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# Asymptotic Behavior of Solutions to Nonlinear Parabolic Equations with Nonlocal Terms

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## Abstract

We consider nonlinear parabolic equations with two classes of nonlocal terms. We especially investigate the asymptotic behavior of the solutions as time goes to infinity.

**Keywords:** nonlinear parabolic equations, nonlocal term, asymptotic behavior.

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## 1 Introduction.

In this paper we consider the asymptotic behavior of solutions to the following nonlinear parabolic equations with nonlocal terms:

$$u_t - a\Delta u = f(x), \quad (x, t) \in \Omega \times R^+ \tag{1.1}$$

subject to the following Dirichlet boundary condition

$$u|_{\Gamma} = 0, \tag{1.2}$$

and the initial condition

$$u|_{t=0} = u_0(x). \quad (1.3)$$

In (1.1),  $\Omega$  is a bounded domain in  $R^n$  with smooth boundary  $\Gamma$ , and  $a$  is a nonlinear nonlocal form in  $u$ . In this paper we consider the following two cases:

**Case (1):**  $a$  depends on  $\|\nabla u(\cdot, t)\|^2$ , i.e.,

$$a = a(\|\nabla u(\cdot, t)\|^2) \quad (1.4)$$

where  $a(s)$ ,  $s \in R$  is a  $C^1$  function such that there is a positive constant  $\alpha > 0$ ,

$$\alpha \leq a(s), \quad \forall s \in R. \quad (1.5)$$

$\| \cdot \|$  denotes the usual  $L^2(\Omega)$ -norm in such a way that it holds that

$$\|\nabla u(\cdot, t)\|^2 = \int_{\Omega} |\nabla u(x, t)|^2 dx.$$

**Case (2):**  $a$  depends on a linear functional  $l(u)$ , i.e.,

$$a = a(l(u)) \quad (1.6)$$

with

$$l(u) = \int_{\Omega} g(x)u(x, t)dx. \quad (1.7)$$

where  $g(x)$  is a given function in  $L^2(\Omega)$  and  $a(s)$  satisfies the assumptions as above.

It is easy to see that for the case (1), the problem (1.1)–(1.3) has a Lyapunov functional

$$E(u) = \frac{1}{2} \int_0^{\|\nabla u\|^2} a(s)ds - \int_{\Omega} ufdx. \quad (1.8)$$

It turns out that we may use results of dynamical systems to study the asymptotic behavior of solutions to the problem (1.1)–(1.3). However, it seems that for the case (2), there is no Lyapunov functional, and we have to use some other methods.

In recent years nonlinear parabolic equations with nonlocal terms have been extensively studied; e.g., see [2]–[4], [5]–[6], [8], and [9]. In particular, it is shown in [9] (for the case (1)) and in [8] (for the case (2)) that if the stationary problem has a unique solution, then, under some additional assumptions, convergence to this unique equilibrium occurs. In this paper we use different approaches to obtain convergence to one of the equilibria without assuming that the stationary problem admits a unique solution. To be more specific in the case of (2) the methods used up to now were restricted to the case of a nonnegative  $f$ . Our technique allows us to drop this hypothesis. However this is at the expense of some smoothness on  $a$  and smallness assumptions on the data. Roughly speaking, when two of the data  $a'$ ,  $\|f\|$ ,  $\|g\|$  are fixed the third one has to be small (see below).

The main results of this note are the following.

**Theorem 1.1** *For the problem (1.1)–(1.3) in case (1), for any given  $f \in L^2$  and any initial datum  $u_0 \in H_0^1$ , the solution  $u(x, t)$  converges in  $H^2$  to a stationary solution as time goes to infinity.*

Let  $C_s$  the best Sobolev constant such that for all  $u \in H_0^1$  the following Poincaré inequality holds:

$$\|u\| \leq C_s \|\nabla u\|.$$

Then we have the following result.

**Theorem 1.2** *For the problem (1.1)–(1.3) in case (2), suppose that*

$$\frac{2C_s^2}{\alpha} \|g\| \|f\| \cdot \sup_{|s| \leq \frac{2C_s^2}{\alpha} \|g\| \|f\|} |a'(s)| < \alpha, \quad (1.9)$$

*then for any given initial datum  $u_0$ , the global solution  $u$  to problem (1.1)–(1.3) converges in  $H^2$  to a stationary solution as time goes to infinity.*

This paper is organized as follows. In the next section, the global existence and uniqueness of strong solutions to problems (1.1)–(1.3) in case (1) is proved, using the Faedo-Galerkin method. Furthermore, the compactness of the orbit is proved. Finally, the proof of Theorem 1.1 is given. In section 3, we give the proof for Theorem 1.2. All along we denote  $L^2(\Omega)$ ,  $H_0^1(\Omega)$ ,  $H^2(\Omega)$  by  $L^2$ ,  $H_0^1$ ,  $H^2$ , and we use  $\|\cdot\|$  to denote the  $L^2(\Omega)$  norm.

## 2 Proof of Theorem 1.1

In this section we first use the Faedo-Galerkin method to prove global existence and uniqueness of strong solutions to problem (1.1)–(1.3). More precisely, we have the following result.

**Theorem 2.1** *Suppose that  $u_0 \in H_0^1(\Omega)$ ,  $f \in L^2(\Omega)$ . Then for any  $T > 0$  problem (1.1)–(1.3) admits a unique strong solution  $u$  such that*

$$u \in C([0, T], H_0^1) \cap L^2([0, T], H^2), \quad u_t \in L^2([0, T], L^2).$$

*Furthermore, there is a positive constant  $C$  depending only on  $\|u_0\|_{H^1}$ ,  $\|f\|$  such that*

$$\|u(t)\|_{H^1} \leq C, \quad \int_0^t \|u_t\|^2 d\tau \leq C. \quad (2.1)$$

*Proof.* The global existence and uniqueness of weak solutions in the class

$$u \in C([0, T], L^2) \cap L^2([0, T], H_0^1), \quad u_t \in L^2([0, T], H^{-1})$$

to both problems (case (1) and case (2)) has been proved in [8] and [9], respectively. To prove the existence of a strong solution to both problems, we use the Faedo-Galerkin approximation method and a compactness argument (see e.g. Lions, [10]). In what follows we only give the detailed proof for the case (1). For the case (2), the proof is essentially the same, and we omit the detail here.

Let  $\{\varphi_k\}$  be the normalized eigenfunctions of the Laplace operator subject to the Dirichlet boundary condition, and  $\lambda_k$  be the corresponding eigenvalues. We look for an approximate solution  $u_m(x, t)$ , ( $m = 1, 2, \dots$ ) in the form

$$u_m(x, t) = \sum_{i=1}^m g_{im}(t)\varphi_i(x) \quad (2.2)$$

with

$$(u'_m, \varphi_k) - a\left(\sum_{i=1}^m \|g_{im}\nabla\varphi_i\|^2\right)(\Delta u_m, \varphi_k) = (f, \varphi_k), \quad k = 1, 2, \dots, m, \quad (2.3)$$

i.e.,

$$g'_{km} + \lambda_k a\left(\sum_{i=1}^m \lambda_i g_{im}^2\right)g_{km} = (f, \varphi_k), \quad k = 1, 2, \dots, m. \quad (2.4)$$

Since  $\{\varphi_k\}$  is dense in  $H_0^1$ , for given  $u_0 \in H_0^1$ , there is a sequence  $\xi_{km}$  such that

$$\sum_{i=1}^m \xi_{km}\varphi_k \rightarrow u_0 \quad \text{in } H_0^1. \quad (2.5)$$

We require that the approximate solutions  $u_m$  satisfy the following initial conditions:

$$g_{km}(0) = \xi_{km}, \quad k = 1, \dots, m. \quad (2.6)$$

By the local existence and uniqueness theorem for ordinary differential equations, there is a positive constant  $\delta > 0$  such that the problem (2.4), (2.6) admits a local solution  $g_{km}(t) \in C^2[0, \delta)$ . To prove the global existence, we multiply (2.4) by  $g'_{km}$ , then sum up with respect to  $k$  to obtain

$$\|u'_m\|^2 + \frac{dE(u_m)}{dt} = 0, \quad (2.7)$$

where  $E(u)$  is defined by (1.8). Integrating (2.7) with respect to  $t$  yields that

$$E(u_m(t)) + \int_0^t \|u'_m\|^2 d\tau = E(u_m(0)). \quad (2.8)$$

Noticing (1.5), (2.5), we deduce from (2.8) that

$$\|u_m(t)\|_{H_0^1} \leq C_1, \quad \int_0^t \|u'_m\|^2 \leq C_1 \quad (2.9)$$

where  $C_1$  is a positive constant depending only on  $\|u_0\|_{H_0^1}$  and  $\|f\|$ . It turns out from (2.9) that  $|g_{km}|$  are uniformly bounded with respect to  $t$ . Thus the local solutions  $g_{km}(t)$  can be continuously extended to the whole interval  $[0, T]$  with  $T > 0$  being any given constant. Furthermore, for all  $t \in [0, T]$ , the inequalities (2.9) hold.

Next, multiplying (2.3) by  $\lambda_k g_{km}$ , and summing up, we can easily get

$$\int_0^T \|u_m\|_{H^2}^2 d\tau \leq C_T \quad (2.10)$$

where  $C_T$  is a positive constant depending on  $\|u_0\|_{H_0^1}$ ,  $\|f\|$  and  $T$ . It is shown by (2.9), (2.10) that  $u_m$  is uniformly bounded in  $L^\infty([0, T], H_0^1) \cap L^2([0, T], H^2)$ , and  $u'_m$  is uniformly bounded in  $L^2([0, T], L^2)$ . Therefore, there is a subsequence, still denoted by  $u_m$  such that

$$u_m \rightharpoonup u \quad \text{weakly } * \text{ in } L^\infty([0, T], H_0^1), \quad (2.11)$$

$$u_m \rightharpoonup u \quad \text{weakly in } L^2([0, T], H^2), \quad (2.12)$$

$$u'_m \rightharpoonup u' \quad \text{weakly in } L^2([0, T], L^2), \quad (2.13)$$

and by Aubin's compactness theorem,

$$u_m \rightarrow u \quad \text{strongly in } L^2([0, T], H_0^1). \quad (2.14)$$

Thus,

$$\|\nabla u_m\|^2 \rightarrow \|\nabla u\|^2 \quad \text{strongly in } L^1[0, T], \quad (2.15)$$

and -up to a subsequence:

$$a(\|\nabla u_m(t)\|^2) \rightarrow a(\|\nabla u(t)\|^2) \quad \text{almost everywhere in } [0, T]. \quad (2.16)$$

Passing to the limit in (2.4) yields

$$(u'(t), \varphi_k) - a(\|\nabla u(t)\|^2)(\Delta u, \varphi_k) = (f, \varphi_k) \quad \forall k = 1, \dots, m. \quad (2.17)$$

Since  $\{\varphi_k\}$  forms a basis in  $L^2$ , it follows that  $u$  satisfies the equation (1.1) in the sense  $L^2([0, T], L^2)$ . By (2.5), the initial condition (1.3) is satisfied. The uniqueness of the strong solution follows directly from the corresponding result for the weak solution (see [9]). It remains to show that  $u \in C([0, T], H_0^1)$ . This can be seen from standard arguments as in [11], or it can also be seen by writing the equation (1.1) in the form

$$u_t - \Delta u = F \quad (2.18)$$

where

$$F = f + (a - 1)\Delta u \in L^2([0, T], L^2) \quad (2.19)$$

and using the uniqueness of the solution and standard result for the heat equation. Thus, the proof is complete.  $\square$

**Remark 2.1** *The previous theorem shows that the solution  $u$  defines a continuous semi-flow in  $H_0^1$ .*

**Remark 2.2** *For the case (2), the global existence and uniqueness of the strong solution still holds. However, the problem now does not have a Lyapunov functional, and it is not clear for the time being whether the constant  $C$  in (2.1) is still independent of  $T$ . We will discuss this matter in the next section.*

**Remark 2.3** *Recently H. Amann [1] has established a general theory for the local solvability of quasilinear parabolic initial boundary value problems with applications to quasilinear parabolic equations with nonlocal terms.*

The following result shows that for any  $\delta > 0$ , the orbit defined by the solution  $u$  is uniformly bounded in  $H^2$ .

**Theorem 2.2** *For any  $\delta > 0$ , there is a positive constant  $C_\delta$  depending only on  $\delta$ ,  $\|u_0\|_{H_0^1}$  and  $\|f\|$  such that*

$$\|u\|_{H^2} \leq C_\delta \quad \forall t \geq \delta. \quad (2.20)$$

*Proof.* First, we notice that if the initial datum  $u_0$  belongs to  $H^2 \cap H_0^1$ , then the strong solution  $u$  to the problem (1.1)–(1.3) has more regularity:

$$u \in C([0, T], H^2), \quad u_t \in C([0, T], L^2) \cap L^2([0, T], H_0^1). \quad (2.21)$$

This can be easily seen by differentiating (2.3) with respect to  $t$ , then multiplying the resultant by  $g'_{km}(t)$  to get the higher order energy estimates i.e. the second part of (2.21) which, by the equation, easily leads to the first part of (2.21).

Having seen this regularity result, we now use a density argument. For any initial datum  $u_0 \in H_0^1$ , we have a sequence of initial data  $u_{0n} \in H^2 \cap H_0^1$  such that

$$u_{0n} \rightarrow u_0 \quad \text{in } H_0^1.$$

In what follows we show that for the corresponding solutions  $u_n$ , the estimate (2.20) holds. Then passing to the limit yields the desired result.

For simplicity of notation, we denote  $u_n$  by  $u$  and  $u_{0n}$  by  $u_0$ . Multiplying the equation (1.1) by  $u_t$ , then integrating over  $\Omega$  yields

$$\frac{dE}{dt} + \|u_t\|^2 = 0. \quad (2.22)$$

Thus,

$$E(u) \leq E(u_0), \quad \int_0^t \|u_t\|^2 d\tau \leq E(u_0). \quad (2.23)$$

Differentiating the equation (1.1) with respect to  $t$ , then taking the dual product with  $u_t$  yields that

$$\frac{1}{2} \frac{d}{dt} \|u_t\|^2 + a(\|\nabla u\|^2) \|\nabla u_t\|^2 = 2a' \int_\Omega \nabla u \cdot \nabla u_t dx \int_\Omega \Delta u \cdot u_t dx \quad (2.24)$$

where  $a'(s)$  denotes the first order derivative of  $a(s)$  with respect to  $s$ . We can get the expression of  $\Delta u$  from the equation (1.1):

$$\Delta u = \frac{u_t - f}{a}, \quad (2.25)$$

then plug it into (2.24) to get the estimate

$$\frac{1}{2} \frac{d}{dt} \|u_t\|^2 + \alpha \|\nabla u_t\|^2 \leq C_3 \|\nabla u_t\| (\|u_t\|^2 + \|u_t\| \|f\|), \quad (2.26)$$

using (2.1) and the assumption that  $a(s) \geq \alpha$ . Applying the Young inequality, we deduce from (2.26) that

$$\frac{d}{dt} \|u_t\|^2 + \alpha \|\nabla u_t\|^2 \leq C_4 \|u_t\|^4 + C_5, \quad (2.27)$$

where  $C_4, C_5$  are positive constants depending only on  $\alpha, \|u_0\|_{H_0^1}$ , and  $\|f\|$ . Let

$$y(t) = \|u_t\|^2.$$

Then we see that  $y(t)$  satisfies (2.1) and

$$\frac{dy}{dt} \leq C_4 y^2 + C_5.$$

Applying a lemma in analysis which was first established in [13] (Lemma 3.1 in [13]; see also [14] and [15]) yields that

$$y(t) = \|u_t\|^2 \leq \left( \frac{C}{\delta} + C_5 \delta \right) e^{C_4 C} \quad \forall t \geq \delta, \quad (2.28)$$

and as  $t \rightarrow +\infty$ ,

$$u_t(\cdot, t) \rightarrow 0 \quad \text{in } L^2. \quad (2.29)$$

Then, estimate (2.20) follows from (2.25) and the standard elliptic estimates. Thus, the proof is complete.  $\square$

Define the  $\omega$ -limit set  $\omega(u_0)$  as follows

$$\omega(u_0) = \{ \psi \mid \exists t_n, t_n \rightarrow +\infty \text{ such that } u(\cdot, t_n) \rightarrow \psi \text{ in } H_0^1 \}.$$

Since the problem (1.1)–(1.3) has a Lyapunov functional  $E$  given by (1.8), it follows from previous theorems and the well known results for the infinite-dimensional dynamical systems that the following result holds.

**Theorem 2.3** *For every  $u_0 \in H_0^1$  the  $\omega$ -limit set  $\omega(u_0)$  is a compact, connected subset of  $H_0^1$ . Furthermore, it consists of equilibria.*

In what follows we study the stationary problem in case (1) (see [9]). Let  $\psi(x)$  be the unique solution to the following problem:

$$-\Delta \psi = f, \quad (2.30)$$

$$\psi|_{\Gamma} = 0. \quad (2.31)$$

Then a solution  $v$  to the stationary problem

$$-a(\|\nabla v\|^2) \Delta v = f, \quad (2.32)$$

$$v|_{\Gamma} = 0 \quad (2.33)$$

can be expressed as

$$v = \sqrt{\xi} \frac{\psi}{\|\nabla \psi\|} \quad (2.34)$$



where  $\xi$  is a root to the following equation:

$$a(\xi) = \frac{\|\nabla\psi\|}{\sqrt{\xi}}. \quad (2.35)$$

More precisely (we refer to [9]) the mapping

$$v \rightarrow \|\nabla v\|^2$$

is a one-to-one mapping from the set of equilibria onto the set  $E$  defined by

$$E = \{\xi \mid a(\xi) = \frac{c}{\sqrt{\xi}}, c = \|\nabla\psi\|\}. \quad (2.36)$$

We can now turn to the proof of Theorem 1.1.

### Proof of Theorem 1.1.

If the set of equilibria is discrete, then by Theorem 2.3, the  $\omega$ -limit set  $\omega(u_0)$  must be a single point, i.e., the solution  $u$  must converge to an equilibrium as time goes to infinity. Therefore, it remains to prove Theorem 1.1 when set of equilibria contain a continued set, i.e.,  $E$  contains an interval.

We first prove the following result.

**Lemma 2.1** *If  $\varphi$  is an equilibrium and if  $\|\nabla u\|^2 \in E$ , then it holds that*

$$\frac{1}{2} \frac{d}{dt} \|u - \varphi\|^2 \leq 0. \quad (2.37)$$

*Proof.* If  $\varphi$  is an equilibrium,  $\|\nabla\varphi\|^2 \in E$ . Thus we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u - \varphi\|^2 &= (u_t, u - \varphi) = (a(\|\nabla u\|^2)\Delta u - a(\|\nabla\varphi\|^2)\Delta\varphi, u - \varphi) \\ &= \left( \frac{c}{\|\nabla u\|} \Delta u - \frac{c}{\|\nabla\varphi\|} \Delta\varphi, u - \varphi \right) \\ &= -c[\|\nabla u\| + \|\nabla\varphi\| - (\nabla u \cdot \nabla\varphi) \left\{ \frac{1}{\|\nabla u\|} + \frac{1}{\|\nabla\varphi\|} \right\}] \\ &\leq -c[\|\nabla u\| + \|\nabla\varphi\| - \|\nabla u\| \|\nabla\varphi\| \left\{ \frac{1}{\|\nabla u\|} + \frac{1}{\|\nabla\varphi\|} \right\}] \leq 0. \end{aligned}$$

This completes the proof of the lemma. □

We now use a contradiction argument to complete the remaining proof of Theorem 1.1. If  $\omega(u_0)$  is not a single point, then by Theorem 2.3 it must be a connected set, i.e., there exist  $0 < \xi_1 < \xi_2$  such that  $[\xi_1, \xi_2] \subset E$ , and for all  $\xi \in [\xi_1, \xi_2]$ ,  $\sqrt{\xi} \frac{\psi}{\|\nabla\psi\|} \subset \omega(u_0)$ . For any interior point  $\xi \in (\xi_1, \xi_2)$  there is a sequence  $t_n$  such that as  $t_n \rightarrow +\infty$ ,

$$u(t_n) \rightarrow \varphi = \sqrt{\xi} \frac{\psi}{\|\nabla\psi\|} \text{ in } H^1. \quad (2.38)$$

In what follows we show that the whole  $u(t)$  converges toward it, a contradiction to

$$\sqrt{\xi} \frac{\psi}{\|\nabla\psi\|} \in \omega(u_0), \forall \xi \in [\xi_1, \xi_2].$$

Let

$$\sigma = \min(\sqrt{\xi_2} - \sqrt{\xi}, \sqrt{\xi} - \sqrt{\xi_1}).$$

Then when  $\|\nabla u - \nabla\varphi\| \leq \sigma$ ,

$$\sqrt{\xi_1} = \|\nabla\varphi\| - (\sqrt{\xi} - \sqrt{\xi_1}) \leq \|\nabla u\| \leq \|\nabla\varphi\| + \sqrt{\xi_2} - \sqrt{\xi} = \sqrt{\xi_2},$$

i.e.,  $\|\nabla u\|^2 \in E$ . By Lemma 2.1, we have

$$\|\nabla(u(t) - \varphi)\| \leq \sigma \Rightarrow \|\nabla u\|^2 \in E \Rightarrow \frac{d}{dt}\|u(t) - \varphi\|^2 \leq 0.$$

It follows from (2.38) that there exists  $N$  such that  $n \geq N$  implies  $\|\nabla(u(t_n) - \varphi)\| < \sigma$ . Set

$$t'_n = \text{Sup}\{t \mid \|\nabla(u(s) - \varphi)\| \leq \sigma \text{ on } [t_n, t]\}.$$

If for some  $n$ ,  $t'_n = +\infty$ , we are done since  $|u(t) - \varphi|^2$  is decreasing for  $t > t_n$  and thus  $u(t)$  converges toward  $\varphi$  in  $L^2$  and also in  $H_0^1$ . Otherwise, one has

$$\|\nabla(u(t'_n) - \varphi)\| = \sigma. \quad (2.39)$$

But by Lemma 2.1,

$$\|u(t'_n) - \varphi\| \leq \|u(t_n) - \varphi\| \rightarrow 0.$$

Thus for a subsequence of  $t'_n$ , still denoted by  $t'_n$ , we deduce from Theorem 2.2 that

$$\|\nabla(u(t'_n) - \varphi)\| \rightarrow 0$$

which contradicts (2.39). Thus  $\omega(u_0)$  reduces to a point. Convergence of  $u$  towards  $\varphi$  in  $H^2$  follows from (2.29) and equation (1.1). This completes the proof of Theorem 1.1.  $\square$

### 3 Proof of Theorem 1.2

For the problem (1.1)–(1.3) in case (2), the global existence and uniqueness of strong solutions can be obtained, using the same Faedo-Galerkin method. However, since now we do not have a Lyapunov functional, the constant appearing in (2.1) may depend on  $T$ . In what follows we get some uniform a priori estimates.

**Lemma 3.1** *For any initial datum  $u_0 \in H_0^1$ , there is a positive constant  $t_0 \geq 0$  depending on  $u_0$  such that the following holds.*

$$\|u(t)\| \leq \frac{2C_s^2}{\alpha}\|f\|, \quad \|\nabla u\| \leq \frac{2C_s}{\alpha}\|f\| \quad \forall t \geq t_0. \quad (3.1)$$

*Proof.* Taking the inner product of the equation (1.1) with  $-\Delta u$  in  $L^2$  yields

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \alpha \|\Delta u\|^2 \leq \|\Delta u\| \|f\| \leq \frac{\alpha}{2} \|\Delta u\|^2 + \frac{\|f\|^2}{2\alpha}, \quad (3.2)$$

i.e.,

$$\frac{d}{dt} \|\nabla u\|^2 + \alpha \|\Delta u\|^2 \leq \frac{\|f\|^2}{\alpha}. \quad (3.3)$$

Since

$$\|\nabla u\|^2 = \int_{\Omega} -u \cdot \Delta u dx \leq \|u\| \|\Delta u\| \leq C_s \|\nabla u\| \|\Delta u\|, \quad (3.4)$$

i.e.,

$$\|\nabla u\| \leq C_s \|\Delta u\|, \quad (3.5)$$

we deduce from (3.3) that

$$\|\nabla u\|^2 \leq e^{-\frac{\alpha}{C_s^2} t} \|\nabla u_0\|^2 + \frac{C_s^2 \|f\|^2}{\alpha^2}, \quad (3.6)$$

and the second estimate in (3.1) follows. the first estimate in (3.1) directly follows from the Poincaré inequality. Thus, the proof is complete.  $\square$

We now give the proof of Theorem 1.2.

### **Proof of Theorem 1.2.**

Differentiating the equation (1.1) with respect to  $t$ , then taking the dual product of the resultant with  $u_t$  yields

$$\frac{1}{2} \frac{d}{dt} \|u_t\|^2 + a \|\nabla u_t\|^2 = -a'(l(u)) \int_{\Omega} u_t g dx \int_{\Omega} \nabla u \cdot \nabla u_t dx. \quad (3.7)$$

We deduce that it holds

$$\frac{d}{dt} \|u_t\|^2 + 2\alpha \|\nabla u_t\|^2 \leq 2|a'(l(u))| \|g\| \|u_t\| \|\nabla u\| \|\nabla u_t\| \quad \forall t \geq 0. \quad (3.8)$$

Since

$$|l(u)| \leq \|g\| \|u\|,$$

by Lemma 3.1, we get

$$\frac{d}{dt} \|u_t\|^2 + 2\alpha \|\nabla u_t\|^2 \leq \frac{4C_s^2}{\alpha} \sup_{|s| \leq \frac{2C_s^2}{\alpha} \|g\| \|f\|} |a'(s)| \cdot \|g\| \|f\| \|\nabla u_t\|^2 \quad \forall t \geq t_0. \quad (3.9)$$

From (1.9) we deduce that for some  $\epsilon$  it holds that

$$\frac{d}{dt} \|u_t\|^2 + 2\alpha \|\nabla u_t\|^2 \leq (2\alpha - \epsilon) \|\nabla u_t\|^2 \quad \forall t \geq t_0. \quad (3.10)$$

Hence

$$\frac{d}{dt} \|u_t\|^2 + \frac{\epsilon}{C_s^2} \|u_t\|^2 \leq 0 \quad \forall t \geq t_0. \quad (3.11)$$

This implies that  $\|u_t\|$  decays exponentially to zero as time goes to infinity. From

$$\|u(t) - u(s)\| \leq \int_s^t \|u_\tau\| d\tau \quad (3.12)$$

we easily derive that  $u(t)$  is a Cauchy sequence in  $L^2$ . It follows from (3.12) that as time goes to infinity,

$$u(\cdot, t) \rightarrow v(x) \quad \text{in } L^2(\Omega). \quad (3.13)$$

Therefore,

$$a(l(u)) \rightarrow a(l(v)). \quad (3.14)$$

Since  $u_t \rightarrow 0$  in  $L^2$ , it is easy to derive from (1.1) that  $u(t)$  also converges to  $v$  in  $H^2$  and  $v$  is a stationary point. Thus the proof is complete.  $\square$

**Remark 3.1** *One could extend the above results to the case where*

$$a(l(u))\Delta$$

*is replaced by a general elliptic operator of the type*

$$\partial_{x_i} \{a_{ij}(l(u)) \partial_{x_j}\}$$

*under similar assumptions on  $a'_{ij}$ ,  $\|f\|$ ,  $\|g\|$ . In the case (1) such an extension remains to be done.*

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