continuous in the interior of its effective domain.*

Proof. It follows from [10, Lemma 2.16, Theorem 2.17 and Proposition 2.3]. □

Lemma 5.1(i) (resp. Lemma 5.1(ii)) will be used in the proof of Proposition 5.2 below (resp. Proposition 2.10).

Proposition 5.2. *If $(\overline{\mathrm{C}}_2)$ holds then $\mathcal{Z}_2 W_0(\xi) \leq c(1 + |\xi|^p)$ for all $\xi \in \mathbb{M}^{3\times 2}$ and some $c > 0$.*

Proposition 5.2 will play an essential role in the proof of Propositions 2.8 and 2.10. To show Proposition 5.2 we need the following lemma.

Lemma 5.3. *If $(\overline{\mathrm{C}}_2)$ holds then for every $\delta > 0$, there exists $r_\delta > 0$ such that for every $\xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3\times 2}$,*

$$\textit{if } \min\big\{|\xi_1 + \xi_2|, |\xi_1 - \xi_2|\big\} \geq \delta \textit{ then } \mathcal{Z}_1 W_0(\xi) \leq r_\delta\big(1 + |\xi|^p\big).$$

Proof. Let $\delta > 0$ and $\xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3\times 2}$ be such that $\min\{|\xi_1 + \xi_2|, |\xi_1 - \xi_2|\} \geq \delta$. Set

$$D := \Big\{(x_1, x_2) \in \mathbb{R}^2 : x_1 - 1 < x_2 < x_1 + 1 \text{ and } -x_1 - 1 < x_2 < 1 - x_1\Big\}.$$

For each $t \in \mathbb{R}$, define $\varphi_t \in \mathrm{Aff}_0(D;\mathbb{R})$ by

$$\varphi_t(x_1, x_2) := \begin{cases} -tx_1 + t(x_2 + 1) & \text{if } (x_1, x_2) \in \Delta_1 \\ t(1 - x_1) - tx_2 & \text{if } (x_1, x_2) \in \Delta_2 \\ tx_1 + t(1 - x_2) & \text{if } (x_1, x_2) \in \Delta_3 \\ t(x_1 + 1) + tx_2 & \text{if } (x_1, x_2) \in \Delta_4 \end{cases}$$

with

$$\begin{aligned} \Delta_1 &:= \big\{(x_1, x_2) \in D : x_1 \geq 0 \text{ and } x_2 \leq 0\big\}; \\ \Delta_2 &:= \big\{(x_1, x_2) \in D : x_1 \geq 0 \text{ and } x_2 \geq 0\big\}; \\ \Delta_3 &:= \big\{(x_1, x_2) \in D : x_1 \leq 0 \text{ and } x_2 \geq 0\big\}; \\ \Delta_4 &:= \big\{(x_1, x_2) \in D : x_1 \leq 0 \text{ and } x_2 \leq 0\big\}. \end{aligned}$$

Assume first $|\xi_1\wedge\xi_2|\neq 0$. Define $\phi\in\mathrm{Aff}_0(D;\mathbb{R}^3)$ by $\phi:=(\varphi_{\nu_1},\varphi_{\nu_2},\varphi_{\nu_3})$ with $\nu:=\frac{\xi_1\wedge\xi_2}{|\xi_1\wedge\xi_2|}$, where ν_1, ν_2, ν_3 are the components of the vector ν. Then,

$$\xi+\nabla\phi(x)=\begin{cases}(\xi_1-\nu\mid\xi_2+\nu) & \text{if } x\in\Delta_1\\ (\xi_1-\nu\mid\xi_2-\nu) & \text{if } x\in\Delta_2\\ (\xi_1+\nu\mid\xi_2-\nu) & \text{if } x\in\Delta_3\\ (\xi_1+\nu\mid\xi_2+\nu) & \text{if } x\in\Delta_4.\end{cases}$$

Taking Lemma 5.1(i) into account, it follows that

$$\begin{aligned}\mathcal{Z}_1W_0(\xi)\ \le\ & \frac{1}{4}\Big(W_0\big(\xi_1-\nu\mid\xi_2+\nu\big)+W_0\big(\xi_1-\nu\mid\xi_2-\nu\big)\\ &+W_0\big(\xi_1+\nu\mid\xi_2-\nu\big)+W_0\big(\xi_1+\nu\mid\xi_2+\nu\big)\Big).\end{aligned}\tag{17}$$

But

$$\begin{aligned}\big|(\xi_1-\nu)\wedge(\xi_2+\nu)\big|^2 &= \big|\xi_1\wedge\xi_2+(\xi_1+\xi_2)\wedge\nu\big|^2\\ &= \big|\xi_1\wedge\xi_2\big|^2+\big|(\xi_1+\xi_2)\wedge\nu\big|^2\\ &\ge \big|(\xi_1+\xi_2)\wedge\nu\big|^2,\end{aligned}$$

and so $|(\xi_1+\nu)\wedge(\xi_2-\nu)|\ge|(\xi_1+\xi_2)\wedge\nu|=|\xi_1+\xi_2|$. Similarly, we obtain:

$$\begin{aligned}&\big|(\xi_1-\nu)\wedge(\xi_2-\nu)\big|\ge\big|\xi_1-\xi_2\big|;\\ &\big|(\xi_1+\nu)\wedge(\xi_2-\nu)\big|\ge\big|\xi_1+\xi_2\big|;\\ &\big|(\xi_1+\nu)\wedge(\xi_2+\nu)\big|\ge\big|\xi_1-\xi_2\big|.\end{aligned}$$

Thus, $|(\xi_1-\nu)\wedge(\xi_2+\nu)|\ge\delta$, $|(\xi_1-\nu)\wedge(\xi_2-\nu)|\ge\delta$, $|(\xi_1+\nu)\wedge(\xi_2-\nu)|\ge\delta$ and $|(\xi_1+\nu)\wedge(\xi_2+\nu)|\ge\delta$, because $\min\big\{|\xi_1+\xi_2|,|\xi_1-\xi_2|\big\}\ge\delta$. Using $(\overline{\mathrm{C}}_2)$, it follows that

$$\begin{aligned}W_0\big(\xi_1-\nu\mid\xi_2+\nu\big) &\le c_\delta\big(1+\big|(\xi_1-\nu\mid\xi_2+\nu)\big|^p\big)\\ &\le c_\delta 2^p\big(1+\big|(\xi_1\mid\xi_2)\big|^p+\big|(-\nu\mid\nu)\big|^p\big)\\ &\le c_\delta 2^{2p+1}\big(1+\big|\xi\big|^p\big).\end{aligned}$$

In the same manner, we have:

$$\begin{aligned}&W_0\big(\xi_1-\nu\mid\xi_2-\nu\big)\le c_\delta 2^{2p+1}\big(1+\big|\xi\big|^p\big);\\ &W_0\big(\xi_1+\nu\mid\xi_2-\nu\big)\le c_\delta 2^{2p+1}\big(1+\big|\xi\big|^p\big);\\ &W_0\big(\xi_1+\nu\mid\xi_2+\nu\big)\le c_\delta 2^{2p+1}\big(1+\big|\xi\big|^p\big),\end{aligned}$$

and, from (17), we conclude that $\mathcal{Z}_1W_0(\xi)\le c_\delta 2^{2p+1}(1+\big|\xi\big|^p)$.

Assume now $|\xi_1\wedge\xi_2|=0$. Then, one the three following possibilities holds:

(i) $\xi_1\neq 0$ and $\xi_2=0$;
(ii) $\xi_1=0$ and $\xi_2\neq 0$;
(iii) there exists $\lambda\in\mathbb{R}$ such that $\xi_1=\lambda\xi_2$ (with $\xi_1\neq 0$ and $\xi_2\neq 0$).

Let $\sigma\in\mathbb{R}^3$ be such that

$$|\sigma|=1 \text{ and } \begin{cases}\langle\xi_1,\sigma\rangle=0 & \text{if either (i) or (iii) is satisfied}\\ \langle\xi_2,\sigma\rangle=0 & \text{if (ii) is satisfied.}\end{cases}$$

Defining $\psi \in \mathrm{Aff}_0(D;\mathbb{R}^3)$ by $\psi := (\varphi_{\sigma_1}, \varphi_{\sigma_2}, \varphi_{\sigma_3})$, and using Lemma 5.1(i) we see that

$$\begin{aligned}(18)\qquad \mathcal{Z}_1 W_0(\xi) \quad &\leq \quad \frac{1}{4}\Big(W_0\big(\xi_1-\sigma \mid \xi_2+\sigma\big)+W_0\big(\xi_1-\sigma \mid \xi_2-\sigma\big)\\ &\qquad + W_0\big(\xi_1+\sigma \mid \xi_2-\sigma\big)+W_0\big(\xi_1+\sigma \mid \xi_2+\sigma\big)\Big).\end{aligned}$$

Moreover, we have:

$$\begin{aligned}&|(\xi_1-\sigma)\wedge(\xi_2+\sigma)| = |(\xi_1+\xi_2)\wedge\sigma| = |\xi_1+\xi_2| \geq \delta;\\ &|(\xi_1-\sigma)\wedge(\xi_2-\sigma)| = |\sigma\wedge(\xi_1-\xi_2)| = |\xi_1-\xi_2| \geq \delta;\\ &|(\xi_1+\sigma)\wedge(\xi_2-\sigma)| = |\sigma\wedge(\xi_1+\xi_2)| = |\xi_1+\xi_2| \geq \delta;\\ &|(\xi_1+\sigma)\wedge(\xi_2+\sigma)| = |(\xi_1-\xi_2)\wedge\sigma| = |\xi_1-\xi_2| \geq \delta,\end{aligned}$$

and, by $(\overline{\mathrm{C}}_2)$, we obtain:

$$\begin{aligned}&W_0\big(\xi_1-\sigma \mid \xi_2+\sigma\big) \leq c_\delta 2^{2p+1}\big(1+|\xi|^p\big);\\ &W_0\big(\xi_1-\sigma \mid \xi_2-\sigma\big) \leq c_\delta 2^{2p+1}\big(1+|\xi|^p\big);\\ &W_0\big(\xi_1+\sigma \mid \xi_2-\sigma\big) \leq c_\delta 2^{2p+1}\big(1+|\xi|^p\big);\\ &W_0\big(\xi_1+\sigma \mid \xi_2+\sigma\big) \leq c_\delta 2^{2p+1}\big(1+|\xi|^p\big).\end{aligned}$$

From (18) it follows that $\mathcal{Z}_1 W_0(\xi) \leq c_\delta 2^{2p+1}(1+|\xi|^p)$, and the proof is complete. □

Proof of Proposition 5.2. Let $\xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3\times 2}$. For each $t \in \mathbb{R}$, define $\varphi_t \in \mathrm{Aff}_0(Y;\mathbb{R})$ by

$$\varphi_t(x_1,x_2) := \begin{cases} tx_2 & \text{if } (x_1,x_2)\in\Delta_1\\ t(1-x_1) & \text{if } (x_1,x_2)\in\Delta_2\\ t(1-x_2) & \text{if } (x_1,x_2)\in\Delta_3\\ tx_1 & \text{if } (x_1,x_2)\in\Delta_4\end{cases}$$

with

$$\begin{aligned}&\Delta_1 := \big\{(x_1,x_2)\in Y : x_2 \leq x_1 \leq -x_2+1\big\};\\ &\Delta_2 := \big\{(x_1,x_2)\in Y : -x_1+1 \leq x_2 \leq x_1\big\};\\ &\Delta_3 := \big\{(x_1,x_2)\in Y : -x_2+1 \leq x_1 \leq x_2\big\};\\ &\Delta_4 := \big\{(x_1,x_2)\in Y : x_1 \leq x_2 \leq -x_1+1\big\}.\end{aligned}$$

Assume first $|\xi_1\wedge\xi_2| \neq 0$. Define $\phi \in \mathrm{Aff}_0(Y;\mathbb{R}^3)$ by $\phi := (\varphi_{\nu_1}, \varphi_{\nu_2}, \varphi_{\nu_3})$ with $\nu := \frac{(\xi_1\wedge\xi_2)}{|\xi_1\wedge\xi_2|}$, where ν_1, ν_2, ν_3 are the components of the vector ν. Then,

$$\xi + \nabla\phi(x) = \begin{cases} (\xi_1 \mid \xi_2+\nu) & \text{if } x\in\Delta_1\\ (\xi_1-\nu \mid \xi_2) & \text{if } x\in\Delta_2\\ (\xi_1 \mid \xi_2-\nu) & \text{if } x\in\Delta_3\\ (\xi_1+\nu \mid \xi_2) & \text{if } x\in\Delta_4.\end{cases}$$

But

$$\begin{aligned}\big|\xi_1+(\xi_2+\nu)\big|^2 \quad &= \quad \big|(\xi_1+\xi_2)+\nu\big|^2\\ &= \quad \big|\xi_1+\xi_2\big|^2+\big|\nu\big|^2\\ &= \quad \big|\xi_1+\xi_2\big|^2+1\\ &\geq \quad 1,\end{aligned}$$

hence $\big|\xi_1+(\xi_2+\nu)\big| \geq 1$. Similarly, we obtain $\big|\xi_1-(\xi_2+\nu)\big| \geq 1$, and consequently $\min\{|\xi_1+(\xi_2+\nu)|, |\xi_1-(\xi_2+\nu)|\} \geq 1$. In the same manner, we have:

$$\min\big\{|(\xi_1-\nu)+\xi_2|, |(\xi_1-\nu)-\xi_2|\big\} \geq 1;$$

$$\min\left\{|\xi_1+(\xi_2-\nu)|,|\xi_1-(\xi_2-\nu)|\right\}\geq 1;$$
$$\min\left\{|(\xi_1+\nu)+\xi_2|,|(\xi_1+\nu)-\xi_2|\right\}\geq 1.$$

Noticing that

$$\begin{aligned}\mathcal{Z}_2W_0(\xi) &\leq \frac{1}{4}\Big(\mathcal{Z}_1W_0\big(\xi_1\mid\xi_2+\nu\big)+\mathcal{Z}_1W_0\big(\xi_1-\nu\mid\xi_2\big)\\ &\quad+\mathcal{Z}_1W_0\big(\xi_1\mid\xi_2-\nu\big)+\mathcal{Z}_1W_0\big(\xi_1+\nu\mid\xi_2\big)\Big),\end{aligned}$$

from Lemma 5.3 we deduce that

$$\begin{aligned}\mathcal{Z}_2W_0(\xi) &\leq \frac{r_1}{4}\Big(4+\big|(\xi_1\mid\xi_2+\nu)\big|^p+\big|(\xi_1-\nu\mid\xi_2)\big|^p\\ &\quad+\big|(\xi_1\mid\xi_2-\nu)\big|^p+\big|(\xi_1+\nu\mid\xi_2)\big|^p\Big)\\ &\leq \frac{r_1}{4}2^p\Big(4+4\big|(\xi_1\mid\xi_2)\big|^p\\ &\quad+\big|(0\mid\nu)\big|^p+\big|(-\nu\mid 0)\big|^p+\big|(0\mid -\nu)\big|^p+\big|(\nu\mid 0)\big|^p\Big)\\ &\leq r_12^{p+1}\big(1+\big|\xi\big|^p\big).\end{aligned}$$

Assume now $|\xi_1\wedge\xi_2|=0$. Then, one the four following possibilities holds:

(i) $\xi_1=0$ and $\xi_2=0$;
(ii) $\xi_1\neq 0$ and $\xi_2=0$;
(iii) $\xi_1=0$ and $\xi_2\neq 0$;
(iv) there exists $\lambda\in\mathbb{R}$ such that $\xi_1=\lambda\xi_2$ (with $\xi_1\neq 0$ and $\xi_2\neq 0$).

Let $\sigma\in\mathbb{R}^3$ be such that

$$\begin{cases}|\sigma|=1 & \text{if (i) is satisfied}\\ |\sigma|=1 \text{ and } \langle\xi_1,\sigma\rangle=0 & \text{if either (ii) or (iv) is satisfied}\\ |\sigma|=1 \text{ and } \langle\xi_2,\sigma\rangle=0 & \text{if (iii) is satisfied.}\end{cases}$$

Defining $\psi\in\mathrm{Aff}_0(Y;\mathbb{R}^3)$ by $\psi:=(\varphi_{\sigma_1},\varphi_{\sigma_2},\varphi_{\sigma_3})$, we have

$$\begin{aligned}\mathcal{Z}_2W_0(\xi) &\leq \frac{1}{4}\Big(\mathcal{Z}_1W_0\big(\xi_1\mid\xi_2+\sigma\big)+\mathcal{Z}_1W_0\big(\xi_1-\sigma\mid\xi_2\big)\\ &\quad+\mathcal{Z}_1W_0\big(\xi_1\mid\xi_2-\sigma\big)+\mathcal{Z}_1W_0\big(\xi_1+\sigma\mid\xi_2\big)\Big).\end{aligned}$$

Moreover, we have:

$$\min\left\{|\xi_1+(\xi_2+\sigma)|,|\xi_1-(\xi_2+\sigma)|\right\}\geq 1;$$
$$\min\left\{|(\xi_1-\sigma)+\xi_2|,|(\xi_1-\sigma)-\xi_2|\right\}\geq 1;$$
$$\min\left\{|\xi_1+(\xi_2-\sigma)|,|\xi_1-(\xi_2-\sigma)|\right\}\geq 1;$$
$$\min\left\{|(\xi_1+\sigma)+\xi_2|,|(\xi_1+\sigma)-\xi_2|\right\}\geq 1,$$

and, by Lemma 5.3, we obtain:

$$\mathcal{Z}_1W_0\big(\xi_1\mid\xi_2+\sigma\big)\leq r_1\big(1+|(\xi_1\mid\xi_2+\sigma)|^p\big);$$
$$\mathcal{Z}_1W_0\big(\xi_1-\sigma\mid\xi_2\big)\leq r_1\big(1+|(\xi_1-\sigma\mid\xi_2)|^p\big);$$
$$\mathcal{Z}_1W_0\big(\xi_1\mid\xi_2-\sigma\big)\leq r_1\big(1+|(\xi_1\mid\xi_2-\sigma)|^p\big);$$
$$\mathcal{Z}_1W_0\big(\xi_1+\sigma\mid\xi_2\big)\leq r_1\big(1+|(\xi_1+\sigma\mid\xi_2)|^p\big).$$

It follows that $\mathcal{Z}_2W_0(\xi)\leq r_12^{p+1}(1+|\xi|^p)$, and the proof is complete. □

To prove Proposition 2.8 we will also need Proposition 5.4 below. Set $\mathcal{Z}_0 W_0 := W_0$ and, for each integer $k \geq 0$, define $\mathfrak{J}_k : W^{1,p}(\Sigma;\mathbb{R}^3) \to [0,+\infty]$ by

$$\mathfrak{J}_k(v) := \inf\left\{\liminf_{n\to+\infty}\int_\Sigma \mathcal{Z}_k W_0\big(\nabla v_n(x)\big)dx : \mathrm{Aff}(\Sigma;\mathbb{R}^3) \ni v_n \rightharpoonup v \text{ in } W^{1,p}(\Sigma;\mathbb{R}^3)\right\}.$$

Proposition 5.4. $\mathfrak{J}_k = \mathcal{J}$ *for all integers* $k \geq 0$.

To prove Proposition 5.4 we need the following lemma.

Lemma 5.5. *For every integer* $k \geq 0$ *and every* $v \in \mathrm{Aff}(\Sigma;\mathbb{R}^3)$,

$$\mathfrak{J}_k(v) \leq \int_\Sigma \mathcal{Z}_{k+1} W_0\big(\nabla v(x)\big)dx. \tag{19}$$

Proof. Let $k \geq 0$ be an integer and let $v \in \mathrm{Aff}(\Sigma;\mathbb{R}^3)$. By definition, there exists a finite family $(V_i)_{i\in I}$ of open disjoint subsets of Σ such that $|\Sigma \setminus \cup_{i\in I} V_i| = 0$ and, for every $i \in I$, $\nabla v(x) = \xi_i$ in V_i with $\xi_i \in \mathbb{M}^{3\times 2}$. Given any $\delta > 0$ and any $i \in I$, we consider $\phi_i \in \mathrm{Aff}_0(Y;\mathbb{R}^3)$ such that

$$\int_Y \mathcal{Z}_k W_0(\xi_i + \nabla\phi_i(y))dy \leq \mathcal{Z}_{k+1} W_0(\xi_i) + \delta|\Sigma|^{-1}. \tag{20}$$

Fix any integer $n \geq 1$. By Vitali's covering theorem, there exists a finite or countable family $(a_{i,j} + \alpha_{i,j} Y)_{j\in J_i}$ of disjoint subsets of V_i, where $a_{i,j} \in \mathbb{R}^2$ and $0 < \alpha_{i,j} < n^{-1}$, such that $\big|V_i \setminus \cup_{j\in J_i}(a_{i,j} + \alpha_{i,j} Y)\big| = 0$. Define $\psi_n : \Sigma \to \mathbb{R}^3$ by

$$\psi_n(x) := \alpha_{i,j}\hat{\phi}_i\left(\frac{x - a_{i,j}}{\alpha_{i,j}}\right) \text{ if } x \in a_{i,j} + \alpha_{i,j} Y,$$

where $\hat{\phi}_i$ is the Y-periodic extension of ϕ_i to $\mathbb{R}^2$. We have

$$\begin{aligned}\int_\Sigma |\psi_n(x)|^p dx &= \sum_{i\in I}\int_{V_i} |\psi_n(x)|^p dx\\ &= \sum_{i\in I}\sum_{j\in J_i} (\alpha_{i,j})^N \int_Y |\alpha_{i,j}\phi_i(y)|^p dy\\ &= \sum_{i\in I}\sum_{j\in J_i} (\alpha_{i,j})^N |\alpha_{i,j}|^p \int_Y |\phi_i(y)|^p dy.\end{aligned}$$

But $|\alpha_{i,j}|^p < n^{-p}$ for all $i \in I$ and all $j \in J_i$, hence

$$\begin{aligned}\int_\Sigma |\psi_n(x)|^p dx &\leq n^{-p}\sum_{i\in I}\sum_{j\in J_i} (\alpha_{i,j})^N \int_Y |\phi_i(y)|^p dy\\ &= n^{-p}\sum_{i\in I} |V_i| \int_Y |\phi_i(y)|^p dy,\end{aligned}$$

and so $\psi_n \to 0$ in $L^p(\Sigma;\mathbb{R}^3)$. Since $\phi_i \in \mathrm{Aff}_0(Y;\mathbb{R}^3)$, there exists a finite family $(Y_{i,l})_{l\in L_i}$ of open disjoint subsets of Y such that $|Y \setminus \cup_{l\in L_i} Y_{i,l}| = 0$ and, for every $l \in L_i$, $\nabla\phi_i(y) = \zeta_{i,l}$ in $Y_{i,l}$ with $\zeta_{i,l} \in \mathbb{M}^{3\times 2}$. Set $U_{i,l,n} := \cup_{j\in J_i} a_{i,j} + \alpha_{i,j} Y_{i,l}$, then $|\Sigma \setminus \cup_{i\in I} \cup_{l\in L_i} U_{i,l,n}| = 0$ and $\nabla\psi_n(x) = \zeta_{i,l}$ in $U_{i,l,n}$, and so $\psi_n \in \mathrm{Aff}_0(\Sigma;\mathbb{R}^3)$ and $\{\nabla\psi_n\}_{n\geq 1}$ is bounded in $L^p(\Sigma;\mathbb{R}^3)$. Consequently, (up to a subsequence) $\psi_n \rightharpoonup 0$

in $W^{1,p}(\Sigma;\mathbb{R}^3)$. Moreover,

$$\begin{aligned}\int_\Sigma \mathcal{Z}_k W_0\left(\nabla v(x)+\nabla\psi_n(x)\right)dx &= \sum_{i\in I}\int_{V_i}\mathcal{Z}_k W_0\left(\xi_i+\nabla\psi_n(x)\right)dx\\ &= \sum_{i\in I}\sum_{j\in J_i}(\alpha_{i,j})^N\int_Y \mathcal{Z}_k W_0\left(\xi_i+\nabla\phi_i(y)\right)dy\\ &= \sum_{i\in I}\left|V_i\right|\int_Y \mathcal{Z}_k W_0\left(\xi_i+\nabla\phi_i(y)\right)dy.\end{aligned}$$

As $v+\psi_n\in \mathrm{Aff}(\Sigma;\mathbb{R}^3)$ for all $n\geq 1$ and $v+\psi_n\rightharpoonup v$, from (20) we deduce that

$$\begin{aligned}\mathfrak{J}_k(v)\leq \liminf_{n\to+\infty}\int_\Sigma \mathcal{Z}_k W_0\left(\nabla v(x)+\nabla\psi_n(x)\right)dx &\leq \sum_{i\in I}\left|V_i\right|\mathcal{Z}_{k+1}W_0(\xi_i)+\delta\\ &= \int_\Sigma \mathcal{Z}_{k+1}W_0(\nabla v(x))dx+\delta,\end{aligned}$$

and (19) follows. □

Proof of Proposition 5.4. Clearly $\mathfrak{J}_0=\mathcal{J}$ and $\mathfrak{J}_k\geq\mathfrak{J}_{k+1}$ for all integers $k\geq 0$. We are thus reduced to prove that for every integer $k\geq 0$,

$$\mathfrak{J}_k\leq\mathfrak{J}_{k+1}. \tag{21}$$

Let $k\geq 0$ be an integer. Fix any $v\in W^{1,p}(\Sigma;\mathbb{R}^3)$ and any sequence $v_n\rightharpoonup v$ with $v_n\in\mathrm{Aff}(\Sigma;\mathbb{R}^3)$. Using Lemma 5.5, we have

$$\mathfrak{J}_k(v_n)\leq\int_\Sigma \mathcal{Z}_{k+1}W_0\big(\nabla v_n(x)\big)dx$$

for all $n\geq 1$. Thus,

$$\mathfrak{J}_k(v)\leq\liminf_{n\to+\infty}\mathfrak{J}_k(v_n)\leq\liminf_{n\to+\infty}\int_\Sigma \mathcal{Z}_{k+1}W_0\big(\nabla v_n(x)\big)dx,$$

and (21) follows. □

5.2. **Proof of Proposition 2.8.** From Proposition 5.2, we see that there exists $c>0$ such that $\mathcal{Z}_2W_0(\xi)\leq c(1+|\xi|^p)$ for all $\xi\in\mathbb{M}^{3\times 2}$. As $\mathrm{Aff}(\Sigma;\mathbb{R}^3)$ is strongly dense in $W^{1,p}(\Sigma;\mathbb{R}^3)$, we deduce that for every $v\in W^{1,p}(\Sigma;\mathbb{R}^3)$,

$$\mathfrak{J}_2(v)=\inf\left\{\liminf_{n\to+\infty}\int_\Sigma \mathcal{Z}_2W_0\big(\nabla v_n\big)dx : W^{1,p}(\Sigma;\mathbb{R}^3)\ni v_n\rightharpoonup v \text{ in } W^{1,p}(\Sigma;\mathbb{R}^3)\right\}.$$

Fix any $v\in W^{1,p}(\Sigma;\mathbb{R}^3)$ and any $\{v_n\}_n\subset W^{1,p}(\Sigma;\mathbb{R}^3)$ such that $v_n\rightharpoonup v$ in $W^{1,p}(\Sigma;\mathbb{R}^3)$. As $\mathcal{Z}_2W_0\leq W_0$ we have

$$\int_\Sigma \mathcal{Z}_2W_0(\nabla v_n(x))dx\leq\int_\Sigma W_0(\nabla v_n(x))dx$$

for all $n\geq 1$. Then,

$$\mathfrak{J}_2(v)\leq\liminf_{n\to+\infty}\int_\Sigma \mathcal{Z}_2W_0(\nabla v_n(x))dx\leq\liminf_{n\to+\infty}\int_\Sigma W_0(\nabla v_n(x))dx,$$

and so $\mathfrak{J}_2\leq\mathcal{I}$. But $\mathcal{I}\leq\mathcal{J}$ and $\mathcal{J}=\mathfrak{J}_2$ by Proposition 5.4, hence $\mathcal{I}=\mathcal{J}$. □

5.3. **Proof of Proposition 2.10.** By Proposition 5.2, there exists $c > 0$ such that $\mathcal{Z}_3 W_0(\xi) \leq \mathcal{Z}_2 W_0(\xi) \leq c(1 + |\xi|^p)$ for all $\xi \in \mathbb{M}^{3\times 2}$. Then $\mathcal{Z}_2 W_0$ is finite, and so $\mathcal{Z}_2 W_0$ is continuous by Lemma 5.1(ii). From Dacorogna's quasiconvexification formula [8, Theorem 1.1 p. 201], it follows that $\mathcal{Q}[\mathcal{Z}_2 W_0] = \mathcal{Z}_3 W_0$. On the other hand, as $\mathcal{Z}_2 W_0 \leq W_0$ we have $\mathcal{Q}[\mathcal{Z}_2 W_0] \leq \mathcal{Q} W_0$. Moreover, it is clear that if $g : \mathbb{M}^{3\times 2} \to [0, +\infty]$ is quasiconvex, then $\mathcal{Z}_k g = g$ for all integers $k \geq 1$. Hence $\mathcal{Q} W_0 \leq \mathcal{Q}[\mathcal{Z}_2 W_0]$, and the proof is complete. □

6. Proof of Theorem 2.7

In this section we prove Theorem 2.7. To this end, we will need the following approximation theorem by Ben Belgacem and Bennequin. Set

$$\mathrm{Aff}_*(\Sigma;\mathbb{R}^3) := \Big\{ v \in \mathrm{Aff}(\Sigma;\mathbb{R}^3) : \partial_1 v(x) \wedge \partial_2 v(x) \neq 0 \text{ a.e. in } \Sigma \Big\}.$$

Ben Belgacem-Bennequin's approximation theorem ([6, Lemme 8 p. 114]). *For every $v \in \mathrm{Aff}_*(\Sigma;\mathbb{R}^3)$, there exists $\{v_n\}_{n\geq 1} \subset C^1_*(\overline{\Sigma};\mathbb{R}^3)$ such that:*

- (A_1) *$v_n \to v$ in $W^{1,p}(\Sigma;\mathbb{R}^3)$;*
- (A_2) *there exists $\delta > 0$ such that* $\big|\partial_1 v_n(x) \wedge \partial_2 v_n(x)\big| \geq \delta$ for all $x \in \overline{\Sigma}$ and all $n \geq 1$.

Proof of Theorem 2.7. As $C^1_*(\overline{\Sigma};\mathbb{R}^3) \subset W^{1,p}(\Sigma;\mathbb{R}^3)$ we have $\mathcal{I} \leq \overline{\mathcal{E}}$. Since $\mathcal{I} = \mathcal{J}$ by Proposition 2.8, for the opposite inequality, it is sufficient to show that for every $v \in \mathrm{Aff}(\Sigma;\mathbb{R}^3)$ with

$$\int_\Sigma W_0(\nabla v(x))dx < +\infty, \tag{22}$$

there exists $\{v_n\}_{n\geq 1} \subset C^1_*(\overline{\Sigma};\mathbb{R}^3)$ such that

$$\lim_{n\to+\infty} \int_\Sigma W_0(\nabla v_n(x))dx = \int_\Sigma W_0(\nabla v(x))dx. \tag{23}$$

Fix $v \in \mathrm{Aff}(\Sigma;\mathbb{R}^3)$ such that (22) holds. Then, using Lemma 2.5(ii), it easy to see that $v \in \mathrm{Aff}_*(\Sigma;\mathbb{R}^3)$. From Ben Belgacem-Bennequin's approximation theorem, we obtain the existence of $\{v_n\}_{n\geq 1} \subset C^1_*(\overline{\Sigma};\mathbb{R}^3)$ such that (A_1) and (A_2) holds and $\nabla v_n(x) \to \nabla v(x)$ a.e. in Σ. As W_0 is continuous (cf. Lemma 2.5(i)), we have

$$\lim_{n\to+\infty} W_0\big(\nabla v_n(x)\big) = W_0\big(\nabla v(x)\big) \text{ a.e. in } \Sigma.$$

Using ($\overline{C}_2$) together with (A_2), we deduce that there exists $c > 0$ such that for every $n \geq 1$ and every measurable set $A \subset \Sigma$,

$$\int_A W_0\big(\nabla v_n(x)\big)dx \leq c\Big(|A| + \int_A |\nabla v_n(x) - \nabla v(x)|^p dx + \int_A |\nabla v(x)|^p dx\Big).$$

But $\nabla v_n \to \nabla v$ in $L^p(\Sigma;\mathbb{M}^{3\times 2})$ by (A_1), hence $\{W_0(\nabla v_n(\cdot))\}_{n\geq 1}$ is absolutely uniformly integrable, and (23) follows by Vitali's theorem. □

7. Proof of Theorem 2.11

Theorem 2.11 is in fact contained in our paper [3]. For the convenience of the reader, we give the proof.

Proof of Theorem 2.11. According to Proposition 2.10, it is sufficient to prove that for every $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$,

$$\mathcal{I}(v) = \int_\Sigma \mathcal{Q}[\mathcal{Z}_2 W_0]\big(\nabla v(x)\big)dx. \tag{24}$$

Using Proposition 5.2 (resp. Propositions 2.8 and 5.4), we see that there exists $c > 0$ such that $\mathcal{Z}_2W_0(\xi) \le c(1+|\xi|^p)$ for all $\xi \in \mathbb{M}^{3\times 2}$ (resp. $\mathcal{I} = \mathfrak{J}_2$). As $\mathrm{Aff}(\Sigma;\mathbb{R}^3)$ is strongly dense in $W^{1,p}(\Sigma;\mathbb{R}^3)$, it follows that for every $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$,

$$\mathcal{I}(v) = \inf\left\{\liminf_{n\to+\infty}\int_\Sigma \mathcal{Z}_2W_0\big(\nabla v_n\big)dx : W^{1,p}(\Sigma;\mathbb{R}^3) \ni v_n \rightharpoonup v \text{ in } W^{1,p}(\Sigma;\mathbb{R}^3)\right\},$$

and (24) follows from the (classical) integral representation theorem below.

Integral representation theorem. *Let $f : \mathbb{M}^{3\times 2} \to [0,+\infty]$ be a Borel measurable function and let $\mathcal{F} : W^{1,p}(\Sigma;\mathbb{R}^3) \to [0,+\infty]$ be defined by*

$$\mathcal{F}(v) := \inf\left\{\liminf_{n\to+\infty}\int_\Sigma f\big(\nabla v_n(x)\big)dx : W^{1,p}(\Sigma;\mathbb{R}^3) \ni v_n \rightharpoonup v \text{ in } W^{1,p}(\Sigma;\mathbb{R}^3)\right\}.$$

If f is continuous and $f(\xi) \le c(1+|\xi|^p)$ for all $\xi \in \mathbb{M}^{3\times 2}$ and some $c > 0$, then for every $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$,

$$\mathcal{F}(v) = \int_\Sigma \mathcal{Q}f\big(\nabla v(x)\big)dx.$$

For a proof of this result we refer the reader to the book [8]. □

References

[1] Acerbi, E., Buttazzo, G., Percivale, D.: *A variational definition for the strain energy of an elastic string*, J. Elasticity **25**, 137-148 (1991)

[2] Anza Hafsa, O., Mandallena, J.-P.: *Interchange of infimum and integral*, Calc. Var. Partial Differential Equations **18**, 433-449 (2003)

[3] Anza Hafsa, O., Mandallena, J.-P.: *Relaxation of variational problems in two-dimensional nonlinear elasticity*, to appear on Ann. Mat. Pura Appl.

[4] Anza Hafsa, O., Mandallena, J.-P.: *The nonlinear membrane energy: variational derivation under the constraint "*$\det\nabla u \neq 0$*"*, submitted

[5] Anzellotti, G., Baldo, S., Percivale, D.: *Dimension reduction in variational problems, asymptotic development in Γ-convergence and thin elastic structures in elasticity*, Asymptot. Anal. **9**, 61-100 (1994)

[6] Ben Belgacem, H: *Modélisation de structures minces en élasticité non linéaire*, Thèse de Doctorat, Université Pierre et Marie Curie, 1996

[7] Ben Belgacem, H: *Une méthode de Γ-convergence pour un modèle de membrane non linéaire*, C. R. Acad. Sci. Paris Série I **323**, 845-849 (1996)

[8] Dacorogna, B.: *Direct Methods in the Calculus of Variations*, Springer-Verlag, Berlin 1989

[9] Dal Maso, G.: *An introduction to Γ-convergence*, Birkäuser, Boston, 1993

[10] Fonseca, I.: *The lower quasiconvex envelope of the stored energy function for an elastic crystal*, J. Math. Pures et Appl. **67**, 175-195 (1988)

[11] Le Dret, H., Raoult, A.: *The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity*, J. Math. Pures Appl. **74**, 549-578 (1995)

[12] Michael, E.: Continuous selections. I, Ann. Math. **63**, 361-382 (1956)

[13] Percivale, D.: *The variational method for tensile structures*, preprint n°16, Dipartimento di Matematica Politecnico di Torino (1991)

[14] Trabelsi, K.: *Sur la modélisation des plaques minces en élasticité non linéaire*, Thèse de Doctorat, Université Pierre et Marie Curie, 2004

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland.
E-mail address: anza@math.unizh.ch

"Equipe Mathématiques Appliquées", Centre Universitaire de Formation et de Recherche de Nîmes, Site des Carmes, Place Gabriel Péri - Cedex 01 - 30021 Nîmes, France.
I3M (Institut de Mathématiques et Modélisation de Montpellier) UMR - CNRS 5149, Université Montpellier II, Place Eugène Bataillon, 34090 Montpellier, France.
E-mail address: jean-philippe.mandallena@unimes.fr