



**University of  
Zurich**<sup>UZH</sup>

**Zurich Open Repository and  
Archive**

University of Zurich  
University Library  
Strickhofstrasse 39  
CH-8057 Zurich  
[www.zora.uzh.ch](http://www.zora.uzh.ch)

---

Year: 2006

---

## **Some random times and martingales associated with $BES_0(\delta)$ processes ( $0 < \delta < 2$ )**

Nikeghbali, A

Other titles: Some random times and martingales associated with  $BES_0(\delta)$  processes ( $0 < \delta < 2$ )

Posted at the Zurich Open Repository and Archive, University of Zurich

ZORA URL: <https://doi.org/10.5167/uzh-21634>

Journal Article

Originally published at:

Nikeghbali, A (2006). Some random times and martingales associated with  $BES_0(\delta)$  processes ( $0 < \delta < 2$ ). *ALEA Latin American Journal of Probability and Mathematical Statistics*, 2:67-89 (electronic).



## Some random times and martingales associated with $BES_0(\delta)$ processes ( $0 < \delta < 2$ )

**Ashkan Nikeghbali**

Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, et CNRS UMR 7599, 175 rue du Chevaleret F-75013 Paris, France.

*Current address:* American Institute of Mathematics 360 Portage Ave Palo Alto, CA 94306-2244 USA

*E-mail address:* [ashkan@aimath.org](mailto:ashkan@aimath.org)

**Abstract.** In this paper, we study Bessel processes of dimension  $\delta \equiv 2(1 - \mu)$ , with  $0 < \delta < 2$ , and some related martingales and random times. Our approach is based on martingale techniques and the general theory of stochastic processes (unlike the usual approach based on excursion theory), although for  $0 < \delta < 1$ , these processes are even not semimartingales. The last time before 1 when a Bessel process hits 0, called  $g_\mu$ , plays a key role in our study: we characterize its conditional distribution and extend Paul Lévy's arc sine law and a related result of Jeulin about the standard Brownian Motion. We also introduce some remarkable families of martingales related to the Bessel process, thus obtaining in some cases a one parameter extension of some results of Azéma and Yor in the Brownian setting: martingales which have the same set of zeros as the Bessel process and which satisfy the stopping theorem for  $g_\mu$ , a one parameter extension of Azéma's second martingale, etc. Throughout our study, the local time of the Bessel process also plays a central role and we shall establish some of its elementary properties.

### 1. Introduction

Bessel processes afford basic examples of diffusion processes and their systematic study was initiated in McKean Jr. (1960). There exist many results for Bessel processes of dimension  $\delta > 2$ , obtained thanks to stochastic calculus and martingale techniques; when  $\delta \in (0, 2)$ , the same methods do not apply anymore: indeed, for  $\delta \in (0, 1)$ , Bessel processes are even not semimartingales (see Revuz and Yor (1999), chapter XI for more details and references).

The main tool to study Bessel processes of dimension  $\delta \in (0, 2)$  is the powerful theory of excursions of Markov processes, as developed e.g. by Gettoor (1979) and

---

*Received by the editors 21/08/2005, accepted 16/12/2005 .*

*2000 Mathematics Subject Classification.* 05C38, 15A15; 05A15, 15A18.

*Key words and phrases.* Bessel processes, Random Times, General theory of stochastic processes, Optional stopping theorem, Martingales, zeros of Bessel processes.

previously initiated by Itô (1972). Among other works related to the study of Bessel processes, we can cite the seminal work of Bertoin (1990) who developed an excursion theory for the Bessel process of dimension  $\delta \in (0, 1)$  and its drift term and the work of Barlow et al. (1989) who proved a multidimensional extension of the arc sine law, for Bessel processes of dimension  $\delta \in (0, 2)$ , using excursion theory. We should also mention here the work of Azéma (1985), who developed a theory of excursions for closed optional sets, which led him to discover the so called first and second Azéma's martingales (the reader can refer to Dellacherie et al. (1992) for an introduction and more references on this topic). This approach is interesting when, for example, one studies martingales or submartingales in the filtration of the zeros of a diffusion process.

In this paper, we shall consider the couple  $(R_t, L_t)$ , where  $(R_t)$  is a Bessel process of dimension  $\delta \equiv 2(1 - \mu) \in (0, 2)$ , starting from 0, and  $(L_t)$  a choice of its local time at level zero. We shall associate with  $R$  the honest time:

$$g_\mu(T) \equiv \sup \{u \leq T : R_u = 0\},$$

where  $T > 0$  is a fixed time (we shall simply note  $g_\mu$  when  $T = 1$ ). The case  $\mu = \frac{1}{2}$  has received much attention in the literature: the distribution of  $g_{1/2}$  was obtained by Lévy (1939), the supermartingale  $Z_t \equiv \mathbb{P}[g_{1/2} > t \mid \mathcal{F}_t]$  and the dual predictable projection of  $\mathbf{1}_{(g_{1/2} \leq t)}$ , which play a key role in the general theory of stochastic processes (see Azéma (1972)), were obtained by Jeulin (1980). Moreover, the latter quantities computed by Jeulin have revealed to be useful on the one hand to Marc Yor in the study of martingales which have the same zeros as the standard Brownian Motion (see Yor (1997), Mansuy and Yor (2005)), and on the other hand to Azéma et al. (1993) in the study of the stopping theorem when stopping times are replaced with  $g_{1/2}$  (see also Nikeghbali (2005)). The computations performed by Jeulin have also been very useful in the mathematical models of default times (see Elliott et al. (2000) for examples and more references). This paper has three main aims:

- to extend the computations of Jeulin to the case of Bessel processes of dimension  $\delta \in (0, 2)$  and then to generalize the above mentioned results of Yor;
- to illustrate the results in Azéma et al. (1993) and Nikeghbali (2005) on the stopping theorem when considered at an honest time and more generally to give a larger class of examples than the usual example of the standard Brownian Motion;
- to illustrate martingale techniques in a setting where only excursion theory is used (except for a forthcoming paper by Roynette et al. (2006) which deals with the penalization of Bessel process of dimension  $0 < \delta < 2$  with a function of its local time at 0).

More precisely, the paper is organized as follows:

In Section 2, we recall and establish some basic facts about the local time of a Bessel process and then we compute explicitly the supermartingale  $Z_t^{g_\mu} \equiv \mathbb{P}[g_\mu > t \mid \mathcal{F}_t]$  and the dual predictable projection of  $\mathbf{1}_{(g_\mu \leq t)}$ . We then use them to obtain a one parameter extension of Lévy's arc sine law. We thus recover a result which Barlow et al. (1989) have found using excursion theory (in fact, this result is due to Dynkin (1961) by completely different means).

In Section 3, we investigate further for the conditional distribution of  $g_\mu$  and then test the stopping theorem on martingales of the form:

$$M_t^h \equiv \mathbb{E}[h(g_\mu) \mid \mathcal{F}_t].$$

More precisely, we show how  $\mathbb{E}[M_\infty^h \mid \mathcal{F}_{g_\mu}]$  and  $M_{g_\mu}^h$  differ (in particular we recover the results in Yor (1997) when we take  $\mu = \frac{1}{2}$ ).

In Section 4, we characterize the martingales  $(M_t)$  which have the same set of zeros as  $R$  and use this characterization to give many examples of martingales which satisfy the stopping theorem with  $g_\mu$ , i.e.:

$$\mathbb{E}[M_\infty \mid \mathcal{F}_{g_\mu}] = M_{g_\mu}^h.$$

Here again, we obtain some natural extensions of the results of Yor in the Brownian setting (Yor (1997), Mansuy and Yor (2005)).

Eventually, in Section 5, using some not so well known results of Yor (1992) about Bessel meanders of dimension  $0 < \delta < 2$ , we obtain a one dimensional extension of Azéma's second martingale. We compare our result with some results of Azéma (1985) and Rainer (1996) about projections of a diffusion on its slow filtration to recover the Lévy measure of the zeros of  $R$ . More generally, we give many examples of  $(\mathcal{F}_{g_\mu(t)})$  martingales by computing the projections of some carefully selected martingales, reminiscent of the Azéma-Yor martingales. We shall also combine these arguments with Doob's maximal identity to obtain some local time estimates; more precisely, inspired by the work of Knight (1973, 1978), and some recent lecture given by Marc Yor at Columbia University (Mansuy and Yor (2005)), we compute explicitly the following probabilities:

$$\mathbb{P}(\exists t \geq 0, R_t > \varphi(L_t)),$$

and

$$\mathbb{P}(\exists t \leq \tau_u, R_t > \varphi(L_t)),$$

where  $\varphi$  is a positive Borel function and  $(\tau_u)$  the right-continuous inverse of the local time  $(L_u)$ .

## 2. The local time of the Bessel process and an extension of the arc sine law

2.1. *Basic facts about the local time of a Bessel process.* We shall now precisely define what we mean by the local time of a Bessel process: indeed, in the literature, one can find different normalizations for the local time (see the forthcoming work Donati-Martin et al. (2006)). Our approach is based on a result of Biane and Yor (1987) about powers of Bessel processes.

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$  be a filtered probability space, where the filtration  $(\mathcal{F}_t)$  is generated by a Bessel process  $(R_t)_{t \geq 0}$ , of dimension  $\delta \in (0, 2)$ , starting from 0. We associate with  $\delta$  two other parameters  $\nu$  and  $\mu$ :

$$\nu \equiv \frac{\delta}{2} - 1, \quad \mu \equiv -\nu.$$

We note that  $\nu \in (-1, 0)$  and  $\mu \in (0, 1)$ . We first recall some basic facts about Bessel processes of index  $\nu \in (-1, 0)$  (see for example Borodin and Salminen (2002), Itô

and McKean Jr. (1974), Revuz and Yor (1999)). The process  $(R_t)$  is an  $\mathbb{R}_+$ -valued diffusion whose infinitesimal generator is defined by:

$$\mathcal{L}f(r) = \frac{1}{2} \frac{d^2 f}{dr^2} + \frac{1-2\mu}{2r} \frac{df}{dr},$$

on the domain:

$$\mathcal{D} = \left\{ f : \mathbb{R}_+ \rightarrow \mathbb{R}; \mathcal{L}f \in \mathcal{C}_b(\mathbb{R}_+), \lim_{r \rightarrow 0} r^{1-2\mu} f'(r) = 0 \right\}.$$

$(R_t)$  is a recurrent diffusion, and  $\{0\}$  is a reflecting point. Its scale function is given by:

$$s(x) = x^{-2\nu} (= x^{2\mu}),$$

and its speed measure by:

$$m_\nu(dx) = \frac{x^{2\nu+1}}{|\nu|} dx (= \frac{x^{1-2\mu}}{\mu} dx).$$

The semigroup of  $(R_t)$ , with respect to the speed measure, is given by:

$$\begin{aligned} p^{(\nu)}(t; x, y) &= \frac{|\nu|}{t} \exp\left(-\frac{x^2 + y^2}{2t}\right) I_\nu(xy) (xy)^{-\nu}, \quad x > 0 \\ p^{(\nu)}(t; 0, y) &= \frac{|\nu|}{2^\nu t^{\nu+1} \Gamma(\nu+1)} \exp\left(-\frac{y^2}{2t}\right), \end{aligned}$$

where  $I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{k! \Gamma(\nu+k+1)}$  is the modified Bessel function and  $\Gamma$  is Euler's gamma function.

Now, we shall look at a conveniently chosen power of the Bessel process to define the local time at 0 of  $R$ :

**Proposition 2.1** (and Definition). *There exists a reflected Brownian Motion  $(\gamma_t)$ , on the same probability space, such that:*

$$R_t^{2\mu} = 2\mu \gamma_{\int_0^t R_u^{2(2\mu-1)} du}. \quad (2.1)$$

*Inspired by the Brownian case ( $\mu = \frac{1}{2}$ ), we shall take as a definition for  $(L_t)$ , the local time at 0 of  $(R_t)$ , the unique increasing process  $(L_t)$  such that*

$$N_t \equiv R_t^{2\mu} - L_t,$$

*is a martingale. Moreover, we have:  $\langle N \rangle_t = 4\mu^2 \int_0^t du R_u^{2(2\mu-1)}$ , and*

$$L_t = \ell_{4\mu^2} \int_0^t R_u^{2(2\mu-1)} du,$$

*where  $(\ell_u)$  is chosen such that  $(\gamma_u - \ell_u)_{u \geq 0}$  is an  $(\sigma\{\gamma_s, s \leq u\})_{u \geq 0}$  martingale.*

**Proof.** It suffices to prove (2.1); it is a consequence of a result of Biane and Yor (1987) about powers of Bessel processes or Proposition 1.11 p.447 in Revuz and Yor (1999): if  $R$  is a Bessel process of index  $\nu$ , and if  $p$  and  $q$  are two conjugate numbers such that  $\frac{1}{p} + \frac{1}{q} = 1$ , with  $\nu > -\frac{1}{q}$ , then there exists a Bessel process  $\widehat{R}$ , of index  $q\nu$ , such that:

$$qR_u^{1/q} = \widehat{R}_{\int_0^t du R_u^{-2/p}}.$$

□

One should be very careful with the choice or the normalization of the local time: indeed, from the general theory of local times for diffusion processes (see for example Borodin and Salminen (2002)), there exists a jointly continuous family  $(L_t^x; x \geq 0, t \geq 0)$ , such that the following occupation formula holds:

$$\int_0^t h(R_u) du = C \int_0^\infty h(x) L_t^x x^{1-2\mu} dx, \quad (2.2)$$

for every Borel function  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ . The choice of  $L_t \equiv L_t^0$  determines the constant  $C$  (see Donati-Martin et al. (2006) for a detailed discussion about the different normalizations found in the literature). More precisely:

**Proposition 2.2.** *Let  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a Borel function. Then, with our choice for  $(L_t)$ , the following occupation formula holds:*

$$\int_0^t h(R_u) du = \frac{1}{\mu} \int_0^\infty h(x) L_t^x x^{1-2\mu} dx; \quad (2.3)$$

$$= \int_0^\infty h(x) L_t^x m(dx). \quad (2.4)$$

Consequently,  $C = \frac{1}{\mu}$ .

**Proof.** Taking the expectation of both sides in (2.2) yields:

$$\frac{1}{\mu} \int_0^t du p^{(\nu)}(u; 0, x) = C \mathbb{E}[L_t^x].$$

Now, letting  $x \rightarrow 0$  in the above equation, we obtain:

$$\frac{2^\mu t^\mu}{\mu \Gamma(1-\mu)} = C \mathbb{E}[R_t^{2\mu}].$$

But since

$$R_t^2 \sim 2t \text{ gam}(1-\mu),$$

where  $\text{gam}(1-\mu)$  denotes a standard Gamma variable of parameter  $1-\mu$ , it follows that:

$$C = \frac{1}{\mu}.$$

□

**Remark 2.3.** *We could obtain the occupation formula without using general results about diffusion processes, but just using Proposition 2.1, the occupation formula for the standard Brownian Motion and time change arguments. Indeed, for any Borel function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , we have:*

$$\int_0^t f(R_u) du = \int_0^t g(\gamma_{A_u}) du,$$

where  $\gamma$  denotes a reflected Brownian Motion,  $A_t \equiv \langle N \rangle_t = 4\mu^2 \int_0^t du R_u^{2(2\mu-1)}$  and  $g(x) \equiv f(x^{1/2\mu})$ . If we note  $(\rho_t)$  the right continuous inverse of  $(A_t)$ , we have:

$$\int_0^t g(\gamma_{A_u}) du = \int_0^t g(\gamma_{A_u}) \frac{R_u^{2(1-2\mu)}}{4\mu^2} dA_u = \int_0^{A_t} g(\gamma_u) \frac{R_{\rho_u}^{2(1-2\mu)}}{4\mu^2} du.$$

But  $R_{\rho_u}^{2\mu} = \gamma_u$ , hence:

$$\int_0^t g(\gamma_{A_u}) du = \frac{1}{4\mu^2} \int_0^{A_t} g(\gamma_u) \gamma_u^{\frac{1-2\mu}{\mu}} du.$$

Now, the occupation formula for the reflected Brownian Motion yields:

$$\frac{1}{4\mu^2} \int_0^{A_t} g(\gamma_u) \gamma_u^{\frac{1-2\mu}{\mu}} du = \frac{1}{4\mu^2} \int_0^\infty f(x^{1/2\mu}) x^{\frac{1-2\mu}{\mu}} \lambda_{A_t}^x dx,$$

where  $(\lambda_t^x, t \geq 0, x \geq 0)$  denotes the family of local times of  $\gamma$ . A straightforward change of variables in the previous integral yields:

$$\frac{1}{4\mu^2} \int_0^\infty f(x^{1/2\mu}) x^{\frac{1-2\mu}{\mu}} \lambda_{A_t}^x dx = \frac{1}{2\mu} \int_0^\infty f(x) x^{1-2\mu} \lambda_{A_t}^{x^{2\mu}} dx.$$

Now, plugging all these informations together, we obtain:

$$\int_0^t f(R_u) du = \frac{1}{2\mu} \int_0^\infty f(x) x^{1-2\mu} \lambda_{A_t}^{x^{2\mu}} dx.$$

Hence,

$$\int_0^t f(R_u) du = \frac{1}{\mu} \int_0^\infty f(x) x^{1-2\mu} L_t^x dx,$$

where

$$L_t^x = \frac{\lambda_{A_t}^{x^{2\mu}}}{2}.$$

This result is consistent with our choice for the local time since  $\lambda_t^0 = 2\ell_t$ . Consequently, with this time change method, we have an explicit expression for the local time in both variables  $(t, x)$ , with respect to the local time of the reflected Brownian Motion. This completes the result in Proposition 2.1.

**Remark 2.4.** The occupation formula and the scaling property for the Bessel processes (Bessel processes have the Brownian scaling property, Revuz and Yor (1999), chapter XI) entail that  $L_t$  is distributed as  $t^\mu L_1$ .

We now give as a corollary of Proposition 2.2 or Remark 2.3 a limit theorem for some integrals of Bessel processes:

**Corollary 2.5.** If  $f$  is a Borel function such that

$$\int_0^\infty dx |f(x)| x^{1-2\mu} < \infty,$$

then:

$$\lim_{n \rightarrow \infty} n^\delta \int_0^t f(nR_u) du = \left( \frac{1}{\mu} \int_0^\infty dx f(x) x^{1-2\mu} \right) L \text{ a.s.}$$

**Proof.** By the occupation formula, for every  $t \geq 0$ :

$$n^\delta \int_0^t f(nR_u) du = \frac{1}{\mu} \int_0^\infty f(x) L_t^{x/n} x^{1-2\mu} dx.$$

Now, for fixed  $t$ , the result follows from the fact that  $x \mapsto L_t^x$  is a.s. continuous and has compact support and from the dominated convergence theorem. The result holds simultaneously for every rational  $t$ , and if  $f$  is positive, the result follows by increasing limits. In the general case, it suffices to decompose  $f$  as the difference of its positive and negative parts.  $\square$

We shall also need the following extension of the occupation times formula:

**Lemma 2.6.** *Let  $h : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a Borel function. Then, almost-surely,*

$$\int_0^t du h(u, R_u) = \frac{1}{\mu} \int_0^\infty da a^{1-2\mu} \int_0^t dL_u^a h(u, a).$$

**Proof.** The formula is easily checked for  $h(u, a) \equiv \mathbf{1}_{]a, \beta]}(u)f(a)$ , and the general result follows by a monotone class argument.  $\square$

2.2. *A one parameter extension of the arc sine law.* Now, define:

$$g_\mu \equiv \sup \{t \leq 1 : R_t = 0\},$$

and more generally, for  $T > 0$ , a fixed time,

$$g_\mu(T) \equiv \sup \{t \leq T : R_t = 0\}.$$

**Proposition 2.7.** *Let  $\mu \in (0, 1)$ , and let  $(R_t)$  be a Bessel process of dimension  $\delta = 2(1 - \mu)$ . Then, we have:*

(1)

$$Z_t^{g_\mu} \equiv \mathbb{P}[g_\mu > t \mid \mathcal{F}_t] = \frac{1}{2^{\mu-1}\Gamma(\mu)} \int_{\frac{R_t}{\sqrt{1-t}}}^\infty dy y^{2\mu-1} \exp\left(-\frac{y^2}{2}\right);$$

(2) *The dual predictable projection  $A_t^{g_\mu}$  of  $\mathbf{1}_{(g_\mu \leq t)}$  is:*

$$A_t^{g_\mu} = \frac{1}{2^\mu \Gamma(1 + \mu)} \int_0^{t \wedge 1} \frac{dL_u}{(1-u)^\mu},$$

*i.e. for every nonnegative predictable process  $(x_t)$ ,*

$$\mathbb{E}[x_{g_\mu}] = \frac{1}{2^\mu \Gamma(1 + \mu)} \mathbb{E}\left[\int_0^1 dL_u \frac{x_u}{(1-u)^\mu}\right].$$

**Remark 2.8.** *Some straightforward change of variables allows us to rewrite  $Z_t^{g_\mu}$  as:*

$$Z_t^{g_\mu} = \frac{1}{\Gamma(\mu)} \int_{\frac{R_t^2}{2(1-t)}}^\infty dz z^{\mu-1} \exp(-z) \quad (2.5)$$

$$= \frac{R_t^{2\mu}}{2^\mu \Gamma(\mu) (1-t)^\mu} \int_1^\infty dz z^{\mu-1} \exp\left(-\frac{z R_t^2}{2(1-t)}\right). \quad (2.6)$$

**Proof.** (1). We have:

$$Z_t^{g_\mu} = 1 - \mathbb{P}[g_\mu \leq t \mid \mathcal{F}_t] = 1 - \mathbb{P}[d_t > 1 \mid \mathcal{F}_t],$$

where (using the Markov property),

$$d_t = \inf \{u \geq t; R_u = 0\} = t + \inf \{u \geq 0; R_{t+u} = 0\} = t + \theta \circ H_0,$$

where  $\theta$  is the shift operator and  $H_0$  is the first hitting time of 0. Consequently, using the Markov property, we obtain:

$$Z_t^{g_\mu} = 1 - \mathbb{P}_{R_t}[H_0 > 1 - t]. \quad (2.7)$$



Now, following Borodin and Salminen (p. 70-71), if for  $-\nu > 0$ ,  $\mathbb{P}_0^{(-\nu)}$  denotes the law of a Bessel process of parameter  $-\nu$ , starting from 0, then the law of  $L_y \equiv \sup\{t : R_t = y\}$ , is given by:

$$\mathbb{P}_0^{(-\nu)}(L_y \in dt) = \frac{y^{-2\nu}}{2^{-\nu}\Gamma(-\nu)t^{-\nu+1}} \exp\left(-\frac{y^2}{2t}\right) dt.$$

Now, from the time reversal property for Bessel processes (Borodin and Salminen (2002) p.70), we have:

$$\mathbb{P}_x[H_0 \in dt] = \mathbb{P}_0^{(-\nu)}(L_x \in dt);$$

consequently, from (2.7), we have (recall  $\mu = -\nu$ ):

$$Z_t^{g_\mu} = 1 - \frac{R_t^{2\mu}}{2^\mu \Gamma(\mu)} \int_{1-t}^{\infty} du \frac{\exp\left(-\frac{R_t^2}{2u}\right)}{u^{1+\mu}},$$

and the desired result is obtained by straightforward change of variables in the above integral.

(2) is a consequence of Itô's formula applied to  $Z_t^{g_\mu}$  and the fact that  $N_t \equiv R_t^{2\mu} - L_t$  is a martingale and  $(dL_t)$  is carried by  $\{t : R_t = 0\}$ .  $\square$

**Remark 2.9.** *The previous proof can be applied mutatis mutandis to obtain:*

$$\mathbb{P}[g_\mu(T) > t \mid \mathcal{F}_t] = \frac{1}{2^{\mu-1}\Gamma(\mu)} \int_{\frac{R_t}{\sqrt{T-t}}}^{\infty} dy y^{2\mu-1} \exp\left(-\frac{y^2}{2}\right);$$

and

$$A_t^{g_\mu(T)} = \frac{1}{2^\mu \Gamma(1+\mu)} \int_0^{t \wedge T} \frac{dL_u}{(T-u)^\mu}.$$

When  $\mu = \frac{1}{2}$ ,  $R_t$  can be viewed as  $|B_t|$ , the absolute value of a standard Brownian Motion. Thus, we recover as a particular case of our framework the celebrated example of the last zero before 1 of a standard Brownian Motion (see Jeulin (1980) p.124, or Yor (1997) for more references).

**Corollary 2.10.** *Let  $(B_t)$  denote a standard Brownian Motion and let*

$$g \equiv \sup\{t \leq 1 : B_t = 0\}.$$

Then:

$$\mathbb{P}[g > t \mid \mathcal{F}_t] = \sqrt{\frac{2}{\pi}} \int_{\frac{|B_t|}{\sqrt{1-t}}}^{\infty} dy \exp\left(-\frac{y^2}{2}\right),$$

and

$$A_t^g = \sqrt{\frac{2}{\pi}} \int_0^{t \wedge 1} \frac{dL_u}{\sqrt{1-u}}.$$

**Proof.** It suffices to take  $\mu \equiv \frac{1}{2}$  in Proposition 2.7.  $\square$

It is well known that  $g$  is arc sine distributed (see for example Revuz and Yor (1999)); with the help of proposition 2.7, we can recover this result and extend it to the case of any Bessel process with dimension  $2(1-\mu)$ , thus recovering a result also obtained and proved by Barlow et al. (1989) using excursion theory (see also Dynkin (1961)):

**Corollary 2.11.** *The variable  $g_\mu$  follows the law:*

$$\mathbb{P}(g_\mu \in dt) = \frac{\sin(\mu\pi)}{\pi} \frac{dt}{t^{1-\mu}(1-t)^\mu}, \quad 0 < t < 1,$$

*i.e. the Beta law with parameters  $(\mu, 1-\mu)$ . In particular,  $\mathbb{P}(g \in dt) = \frac{1}{\pi} \frac{dt}{\sqrt{t(1-t)}}$ ,*

*i.e.  $g$  is arc sine distributed.*

**Proof.** From Proposition 2.7, (2), for every Borel function  $f : [0, 1] \rightarrow \mathbb{R}_+$ , we have:

$$\mathbb{E}[f(g_\mu)] = \frac{1}{2^\mu \mu \Gamma(\mu)} \mathbb{E} \left[ \int_0^1 dL_u \frac{f(u)}{(1-u)^\mu} \right] = \frac{1}{2^\mu \mu \Gamma(\mu)} \int_0^1 d_u \mathbb{E}[L_u] \frac{f(u)}{(1-u)^\mu}. \quad (2.8)$$

By the scaling property of  $(L_t)$ ,

$$\mathbb{E}[L_u] = u^\mu \mathbb{E}[L_1].$$

Moreover, by definition of  $(L_t)$ ,

$$\mathbb{E}[L_1] = \mathbb{E}[R_1^{2\mu}];$$

since  $R_1^2$  is distributed as  $2 \text{ gam}(1-\mu)$ , we have

$$\mathbb{E}[R_1^{2\mu}] = \frac{2^\mu}{\Gamma(1-\mu)}.$$

Now, plugging this in (2.8) yields:

$$\mathbb{E}[f(g_\mu)] = \frac{1}{\Gamma(\mu) \Gamma(1-\mu)} \int_0^1 du \frac{f(u)}{u^{1-\mu}(1-u)^\mu}.$$

To conclude, it suffices to use the duplication formula for the Gamma function (Andrews et al. (1999)):

$$\Gamma(\mu) \Gamma(1-\mu) = \frac{\pi}{\sin(\mu\pi)}.$$

□

We now state a lemma that we shall often use in the sequel:

**Lemma 2.12** (Azéma (1972)). *Let  $L$  be an honest time that avoids  $(\mathcal{F}_t)$  stopping times, i.e. for every  $(\mathcal{F}_t)$  stopping time  $T$ , we have  $\mathbb{P}[L = T] = 0$ , and let*

$$Z_t^L \equiv \mathbb{P}[L > t \mid \mathcal{F}_t].$$

*Let*

$$Z_t^L = M_t^L - A_t,$$

*denote its Doob-Meyer decomposition. Then  $A_\infty$  follows the exponential law with parameter 1 and the measure  $dA_t$  is carried by the set  $\{t : Z_t^L = 1\}$ . Moreover,  $A$  does not increase after  $L$ , i.e.  $A_L = A_\infty$ . We also have:*

$$L = \sup \{t : 1 - Z_t^L = 0\}.$$

**Corollary 2.13.** *The variable*

$$\frac{1}{2^\mu \Gamma(1 + \mu)} \int_0^1 \frac{dL_u}{(1-u)^\mu}$$

*is exponentially distributed with expectation 1; consequently, its law is independent of  $\mu$ .*

**Proof.** The random time  $g_\mu$  is honest by definition (it is the end of a predictable set). It also avoids stopping times since  $A_t^{g_\mu}$  is continuous (this can also be seen as a consequence of the strong Markov property for  $R$  and the fact that 0 is instantaneously reflecting). Thus the result of the corollary is a consequence of Proposition 2.7 and Lemma 2.12.  $\square$

We can also use Proposition 2.7 to give an example of a remarkable random time, called pseudo-stopping time (Nikeghbali and Yor (2005)), which is not a stopping time but which satisfies the stopping theorem:

**Corollary 2.14.** *Define:*

$$\rho \equiv \sup \left\{ t < g_\mu : \frac{R_t}{\sqrt{1-t}} = \sup_{u < g_\mu} \frac{R_u}{\sqrt{1-u}} \right\}.$$

*Then,  $\rho$  is a pseudo-stopping time, i.e. for every  $(\mathcal{F}_t)$  uniformly integrable martingale  $(M_t)$ , we have:*

$$\mathbb{E}[M_\rho] = \mathbb{E}[M_\infty].$$

**Proof.** It is an easy consequence of Proposition 2.7 and Proposition 5 in Nikeghbali and Yor (2005).  $\square$

### 3. The conditional law of the last zero before time 1

In this section, for simplicity, we consider  $(R_t)_{t \leq 1}$ ; one could easily replace 1 with any fixed time  $T$ . Our aim in this section is twofold:

- to investigate further for the distribution of  $g_\mu$  by computing its conditional distribution given  $\mathcal{F}_t$ ;
- to test the stopping theorem on martingales whose terminal values are  $\sigma(g_\mu)$  measurable, extending thus the work by Yor for the Brownian Motion in Yor (1997) (i.e.  $\mu = \frac{1}{2}$ ).

We shall need the following lemma which also appears in Nikeghbali (2005) for the resolution of some martingale equations:

**Lemma 3.1.** *Let  $L$  be an honest time that avoids stopping times and let  $(K_t)_{t \geq 0}$  be a predictable process such that  $\mathbb{E}[|K_L|] < \infty$ . Then for every  $t \geq 0$ :*

$$\mathbb{E}[K_L | \mathcal{F}_t] = K_{L_t} \mathbb{P}(L \leq t | \mathcal{F}_t) + \mathbb{E} \left[ \int_t^\infty K_s dA_s | \mathcal{F}_t \right], \quad (3.1)$$

where  $A_t$  is the dual predictable projection of  $\mathbf{1}_{(L \leq t)}$  and

$$L_t = \sup \{ s \leq t : 1 - Z_s^L = 0 \}.$$

Moreover, the latter martingale can also be written as:

$$\mathbb{E}[K_L | \mathcal{F}_t] = - \int_0^t K_{L_s} dM_s^L + \mathbb{E} \left[ \int_0^\infty K_s dA_s | \mathcal{F}_t \right],$$

where  $(M_s^L)$  is defined in Lemma 2.12.

**Proof.**

$$\begin{aligned}\mathbb{E}[K_L | \mathcal{F}_t] &= \mathbb{E}[K_L \mathbf{1}_{L \leq t} | \mathcal{F}_t] + \mathbb{E}[K_L \mathbf{1}_{L > t} | \mathcal{F}_t] \\ &= K_{L_t} \mathbb{P}(L \leq t | \mathcal{F}_t) + \mathbb{E}[K_L \mathbf{1}_{L > t} | \mathcal{F}_t].\end{aligned}$$

Now, let  $\Gamma_t$  be an  $(\mathcal{F}_t)$  measurable set;

$$\mathbb{E}[K_L \mathbf{1}_{L > t} \mathbf{1}_{\Gamma_t}] = \mathbb{E}\left[\int_t^\infty K_s dA_s \mathbf{1}_{\Gamma_t}\right];$$

hence

$$\mathbb{E}[K_L \mathbf{1}_{L > t} | \mathcal{F}_t] = \mathbb{E}\left[\int_t^\infty K_s dA_s | \mathcal{F}_t\right],$$

and this completes the proof of the first part of the lemma. The second part follows from balayage arguments; indeed:

$$\begin{aligned}K_{L_t} \mathbb{P}(L \leq t | \mathcal{F}_t) &= K_{L_t} (1 - Z_t^L) \\ &= -\int_0^t K_{L_s} dM_s^L + \int_0^t K_s dA_s.\end{aligned}$$

Now, since  $\mathbb{E}\left[\int_t^\infty K_s dA_s | \mathcal{F}_t\right] = \mathbb{E}\left[\int_0^\infty K_s dA_s | \mathcal{F}_t\right] - \int_0^t K_s dA_s$ , we have

$$K_{L_t} \mathbb{P}(L \leq t | \mathcal{F}_t) + \mathbb{E}\left[\int_t^\infty K_s dA_s | \mathcal{F}_t\right] = -\int_0^t K_{L_s} dM_s^L + \mathbb{E}\left[\int_0^\infty K_s dA_s | \mathcal{F}_t\right],$$

and the proof of the lemma is now complete.  $\square$

Now we shall obtain some closed formulae for martingales whose terminal values are  $g_\mu$  measurable and hence obtain the conditional laws of  $g_\mu$  given  $\mathcal{F}_t$ ,  $t \geq 0$ . Let  $h : [0, 1] \rightarrow \mathbb{R}_+$  be a Borel function and define:

$$M_t^h \equiv \mathbb{E}[h(g_\mu) | \mathcal{F}_t].$$

The problem of computing the martingales  $(M_t^h)$  can be dealt with Lemma 3.1, which takes here the following form (recall our processes are stopped at 1): for any nonnegative predictable process  $(K_t)_{t \leq 1}$  (such that  $\mathbb{E}[K_{g_\mu}] < \infty$ ),

$$\mathbb{E}[K_{g_\mu} | \mathcal{F}_t] = K_{g_\mu(t)} \mathbb{P}(g_\mu \leq t | \mathcal{F}_t) + \mathbb{E}\left[\int_t^1 K_s dA_s^{g_\mu} | \mathcal{F}_t\right], \quad (3.2)$$

where

$$g_\mu(t) = \sup\{s \leq t : Z_s^{g_\mu} = 1\} = \sup\{s \leq t : R_s = 0\}.$$

The fact that  $\sup\{s \leq t : Z_s^{g_\mu} = 1\} = \sup\{s \leq t : R_s = 0\}$  can be seen on the expression of  $Z_t^{g_\mu}$  in Proposition 2.7.

The following lemma will help us to compute explicitly the quantity

$$\mathbb{E}\left[\int_t^1 K_s dA_s^{g_\mu} | \mathcal{F}_t\right]$$

when  $K$  is a deterministic function:

**Lemma 3.2.** *Let  $(L_t^a)$  denote the local time at  $a \in \mathbb{R}_+$  of the Bessel Process  $(R_t)$ .*

(1) *For every Borel function  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ,*

$$\mathbb{E}\left[\int_t^1 dL_u^a h(u) | \mathcal{F}_t\right] = \int_t^1 du h(u) p^{(\nu)}(u - t; R_t, a); \quad (3.3)$$

(2) Consequently, for every Borel function  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , we have:

$$\mathbb{E} \left[ \int_t^1 dA_u^{g_\mu} h(u) \mid \mathcal{F}_t \right] = \frac{\sin(\pi\mu)}{\pi} \int_0^1 dz \frac{h(t+z(1-t))}{(1-z)^\mu z^{1-\mu}} \exp\left(-\frac{R_t^2}{2z(1-t)}\right). \quad (3.4)$$

**Proof.** (1). First, by the generalized occupation density formula (2.6), for every nonnegative Borel function  $f$ , we have:

$$\int_t^1 du f(R_u) h(u) = \int_0^\infty m(da) f(a) \int_t^1 dL_u^a h(u). \quad (3.5)$$

We also have:

$$\begin{aligned} \mathbb{E} \left[ \int_t^1 du f(R_u) h(u) \mid \mathcal{F}_t \right] &= \int_t^1 du h(u) \mathbb{E}[f(R_u) \mid \mathcal{F}_t] \\ &= \int_0^\infty m(da) f(a) \int_t^1 du h(u) p^{(\nu)}(u-t; R_t, a) \end{aligned}$$

and from (3.5) we obtain:

$$\mathbb{E} \left[ \int_t^1 dL_u^a h(u) \mid \mathcal{F}_t \right] = \int_t^1 du h(u) p^{(\nu)}(u-t; R_t, a).$$

(2). We know from Proposition (2.7) that:  $A_t^{g_\mu} = \frac{1}{2^\mu \mu \Gamma(\mu)} \int_0^{t \wedge 1} \frac{dL_u}{(1-u)^\mu}$ . Plugging this into (3.3) yields ( $a=0$ ):

$$\mathbb{E} \left[ \int_t^1 dA_u^{g_\mu} h(u) \mid \mathcal{F}_t \right] = \int_t^1 du h(u) \frac{\exp\left(-\frac{R_t^2}{2(u-t)}\right)}{\Gamma(\mu) \Gamma(1-\mu) (1-u)^\mu (t-u)^{1-\mu}}.$$

Now, using the duplication formula:

$$\Gamma(\mu) \Gamma(1-\mu) = \frac{\pi}{\sin(\mu\pi)},$$

and making the change of variable  $u = t + z(1-t)$ , we obtain:

$$\mathbb{E} \left[ \int_t^1 dA_u^{g_\mu} h(u) \mid \mathcal{F}_t \right] = \frac{\sin(\pi\mu)}{\pi} \int_0^1 dz \frac{h(t+z(1-t))}{(1-z)^\mu z^{1-\mu}} \exp\left(-\frac{R_t^2}{2z(1-t)}\right),$$

which completes the proof of the lemma.  $\square$

Now, we shall give three nice corollaries of Lemma 3.2. The last two corollaries extend naturally some results of Mansuy and Yor (2005) for the Brownian Motion to any Bessel process of dimension  $\delta \in (0, 2)$ .

**Corollary 3.3.** Let  $h : [0, 1] \rightarrow \mathbb{R}_+$ , be a Borel function, then:

$$\mathbb{E}[h(g_\mu) \mid \mathcal{F}_t] = h(g_\mu(t)) (1 - Z_t^{g_\mu}) + \mathbb{E}[h(g_\mu) \mathbf{1}_{(g_\mu > t)} \mid \mathcal{F}_t];$$

with

$$\mathbb{E}[h(g_\mu) \mathbf{1}_{(g_\mu > t)} \mid \mathcal{F}_t] = \frac{\sin(\pi\mu)}{\pi} \int_0^1 dz \frac{h(t+z(1-t))}{(1-z)^\mu z^{1-\mu}} \exp\left(-\frac{R_t^2}{2z(1-t)}\right). \quad (3.6)$$

Consequently, the law of  $g_\mu$  given  $\mathcal{F}_t$ , which we note  $\lambda_t(dz)$ , is given by:

$$\lambda_t(dz) = (1 - Z_t^{g_\mu}) \varepsilon_{g_\mu(t)}(dz) + \mathbf{1}_{(t,1)}(z) \frac{\sin(\pi\mu)}{\pi} \frac{\exp\left(-\frac{R_t^2}{2(z-t)}\right)}{(1-z)^\mu (z-t)^{1-\mu}} dz,$$

and taking  $t = 0$ , we recover the generalized arc sine law.

As a consequence of this corollary, we can also see how the stopping theorem fails to hold for the family of martingales  $M_t^h \equiv \mathbb{E}[h(g_\mu) | \mathcal{F}_t]$ , thus completing the examples in Yor (1997) and Nikeghbali (2005):

**Corollary 3.4.** *Let  $h : [0, 1] \rightarrow \mathbb{R}_+$ , be a Borel function, and define  $M_t^h = \mathbb{E}[h(g_\mu) | \mathcal{F}_t]$ ; then*

$$\mathbb{E}[M_\infty^h | \mathcal{F}_{g_\mu}] = h(g_\mu), \quad (3.7)$$

whilst

$$M_{g_\mu}^h = \frac{\sin(\pi\mu)}{\pi} \int_0^1 dz \frac{1}{(1-z)^\mu z^{1-\mu}} h(g_\mu + z(1-g_\mu)). \quad (3.8)$$

We can also compute explicitly the martingale  $\mathbb{E}[L_1 | \mathcal{F}_t]$ :

**Corollary 3.5.** *Let*

$$X_t \equiv \mathbb{E}[L_1 | \mathcal{F}_t].$$

Then,

$$X_1 \equiv X_\infty = L_1,$$

whilst

$$X_{g_\mu} = L_1 + \frac{2^\mu}{\Gamma(1-\mu)} (1-g_\mu)^\mu.$$

**Proof.** It suffices to take  $h \equiv 1$  in (3.3).  $\square$

Taking  $\mu = \frac{1}{2}$ , we recover the following results obtained for Brownian Motion in Mansuy and Yor (2005):

**Corollary 3.6.** *Let  $(B_t)$  be the standard Brownian Motion, and denote  $(\ell_t)$  its local time at zero. Let  $g \equiv g_{1/2}$ . Then, for any Borel function  $h : [0, 1] \rightarrow \mathbb{R}_+$ , the following identities hold:*

(1)

$$\mathbb{E}[h(g) | \mathcal{F}_t]_{t=g} = \frac{1}{\pi} \int_0^1 \frac{dz}{\sqrt{z(1-z)}} h(g + z(1-g)).$$

(2)

$$\mathbb{E}[\ell_1 | \mathcal{F}_t]_{t=g} = \ell_1 + \sqrt{\frac{2}{\pi}} \sqrt{1-g}.$$

#### 4. Some martingales with the same set of zeros as $R_t$ and which satisfy the stopping theorem with respect to $g_\mu$

In this section, we shall illustrate with some examples related to Bessel processes the seminal work of Azéma and Yor (1992) on zeros of continuous martingales. Yor has specialized further this work to the important case of the standard Brownian Motion, giving explicit examples of martingales which have the same zeros as  $(B_t)_{t \leq 1}$  (Yor (1997), chapter 14). Quite unexpectedly, the study of martingales which have same zeros leads to some discussion on the stopping theorem when stopping times are replaced with honest times (Azéma et al. (1993), Nikeghbali (2005)). Our aim here is to provide a one parameter ( $\mu$ ) extension of some results in Yor (1997), chapter 14, and to give more examples of martingales associated with honest times satisfying the stopping theorem (which is, regarding the results of the

previous section, quite exceptional). Even though  $(R_t)$  is not a martingale (and not even a semimartingale when  $\mu > \frac{1}{2}$ ), the methods of Azéma and Yor apply remarkably here. More precisely:

**Proposition 4.1.** *Define:*

$$\mathcal{Z}_1 = \{t \in [0, 1] : R_t = 0\},$$

and let  $(M_t)$  be an  $(\mathcal{F}_t)$  martingale. Then the following are equivalent:

- (1) for all  $t \in \mathcal{Z}_1$ ,  $M_t = 0$ ;
- (2)  $M_{g_\mu} = 0$ .

If furthermore  $(M_t)$  is uniformly integrable, then the previous assertions are equivalent to:

- (3)  $\mathbb{E}[M_\infty | \mathcal{F}_{g_\mu}] = 0$ ; consequently, in this case, we have:

$$\mathbb{E}[M_\infty | \mathcal{F}_{g_\mu}] = M_{g_\mu}. \quad (4.1)$$

**Proof.** We only prove (1)  $\Leftrightarrow$  (2), as the proof of the equivalence with (3) can be found in Azéma and Yor (1992); Azéma et al. (1993); Nikeghbali (2005).

Only the implication (2)  $\Rightarrow$  (1) is not obvious. Assume that  $M_{g_\mu} = 0$ ; then, we also have  $|M_{g_\mu}| = 0$ . But from Proposition 2.7,

$$0 = \mathbb{E}[|M_{g_\mu}|] = \mathbb{E}\left[\int_0^1 |M_u| dA_u^{g_\mu}\right],$$

and consequently

$$|M_u| = 0, \quad dA_u^{g_\mu} d\mathbb{P} \text{ a.s.}$$

and from Lemma 2.12 and Proposition 2.7, this means that  $M_t = 0$  if and only if  $t \in \mathcal{Z}_1$ .  $\square$

Now, as a consequence of Proposition 4.1, we can associate canonically with a uniformly integrable martingale another uniformly integrable martingale whose zeros are in  $\mathcal{Z}_1$  and which satisfy (4.1). More precisely, if  $(M_t)$  is a uniformly integrable martingale, then the uniformly integrable martingale  $(\widehat{M}_t)$  defined by:

$$\widehat{M}_t \equiv \mathbb{E}[(M_\infty - \mathbb{E}[M_\infty | \mathcal{F}_{g_\mu}]) | \mathcal{F}_t],$$

satisfies the equivalent assertions of Proposition 4.1 ((3)). We shall now illustrate this fact on an example reminiscent of Yor's study for the standard Brownian Motion (Yor (1997); Mansuy and Yor (2005)). More precisely, with  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , a Borel function, we associate the martingale:

$$M_t^f \equiv \mathbb{E}[f(R_1) | \mathcal{F}_t].$$

To give an explicit expression for the associated martingale  $\widehat{M}_t^f$ , we shall need some results of Yor about generalized meanders as exposed in Yor (1992). For a fixed time  $T > 0$ , we call:

$$m_\mu^T(u) = \frac{1}{\sqrt{T - g_\mu(T)}} R_{g_\mu(T) + u(T - g_\mu(T))}, \quad u \leq 1$$

the Bessel meander associated to the Bessel process  $R$  of dimension  $2(1 - \mu)$ ;  $\mu \in (0, 1)$ .  $(m_\mu^T(u))_{u \leq 1}$  is independent of  $\mathcal{F}_{g_\mu(T)}$ , and the law of  $m_\mu^T \equiv m_\mu^T(1)$  does

not depend on  $\mu$  and  $T$ , and is distributed as the two dimensional Bessel process at time 1, i.e.:

$$\mathbb{P}(m_\mu^T \in dx) = x \exp\left(-\frac{x^2}{2}\right) dx. \quad (4.2)$$

In the sequel, we shall simply note  $m_\mu$  for  $m_\mu^1(1)$ .

Now, let us come back to the case of the martingales  $M_t^f = \mathbb{E}[f(R_1) | \mathcal{F}_t]$ . From the Markov property, we have:

$$M_t^f = \int_0^\infty m(dz) f(z) p^{(\nu)}(1-t; R_t, z),$$

and from the properties recalled above about Bessel meanders, we have:

$$\begin{aligned} \mathbb{E}[f(R_1) | \mathcal{F}_{g_\mu}] &= \mathbb{E}[f(m_\mu \sqrt{1-g_\mu}) | \mathcal{F}_{g_\mu}] \\ &= \int_0^\infty dz f(z \sqrt{1-g_\mu}) z \exp\left(-\frac{z^2}{2}\right). \end{aligned}$$

Now, with the help of Corollary 3.3, we are able to compute  $\mathbb{E}[\mathbb{E}[f(R_1) | \mathcal{F}_{g_\mu}] | \mathcal{F}_t]$ :

**Proposition 4.2.** *Define as above:*

$$\widehat{M}_t^f \equiv \mathbb{E}[(f(R_1) - \mathbb{E}[f(R_1) | \mathcal{F}_{g_\mu}]) | \mathcal{F}_t].$$

*Define:*

$$\begin{aligned} g_\mu(t) &\equiv \sup\{s \leq t : R_s = 0\}; \\ \theta_\mu(x) &= \frac{1}{2^{\mu-1} \Gamma(\mu)} \int_x^\infty dz z^{2\mu-1} \exp\left(-\frac{z^2}{2}\right). \end{aligned}$$

Then  $(\widehat{M}_t^f)$  satisfies the equivalent assertions of Proposition 4.1 and can be expressed as:

$$\widehat{M}_t^f = \widehat{M}_t^{(1)} - \widehat{M}_t^{(2)} - \widehat{M}_t^{(3)},$$

where:

$$\begin{aligned} \widehat{M}_t^{(1)} &= \int_0^\infty m(dz) f(z) p^{(\nu)}(1-t; R_t, z), \\ \widehat{M}_t^{(2)} &= \theta_\mu\left(\frac{R_t}{\sqrt{1-t}}\right) \int_0^\infty dz z f(z \sqrt{1-g_\mu(t)}) \exp\left(-\frac{z^2}{2}\right), \\ \widehat{M}_t^{(3)} &= \frac{\sin(\pi\mu)}{\pi} \int_0^\infty dz z \exp\left(-\frac{z^2}{2}\right) \int_0^1 dw \frac{f(z \sqrt{1-t} \sqrt{1-w})}{(1-w)^\mu w^\mu} \exp\left(-\frac{R_t^2}{2w(1-t)}\right). \end{aligned}$$

**Proof.** The proof is obtained with the help of Corollary 3.3, Proposition 2.7 and a few elementary computations.  $\square$

**Remark 4.3.** *The above proposition tells us that though Proposition 4.1 is simple, obtaining the explicit expression of the projections on  $\mathcal{F}_{g_\mu}$  and then on  $\mathcal{F}_t$  can be difficult in practice.*

Now, we shall use the remarkable fact that the density of the generalized meander at time 1 does not depend on  $\mu$  and satisfies a functional equation to build a another family of martingales with the same properties as the martingales  $(\widehat{M}_t^f)$ .



**Proposition 4.4.** *Let  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ , be a function of class  $\mathcal{C}^2$ , with compact support. Define*

$$X^f \equiv f(R_1) - f(0) - R_1 f'(R_1) + (1 - g_\mu) f''(R_1),$$

and

$$X_t^f \equiv \mathbb{E}[X^f | \mathcal{F}_t].$$

Then,  $(X_t^f)$  satisfies (4.1), i.e.

$$\mathbb{E}[X_\infty^f | \mathcal{F}_{g_\mu}] = X_{g_\mu}^f,$$

and

$$X_t^f = 0 \Leftrightarrow t \in \mathcal{Z}_1.$$

**Proof.** As recalled at the beginning of the previous subsection,  $\mathbb{P}(m_\mu \in d\rho) = \rho \exp\left(-\frac{\rho^2}{2}\right) d\rho$ . It is not difficult to show that for  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ , a function of class  $\mathcal{C}^2$ , which is compactly supported, we have:

$$\mathbb{E}[f(m_\mu)] = \mathbb{E}[f(0) + m_\mu f'(m_\mu) - f''(m_\mu)].$$

Now, replacing  $f$  with  $f(\kappa \bullet)$  in the above, with  $\kappa \in \mathbb{R}$ , we obtain:

$$\mathbb{E}[f(\kappa m)] = \mathbb{E}[f(0) + \kappa m f'(\kappa m) - \kappa^2 f''(\kappa m)].$$

Next, as  $R_1 = \sqrt{1 - g_\mu} m_\mu$ , with  $m_\mu$  independent from  $\mathcal{F}_{g_\mu}$ , we have:

$$\mathbb{E}[X | \mathcal{F}_{g_\mu}] = 0,$$

and the result of the Proposition follows from Proposition 4.1.  $\square$

**Remark 4.5.** *In fact, the result of the Proposition is still true if we only assume that:  $\mathbb{E}[|f(m_\mu)|] < \infty$ ,  $\mathbb{E}[m_\mu |f'(m_\mu)|] < \infty$ ,  $\mathbb{E}[|f''(m_\mu)|] < \infty$ .*

**Remark 4.6.** *More generally, if  $\varphi : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}_+$ , then, using the properties of the meander, we obtain that:*

$$\mathbb{E}[\varphi(R_1, g_\mu) | \mathcal{F}_{g_\mu}] = 0,$$

if and only if:

$$\int_0^\infty dz z \exp\left(-\frac{z^2}{2(1-g_\mu)}\right) \varphi(z, g_\mu) = 0.$$

## 5. A one parameter extension of Azéma's second martingale

In this section, we shall associate with the pair  $(R_t, L_t)$  a family of local martingales reminiscent of the Azéma-Yor martingales and use them to compute the distribution of the local time at some stopping times. Furthermore, we shall project these martingales on the filtration of the zeros of  $(R_t)$  to obtain some remarkable martingales; in particular, we prove a one parameter extension of Azéma's second martingale, i.e. we find a submartingale in the filtration of the zeros, which has the same local time  $(L_t)$  as  $(R_t)$ .

Let  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  be a locally bounded Borel function and let  $F(x) = \int_0^x dz f(z)$ .

**Proposition 5.1.** *Let  $f$  and  $F$  be defined as above. Then the process*

$$F(L_t) - f(L_t) R_t^{2\mu},$$

*is a local martingale.*

**Proof.** We have:

$$R_t^{2\mu} = N_t + L_t,$$

and  $dL_t$  is carried by  $\{t : R_t = 0\}$ . Hence as an application of Skorokhod's reflection lemma (see Revuz and Yor (1999), chapter VI) yields:

$$L_t = \sup_{s \leq t} (-N_s)$$

Then applying the balayage formula (see Revuz and Yor (1999), chapter VI, p.262) we obtain:

$$F(L_t) - f(L_t) R_t^{2\mu} \tag{5.1}$$

is a local martingale.  $\square$

Now, for  $a > 0$ , let:

$$T_a \equiv \inf \{t : R_t = a\}.$$

**Corollary 5.2.** *The random variable  $L_{T_a}$  is distributed as an exponential variable of parameter  $a^{2\mu}$ .*

**Proof.** Taking  $F(x) \equiv \exp(-\theta x)$ ,  $\theta > 0$  in Proposition 5.1, we obtain that:

$$M_t \equiv \exp(-\theta L_{t \wedge T_a}) \left(1 + \theta R_{t \wedge T_a}^{2\mu}\right),$$

is a local martingale. Since it is bounded, it is a uniformly integrable martingale and the optional stopping theorem yields:

$$\mathbb{E}[\exp(-\theta L_{T_a})] = \frac{1}{1 + \theta a^{2\mu}},$$

and thus  $L_{T_a}$  is distributed as an exponential variable of parameter  $a^{2\mu}$ .  $\square$

**Remark 5.3.** *When  $\mu = \frac{1}{2}$ , we recover a well known result for the standard Brownian Motion: in this special case,  $T_a \equiv \inf \{t : |B_t| = a\}$ .*

Now, we shall project the martingales of Proposition 5.1 on the smaller filtration  $(\mathcal{F}_{g_\mu(t)})$ .

**Lemma 5.4.** *For  $u \leq v$ , we have:*

$$\mathcal{F}_{g_\mu(u)} \subset \mathcal{F}_{g_\mu(v)},$$

and for every  $t \geq 0$ , we have:

$$\mathcal{F}_{g_\mu(t)} \subset \mathcal{F}_t.$$

**Proof.** This is a consequence of Theorem 27, p. 142 in Dellacherie et al. (1992) about the sigma algebras associated with honest times.  $\square$

Our aim now is to find martingales for the filtration  $(\mathcal{F}_{g_\mu(t)})$ , the filtration of the zeros of  $R$ . There exist some general results, essentially due to Azéma, on excursion theory for closed optional random sets (Azéma (1985), Dellacherie et al. (1992)), and our framework would fit in that general setting. However, the computations are not always easy to carry in the latter setting and we shall use more elementary arguments to deal with this problem. Azéma did some exact computations in the Brownian setting and obtained the celebrated Azéma's martingales. We shall obtain here a one parameter generalization of Azéma's second martingale.

We are interested in local martingales of Proposition 5.1 which are true martingales. This happens for example if  $F$  and  $F' = f$  are with compact support, or if  $f$  is a probability density on  $\mathbb{R}_+$ .

**Proposition 5.5.** *Let  $f$  be chosen such that the local martingale  $M_t = f(L_t) R_t^{2\mu} - F(L_t)$  is a true martingale. Define:*

$$\Lambda_t \equiv \mathbb{E} [M_t \mid \mathcal{F}_{g_\mu(t)}].$$

Then  $(\Lambda_t)$  is a martingale in the filtration  $(\mathcal{F}_{g_\mu(t)})$  and we have:

$$\Lambda_t = 2^\mu \Gamma(1 + \mu) f(L_t) (t - g_\mu(t))^\mu - F(L_t).$$

**Proof.** First, we note that  $L_{g_\mu(t)} = L_t$  ( $L$  increases on the set of zeros of  $R$ ). Consequently, we have:

$$\mathbb{E} [M_t \mid \mathcal{F}_{g_\mu(t)}] = f(L_t) \mathbb{E} [R_t^{2\mu} \mid \mathcal{F}_{g_\mu(t)}] - F(L_t).$$

Now, from the properties of the generalized meander, we have:

$$\mathbb{E} [R_t^{2\mu} \mid \mathcal{F}_{g_\mu(t)}] = (t - g_\mu(t))^\mu \mathbb{E} \left[ \left( m_\mu^{(t)} \right)^{2\mu} \right].$$

To conclude, it suffice to note that:

$$\mathbb{E} \left[ \left( m_\mu^{(t)} \right)^{2\mu} \right] = \int_0^\infty dz z^{2\mu+1} \exp\left(-\frac{z^2}{2}\right) = 2^\mu \Gamma(1 + \mu).$$

□

A nice consequence of Proposition 5.5 is the existence of a remarkable submartingale in the filtration  $(\mathcal{F}_{g_\mu(t)})$ , which vanishes on  $\mathcal{Z}_1$  and which has the same local time as  $(R_t)$ . The existence of such a submartingale leads to the generalization of the so called Azéma's second martingale: by analogy with the Brownian case (see Azéma (1985), p.462), we shall project the martingale  $(R_t^{2\mu} - L_t)$  on the filtration  $(\mathcal{F}_{g_\mu(t)})$ .

**Corollary 5.6.** *The stochastic process*

$$\nu_t \equiv (t - g_\mu(t))^\mu - c_\mu L_t,$$

where

$$c_\mu = \frac{1}{2^\mu \Gamma(1 + \mu)},$$

is an  $(\mathcal{F}_{g_\mu(t)})$  martingale. Consequently  $Y_t \equiv 2^\mu \Gamma(1 + \mu) (t - g_\mu(t))^\mu$  is an  $(\mathcal{F}_{g_\mu(t)})$  submartingale which vanishes on  $\mathcal{Z}_1$  and whose local time is  $(L_t)$ . In Azéma's terminology,  $(Y_t)$  is called the equilibrium submartingale of  $\mathcal{Z}_1$ , and taking  $\mu = \frac{1}{2}$ , we recover the calculations of Azéma in the Brownian setting (we drop the index  $\mu$  for notational convenience):

$$\begin{aligned} Y_t &= \sqrt{\frac{\pi}{2}} \sqrt{t - g_t} \\ \nu_t &= \sqrt{t - g_t} - \sqrt{\frac{2}{\pi}} \ell_t, \end{aligned}$$

the latter martingale being the celebrated Azéma's second martingale (Yor (1997)).

**Proof.** It suffices to take  $f \equiv 1$  in Proposition 5.5.  $\square$

**Remark 5.7.** *Rainer (1996) has some projection formulae for real diffusions on the slow filtration; our result is not contained in her work but we could use her results, based on excursion theory, to extend some of our results to some diffusions on natural scale.*

**Corollary 5.8.** *Let  $n_\mu$  be the Lévy measure of the inverse local time of  $R$  or the distribution of the lifetime of excursions under the Itô measure. Then, we have:*

$$n_\mu(dx) = \frac{1}{2^\mu \Gamma(\mu)} \frac{dx}{x^{1+\mu}}.$$

**Proof.** This is a consequence of Corollary 5.6 and the following result of Azéma (1985), p.452, which takes in our setting the following form: the stochastic process

$$\overline{M}_t \equiv \frac{1}{n_\mu([t - g_\mu(t)])} - L_t \equiv 2^\mu \Gamma(\mu + 1) (t - g_\mu(t))^\mu - L_t,$$

is a local martingale for the filtration  $(\mathcal{F}_{g_\mu(t)})$  (in fact we have proved it is a true martingale).  $\square$

Now, to conclude, we shall give a local time estimate for  $R$ . What follows is reminiscent of some studies by Knight (Knight (1973, 1978)), Shi (Shi (1996)) and Khoshnevisan (Khoshnevisan (1996)), and is inspired by some recent lectures given by Marc Yor at Columbia University (Mansuy and Yor (2005)). It should be mentioned that our results, which generalize some similar results in the Brownian setting, can be in fact generalized to a much wider class of stochastic processes studied in a forthcoming paper Nikeghbali (2006), without assuming any Markov nor scaling property. This is in fact possible thanks to the martingale techniques we shall use. More precisely, we shall need the following result, called Doob's maximal identity. We only mention it without proving it; the reader can refer to Nikeghbali and Yor (2006) for a proof (which is essentially an application of Doob's optional stopping theorem) and for some nice applications to enlargements of filtrations and path decompositions for some large classes of diffusion processes.

**Lemma 5.9** (Doob's maximal identity). *Let  $(N_t)$  be a continuous and positive local martingale which satisfies:*

$$N_0 = x, \quad x > 0; \quad \lim_{t \rightarrow \infty} N_t = 0.$$

*If we note*

$$S_t \equiv \sup_{u \leq t} N_u,$$

*then, for any  $a > 0$ , we have:*

(1)

$$\mathbf{P}(S_\infty > a) = \left(\frac{x}{a}\right) \wedge 1. \quad (5.2)$$

*Hence,  $\frac{1}{S_\infty}$  is a uniform random variable on  $(0, 1/a)$ .*

(2) For any stopping time  $T$ :

$$\mathbf{P}(S^T > a \mid \mathcal{F}_T) = \left(\frac{N_T}{a}\right) \wedge 1, \quad (5.3)$$

where

$$S^T = \sup_{u \geq T} N_u.$$

Hence  $\frac{N_T}{S^T}$  is also a uniform random variable on  $(0, 1)$ , independent of  $\mathcal{F}_T$ .

Now, we can state and prove our result about local time estimates:

**Proposition 5.10.** *Let  $R$  be a Bessel process of dimension  $2(1-\mu)$ , with  $\mu \in (0, 1)$ ,  $L$  its local time at 0 (as defined in Section 2). Define  $\tau$  the right continuous inverse of  $L$ :*

$$\tau_u = \inf \{t \geq 0; L_t > u\}.$$

Then, for any  $u > 0$ , and any positive Borel function  $\varphi$ , we have:

$$\mathbb{P}(\exists t \leq \tau_u, R_t > \varphi(L_t)) = 1 - \exp\left(-\int_0^u \frac{dx}{\varphi^{2\mu}(x)}\right),$$

and consequently:

$$\mathbb{P}(\exists t \geq 0, R_t > \varphi(L_t)) = 1 - \exp\left(-\int_0^\infty \frac{dx}{\varphi^{2\mu}(x)}\right).$$

**Proof.** For  $u > 0$ , define:

$$\varphi_u(x) = \varphi(x), \text{ if } x < u; \text{ and } \varphi_u(x) = \infty \text{ otherwise.}$$

Now, it is clear that:

$$\mathbb{P}(\exists t \leq \tau_u, R_t > \varphi(L_t)) = \mathbb{P}(\exists t \geq 0, R_t > \varphi_u(L_t)).$$

Consider now the local martingale:

$$M_t \equiv F(L_t) - f(L_t) R_t^{2\mu},$$

where we choose (with the convention  $\frac{1}{\infty} = 0$ ):

$$F(x) \equiv 1 - \exp\left(-\int_x^\infty \frac{dz}{\varphi_u^{2\mu}(z)}\right),$$

and  $f = F'$ , the Lebesgue derivative of  $F$ . Now, it is easily checked that  $M$  is a positive local martingale. Moreover,  $\lim_{t \rightarrow \infty} M_t = 0$ . Indeed, since  $M$  is a positive local martingale, it converges almost surely to a limit  $M_\infty$ . To see that in fact  $M_\infty = 0$ , we look at  $\lim_{v \rightarrow \infty} M_{\tau_v}$ . Since  $L_{\tau_v} = v$  and  $R_{\tau_v} = 0$ , we easily find that:  $\lim_{v \rightarrow \infty} M_{\tau_v} = 0$ , and consequently,  $M_\infty = 0$ .

Now let us note that if for a given  $t_0 < \infty$ , we have  $R_{t_0} > \varphi_u(L_{t_0})$ , then we must have:

$$M_{t_0} > F(L_{t_0}) - f(L_{t_0}) \varphi_u^{2\mu}(L_{t_0}) = 1,$$

and hence we easily deduce from this that:

$$\begin{aligned} \mathbb{P}(\exists t \geq 0, R_t > \varphi_u(L_t)) &= \mathbb{P}\left(\exists t \geq 0, R_t^{2\mu} > \varphi_u^{2\mu}(L_t)\right) \\ &= \mathbb{P}\left(\sup_{t \geq 0} M_t > 1\right) \\ &= \mathbb{P}\left(\sup_{t \geq 0} \frac{M_t}{M_0} > \frac{1}{M_0}\right) \\ &= M_0, \end{aligned}$$

where the last equality is obtained by an application of Doob's maximal identity (Lemma 5.9). To conclude, it suffices to note that

$$M_0 = 1 - \exp\left(-\int_0^\infty \frac{dx}{\varphi_u^{2\mu}(x)}\right) = 1 - \exp\left(-\int_0^u \frac{dx}{\varphi_u^{2\mu}(x)}\right).$$

□

**Corollary 5.11.** *With the hypotheses of the above proposition, the following integral criterion holds: if  $\int_0^\infty \frac{dx}{\varphi^{2\mu}(x)} = \infty$ , then:*

$$\mathbb{P}(\forall A > 0, \exists t \geq A, R_t > \varphi(L_t)) = 1;$$

if  $\int_0^\infty \frac{dx}{\varphi^{2\mu}(x)} < \infty$ , then:

$$\mathbb{P}(\forall A > 0, \exists t \geq A, R_t > \varphi(L_t)) = 0.$$

**Proof.** The event  $\{\forall A > 0, \exists t \geq A, R_t > \varphi(L_t)\}$  is in the tail sigma field, so its probability is 0 or 1. From Proposition 5.10, if

$$\int_0^\infty \frac{dx}{\varphi^{2\mu}(x)} < \infty,$$

then  $\mathbb{P}(\exists t \geq 0, R_t > \varphi(L_t)) < 1$ , and hence  $\mathbb{P}(\forall A > 0, \exists t \geq A, R_t > \varphi(L_t)) = 0$ . Next, if  $\int_0^\infty \frac{dx}{\varphi^{2\mu}(x)} = \infty$ , then, again from Proposition 5.10, we have, for all  $u \geq 0$ :

$$\mathbb{P}(\exists t \geq \tau_u, R_t > \varphi(L_t)) = 1.$$

To conclude, it suffices to notice that  $\{\forall A > 0, \exists t \geq A, R_t > \varphi(L_t)\}$  is the decreasing limit, as  $u \rightarrow \infty$ , of the events  $\{\exists t \geq \tau_u, R_t > \varphi(L_t)\}$ . □

**Remark 5.12.** *The results of Proposition 5.10 and its corollary can also be obtained with the help of excursion theory for the standard Brownian Motion and then for the Bessel processes by time change techniques.*

## Acknowledgements

I would like to thank Marc Yor for many fruitful discussions and for sharing with me some of his unpublished notes (Donati-Martin et al. (2006)).

## References

- G. E. Andrews, R. Askey and R. Roy. *Special functions*. Cambridge University Press. Encyclopedia of Mathematics and its Applications, 71, Cambridge (1999).
- J. Azéma. Quelques applications de la théorie générale des processus. *Invent. Math.* **18**, 293–336 (1972).
- J. Azéma. Sur les fermés aléatoires. *Lecture Notes in Math.*, 1123 **39**, 397–495 (1985). Séminaire de probabilités, XIX.
- J. Azéma, T. Jeulin, F. Knight and M. Yor. Le théorème d'arrêt en une fin d'ensemble prévisible. *Lecture Notes in Math.*, 1557 pages 133–158 (1993). Séminaire de Probabilités, XXVII.
- J. Azéma and M. Yor. Sur les zéros des martingales continues. *Lecture Notes in Math.*, 1526 pages 248–306 (1992). Séminaire de Probabilités, XXVI.
- M. Barlow, J. Pitman and M. Yor. Une extension multidimensionnelle de la loi de l'arc sinus. *Lecture Notes in Math.*, 1372 pages 294–314 (1989). Sminaire de Probabilités, XXIII.
- J. Bertoin. Excursions of a  $bes_0(d)$  and its drift term ( $0 < d < 1$ ). *Prob. Theory Related Fields* **84**, 231–250 (1990).
- Ph. Biane and M. Yor. Valeurs principales associées aux temps locaux browniens. *Bull. Sci. Math. (2)* **111**, 23–101 (1987).
- A. N. Borodin and P. Salminen. *Handbook of Brownian motion—facts and formulae*. Birkhuser Verlag. Probability and its Applications, Basel, second edition edition (2002).
- C. Dellacherie, B. Maisonneuve and P. Meyer. *Probabilités et Potentiel. Chapitres XVII–XXIV*. Publications de l'Institut de Mathématiques de l'Université de Strasbourg, XX. Hermann, Paris (1992).
- C. Donati-Martin, B. Roynette, P. Vallois and M. Yor. On constants related to the choice of the local time at 0, and the corresponding itô measure for bessel processes with dimension  $d = 2(1 - \alpha)$ ,  $0 < \alpha < 1$  (2006). In preparation.
- E. B. Dynkin. Some limit theorems for sums of independent random variables with infinite mathematical expectations. *Inst. Math. Statist. and Amer. Math. Soc., Providence, R.I.* **1**, 171–189 (1961).
- R. J. Elliott, M. Jeanblanc and M. Yor. On models of default risk. *INFORMS Applied Probability Conference* **10 no. 2**, 179–195 (2000).
- R. K. Gettoor. Excursions of a markov process. *Ann. Probab.* **7 no. 2**, 244–266 (1979).
- K. Itô. Poisson point processes attached to markov processes. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* **1 no. 2**, 225–239 (1972).
- K. Itô and H. P. McKean Jr. *Diffusion processes and their sample paths. Second printing, corrected*. Die Grundlehren der mathematischen Wissenschaften, Band 125. Springer-Verlag, Berlin (1974).
- Th. Jeulin. *Semi-martingales et grossissement d'une filtration*. Lecture Notes in Mathematics, 833. Springer, Berlin (1980).
- D. Khoshnevisan. Lévy classes and self-normalization. *Electron. J. Probab.* **1** 1, 18pp.(eletic) (1996).
- F. B. Knight. Local variation of diffusion in local time. *Ann. Probability* **1**, 1026–1034 (1973).

- F. B. Knight. On the sojourn times of killed brownian motion. *Lecture Notes in Math.*, 649 **42**, 428–445 (1978). Séminaire de Probabilités, XII (Univ. Strasbourg, Strasbourg, 1976/1977).
- P. Lévy. Sur un problème de m. marcinkiewicz. *C.R.A.S.* **208**, 318–321 (1939). Errata p. 776.
- R. Mansuy and M. Yor. Random times and (enlargement of filtrations) in a brownian setting (2005). To be published in *Lecture Notes in Mathematics*, Springer.
- H. P. McKean Jr. The besel motion and a singular integral equation. *Mem. Coll. Sci. Univ. Kyoto. Ser. A Math.* **33**, 317–322 (1960).
- A. Nikeghbali. Non stopping times and stopping theorems (2005). Available on the ArXiv.
- A. Nikeghbali. A class of remarkable submartingales (2006). To appear in *Stochastic Process. Appl.*
- A. Nikeghbali and M. Yor. A definition and some characteristic properties of pseudo-stopping times. *Ann. Probab.* **33**, 1804–1824 (2005).
- A. Nikeghbali and M. Yor. Doob's maximal identity, multiplicative decompositions and enlargements of filtrations (2006). To appear in *Illinois J. Math.*
- C. Rainer. Projection d'une diffusion sur sa filtration lente. *Lecture Notes in Math.*, 1626 **1**, 228–242 (1996). Séminaire de Probabilités, XXX.
- D. Revuz and M. Yor. *Continuous martingales and Brownian motion*. Grundlehren der Mathematischen Wissenschaften 293. Springer-Verlag, Berlin, third edition edition (1999).
- B. Roynette, P. Vallois and M. Yor. Penalizing a  $bes(d)$  process ( $0 < d < 2$ ) with a function of its local time at 0;  $v$  (2006). In preparation.
- Z. Shi. Brownian motion normalized by maximum local time. *Stochastic Process. Appl.* **65**, 217–231 (1996).
- M. Yor. *Some aspects of Brownian motion. Part I. Some special functionals*. Birkhäuser Verlag. Lectures in Mathematics ETH Zürich., Basel, third edition edition (1992).
- M. Yor. *Some aspects of Brownian motion. Part II. Some recent martingale problems*. Birkhäuser Verlag. Lectures in Mathematics ETH Zürich., Basel, third edition edition (1997).