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Pseudocodeword weights for non-binary LDPC codes

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Abstract—Pseudocodewords of q -ary LDPC codes are examined and the weight of a pseudocodeword on the q -ary symmetric channel is defined. The weight definition of a pseudocodeword on the AWGN channel is also extended to two-dimensional q -ary modulation such as q -PAM and q -PSK. The tree-based lower bounds on the minimum pseudocodeword weight are shown to also hold for q -ary LDPC codes on these channels.

I. INTRODUCTION

Low density parity check (LDPC) codes have been shown to achieve near-capacity performance over several communication channels. Typically, they are binary linear codes described by sparse, randomly, generated parity-check matrices. In [3] and [4], the performance of non-binary LDPC codes, defined over larger finite fields and over integer rings, is investigated and compared with that of binary LDPC codes. For several applications such as coded-modulation, codes over higher alphabets are more appropriate for system design. The popularity of LDPC codes is due to their efficient and simple decoding. Graph-based message passing iterative decoders have been shown to achieve near-capacity performance with complexity only linear in the length of the code. However, these iterative decoders are sub-optimal and discrepancies between iterative and maximum-likelihood (ML) decoding performance of short to moderate block length binary LDPC codes has been attributed to the presence of pseudocodewords of the LDPC constraint graphs (or, Tanner graphs) [8]. Analogous to the role of minimum Hamming distance, d_{\min} , in ML-decoding, the minimum pseudocodeword weight, w_{\min} , has been shown to be a leading predictor of performance in iterative decoding [8]. Furthermore, it has been observed that pseudocodewords with weight $w_{\min} < d_{\min}$ are especially problematic for iterative decoding [6]. In this paper, we define pseudocodeword weights for q -ary LDPC codes when the channel is a AWGN channel or a q -ary symmetric channel and obtain lower bounds for the minimum pseudocodeword weight.

The following section shows a tree-based lower bound on the minimum pseudocodeword weight of binary LDPC codes. In Section III, the pseudocodeword weight of q -ary LDPC codes is defined for the AWGN and the q -ary symmetric channels. Subsequently, the tree-based lower bound for binary LDPC codes is extended to the q -ary setting. We note here that we restrict our analysis to pseudocodewords arising from

finite-degree graph covers as described in [8]. Since these pseudocodewords are the same as those occurring in the context of linear programming (LP) decoding, the results obtained here are applicable to pseudocodewords of LP decoding as well. Section IV summarizes the paper and outlines some other techniques that are being investigated for bounding the pseudocodeword weight of q -ary LDPC codes.

II. BINARY LDPC CODES

Definition 2.1: The tree bound of a d left (variable node) regular bipartite LDPC constraint graph with girth g is defined as

$$T(d, g) := \begin{cases} 1 + d + d(d-1) + d(d-1)^2 + \dots + d(d-1)^{\frac{g-6}{4}}, & \frac{g}{2} \text{ odd}, \\ 1 + d + d(d-1) + \dots + d(d-1)^{\frac{g-8}{4}} + (d-1)^{\frac{g-4}{4}}, & \frac{g}{2} \text{ even}. \end{cases} \quad (1)$$

Theorem 2.1: Let G be a bipartite LDPC constraint graph with smallest left (variable node) degree d and girth g . Then the minimum pseudocodeword weight w_{\min} is lower bounded by

$$w_{\min} \geq T(d, g).$$

on the additive white Gaussian noise (AWGN) channel and the binary symmetric channel (BSC).

The proof of this result is presented in [6]. The tree bound was originally derived by Tanner in [10] to lower-bound the minimum distance of the code. Since the set of pseudocodewords includes all codewords, we have $w_{\min} \leq d_{\min}$.

III. NON-BINARY LDPC CODES

Let H be a parity check matrix representing a q -ary LDPC code C . Thus, H is sparse in the number of non-zero entries. The corresponding LDPC constraint graph G that represents H is an incidence graph of the parity check matrix as in the binary case. However, each edge of G is now assigned a weight which is the value of the corresponding non-zero entry in H . (In [3], [2], LDPC codes over $GF(q)$ are considered for transmission over binary modulated channels, whereas in [4], LDPC codes over integer rings are considered for higher-order modulation signal sets.) For convenience, we consider the special case wherein each of these edge weights are equal to one. This is the case when the parity check matrix has only zeros and ones.

Furthermore, whenever the LDPC graphs have edge weights of unity for all the edges, we refer to such a graph as a binary LDPC constraint graph representing a q -ary LDPC code \mathcal{C} .

A. Bound on minimum distance

We first show that if the LDPC graph corresponding to H is d -left (variable-node) regular, then the same tree bound of Theorem 2.1 holds. That is,

Lemma 3.1: *If G is a d -left regular bipartite LDPC constraint graph with unity edge weights, girth g , and represents a q -ary LDPC code \mathcal{C} . Then the minimum distance of the q -ary LDPC code \mathcal{C} is lower bounded as*

$$d_{\min} \geq T(d, g).$$

Proof: The proof is essentially the same as in the binary case. Enumerate the graph as a tree starting at an arbitrary variable node. Furthermore, assume that a codeword in \mathcal{C} contains the root node in its support. The root variable node (at layer L_0 of the tree) connects to d constraint nodes in the next layer (layer L_1) of the tree. These constraint nodes are each connected to some sets of variable nodes in layer L_2 , and so on. Since the graph has girth g , the nodes enumerated up to layer $L_{\frac{g-2}{2}}$ when $\frac{g}{2}$ is odd (respectively, $L_{\frac{g}{2}}$ when $\frac{g}{2}$ is even) are all distinct. Since the root node belongs to a codeword, say \mathbf{c} , it assumes a non-zero value in \mathbf{c} . Since the constraints must be satisfied at the nodes in layer L_1 , at least one node in Layer L_2 for each constraint node in L_1 must assume a non-zero value in \mathbf{c} . (This is true under the assumption that an edge weight times a (non-zero) value, assigned to the corresponding variable node, is non-zero in the code alphabet.)

Under the above assumption, there are at least d variable nodes (i.e., at least one for each node in layer L_1) in layer L_2 that are non-zero in \mathbf{c} . Continuing this argument, it is easy to see that the number of non-zero components in \mathbf{c} is at least $1 + d + d(d-1) + \dots + d(d-1)^{\frac{g-6}{4}}$ when $\frac{g}{2}$ is odd, and $1 + d + d(d-1) + \dots + d(d-1)^{\frac{g-8}{4}} + (d-1)^{\frac{g-4}{4}}$ when $\frac{g}{2}$ is even. This proves the desired lower bound. ■

Remark 3.1: A non-zero edge-weight times a (non-zero) value, assigned to the corresponding variable node, may be zero in certain code alphabets. Since we have chosen the edge weights to be unity, such a case will not arise here. But also more generally, such cases will not arise when the alphabet and the arithmetic operations correspond to finite-field operations. However, when working over other structures, such as finite integer rings and more general groups, such cases could arise.

We note here that in general this lower bound is not met and typically q -ary LDPC codes that have the above graph representation have minimum distances larger than the above lower bound.

B. Pseudocodewords of q -ary LDPC codes

Recall from [8], [6] that a pseudocodeword of an LDPC constraint graph G is a valid codeword in some finite cover

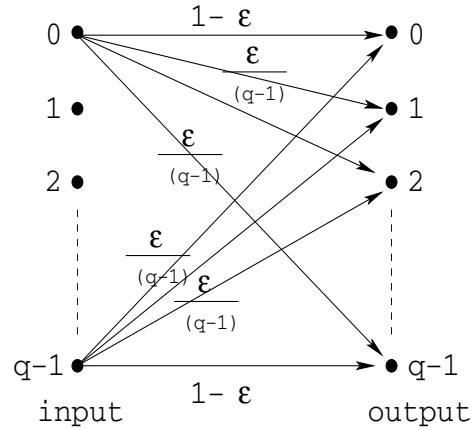


Fig. 1. A q -ary symmetric channel.

of G . To define a pseudocodeword for a q -ary LDPC code, we will restrict the discussion to LDPC constraint graphs that have edge weights of unity among all their edges – in other words, binary LDPC constraint graphs that represent q -ary LDPC codes. A finite cover of a graph is defined in a natural way as in [8] wherein all edges in the finite cover also have an edge weight of unity. For the rest of this section, let G be a LDPC constraint graph of a q -ary LDPC code \mathcal{C} of block length n , and let the weights on every edge of G be unity. We define a pseudocodeword F of G as a $n \times q$ matrix of the form

$$F = \begin{bmatrix} f_{0,0} & f_{0,1} & f_{0,2} & \dots & f_{0,q-1} \\ f_{1,0} & f_{1,1} & f_{1,2} & \dots & f_{1,q-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{n-1,0} & f_{n-1,1} & f_{n-1,2} & \dots & f_{n-1,q-1} \end{bmatrix},$$

where the pseudocodeword F forms a valid codeword $\hat{\mathbf{c}}$ in a finite cover \hat{G} of G and $f_{i,j}$ is the fraction of variable nodes in the i^{th} variable node cloud, for $0 \leq i \leq n-1$, of \hat{G} that have the assignment (or, value) equal to j , for $0 \leq j \leq q-1$, in $\hat{\mathbf{c}}$.

A q -ary symmetric channel is shown in Figure 1. The input and the output of the channel are random variables belonging to a q -ary alphabet that can be denoted as $\{0, 1, 2, \dots, q-1\}$. An error occurs with probability ϵ , which is parameterized by the channel, and in the case of an error, it is equally probable for an input symbol to be altered to any one of the remaining symbols.

Following the definition of pseudocodeword weight for the binary symmetric channel [5], we provide the following definition for the weight of a pseudocodeword on the q -ary symmetric channel. For a pseudocodeword F , let F' be the sub-matrix obtained by removing the first column in F . (Note that the first column in F contains the entries $f_{0,0}, f_{1,0}, f_{2,0}, \dots, f_{n-1,0}$.) Then the weight of a pseudocodeword F on the q -ary symmetric channel is defined as follows.

Definition 3.1: Let e be the smallest number such that the sum of the e largest components in the matrix F' , say, $f_{i_1, j_1}, f_{i_2, j_2}, \dots, f_{i_e, j_e}$, exceeds $\sum_{i \neq i_1, i_2, \dots, i_e} (1 - f_{i, 0})$. Then the weight of F on the q -ary symmetric channel is defined as

$$w_{qSC}(F) = \begin{cases} 2e, & \text{if } f_{i_1, j_1} + \dots + f_{i_e, j_e} = \sum_{i \neq i_1, i_2, \dots, i_e} (1 - f_{i, 0}), \\ 2e - 1, & \text{if } f_{i_1, j_1} + \dots + f_{i_e, j_e} > \sum_{i \neq i_1, i_2, \dots, i_e} (1 - f_{i, 0}). \end{cases}$$

Note that in the above definition, none of the j_k 's, for $k = 1, 2, \dots, e$, are equal to zero, and all the i_k 's, for $k = 1, 2, \dots, e$, are distinct. That is, we choose at most one component from every row of F' when choosing the e largest components. The following sub-section provides an explanation for the above definition of weight.

C. PSEUDOCODEWORD WEIGHT FOR q -ARY LDPC CODES ON THE q -ARY SYMMETRIC CHANNEL

Suppose the all-zero codeword is sent across a q -ary symmetric channel and the vector $\mathbf{r} = (r_0, r_1, \dots, r_{n-1})$ is received. Then errors occur in positions where $r_i \neq 0$. Let $S = \{i \mid r_i \neq 0\}$ and let $S^c = \{i \mid r_i = 0\}$. The distance between \mathbf{r} and a pseudocodeword F is defined as

$$d(\mathbf{r}, F) = \sum_{i=0}^{n-1} \sum_{k=0}^{q-1} \chi(r_i \neq k) f_{i, k}, \quad (2)$$

where $\chi(P)$ is an indicator function that is equal to 1 if the proposition P is true and is equal to 0 otherwise.

The distance between \mathbf{r} and the all-zero codeword $\mathbf{0}$ is

$$d(\mathbf{r}, \mathbf{0}) = \sum_{i=0}^{n-1} \chi(r_i \neq 0)$$

which is the Hamming weight of \mathbf{r} and can be obtained from equation (2).

The iterative decoder chooses in favor of F instead of the all-zero codeword $\mathbf{0}$ when $d(\mathbf{r}, F) \leq d(\mathbf{r}, \mathbf{0})$. That is, if

$$\sum_{i \in S^c} (1 - f_{i, 0}) + \sum_{i \in S} (1 - f_{i, r_i}) \leq \sum_{i \in S} 1$$

The condition for choosing F over the all-zero codeword reduces to

$$\left\{ \sum_{i \in S^c} (1 - f_{i, 0}) \leq \sum_{i \in S} f_{i, r_i} \right\}$$

Hence, we define the weight of a pseudocodeword F in the following manner.

Let e be the smallest number such that the sum of the e largest components in the matrix F' , say, $f_{i_1, j_1}, f_{i_2, j_2}, \dots, f_{i_e, j_e}$, exceeds $\sum_{i \neq i_1, i_2, \dots, i_e} (1 - f_{i, 0})$. Then the weight of F on the p -ary symmetric channel is defined as

$$w_{qSC}(F) = \begin{cases} 2e, & \text{if } f_{i_1, j_1} + \dots + f_{i_e, j_e} = \sum_{i \neq i_1, i_2, \dots, i_e} (1 - f_{i, 0}), \\ 2e - 1, & \text{if } f_{i_1, j_1} + \dots + f_{i_e, j_e} > \sum_{i \neq i_1, i_2, \dots, i_e} (1 - f_{i, 0}). \end{cases}$$

Note that in the above definition, none of the j_k 's, for $k = 1, 2, \dots, e$, are equal to zero, and all the i_k 's, for $k = 1, 2, \dots, e$, are distinct. That is, we choose at most one component in every row of F' when picking the e largest components. The received vector $\mathbf{r} = (r_0, r_1, \dots, r_{n-1})$ that

has the following components: $r_{i_1} = j_1, r_{i_2} = j_2, \dots, r_{i_e} = j_e, r_i = 0$, for $i \notin \{i_1, i_2, \dots, i_e\}$, will cause the decoder to make an error and choose F over the all-zero codeword.

Observe that for a codeword, the above weight definition reduces to the Hamming weight. If F represents a codeword \mathbf{c} , then exactly $w = wt_H(\mathbf{c})$, the Hamming weight of \mathbf{c} , rows in F' contain the entry 1 in some column, and the remaining entries in F' are zero. Furthermore, the matrix F has the entry 0 in the first column of these w rows and has the entry 1 in the first column of the remaining rows. Therefore, from the weight definition of F , $e = \frac{w}{2}$ and the weight of F is $2e = w$.

D. TREE BOUND ON THE q -ARY SYMMETRIC CHANNEL

We define the q -ary minimum pseudocodeword weight of G (or, minimum pseudoweight) as in the binary case, i.e., as the minimum weight of a pseudocodeword among all finite covers of G , and denote this as $w_{\min}(G)$ or w_{\min} when it is clear that we are referring to the graph G .

Theorem 3.1: Let G be a d -left regular bipartite graph with girth g that represents a q -ary LDPC code \mathcal{C} . Then the minimum pseudocodeword weight w_{\min} on the q -ary symmetric channel is lower bounded as

$$w_{\min} \geq T(d, g)$$

Proof:

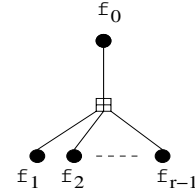


Fig. 2. Single constraint code.

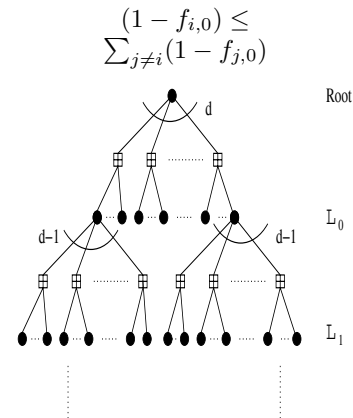


Fig. 3. Local tree structure for a d -left regular graph.

$$\begin{aligned} d(1 - f_{0,0}) &\leq \sum_{j \in L_0} (1 - f_{j,0}), \\ d(d-1)(1 - f_{0,0}) &\leq \sum_{j \in L_1} (1 - f_{j,0}) \\ &\vdots \end{aligned}$$

Case: $\frac{q}{2}$ odd. Consider a single constraint node with r variable node neighbors as shown in Figure 2. Then, for $i = 0, 1, \dots, r-1$ and $k = 0, 1, \dots, p-1$, the following inequality holds:

$$(1 - f_{i,0}) \leq \sum_{j \neq i} (1 - f_{j,0}) \quad (3)$$

Now let us consider a d -left regular LDPC constraint graph representing a q -ary LDPC code. We will enumerate the LDPC constraint graph as a tree from an arbitrary root variable node, as shown in Figure 3. Let F be a pseudocodeword matrix for this graph. Without loss of generality, let us assume that the component $(1 - f_{0,0})$ corresponding to the root node is the maximum among all $(1 - f_{i,0})$ over all i .

Applying the inequality in (3) at every constraint node in first constraint node layer of the tree, we obtain

$$d(1 - f_{0,0}) \leq \sum_{j \in L_0} (1 - f_{j,0}),$$

where L_0 corresponds to variable nodes in first level of the tree. Subsequent application of the inequality in (3) to the second layer of constraint nodes in the tree yields

$$d(d-1)(1 - f_{0,0}) \leq \sum_{j \in L_1} (1 - f_{j,0}),$$

Continuing this process until layer $L_{\frac{q-6}{4}}$, we obtain

$$d(d-1)^{\frac{q-6}{4}}(1 - f_{0,0}) \leq \sum_{j \in L_{\frac{q-6}{4}}} (1 - f_{j,0})$$

Since the LDPC graph has girth g , the variable nodes up to level $L_{\frac{q-6}{4}}$ are all distinct. The above inequalities yield:

$$\begin{aligned} & [1 + d + d(d-1) + \dots + d(d-1)^{\frac{q-6}{4}}](1 - f_{0,0}) \\ & \leq \sum_{i \in \{0\} \cup L_0 \cup \dots \cup L_{\frac{q-6}{4}}} (1 - f_{i,0}) \leq \sum_{\text{all } i} (1 - f_{i,0}) \end{aligned} \quad (4)$$

Let e the smallest number such that there are e maximal components $f_{i_1, j_1}, f_{i_2, j_2}, f_{i_3, j_3}, \dots, f_{i_e, j_e}$, for i_1, i_2, \dots, i_e all distinct and $j_1, j_2, \dots, j_e \in \{1, 2, \dots, q-1\}$, in F' (the submatrix of F excluding the first column in F) such that

$$f_{i_1, j_1} + f_{i_2, j_2} + \dots + f_{i_e, j_e} \geq \sum_{i \notin \{i_1, i_2, i_3, \dots, i_e\}} (1 - f_{i,0})$$

Then, since none of the j_k 's, $k = 1, 2, \dots, e$, are zero, we have

$$\begin{aligned} (1 - f_{i_1,0}) + (1 - f_{i_2,0}) + \dots + (1 - f_{i_e,0}) & \geq f_{i_1, j_1} + \dots + f_{i_e, j_e} \\ & \geq \sum_{i \notin \{i_1, i_2, i_3, \dots, i_e\}} (1 - f_{i,0}) \end{aligned}$$

Hence we have that

$$2((1 - f_{i_1,0}) + (1 - f_{i_2,0}) + \dots + (1 - f_{i_e,0}))$$

$$\geq \sum_{\text{all } i} (1 - f_{i,0})$$

We can then lower bound this further using the inequality in (4) as

$$\begin{aligned} & 2((1 - f_{i_1,0}) + (1 - f_{i_2,0}) + \dots + (1 - f_{i_e,0})) \\ & \geq [1 + d + d(d-1) + \dots + d(d-1)^{\frac{q-6}{4}}](1 - f_{0,0}) \end{aligned}$$

Since we assumed that $(1 - f_{0,0})$ is the maximum among $(1 - f_{i,0})$ over all i , we have

$$\begin{aligned} 2e(1 - f_{0,0}) & \geq 2((1 - f_{i_1,0}) + (1 - f_{i_2,0}) + \dots + (1 - f_{i_e,0})) \\ & \geq [1 + d + d(d-1) + \dots + d(d-1)^{\frac{q-6}{4}}](1 - f_{0,0}) \end{aligned}$$

This yields the desired bound

$$w_{qSC}(F) = 2e \geq 1 + d + d(d-1) + \dots + d(d-1)^{\frac{q-6}{4}}.$$

Since the pseudocodeword F was chosen arbitrary, we also have $w_{\min} \geq 1 + d + d(d-1) + \dots + d(d-1)^{\frac{q-6}{4}}$. The case $\frac{q}{2}$ even is treated similarly. ■

Since the inequality in (3), in the proof of Theorem 3.1, is typically not tight, the above bound is rather loose.

E. PSEUDOCODEWORD WEIGHT ON THE AWGN CHANNEL

Following the definition of effective distance $d_{eff}^2(F, \mathbf{c})$, between a pseudocodeword F and a codeword \mathbf{c} on the AWGN channel, presented in [5], the weight of a pseudocodeword F is given by $d_{eff}^2(F, \mathbf{0})$. On simplifying the expression in [5], the weight of pseudocodeword F on the AWGN channel is given by

$$w_{q-AWGN}(F) = \frac{(\sum_{i=0}^{n-1} \sum_{m=0}^{q-1} f_{i,m} m^2)^2}{\sum_{i=0}^{n-1} (\sum_{m=0}^{q-1} f_{i,m} m)^2} \quad (*)$$

The above weight definition assumes q -ary pulse amplitude modulation, i.e., the symbols sent across the channel belong to the signal set $\{0, 1, 2, \dots, q-1\}$.

Now if we assume a two-dimensional signal set for transmission on the memoryless AWGN channel, then under the assumption that the resulting signal-space code is geometrically uniform [11], we can derive the weight of a pseudocodeword F as the effective distance of F from the all-zero codeword in signal space. The pseudocodeword weight of F is given by

$$w_{q-AWGN}(F) = \frac{(R - M)^2}{V},$$

where (x_m, y_m) is the coordinate in the two-dimensional signal set corresponding to the symbol $m \in \{0, 1, \dots, q-1\}$,

$$R = \sum_j \left[\sum_m f_{j,m} (x_m^2 + y_m^2) - x_0^2 - y_0^2 \right],$$

$$M = 2 \sum_j \left[\left(\sum_m f_{j,m} x_m x_0 \right) - x_0^2 + \left(\sum_m f_{j,m} y_m y_0 \right) - y_0^2 \right],$$

$$V = 4 \sum_j \left[\left(\sum_m f_{j,m} x_m \right) - x_0 \right]^2 + \sum_j \left[\left(\sum_m f_{j,m} y_m \right) - y_0 \right]^2,$$

and $j \in \{0, \dots, n-1\}$.

Note that for q -ary pulse amplitude modulation as described above, this weight definition reduces to the one in (*).

Suppose we assume q -PSK modulation, then we have $x_m = \cos(\frac{2\pi m}{q})$ and $y_m = \sin(\frac{2\pi m}{q})$. Note that $x_0 = \cos(0) = 1$ and $y_0 = \sin(0) = 0$. In addition, $R = 0$. Therefore, the weight of a pseudocodeword F on the AWGN channel under q -PSK modulation is given by: $w_{q-AWGN}(F) = \frac{M^2}{V}$, where

$$M = 2 \sum_j \left(\sum_m f_{j,m} \cos\left(\frac{2\pi m}{q}\right) - 1 \right)$$

$$V = 4 \sum_j \left[\sum_m f_{j,m}^2 + 2 \left(\sum_{m,m'; m \neq m'} f_{j,m} f_{j,m'} \cos\left(\frac{2\pi(m-m')}{q}\right) \right) - 2 \sum_m f_{j,m} \cos\left(\frac{2\pi m}{q}\right) + 1 \right].$$

F. TREE-BOUND OF q -ARY LDPC CODES ON THE AWGN CHANNEL UNDER q -PAM

Theorem 3.2 (q -ary pulse amplitude modulation): Let G be a d -left regular bipartite graph with girth g that represents a q -ary LDPC code C . Then the minimum pseudocodeword weight w_{\min} on the AWGN channel is lower bounded as

$$w_{\min} \geq T(d, g).$$

(Note that we assume a slightly unconventional definition of q -ary PAM in that the symbol m is mapped to the point m rather than to the point $2m-1$ as in the conventional definition, for $m \in \{0, 1, 2, \dots, q-1\}$.)

Proof: Let F be a pseudocodeword in G . Without loss of generality, let $(1 - f_{0,0})$ be the maximum of $(1 - f_{0,i})$ over all i . We will first lower bound the weight $w_{q-AWGN}(F)$ as

$$w_{q-AWGN}(F) = \frac{(\sum_{i=0}^{n-1} \sum_{m=0}^{q-1} f_{i,m} m^2)^2}{\sum_{i=0}^{n-1} (\sum_{m=0}^{q-1} f_{i,m} m)^2}$$

$$\geq \frac{(\sum_{i=0}^{n-1} \sum_{m=0}^{q-1} f_{i,m} m^2)}{1 - f_{0,0}} \quad (**)$$

This lower bound is obtained by showing that the denominator in the weight expression can be upper bounded by using the Cauchy-Schwartz inequality as follows

$$\sum_{i=0}^{n-1} \left(\sum_{m=0}^{q-1} f_{i,m} m \right)^2$$

$$\leq \left(\sum_{i=0}^{n-1} (f_{i,1} + f_{i,2} + \dots + f_{i,q-1}) \right) \left(\sum_{i=0}^{n-1} \sum_{m=0}^{q-1} f_{i,m} m^2 \right).$$

Further, since $f_{i,1} + f_{i,2} + \dots + f_{i,q-1} = 1 - f_{i,0} \leq 1 - f_{0,0}$, we obtain the lower bound in (**).

Since $\sum_{i=0}^{n-1} \sum_{m=0}^{q-1} f_{i,m} m^2 \geq \sum_{i=0}^{n-1} (f_{i,1} + \dots + f_{i,q-1}) = \sum_{i=0}^{n-1} (1 - f_{i,0})$, we have

$$w_{q-AWGN}(F) \geq \frac{\sum_{i=0}^{n-1} (1 - f_{i,0})}{1 - f_{0,0}}$$

Now, the inequality (4) from the proof of Theorem 3.1 yields the desired lower bound $w_{q-AWGN}(F) \geq 1 + d + d(d-1) + \dots + d(d-1)^{\frac{g-6}{2}}$ for the case $g/2$ odd. (The case $g/2$ even follows similarly.) ■

IV. CONCLUSIONS

This paper examined the pseudocodeword weight of q -ary LDPC codes on the q -ary symmetric channel and the AWGN channel. A definition for the pseudocodeword weight was derived on the q -ary symmetric channel and the AWGN channel with two-dimensional q -ary modulation. The tree bound from [6] for binary LDPC codes was extended to the q -ary case. More sophisticated bounding techniques for the pseudocodeword weight of q -ary LDPC codes remains an open problem. It would be useful to also derive a cost-function of the min-sum decoder for q -ary LDPC codes to give an insight into which pseudocodewords are problematic for iterative decoding.

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