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ON THE ACOUSTIC SINGLE LAYER POTENTIAL: STABILIZATION AND FOURIER ANALYSIS*

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Abstract. In this paper, we propose a general approach for stabilizing the single layer potential for the Helmholtz boundary integral equation and prove its stability. We consider Galerkin boundary element discretizations and analyze their convergence. Furthermore, we derive quantitative error bounds for the Galerkin discretization which are explicit with respect to the mesh width *and* the wave number for the special case that the surface is the unit sphere in $\mathbb{R}^3$. We perform then a qualitative analysis which allows us to choose the stabilization such that the (negative) influence of the wave number in the stability and convergence estimates attains its minumum.

Key words. acoustic scattering, Galerkin boundary element method, stabilization of Helmholtz equation, high wave number

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1. Introduction. In this paper, we will address problems related to the discretization of boundary integral equations for the Helmholtz problem outside of a reflecting obstacle Ω^-, where $\Omega^- \subset \mathbb{R}^3$ is a bounded Lipschitz domain. Let $\Omega^+ = \mathbb{R}^3 \setminus \bar{\Omega}^-$ and $\mathcal{L}_k := -\Delta - k^2$. We consider the following problem: Find $u^+ \in H^1_{\mathrm{loc}}(\Omega^+)$ such that the Helmholtz problem

$$\begin{aligned} \mathcal{L}_k u^+ &= 0 && \text{in } \Omega^+, \\ u^+ &= g && \text{on } \Gamma := \partial\Omega^-, \\ \left| \frac{\partial u}{\partial r} - \mathrm{i}\, k u \right| &\le C \left\| \mathbf{x} \right\|^{-2} && \left\| \mathbf{x} \right\| \to \infty \end{aligned} \tag{1.1}$$

is satisfied in a weak sense (cf. [26]). Here, $\partial/\partial r$ denotes the derivative in radial direction $\mathbf{x}/\left\|\mathbf{x}\right\|$.

Our goal is to solve these equations by the method of integral equations. A potential ansatz leads to a boundary integral equation on Γ for the unknown density φ which is of the form $R_k \varphi = g$. Here, R_k is the trace V_k of the single layer potential associated to $\mathcal{L}_k$ on Γ or a stabilized version of it. We will consider the Galerkin boundary element method for its discretization. It is well known that the V_k is not invertible on a countable set of frequencies k (see, e.g., [11]) and we will introduce a class of stabilizations such that the boundary integral equation is well posed for all frequencies $k > 0$.

Alternatively, the Helmholtz equation (1.1) can be solved numerically by finite element discretizations where the problem related to the unbounded domain Ω^+ is treated either by *infinite* elements or by introducing an artificial outer boundary far away from the scatterer. It is well known (see, e.g., [3]) that finite element discretiza-

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tions for the Helmholtz problem suffer from the *pollution effect*; i.e., the constants in the Galerkin error estimates deteriorate to infinity with increasing wave number $k > 0$. Hence, the question arises whether this pollution effect is possibly reduced by solving the boundary integral equation for the Helmholtz problem via the Galerkin boundary element method. In order to address this question, it is mandatory to first remove the forbidden frequencies of the single layer potential through a suitable stabilization.

In the literature, various approaches exist for stabilizing these integral equations (cf. [31], [4], [10], [2], [14], [20], [9]). Among them the so-called Brakhage–Werner formulation for the stabilization of the acoustic double layer potential is one of the most popular.

Here we introduce a class of stabilizations for the single layer potential for which the well-posedness of the resulting continuous and discrete equations (for mesh size sufficiently small) can be proved (a related work is [8]). The results can be summarized as follows:

a. The stabilized acoustic single layer potential, on the continuous level, admits a unique solution which depends continuously on the data for general Lipschitz surfaces. This is a strong advantage compared to the Brakhage–Werner stabilization, where the question of existence and uniqueness is open for general Lipschitz surfaces and even for piecewise smooth surfaces.
b. The Galerkin method converges for "sufficiently small" step size on general triangulated surfaces with optimal rate.

Indeed, as in all stabilization approaches, well-posedness and quasi-optimality can be proved provided the step size is "sufficiently small." More precisely, the threshold for the maximal step size such that the Galerkin discretization is stable depends on the wave number, and the "constant" in the quasi-optimality error estimate typically deteriorates to infinity as the wave number increases.

Consequently, in order to compare different approaches from the viewpoint of numerical efficiency the following questions have to be addressed:

1. How does the threshold for the stability of the Galerkin discretization quantitatively depend on the wave number?
2. How does the Galerkin error quantitatively depend on the mesh width *and* the wave number?
3. Can the stabilization approach be implemented efficiently in a boundary element code? What is the computational complexity?

These questions have been discussed for the Brakhage–Werner stabilization in [17] (see also [24]). Here, we analyze quantitatively the dependence of the constants entering the stability and convergence estimates for our class of stabilized single layer potentials, in the case that the surface is the unit sphere in $\mathbb{R}^3$. For this case, we obtain the following:

c. The condition for the stability of Galerkin method (related to the condition "the step size has to be sufficiently small") is slightly more restrictive as for the stabilization in the Brakhage–Werner approach.
d. The constant of quasi-optimality in the Galerkin error estimates which amplifies the error of the best approximation is $k^{1/3}$ for both the stabilized acoustic single layer potential and the Brakhage–Werner stabilization.

In this light, the Fourier analysis in this paper shows that, for the surface of the unit sphere, the stabilized acoustic single layer potential has similar convergence properties as the Brakhage–Werner stabilization.

However, we consider the result that the stabilized acoustic single layer potential is stable also on surfaces of general Lipschitz polyhedra as the essential advantage compared to the Brakhage–Werner formulation, where the stability on general Lipschitz polyhedra is still an open question.

The paper is organized as follows. In section 2, we will formulate the boundary integral equation for the Helmholtz problem and introduce our abstract stabilization approach. Concrete stabilization operators which satisfy the abstract assumptions will be presented as examples. As a side result, we will prove, for some sesquilinear forms associated to integral operators of general fractional order, continuity and ellipticity in appropriate Sobolev spaces.

In section 3, we will introduce the Galerkin boundary element method with piecewise constant boundary elements for the stabilized single layer integral equation.

In section 4, the stability and convergence of the Galerkin boundary element method will be analyzed. It will be proved that the discretization is stable on general Lipschitz polyhedrons and the Galerkin solution converges at an optimal rate, provided the step size is sufficiently small.

In section 5, we will employ Fourier analysis to explicitly analyze the dependence of the stability and convergence of the stabilized Galerkin method with respect to both the mesh size and the wave number. We will discuss three parameter constellations by asymptotic analysis in a rigorous way. The intermediate ranges of the parameters are studied by systematic computer experiments and show that the asymptotic cases are relevant for the estimates of the constants of interest. In this light, the asymptotic analysis *proves* that the estimates of the constants of interest cannot be improved while the computer experiments *indicate* that these constants do not behave worse in the intermediate ranges of the parameters.

Section 6 briefly discusses the computational complexity of the proposed stabilization approaches.

2. Boundary integral formulation.

2.1. Sobolev spaces on Lipschitz manifolds and trace operators. In this section, we will introduce some notation related to Sobolev spaces and recall some of their properties. As before, Ω^- denotes a bounded Lipschitz domain in $\mathbb{R}^3$, Ω^+ its complement, and $\mathbf{n}$ the unit normal vector pointing from Ω^- to Ω^+.

We make use of standard complex Sobolev spaces in the whole space $H^s(\mathbb{R}^3)$, in the domains $H^s(\Omega^\pm)$, $s \in \mathbb{R}$, and on the boundary Γ, $H^\ell(\Gamma)$, $\ell \in [-1,1]$. On the boundary, $(\cdot,\cdot)_0$ is the $L^2(\Gamma)$ scalar product, i.e., $(u,v)_0 = \int_\Gamma u\bar{v}$, which is identified with its continuous extension to the duality pairing between $H^s(\Gamma)$ and $H^{-s}(\Gamma)$, $s \in (0,1]$.

Moreover, for any positive s, we denote by $H^s_{\mathrm{loc}}(\mathbb{R}^3)$, $H^s_{\mathrm{loc}}(\Omega^+)$ the space of functions which are locally in H^s and by $H^{-s}_{\mathrm{comp}}(\mathbb{R}^3)$, $H^{-s}_{\mathrm{comp}}(\Omega^+)$ their dual spaces. Furthermore, we introduce

$$H^{s+1}_\Delta(\Omega^\pm) := \{u \in H^{s+1}(\Omega^\pm) \ : \ \Delta u \in H^s(\Omega^\pm)\}. \tag{2.1}$$

The standard one-sided trace operators are denoted by γ_0^+, γ_0^- and they map $\gamma_0^\pm : H^s(\Omega^\pm) \to H^{s-1/2}(\Gamma)$ continuously for all $s \in (1/2, 3/2)$. The one-sided normal derivative trace operators associated to the mapping $u \mapsto \partial_{\mathbf{n}} u$ are denoted by γ_1^+, γ_1^- and they are continuous operators from $H^{s+1}_\Delta(\Omega^\pm)$ to $H^{s-1/2}(\Gamma)$ for $s \in [0, 1/2)$. Note that the ranges of $\gamma_0^\pm$ and $\gamma_1^\pm$ do not change if the spaces $H^s(\Omega^+)$ are replaced by $H^s_{\mathrm{loc}}(\Omega^+)$ everywhere.

For later use, we will define Sobolev spaces of order $H^{1+s}(\Gamma)$ for $s > 0$ as the ranges of the trace operator γ_0 applied to functions in $H^{3/2+s}(\Omega^-)$. More precisely, for $s \in (0, \frac{1}{2}]$, we define

$$H^{1+s}(\Gamma) := \gamma_0(H^{3/2+s}(\Omega^-)),$$

$$\|v\|_{H^{1+s}(\Gamma)} := \inf_{u \in H^{3/2+s}(\Omega^-), \gamma_0(u)=v} \|u\|_{H^{3/2+s}(\Omega^-)},$$

and we denote by $H^{-(1+s)}(\Gamma)$, $s \in (0, 1/2]$, the dual space of $H^{1+s}(\Gamma)$ with $L^2(\Gamma)$ as pivot space. The corresponding duality pairing is denoted again by $(\cdot, \cdot)_0$. We will often use the shorthand notation $\|\cdot\|_s$ for $\|\cdot\|_{H^s(\Gamma)}$.

Since we shall deal with the numerical discretization of boundary integral operators via the boundary element method, it is reasonable to assume that the Lipschitz surface Γ is piecewise smooth.

NOTATION 2.1. *We say that Γ is a polyhedral surface if Γ is the surface of a bounded Lipschitz polyhedron; i.e., there exist finitely many smooth, nonoverlapping, and open subsets $\Gamma_j \subset \Gamma$, $1 \le j \le J < \infty$, such that $\Gamma = \bigcup_{j=1}^{J} \overline{\Gamma}_j$.*

PROPOSITION 2.2. *Let Γ be a polyhedral surface. The Sobolev spaces $H^{1+s}(\Gamma_j)$, $s \in (0, \frac{1}{2}]$, are well defined for all j and the following holds:*

$$H^{1+s}(\Gamma) \equiv \{v \in H^1(\Gamma) \ : \ v_{|\Gamma_j} \in H^{1+s}(\Gamma_j)\}, \quad s \in (0, \tfrac{1}{2}]. \tag{2.2}$$

The proof can be found in, e.g., [13] or [7].

2.2. Boundary integral operators. For $z \in \mathbb{R}^3 \setminus \{0\}$, let $G_k(z) := \frac{e^{\mathrm{i}\, k \|z\|}}{4\pi \|z\|}$ and define the associated single layer and double potential by

$$(S_k\varphi)(x) := \int_\Gamma G_k(x-y)\, \varphi(y)\, ds_y, \qquad x \in \mathbb{R}^3 \setminus \Gamma,$$

$$(D_k\varphi)(x) := \int_\Gamma \left(\frac{\partial}{\partial n_y} G_k(x-y) \right) \varphi(y)\, ds_y, \qquad x \in \mathbb{R}^3 \setminus \Gamma.$$

The restrictions of these operators to Ω^+ (resp., Ω^-) are denoted by S_k^+, D_k^+ (resp., S_k^-, D_k^-).

The boundary integral operators associated with the single and double layer potentials are given by $V_k := \gamma_0 S_k$ and $K_k := \{\gamma_0\} D_k := \frac{1}{2}\left(\gamma_0^+ D_k + \gamma_0^- D_k\right)$.

It is well known that every solution $\varphi \in H^{-1/2}(\Gamma)$ of

$$V_k \varphi = g \tag{2.3}$$

has the property that $u^- = S_k^- \varphi$, respectively, $u^+ = S_k^+ \varphi$, satisfies the homogeneous interior, respectively, exterior, Helmholtz problem.

However, for countably many wave numbers k, the boundary integral equation (2.3) is not injective and, hence, does not admit a solution for all right-hand sides g (although, e.g., the exterior Helmholtz problem (1.1) admits a unique solution for all boundary data $g \in H^{1/2}(\Gamma)$).

Our goal is to modify the boundary integral equation so that the problem admits a unique solution for all wave numbers. Our approach follows the framework of [31].

We start with the formal ansatz

$$R_k = V_k + \mathrm{i}\,\eta(\tfrac{1}{2}I + K_k)\gamma_0 B, \tag{2.4}$$

where $B : H^{-1/2}(\Gamma) \to H^{1/2}(\mathbb{R}^3)$ is, for the moment, any linear and continuous operator. We consider the equation

$$R_k\varphi = g. \tag{2.5}$$

In Proposition 2.3, we will prove that, under suitable conditions on B, (2.5) admits a unique solution for all wave numbers, and the functions

$$u^+ = \left(S_k^+ + \mathrm{i}\,\eta D_k^+\gamma_0 B\right)\varphi, \quad \text{resp.,} \quad u^- = \left(S_k^- + \mathrm{i}\,\eta D_k^-\gamma_0 B\right)\varphi \tag{2.6}$$

satisfy the homogeneous exterior, respectively, interior, Helmholtz problem.

PROPOSITION 2.3. *Assume that $\gamma_0 B : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is a compact operator and the associated sesquilinear form $(\cdot, \gamma_0 B\cdot)_0 : H^{-1/2}(\Gamma)\times H^{-1/2}(\Gamma) \to \mathbb{C}$ is hermitian and satisfies the following: For every $\varphi \in H^{-1/2}(\Gamma)$, there holds $(\varphi, \gamma_0 B\varphi)_0 > 0 \;\Leftrightarrow\; \varphi \neq 0$. Then, (2.5) admits a unique solution for all $k \in \mathbb{R}_+$ and $\eta \in \mathbb{R}\setminus\{0\}$. Moreover, u^+ (resp., u^-) as defined in (2.6) satisfy Helmholtz' equations in Ω^+ (resp., Ω^-).*

For the proof, see [11] or [8].

There are various ways of choosing an operator B in (2.4) satisfying the assumptions in Proposition 2.3. However, to obtain an efficient numerical scheme it is essential that the complexity of the numerical realization of B is moderate and the implementation does not cause too much extra work. We will present in section 2.3 some choices of B and will comment on the numerical complexity in section 6. All these choices will satisfy the following assumption.

Assumption 2.4. There exists $0 < \varepsilon \leq 1$, such that the following hold:

1. The operator $\gamma_0 B : H^{-1/2-\varepsilon}(\Gamma) \to H^{1/2+\varepsilon}(\Gamma)$ is continuous.
2. The sesquilinear form $(\cdot, \gamma_0 B\cdot)_0 : H^{-1/2-\varepsilon}(\Gamma) \times H^{-1/2-\varepsilon}(\Gamma) \to \mathbb{C}$ is hermitian and $H^{-1/2-\varepsilon}(\Gamma)$-elliptic: There exists $\alpha_B > 0$ such that

$$(\varphi, \gamma_0 B\varphi)_0 \geq \alpha_B \|\varphi\|^2_{-1/2-\varepsilon} \qquad \forall \varphi \in H^{-1/2-\varepsilon}(\Gamma).$$

Note that Assumption 2.4 implies that the assumptions in Proposition 2.3 are satisfied.

2.3. Choices of B. In this section, we will present various choices for the operator B.

2.3.1. $B = S_0 V_0$. The single layer potential for the operator $-\Delta$ is given by

$$(S_0\varphi)(x) := \int_\Gamma \frac{\varphi(y)}{4\pi\|x-y\|} ds_y, \qquad x \in \mathbb{R}^3.$$

It is well known (see [12], [21]) that $V_0 := \gamma_0 S : H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)$ is a linear and continuous isomorphism for all $s \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and, for $s = 0$, the associated sesquilinear form $(\cdot, V_0\cdot)_0 : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to \mathbb{C}$ is elliptic. Hence,

$$V_0^2 : H^{-1}(\Gamma) \to H^1(\Gamma)$$

is continuous and

$$\left(\varphi, V_0^2\varphi\right)_0 = \|V_0\varphi\|_0^2 \geq c\|\varphi\|^2_{H^{-1}(\Gamma)},$$

where we have used that $V_0 : L^2(\Gamma) \to H^1(\Gamma)$ is an isomorphism.

Hence, the choice $B = S_0 V_0$ leads to $\gamma_0 B = V_0^2$ which satisfies Assumption 2.4 with $\varepsilon = 1/2$.

2.3.2. Single layer potential for $(I-\Delta)^{1+\varepsilon}$. The fundamental solution of the pseudodifferential operator $(I-\Delta)^{1+\varepsilon}$, $0 < \varepsilon \leq 1$, is given by (cf. [30, Example 2.2])

$$\mathcal{G}_\varepsilon(z) := \frac{2^{-\varepsilon}}{(2\pi)^{3/2}\,\Gamma(1+\varepsilon)}\,\|z\|^{\varepsilon-1/2}\,\mathrm{K}_{\varepsilon-1/2}(\|z\|), \tag{2.7}$$

where K_ν is the modified Bessel function (cf. [1, sec. 9.6]) and $\Gamma(\cdot)$ denotes the Gamma function. The corresponding potential is given by

$$(B_\varepsilon \varphi)(x) := \int_\Gamma \mathcal{G}_\varepsilon(x-y)\,\varphi(y)\,ds_y, \qquad x \in \mathbb{R}^3. \tag{2.8}$$

Next, we will prove that the operator $\gamma_0 B_\varepsilon : H^{-1/2-\varepsilon}(\Gamma) \to H^{1/2+\varepsilon}(\Gamma)$ is continuous and the associated sesquilinear form $(\cdot, \gamma_0 B_\varepsilon \cdot)_0 : H^{-1/2-\varepsilon}(\Gamma) \times H^{-1/2-\varepsilon}(\Gamma) \to \mathbb{C}$ is hermitian and $H^{-1/2-\varepsilon}(\Gamma)$-elliptic, i.e., satisfies Assumption 2.4 for any chosen value of $\varepsilon \in]0,1] \setminus \{1/2\}$. (The case $\varepsilon = 1/2$ is exceptional only for nonsmooth surfaces and this problem is related to the mapping properties of γ_0 applied to functions in $H^{3/2}(\mathbb{R}^3)$.)

THEOREM 2.5 (mapping properties). *Let $1/2 < \ell \leq 2$, $\ell \neq 3/2$. For any $\varepsilon \in]0,1]$, the operator $B_\varepsilon : H^{1/2-\ell}(\Gamma) \to H^{2+2\varepsilon-\ell}(\mathbb{R}^3)$ is continuous. For any $\varepsilon \in]0,1] \setminus \{1/2\}$, the operator $\gamma_0 B_\varepsilon : H^{-1/2-\varepsilon}(\Gamma) \to H^{1/2+\varepsilon}(\Gamma)$ is continuous. If the surface is smooth this holds also for $\varepsilon = 1/2$.*

Proof. We introduce the Bessel potential for the operator $(I-\Delta)^{1+\varepsilon}$:

$$N_\varepsilon u(x) := \int_{\mathbb{R}^3} \mathcal{G}_\varepsilon(x-y)\,u(y)\,dy, \qquad x \in \mathbb{R}^3, \tag{2.9}$$

with the fundamental solution $\mathcal{G}_\varepsilon$ as in (2.7). Its symbol is given by $\sigma(\xi) = (\|\xi\|^2 + 1)^{m/2}$ with $m = -(2+2\varepsilon)$. Hence, Theorem 1.4' in [15] implies the mapping property $N_\varepsilon : H^s(\mathbb{R}^3) \to H^{s+2+2\varepsilon}(\mathbb{R}^3)$ for all $s \in \mathbb{R}$. Since, for all $1/2 < \ell \leq 2$, $\ell \neq 3/2$, the trace operator $\gamma_0 : H^\ell(\mathbb{R}^3) \to H^{-1/2+\ell}(\Gamma)$ is continuous, its dual $\gamma_0' := H^{1/2-\ell}(\Gamma) \to H^{-\ell}(\mathbb{R}^3)$ is continuous in the same range of ℓ. Thus, the representation $B_\varepsilon = N_\varepsilon \gamma_0'$ implies the continuity of

$$B_\varepsilon : H^{1/2-\ell}(\Gamma) \to H^{2+2\varepsilon-\ell}(\mathbb{R}^3). \tag{2.10}$$

The mapping property of $\gamma_0 B_\varepsilon$ follows from this and the mapping properties of the trace operator γ_0. ∎

THEOREM 2.6. *For any $\varepsilon \in]0,1] \setminus \{1/2\}$, the sesquilinear form $(\cdot, \gamma_0 B_\varepsilon \cdot)_0 : H^{-1/2-\varepsilon}(\Gamma) \times H^{-1/2-\varepsilon}(\Gamma) \to \mathbb{C}$ is hermitian and $H^{-1/2-\varepsilon}(\Gamma)$-elliptic. For smooth surfaces, this holds also for $\varepsilon = 1/2$.*

Proof. Let $\varepsilon \in]0,1]$ with the exclusion of the case $\varepsilon = 1/2$ for nonsmooth surfaces. Let $(\cdot,\cdot)_{0,\mathbb{R}^3}$ denote the continuous extension of the $L^2(\mathbb{R}^3)$-scalar product to the duality pairing $H^{-1-\varepsilon}(\mathbb{R}^3) \times H^{1+\varepsilon}(\mathbb{R}^3)$. The relation $B_\varepsilon = N_\varepsilon \gamma_0'$ and the mapping properties of B_ε imply

$$(\varphi, \gamma_0 B_\varepsilon \varphi)_0 = (\varphi, \gamma_0 N_\varepsilon \gamma_0' \varphi)_0 = (\gamma_0' \varphi, N_\varepsilon \gamma_0' \varphi)_{0,\mathbb{R}^3} \qquad \forall \varphi \in H^{-1/2-\varepsilon}(\Gamma). \tag{2.11}$$

Simple properties of the Fourier transform lead to

(2.12)
$$(\gamma_0'\varphi, N_\varepsilon\gamma_0'\varphi)_{0,\mathbb{R}^3} = \left(\widehat{\gamma_0'\varphi}, \frac{\widehat{\gamma_0'\varphi}}{\left(1+\|\cdot\|^2\right)^{1+\varepsilon}}\right)_{0,\mathbb{R}^3}$$
$$= \left(\frac{\widehat{\gamma_0'\varphi}}{\left(1+\|\cdot\|^2\right)^{\frac{1+\varepsilon}{2}}}, \frac{\widehat{\gamma_0'\varphi}}{\left(1+\|\cdot\|^2\right)^{\frac{1+\varepsilon}{2}}}\right)_{0,\mathbb{R}^3} \geq c\,\|\gamma_0'\varphi\|^2_{H^{-1-\varepsilon}(\mathbb{R}^3)}.$$

Let $R := \left\{\gamma_0'(\varphi) : \varphi \in H^{-1/2-\varepsilon}(\Gamma)\right\}$ denote the range of γ_0'. Next we will show that $\gamma_0' : H^{-1/2-\varepsilon}(\Gamma) \to R$ is an isomorphism.

It is well known that $\gamma_0 : H^{t+1/2}(\mathbb{R}^3) \to H^t(\Gamma)$ is surjective for $t \in (0, 3/2] \setminus \{1\}$, and hence, trivially, γ_0 has closed range. From the surjectivity of γ_0 we conclude the injectivity of γ_0' and the closed range theorem [34, sec. VII.5] implies that R is closed in $H^{-1-\varepsilon}\left(\mathbb{R}^3\right)$. Thus, $\gamma_0' : H^{-1/2-\varepsilon}(\Gamma) \to R$ is bijective and has closed range in $H^{-1-\varepsilon}\left(\mathbb{R}^3\right)$. The open mapping theorem implies that $\gamma_0' : H^{-1/2-\varepsilon}(\Gamma) \to R$ is an isomorphism. Thus,

$$\|\gamma_0'\varphi\|^2_{H^{-1-\varepsilon}(\mathbb{R}^3)} \geq c\,\|\varphi\|^2_{-1/2-\varepsilon} \qquad \forall \varphi \in H^{-1/2-\varepsilon}(\Gamma),$$

and this, in combination with (2.11) and (2.12), yields the $H^{-1/2-\varepsilon}(\Gamma)$-ellipticity of $(\cdot, \gamma_0 B_\varepsilon \cdot)_{0,\Gamma}$.

The sesquilinear form $(\cdot, \gamma_0 B_\varepsilon \cdot)_{0,\Gamma}$ is hermitian because the kernel function $\mathcal{G}_\varepsilon$ in (2.7) is real valued (cf. [1, sec. 9.6.1]) and symmetric; i.e., $\mathcal{G}_\varepsilon(x-y) = \mathcal{G}_\varepsilon(y-x)$. ∎

Remark 2.7. The choice $\varepsilon = 1$ in (2.3.2) yields that B_1 is the single layer potential for the biharmonic operator $\mathcal{L}_{\mathrm{i}}^2 = (-\Delta + I)(-\Delta + I)$, and in this case the fundamental solution becomes

$$\mathcal{G}_1(z) = \frac{e^{-\|z\|}}{8\pi}. \tag{2.13}$$

Remark 2.8. A further choice for the operator $\gamma_0 B$ in (2.4) is the inverse of the Laplace–Beltrami operator. This operator satisfies, as V_0^2, Assumption 2.4 with $\varepsilon = 1/2$ (cf. [9] and [8]).

3. Galerkin boundary element method. The continuous problem which we are going to solve numerically is given by seeking $\varphi \in H^{-1/2}(\Gamma)$ such that

$$(R_k\varphi, \psi)_0 = (g, \psi)_0 \qquad \forall \psi \in H^{-1/2}(\Gamma), \tag{3.1}$$

where R_k is as in (2.4) and $g \in H^{1/2}(\Gamma)$ is a given right-hand side.

Our goal is to solve the problem (3.1) by the Galerkin boundary element method and we start by defining the relevant boundary element space.

Let Γ be a polyhedral surface. Let $\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_N\}$ denote a shape-regular triangulation of Γ, and let X_h be the boundary element space

$$X_h := \operatorname{span}\{b_\tau : \tau \in \mathcal{T}\} \tag{3.2}$$

with the indicator function $b_\tau : \Gamma \to \mathbb{R}$ for the triangle τ. The mesh width is denoted by

$$h := \max_{\tau \in \mathcal{T}} h_\tau \quad \text{with} \quad h_\tau := \operatorname{diam} \tau.$$

The discrete problem is given by seeking $\varphi_h \in X_h$ such that

$$(R_k \varphi_h, \psi)_0 = (g, \psi)_0 \qquad \forall \psi \in X_h. \tag{3.3}$$

4. Stability and error analysis for the Galerkin boundary element method. We start with the well-known approximation property of piecewise constant boundary elements on a shape-regular triangulation. In this light, we assume from now on that Γ is a polyhedral Lipschitz surface.

THEOREM 4.1. *Let $\mathcal{T}$ denote a shape-regular triangulation of the polyhedral surface Γ with maximal mesh width h. Then, there exists a constant C_A depending only on the minimal angle in the triangles in $\mathcal{T}$ such that, for all $-1/2 \le s \le 0$ and $s \le t \le 1$, the approximation property holds:*

$$\inf_{\psi \in X_h} \|\varphi - \psi\|_s \le C_A h^{t-s} \|\varphi\|_t \qquad \forall \varphi \in H^t(\Gamma).$$

We need now to recall the well-known mapping properties for the operators V_k and $\gamma_0^+ D_k$.

For polyhedral surfaces, we denote by s_Γ the regularity exponent associated with Γ such that for all s, $|s| < s_\Gamma$, the operators

$$V_k : H^{-1/2+s}(\Gamma) \to H^{+1/2+s}(\Gamma) \quad \text{and} \quad \gamma_0^+ D_k = \tfrac{1}{2} I + K_k : H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma) \tag{4.1}$$

are continuous. Note that, for polyhedral surfaces, we may choose $s_\Gamma = 1/2$. For smooth surfaces of class C^∞, the choice $s_\Gamma = \infty$ is allowed and, moreover, the operator K_k is of order -1, namely,

$$K_k : H^{1/2+s}(\Gamma) \to H^{3/2+s}(\Gamma) \quad \forall\, s \in \mathbb{R},\ \Gamma \text{ in } C^\infty. \tag{4.2}$$

As a consequence of these facts and of Proposition 2.3, the operator

$$R_k : H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma) \text{ is an isomorphism } \forall |s| < s_\Gamma. \tag{4.3}$$

The stability and convergence analysis will be based on a splitting of R_k into its principal part $V_0 + \frac{\mathrm{i}\eta}{2}\gamma_0 B_\varepsilon$ and the compact perturbation $\tilde{R}_k := V_0 + \frac{\mathrm{i}\eta}{2}\gamma_0 B_\varepsilon - R_k$. It is well known that the boundary integral operator V_0 for the single layer potential of the Laplacian is coercive, i.e., there is a constant $\alpha_0 > 0$ such that

$$(V_0 \varphi, \varphi)_0 \ge \alpha_0 \|\varphi\|^2_{-1/2} \qquad \forall \varphi \in H^{-1/2}(\Gamma). \tag{4.4}$$

Assumption 2.4 implies that the sesquilinear form $\left(\cdot, \frac{1}{2}\gamma_0 B_\varepsilon \cdot\right)_0$ is hermitian and

$$\left(\varphi, \frac{1}{2}\gamma_0 B_\varepsilon \varphi\right)_0 \ge \alpha_B \|\varphi\|^2_{-1/2-\varepsilon} \ge 0 \qquad \forall \varphi \in H^{-1/2}(\Gamma).$$

We conclude that $V_0 + \frac{1}{2}\gamma_0 B_\varepsilon$ is $H^{-1/2}$-coercive

$$\operatorname{Re}\left(\left(V_0 + \frac{\mathrm{i}\eta}{2}\gamma_0 B_\varepsilon\right)\varphi, \varphi\right)_0 \ge \alpha_0 \|\varphi\|^2_{-1/2} \qquad \forall \varphi \in H^{-1/2}(\Gamma), \tag{4.5}$$

where α_0 is as in (4.4) and, in particular, independent of ε.

In order to describe the mapping properties of the compact perturbation $\tilde{R}_k$ we introduce the interval I_{shift} which depends on the smoothness of the surface by

$$I_{\text{shift}} := \begin{cases} [0, 2\varepsilon] \cap [0, s_\Gamma[& \text{if } \Gamma \text{ is a general polyhedral surface,} \\ \left[0, \min\left\{\frac{3}{2}, 1+2\varepsilon\right\}\right] & \text{if } \Gamma \text{ is of class } C^\infty. \end{cases} \tag{4.6}$$

PROPOSITION 4.2. *The operator* $\tilde{R}_k : H^{-1/2}(\Gamma) \to H^{1/2+\mu}(\Gamma)$ *is continuous for all* $\mu \in I_{\text{shift}}$.

Proof. For polyhedral surfaces, the combination of (2.10) and (4.1) yields

$$K_k \gamma_0 B : H^{-1/2}(\Gamma) \to H^{1/2+\mu}\left(\mathbb{R}^3\right) \qquad \forall \mu \in I_{\text{shift}}, \tag{4.7}$$

whereas, for surfaces of class C^∞, the combination of (2.10) and (4.2) yields

$$K_k \gamma_0 B : H^{-1/2}(\Gamma) \to H^{1/2+\mu}\left(\mathbb{R}^3\right) \qquad \forall \mu \in [0, 1+2\varepsilon]. \tag{4.8}$$

The difference $V_0 - V_k$ can be written in the form $\gamma_0 \mathcal{N}_{0,k} \gamma_0'$, where $\mathcal{N}_{0,k} : H^s_{\text{comp}}\left(\mathbb{R}^3\right) \to H^{s+4}_{\text{loc}}\left(\mathbb{R}^3\right)$ is a pseudodifferential operator of order -4 (cf. [32, Bemerkung 3.1.3]). Hence, the mapping properties of the trace operator and its dual imply the continuity of

$$V_0 - V_k : H^{-1/2}(\Gamma) \to H^{1/2+s}(\Gamma) \qquad \forall s \ : \ s \le 2, s < s_\Gamma. \qquad \square \tag{4.9}$$

Note that the proof of Proposition 4.2 allows us to replace $\frac{3}{2}$ by 2 in (4.6) for smooth surfaces. However, the proof of Theorem 4.3 will further restrict the set of admissible shifts to $\mu \le 3/2$ and we have taken this fact into account already in (4.6).

We denote by C_X the continuity constant of R_k; i.e.,

$$C_X := \sup\left\{ |(\varphi, R_k \psi)_0| : \varphi, \psi \in H^{-1/2}(\Gamma) : \|\varphi\|_{-1/2} = \|\psi\|_{-1/2} = 1 \right\}. \tag{4.10}$$

The combination of (4.3) with Proposition 4.2 yields that

$$C_\mu := \left\| (R_k^\star)^{-1} \tilde{R}_k \right\|_{H^{-1/2+\mu}(\Gamma) \leftarrow H^{-1/2}(\Gamma)} \tag{4.11}$$

is bounded for all $\mu \in I_{\text{shift}}$. The operator R_k satisfies a Gårding inequality but is not coercive. Hence, we may expect the existence of a discrete solution of (3.3) only for sufficiently small mesh width h. In this light, we define h_0 depending on $\mu \in I_{\text{shift}}$ by

$$h_0 := \left(\frac{\alpha_0}{2C_A} \frac{1}{C_X C_\mu} \right)^{1/\mu}. \tag{4.12}$$

THEOREM 4.3. *Let Assumption* 2.4 *be satisfied. Then, for all* $0 < h < h_0$ *with* h_0 *as in* (4.12) *for some* $\mu \in I_{\text{shift}}$, *the Galerkin discretization* (3.3) *has a unique solution which satisfies the quasi-optimal error estimate*

$$\|\varphi - \varphi_h\|_{-1/2} \le \frac{2C_X}{\alpha_0} \inf_{\psi \in X_h} \|\varphi - \psi\|_{-1/2}. \tag{4.13}$$

The dense embedding $\bigcup X_h \hookrightarrow H^{-1/2}(\Gamma)$ *implies the convergence as* $h \to 0$.

Proof. This proof is inspired by the proof of the analogue theorem for elliptic partial differential equations (see, e.g., [6, sec. 5.7]).

(a) *Convergence estimate.* Assume that a discrete solution φ_h exists. The error is denoted by $e := \varphi - \varphi_h$ and can be estimated by using the Galerkin orthogonality

$$\begin{aligned}(4.14)\quad \alpha_0 \|e\|^2_{-1/2} &\le (R_k e, e)_0 + \left(\tilde{R}_k e, e\right)_0 \le |(R_k e, \varphi - \phi_h)_0| + \left|\left(\tilde{R}_k e, e\right)_0\right| \\ &\le C_X \|e\|_{-1/2} \|\varphi - \phi_h\|_{-1/2} + \left|\left(\tilde{R}_k e, e\right)_0\right|,\end{aligned}$$

where $\phi_h \in X_h$ is the best approximation of φ with respect to the $\|\cdot\|_{-1/2}$-norm. In order to estimate the modulus of $(\tilde{R}_k e, e)_0$ we use a duality argument. Let ψ_e be the solution of the adjoint problem:

$$R_k^\star \psi_e = \tilde{R}_k e \qquad \text{in } H^{1/2}(\Gamma).$$

By the definition of the constant C_μ, we conclude that

$$\|\psi_e\|_{-1/2+\mu} \le C_\mu \|e\|_{-1/2} \qquad \forall \mu \in I_{\text{shift}}$$

holds.

Let $\psi_h \in X_h$ denote the best approximation of ψ_e with respect to the $\|\cdot\|_{-1/2}$-norm. The approximation property of piecewise constant boundary elements yields

$$\|\psi_e - \psi_h\|_{-1/2} \le C_A C_\mu \|e\|_{-1/2} h^\mu \qquad \forall \mu \in I_{\text{shift}}.$$

Galerkin's orthogonality implies

$$\begin{aligned}\left|\left(\tilde{R}_k e, e\right)_0\right| &= |(R_k^\star \psi_e, e)_0| = |(\psi_e, R_k e)_0| = |(R_k e, \psi_e - \psi_h)_0| \\ &\le C_X \|e\|_{-1/2} \|\psi_e - \psi_h\|_{-1/2} \le C_A C_X C_\mu h^\mu \|e\|^2_{-1/2} \qquad \forall \mu \in I_{\text{shift}}.\end{aligned}$$

Using this estimate in (4.14) we obtain

$$\alpha_0 \|e\|^2_{-1/2} \le C_X \|e\|_{-1/2} \|\varphi - \phi_h\|_{-1/2} + C_A C_X C_\mu h^\mu \|e\|^2_{-1/2}.$$

Let h_0 be as in (4.12). Then, for $h < h_0$, the estimate (4.13) holds. Note that, when Γ is a C^∞-surface, the choice $\mu = \min\left\{\frac{3}{2}, 1 + 2\varepsilon\right\}$ is allowed.

(b) *Existence.* Since the discrete problem is finite dimensional it suffices to prove uniqueness. For $g = 0$, Proposition 2.3 implies that the exact solution is $\varphi = 0$. Equation (4.13) implies $\varphi_h = 0$ and this proves the uniqueness. ∎

Remark 4.4. Theorem 4.3 provides convergence of the Galerkin method for polyhedral surfaces Γ. This is a much stronger result as for the Brakhage–Werner stabilization where the stability of the resulting method is still open for polyhedral surfaces.

Remark 4.5. In section 5 we will investigate the dependence of the maximal mesh width h_0 on the wave number k for the special case that Γ is the unit sphere. It turns out that the operator splitting

$$C_\mu \le \left\|(R_k^\star)^{-1}\right\|_{H^{-1/2+\mu}(\Gamma) \leftarrow H^{1/2+\mu}(\Gamma)} \left\|\tilde{R}_k\right\|_{H^{1/2+\mu}(\Gamma) \leftarrow H^{-1/2}(\Gamma)}$$

and the estimate of the two factors in the right-hand side lead to too pessimistic estimates of the dependence of C_μ on k. Thus, we estimate the constant C_μ directly.

In the following, we will analyze the dependence of the Galerkin error on the wave number k for the case, where we have full regularity. In this light, we introduce the set of functions having the property that the derivatives grow proportionally with respect to the wave number k.

DEFINITION 4.6. *For given $\rho > 0$, the set $\mathcal{O}_{\rho,k}$ contains all functions $\varphi \in H^1(\Gamma)$ such that*

$$\|\varphi\|_1 \leq \rho k^{3/2} \|\varphi\|_{-1/2}. \tag{4.15}$$

THEOREM 4.7. *Let Assumption 2.4 be satisfied. Assume that Γ is smooth and let the solution φ of (2.5) be in $\mathcal{O}_{\rho,k}$ for some $\rho > 0$. For all $0 < h < h_0$ with h_0 as in (4.12) for some $\mu \in I_{\text{shift}}$ and solutions $\varphi \neq 0$, the relative error can be estimated by*

$$\frac{\|\varphi - \varphi_h\|_{-1/2}}{\|\varphi\|_{-1/2}} \leq \left(\frac{2C_X}{\alpha_0}\right) C_A \rho (kh)^{3/2}.$$

Proof. Using (4.13) together with Theorem 4.1, we obtain

$$\|\varphi - \varphi_h\|_{-1/2} \leq \left(\frac{C_X}{2\alpha_0}\right) C_A h^{3/2} \|\varphi\|_1.$$

Taking into account the oscillation condition (4.15) yields the proof. ∎

In subsection 5.7 we will consider the question of under which conditions the solution of the integral equation (2.5) belongs to the set $\mathcal{O}_{\rho,k}$.

5. The special case of $\Gamma = \mathbb{S}^2$. In this section, we will investigate the dependence of the constant C_X and the minimal mesh width h_0 (see (4.12)) upon η and k for the special case that Ω is the unit ball

$$\Omega := \left\{x \in \mathbb{R}^3 : \|x\| < 1\right\}.$$

It is well known that for $\Gamma = \mathbb{S}^2$, the Sobolev spaces can be defined via the decay properties of the Fourier coefficients.

5.1. Spherical harmonics. For a function $f \in L^2(\mathbb{S}^2)$ the Fourier coefficients are defined by

$$f_n^m := \int_{\mathbb{S}^2} Y_n^m(\hat{x}) \overline{f(\hat{x})} ds_x, \tag{5.1}$$

where Y_n^m are the spherical harmonics. We defer the reader to [1].

DEFINITION 5.1. *A function $f \in L^2(\mathbb{S}^2)$ is in the Sobolev space $\mathcal{H}^s(\mathbb{S}^2)$, $s \geq 0$, if the Fourier coefficients satisfy*

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} |f_n^m|^2 (1+n^2)^s < \infty. \tag{5.2}$$

With the inner product

$$\langle f, g\rangle_s := \sum_{n=0}^{\infty} (1+n^2)^s \sum_{m=-n}^{n} f_n^m \overline{g_n^m} \tag{5.3}$$

and the induced norm, the space $\mathcal{H}^s(\mathbb{S}^2)$ is a Hilbert space. For negative $s < 0$, the space $\mathcal{H}^s(\mathbb{S}^2)$ is the dual space of $\mathcal{H}^{-s}(\mathbb{S}^2)$. For a functional $F \in \mathcal{H}^s(\mathbb{S}^2)$, its norm is given by (5.2), where the Fourier coefficients f_n^m are given by $f_n^m := F(Y_n^m)$.

The eigenfunctions of all arising boundary integral operator on $\mathbb{S}^2$ are given by the spherical harmonics. The eigenvalues can be expressed by Bessel and related functions. Let j_n, respectively, $h_n^{(1)}$, denote the spherical Bessel functions of first and third kind (cf. [1]).

LEMMA 5.2.

a. *The spherical harmonics form a complete orthogonal system in* $\mathcal{H}^s\left(\mathbb{S}^2\right)$, $s \in \mathbb{R}$, *and*

$$\left\langle Y_n^m, Y_{n'}^{m'} \right\rangle_s = \delta_{n,n'}\delta_{m,m'}\left(1+n^2\right)^s$$

for all $n, n' \in \mathbb{N}$ *and* $|m| \le n$ *and* $|m'| \le n'$.

b. *The spherical harmonics are the eigenfunctions of the operator* R_k. *More precisely, we have*

i.

$$V_k Y_n^m = \lambda_{n,k}^{(V)} Y_n^m \quad \textit{with} \quad \lambda_{n,k}^{(V)} := 2\,\mathrm{i}\,k h_n^{(1)}(k)\, j_n(k).$$

ii.

$$\left(\frac{1}{2}I + K_k\right) Y_n^m = \lambda_{n,k}^{(K)} Y_n^m \quad \textit{with} \quad \lambda_{n,k}^{(K)} := \mathrm{i}\,k^2 h_n^{(1)}(k)\, j_n'(k).$$

c. *The sets* $\mathcal{H}^s\left(\mathbb{S}^2\right)$ *and* $H^s\left(\mathbb{S}^2\right)$ *coincide and their norms are equivalent: For* $\varphi \in \mathcal{H}^s\left(\mathbb{S}^2\right)$, *let* $\varphi := (\varphi_n^m)_{\substack{n\in\mathbb{N}_0 \\ -n\le m\le n}}$ *denote its Fourier coefficients. Define the (infinite) diagonal matrix* $\mathbf{H}_s := \left(\operatorname{diag}\left(1+n^2\right)^s\right)_{n\in\mathbb{N}_0}$. *Then*

$$\|\varphi\|_{H^s(\mathbb{S})}^2 \sim \left|\!\left|\!\left|(\varphi_n^m)_{n,m}\right|\!\right|\!\right|_s^2 := \sum_{n=0}^{\infty} (\mathbf{H}_s)_n \sum_{m=-n}^{n} |\varphi_n^m|^2. \tag{5.4}$$

Proof. Ad a. The result follows by using the $L^2(\mathbb{S})$-orthogonality of the spherical harmonics (cf., e.g., [29, Theorem 2.4.1]) for $f = Y_{n'}^{m'}$ in (5.1) and the definition of the scalar product (5.3).

Ad b. See [22].

Ad c. See [28, Chapter X, Theorem 6.4]. ∎

In view of Lemma 5.2 (c), we will use the same notation $H^s(\mathbb{S})$ for both $H^s(\mathbb{S})$ and $\mathcal{H}^s(\mathbb{S})$.

The space of sequences $(\varphi_n^m)_{\substack{n\in\mathbb{N} \\ -n\le m\le n}}$ where the right-hand side in (5.4) is finite is denoted by $\mathbf{h}_s$ and the right-hand side in (5.4) defines (the square of) the norm in $\mathbf{h}_s$.

Assumption 2.4 implies that $\gamma_0 B : H^{-1/2-\varepsilon}(\Gamma) \to H^{1/2+\varepsilon}(\Gamma)$ is continuous and elliptic. In order to develop a spectral analysis of the operator R_k we furthermore assume that, for $\Gamma = \mathbb{S}^2$, the spherical harmonics are the eigenfunctions also for the operator $\gamma_0 B$.

Assumption 5.3. Let $\Gamma = \mathbb{S}^2$ and let Assumption 2.4 be satisfied for some $0 < \varepsilon \le 1$. The spherical harmonics are the eigenfunctions of $\gamma_0 B$:

$$(\gamma_0 B)\, Y_n^m = \lambda_n^{(B)} Y_n^m \qquad \forall n \in \mathbb{N}, -m \le n \le m.$$

There exist constants $0 < c_1 \le C_1 < \infty$ independent of n such that

$$\frac{c_1}{(n+1)^{1+2\varepsilon}} \le \lambda_n^{(B)} \le \frac{C_1}{(n+1)^{1+2\varepsilon}} \qquad \forall n \in \mathbb{N}. \tag{5.5}$$

For the choice $\gamma_0 B = V_0^2$ (cf. subsection 2.3.1) Assumption 5.3 with $\varepsilon = 1/2$ simply follows from the well-known relation $V_0 Y_n^m = (2n+1)^{-1} Y_n^m$ (cf. [29]).

THEOREM 5.4. *For any fixed $\varepsilon \in]0,1]$, the operator B_ε as defined in* (2.8) *satisfies Assumption* 5.3.

Proof. First, we will prove that

$$\gamma_0 B_\varepsilon Y_n^m = \lambda_n Y_n^m$$

holds. Note that

$$\gamma_0 B_\varepsilon Y_n^m = \gamma_0 N_\varepsilon \left(Y_n^m \delta_\Gamma \right), \tag{5.6}$$

where N_ε denotes the Bessel potential (cf. (2.9)) and δ_Γ is the Dirac function concentrated on Γ. For $\xi \in \mathbb{S}^2$, we write for short $Y_n^m(\xi)$ instead of $Y_n^m(\alpha,\beta)$, where α,β are the spherical angles of ξ. We employ the Fourier transform to evaluate (5.6) and obtain

$$\begin{aligned}
\widehat{N_\varepsilon (Y_n^m \delta_\Gamma)}(\hat{x}) &= \frac{1}{\left(1+\|\hat{x}\|^2\right)^{1+\varepsilon} (2\pi)^{3/2}} \int_{\mathbb{R}^3} Y_n^m\left(\frac{x}{\|x\|}\right) \delta_\Gamma(x)\, e^{-i\langle \hat{x}, x\rangle} dx \\
&= \frac{1}{\left(1+\|\hat{x}\|^2\right)^{1+\varepsilon} (2\pi)^{3/2}} \int_{\mathbb{S}^2} Y_n^m(\xi)\, e^{-i\langle \hat{x}, \xi\rangle} d\xi.
\end{aligned}$$

We make use of the formula

$$\int_{\mathbb{S}^2} e^{-i\langle x, \hat{x}\rangle} Y_\ell^m(x)\, dx = g_\ell(\|\hat{x}\|)\, Y_\ell^m\left(\frac{\hat{x}}{\|\hat{x}\|}\right) \quad \text{with} \quad g_\ell(r) = (-i)^\ell 4\pi j_\ell(r)$$

which follows by a comparison of [29, section 3.2.4, formula (3.2.44)] and [29, section 3.2.4, formula (3.2.54)] . Let $\mathcal{F}^{-1}$ denote the inverse Fourier transform. Then

$$\begin{aligned}
N_\varepsilon (Y_n^m \delta_\Gamma)(x) &= \left(\mathcal{F}_{\hat{x}}^{-1} \frac{g_n(\|\hat{x}\|)\, Y_n^m\left(\frac{\hat{x}}{\|\hat{x}\|}\right)}{\left(1+\|\hat{x}\|^2\right)^{1+\varepsilon} (2\pi)^{3/2}} \right)(x) \\
&= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{g_n(\|\hat{x}\|)\, Y_n^m\left(\frac{\hat{x}}{\|\hat{x}\|}\right)}{\left(1+\|\hat{x}\|^2\right)^{1+\varepsilon}} e^{i\langle x, \hat{x}\rangle} d\hat{x} \\
&\overset{\hat{x} \leftarrow r\zeta}{=} \frac{1}{(2\pi)^{3/2}} \int_0^\infty \frac{r^2 g_n(r)}{(1+r^2)^{1+\varepsilon}} \left(\int_{\mathbb{S}^2} Y_n^m(\zeta)\, e^{i\langle x, r\zeta\rangle} d\zeta \right) dr \\
&= Y_n^m\left(\frac{-x}{\|x\|}\right) \frac{1}{(2\pi)^{3/2}} \int_0^\infty \frac{r^2 g_n(r)\, g_n(-\|x\| r)}{(1+r^2)^{1+\varepsilon}} dr.
\end{aligned}$$

Note that $Y_n^m(-\xi) = (-1)^n Y_n^m(\xi)$. Applying the trace operator to this equation yields

$$\gamma_0 B_\varepsilon Y_n^m = \lambda_n Y_n^m \quad \text{with} \quad \lambda_n := \frac{(-1)^n}{(2\pi)^{3/2}} \int_0^\infty \frac{r^2 g_n(r)\, g_n(-r)}{(1+r^2)^{1+\varepsilon}} dr,$$

where the asymptotic behavior of the spherical Bessel functions (see [1, (10.1)]) implies that λ_n is finite.

In summary, we have proved that Y_n^m are the eigenfunctions of the operator $\gamma_0 B_\varepsilon$ with eigenvalues λ_n. We already proved (cf. Theorem 2.5) that $B_\varepsilon : H^{-1/2-\varepsilon}(\Gamma) \to H^{1/2+\varepsilon}(\Gamma)$ is continuous and (cf. Theorem 2.6) that the sesquilinear form associated with $\gamma_0 B_\varepsilon$ is $H^{-1/2-\varepsilon}(\Gamma)$-elliptic and, thus, $\lambda_n > 0$. Hence,

$$\frac{1}{C}\|Y_n^m\|^2_{H^{-1/2-\varepsilon}(\Gamma)} \le \underbrace{(\gamma_0 B_\varepsilon Y_n^m, Y_n^m)_{0,\Gamma}}_{=\lambda_n} \le C\|Y_n^m\|^2_{H^{-1/2-\varepsilon}(\Gamma)} \quad \forall n \in \mathbb{N},\ -n \le m \le n,$$

where $C > 0$ is independent of n, m. For $\Gamma = \mathbb{S}^2$, the $H^{-1/2-\varepsilon}(\Gamma)$-norm is equivalent to the Fourier norm (cf. Lemma 5.2) and the estimate

$$\tilde{c}\,(1+n)^{-1-2\varepsilon} \le \lambda_n \le \tilde{C}\,(1+n)^{-1-2\varepsilon} \qquad \forall n \in \mathbb{N}$$

directly follows. ∎

5.2. Evaluation of the constants of interest in terms of eigenvalues of the underlying integral operators. The eigenvalues of the operator R_k are given by

$$\lambda_{n,k}^{(R)} := \lambda_{n,k}^{(V)} + \mathrm{i}\,\eta \lambda_n^{(B)} \lambda_{n,k}^{(K)}. \tag{5.7}$$

Remark 5.5. Assumption 5.3 implies that the qualitative dependence of $\lambda_n^{(B)}$ on n can be studied by replacing $\lambda_n^{(B)}$ by $\tilde{\lambda}_n^{(B)} := (n+1)^{-1-2\varepsilon}$. For all numerical experiments, we have employed this simplification and indicated this by a superscript $\tilde{\lambda}$.

Since we are concerned in this section with the smooth surface of the unit ball we may choose the parameter μ in the definition of h_0 (cf. (4.12)) by

$$\mu := \min\left\{\frac{3}{2}, 1+2\varepsilon\right\}. \tag{5.8}$$

In view of Theorem 4.3, we see that the maximal mesh width h_0 which guarantees existence and uniqueness is given by (4.12) and depends on α_0, ε, C_μ, C_X, C_A. Our goal is to analyze the dependence of h_0 on k and h. Since α_0 and C_A (cf. (4.5), Theorem 4.1) are independent of k and h, we are left with the analysis of C_μ (cf. (4.11)), C_X (cf. (4.10)). Note that such estimates directly imply an estimate of the constant $2C_X/\alpha_0$ in the error estimate (4.13).

i. The continuity constant C_X can be estimated in terms of eigenvalues by

$$\begin{aligned} C_X = \|R_k\|_{H^{1/2}(\Gamma)\leftarrow H^{-1/2}(\Gamma)} &\le \sup_{n\in\mathbb{N}} \sqrt{1+n^2}\left|\lambda_{n,k}^{(R)}\right| \\ &\le \sup_{n\in\mathbb{N}} \sqrt{1+n^2}\left(\left|\lambda_{n,k}^{(V)}\right| + \eta\left|\lambda_n^{(B)}\lambda_{n,k}^{(K)}\right|\right). \end{aligned} \tag{5.9}$$

ii. The constant C_μ can be expressed in terms of the eigenvalues as follows. Let $\varphi = \sum_{n=0}^\infty \sum_{m=-n}^n \varphi_n^m Y_n^m$ be such that $\sum_{n=0}^\infty \sum_{m=-n}^n (1+n^2)^{-1/2}|\varphi_n^m|^2 = 1$; then $(R_k^\star)^{-1}\tilde{R}_k\varphi$ reads

$$(R_k^\star)^{-1}\tilde{R}_k\varphi = \sum_{n=0}^\infty \frac{1}{\overline{\lambda}_{n,k}^{(R)}}\left(\lambda_{n,k}^{(R)} - \frac{2}{2n+1} - \frac{\mathrm{i}\,\eta}{2}\lambda_n^{(B)}\right)\sum_{m=-n}^n \varphi_n^m Y_n^m$$

and C_μ can be expressed by

$$C_\mu = \sup_{n\in\mathbb{N}}\left\{\frac{\left(1+n^2\right)^{\frac{\mu+1}{2}}\left|\lambda_{n,k}^{(R)} - \frac{2}{2n+1} - \frac{\mathrm{i}\,\eta}{2}\lambda_n^{(B)}\right|}{|\lambda_{n,k}^{(R)}|\sqrt{1+n^2}}\right\}. \tag{5.10}$$

The rest of the section is structured as follows: In subsection 5.3 we will study the behavior (in k and n) of the eigenvalues in the asymptotic ranges of k and n; the choice of η in subsection 5.4 will be based on these asymptotics; in subsection 5.5 we will study the behavior of the eigenvalues in the nonasymptotic range of k and n via computer experiments. In subsection 5.6 we will derive bounds for the quantities C_X and h_0 which are *explicit* in the wave number k. Finally, in subsection 5.7 we will investigate the oscillation hypothesis made in Theorem 4.7.

5.3. Asymptotic analysis of the eigenvalues. The following lemma concerns the asymptotic behavior of the eigenvalues $\lambda_{n,k}^{(V)}$, $\lambda_n^{(B)}$, and $\lambda_{n,k}^{(K)}$. Let

$$\operatorname{trig}(k) := \begin{cases} \sin k & \text{if } n \text{ is even,} \\ \mathrm{i}\cos k & \text{if } n \text{ if odd,} \end{cases}$$

where we suppress the dependence on n in the notation of trig. Note that

$$\operatorname{trig}\left(k+\frac{\pi}{2}\right) = \operatorname{trig}'(k) = \begin{cases} \cos k & \text{if } n \text{ is even,} \\ -\mathrm{i}\sin k & \text{if } n \text{ if odd.} \end{cases}$$

LEMMA 5.6. *The eigenvalues $\lambda_{n,k}^{(V)}$, $\lambda_{n,k}^{(K)}$ have the following asymptotic behavior*

1. *For fixed n and $k \to \infty$ we have*

$$\lambda_{n,k}^{(V)} = 2\frac{e^{\mathrm{i}k}}{k}\operatorname{trig}(k) + O\left(\frac{1}{k^2}\right),$$

$$\lambda_{n,k}^{(K)} = e^{\mathrm{i}k}\operatorname{trig}'(k) + O\left(\frac{1}{k}\right).$$

2. *Let $k = n + 1/2$. Then,*

$$\lambda_{n,k}^{(V)} = \pi\left(\frac{2^{1/3}}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}\right)^2 k^{-2/3}\left(\sqrt{3}+\mathrm{i}\right)(1+o(1)),$$

$$\lambda_{n,k}^{(K)} = \frac{\mathrm{i}+\sqrt{3}}{2\sqrt{3}}(1+o(1)).$$

3. *Let k be fixed and $n \to \infty$. Then, we obtain*

$$\lambda_{n,k}^{(V)} = \frac{2}{2n+1}\left(1+O\left(\frac{1}{n^2}\right)\right),$$

$$\lambda_{n,k}^{(K)} = \frac{1}{2} + O\left(\frac{1}{n}\right).$$

Proof. Ad 1. The *spherical* Bessel functions j_n, h_n can be expressed via the Bessel functions of first and third kind J_n and H_n (cf. [1, (10.1.1)]):

$$j_n(z) = \sqrt{\frac{\pi}{2z}}J_{n+1/2}(z) \quad \text{and} \quad h_n(z) = \sqrt{\frac{\pi}{2z}}H_{n+1/2}^{(1)}(z). \tag{5.11}$$

We combine (5.11) with the asymptotic expansion (cf. [1, (9.2.1) and (9.2.3)]) to obtain

$$
\begin{aligned}
h_n^{(1)}(k)\, j_n(k) &\overset{[1,\ (10.1.1)]}{=} \frac{\pi}{2k} J_{n+1/2}(k)\, H_{n+1/2}^{(1)}(k) \\
&\overset{[1,\ (9.2.1),\ (9.2.3)]}{\sim} \frac{1}{k^2} e^{\mathrm{i}\left(k-\frac{\pi}{2}(n+1)\right)} \cos\left(k-\frac{\pi}{2}(n+1)\right) + O\left(\frac{1}{k^3}\right) \\
&= \frac{e^{\mathrm{i}\,k}}{\mathrm{i}\,k^2}\,\mathrm{trig}(k) + O\left(\frac{1}{k^3}\right)
\end{aligned} \tag{5.12}
$$

and

$$
\begin{aligned}
h_n^{(1)}(k)\, j_n'(k) &\overset{[1,\ (10.1.20)]}{=} \sqrt{\frac{\pi}{2k}} H_{n+1/2}^{(1)}(k)\, \frac{\left(n j_{n-1}(k) - (n+1)\, j_{n+1}(k)\right)}{2n+1} \\
&\overset{[1,\ (10.1.1)]}{=} \frac{\pi}{2k} H_{n+1/2}^{(1)}(k)\, \frac{\left(n J_{n-1/2}(k) - (n+1)\, J_{n+3/2}(k)\right)}{2n+1} \qquad (5.13) \\
&\overset{[1,\ (9.2.1),\ (9.2.3)]}{\sim} \frac{e^{\mathrm{i}\,k}}{k^2} e^{-\mathrm{i}\,\frac{\pi}{2}(n+1)} \frac{n\cos\left(k-\frac{\pi n}{2}\right) - (n+1)\cos\left(k-\frac{\pi(n+2)}{2}\right)}{2n+1} \\
&\quad + O\left(\frac{1}{k^3}\right) \\
&= \frac{e^{\mathrm{i}\,k}}{\mathrm{i}\,k^2}\,\mathrm{trig}'(k) + O\left(\frac{1}{k^3}\right).
\end{aligned}
$$

Hence

$$
\lambda_{n,k}^{(V)} = 2\,\mathrm{i}\,k h_n^{(1)}(k)\, j_n(k) = 2\frac{e^{\mathrm{i}\,k}}{k}\,\mathrm{trig}(k) + O\left(\frac{1}{k^2}\right), \tag{5.14}
$$

$$
\lambda_{n,k}^{(K)} = \mathrm{i}\,k^2 h_n^{(1)}(k)\, j_n'(k) = e^{\mathrm{i}\,k}\,\mathrm{trig}'(k) + O\left(\frac{1}{k}\right). \tag{5.15}
$$

Ad 2. Let $k = n + 1/2$. Then,

$$
j_n\left(n+\frac{1}{2}\right) \overset{(5.11)}{=} \sqrt{\frac{\pi}{2k}} J_k(k) \quad \text{and} \quad h_n^{(1)}(k) = \sqrt{\frac{\pi}{2k}} H_k^{(1)}(k).
$$

We employ the asymptotics for $J_\nu(\nu)$ and $H_\nu(\nu)$ to obtain

$$
\begin{aligned}
\lambda_{n,k}^{(V)} &= 2\,\mathrm{i}\,k j_n\left(n+\frac{1}{2}\right) h_n^{(1)}\left(n+\frac{1}{2}\right) \\
&\overset{[1,\ (9.3.31),\ (9.3.32)]}{=} \pi\left(\frac{2^{1/3}}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}\right)^2 k^{-2/3}\left(\sqrt{3}+\mathrm{i}\right)(1+o(1))
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_{n,k}^{(K)} &= \mathrm{i}\,k^2 h_n^{(1)}(k)\, j_n'(k) \overset{(5.13)}{=} \mathrm{i}\,k^2 \frac{\pi}{2k} H_k^{(1)}(k)\, \frac{\left(\left(k-\frac{1}{2}\right) J_{k-1}(k) - \left(k+\frac{1}{2}\right) J_{k+1}(k)\right)}{2k} \\
&\overset{[1,\ (9.1.27)(1)\text{-}(2)]}{=} \mathrm{i}\,k\frac{\pi}{2} H_k^{(1)}(k)\left(J_k'(k) - \frac{1}{2k} J_k(k)\right) \\
&\overset{[1,\ (9.3.31\text{-}34)]}{\sim} \frac{\mathrm{i}+\sqrt{3}}{2\sqrt{3}}(1+o(1)).
\end{aligned}
$$

Ad 3. Let k be fixed and $n \to \infty$.

The definition of the spherical Bessel and Hankel functions implies in that case

$$\lambda_{n,k}^{(V)} = \frac{2\,\mathrm{i}\,k \cdot k^n \left\{1 - \frac{k^2/2}{2n+3} + O\left(n^{-2}\right)\right\}}{1 \cdot 3 \cdot \cdots \cdot (2n+1)} \times \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)\left\{1 - \frac{k^2/2}{1-2n} + O\left(n^{-2}\right)\right\}}{\mathrm{i}\,k^{n+1}}$$
$$= \frac{2}{2n+1}\left(1 + O\left(n^{-2}\right)\right)$$

and

$$\lambda_{n,k}^{(K)} = \mathrm{i}\,k^2 h_n^{(1)}(k)\, j_n'(k) = \frac{n}{2n+1}\left(1 + O\left(\frac{1}{n}\right)\right) = \frac{1}{2} + O\left(\frac{1}{n}\right). \qquad \square$$

Lemma 5.6 leads to the following asymptotic behavior of the real and imaginary parts of the eigenvalues $\lambda_{n,k}^{(R)}$. We restrict ourselves to the case $k \geq k_0 > 0$. The "$\sim$"-notation indicates that we neglect the higher order terms in Lemma 5.6. The numbers $a_0, \ldots, a_4$ below are positive and may depend on k_0 but not on k.

- $n = 0$, $k \to \infty$:

  (5.16a)
  $$\operatorname{Re}\lambda_{0,k}^{(R)} \sim \sin k \cos k \left(\frac{2}{k} - a_0\eta\right) \quad \text{and} \quad \operatorname{Im}\lambda_{0,k}^{(R)} \sim 2\frac{\sin^2 k}{k} + a_0\eta\cos^2 k.$$

- $n + \frac{1}{2} = k$:

  (5.16b)
  $$\sqrt{1+n^2}\operatorname{Re}\lambda_{n,k}^{(R)} \sim a_1 k^{1/3} - \frac{a_2\eta}{k^{2\varepsilon}} \quad \text{and} \quad \sqrt{1+n^2}\operatorname{Im}\lambda_{n,k}^{(R)} \sim a_3 k^{1/3} + \frac{a_4\eta}{k^{2\varepsilon}}.$$

- k fixed and $n \to \infty$:

  (5.16c)
  $$\sqrt{1+n^2}\operatorname{Re}\lambda_{n,k}^{(R)} \sim 1 \quad \text{and} \quad \operatorname{Im}\lambda_{n,k}^{(R)} \to 0.$$

5.4. Choice of η. Our goal is to stabilize the single layer potential V_k such that the constant governing the convergence behavior, i.e., $2C_X/\alpha_0$, and the bound for the maximal step width h_0 are as close as possible to those of the pure single layer potential away from the forbidden frequencies. The asymptotic behavior of the eigenvalues in (5.16b) indicates that the continuity constant C_X in (5.9) cannot behave better than $O\left(k^{1/3}\right)$. This leads to the heuristics to choose the stabilization parameter η maximal under the side condition that C_X still is bounded by $Ck^{1/3}$. The asymptotics (5.16a)–(5.16c) show that the choice $\eta = k^{1/3}$ leads to the bound

$$\sqrt{1+n^2}\left|\lambda_{n,k}^{(R)}\right| \leq Ck^{1/3}$$

for the three asymptotic cases, while the choice $\eta = k^{1/3+\delta}$ for any $\delta > 0$ would lead to an increased constant $C_X = O\left(k^{1/3+\delta}\right)$; cf. (5.16a).

We have performed computer experiments to study the behavior of $\sqrt{1+n^2}|\lambda_{n,k}^{(R)}|$ in the intermediate ranges.

a. Figure 5.1 indicates that the case $n + \frac{1}{2} = k$ (Lemma 5.6(2)) is relevant for the upper bound of $\sqrt{1+n^2}|\lambda_{n,k}^{(V)}|$ and this choice leads to

(5.17)
$$\sqrt{1 + \left(k - \frac{1}{2}\right)^2}\left|\lambda_{k-1/2,k}^{(V)}\right| \overset{\text{Lemma 5.6(2)}}{\leq} C' k^{1/3}.$$

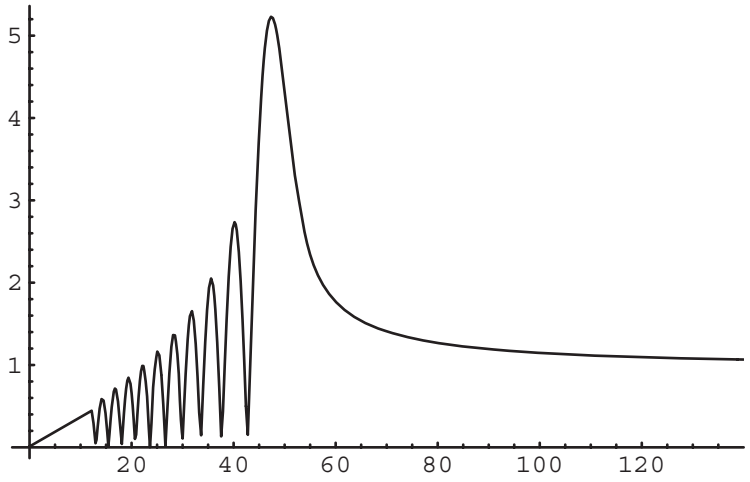

FIG. 5.1. *Plot of* $\sqrt{1+n^2}\left|\lambda_{n,k}^{(V)}\right|$ *for* $k = 50$ *in the range of* $1 \le n \le 300$. *The maximum is achieved at about* $n = k - 1/2$. *Pictures for different values of* k *show the same behavior.*

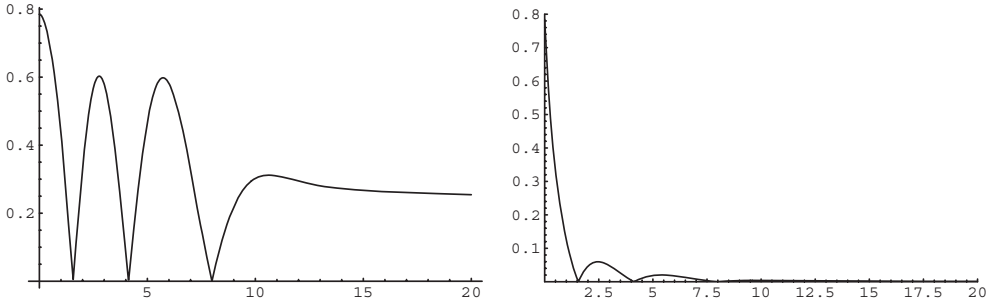

FIG. 5.2. *Plot of* $\sqrt{1+n^2}|\tilde{\lambda}_n^{(B)}\lambda_{n,k}^{(K)}|$ *for* $k = 10$ *and* $\varepsilon = \frac{1}{10}$ *(left picture),* $\varepsilon = 1$ *(right picture) in the range of* $0 \le n \le 20$. *The function decreases as* $n \to \infty$. *The maximum is achieved at* $n = 0$. *Pictures for different values of* k *show qualitatively the same behavior.*

b. We have plotted the upper bound of $|\sqrt{1+n^2}\tilde{\lambda}_n^{(B)}\lambda_{n,k}^{(K)}|$ for a small value $\varepsilon = 0.1$ and the maximal value $\varepsilon = 1$ to study its qualitative behavior. Figure 5.2 indicates that the case $n = 0$ is relevant for the upper bound and this choice leads to the (rough) estimate (cf. Lemma 5.2 (b.ii) for $n = 0$)

$$\left|\lambda_0^{(B)}\lambda_{0,k}^{(K)}\right| \le C\left|\cos k - \frac{\sin k}{k}\right| \le 2C. \tag{5.18}$$

Hence the estimates (5.17), (5.18) along with the asymptotics (5.16a)–(5.16c) suggest $\eta \le k^{1/3}$ in order not to destroy the upper bound for the continuity constant C_X.

DEFINITION 5.7. *The stabilization parameter* η *in* (2.4) *is chosen to be*

$$\eta := k^{1/3}.$$

5.5. Computer based analysis of the constant C_μ. Recall the choice of the shift parameter μ as in (5.8) and the formula (5.10) for the constant $C_\mu = \sup_{n\in\mathbb{N}} \frac{\beta_{n,k}}{\gamma_{n,k}}$,

where

$$\beta_{n,k} := \left(1+n^2\right)^{\frac{\mu+1}{2}} \left|\lambda_{n,k}^{(R)} - \frac{2}{2n+1} - \frac{\mathrm{i}\,\eta}{2}\lambda_n^{(B)}\right| \quad \text{and} \quad \gamma_{n,k} := \sqrt{1+n^2}\left|\lambda_{n,k}^{(R)}\right|.$$

First, we will estimate the behavior of $\beta_{n,k}$. The results of the asymptotic analysis (see Lemma 5.6 and (5.16)) together with (5.5) yield the following:

1. $n = 0$ and $k \to \infty$,

$$\left(1+n^2\right)^{\frac{\mu+1}{2}} \left|\lambda_{n,k}^{(R)} - \frac{2}{2n+1} - \frac{\mathrm{i}\,\eta}{2}\lambda_n^{(B)}\right| \le a_5 k^{1/3}; \tag{5.19a}$$

2. for $n + \frac{1}{2} = k$,

$$\left(1+n^2\right)^{\frac{\mu+1}{2}} \left|\lambda_{n,k}^{(R)} - \frac{2}{2n+1} - \frac{\mathrm{i}\,\eta}{2}\lambda_n^{(B)}\right| \le a_6 k^{\mu+1/3}; \tag{5.19b}$$

3. for k fixed and $n \to \infty$,

$$\left(1+n^2\right)^{\frac{\mu+1}{2}} \left|\lambda_{n,k}^{(R)} - \frac{2}{2n+1} - \frac{\mathrm{i}\,\eta}{2}\lambda_n^{(B)}\right| \le a_7 k^{1/3}. \tag{5.19c}$$

In Figure 5.3, the function $\left(1+n^2\right)^{\frac{\mu+1}{2}} |\lambda_{n,k}^{(R)} - \frac{2}{2n+1} - \frac{\mathrm{i}\,\eta}{2}\lambda_n^{(B)}|$ is depicted for $k = 20$ indicating that the bounds derived from asymptotic analysis are valid in neighborhoods of $n = 0$, $n = k - 1/2$ as well.

Next, we will investigate the behavior of $\gamma_{n,k}$. The asymptotic behavior of $\gamma_{n,k}$ as $n \to \infty$ is derived by using (5.16c)

$$\left|\gamma_{n,k}\right| \ge \sqrt{1+n^2}\left|\operatorname{Re}\lambda_{n,k}^{(R)}\right| \sim 1 \quad \text{for } k \text{ fixed and } n \to \infty.$$

By choosing $\eta = k^{1/3}$, the function $\operatorname{Re}\lambda_{n,k}^{(R)}$ is oscillating as a function of $n < k$ about zero and bounded from below properly away from zero for $n \ge k$ (cf. Figure 5.4). This supports the heuristics

$$\inf_{n\ge k} \gamma_{n,k} \ge \inf_{n\ge k} \left\{\sqrt{1+n^2}\operatorname{Re}\lambda_{n,k}^{(R)}\right\} \ge 1. \tag{5.20a}$$

The behavior of $\gamma_{n,k}$ in the range $0 \le n < k$ is more complicated.

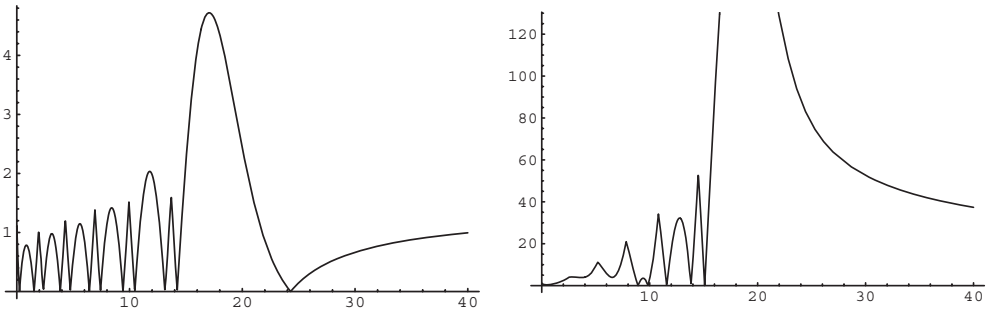

FIG. 5.3. *Plot of* $\left(1+n^2\right)^{\frac{s+1}{2}} |\lambda_{n,k}^{(R)} - \frac{2}{2n+1} - \frac{\mathrm{i}\,\eta}{2}\tilde{\lambda}_n^{(B)}|$ *for* $k = 20$ *and* $\varepsilon = \frac{1}{10}$ *(left),* $\varepsilon = 1$ *(right) in the range of* $n \in [0, 40]$.

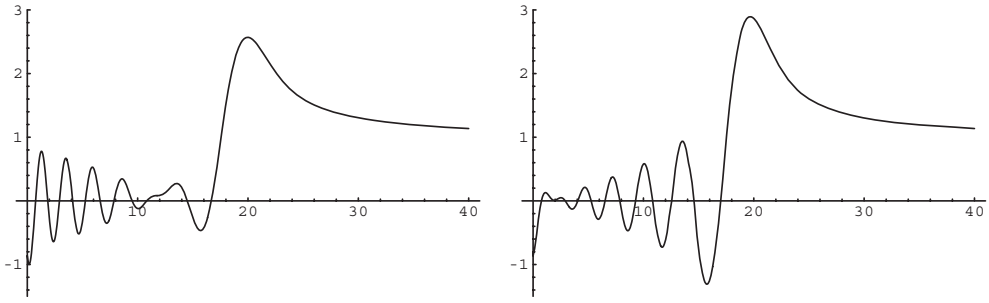

FIG. 5.4. *Function* $\sqrt{1+n^2}\,\mathrm{Re}\,\tilde{\lambda}_{n,k}^{(R)}$ *for fixed* $k = 20$ *and* $\varepsilon = \frac{1}{10}$ *(left picture),* $\varepsilon = 1$ *(right picture) in the range* $n \in [0, 40]$. *The function values are oscillating for* $n \in [0, k[$ *about zero and bounded from below away from zero in the range* $n \geq k$.

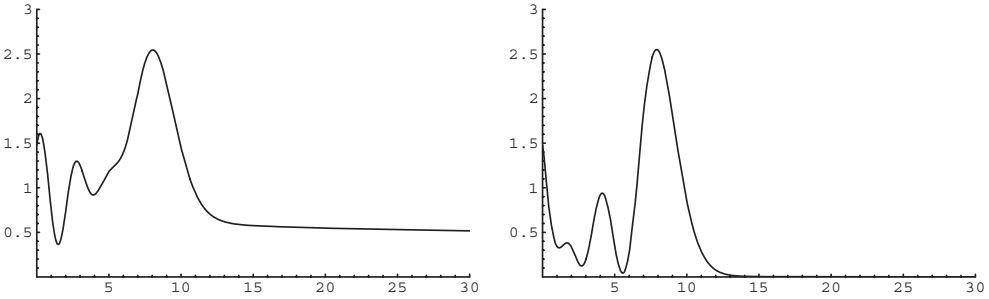

FIG. 5.5. *Plot of* $\sqrt{1+n^2}\,\mathrm{Im}\,\tilde{\lambda}_{n,k}^{(R)}$ *for* $k = 10$ *and* $\varepsilon = \frac{1}{10}$ *(left picture),* $\varepsilon = 1$ *(right picture) in the range* $0 \leq n \leq 30$. *The function values are all positive and decreasing as* $n \to \infty$.

By choosing $\eta = k^{1/3}$, the function $\mathrm{Im}\,\lambda_{n,k}^{(R)}$ is positive for all $n \geq 0$ and $k \geq 0$ (cf. Figure 5.5).

Since the quantities $\mathrm{Re}\,\lambda_{n,k}^{(R)}$ are oscillating in the range $0 \leq n \leq k$, we employ the imaginary part to bound $\gamma_{n,k}$ from below

$$\gamma_{n,k} \geq \sqrt{1+n^2}\,\mathrm{Im}\,\lambda_{n,k}^{(R)} = f_1(n,k) + f_2(n,k,\varepsilon)\,, \tag{5.21}$$

where

$$f_1(n,k) := \sqrt{1+n^2}\,\mathrm{Im}\,\lambda_{n,k}^{V} \quad \text{and} \quad f_2(n,k,\varepsilon) := \sqrt{1+n^2}\,\mathrm{Im}\left(\mathrm{i}\,\eta\lambda_n^{(B)}\lambda_{n,k}^{(K)}\right).$$

We have plotted both summands in the right-hand side of (5.21) for $\eta = k^{1/3}$ in the range $0 \leq n \leq k$ separately (cf. Figure 5.6).

In the range $0 \leq n \leq k$, the function f_1 is large when f_2 is zero or close to zero while, vice versa, f_2 is large when f_1 is zero or close to zero. The maxima of f_1 are monotonously increasing while the maxima of f_2 are monotonously decreasing. These observations lead to the following heuristics.

- If n is small, then

$$\gamma_{n,k} \geq \sqrt{1+n^2}\,\mathrm{Im}\,\lambda_{n,k}^{(R)} \geq a_8 f_1(n_0,k) \overset{(5.16a)}{\geq} a_9 k^{-1}, \tag{5.20b}$$

  where n_0 denotes the first local maximum of f_1 and a_8, a_9 depend only on k_0.

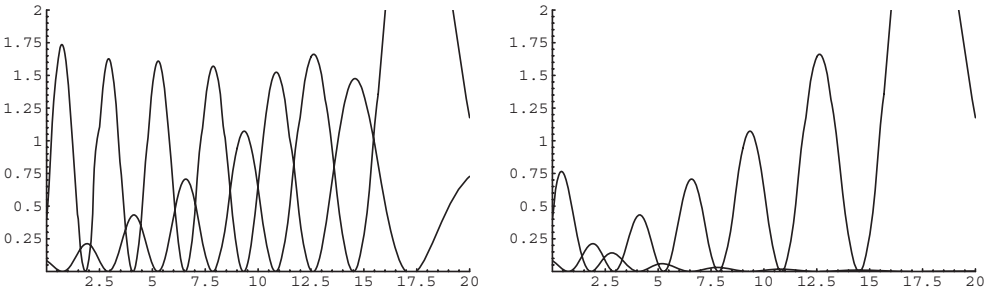

FIG. 5.6. *Functions* $\sqrt{1+n^2}\,\mathrm{Im}\,\lambda_{n,k}^V$ *and* $\sqrt{1+n^2}\,\mathrm{Im}(\mathrm{i}\,\eta\tilde{\lambda}_n^{(B)}\lambda_{n,k}^{(K)})$ *for* $k=20$ *and* $\varepsilon=\frac{1}{10}$ *(left picture),* $\varepsilon=1$ *(right picture) as a function of* n *in the range* $0\le n\le 20$. *The zeros and extrema are properly separated.*

- If n is close to k but smaller than k, then

  (5.20c)
  $$\gamma_{n,k}\ge\sqrt{1+n^2}\,\mathrm{Im}\,\lambda_{n,k}^{(R)}\ge a_{10}f_2\left(k-\frac{1}{2},k,\varepsilon\right)\overset{(5.5),\ \text{Lemma 5.6(2)},}{\ge}\ a_{11}k^{1/3-2\varepsilon},$$

  where a_{10}, a_{11} depend only on k_0.

We combine the estimates (5.19) and (5.20) and obtain the following heuristics.

- If n is small, then
  $$\beta_{n,k}/\gamma_{n,k}\le a_{12}k^{4/3}.$$
- If n is close to k but smaller than k, then
  $$\beta_{n,k}/\gamma_{n,k}\le a_{13}k^{\mu+2\varepsilon}.$$
- If k is fixed and n becomes large, then
  $$\beta_{n,k}/\gamma_{n,k}\le a_{14}k^{1/3}.$$

This leads to the heuristics

$$C_\mu=\sup_{n\ge0}\frac{\beta_{n,k}}{\gamma_{n,k}}\le a_{15}k^{\max\{4/3,\mu+2\varepsilon\}}. \tag{5.22}$$

To verify the qualitative dependence of C_μ on k numerically in the whole range of n we have evaluated the quantity $k^{-\max\{4/3,\mu+2\varepsilon\}}\beta_{n,k}/\gamma_{n,k}$ as a function of n for different values of k and ε by the software *Mathematica* . In Table 5.1, the results are depicted which strongly support the heuristics that the asymptotic behavior as in (5.22) is valid in the whole range of n.

5.6. Estimates on the constants of interest. We combine the general error estimate (cf. Theorem 4.3) with the estimate for the continuity constant C_X (cf. subsection 5.4) to obtain an estimate of the Galerkin error which is explicit with respect to the wave number k.

THEOREM 5.8. *Let all assumptions and hypotheses be satisfied and assume that the step size* h *satisfies* $h<h_0$, h_0 *given in* (4.12). *Then, there exist positive constants* $\hat{C}$, $k_0>0$ *such that for all* $k\ge k_0$

$$\frac{\|\varphi-\varphi_h\|_{H^{-1/2}(\Gamma)}}{\|\varphi\|_{H^{-1/2}(\Gamma)}}\le\hat{C}k^{1/3}\,(hk)^{3/2}\,. \tag{5.23}$$

TABLE 5.1
Numerical verification of the heuristics (5.22) *for the constant* C_μ. *The table lists the numerical determined value of* $k^{-\max\{4/3,\mu+2\varepsilon\}} \sup_{n\geq 0} \beta_{n,k}/\gamma_{n,k}$ *which we expect to be bounded by a constant independent of* k.

ε	$k=10$	$k=20$	$k=40$	$k=80$	$k=160$
$\frac{1}{20}$	0.307	0.301	0.284	0.258	0.240
$\frac{1}{5}$	0.251	0.278	0.268	0.238	0.201
1	0.0917	0.142	0.164	0.162	0.148

Finally, we will determine the dependence of the maximal step size h_0 on the wave number k such that existence and uniqueness of the Galerkin solution is guaranteed. The combination of (4.12) with the results of subsection 5.4 and (5.22) leads to the condition

$$h_0^\mu k^{1/3} k^{\max\{4/3,\mu+2\varepsilon\}} \leq C, \tag{5.24}$$

where the constant C is independent of k and ε. Recall that $\mu = \min\{3/2, 1+2\varepsilon\}$. We distinguish the following cases.

1. $0 \leq \varepsilon \leq \frac{1}{12}$. Condition (5.24) takes the form
$$h^{1+2\varepsilon} k^{5/3} \leq C.$$
2. $\frac{1}{12} \leq \varepsilon \leq \frac{1}{4}$. Condition (5.24) takes the form
$$h^{1+2\varepsilon} k^{4/3+4\varepsilon} \leq C.$$
3. $\frac{1}{4} \leq \varepsilon \leq 1$. Condition (5.24) takes the form
$$h^{3/2} k^{11/6+2\varepsilon} \leq C.$$

By inspecting these cases we derive that the choice $\varepsilon = 1/12$ is optimal and the condition

$$(hk)\, k^{\frac{3}{7}} \leq C$$

ensures existence and uniqueness. In Table 5.2, the Galerkin error estimate and the stability condition are listed for different choices of $\gamma_0 B$.

We finish this section by comparing these results with the quantitative analysis of the Brakhage–Werner stabilization which we briefly recall. For the solution of the Helmholtz problem, the ansatz

$$u_+ = D_k \psi - i\eta S_k \psi$$

TABLE 5.2
Stability condition and estimate of the relative Galerkin error for different choices of $\gamma_0 B$.

$\gamma_0 B$	ε	Stability condition	Rel. Galerkin error is smaller than
$(V_0)^2$	$\frac{1}{2}$	$hk^{1+8/9} \leq C$	$\hat{C} k^{1/3} (hk)^{3/2}$
B_1	1	$hk^{1+14/19} \leq C$	$\hat{C} k^{1/3} (hk)^{3/2}$
$B_{1/12}$	$\frac{1}{12}$	$hk^{1+3/7} \leq C$	$\hat{C} k^{1/3} (hk)^{3/2}$

is employed and the density ψ is determined by solving the following boundary integral equation:

Find $\psi \in L^2(\Gamma)$ such that

$$\left(\left(\frac{1}{2} I + K_k - i\eta V_k \right) \psi, \sigma \right)_{0,\Gamma} = (g, \sigma)_{0,\Gamma} \qquad \forall \sigma \in L^2(\Gamma).$$

Let ψ_h denote the corresponding Galerkin solution where $L^2(\Gamma)$ is replaced by the piecewise constant boundary element space S as in (3.2). By choosing $\eta \sim k/4$, it was proved in [17]—for Γ being the surface of the unit sphere—that the relative error can be estimated by

$$\frac{\|\psi - \psi_h\|_{L^2(\Gamma)}}{\|\psi\|_{L^2(\Gamma)}} \leq \hat{C} k^{1/3} (hk).$$

By comparing this result with Table 5.2 it is obvious that the pollution factor which amplifies the error of the best approximation is $k^{1/3}$ for both cases, the Brakhage–Werner approach and the stabilized acoustic single layer potential. On the other hand, the convergence rate $(hk)^{3/2}$ compared to (kh) is higher for the stabilized single layer ansatz. In addition, we emphasize that the stability behavior of the acoustic single layer potential can be proved for general Lipschitz polyhedra while the proof of the well-posedness of the Brakhage–Werner approach is restricted to smooth surfaces. For the special case of a sphere, the stability condition $hk \lesssim 1$ for the Brakhage–Werner formulation is better compared to the stability condition $hk^{1+3/7} \lesssim 1$ for the stabilized single layer potential.

5.7. Oscillation condition. In this subsection, we will investigate the oscillation condition (4.15) in the case of the unit sphere.

Any right-hand side $g \in H^{1/2}(\Gamma)$ in (2.5) has a Fourier representation

$$g = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} g_n^m Y_n^m.$$

Due to the orthogonality of the spherical harmonics, the solution of (2.5) has the representation

$$\varphi = \sum_{n=0}^{\infty} \frac{1}{\lambda_{n,k}^{(R)}} \sum_{m=-n}^{n} g_n^m Y_n^m.$$

The ratio of the $H^1(\mathbb{S}^2)$- and the $H^{-1/2}(\mathbb{S}^2)$-norms takes the form

$$Q(g) := \frac{\displaystyle\sum_{n=0}^{\infty} \frac{1+n^2}{\left|\lambda_{n,k}^{(R)}\right|} |g_n|^2}{\displaystyle\sum_{n=0}^{\infty} \frac{|g_n|^2}{\sqrt{1+n^2}\left|\lambda_{n,k}^{(R)}\right|}} = \frac{\|\varphi\|_{H^1(\Gamma)}^2}{\|\varphi\|_{H^{-1/2}(\Gamma)}^2} \quad \text{with} \quad |g_n|^2 := \sum_{n=-m}^{m} |g_n^m|^2.$$

By the substitution $w_n := \frac{g_n}{(1+n^2)^{1/4} |\lambda_{n,k}^{(R)}|^{1/2}}$ we obtain

$$Q(g) = \left(\sum_{n=0}^{\infty} \left(1+n^2\right)^{3/2} |w_n|^2 \right) / \|w_n\|_0^2.$$

It is not our goal to derive sharp decay conditions for the Fourier coefficients of the right-hand side $g \in H^{1/2}(\Gamma)$ so that $Q(g)$ is bounded by $C\left(1+k^2\right)^{3/2}$ and the oscillation property (4.15) is valid with properly chosen $\rho = O(1)$. Instead we will discuss the characteristic case that all oscillations in g vanish starting from $n \geq k$; i.e., all oscillations in the solution stem from the boundary integral equation itself and not from the right-hand side.

PROPOSITION 5.9. *Let $\Gamma = \mathbb{S}^2$ and let the Fourier coefficients g_n^m of the right-hand side $g \in H^{1/2}(\Gamma)$ in* (3.1) *satisfy*

$$g_n^m = 0 \qquad \forall n > k \text{ and } -n \leq m \leq n.$$

Then,

$$Q(g) \leq \left(1+k^2\right)^{3/2}$$

and (4.15) *holds with a properly chosen constant $\rho = O(1)$.*

6. Computational complexity. In this section, we will remark on the computational complexity related to the different choices of the operator B.

The computation of the Galerkin system matrix $\mathbf{R}_k$ for (3.3) requires the evaluation of the integrals $(R_k b_\tau, b_t)_{\tau,t\in\mathcal{T}}$. Besides the additive term $K_k\gamma_0 B$ in (2.4), the discretization of the arising integral operators is standard. The numerical treatment of the composition $K_k\gamma_0 B$ is discussed in the following remark.

Remark 6.1. In the context of the Galerkin boundary element method, the exact computation of the system matrix entries is not possible in general and, e.g., numerical quadrature and panel clustering have to be employed. This leads to a perturbed Galerkin method and the accuracy requirements are determined via Strang's lemma [18]. In this light, we recommend replacing the matrix entries $\mathbf{W}_{\tau,t} := (K_k\gamma_0 B b_\tau, b_t)_0$ by $\tilde{\mathbf{W}}_{\tau,t} := (K_k P\gamma_0 B b_\tau, b_t)_0$, where P denotes the L^2-orthogonal projection onto $\mathcal{S}$. The advantage is that $\tilde{\mathbf{W}}$ has the representation

$$\tilde{\mathbf{W}} = \mathbf{K}_k\mathbf{M}^{-1}\mathbf{B},$$

where $\mathbf{K}_k := ((K_k b_\tau, b_t)_0)_{\tau,t\in\mathcal{T}}$, respectively, $\mathbf{B} := ((\gamma_0 B b_\tau, b_t)_0)_{\tau,t\in\mathcal{T}}$, are the standard Galerkin system matrices of K_k, respectively, $\gamma_0 B$, or approximations to it. The mass matrix $\mathbf{M} = ((b_\tau, b_t)_0)_{\tau,t\in\mathcal{T}}$ is diagonal and its inversion is trivial.

In other words, the Galerkin system matrix $\mathbf{R}_k$ for the operator R_k can be approximately computed from the system matrices $\mathbf{V}_k$, $\mathbf{K}_k$, $\mathbf{B}$ (or approximations thereof) via

$$\widehat{\mathbf{R}}_k := \mathbf{V}_k + \mathrm{i}\,\eta\left(\frac{1}{2}\mathbf{B} + \mathbf{K}_k\mathbf{M}^{-1}\mathbf{B}\right).$$

Some comments concerning the complexity are listed below.

- Since for large problems the linear system should be solved iteratively, the matrix entries of $\widehat{\mathbf{R}}_k$ need not be computed explicitly; only a procedure for a matrix-vector multiplication has to be provided. Hence, the matrix-matrix multiplication $\mathbf{K}_k\mathbf{M}^{-1}\mathbf{B}$ does not have to be carried out, but $\mathbf{K}_k\mathbf{M}^{-1}\mathbf{B}\mathbf{v}$ can be realized by a matrix-vector multiplication with $\mathbf{K}_k$, one diagonal scaling, and the evaluation of $\mathbf{B}\mathbf{v}$. Table 6.1 lists the different strategies for this evaluation.

TABLE 6.1
Evaluation of $\mathbf{Bv}$.

$\gamma_0 B$	Storage	Matrix-vector multiplication
$(V_0)^2$	$\mathbf{V}_0 = \big((V_0 b_\tau, b_t)_0\big)_{\tau,t\in\mathcal{T}}$	$\mathbf{Bv} = \mathbf{V}_0\mathbf{M}^{-1}\mathbf{V}_0\mathbf{v}$
B_1	$\mathbf{B}_1 := \big((B_1 b_\tau, b_t)_0\big)_{\tau,t\in\mathcal{T}}$	$\mathbf{Bv} = \mathbf{B}_1\mathbf{v}$
$B_{1/12}$	$\mathbf{B}_{1/12} := \Big(\big(B_{1/12} b_\tau, b_t\big)_0\Big)_{\tau,t\in\mathcal{T}}$	$\mathbf{Bv} = \mathbf{B}_{1/12}\mathbf{v}$

- Since $\mathbf{B}$ is independent of the possibly high wave number k, the accuracy requirements for numerical quadrature, panel-clustering, and iterative solution are reduced compared to $\mathbf{V}_k$, $\mathbf{K}_k$. Hence, we expect that the extra cost for the stabilization is moderate.
- The choices $\mathbf{V}_0$, $\mathbf{B}_1$, $\mathbf{B}_{1/12}$ correspond to the discretization of a boundary integral operator (such as $\mathbf{V}_k$ and $\mathbf{K}_k$). Hence, the numerical implementation does *not* require any new data structures. By using the blackbox quadrature methods (cf. [16]) and/or the panel-clustering based on interpolation (cf. [5], [25]) the discretization of $\gamma_0 B$ requires only a subroutine for the evaluation of the integral kernel in (2.13).

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