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ESTIMATES ON PATH DELOCALIZATION FOR COPOLYMERS AT SELECTIVE INTERFACES

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Abstract. Starting from the simple symmetric random walk $\{S_n\}_n$, we introduce a new process whose path measure is weighted by a factor $\exp\left(\lambda \sum_{n=1}^N (\omega_n + h) \operatorname{sign}(S_n)\right)$, with $\lambda, h \geq 0$, $\{\omega_n\}_n$ a typical realization of an IID process and N a positive integer. We are looking for results in the large N limit. This factor favors $S_n > 0$ if $\omega_n + h > 0$ and $S_n < 0$ if $\omega_n + h < 0$. The process can be interpreted as a model for a random heterogeneous polymer in the proximity of an interface separating two selective solvents. It has been shown [6] that this model undergoes a (de)localization transition: more precisely there exists a continuous increasing function $\lambda \longmapsto h_c(\lambda)$ such that if $h < h_c(\lambda)$ then the model is localized while it is delocalized if $h \geq h_c(\lambda)$. However, localization and delocalization were not given in terms of path properties, but in a free energy sense. Later on it has been shown that free energy localization does indeed correspond to a (strong) form of path localization [3]. On the other hand, only weak results on the delocalized regime have been known so far.
We present a method, based on concentration bounds on *suitably restricted* partition functions, that yields much stronger results on the path behavior in the interior of the delocalized region, that is for $h > h_c(\lambda)$. In particular we prove that, in a suitable sense, one cannot expect more than $O(\log N)$ visits of the walk to the lower half plane. The previously known bound was $o(N)$. Stronger $O(1)$–type results are obtained deep inside the delocalized region.

The same approach is also helpful for a different type of question: we prove in fact that the limit as λ tends to zero of $h_c(\lambda)/\lambda$ exists and it is independent of the law of ω_1, at least when the random variable ω_1 is bounded or it is Gaussian. This is achieved by interpolating between this class of variables and the particular case of ω_1 taking values ± 1 with probability 1/2, treated in [6].

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1. Introduction

1.1. **The model and its free energy.** Let $S = \{S_n\}_{n=0,1,\ldots}$ be a simple random walk: $S_0 = 0$ and $\{S_j - S_{j-1}\}_{j \in \mathbb{N}}$ a sequence of IID random variables with $\mathbf{P}(S_1 = \pm 1) = 1/2$. We denote by Ω the set of all random walk trajectories. For $\lambda \geq 0$, $h \geq 0$, $N \in 2\mathbb{N}$ and $\omega = \{\omega_n\}_{n=1,2,\ldots} \in \mathbb{R}^{\mathbb{N}}$ we introduce the *copolymer* measures

$$\frac{\mathrm{d}\mathbf{P}^a_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) = \frac{1}{\widetilde{Z}^a_{N,\omega}} \exp\left(\lambda \sum_{n=1}^N (\omega_n + h) \operatorname{sign}(S_n)\right) \mathbf{1}_{\Omega^a_N}, \tag{1.1}$$

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with $a = \mathtt{f}$ (free case) or $a = \mathtt{c}$ (constrained case), $\Omega_N^{\mathtt{f}} = \Omega$, $\Omega_N^{\mathtt{c}} = \{S \in \Omega : S_N = 0\}$. $\widetilde{Z}_{N,\omega}^a$ is the partition function and $\operatorname{sign}(S_{2n})$ is set to be equal to $\operatorname{sign}(S_{2n-1})$ for any n such that $S_{2n} = 0$.

The sequence ω is chosen as a typical realization of an IID sequence of random variables, still denoted by $\omega = \{\omega_n\}_n$. We call $\mathbb{P}$ the law of ω. Further hypotheses on ω are summed up by:

Definition 1.1.

- **Basic assumptions**: *$\omega_1 \sim -\omega_1$ and $\mathrm{M}(t) := \mathbb{E}[\exp(t\omega_1)] < \infty$ for t in a neighborhood of zero. Without loss of generality we assume $\mathbb{E}\left[{\omega_1}^2\right] = 1$.*
- **Deviation inequality above the mean**: *there exists a positive constant C such that for every N, for every Lipschitz and convex function $g : \mathbb{R}^N \to \mathbb{R}$ with $g(\omega) := g(\omega_1, \ldots, \omega_N) \in \mathbb{L}^1(\mathbb{P})$ and $t \geq 0$*

$$\mathbb{P}\left(g(\omega) - \mathbb{E}[g(\omega)] \geq t\right) \leq C \exp\left(-\frac{t^2}{C\|g\|_{\mathrm{Lip}}^2}\right), \tag{1.2}$$

where $\|g\|_{\mathrm{Lip}}$ is the Lipschitz constant of g with respect to the Euclidean distance.

The deviation inequality (1.2) is known to hold with a certain generality: its validity for the Gaussian case $\omega_1 \sim \mathcal{N}(0,1)$ and for the case of bounded random variables is by now a classical result, see [19], [15] and [21]. However one can go beyond: it holds in particular whenever the law of ω_1 satisfies the log–Sobolev inequality [15] and in that case of course C depends on the log–Sobolev constant. As a matter of fact, in all the cases we have mentioned not only a deviation inequality above the mean holds, but also below, and therefore one has the full concentration inequality. A necessary and sufficient condition for the log–Sobolev inequality to hold can be found in [4]. In order to be more explicit we point out that if ω_1 has a density of the type $\exp(-V)$, with V bounded from below and strictly convex outside a finite interval, the law of ω_1 satisfies the log–Sobolev inequality with a finite constant and therefore (1.2) holds.

Under the basic assumptions on ω the *quenched free energy* of the system exists, namely the limit

$$f(\lambda, h) := \lim_{N\to\infty} \frac{1}{N} \log \widetilde{Z}_{N,\omega}^a, \tag{1.3}$$

exists in the $\mathbb{P}(\mathrm{d}\omega)$–almost sure sense and in the $\mathbb{L}^1(\mathbb{P})$ sense. This existence result can be proven via super–additivity arguments (we refer to [12] for the details) and the method shows also that $f(\lambda, h)$ is non-random and independent of the choice of a.

We observe that

$$f(\lambda, h) \geq \lambda h. \tag{1.4}$$

The proof of such a result is elementary: if we set $\Omega_N^+ = \{S \in \Omega : S_n > 0 \text{ for } n = 1, 2, \ldots, N\}$ we have

$$\frac{1}{N} \log \widetilde{Z}_{N,\omega}^{\mathtt{f}} \geq \frac{1}{N} \log \mathbf{E}\left[\exp\left(\lambda \sum_{n=1}^N (\omega_n + h) \operatorname{sign}(S_n)\right); \Omega_N^+\right]$$

$$= \frac{\lambda}{N} \sum_{n=1}^N (\omega_n + h) + \frac{1}{N} \log \mathbf{P}\left(\Omega_N^+\right) \stackrel{N\to\infty}{\longrightarrow} \lambda h, \tag{1.5}$$

where the limit is taken in the almost sure sense: we have applied the strong law of large numbers and the well known fact that $\mathbf{P}\left(\Omega_N^+\right)$ behaves like $N^{-1/2}$ for N large [10, Ch. 3]. The observation (1.4), above all if viewed in the light of its proof, suggests the following partition of the parameter space (or *phase diagram*):

- The localized region: $\mathcal{L} = \{(\lambda, h) : f(\lambda, h) > \lambda h\}$;
- The delocalized region: $\mathcal{D} = \{(\lambda, h) : f(\lambda, h) = \lambda h\}$.

We sum up the known results on the phase diagram:

Theorem 1.2. *Under the basic assumptions on ω there exists an increasing function $h_c : [0, \infty) \longrightarrow [0, \infty]$ such that*

$$\mathcal{L} = \{(\lambda, h) : h < h_c(\lambda)\} \quad \text{and} \quad \mathcal{D} = \{(\lambda, h) : h \geq h_c(\lambda)\}. \tag{1.6}$$

$h_c(\cdot)$ is continuous if it takes values in $[0, \infty)$, otherwise it is continuous in $[0, \sup\{\lambda : h_c(\lambda) < \infty\})$. Moreover

$$\underline{h}(\lambda) := \frac{1}{4\lambda/3} \log \mathrm{M}\left(4\lambda/3\right) \leq h_c(\lambda) \leq \frac{1}{2\lambda} \log \mathrm{M}\left(2\lambda\right) =: \overline{h}(\lambda). \tag{1.7}$$

Part of the results in Theorem 1.2 have been proven in [6]. The present version takes into account the improvements brought by [5]. For the rest of the paper we will refer to $\{(\lambda, h) : h > \overline{h}(\lambda)\} \subset \mathcal{D}$ as *strongly delocalized region*.

The bounds in (1.7) yield that $2/3 \leq \liminf_{\lambda \searrow 0} h_c(\lambda)/\lambda$ and $\limsup_{\lambda \searrow 0} h_c(\lambda)/\lambda \leq 1$. In [6] it has been shown that the limit of $h_c(\lambda)/\lambda$ exists in the particular case of ω_1 taking values ± 1 and it can be expressed in terms of a suitable Brownian copolymer, suggesting thus a universality of this result. The techniques we develop allow to interpolate between the ± 1 case and more general cases, namely:

Theorem 1.3. *The slope of the critical curve at the origin,*

$$m_c := \lim_{\lambda \searrow 0} \frac{h_c(\lambda)}{\lambda}, \tag{1.8}$$

exists and does not depend on the law of ω_1, provided that ω_1 is either a bounded symmetric variable of unit variance or a standard Gaussian variable.

Theorem 1.3 is proven in Section 3. It is in the line of the interpolation results [13] and [7], but here one needs to have a more explicit control of the λ dependence of the error made in the interpolation procedure. It turns out that the approach that we propose here for path estimates yields also this control. It would be interesting to investigate whether a suitable refinement of the strategy we propose or an extension of the approach in [6], or possibly a combination of both, would allow to obtain a better result, removing the rather unnatural boundedness requirement on the random variables, which arises from our application of the interpolation method.

1.2. **From free energy to path behavior.** The polymer measures $\mathbf{P}^a_{N,\omega}$ have been introduced in [18] and [6] motivated by earlier theoretical physics works, in particular by [11] (for updated physics developments see [16] and references therein). It is a model for an heterogeneous polymer, constituted by charged units (*monomers*). The polymer lives in a solvent which is also heterogeneous: it is made of two solvents in a state in which a flat interface is present (an example familiar to everybody is the case of an oil/water interface). The sign of the charge determines the preference of a monomer for one solvent or the other and the absolute value of the charge plays a role in the intensity of such a preference. Moreover, in general the situation may be asymmetric: there may be more charges of a certain sign or the intensity of the solvent–monomer interaction may not be invariant under the change of sign of the charge (we are modeling this second situation and h is the asymmetry parameter). What we want to analyze is which of the two following scenarios prevails:

(1) The polymer places *most of* the monomers in their preferred solvent (in the model the n^{th}–monomer is preferably above the x–axis, that plays the role of the interface, if $\omega_n + h > 0$, and below if $\omega_n + h < 0$). This forces of course the polymer to stick close to the interface and this is the intuitive concept of a localized polymer path.
(2) The polymer lies almost fully in one of the two solvents. Intuitively that may happen in an asymmetric case. In such a situation one expects the polymer to wander away from the interface, since it would be undergoing a repulsion effect of *entropic origin*: the trajectories staying close to the interface are very few with respect to the trajectories exploring freely a half–space. This is for us a delocalized behavior.

In principle there is a third reasonable scenario: the case in which the polymer has large fluctuations between the two solvents. It turns out that, at least if we disregard the critical case $h = h_c(\lambda)$, this situation is possible only in the trivial $\lambda = 0$ case. Moreover scenario (1) is effectively observed if $(\lambda, h) \in \mathcal{L}$ and scenario (2) is verified at least in the interior of $\mathcal{D}$. But let us be more precise and let us sum up the state of the art on this issue:

(1) If $(\lambda, h) \in \mathcal{L}$ then very strong localization results are available. The keyword in this case is *tightness* and one should really think of a path essentially as being at distance $O(1)$ from the interface. The precise statements are rather involved, due to the presence of atypical finite stretches in any typical ω, and we prefer to refer to [18], [1] and [3].
(2) The results in the delocalized regime are much more meager. All the same the following result is available [3]: if $(\lambda, h) \in \overset{\circ}{\mathcal{D}}$ then for every L

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{P}^a_{N,\omega} \left(S_n > L\right) = 1, \qquad \mathbb{P}(\,\mathrm{d}\omega) - \text{a.s.}. \tag{1.9}$$

While the delocalization result (1.9) is in sharp contrast with the localized scenario (1), it is far from matching the very strong delocalization results available in polymer models without disorder (for example in the well known $(1 + 1)$–dimensional wetting models, see, e.g., [12], [14] and [9]): loosely stated one expects that a typical delocalized path in the limit of $N \to \infty$ has only a finite number of visits to the lower half–plane and, as a consequence, a Brownian scaling result should hold with convergence to well known processes like the Brownian meander or the Bessel(3) bridge according to whether $a = \mathtt{f}$ or $a = \mathtt{c}$ (see Section 4 for more precision on this issue). These are reasonable conjectures, supported also by the fact that in the localized regime the results in the disordered model match what one observes in the non-disordered case. One should however stress the essential difference between the localized and delocalized regions: in the first case one is in a large deviation regime – $\mathbf{P}^a_{N,\omega}$ charges a set of trajectories which has exponentially small probability with respect to $\mathbf{P}$ – while this is not the case in the delocalized regime. The large deviation machinery does not seem to go beyond results of the type (1.9).

The purpose of this paper is to present an approach, based on concentration inequalities, that yields results that go well beyond the *density* result (1.9).

In order to state our main theorem we need some notation: we set $\Delta_n = (1-\text{sign}(S_n))/2$ and introduce the random variable $\mathcal{N} = \sum_{n=1}^N \Delta_n$, counting how many monomers are in the lower half–plane (unfavorable solvent). We introduce also the random set $\mathcal{A} := \{n \le N : \Delta_n = 1\} \cup \{0\}$ and note that the (even) number $\max \mathcal{A}$ identifies the point of last exit of S from the lower half–plane.

Theorem 1.4. *Under the basic assumptions on ω we have that*

(1) *if $h > \overline{h}(\lambda)$ there exists c such that*

$$\mathbb{E}\,\mathbf{P}^{\mathtt{f}}_{N,\omega}\left(\max \mathcal{A} \le \ell\right) \ge 1 - c/\sqrt{\ell+1}, \tag{1.10}$$

for every N and every non-negative integer $\ell \le N$. Analogously,

$$\mathbb{E}\,\mathbf{P}^{\mathtt{c}}_{N,\omega}\left(\max\{\mathcal{A}\cap[0,N/2]\} \le \ell_1 \, or\ \min\{\mathcal{A}\cap[N/2,N]\} \ge N-\ell_2\right) \ge 1 - \frac{c}{\sqrt{\ell_1\ell_2+1}}, \tag{1.11}$$

for every N, every $\ell_1 \le N/2$ and $\ell_2 \le N/2$, with the convention that $\min(\emptyset) = N$. *Moreover*

$$\mathbb{E}\,\mathbf{P}^{a}_{N,\omega}\left(\mathcal{N} \ge m\right) \le \frac{1}{c}\exp\left(-cm\right), \tag{1.12}$$

both for $a = \mathtt{f}$ and $a = \mathtt{c}$, for every N and every $m \in \mathbb{N}$.

(2) *If the deviation inequality holds then for $h > h_c(\lambda)$ there exist two positive constants c and q such that*

$$\mathbb{E}\,\mathbf{P}^{a}_{N,\omega}\left(\mathcal{N} \ge m\right) \le \exp\left(-cm\right), \tag{1.13}$$

both for $a = \mathtt{f}$ and $a = \mathtt{c}$, for every N and every $m \ge q \log N$.

We refer to Section 4 for a thorough discussion on how these results relate to what is expected to happen, with a particular attention to scaling limits and almost sure results. In the same section one finds also some further considerations on the delocalized path behavior.

Remark 1.5. The methods of proof of Theorem 1.4 are applicable in more general contexts. We mention in particular the case of disordered pinning or wetting. Consider in particular the case of a model defined like in (1.1), but with $\text{sign}(S_n)$ replaced by $\mathbf{1}_{\{0\}}(S_n)$. In spite of the formal resemblance, this is a profoundly different model and, in order to deal with interesting phenomena, one has to allow h to take negative values too. In [2] it is proven that the free energy of the model exists and it is non negative and, exactly in analogy with $f(\lambda,h) - \lambda h$ in our setting, one defines the localization and delocalization regimes depending on whether the free energy is positive or zero. Moreover for any $\lambda > 0$ the transition takes place at a critical value h_c of the parameter h and $h_c \in [h_c^a, 0)$, where $h_c^a < 0$ is the critical value for the corresponding annealed model, a homopolymer model that can be solved exactly. It is expected, but not proven, that $h_c > h_c^a$ (see references in [2]). Theorem 1.4 holds for this disordered pinning model provided one changes the definition of $\mathcal{N}$ to $\mathcal{N} := \sum_{n=1}^N \mathbf{1}_{\{0\}}(S_n)$.

2. Proof of Theorem 1.4

It is convenient to consider a modified partition function. To this purpose we observe that we may write

$$\frac{\mathrm{d}\mathbf{P}^a_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) \,=\, \frac{1}{Z^a_{N,\omega}} \exp\left(-2\lambda \sum_{n=1}^N (\omega_n + h)\,\Delta_n\right) \mathbf{1}_{\Omega^a_N}(S)\,, \tag{2.1}$$

where $Z^a_{N,\omega} = Z_{N,\omega}(\Omega^a_N)$, with the notation

$$Z_{N,\omega}\left(\widetilde{\Omega}\right) \,=\, \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^N (\omega_n + h)\,\Delta_n\right); \widetilde{\Omega}\right], \tag{2.2}$$

for $\widetilde{\Omega} \subset \Omega$. Likewise we introduce $\mathrm{F}_{N,\omega}\left(\widetilde{\Omega}\right) := (1/N)\log Z_{N,\omega}\left(\widetilde{\Omega}\right)$ and $\mathrm{F}^a_{N,\omega} := \mathrm{F}_{N,\omega}(\Omega^a_N)$. Notice that $\mathbb{P}(\mathrm{d}\omega)$–a.s. we have that $Z^a_{N,\omega} \asymp \widetilde{Z}^a_{N,\omega} \exp(-\lambda h N)$, where $\asymp$ denotes the Laplace asymptotic equivalence, which means that the $\mathbb{P}(\mathrm{d}\omega)$–a.s. limit of $\mathrm{F}^a_{N,\omega}$ equals $f(\lambda,h) - \lambda h =: \mathrm{F}(\lambda,h)$.

2.1. **The concentration lemma.** For $m \in \{0,2,\dots,N\}$ let us consider an event $\Omega_m \subset \Omega$ such that $\mathbf{P}(\Omega_m) > 0$ and such that $\mathcal{N} = m$ for every $S \in \Omega_m$. If the distribution of ω satisfies the deviation inequality, we have

Lemma 2.1. *For every N, every $m \in \{0,2,\dots,N\}$ and every $u \geq 0$ we have*

$$\mathbb{P}\left(\mathrm{F}_{N,\omega}(\Omega_m) - \mathbb{E}\left[\mathrm{F}_{N,\omega}(\Omega_m)\right] \geq u\right) \leq C \exp\left(-\frac{u^2 N^2}{4C\lambda^2 m}\right). \tag{2.3}$$

Proof of Lemma 2.1. By the deviation inequality it suffices to show that for every $\omega, \omega' \in \mathbb{R}^N$ we have

$$\left|\mathrm{F}_{N,\omega}(\Omega_m) - \mathrm{F}_{N,\omega'}(\Omega_m)\right| \,\leq\, \frac{2\lambda\sqrt{m}}{N}\left\|\omega - \omega'\right\|, \tag{2.4}$$

where $\|\cdot\|$ is the Euclidean norm of $\cdot$. In order to establish (2.4) we introduce $\omega_t = t\omega + (1-t)\omega'$ and, taking the derivative with respect to t and integrating back, after the use of the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\left|\mathrm{F}_{N,\omega}(\Omega_m) - \mathrm{F}_{N,\omega'}(\Omega_m)\right| &= \left|\int_0^1 \sum_{n=1}^N \frac{(-2\lambda)}{N} \mathbf{E}_{N,\omega_t}\left[\Delta_n \middle| \Omega_m\right]\left(\omega_n - \omega'_n\right)\,\mathrm{d}t\right| \\ &\leq \frac{2\lambda}{N}\sqrt{\sup_t \sum_{n=1}^N \left(\mathbf{E}_{N,\omega_t}\left[\Delta_n \middle| \Omega_m\right]\right)^2 \sum_{n=1}^N \left(\omega_n - \omega'_n\right)^2}.\end{aligned} \tag{2.5}$$

Since $\sum_{n=1}^N (\mathbf{E}_{N,\omega_t}[\Delta_n|\Omega_m])^2 \leq m$, the proof is complete. $\overset{\text{Lemma 2.1}}{\square}$

2.2. **The delocalized region.** *Proof of Theorem 1.4, part (2).* In this proof we set $\mathrm{F}^a_{N,\omega}(\lambda,h) := (1/N)\log Z^a_{N,\omega}$ and

$$\mathrm{F}^a_{N,\omega}(\lambda,h;m) := \frac{1}{N}\log Z_{N,\omega}\left(\Omega^a_N \cap \{\mathcal{N} = m\}\right). \tag{2.6}$$

Since for $(\lambda, h) \in \mathcal{D}$ we have $\textsc{f}(\lambda, h) = 0$ and since $\left\{ N\mathbb{E}\left[\textsc{f}^{\mathtt{c}}_{N,\omega}(\lambda,h)\right]\right\}_N$ is superadditive, so that $\lim_{N\to\infty} \mathbb{E}\left[\textsc{f}^{\mathtt{c}}_{N,\omega}(\lambda,h)\right] = \sup_N \mathbb{E}\left[\textsc{f}^{\mathtt{c}}_{N,\omega}(\lambda,h)\right]$, we have that

$$\mathbb{E}\left[\textsc{f}^{\mathtt{c}}_{N,\omega}(\lambda,h)\right] \le 0, \tag{2.7}$$

for every N. The superadditivity is a direct consequence of the Markovian character of S, see [6] or [12] for the details.

Now let us fix $(\lambda, h) \in \overset{\circ}{\mathcal{D}}$ and $\varepsilon > 0$ such that $(\lambda, h-\varepsilon) \in \mathcal{D}$. Observe that for every ω

$$\textsc{f}^{\mathtt{c}}_{N,\omega}(\lambda,h;m) \ge -\lambda\varepsilon m/N \quad \Longleftrightarrow \quad \textsc{f}^{\mathtt{c}}_{N,\omega}(\lambda,h-\varepsilon;m) \ge \lambda\varepsilon m/N, \tag{2.8}$$

but $\mathbb{E}\left[\textsc{f}^{\mathtt{c}}_{N,\omega}(\lambda,h-\varepsilon;m)\right] \le \mathbb{E}\left[\textsc{f}^{\mathtt{c}}_{N,\omega}(\lambda,h-\varepsilon)\right] \le 0$, so that, by Lemma 2.1, we have

$$\begin{aligned}\mathbb{P}\left(\textsc{f}^{\mathtt{c}}_{N,\omega}(\lambda,h;m) \ge -\lambda\varepsilon m/N\right) &= \mathbb{P}\left(\textsc{f}^{\mathtt{c}}_{N,\omega}(\lambda,h-\varepsilon;m) \ge \lambda\varepsilon m/N\right)\\ &\le C\exp\left(-\varepsilon^2 m/4C\right).\end{aligned} \tag{2.9}$$

From this we directly obtain that if we set $E_{\overline{m}} = \{$there exists $m \ge \overline{m}$ such that $\textsc{f}^{\mathtt{c}}_{N,\omega}(\lambda,h;m) \ge -\lambda\varepsilon m/N\}$ then

$$\mathbb{P}\left(E_{\overline{m}}\right) \le c_1 \exp\left(-c_2\overline{m}\right). \tag{2.10}$$

We can now evaluate the tail of $\mathcal{N}$. For $\omega \in E^{\complement}_{\overline{m}}$, with $\Omega_m = \{\mathcal{N} = m,\, S_N = 0\}$, we have

$$\mathbf{P}^{\mathtt{c}}_{N,\omega}\left(\mathcal{N} \ge \overline{m}\right) = \frac{\sum_{m\ge\overline{m}} Z_{N,\omega}\left(\Omega_m\right)}{Z^{\mathtt{c}}_{N,\omega}} \le c_3 N^{3/2} \sum_{m\ge\overline{m}} \exp\left(-\lambda\varepsilon m\right) \le c_4 N^{3/2}\exp\left(-\lambda\varepsilon\overline{m}\right), \tag{2.11}$$

where we have used that $Z^{\mathtt{c}}_{N,\omega} \ge \mathbf{P}(S_n > 0,\, n = 1,\ldots,N-1,\, S_N = 0) \ge 1/(c_3 N^{3/2})$ [10, Ch. 3]. The estimates (2.10) and (2.11) readily imply

$$\mathbb{E}\,\mathbf{P}^{\mathtt{c}}_{N,\omega}\left(\mathcal{N} \ge \overline{m}\right) \le c_5 N^{3/2}\exp\left(-c_6\overline{m}\right). \tag{2.12}$$

The choice of $\overline{m} \ge q\log N$, for q sufficiently large completes the proof for the case of $\mathbf{P}^{\mathtt{c}}_{N,\omega}$.

For the free endpoint case $\mathbf{P}^{\mathtt{f}}_{N,\omega}$ one recalls that in [6] (or in [12]) it is proven that there exists a positive constant c such that

$$Z^{\mathtt{f}}_{N,\omega} \le cNZ^{\mathtt{c}}_{N,\omega}, \tag{2.13}$$

for every ω and every N. Therefore, by (2.7), we have

$$\mathbb{E}\left[\textsc{f}^{\mathtt{f}}_{N,\omega}(\lambda,h)\right] \le \frac{1}{N}\log(cN), \tag{2.14}$$

and therefore formulas (2.8), (2.9) and (2.10) hold if we replace the quantity $\textsc{f}^{\mathtt{c}}_{N,\omega}(\lambda,h;m)$ with $\textsc{f}^{\mathtt{f}}_{N,\omega}(\lambda,h;m) - (\log cN)/N$. It suffices therefore to observe that $\inf_{N,\omega} N^{1/2} Z^{\mathtt{f}}_{N,\omega} \ge \inf_N N^{1/2}\mathbf{P}(S_n > 0,\, n = 1,\ldots,N) > 0$ to conclude that (2.11) holds unchanged if $a = \mathtt{c}$ is replaced by $a = \mathtt{f}$ and $\Omega_m = \{\mathcal{N} = m\}$, apart for the explicit values of the multiplicative constants (which we have not tracked anyway). The proof is therefore complete. Theorem 1.4(2)

□

2.3. **The strongly delocalized region.** *Proof of Theorem 1.4, part (1).* We start by proving (1.12). We compute by means of the Fubini–Tonelli theorem:

$$\begin{aligned}\mathbb{E}\left[Z_{N,\omega}\left(\Omega_m\right)\right] &= \mathbf{E}\mathbb{E}\left[\exp\left(-2\lambda\sum_{n=1}^{N}\left(\omega_n+h\right)\Delta_n\right);\,\Omega_m\right]\\ &= \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}\left(\log M(2\lambda\Delta_n)-2\lambda h\Delta_n\right)\right);\,\Omega_m\right]\\ &= \mathbf{E}\left[\exp\left(-2\lambda\left(h-\log M(2\lambda)/2\lambda\right)\sum_{n=1}^{N}\Delta_n\right)\bigg|\Omega_m\right]\mathbf{P}\left(\Omega_m\right),\end{aligned} \tag{2.15}$$

where Ω_m is like in Lemma 2.1. Since $\mathcal{N}=\sum_{n=1}^{N}\Delta_n=m$ on Ω_m we have

$$\mathbb{E}\left[Z_{N,\omega}\left(\Omega_m\right)\right] = \mathbf{P}\left(\Omega_m\right)\exp\left(-\beta m\right), \tag{2.16}$$

with $\beta := 2\lambda\left(h-\log M(2\lambda)/2\lambda\right)$, so $\beta>0$ in the strongly delocalized regime.

We can now estimate the tail behavior of $\mathcal{N}$, averaged over the disorder ω. We first consider the free case: set $\Omega_m=\{\mathcal{N}=m\}$ and $P_N(m):=\mathbf{P}\left(\Omega_m\right)$. We have

$$\mathbb{E}\,\mathbf{P}^{\mathtt{f}}_{N,\omega}\left(\mathcal{N}\geq\overline{m}\right) = \mathbb{E}\left[\frac{\sum_{m\geq\overline{m}}Z_{N,\omega}\left(\Omega_m\right)}{Z^{\mathtt{f}}_{N,\omega}}\right] \leq \sum_{m\geq\overline{m}}\exp\left(-\beta m\right)\frac{P_N(m)}{P_N(0)}, \tag{2.17}$$

where we have used once again that $Z^{\mathtt{f}}_{N,\omega}\geq\mathbf{P}\left(\mathcal{N}=0\right)=P_N(0)$ and we have applied (2.16). The proof of (1.12) in the free endpoint case is completed once we observe that $P_N(m)\leq P_N(0)$, a fact that can be easily extracted from the exact expression of $P_N(m)$ [10, Ch. 3].

In the constrained endpoint case one takes $\Omega_m=\{\mathcal{N}=m,\,S_N=0\}$ and the steps are then identical. Notice however that in this case $P_N(m)=P_N(0)$ every m [10, Ch. 3].

We turn now to the proof of (1.10) and (1.11). Like in (2.15), by explicit computation we have

$$\mathbb{E}\left[Z^a_{N,\omega}\right] = \mathbf{E}\left[\exp\left(-\beta\mathcal{N}\right);\,\Omega^a_N\right]. \tag{2.18}$$

Let us observe preliminarily that we have [10, Ch. 3]:

$$\mathbf{P}\left(\mathcal{N}=k\right)\leq\frac{c}{\sqrt{N}}\quad\text{ and }\quad\mathbf{P}\left(\mathcal{N}=k,S_N=0\right)\leq\frac{c}{N^{3/2}}, \tag{2.19}$$

and from this one easily finds a constant $C>0$ such that

$$\mathbb{E}\left[Z^{\mathtt{f}}_{N,\omega}\right]\leq\frac{C}{N^{1/2}}\quad\text{ and }\quad\mathbb{E}\left[Z^{\mathtt{c}}_{N,\omega}\right]\leq\frac{C}{N^{3/2}}. \tag{2.20}$$

On the other hand we know (and used several times by now) that there exists $c>0$ such that

$$Z^{\mathtt{f}}_{N,\omega}\geq c/N^{1/2}\quad\text{ and }\quad Z^{\mathtt{c}}_{N,\omega}\geq c/N^{3/2} \tag{2.21}$$

for every ω.

We focus now on the proof of (1.10). Let us call F_ℓ the event of the random walk trajectories for which there exists $n\in\{\ell,\dots,N\}$ such that $S_\ell=0$. By conditioning on the last hitting time of zero before time N we obtain

$$\mathbf{P}^{\mathtt{f}}_{N,\omega}\left(F_\ell\right) = \frac{1}{Z^{\mathtt{f}}_{N,\omega}} \sum_{l=\ell}^{N} Z^{\mathtt{c}}_{l,\omega} \mathbf{P}\left(S_n > 0 \text{ for } n = 1,2,\ldots,(N-l)\right) \left(1 + \exp\left(-2\lambda \sum_{n=l+1}^{N} (\omega_n + h)\right)\right). \tag{2.22}$$

Since the denominator can be bounded below uniformly in ω, cf. (2.21), by integrating with respect to ω we obtain

$$\begin{aligned}\mathbb{E}\left[\mathbf{P}^{\mathtt{f}}_{N,\omega}\left(F_\ell\right)\right] &\le cN^{1/2} \sum_{l=\ell}^{N} \mathbb{E}\left[Z^{\mathtt{c}}_{l,\omega}\right] \mathbf{P}\left(S_n > 0,\, n = 1,2,\ldots,N-l\right)\left(1+\exp(-\beta(N-l))\right) \\ &\le c_1 N^{1/2} \sum_{l=\ell}^{N} \frac{1}{(l+1)^{3/2}} \frac{1}{(N+1-l)^{1/2}} \le \frac{c_2}{(\ell+1)^{1/2}}. \end{aligned}\tag{2.23}$$

In order to complete the proof of (1.10) we need to exclude that the last excursion of the polymer is in the lower half plane. However one directly verifies that:

$$\mathbb{E}\,\mathbf{P}^{\mathtt{f}}_{N\omega}\left(F_\ell^{\complement},\, S_n < 0 \text{ for } n \ge \ell\right) \le \exp(-\beta(N-\ell)), \tag{2.24}$$

and this suffices to conclude the proof of (1.10).

The proof of (1.11) is conceptually very close to the proof of (1.10). We introduce the event F_{ℓ_1,ℓ_2} of the polymer trajectories hitting 0 in the set $\{\ell_1,\ldots,N/2\}$ and in $\{N/2,\ldots,N-\ell_2\}$. We may write

$$\mathbf{P}^{\mathtt{c}}_{N,\omega}\left(F_{\ell_1,\ell_2}\right) = \frac{1}{Z^{\mathtt{c}}_{N,\omega}} \sum_{j_1=\ell_1}^{N/2} \sum_{j_2=\ell_2}^{N/2} Z^{\mathtt{c}}_{j_1,\omega} Z^{\pm}_{N,\omega}(j_1,j_2) Z^{\mathtt{c}}_{j_2,\tau_{N-j_2}\omega}, \tag{2.25}$$

where τ_k is the k–shift, i.e., $(\tau_k\omega)_n = \omega_{n+k}$, and

$$\begin{aligned} Z^{\pm}_{N,\omega}(j_1,j_2) &:= \mathbf{P}\left(S_n > 0,\, n = 1,2,\ldots,N-j_1-j_2-1,\, S_{N-j_1-j_2} = 0\right) \\ &\quad\times \left(1 + \exp\left(-2\lambda \sum_{n=j_1+1}^{N-j_2} (\omega_n + h)\right)\right). \end{aligned}\tag{2.26}$$

Once again we estimate the denominator uniformly with respect to ω, cf. (2.21), and then take expectation. By applying (2.19) and (2.20) we obtain

$$\mathbb{E}\,\mathbf{P}^{\mathtt{c}}_{N,\omega}\left(F_{\ell_1,\ell_2}\right) \le c_1 \sum_{j_1=\ell_1}^{N/2} \sum_{j_2=\ell_2}^{N/2} \frac{1}{(j_1+1)^{3/2}} \frac{1}{(j_2+1)^{3/2}} \left(\frac{N}{N-j_1-j_2+1}\right)^{3/2}. \tag{2.27}$$

Since the right–hand side can be bounded above by $c_2/\sqrt{\ell_1\ell_2+1}$ the proof is easily completed. _{Theorem 1.4(3)} □

3. Universality of the slope at the origin

Proof of Theorem 1.3. Let us first of all prove the theorem when the random variable ω_1 is bounded.

Let $\mathbb{P}^{(1)}$ be the law of IID centered Bernoulli random variables $\omega_n = \pm 1$. Also, consider a law $\mathbb{P}^{(2)}$ for the IID symmetric bounded random variables $\{\omega_n\}_n$, and recall that by convention $\mathbb{E}^{(2)}[\omega_1^2] = 1$.

Let $m_c^{(1)}$ be the slope at the origin of the critical curve of the copolymer model with Bernoulli disorder, whose existence was proven in [6]. By definition of $m_c^{(1)}$, for any $v > m_c^{(1)}$ and λ sufficiently small one has

$$\mathrm{F}^{(1)}(\lambda, v\lambda) := \sup_N \mathbb{E}^{(1)} \mathrm{F}^{\mathtt{c}}_{N,\omega}(\lambda, v\lambda) = 0. \tag{3.1}$$

This implies that, for any $\varepsilon > 0$, $N \in 2\mathbb{N}$ and $m \in \{0, 2, \cdots, N\}$,

$$\mathbb{E}^{(1)} \mathrm{F}^{\mathtt{c}}_{N,\omega}(\lambda, (v+\varepsilon)\lambda; m) \le -\frac{2\varepsilon\lambda^2 m}{N}, \tag{3.2}$$

where we use the same notation as in equation (2.6). On the other hand, one has the following lemma, proven below.

Lemma 3.1. *If the laws $\mathbb{P}^{(\ell)}$, $\ell = 1, 2$ correspond to IID centered bounded random variables, there exists a constant $c > 0$ such that for any $h \ge 0$, $0 \le \lambda \le 1$, $N \in 2\mathbb{N}$ and $m \in \{0, 2, \cdots, N\}$,*

$$\left| \mathbb{E}^{(1)} \mathrm{F}^{a}_{N,\omega}(\lambda, h; m) - \mathbb{E}^{(2)} \mathrm{F}^{a}_{N,\omega}(\lambda, h; m) \right| \le c \frac{m\lambda^3}{N}, \tag{3.3}$$

and

$$\left| \mathbb{E}^{(1)} \mathrm{F}^{a}_{N,\omega}(\lambda, h) - \mathbb{E}^{(2)} \mathrm{F}^{a}_{N,\omega}(\lambda, h) \right| \le c\lambda^3. \tag{3.4}$$

Thanks to equations (3.2) and (3.3), one has

$$\mathbb{E}^{(2)} \mathrm{F}^{\mathtt{c}}_{N,\omega}(\lambda, (v+\varepsilon)\lambda; m) \le -\frac{2\varepsilon\lambda^2 m}{N} + c\frac{m\lambda^3}{N} \le -\frac{\varepsilon\lambda^2 m}{N} \tag{3.5}$$

provided that $\lambda \le \min(1, \varepsilon/c)$. Using the deviation inequality (1.2), which is applicable since the random variables are bounded, it is then possible to deduce that

$$\mathrm{F}^{(2)}(\lambda, (v+\varepsilon)\lambda) := \lim_{N\to\infty} \mathbb{E}^{(2)} \mathrm{F}^{\mathtt{c}}_{N,\omega}(\lambda, (v+\varepsilon)\lambda) = 0. \tag{3.6}$$

This point is discussed in greater detail at the end of this section, in a more general context where the random variables $\omega_n^{(2)}$ are not necessarily bounded. Therefore, one has $\limsup_{\lambda\searrow 0} h_c^{(2)}(\lambda)/\lambda \le v + \varepsilon$ and, thanks to the arbitrariness of $\varepsilon > 0$ and of $v > m_c^{(1)}$,

$$\limsup_{\lambda\searrow 0} \frac{h_c^{(2)}(\lambda)}{\lambda} \le m_c^{(1)}. \tag{3.7}$$

To obtain the opposite bound, observe that from Theorem 6 of [6] follows that for any $v < m_c^{(1)}$, there exists $c(v) > 0$ such that

$$\mathrm{F}^{(1)}(\lambda, v\lambda) \ge c(v)\lambda^2 \tag{3.8}$$

for λ sufficiently small. On the other hand, thanks to (3.4), for λ sufficiently small one has

$$\mathrm{F}^{(2)}(\lambda, v\lambda) \geq \frac{c(v)}{2}\lambda^2 \tag{3.9}$$

which implies

$$\liminf_{\lambda\searrow 0} \frac{h_c^{(2)}(\lambda)}{\lambda} \geq m_c^{(1)} \tag{3.10}$$

and the statement of the theorem in the bounded case. Theorem 1.3, ω_1 bounded □

Proof of Lemma 3.1 This is based on an interpolation argument, of the type of the one showing that the free energy of the Sherrington-Kirkpatrick spin glass model does not depend on the distribution of the couplings (see [20], [13] and the more recent [7]).

For definiteness, we give the proof of (3.3) in the pinned case $a = \mathtt{c}$. For $0 \leq t \leq 1$, consider the auxiliary free energy

$$F_N(t) = \frac{1}{N}\mathbb{E}^{(1,2)} \log \mathbf{E}\left[\exp\left(-2\lambda\sum_{n=1}^N(\sqrt{t}\omega_n^{(1)} + \sqrt{1-t}\omega_n^{(2)} + h)\Delta_n\right); \mathcal{N} = m, S_N = 0\right] \tag{3.11}$$

where $\omega^{(1)}, \omega^{(2)}$ are independent and distributed according to the laws $\mathbb{P}^{(1)}, \mathbb{P}^{(2)}$ respectively. Then, one has immediately

$$F_N(1) = \mathbb{E}^{(1)}\mathrm{F}^{\mathtt{c}}_{N,\omega}(\lambda, h; m) \tag{3.12}$$

$$F_N(0) = \mathbb{E}^{(2)}\mathrm{F}^{\mathtt{c}}_{N,\omega}(\lambda, h; m). \tag{3.13}$$

Therefore, one has to estimate the t-derivative of the free energy, which is easily computed:

$$\frac{\mathrm{d}F_N(t)}{\mathrm{d}t} = -\frac{\lambda}{N}\mathbb{E}^{(1,2)}\sum_{n=1}^N \mathbf{E}_{N,\omega_t}(\Delta_n|\mathcal{N} = m, S_N = 0)\left(\frac{1}{\sqrt{t}}\omega_n^{(1)} - \frac{1}{\sqrt{1-t}}\omega_n^{(2)}\right). \tag{3.14}$$

This expression can be manipulated by means of the identity

$$\mathbb{E}\,\eta G(\eta) = \mathbb{E}\,G'(\eta) + \mathbb{E}\left((\eta^2-1)\int_0^\eta G''(u)\,\mathrm{d}u\right) - \frac{1}{4}\mathbb{E}\,|\eta|\int_{-|\eta|}^{+|\eta|}(\eta^2-u^2)G'''(u)\,\mathrm{d}u, \tag{3.15}$$

which holds for any symmetric random variable η with $\mathbb{E}\,[\eta^2] = 1$ and for sufficiently regular functions G. In our case, the idea is that every derivative with respect to $\omega_n^{(\ell)}$ carries a (small) factor λ, so that the first term in the r.h.s. of (3.15) is the dominant one. Applying this identity to (3.14), one finds that the *dominant terms* cancel exactly, and one is left with terms involving derivatives of $\mathbf{E}_{N,\omega_t}(\Delta_n|\mathcal{N} = m, S_N = 0)$ of order higher than one. Indeed, denoting

$$0 \leq X_n^{(\ell)}(u) := \mathbf{E}_{N,\omega_t}(\Delta_n|\mathcal{N} = m, S_N = 0)\big|_{\omega_n^{(\ell)}=u},$$

and noting that for $k \geq 1$

$$0 \leq (X_n^{(\ell)}(u))^k \leq X_n^{(\ell)}(u),$$

one has

$$\left|\frac{\mathrm{d}F_N(t)}{\mathrm{d}t}\right| \le \sum_{\ell=1}^{2}\frac{12\lambda^3}{N}\sum_{n=1}^{N}\mathbb{E}^{(1,2)}((\omega_n^{(\ell)})^2+1)\left|\int_0^{\omega_n^{(\ell)}} X_n^{(\ell)}(u)\,\mathrm{d}u\right| \tag{3.16}$$

$$+\sum_{\ell=1}^{2}\frac{26\lambda^4}{N}\sum_{n=1}^{N}\mathbb{E}^{(1,2)}|\omega_n^{(\ell)}|^3\int_{-|\omega_n^{(\ell)}|}^{+|\omega_n^{(\ell)}|} X_n^{(\ell)}(u)\,\mathrm{d}u. \tag{3.17}$$

Below, we consider only the terms with $\ell = 1$, the other case requiring only minimal modifications. Let us first consider the term in (3.17). Observe that

$$-2\lambda X_n^{(1)}(u) \le -2\lambda\sqrt{t}\left(X_n^{(1)}(u) - (X_n^{(1)}(u))^2\right) = \frac{d}{du}X_n^{(1)}(u) \le 0 \tag{3.18}$$

so that for any u, u'

$$X_n^{(1)}(u) \le X_n^{(1)}(u')e^{2\lambda|u-u'|}. \tag{3.19}$$

Therefore, the term in (3.17) can be bounded above by

$$\frac{26\lambda^4}{N}\sum_{n=1}^{N}\mathbb{E}^{(1,2)}|\omega_n^{(1)}|^3\int_{-|\omega_n^{(1)}|}^{+|\omega_n^{(1)}|}\mathbf{E}_{N,\omega_t}(\Delta_n|\mathcal{N}=m,S_N=0)e^{2\lambda|u-\omega_n^{(1)}|}\,\mathrm{d}u \le c\frac{m\lambda^4}{N} \tag{3.20}$$

where we made use of the boundedness of $\omega_n^{(1)}$ and of the fact that $\mathbf{E}_{N,\omega_t}(\Delta_n|\mathcal{N}=m,S_N=0)$ does not depend on u. An analogous bound can be obtained for the term in (3.16). Indeed, it is bounded above by

$$c\frac{\lambda^3}{N}\sum_{n=1}^{N}\mathbb{E}^{(1,2)}\left|\int_0^{\omega_n^{(1)}}\mathbf{E}_{N,\omega_t}(\Delta_n|\mathcal{N}=m,S_N=0)e^{2\lambda|u-\omega_n^{(1)}|}\,\mathrm{d}u\right|$$

$$\le c'\frac{\lambda^3}{N}\sum_{n=1}^{N}\mathbb{E}^{(1,2)}\mathbf{E}_{N,\omega_t}(\Delta_n|\mathcal{N}=m,S_N=0) = c'\frac{m\lambda^3}{N}. \tag{3.21}$$

The proof of (3.4) is much simpler. In this case one removes the constraint on $\mathcal{N}$ in the definition of $F_N(t)$ and then it is immediate to realize that (3.16) and (3.17) are of order $O(\lambda^3)$ and $O(\lambda^4)$, respectively.
Lemma 3.1
□

Proof of Theorem 1.3, $\omega_1 \sim \mathcal{N}(0,1)$. It remains to show that the proof covers also the case when $\mathbb{P}^{(2)}$ is the law of IID centered Gaussian variables. One easily verifies that Lemma 3.1 still holds if one of the two laws is replaced by the Gaussian one. To this purpose, it is sufficient to observe that, if η is a $\mathcal{N}(0,1)$ random variable, identity (3.15) can be replaced by the integration by parts formula

$$\mathbb{E}\,\eta G(\eta) = \mathbb{E}\,G'(\eta).$$

Therefore, one still obtains the uniform bound (3.5). In order to deduce (3.6) from (3.5), one proceeds as follows. For any $\overline{m} \ge 0$, one can decompose the partition function and write with obvious notation

$$Z^{\mathsf{c}}_{N,\omega}(\lambda,h) = Z^{\mathsf{c}}_{N,\omega}(\lambda,h;m\le\overline{m}) + Z^{\mathsf{c}}_{N,\omega}(\lambda,h;m>\overline{m}). \tag{3.22}$$

From now until the end of the proof we set $h = (v+\varepsilon)\lambda$. Then, using the inequality

$$\log(a+b) \le \log 2 + \log a + \log b, \tag{3.23}$$

which holds whenever $a, b \geq 1$, and the fact that $Z^{\mathtt{c}}_{N,\omega}(\lambda, h; m \leq \overline{m}) \geq c_1 N^{-3/2}$ for some constant c_1 independent of ω and N, one has

$$\begin{aligned}\mathbb{E}^{(2)} \mathrm{F}^{\mathtt{c}}_{N,\omega}(\lambda, h) \quad &\leq \quad \frac{1}{N}\mathbb{E}^{(2)} \log Z^{\mathtt{c}}_{N,\omega}(\lambda, h; m \leq \overline{m}) \\ &\quad + \frac{1}{N}\mathbb{E}^{(2)} \log \max\left(1, Z^{\mathtt{c}}_{N,\omega}(\lambda, h; m > \overline{m}) N^{3/2}/c_1\right) + \frac{\log 2}{N}.\end{aligned} \tag{3.24}$$

The first term in (3.24) can be bounded above via Jensen's inequality by $c_2\overline{m}/N$, where c_2 is a constant independent of $\overline{m}$ and N. As for the second term, define the event $E_{\overline{m}}$ as

$$E_{\overline{m}} = \left\{ \text{ there exists } \; m \geq \overline{m} \text{ such that } \; \mathrm{F}^{\mathtt{c}}_{N,\omega}(\lambda, h; m) \geq -\frac{\varepsilon m \lambda^2}{2N} \right\} \tag{3.25}$$

whose probability, thanks to the deviation inequality (1.2) and to (3.5), satisfies

$$\mathbb{P}^{(2)}\left(E_{\overline{m}}\right) \leq \frac{1}{c_3} e^{-c_3 \overline{m}}.$$

The second term in (3.24) can be therefore bounded above by

$$c_4 \frac{\log N}{N} + \frac{1}{N}\sqrt{\mathbb{P}^{(2)}\left(E_{\overline{m}}\right) \mathbb{E}^{(2)} \left(\log \max\left(1, Z^{\mathtt{c}}_{N,\omega}(\lambda, h; m > \overline{m}) N^{3/2}/c_1\right)\right)^2}, \tag{3.26}$$

where the first term comes from the average restricted to the event $E^{\complement}_{\overline{m}}$ and in the second we applied Cauchy-Schwarz inequality. Observing that

$$Z^{\mathtt{c}}_{N,\omega}(\lambda, h; m > \overline{m}) \leq \exp\left(2\lambda \sum_{n=1}^{N} (h + |\omega_n|)\right)$$

and putting everything together, one obtains finally

$$\mathbb{E}^{(2)} \mathrm{F}^{\mathtt{c}}_{N,\omega}(\lambda, h) \quad \leq \quad c_5 \left(\frac{\overline{m}}{N} + \frac{\log N}{N} + e^{-c_3\overline{m}/2}\right), \tag{3.27}$$

from which (3.6) follows choosing for instance $\overline{m} = \sqrt{N}$.

From this point on, the proof proceeds exactly like in the bounded case. Theorem 1.3, $\omega_1 \sim \mathcal{N}(0,1)$ □

4. Further results and considerations

4.1. **What does one expect on delocalized paths.** In the previous section, we have given delocalization results that hold in average with respect to the $\mathbb{P}$-probability. On the other hand, one would like to prove $\mathbb{P}(\,\mathrm{d}\omega)$–almost sure results. In this respect, based on what is known on non-disordered models, see e.g. [9] and [14], it is tempting to conjecture the following scenario: for $h > h_c(\lambda)$

C.1 there are only a finite number of visits to the unfavorable solvent, that is $\mathbb{P}(\,\mathrm{d}\omega)$–a.s.

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} \mathbf{P}^{\mathtt{f}}_{N,\omega}\left(\max \mathcal{A} > \ell\right) = 0. \tag{4.1}$$

C.2 there is a diffusive scaling limit to a *Brownian meander*. In other terms if we set $B^{(N)}_t := S_{Nt}/\sqrt{N}$ for $N \in \{0, 1, \ldots, N\}$ and we extend the definition of $B^{(N)}_\cdot$ to a function in $C^0([0,1];\mathbb{R})$ by linear interpolation for $S_{N\cdot}$, $\mathbb{P}(\,\mathrm{d}\omega)$–a.s. we have that the law of $B^{(N)}_\cdot$, with S distributed according to $\mathbf{P}^{\mathtt{f}}_{N,\omega}$, converges weakly as $N \to \infty$ to the law of the Brownian meander, that is the law of a standard Brownian process conditioned not to enter the lower half plane. The standard reference for the Brownian meander is [17].

With the same level of confidence one might formulate the analogous statements for the constrained case: in particular, in C.2 the expected scaling limit would the Brownian bridge conditioned to stay positive, a process that normally goes under the name of *Bessel(3) bridge*, see [17].

As we will see, the scenario outlined above cannot hold if taken literally, though we expect the qualitative picture to be correct. To start with, observe that Theorem 1.4 gives partial support to the conjectures, at least for $h \geq \overline{h}(\lambda)$. Indeed, for example if we choose a sequence $\{\ell_N\}_N$ with $\lim_N \ell_N = \infty$, then by (1.10) we have that

$$\lim_N \mathbf{P}^{\mathbf{f}}_{N,\omega} \left(\max \mathcal{A} > \ell_N\right) \,=\, 0, \tag{4.2}$$

in $\mathbb{P}$–probability, or $\mathbb{P}(\,\mathrm{d}\omega)$–a.s. by subsequences. This of course falls a bit short of proving C.1, even in the strongly delocalized region. Just about the same is true for C.2. Let us set $\zeta_N := \max \mathcal{A}$. By the result we just stated, for $h > \overline{h}(\lambda)$ there exists a sequence $\{N_j\}_j$ such that $\mathbb{P}(\,\mathrm{d}\omega)$–a.s. the random variable ζ_{N_j}/N_j tends to zero as j tends to infinity, in $\mathbf{P}_{N_j,\omega}$–probability. Since it is not difficult to see that, conditionally to $\zeta_N = k$, the law of $\{S_{\zeta_N+n}\}_{n=0,1,\ldots}$ coincides with the law of a simple random walk constrained not to enter the lower half–plane up to time $N-k$, we are in the framework already considered for example in [14] or [9]. Therefore, by proceeding like in [9], one can show that for a $\mathbb{P}$–typical ω the sequence of random functions $\left\{B^{(N_j)}_{\cdot}\right\}_j$, with S distributed according to $\mathbf{P}^{\mathbf{f}}_{N_j,\omega}$, converges weakly as $j \to \infty$ to the law of the Brownian meander.

On the other hand, Theorem 1.4 does not say much in the direction of C.1 and C.2 for $h \in (h_c(\lambda), \overline{h}(\lambda)]$. This is due to the fact that, in spite of knowing that there are few visits to the unfavorable solvent, we do not know that they are close to the origin (or to N, in the constrained case).

4.2. **On the size of $Z^a_{N,\omega}$.** Some further insight on the behavior of paths in the delocalized phase may be obtained by looking at the size of $Z^a_{N,\omega}$.

Observe that, by (2.20), under the basic assumptions on the disorder distribution and in the strongly delocalized regime $h \geq \overline{h}(\lambda)$, $Z^a_{N,\omega}$ is of the order of $N^{-1/2}$ for $a = \mathtt{f}$, and $N^{-3/2}$ for $a = \mathtt{c}$, in the evident $\mathbb{P}$–probability sense. Recalling (2.21), the result is somewhat sharp. We lack however an almost sure result going beyond the fact that $Z^a_{N,\omega}$ tends to 0 $\mathbb{P}(\,\mathrm{d}\omega)$–a.s. in the strongly delocalized region (this is an immediate consequence of the convergence in probability along with the fact that $\{Z^a_{N,\cdot}\}_N$ is a positive supermartingale for $h \geq \overline{h}(\lambda)$ with respect to the natural filtration of ω and therefore it converges $\mathbb{P}(\,\mathrm{d}\omega)$–a.s.).

On the other hand, the result that we are going to present now says that something qualitatively different happens for $h_c(\lambda) \leq h \leq \overline{h}(\lambda)$. As we will discuss at the end of the section, this phenomenon reflects on the behavior of the paths.

Proposition 4.1. *Under the basic assumptions on ω one can construct a sequence $\{\tau_N\}_N$ of stopping times, with respect to the natural filtration of the sequence ω, with the property that $\log \tau_N(\omega)/\log N \stackrel{N\to\infty}{\longrightarrow} 1$ $\mathbb{P}(\,\mathrm{d}\omega)$–a.s. and we can find a number $\delta = \delta(\lambda, h)$, explicitly* *given below, such that* $\delta > 0$ *if* $h < \overline{h}(\lambda)$, *and that*

$$\lim_{N\to\infty} N^{1/2-\delta'} Z^{\mathtt{f}}_{\tau_N(\omega),\omega} = +\infty, \qquad \mathbb{P}(\,\mathrm{d}\omega) - a.s.. \tag{4.3}$$

for every $\delta' < \delta$.

Proof. Set $\widetilde{\omega}_n := \omega_n + h$, choose a real number $q < h$ and define

$$\tau_N \;:=\; \inf\left\{ n \in 2\mathbb{N} : \frac{\sum_{j=k+1}^{n} \widetilde{\omega}_j}{n-k} \le q \text{ for some } k \in \{0, 2, \ldots, n - r_N\} \right\} \tag{4.4}$$

with r_N the largest even integer smaller than $(\log N)/\Sigma_h(q)$, where $\Sigma_h(\cdot)$ is the Cramer Large Deviation functional of $\widetilde{\omega}$:

$$\lim_{\ell\to\infty} \frac{1}{\ell} \log \mathbb{P}\left(\sum_{j=1}^{\ell} \widetilde{\omega}_j \le q\ell \right) \;=\; -\Sigma_h(q). \tag{4.5}$$

We note that τ_N is the first moment at which an *atypical* stretch of length at least r_N appears along the sequence ω. By Theorem 3.2.1 in [8, §3.2] we have that $\log \tau_N / \log N$ tends to 1 $\mathbb{P}(\,\mathrm{d}\omega)$–a.s.. The same theorem tells us that, if

$$R_n \;:=\; \max\left\{ \ell - k : \, k \text{ and } \ell \text{ even }, \, 0 \le k < \ell \le n, \, \frac{\sum_{j=k+1}^{\ell} \widetilde{\omega}_j}{\ell - k} \le q \right\}, \tag{4.6}$$

then $R_n / \log n \stackrel{n\to\infty}{\longrightarrow} 1/\Sigma_h(q)$ $\mathbb{P}(\,\mathrm{d}\omega)$–a.s. and therefore

$$\lim_{N\to\infty} \frac{R_{\tau_N}}{\log N} \;=\; \frac{1}{\Sigma_h(q)}, \quad \mathbb{P}(\,\mathrm{d}\omega) - \text{a.s..} \tag{4.7}$$

Notice that the longest atypical stretch, in the sense of (4.6), for $n = \tau_N$ ranges from $\tau_N - R_{\tau_N}$ to τ_N, so $\sum_{j=\tau_N - R_{\tau_N}+1}^{\tau_N} \widetilde{\omega}_j \le q R_{\tau_N}$.

Choose now any $\varepsilon > 0$ and a typical ω. We have

$$\begin{aligned} Z^{\mathtt{f}}_{\tau_N,\omega} \;&\ge\; \mathbf{P}\left(S_n > 0, \, n = 1, 2, \ldots, \tau_N - R_{\tau_N} - 1, \, S_{\tau_N - R_{\tau_N}} = 0 \right) \\ &\qquad\qquad \times \exp\left(-2\lambda q R_{\tau_N}\right) \mathbf{P}\left(S_n < 0, \, n = 1, 2, \ldots, R_{\tau_N}\right) \\ &\ge\; c \frac{1}{N^{3/2+\varepsilon}} N^{-2\lambda q / \Sigma_h(q)}. \end{aligned} \tag{4.8}$$

The second inequality holds for N sufficiently large. Now we set

$$\delta \;:=\; \sup_{q<h} \frac{-2\lambda q - \Sigma_h(q)}{\Sigma_h(q)}, \tag{4.9}$$

and observe that $\sup_{q<h}\left(-2\lambda q - \Sigma_h(q)\right)$ is positive if and only if $\sup_{q\in\mathbb{R}}\left(-2\lambda q - \Sigma_h(q)\right)$ is positive. The latter expression is the Legendre transform of $\Sigma_h(\cdot)$ and therefore it coincides with $-2\lambda h + \log \mathrm{M}(2\lambda)$, which is positive for $h < \overline{h}(\lambda)$. We have therefore proven (4.3). □

Two remarks are in order:

Remark 4.2. It is immediate to see that, at least in the case in which $\operatorname{ess\,sup} \omega_1 =: \omega^\star$ is finite, Proposition 4.1 yields that C.1 cannot hold in the free endpoint case below $\overline{h}(\lambda)$. Even more, it is in contradiction with the possibility of having $o(\log N)$ visits to the unfavorable solvent, if one insists on having $\mathbb{P}(\,\mathrm{d}\omega)$–a.s. results. This is easily seen by considering copolymers of length τ_N: one estimates for the numerator the partition function restricted to trajectories of the walk visiting $o(\log N)$ times the lower half plane and for the denominator one uses (4.3).

Remark 4.3. The argument in the proof of Proposition 4.1 may be repeated for the constrained case and one obtains that, as long as $h < \omega^\star$ there exists $\delta > 0$ such that $N^{3/2-\delta} Z^{\mathtt{c}}_{N,\omega}$ does not vanish. This once again contradicts C.1: it is however a more evident phenomenon, since the polymer is forced in any case to visit all atypical ω–stretches when its endpoint encounters them.

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