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LEBESGUE MEASURE OF FEIGENBAUM JULIA SETS

ARTUR AVILA AND MIKHAIL LYUBICH

ABSTRACT. We construct Feigenbaum quadratic-like maps with a Julia set of positive Lebesgue measure. Indeed, in the quadratic family $P_c : z \mapsto z^2 + c$ the corresponding set of parameters c is shown to have positive Hausdorff dimension. Our examples include renormalization fixed points, and the corresponding quadratic polynomials in their stable manifold are the first known rational maps for which the hyperbolic dimension is different from the Hausdorff dimension of the Julia set.

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1. INTRODUCTION

One of the major successes of the theory of one-dimensional dynamical systems was the conceptual explanation, in terms of the dynamics of a renormalization operator, of the striking universality phenomena discovered by Feigenbaum and Coulet-Tresser in 1970's. At the center of the picture lies the concept of a *Feigenbaum map*, which is a quadratic-like map that can be renormalized infinitely many times with bounded combinatorics and *a priori* bounds (a certain control on the nonlinearity). The successive renormalizations are then exponentially asymptotic to a *renormalization attractor*, see [S2, McM3, L5]. In the simplest case of stationary combinatorics, the renormalization attractor consists of a single renormalization fixed point. As a consequence, the dynamics of such Feigenbaum maps display remarkable self-similarity reflected in the geometry of the corresponding Julia sets.

In fact, understanding the geometry of Feigenbaum Julia sets already played a key role in the first proof of exponential convergence of the renormalization [McM3]. However, for a long time the theory had been unable to tackle natural geometric problems: do Feigenbaum Julia sets have full Hausdorff dimension or even positive area? (See [McM3], page 177, question 3). In [AL1], a new approach to these problems was developed, which allowed us to show, in particular, that Feigenbaum Julia sets can have Hausdorff dimension strictly less than two, while leaving open

the problem of whether they can ever have positive area. The goal of this work is to settle the latter question affirmatively. Namely, we will show that Julia sets of positive area appear already among Feigenbaum quadratic polynomials with stationary combinatorics (note that there are only countably many such polynomials). At the same time, we construct a set of parameters c of positive Hausdorff dimension such that the quadratic polynomials $P_c : z \mapsto z^2 + c$ are Feigenbaum maps with Julia sets of positive area.

Note that our results (as well as the earlier results of [AL1]) go against intuition coming from hyperbolic geometry. Indeed, according to the philosophy known as *Sullivan's dictionary*, there is a correspondence between certain objects and results in complex dynamics and hyperbolic geometry. As McMullen suggested in [McM3] (see especially the last paragraph on page 177), Feigenbaum maps are analogous to 3-manifolds with two ends, one of which is geometrically finite, while the other one is asymptotically fibered over the circle. The limit sets $\Lambda(\Gamma)$ of the corresponding Kleinian groups have zero area but full Hausdorff dimension (see Thurston [Th] and Sullivan[S1]). So, it may look like the dictionary completely breaks down at this point, though in fact there is a way to rehabilitate it (see §1.2.6 below).

1.1. Feigenbaum maps. Let us begin with reminding briefly the main concepts of the complex renormalization theory. (See §2 for a precise account.) A *quadratic-like map* is a holomorphic double covering $f : U \rightarrow V$ where U and V are quasidisks with U compactly contained in V . The *filled-in Julia set* of f is the set $K(f)$ of points z with $f^n(z) \in U$ for all $n \geq 0$; its boundary is the *Julia set* $J(f)$. The filled-in Julia set is always a full compact set which is either connected or totally disconnected, according to whether or not it contains the critical point.

Simplest examples of quadratic-like maps are given by restrictions of quadratic maps $P_c : z \mapsto z^2 + c$ to suitable neighborhoods of $K(P_c)$. The precise choice of the restriction is dynamically inessential, which is expressed by saying they all define the same *quadratic-like germ*. The *Mandelbrot set* \mathcal{M} can then be defined as the set of parameters $c \in \mathbb{C}$ for which $K(P_c)$ is connected.

The central role of the quadratic family is made clear by Douady-Hubbard's *Straightening Theorem* that states that each quadratic-like map with connected Julia set is *hybrid conjugate* to a unique quadratic map P_c , i.e., there exists a quasiconformal map $h : (\mathbb{C}, K(f)) \rightarrow (\mathbb{C}, K(P_c))$ satisfying $h \circ f = P_c \circ h$ near $K(f)$ and with $\bar{\partial}h|_{K(f)} = 0$ a.e. We say that P_c is the *straightening* of f , and write $c = \chi(f)$.

A quadratic-like map $f : U \rightarrow V$ is said to be *renormalizable* with period $\mathfrak{p} \geq 2$ if the \mathfrak{p} -th iterate of f can be restricted to a quadratic-like map $g : U' \rightarrow V'$ such that the *little Julia sets* $K_j := f^j(K(g))$, $0 \leq j \leq \mathfrak{p} - 1$, are connected and do not cross each other (meaning that $K_j \setminus K_i$ are connected for $i \neq j$). We can always choose g to have the same critical point as f , and such a g is called the *pre-renormalization of period \mathfrak{p} of f* . The smallest possible value of \mathfrak{p} is called the *renormalization period* of f , and the corresponding pre-renormalization, considered up to affine conjugacy, is called the *renormalization* of f and denoted by $R(f)$. The *renormalization operator* $f \mapsto Rf$ is then well defined at the level of affine conjugacy classes of quadratic-like germs.

The set of parameter values corresponding to renormalizable quadratic maps is disconnected. Its connected components are called (maximal) *Mandelbrot copies*, which can be of two types, *primitive* or *satellite*, according to whether they are

canonically homeomorphic (via the straightening map $c \mapsto \chi(R(P_c))$) to the full Mandelbrot set or to $\mathcal{M} \setminus \{1/4\}$ (note that $1/4$ is the *cusps* of the *main cardioid* bounding the “largest” component of the interior of \mathcal{M}). Alternatively, (maximal) satellite copies can be distinguished by the property that they are “attached” to the main cardioid at the “missing” cusp. They can also be distinguished dynamically: For the satellite renormalization (with the minimal period), all little Julia sets have a common touching point, while for the primitive renormalization, they are pairwise disjoint.

The *renormalization combinatorics* of a renormalizable quadratic-like map f is the Mandelbrot copy \mathcal{M}' containing $\chi(f)$.¹ The renormalization period only depends on the renormalization combinatorics, but the converse is false (except for period two). There are however only finitely many combinatorics corresponding to each period.

Assume now that f is *infinitely renormalizable*, i.e., the renormalizations $R^j f$ are well defined for all $j \geq 0$. We say that f has *bounded combinatorics* if the renormalization periods of the successive renormalizations $R^j f$, $j \geq 0$, remain bounded. The combinatorics is *stationary* if it is the same for all $R^j f$.

The “analytic quality” of a quadratic-like map $f : U \rightarrow V$ is measured by the *modulus* of the *fundamental annulus* $V \setminus \overline{U}$, denoted by $\text{mod } f$. (The quality is poor if $\text{mod } f$ is small.) An infinitely renormalizable map is said to have *a priori bounds* if all of its renormalizations have definite quality, i.e., the corresponding moduli are bounded away from zero. (*A priori bounds* are equivalent to precompactness of the full renormalization orbit $\{R^j f\}_{j \geq 0}$ in a suitable topology.) While by no means all infinitely renormalizable maps have *a priori bounds*, many do, and in particular it is conjectured that bounded combinatorics implies *a priori bounds* (which has indeed been proved whenever the renormalization combinatorics of all the $R^j f$ are primitive [K]).

Recall that a *Feigenbaum map* is an infinitely renormalizable quadratic-like map with bounded combinatorics and *a priori bounds*.

Theorem 1.1. *There exists a Feigenbaum quadratic polynomial P_c with primitive stationary combinatorics whose Julia set J_c has positive area.*

Our methods yields in fact an infinite family of primitive Mandelbrot copies which have the property that all infinitely renormalizable maps whose renormalization combinatorics (for all the renormalizations) belong to this family have Julia sets of positive area. We recall that any finite family \mathcal{F} of primitive Mandelbrot copies with $\#\mathcal{F} \geq 2$ defines an associated *renormalization horseshoe* \mathcal{A} consisting of all quadratic-like maps that belong to the ω -limit of the renormalization operator restricted to those combinatorics, see [AL2]) (complemented with [K]) for a recent account of this result . The dynamics of $R|\mathcal{A}$ is topologically semiconjugate to the shift on $\mathcal{F}^{\mathbb{Z}}$, and the corresponding quadratic parameters in $\chi(\mathcal{A})$ form a Cantor set naturally labeled by $\mathcal{F}^{\mathbb{N}}$. This Cantor set has bounded geometry by [L5], so we can conclude:

¹The renormalization combinatorics can be alternatively encoded by a finite graph, the *Hubbard tree*, which describes the positioning of the little Julia sets (of the first pre-renormalization) inside the full Julia set. It coincides with the Hubbard tree of the superattracting map $\mathbf{f}_{c'}$, $c' \in \mathcal{M}'$, whose period is equal to the renormalization period of f .

Theorem 1.2. *The set of Feigenbaum quadratic maps with Julia sets of positive area has positive Hausdorff dimension in the parameter space.*

1.2. What do we learn about Julia sets of positive area?

1.2.1. *Preamble: Area Problem.* The problem of whether all nowhere dense Julia sets have zero area goes back to the classical Fatou’s memoirs who gave first examples of such Julia sets [F].² In 1980-90’s, broad classes of Julia sets with zero area were given in [L1, L3, Sh2, Yar] and [U, PR, GS]. First examples of a rational maps³ (in fact, quadratic polynomials) with nowhere dense Julia sets with positive area have been recently constructed by Buff and Cheritat [BC] in a remarkable development that successfully brought to completion Douady’s program from mid-1990’s. (See also Yampolsky [Ya3] for an alternative point of view on the final piece of their argument.) An important technical input to this program was supplied by the recent breakthrough in the Parabolic Bifurcation Theory by Inou and Shishikura [IS].

The strategy carried by Buff and Cheritat depends on a *Liouvillean mechanism* of fast rational approximation. It produces three type of examples: *Cremer*, *Siegel*, and *infinitely renormalizable with unbounded satellite combinatorics*.⁴

Feigenbaum Julia sets have a rather different nature, so our work brings new light on the realm of Julia sets of positive area.

1.2.2. *Parameter visibility.* Julia sets of positive area are supposed to be *visible* objects. However, sets of parameters produced by the Liouvillean mechanisms (such as in [BC]) tend to be tiny: they probably have zero Hausdorff dimension. (This is definitely so in the Cremer case as the whole set of Cremer parameters has zero Hausdorff dimension).

By our previous work [AL1], Feigenbaum Julia sets of positive area are more robust: the existence of a single Feigenbaum Julia set of positive area inside some renormalization horseshoe implies that there is a whole “sub-horseshoe” of them, restricted to which the renormalization dynamics is topologically conjugate to an subshift of finite type. This creates a parameter set of positive Hausdorff dimension. The construction we use to proof of Theorem 1.2 is even more precise, providing us with full renormalization horseshoes and allowing us to obtain an effective estimate: the set of parameters c such that P_c is a Feigenbaum map of positive area has Hausdorff dimension at least $1/2$.

We note that it is expected that Lebesgue almost every quadratic map is hyperbolic, and hence has a Julia set of not only zero Lebesgue measure but even of Hausdorff dimension less than two. It is unclear whether the set of all complex Feigenbaum parameters has Hausdorff dimension strictly less than two.⁵ At the moment, it is only known that the Hausdorff dimension of these parameters is at least 1 [L4].

²What Fatou showed is that if $|Df(z)| > \deg f$ for all $z \in J(f)$, then $J(f)$ is a Cantor set of zero length.

³For transcendental entire functions, a class of Julia sets of zero area was described in [EL1], and examples of Julia sets with positive area appeared in [EL2, McM1].

⁴We recall that a quadratic map with a periodic orbit β with irrationally indifferent multiplier $e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, is classified as *Siegel* or *Cremer* according to whether it is locally linearizable near β or not.

⁵The real analogue of this statement is known to be true [AM].

1.2.3. *Poincaré series and Hausdorff dimension.* The notion of *Poincaré series* was transferred from the theory of Kleinian groups to Holomorphic Dynamics by Sullivan [S2], and it became an efficient tool in the study of Hausdorff dimension of Julia sets. Previously to this work, in all known cases the Hausdorff dimension of rational Julia sets coincided with the critical exponent of the Poincaré series (see [U, PR, GS] and [AL1]). On the other hand, it was shown in [AL1] that equality must break down in the case of a Feigenbaum map with periodic combinatorics and positive Lebesgue measure Julia set.⁶

The critical exponent does coincide with the *hyperbolic dimension* for all Feigenbaum Julia sets (and indeed for all known cases of rational maps), so our examples display a definite gap between the Hausdorff dimensions of the Julia set and of its hyperbolic subsets. It is conceivable, however, that *for Julia sets of zero area, the critical exponent, Hausdorff dimension and hyperbolic dimension, are all equal* (without any further assumptions on the rational map.)

1.2.4. *Positive measure vs non-local connectivity.* There was a general feeling that these two phenomena are tightly linked as the examples constructed by Buff and Cheritat are probably all non-locally connected. (Note, in particular, that Cremer Julia sets are never locally connected). On the other hand, all Feigenbaum Julia sets have well behaved geometry and in particular are locally connected, see [HJ, McM2]. Note that local connectivity make a Julia set *topologically tame*: it admits an explicit topological model (see [D3]). Thus, our examples show that positive area is compatible with topological tameness.

Related to this issue is the fact that all Feigenbaum Julia sets constructed here have primitive combinatorics, while the previously known infinitely renormalizable examples have satellite combinatorics. In fact all known examples of infinitely renormalizable maps with non-locally connected Julia set have satellite combinatorics.

1.2.5. *Wild attractors and ergodicity.* The measure-theoretic dynamics on Feigenbaum Julia sets of positive area is well understood. In particular it is *ergodic* with respect to the Lebesgue measure [P], and there is a uniquely ergodic Cantor attractor $A \subset J(f)$ (of Hausdorff dimension strictly less than two) that attracts almost all orbits in the Julia set, see [L1]. Moreover, almost all orbits are equidistributed with respect to the canonical measure on A . Such a structure was known to exist in the real dynamics [BKNS] but not in the complex rational dynamics.⁷

By contrast, the description of the measure-theoretical dynamics of the examples obtained in [BC] is less developed. While it is known that there exists an attractor A properly contained in $J(f)$ such that $\omega(z) \subset A$ for almost all $z \in J(f)$, see [L1, IS, Ch], the structure of A is not fully understood. It is also unknown whether the Lebesgue measure on $J(f)$ is ergodic under the dynamics.

1.2.6. *Sullivan's Dictionary.* A parallel spectacular development in the problem of area and Hausdorff dimension has happened in the Theory of Kleinian groups. However, the outcome appeared to be quite different. In mid 1990's, it was proved by Bishop and Jones [BJ] that the limit set $\Lambda = \Lambda(\Gamma)$ of a (finitely generated)

⁶More recently, such a phenomenon was also observed in the transcendental dynamics [UZ].

⁷See, however, [L2, R] for a related phenomenon in transcendental dynamics.

Kleinian group Γ has full Hausdorff dimension if and only if the group is geometrically infinite. As geometrically finite groups correspond to hyperbolic or parabolic rational maps, we see that the answer for Kleinian groups is much simpler.

As the *area* is concerned, it had been the subject of the long-standing *Ahlfors Area Conjecture* asserting that any limit set $\Lambda(\Gamma)$ has zero area as long as it is different from the whole sphere. Through the work of Thurston [Th], Bonahon [Bo] and Canary [Ca], this conjecture was reduced to *Marden's Tameness Conjecture*, and the latter was recently proved by Agol [Ag] and Calegari-Gabai [CG]. Thus, there are no non-trivial limit sets Λ of positive area: again, the situation is for Kleinian groups is much more definite compared with rational maps.

It does not mean, however, that Sullivan's Dictionary between Kleinian groups and rational maps completely breaks down at this point. Kleinian groups belong to a special class of *reversible* dynamical systems: the corresponding geodesic flow on the hyperbolic 3-manifold M_Γ admits a nice involution that conjugates it to the inverse flow. The analogous flow for a rational map f lives on the hyperbolic 3-lamination \mathcal{H}_f constructed in [LM]. However, this flow is not reversible, which reflects the *unbalanced* property (see the next section) of the underlying maps and bears responsibility for richer geometric properties of Julia sets.

1.3. Basic Trichotomy. To put our result into deeper perspective, let us briefly recall the basic trichotomy of [AL1]. Consider the following alternative for Feigenbaum maps:

Lean case: $\text{HD}(J(f)) < 2$;

Balanced case: $\text{HD}(J(f)) = 2$ but $\text{area } J(f) = 0$;

Black hole case: $\text{area } J(f) > 0$.

In that paper, we showed that if a periodic point of renormalization is either of Lean or Black hole type, then this can be verified “in finite time”, by estimating some geometric quantities associated to some (not necessarily the first) renormalization of f . Namely, let us define two parameters:

- η_n gives the probability for an orbit starting in the domain of f to enter the domain of the n -th pre-renormalization,
- ξ_n gives the probability that an orbit starting in the domain of the n -th pre-renormalization will never come back to it.

We showed that in the Lean case $\eta_n/\xi_n \rightarrow 0$ exponentially, in the Black hole case $\eta_n/\xi_n \rightarrow \infty$ exponentially, and that in the Balanced case η_n/ξ_n remains bounded away from zero and infinity. Moreover, there is an effective constant $C > 1$ (given in terms of some rough geometric parameters, like $\text{mod } f$) such that if $R^n f = f$ then

- $\eta_n/\xi_n > C$ implies the Black hole case,
- $\eta_n/\xi_n < C^{-1}$ implies the Lean case.

Regarding the Balanced case, Theorem 8.2 of [AL1] asserts that the existence of both Lean and Black hole Feigenbaum maps inside some renormalization horseshoe implies that *there exist some Balanced Feigenbaum maps* in this horseshoe, but the construction does not yield a renormalization periodic point. In fact, it seems

unlikely that Balanced maps with periodic combinatorics exist (the geometric parameters would be too fine tuned for it to happen “by chance” given that there are only countably many periodic points of renormalization).⁸

1.4. Strategy. As discussed above, [AL1] gives a probabilistic criterion for the Black Hole property to hold for a fixed point of renormalization: it suffices to check that η_n/ξ_n is sufficiently large for some n . Below we will use this only in the particular case $n = 1$: We will produce a sequence of fixed points of renormalization $f_m : U_m \rightarrow V_m$ such that $\inf \eta(m) > 0$ while $\lim \xi(m) = 0$, where $\eta(m) = \eta_1(f_m)$ and $\xi(m) = \xi_1(f_m)$. We will also verify that the rough initial geometry of the fundamental annuli $V_m \setminus \overline{U}_m$ remains under uniform control. Since the “constant to beat” in the criterion only depends on such a control, this will show that for m sufficiently large the criterion is satisfied so that the Julia set of f_m has positive Lebesgue measure.

It is easy to see that if the sequence $\chi(f_m)$ converges to a parameter c for which $\text{area } K(P_c) = 0$, and the rough initial geometry remains under control, then $\eta(m) \rightarrow 0$. Given this observation, it is natural to consider sequences of renormalization combinatorics which approach a parameter c with either a Siegel disk or a parabolic point. In our argument, we will take c to have a Siegel disk of bounded type. One still has to select the combinatorics very carefully, and a number of natural options we had initially tried had either displayed degeneration of the geometry (for instance, with growing modulus of the fundamental annulus), or could not be treated in a definitive way without computer assistance.

We now describe the idea more precisely. Let us consider a quadratic polynomial P_c that has a Siegel disk S with rotation number $\theta = [N, N, \dots]$, N being big enough. Let $p_m/q_m = [N, \dots, N]$ be the continuous fraction approximands to θ , and let P_{c_m} be the corresponding quadratic maps with a parabolic fixed point with rotation number p_m/q_m . We perturb c_m within the (p_m/q_m) -limb (the connected component of $\mathcal{M} \setminus \{c_m\}$ not containing 0) to a Misiurewicz map P_{a_m} , i.e., one for which the critical orbit is eventually periodic, but not periodic. Then we further perturb a_m to a superattracting parameter b_m . This parameter is the center of some maximal primitive Mandelbrot copy \mathcal{M}_m .

Let $f_m : U_m \rightarrow V_m$ be the corresponding renormalization fixed points with stationary combinatorics \mathcal{M}_m . To control the dynamics of these maps in what follows, we need a good control of the postcritical set after all the perturbations. This has also been crucial in Buff and Cheritat’s work [BC], who proved using the Inou-Shishikura renormalization theory [IS] (which currently is only available for large N , hence the choice above), that the postcritical set of P_{c_m} stays in a small neighborhood of the Siegel disk S . Our further choice of a_m and b_m is in part designed to keep this property for the further perturbations: In particular excursions of the critical orbit away from the Siegel disk must be prevented to avoid excessive expansion (which would again lead to growing fundamental annuli). Thus, the periodic orbit on which the critical point eventually lands must be taken quite close to the Siegel disk. The most natural choice would be the periodic orbit

⁸See also the discussion in [AL1] about a related problem for real maps of the form $x \mapsto |x|^\alpha + c$: therein one can vary the degree α of the critical point continuously to fine-tune the parameters, so the corresponding Balanced case is believed to exist (and a conditional proof is given, subject to a Renormalization Conjecture), but it is unlikely that the fine tuned degrees would ever correspond to an integer number (i.e., to a polynomial).

with combinatorial rotation number p_m/q_m that arises from the bifurcation of P_{c_m} , but for technical implementation reasons we actually use some orbit of rotation number $p_{m-\kappa}/q_{m-\kappa}$, for some big but bounded (as $m \rightarrow \infty$) κ (so that the critical point still only goes a bounded number of levels up in terms of the cylinder Siegel renormalization).

We then fine tune the superattracting parameter b_m to get a suitable control on the initial geometry of the first renormalization. While we want the moduli of fundamental annuli to remain bounded, we'd like them to be sufficiently large to obtain control on the actual renormalization fixed point. Indeed, there is a "threshold" lower bound on the moduli of the fundamental annuli of the first renormalization of a Feigenbaum quadratic map with stationary primitive combinatorics, which, once surpassed, implies uniform control for the associated renormalization fixed point. Below this threshold, current techniques do not give such uniform bounds without further restrictions (which would in particular not apply when approaching Siegel parameters). Thus, we make the critical orbit (after perturbation) follow closely the periodic orbit for large but bounded number of turns around the Siegel disk, picking up the right amount of expansion from the periodic orbit before drifting apart and closing.

Once the geometry of the first renormalization is controlled, we construct a *safe trapping disk* D that it stays away from the postcritical set, captures all orbits that escape from the Siegel disk S to infinity and has the property that a definite portion of D lands in the renormalization domain U . Then a direct Distortion Argument implies that the pullbacks of U occupy a definite proportion of S , which implies that the landing probability η_m stays bounded away from 0.

To control the escaping parameter ξ_m , we make use of the *Siegel Return Machinery* that ensures high probability of returns back to the trapping disk, and hence high probability of eventual landing in the renormalization domain U . (The Return Machinery makes use of the *hyperbolic expansion* outside the postcritical set [McM2] and that was also used by Buff and Cheritat [BC]).

In this construction, there is one free parameter that can be varied without significant impact on the geometry of the first renormalization, which is the time the critical point spends in the *parabolic gate* created when the parabolic map P_{a_m} is perturbed to the Misiurewicz map P_{a_m} . There is a uniform control of this perturbation governed by the limiting transit map (the *geometric limit*). Varying this time produces a sequence of Black Hole combinatorics whose Mandelbrot copies decay quadratically. Alternating these combinatorics creates a Cantor set of Hausdorff dimension $\geq 1/2$ consisting of Black Hole parameters.

To carry out the above strategy, we make use of four Renormalization Theories:

- Renormalization of *quadratic-like maps*, including the probabilistic criterion of [AL1], is discussed in §2.
- Renormalization of *quasicritical circle maps* is developed in §3 (roughly speaking, "quasicritical" means that the map is allowed to lose analyticity at the critical point, but is assumed to be quasiregular over there).
- *Siegel* renormalization theory based upon renormalization of quasicritical circle maps is laid down in §4.
- Finally, in §5 we briefly discuss the *parabolic* renormalization, and particularly, the *Inou-Shishikura Theory*.

With these renormalization tools in hands, we proceed to the main construction (§6).

1.5. Basic Terminology and Notation. $\mathbb{N}_0 = \{0, 1, \dots\}$, $\mathbb{N} \equiv \mathbb{N}_1 = \{1, 2, \dots\}$,

and in general, $\mathbb{N}_\kappa = \{n \in \mathbb{N} : n \geq \kappa\}$;

$\bar{\mathbb{N}}_\kappa = \mathbb{N}_\kappa \cup \infty$ (with the natural topology);

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$;

$\mathbb{D}_R(a) = \{z : |z - a| < R\}$; $\mathbb{D}_R = \mathbb{D}_R(0)$, $\mathbb{D} = \mathbb{D}_1$;

Notation \mathbb{T} will be used for both the unit circle in \mathbb{C} and its angular parametrization by \mathbb{R}/\mathbb{Z} ;

area refers to the Lebesgue measure;

For a set $Z \subset \mathbb{C}$ and a point $z \in Z$, we let $\text{Comp}_z(Z)$ be the component of Z containing z .

For a topological annulus $A \Subset \mathbb{C}$, we let $\partial^o A$ and $\partial^i A$ be its *outer* and *inner* boundaries.

$\text{Dom } f$ is the domain of a map f ;

$\text{orb } z = \text{orb}_f z$ is the forward orbit of a point z ;

\mathcal{O}_f is the postcritical set of a map f , i.e., the closure of the orbit of its critical point;

$\mathbf{f}_\theta : z \mapsto e^{2\pi i \theta} z + z^2$, $\theta \in \mathbb{C}/\mathbb{Z}$;

$\mathcal{F} = (\mathbf{f}_\theta)_{\theta \in \mathbb{C}}$ is the quadratic family;

\mathcal{M} is the Mandelbrot set.

By saying that some quantity, e.g. η , depending on parameters is *definite*, we mean that $\eta \geq \epsilon > 0$ where ϵ is independent of the parameters (or rather, it may depend only on some, explicitly specified, parameters). By saying that a set K is *well inside a domain* $D \Subset \mathbb{C}$ we mean that $K \Subset D$ with a definite $\text{mod}(U \setminus K)$. The meaning of expressions *bounded*, *comparable*, etc. is similar. If we need to specify a constant then we say “ ϵ -definite”, “ C -comparable (\asymp)”, etc.

Given a pointed domain (D, β) , we say that β *lies in the middle of* D , or equivalently, that D *has a bounded shape around* β if

$$\max_{\zeta \in \partial D} |\beta - \zeta| \leq C \min_{\zeta \in \partial D} |\beta - \zeta|,$$

where C is a constant that may depend only on specified parameters.

2. QUADRATIC-LIKE MAPS

2.1. Basic definitions. A *quadratic-like map* $f : U \rightarrow V$ [DH2], which will also be abbreviated as a *q-l map*, is a holomorphic double branched covering between two Jordan disks $U \Subset V \subset \mathbb{C}$. It has a single critical point that we denote c_0 . The annulus $A = U \setminus \bar{V}$ is called the *fundamental annulus* of f . We let $\text{mod } f := \text{mod } A$. The *filled Julia set* $K(f)$ is the set of non-escaping points:

$$K(f) = \{z : f^n z \in U, n = 0, 1, 2, \dots\}$$

Its boundary is called the *Julia set* $J(f)$. The (filled) Julia set is either connected or Cantor, depending on whether the critical point is non-escaping (i.e., $c_0 \in K(f)$) or otherwise.

Two quadratic-like maps $f : U \rightarrow V$ and $\tilde{f} : \tilde{U} \rightarrow \tilde{V}$ are called *hybrid equivalent* if they are conjugate by a quasiconformal map $h : (V, U) \rightarrow (\tilde{V}, \tilde{U})$ such that $\bar{\partial}h = 0$ a.e. on $K(f)$.

A simplest example of a quadratic-like map is provided by a quadratic polynomial $P_c : z \mapsto z^2 + c$ restricted to a disk \mathbb{D}_R of sufficiently big radius. The Douady and Hubbard *Straightening Theorem* asserts that any quadratic-like map f is hybrid equivalent to some restricted quadratic polynomial P_c . Moreover, if $J(f)$ is connected then the parameter $c \in \mathcal{M}$ is unique.

As for quadratic polynomials, two fixed points of a quadratic like maps with connected Julia set have a different dynamical meaning. One of them, called β , is the landing point of a proper arc $\gamma \subset U \setminus K(f)$ such that $f(\gamma) \supset \gamma$. It is either repelling or parabolic with multiplier one. The other fixed point, called α , is either non-repelling or a *cut-point* of the Julia set.

2.1.1. Quadratic-like families. A *quadratic-like family* $\mathbb{F} = (F_\lambda : U_\lambda \rightarrow V_\lambda)$ over a parameter domain $\Lambda \subset \mathbb{C}$ is a family of quadratic-like maps F_λ holomorphically depending on λ . The latter means more precisely that the set

$$\mathbb{U} = \bigcup_{\lambda \in \Lambda} U_\lambda$$

is a domain in \mathbb{C}^2 and the function $f_\lambda(z)$ is holomorphic on \mathbb{U} . Let us normalize it so that $c_0 = 0$ for all f_λ . The associated Mandelbrot set is defined as

$$\mathcal{M}_{\mathbb{F}} = \{\lambda \in \Lambda : J(F_\lambda) \text{ is connected}\}.$$

Let us select a base point λ_0 and let $U_\circ \equiv U_{\lambda_0}$ etc. We say that a quadratic-like family \mathbb{F} is *equipped* if there is a holomorphic motion

$$h_\lambda : \bar{V}_\circ \setminus U_\circ \rightarrow \bar{V}_\lambda \setminus U_\lambda$$

of the fundamental annulus $\bar{V}_\lambda \setminus U_\lambda$ over the pointed domain (Λ, λ_0) which is *equivariant* on the boundary of the annulus, i.e.,

$$h_\lambda(f_\circ(z)) = f_\lambda(h_\lambda(z)), \quad z \in \partial U_\circ.$$

An equipped quadratic-like family \mathbb{F} is called *proper* if $f_\lambda(0) \in \partial V_\lambda$ for $\lambda \in \partial \Lambda$ (which assumes implicitly that the family f_λ is continuous up to $\partial \Lambda$).

A quadratic-like family \mathbb{F} is called *unfolded* if the curve

$$\lambda \mapsto f_\lambda(0), \quad \lambda \in \partial \Lambda,$$

has winding number 1 around 0.

Theorem 2.1. [DH2] *For any equipped proper unfolded quadratic-like family \mathbb{F} , the Mandelbrot set $\mathcal{M}_{\mathbb{F}}$ is canonically homeomorphic to the standard Mandelbrot set \mathcal{M} .*

The proof can be also found in [L6].

2.2. Renormalization. A quadratic-like map $f : U \rightarrow V$ is called *DH renormalizable* (after Douady and Hubbard) if there is a quadratic-like restriction

$$Rf \equiv R_{DH}f = f^{\mathfrak{p}} : U' \rightarrow V'$$

with connected Julia set K' such that the sets $f^i(K')$, $k = 1, \dots, \mathfrak{p} - 1$, are either disjoint from K' or else touch it at its β -fixed point.⁹ In the former case the renormalization is called *primitive*, while in the latter it is called *satellite*.

The map $Rf : U' \rightarrow V'$ is called the *pre-renormalization* of f . If it is considered up to rescaling, it is called the *renormalization* of f .

⁹See [McM2] for a discussion of this condition.

The sets $f^i(K')$, $k = 0, \dots, \mathfrak{p} - 1$, are referred to as the *little (filled) Julia sets*. Their positions in the big Julia set $K(f)$ determines the renormalization *combinatorics*. The set of parameters c for which the quadratic polynomial P_c is renormalizable with a given combinatorics forms a *little Mandelbrot copy* $\mathcal{M}' \subset \mathcal{M}$. In fact, the family of renormalizations $R(P_c)$, $c \in \mathcal{M}'$, with a given combinatorics can be included in a quadratic-like family $\mathbb{F} = (f^{\mathfrak{p}} : U_c \rightarrow V_c)$ over some domain $\Lambda \supset \mathcal{M}'$ so that $\mathcal{M}' = \mathcal{M}_{\mathbb{F}}$. A natural base point $c_{\circ} \in \mathcal{M}'$ in this family is the superattracting parameter with period \mathfrak{p} . It is called the *center* of \mathcal{M}' . Any superattracting parameter in \mathcal{M} with period $\mathfrak{p} > 1$ is the center of some Mandelbrot copy \mathcal{M}' like this. Moreover, in case of primitive combinatorics the quadratic-like family \mathbb{F} is proper and unfolded. (See [DH2, D1, L6] for a discussion of all these facts.)

We can encode the renormalization combinatorics by the corresponding copy \mathcal{M}' itself. Equivalently, it can be encoded by the center c_{\circ} of \mathcal{M}' or the corresponding Hubbard tree H' .

A little Mandelbrot copy is called *primitive* or *satellite* depending on the type of the corresponding renormalization. They can be easily distinguished as any satellite copy is attached to some hyperbolic component of $\text{int } \mathcal{M}$ and does not have the cusp at its root point.

For infinitely renormalizable maps, notions of *stationary/bounded combinatorics*, *a priori* bounds, and *Feigenbaum maps* were defined in the Introduction (§1.1). We say that a Feigenbaum map is *primitive* if all its renormalizations are such.

One says that a family \mathcal{F} of Feigenbaum maps (e.g., for the family of maps with a given combinatorics) has *beau bounds* if there exists $\mu > 0$ such that for any $f \in \mathcal{F}$ we have

$$\text{mod } R^n f \geq \mu \text{ for all } n \geq n(\text{mod } f).$$

It was proved by Kahn [K] that *primitive Feigenbaum maps have beau bounds*, with μ depending only on the combinatorial bound. In fact, μ can be made uniform over some class of combinatorics [KL].¹⁰

The *renormalization fixed point* f_* is a quadratic-like map which is invariant under renormalization: $Rf = f$. In terms of the *pre-renormalization*, there exists a *scaling factor* $\lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}$ such that

$$Rf(z) = \lambda^{-1} f(\lambda z).$$

Theorem 2.2. *For any stationary bounded combinatorics with a beau bound, there exists a unique renormalization fixed point f_* with this combinatorics. Moreover, $\text{mod } f_* \geq \mu$, where $\mu > 0$ is the beau bound.*

This theorem was originally proved by Sullivan [S2]. Other proofs were given by McMullen [McM3], and recently, by the authors [AL2].

2.3. Probabilistic criterion for positive area. Let us now introduce precisely probabilistic parameters η and ξ mentioned in the Introduction. Let $f : U \rightarrow V$ be a Feigenbaum map with *a priori* bound $\mu > 0$ (i.e., $\text{mod } R^n f \geq \mu$ for all $n \in \mathbb{N}$), and let $Rf : U' \rightarrow V'$ be its first pre-renormalization, $A' = U' \setminus \overline{V}'$ be the corresponding fundamental annulus.

¹⁰ We will not use these results as the combinatorics we construct do not fall into the class [KL]. On the other hand, *beau bounds* can be easily supplied for our class.

The *landing parameter* η is the probability of landing in U' . Precisely, let $\mathcal{X} = \bigcup_{n \in \mathbb{N}} f^{-n}U'$ be the set of points in U that eventually land in U' . Then

$$(2.1) \quad \eta = \frac{\text{area } \mathcal{X}}{\text{area } U}.$$

The *escaping parameter* ξ is the probability of escaping from the fundamental annulus A' . Precisely, let \mathcal{Y} be the set of points in A that never return back to V' :

$$\mathcal{Y} = \{z \in A' : f^n z \notin V' \text{ for } n \geq 1 \text{ (as long as } f^n z \text{ is well defined)}\}.$$

Then

$$(2.2) \quad \xi = \frac{\text{area } \mathcal{Y}}{\text{area } A'}.$$

The following result asserts that if the landing probability is much higher than the escaping one, then the Julia set has positive area.

Theorem 2.3 (Black Hole Criterion [AL1]). *There exists $C = C(\mu)$ with the following property. Let f be a primitive Feigenbaum map with stationary combinatorics and a priori bound μ . If $\eta \geq C\xi$ then $\text{area } J(f) > 0$.*

3. QUASICRITICAL CIRCLE MAPS

An (*analytic*) *critical circle map* is an analytic homeomorphism $f : \mathbb{T} \rightarrow \mathbb{T}$ of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with a single critical point c_0 of cubic type (i.e., $f'''(c_0) \neq 0$). It is usually normalized so that $c_0 = 0$ in the angular coordinate.

To study Siegel disks of non-polynomial maps we need to enlarge this class allowing the map be only quasiregular at the critical point.

3.1. Definitions. A *quasicritical circle map* is a homeomorphism $f : \mathbb{T} \rightarrow \mathbb{T}$ of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the following properties:

Q1. f is a real analytic diffeomorphism outside a single critical point c_0 normalized so that $c_0 = 0$ in the angular coordinate; we let $c_n = f^n c_0$;

Q2. Near the critical point, f admits a quasiregular extension to \mathbb{C} of cubic type, i.e, it has a local form¹¹ $h(z)^3 + c_1$ with a quasiconformal $h : (\mathbb{C}, c_0) \rightarrow (\mathbb{C}, 0)$ such that h is holomorphic near z whenever $f(z)$ lies on the same side of \mathbb{T} as z .

It follows, in particular, that $f|_{\mathbb{T}}$ is *quasisymmetric*. Moreover, it admits a quasiregular extension to a neighborhood of \mathbb{T} , symmetric with respect to \mathbb{T} , that is holomorphic in the domain

$$\text{Dom}^h f = \{z \in \text{Dom } f : z \text{ and } f(z) \text{ lie on the same side of } \mathbb{T}\} \cup \mathbb{T} \setminus \{c_0\}.$$

We will also make extra assumptions about exterior structure of f :

Q3. $\text{Dom } f$ is a \mathbb{T} -symmetric annulus, and $\text{Dom}^h f \setminus \mathbb{D}$ is obtained from the outer annulus $\text{Dom } f \setminus \mathbb{D}$ by removing a closed topological triangle

$$\mathcal{T} = \mathcal{T}_f \subset (\text{Dom } f \setminus \bar{\mathbb{D}}) \cup \{c_0\}$$

with a vertex at c_0 and the opposite side on the outer boundary of $\text{Dom } f$;

Q4. $f : \text{Dom}^h f \rightarrow \mathbb{C}$ is an immersion, and $f : \mathcal{T} \rightarrow \mathbb{D} \cup \{c_1\}$ is an embedding.

¹¹We could write $h(z)^\delta + c_1$ with any $\delta > 1$ as well since all the powers are quasisymmetrically equivalent.

Let Cir stand for the space of all quascritical circle maps. The *geometry* of such a map is specified by the dilatation of the map h from $\mathbb{Q}2$, and the size of $\text{Dom } f$. We call f a (K, ϵ) -*quascritical* if $\text{Dil } h \leq K$ and $\text{Dom } f$ contains the (2ϵ) -neighborhood of $[0, 1] \subset \mathbb{C}$. Let $\text{Cir}(\bar{N}, K, \epsilon)$ denote the class of (K, ϵ) -quascritical circle maps of type bounded by \bar{N} .

3.2. Local properties near the critical point.

3.2.1. John Property. Let $\mathcal{N}(K)$ stand for the class of K -quasiregular *normalized* maps $F : (\mathbb{C}, \mathbb{R}, 0, 1) \rightarrow (\mathbb{C}, \mathbb{R}, 0, 1)$ such that $F(z) = H(z)^3$, where $H : (\mathbb{C}, \mathbb{R}, 0, 1) \rightarrow (\mathbb{C}, \mathbb{R}, 0, 1)$ is a K -qc homeomorphism.

Lemma 3.1. *Let $F \in \mathcal{N}(K)$, and let $S_{\pm} = F^{-1}(\mathbb{C} \setminus \mathbb{R}_{\pm})$, where F^{-1} is the branch of the inverse map preserving \mathbb{R}_{\mp} . Then*

$$S_+ \supset \{|\arg z - \pi| \leq \alpha\pi\}, \quad S_- \supset \{|\arg z| \leq \alpha\pi\},$$

where $\alpha > 0$ depends only on K .

Proof. We will deal with S_+ only, as the argument for S_- is the same. The inverse branch $F^{-1} : \mathbb{C} \setminus \mathbb{R}_+ \rightarrow S_+$ is the composition of $z \mapsto z^{1/3}$ with H^{-1} , so

$$S_+ = H^{-1}(T_+), \quad \text{where } T_+ = \{|\arg z - \pi| < \pi/3\}.$$

Since $H^{-1} : (\mathbb{C}, 0, 1) \rightarrow (\mathbb{C}, 0, 1)$ is a normalized K -qc map, it is $L(K)$ -quasisymmetric on the whole plane.

For any $\zeta \in \mathbb{R}_-$, we have: $\text{dist}(\zeta, \partial T_+) = (\sqrt{3}/2)|\zeta|$. Take any $z \in \mathbb{R}_-$ and let $\zeta = H(z)$. By definition of L_0 -quasisymmetry, we have

$$\frac{\text{dist}(z, \partial S_+)}{|z|} < \frac{1}{L} \cdot \frac{\text{dist}(\zeta, \partial T_+)}{|\zeta|} < \frac{\sqrt{3}}{2L},$$

with some L depending only on L_0 (and on $\sqrt{3}/2$). The conclusion follows. \square

Any quascritical circle map $f \in \text{Cir}(K, \epsilon)$ can be non-dynamically normalized without changing its dilatation so that it fixes 0 and 1, Namely for any $t \in (0, 1/2)$, let

$$(3.1) \quad F_t : (\mathbb{C}, \mathbb{R}, 0, 1) \rightarrow (\mathbb{C}, \mathbb{R}, 0, 1), \quad F_t(x) = \frac{f(tx) - c_1}{f(t) - c_1}.$$

Then it can be modified outside the ϵ -neighborhood of $[0, 1]$ to turn it into a map of class $\mathcal{N}(K')$ with some $K' = K'(K, \epsilon)$. Let us call the modified map \hat{F}_t . Applying to it the previous lemma, we immediately obtain:

Proposition 3.2. *For any quascritical circle map $f \in \mathcal{C}(K, \epsilon)$, the domain $\text{Dom}^h f$ contains local sectors*

$$S_+(f) = \{|\arg z - \pi| \leq \alpha\pi, |z| < \epsilon\} \quad \text{and} \quad S_-(f) = \{|\arg z| \leq \alpha\pi, |z| < \epsilon\}$$

with some $\alpha > 0$ depending only on (K, ϵ) .

3.2.2. *Normalized Epstein class and scaling limits.* We say that a map $F \in \mathcal{N}(K)$ belongs to the *Normalized Epstein class* $\mathcal{EN}(K)$ if the inverse maps $F^{-1}|_{\mathbb{R}_{\mp}}$ admits a conformal extension to $\mathbb{C} \setminus \mathbb{R}_{\pm} \rightarrow \mathbb{C} \setminus \mathbb{R}_{\pm}$.

Lemma 3.3. *Let f be a quasicritical circle map. Then the family of modified rescalings \hat{F}_t , $t \in (0, 1/2)$ defined after (3.1) is precompact in the uniform topology on $\hat{\mathbb{C}}$. All limit maps as $t \rightarrow 0$ belong to the normalized Epstein class $\mathcal{NE}(K)$, with K depending only on the geometry of f .*

Proof. The modified rescalings \hat{F}_t form a precompact family since they are normalized degree three uniformly quasiregular maps. Moreover, the inverse maps $F_t^{-1}|_{\mathbb{R}_{\mp}}$ admit a conformal extension to $(\mathbb{C} \setminus \mathbb{R}_{\pm}) \cap \mathbb{D}_{\delta/t}$, with some δ depending on the geometry of f . Hence in the limit we obtain a map of Epstein class. \square

3.2.3. *Schwarzian derivative.* We will now show that quasicritical circle maps have negative Schwarzian derivative near the critical point. Let us begin with maps of Epstein class:

Lemma 3.4. *Any map $F \in \mathcal{NE}$ has negative Schwarzian derivative on the whole punctured line $\mathbb{R} \setminus \{0\}$.*

Proof. Let us consider an open interval $I = (a, d) \subset \mathbb{R} \setminus \{0\}$ as a Poincaré model of the hyperbolic line. Given a subinterval $J = (b, c) \Subset I$, let

$$(3.2) \quad |J : I| = \log \frac{(c-a)(d-b)}{(b-a)(d-c)}$$

stand for its hyperbolic length. Condition of negative Schwarzian derivative for F is equivalent to the property that F^{-1} is a hyperbolic contraction:

$$|F^{-1}(J) : F^{-1}(I)| \leq |J : I|$$

for any pair of intervals I and I as above.

Let us now consider the slit plane $\mathbb{C}(I) := \mathbb{C} \setminus (\mathbb{R} \setminus I)$ endowed with its hyperbolic metric. Then I is a hyperbolic geodesic in $\mathbb{C}(I)$. Let $\mathbb{D}(I)$ be the round disk based upon I as a diameter. It is the hyperbolic neighborhood of I in $\mathbb{C}(I)$ of certain radius r independent of I .

If F belongs to the Epstein class then the inverse map $F^{-1} : I \rightarrow I'$ (where $I' = F^{-1}(I)$) extends to a holomorphic map $F^{-1} : \mathbb{C}(I) \rightarrow \mathbb{C}(I')$. By the Schwarz Lemma, it is a hyperbolic contraction. Since $F^{-1}(I) = I'$, we conclude that $F^{-1}(\mathbb{D}(I)) \subset \mathbb{D}(I')$. Applying the Schwarz Lemma again, we obtain that $F^{-1} : \mathbb{D}(I) \rightarrow \mathbb{D}(I')$ is contracting with respect to the hyperbolic metric in these disks. Since the hyperbolic metrics on I and I' are induced by the hyperbolic metrics in the corresponding disks, we are done. \square

Remark 3.1. In fact, in the applications to the distortion bounds, the contracting property for the cross ratios from (3.2), rather than the Schwarzian derivative, is directly used (see Theorem 3.7).

Proposition 3.5. *Any quasicritical circle map $f \in \text{Cir}(K, \epsilon)$ has negative Schwarzian derivative in δ -neighborhood of the critical point, where $\delta = \delta(K, \epsilon)$ depends only on the geometry of f .*

Proof. By Lemma 3.3, the modified rescalings \hat{F}_t accumulate on a compact set $\mathcal{K} \subset \mathcal{NE}(K')$ of normalized Epstein maps, with $K' = K'(K, \epsilon)$. By Lemma 3.4, the latter have negative Schwarzian derivative. By Proposition 3.2, the maps \hat{F}_t are eventually (for $t < t_0(K, \epsilon)$) holomorphic in definite sectors $\{|\arg z| < \alpha\pi\} \cap \mathbb{D}$ and $\{|\arg z - \pi| < \alpha\pi\} \cap \mathbb{D}$. It follows that $S\hat{F}_t \rightarrow SF \in \mathcal{K}$ uniformly on $\pm[1/2, 1]$, and hence the Schwarzian derivatives SF_t are eventually negative on these two intervals. By the scaling properties of the Schwarzian, we have: $S\hat{F}_t(x) = t^2 Sf(tx)$, and hence $Sf < 0$ on some punctured interval $[-\delta, \delta]$, with $\delta > 0$ depending only on the geometry of f . \square

3.2.4. Power expansion. Let us consider a map $F \in \mathcal{NE}$ of Normalized Epstein class, and let $\text{Dom}^h F = \{z : (\text{Im } z) \cdot (\text{Im } F(z)) > 0\}$. Recall from Lemma 3.1 that it consists of two disjoint topological sectors S_\pm with the axes \mathbb{R}_\mp mapped conformally onto $\mathbb{H} \setminus \mathbb{R}_\pm$ respectively. Let us slightly shrink these sectors, namely for $\beta \in (0, 1)$, let

$$S_+(\beta) = \{z \in S_+ : |\arg F(z)| > \beta\pi\}, \quad S_-(\beta) = \{z \in S_- : |\arg F(z)| < (1 - \beta)\pi\}.$$

Lemma 3.6. *Let us consider a map $F \in \mathcal{NE}(K)$ of Normalized Epstein class, and let $\beta \in (0, 1)$. Then*

$$|F(z)| \geq C|z|^{1+\sigma} \quad \text{for } z \in S_\pm(\beta), \quad |z| \geq 1.$$

where $\sigma > 0$ and $C > 0$ depend only on K and $\beta > 0$.

Proof. Since S_+ contains the sector $\{|\arg z - \pi| < \alpha\pi\}$, we have:

$$S_- \subset \{|\arg z| < (1 - \alpha)\pi\}.$$

Hence the inverse branch $F^{-1} : \mathbb{H} \setminus \mathbb{R}_- \rightarrow S_-$ can be decomposed as $\phi(z)^{1-\alpha}$, where $\phi : (\mathbb{H}, \mathbb{R}, 0, 1) \rightarrow (\mathbb{H}, \mathbb{R}, 0, 1)$ is a conformal embedding. For such a map, we have:

$$(3.3) \quad |\phi(z)| \leq A|z| \quad \text{as long as } |z| \geq 1, \quad |\arg z| < \pi(1 - \beta),$$

where A depends only on $\beta > 0$. Indeed, the hyperbolic distance (in $\mathbb{C} \setminus \mathbb{R}_-$) from z as above to 1 is $\log |z| + O(1)$ (note that by the scaling invariance, the hyperbolic distance from z to $|z|$ depends only on $\arg z$). Since 1 is fixed under ϕ , the Schwarz Lemma implies (3.3). The conclusion for F on S_- follows.

The argument for S_+ is similar, except -1 is not the fixed point any more. But since F is quasiregular, $|\phi(-1)| \asymp 1$, and the Schwarz lemma implies the assertion again. \square

3.3. Real geometry. Due to the above local properties, quascritical circle maps enjoy the same geometric virtues as usual analytic critical circle maps. The main results formulated below are proven in a standard way, see e.g., the monograph by de Melo and van Strien [MvS] for a reference.

3.3.1. Koebe Distortion Bounds. The following statement extends the usual Koebe distortion bounds to quascritical circle maps:

Theorem 3.7. *Let $f \in \text{Cir}(K, \epsilon)$ be a quascritical circle map. Let $J \subset I \subset \mathbb{R}/\mathbb{Z}$ be two nested intervals in \mathbb{T} , with I open. Assume that for some $n, m \in \mathbb{N}$, the intersection multiplicity of the intervals $f^{-k}I$, $k = 0, 1, \dots, n$ is bounded by m and $|f^{-k}I| < \delta/2$ with δ from Proposition 3.5. Then*

$$|f^{-k}J : f^{-k}J| \leq C(K, \epsilon, m) |J : I|.$$

Proof. It is obtained by the standard cross-ratio distortion techniques, see [MvS]. To see the role of various properties of f , let us recall the main ingredients.

- *Denjoy Distortion control outside the $(\delta/2)$ -neighborhood of c_0 .* The distortion bound depends on C^2 -norm of f on \mathbb{T} and on $\sum_{k \in \mathcal{L}} |f^{-k}I|$, where \mathcal{L} is the set of

moments $k \leq n$ for which $f^{-k}I \cap (-\delta/2, \delta/2) = \emptyset$. The C^2 -norm of f depends only on (K, ϵ) by compactness of $\text{Cir}(K, \epsilon)$ and the Cauchy control of the derivatives of holomorphic functions. The total length of the intervals $f^{-k}I$ is bounded m .

- *Contraction of the cross-ratio in the punctured δ -neighborhood of c_0 .* This is concerned with the moments $k \leq n$ when $f^{-k}I \subset (-\delta, \delta) \setminus \{0\}$. At these moments the hyperbolic length $|f^{-k}J : f^{-k}I|$ is contracted under f^{-1} by Proposition 3.5.

- *Quasisymmetric distortion control at the critical moments.* At the moments $k \leq n$ when $f^{-k}I \ni c_0$, we have:

$$|f^{-k-1}J : f^{-k-1}I| \leq C(H, L) \cdot |f^{-k}J : f^{-k}I|$$

where L is an upper bound for $|f^{-k}J : f^{-k}I|$ and $H = H(K, \epsilon)$ is the qs-dilatation of f near c_0 . Since the number of the critical moments is bounded by m , their contribution to the total distortion is bounded. \square

3.3.2. No wandering intervals. Recall that an interval $J \subset I$ is called *wandering* if $f^n J \cap J = \emptyset$ for any $n > 0$. The above Koebe Distortion Bounds lead to the following generalization of Yoccoz's No Wandering Intervals Theorem [Y1]:

Theorem 3.8. *A quasycritical circle map does not have wandering intervals.*

It follows by the classical theory (Poincaré's thesis) if $f \in \text{Cir}$ does not have periodic points then it is topologically conjugate to a rigid rotation

$$T_\theta : x \mapsto x + \theta \pmod{1},$$

where $\theta \in \mathbb{R} \setminus \mathbb{Q} \pmod{\mathbb{Z}}$ is the *rotation number* of f .

When we want to specify the rotation number of circle maps under consideration, we will use notation Cir_θ and $\text{Cir}_\theta(K, \epsilon)$.

3.3.3. Bounded geometry and dynamical scales. The further theory largely depends on the Diophantine properties of θ encoded in its continuous fraction expansion $[N1, N2, \dots]$. Let $p_m/q_m = [N_1, \dots, N_m]$ be the m -fold rational approximand to θ . The rotation number (and the map f itself) is called of *bounded type* if the entries of the expansion are bounded by some \bar{N} . The spaces of circle map with rotation number bounded by \bar{N} will be denoted $\text{Cir}(\bar{N})$, $\text{Cir}_\theta(\bar{N}, K, \epsilon)$, etc. (depending on how many parameters we need to specify).

The Koebe Distortion Bounds also imply a more general version of the Hérmán-Swiątek Theorem [H, Sw]:

Theorem 3.9. *A quasycritical circle map $f \in \text{Cir}(\bar{N}, K, \epsilon)$ of bounded type is H -quasisymmetrically conjugate to the rigid rotation T_θ , with $H = H(\bar{N}, K, \epsilon)$.*

The circle dynamics naturally encodes the continued fraction expansion of the rotation number, as the denominators q_n are the *moments of combinatorially closest approaches*¹² of the critical orbit $\{c_n\}$ back to the critical point c_0 . Let us consider

¹²Meaning that these are the closest approaches for the corresponding circle rotation T_θ .

the corresponding intervals $I^n = [c_0, c_{q_n}]$ (i.e., the combinatorially shortest intervals bounded by c_0 and c_{q_n}). The orbits of two consecutive ones,

$$(3.4) \quad f^k(I^n), \quad k = 1, \dots, q_{n+1} - 1 \quad \text{and} \quad f^k(I^{n+1}), \quad k = 1, \dots, q_n - 1,$$

together with the *central* interval $I_0^n := I^n \cup I^{n+1}$ form a dynamical tiling \mathcal{I}^n of \mathbb{T} . Moreover, these tilings are nested: \mathcal{I}^{n+1} is a refinement of \mathcal{I}^n .

We label the intervals $I_k^n \in \mathcal{I}^n$, $k = 1, \dots, q_n + q_{n+1} - 2$, in an arbitrary way. Each of these intervals is homeomorphically mapped onto either $f^{q_{n+1}}(I^n)$ or $f^{q_n}(I^{n+1})$ by some iterate of f . We call it the *landing map* $L = L_n$ of level n . On the central interval I_0^n , we let $L_n = \text{id}$.

In case of bounded type, Theorem 3.9 ensures that these tilings have bounded geometry,¹³ i.e., the neighboring tiles are comparable, and hence the consecutive nested tiles are also comparable. This gives us a notion of *n-th dynamical scale* at any point $z \in \mathbb{T}$ (well defined up to a constant): it is the size of any tile $I^n(z) \in \mathcal{I}^n$ containing z .

More precisely, let $C_0 = C(\bar{N}, K, \epsilon) \geq 2$ be an upper bound for the ratios of any two neighboring and any two consecutive nested dynamical tiles. We say that a point $\zeta \in \mathbb{C}$ lies in *n-th dynamical scale* around $z \in \mathbb{T}$ if

$$(3.5) \quad C_0^{-1}|I_k^n| \leq |\zeta - z| \leq C_0|I_k^n|$$

for the dynamical tile I_k^n of depth n containing z . Any point $\zeta \in \mathbb{D}_2$ lies in some dynamical scale around any $z \in T$, and the number of such scales is bounded in terms of (\bar{N}, K, ϵ) .

3.4. Renormalization R_{cp} of circle pairs. A quasicritical circle map can be represented as a discontinuous map of the fundamental interval $[c_1 - 1, c_1]$, which motivates the following definition: a *quasicritical circle pair* $F = (\phi_+, \phi_-)$ is a pair of real analytic homeomorphisms

$$(3.6) \quad \phi_- : [\beta_-, 0) \rightarrow [b, \beta_+), \quad \phi_+ : (0, \beta_+] \rightarrow (\beta_-, b]$$

with some $\beta_- \leq 0 \leq \beta_+$, $\beta_+ - \beta_- = 1$. Moreover, $c_0 = 0$ is the only critical point of the ϕ_{\pm} and this point is of quasicubic type in the sense of property Q2 from §3.1. Properties Q3 and Q4 are also easily translated to this setting.

Renormalization R_{cp} of circle pairs is defined as follows. In the degenerate case $\beta_- = 0$ or $\beta_+ = 0$ (so that the critical point is fixed under ϕ_+ or ϕ_-) F is non-renormalizable. In the non-degenerate case, assume for definiteness that $b \in (\beta_-, 0]$ (otherwise, one should change the roles of β_- and β_+). If $\phi_-^N(\beta_-) \leq 0$ for all $N \in \mathbb{N}$ (equivalently, there is a fixed point in $(\beta_-, 0)$) then F is still non-renormalizable.¹⁴ Otherwise, let $N \geq 1$ be the biggest integer such that

$$\beta'_- := \phi_-^N(\beta_-) \leq 0, \quad \beta'_+ = \beta_+,$$

and let

$$\phi'_-|[\beta'_-, 0] = \phi_-, \quad \phi'_+|[0, \beta'_+] = \phi_-^N \circ \phi_+.$$

Rescaling the interval $[\beta'_-, \beta'_+]$ to the unit size by an orientation preserving¹⁵ linear map, we obtain $R_{cp}F$.

¹³This property is also referred to as *real a priori bounds*.

¹⁴In other words, maps with zero rotation number are non-renormalizable.

¹⁵Under the usual convention, the rescaling is orientation reversing. However, in further applications to Siegel maps, this would lead to some inconvenience.

To see how the renormalization acts on the rotation numbers, let us consider the linear case (corresponding to the pure rotation). In this case, a convenient normalization of F is to let $\max(|\beta_-|, \beta_+) = 1$ leaving only one parameter $\beta = \min(|\beta_-|, \beta_+) \in [0, 1]$ (related to the rotation number θ of f by $\theta = \beta_+/(1 + \beta)$). Then N is the biggest integer such that $N\beta \leq 1$, so N is the integer part of $1/\beta$. Under the renormalization, we obtain

$$\beta' = \frac{1 - N\beta}{\beta} = \frac{1}{\beta} \pmod{\mathbb{Z}},$$

which is the Gauss map applied to β . In this way, the continued fraction expansion of β (and hence θ) is directly related to the renormalization dynamics.¹⁶

3.4.1. Epstein class. We say that a quasiconformal circle pair $F = (\Phi_{\pm})$ belongs to *Epstein class* \mathcal{E} if the inverse maps Φ_{\pm}^{-1} admit a conformal extension to the whole upper half-plane $\{\operatorname{Im} z > 0\}$. Letting $I_- = [\beta_-, c_0]$, $I_+ = [c_0, \beta_+]$, we see by symmetry that the maps Φ_{\pm}^{-1} admit a conformal extension to the plane slit along two rays. $\mathbb{C}_{\pm} = \mathbb{C} \setminus (\mathbb{R} \setminus \Phi_{\pm}(I_{\pm}))$. For $\delta > 0$, we let $\mathbb{C}_{\pm}(\delta)$ be a similar domain where the interval $\Phi_{\pm}(I_{\pm})$ is scaled by factor $(1 + \delta)$.

Let us consider another quasiconformal circle map $f = (\phi_{\pm})$ and write its renormalizations as as

$$R_{cp}^m f = (\phi_{m,\pm}) = (\psi_{m,\pm} \circ \phi_{\pm})$$

(splitting off the first iterate of f). We say that they converge (along a subsequence) to $F \in \mathcal{E}$ as above if the latter can be represented as

$$F = (\Phi_{\pm}) = (\Psi_{\pm} \circ \phi_{\pm}),$$

and there exists $\delta > 0$ such that for any domain Ω_{\pm} compactly contained in the slit plane $\mathbb{C}_{\pm}(\delta)$, the inverse maps $(\psi_{m,\pm})^{-1}$ are eventually defined on Ω_{\pm} and uniformly converge to Ψ_{\pm}^{-1} on it (along the subsequence in question).

The real *a priori* bounds imply, in the standard way, precompactness of the renormalizations, with all limits in the Epstein class, see [dFdM]:

Proposition 3.10. *For a quasiconformal circle map $f \in \operatorname{Cir}(\bar{N}, K, \epsilon)$ of bounded type, any sequence of the renormalizations $R_{cp}^n f$ admits a subsequence converging to a quasiconformal circle pair of Epstein class $\mathcal{E}(\delta)$ with $\delta = \delta(\bar{N}, K, \epsilon)$.*

3.5. Complex bounds and butterfly.

3.5.1. Holomorphic circle pairs (butterfly). Let us now complexify the above notions. A *holomorphic circle pair* or a *butterfly map*

$$(3.7) \quad F = (\phi_-, \phi_+) : (\hat{X}_-, \hat{X}_+) \rightarrow \hat{Y}$$

is a holomorphic extension of a real circle pair $(\phi_-, \phi_+) : (I_-, I_+) \rightarrow \mathbb{R}$ with the following properties:

- $\hat{X}_{\pm} \supset \operatorname{int} I_{\pm}$ are disjoint \mathbb{R} -symmetric Jordan disks whose closures touch only at 0; we let $X_{\pm} = \hat{X}_{\pm} \cap \{\operatorname{Im} z > 0\}$;
- \hat{Y} is an \mathbb{R} -symmetric topological disk compactly containing the X_{\pm} ; we let $Y = \hat{Y} \cap \{\operatorname{Im} z > 0\}$;
- Each ϕ_{\pm} maps the corresponding X_{\pm} univalently onto Y ;

¹⁶It is worth noting that in this renormalization scheme, the cases $\beta_+ = 1$ and $\beta_- = -1$ alternate.

- The maps ϕ_{\pm} admit a quasiregular extension to a neighborhood of c_0 with local degree 3.

The configuration of domains $X_+ \cup X_-$ sitting inside Y is called a *butterfly*.

Let us mark in \hat{Y} the critical point $c_0 = 0$, and in \hat{X}_{\pm} the critical value $c_{\mp} = \phi_{\mp}(0)$. We say that a butterfly has a κ -bounded shape if each of the marked domains involved can be mapped onto the unit disk $(\mathbb{D}, 0)$ by a global \mathbb{R} -symmetric κ -qc map.

Let $\text{mod } F = \min(\text{mod}(Y \setminus X_{\pm}))$.

3.5.2. Complex bounds. We are ready to state a quascritical version of de Faria-Yampolsky complex bounds [dF, Ya1] :

Theorem 3.11. *Let $f \in \text{Cir}(\bar{N}, K, \epsilon)$ be a quascritical circle map of bounded type. Then there exists an \underline{l} depending only on (\bar{N}, K, ϵ) such that for all $m \geq \underline{l} - 1$ the renormalizations $R_{cp}^m f$ can be represented as a butterfly $X_-^m \cup X_+^m \rightarrow Y^m$ of bounded shape such that $\hat{Y}^{\underline{l}-1} \ni \hat{Y}^{\underline{l}} \ni \dots$ and*

$$\text{mod}(\hat{Y}^m \setminus \hat{Y}^{m+1}) \geq \mu, \quad \text{mod } R_{cp}^m f \geq \mu > 0.$$

Moreover, the boundary ∂X_+^m near c_0 is a wedge of angle $\pi/3$ obtained by taking f -preimage of $[c_1, c_1 + \delta_m^3]$ with $\delta_m \asymp \text{diam } Y^m$ (and similarly for X_- , using the other half-neighborhood of c_1).

All geometric constants and bounds depend only on (\bar{N}, K, ϵ) .

Proof. The proof is the same as in the analytic case (at the last moment making use of Lemma 3.6). We will remind main steps following the strategy of [LY, Ya1].

Due to Proposition 3.10, it is sufficient to prove the bounds for circle pairs F of Epstein class.

Let us consider an interval $I = I^n \in \mathcal{I}^n$ attached to the critical point, and let $q = q_{n+1}$, $J = f^q(I)$. Then $F^q|I$ can be decomposed as $\psi \circ F$ where $\psi^{-1} : J \rightarrow f(I)$ admits a conformal extension to the slit plane $\mathbb{C}(J)$. Here is the Key Estimate: for any z outside \mathbb{T} , we have:

$$(3.8) \quad \frac{\text{dist}(\psi^{-1}(z), |f(I)|)}{|f(I)|} \leq A \left(\frac{\text{dist}(z, I)}{|I|} \right) + B.$$

The proof uses only the real bounds and the Schwarz lemma for holomorphic maps between slit planes. As both these ingredients are available for our class (as we always apply only holomorphic inverse branches of F), the Key Estimate is valid in this generality.

At the last moment we apply the inverse branch of the cubic quasiregular map F near its critical point. By Lemma 3.6, it is highly contracting in big (rel $|I|$) scales, which implies (3.8).

Take now a big hyperbolic neighborhood Δ of $L_{\delta}(F)$ in the slit plane $\mathbb{C}(L_{\delta}(F))$ and pull it back by f^q . The Key Estimate easily implies that the pullback will be trapped well inside Δ . This produces a butterfly with a definite modulus μ .

Slightly shrinking \hat{Y}^l (using the space in between \hat{Y}^l and the \hat{X}_{\pm}^l) and taking its pullbacks under $R^l f$, we obtain a butterfly with a bounded shape. \square

3.5.3. Expansion. For $z \in \mathbb{C} \setminus \bar{\mathbb{D}}$ near c_0 , we will use notation $\text{ang } z$ for the smallest angle between z and \mathbb{R} in the \mathbb{C}/\mathbb{R} -model with $c_0 = 0$. Together with the Schwarz Lemma, the above complex bounds imply:

Corollary 3.12. *Under the circumstance of Theorem 3.11, the butterfly renormalizations $f_m := R_{cp}^m f$ are expanding in the hyperbolic metric of Y^m . Moreover,*

$$\|Df_m(z)\|_{\text{hyp}} \geq \rho > 1$$

with ρ depending only on (\bar{N}, K, ϵ) and a lower bound on $\text{ang } z$

Proof. Let $z \in X_+^m$, for definiteness. The hyperbolic expanding factor is equal to the inverse of $\|Di(z)\|_{\text{hyp}}$, where $i : X_+^m \rightarrow Y$ is the embedding, and the norm is measured from the hyperbolic metric of X_+^m to the one of Y^m . This norm is bounded in terms of the upper bound on $\text{dist}_{\text{hyp}}(z, \partial X_+^m)$ measured in Y^m , which in turn, is controlled by the relative Euclidean distance, $\text{dist}(z, \partial X_+^m) / \text{dist}(z, Y^m)$. Finally, complex bounds (and in particular, the wedge property of the butterfly) imply that the latter is bounded in terms of the lower bound on $\text{ang } z$. \square

Corollary 3.13. *Let $f \in \text{Cir}(\bar{N}, K, \epsilon)$ be a quasycritical circle map. Then there exist $a > 0$ and $\rho > 1$ depending on (\bar{N}, K, ϵ) only such that if $z \in Y^m \cap \text{Dom}^h f^n$ while $f^n z \in Y^{m-k}$ for some $n \in \mathbb{N}$, $0 < k < m$ (with $m - k > ul$), then*

$$\|Df^n(z)\|_{\text{hyp}} \geq a\rho^k,$$

where the norm is measured in the hyperbolic metric of $\mathbb{C} \setminus \bar{\mathbb{D}}$.

Proof. On its way from Y^m to Y^{m-k} , the orbit of z must land in the middle of $\asymp k$ domains $X_+^{m-i} \cup X_-^{m-i}$, $0 \leq i \leq k$. By Corollary 3.12, the return map $R^{m-1}f$ is definitely expanding at these moments, in the hyperbolic metric of Y^{m-1} . By the complex bounds, this metric restricted to Y^m is boundedly equivalent to the hyperbolic metric on $\mathbb{C} \setminus \bar{\mathbb{D}}$. The conclusion follows. \square

3.5.4. Compactness. Let us normalize a complex pair $F : \hat{X}_+ \cup \hat{X}_- \rightarrow \hat{Y}$ so that $|\hat{Y} \cap \mathbb{R}| = 1$ and introduce the following topology on the space of normalized pairs. A sequence $F_n : \hat{X}_+^n \cup \hat{X}_-^n \rightarrow \hat{Y}^n$ converges to a pair $F : \hat{X}_+ \cup \hat{X}_- \rightarrow \hat{Y}$ if the domains \hat{Y}^n Carathéodory converge to \hat{Y} and the inverse branches $(F_n)^{-1} : \hat{Y}_\pm^n \rightarrow \hat{X}_\pm^n$ converge to the corresponding branches of F^{-1} uniformly on compact subsets of \hat{Y}_\pm .

The *geometry* of a complex commuting pair is controlled by three parameters: μ (a lower bound on the modulus), κ (a bound on the shape of the butterfly), and B , a bound on the geometry of the intervals $\hat{X}_\pm \cap \mathbb{R}$ inside $\hat{Y} \cap \mathbb{R}$. The latter is defined as the best dilatation of a quasisymmetric map $(\hat{Y} \cap \mathbb{R}, 0) \rightarrow ([-1, 1], 0)$ that moves the boundary points of the intervals in question to some standard configuration. Let $\mathcal{P}(\mu, \kappa, B)$ stand for the space of complex pairs with geometry controlled by the specified parameters.

Proposition 3.14. *The space $\mathcal{P}(\mu, \kappa, B)$ is compact. For any $f \in \text{Cir}(\bar{N}, K, \epsilon)$, the renormalizations $R^m f : \hat{X}_+^m \cup \hat{X}_-^m \rightarrow \hat{Y}^m$ of a quasycritical circle map of bounded type eventually (for $m \geq m_0(\bar{N}, K, \epsilon)$) belong to some space $\mathcal{P}(\mu, \kappa, B)$, with all the parameters depending only on \bar{N} and K .*

Proof. The first assertion follows from the standard compactness properties of the Carathéodory topology. The second one is the content of real and complex *a priori* bounds. \square

3.6. Periodic points α^l , collars A^l , and trapping disks D^l .

3.6.1. *Periodic points α^l .* Let us start collecting consequences of the complex bounds.

Proposition 3.15. *For any $l \geq \underline{l} - 1$, a quascritical circle¹⁷ map of bounded type has a repelling periodic point $\alpha^l \in X_-^l \cup X_+^l$ of period q_l . Moreover,*

- (i) $\text{dist}(\alpha^l, \mathbb{T})$ is comparable to the dynamical depth at c_0 at scale l ;
- (ii) the multiplier of α^l is bounded and bounded away from 1 in absolute value.

Proof. Each restriction $R^l f : X_{\pm}^l \rightarrow Y^l$ is a conformal map from a smaller domain onto a bigger one. By the Wolff-Denjoy Theorem (applied to the inverse map) it has a fixed point in the closure \bar{X}_{\pm}^l . However, it does not have fixed points on the boundary since f does not have periodic points on \mathbb{R} , while the image of $\partial X_{\pm}^l \setminus \mathbb{R}$ under $R^l f$ (equal to $\partial Y_{\pm}^l \setminus \mathbb{R}$) is disjoint from itself. So, there is a fixed point $\alpha_{\pm}^l \in X_{\pm}^l$.

Assertions (i) and (ii) follow from compactness (Proposition 3.14).

Finally one of the points α_{\pm}^l has period q_l . □

3.6.2. *Collar Lemma and trapping disks D^l .* For all sufficiently big l , complex *a priori* bounds allow us to construct nice collars A^l around \mathbb{D} and nice trapping disks D^l that capture all orbits that escape beyond the corresponding collars.

We say that a point $z \in \mathbb{C} \setminus \mathbb{D}$ lies on depth l , $d(z) = l$, if

$$C_0^{-1}|I^l(\zeta)| \leq \text{dist}(z, \mathbb{T}) \leq C_0|I^l(\zeta)|,$$

where ζ is the closest to z point of \mathbb{T} , and $C_0 = C_0(\bar{N}, K, \epsilon)$ is the constant from (3.5). Of course, any point can lie on several depths (so $d(z)$ is multivalued), but this number is bounded in term of (\bar{N}, K, ϵ) .

Lemma 3.16. *For any quascritical circle map $f \in \text{Cir}(\bar{N}, K, \epsilon)$ and any $l \geq \underline{l} - 1$, there exists a pair of smooth annuli (“collars”)¹⁸ $A_0^l \Subset A^l$ surrounding \mathbb{D} in $\text{Dom } f \setminus \bar{\mathbb{D}}$, and a smooth quasidisk $D^l \ni \alpha^l$ in Y^l with the following properties:*

(A1) Any boundary point $z \in \partial^{\circ} A_0^l \cup \partial^{\circ} A^l$ of these collars lies on depth $d(z)$ with

$$|d(z) - l| \leq \bar{\iota} = \bar{\iota}(\bar{N}, K, \epsilon);$$

Moreover, for any $z \in \partial^{\circ} A_0^l$, $\text{dist}(z, \partial^{\circ} A^l) \asymp \text{dist}(z, \mathbb{T})$, and similarly for the inner boundaries $\partial^i A_0^l$ and $\partial^i A^l$;

(A2) It is impossible to “jump over the collar”:

$$\text{If } z \in \text{Comp}_0(\mathbb{C} \setminus A_0^l) \setminus \bar{\mathbb{D}} \text{ while } f(z) \notin \text{Comp}_0(\mathbb{C} \setminus A_0^l) \text{ then } f(z) \in A_0^l;$$

(D1) The disk D^l has a bounded shape around α^l ; it has also the hyperbolic diameter of order 1 in $Y^l \setminus \bar{\mathbb{D}}$ and in $\mathbb{C} \setminus \bar{\mathbb{D}}$;

(D2) A definite portion of D^l is contained in $f^{-1}(\mathbb{D}) \setminus \mathbb{D}$; moreover,

$$\text{there is a point } \beta \in f^{-1}(\mathbb{T}) \setminus \bar{\mathbb{D}} \text{ that lies in the middle of } D^l;$$

(D3) If $z \in A^l$ then there exists a moment $k < q_{l+1}$ such that $f^k z$ lies in the middle of D^l .

(D4) There exists $\underline{\iota} = \underline{\iota}(\bar{N}, \mu, K)$ such that for any $\iota > \underline{\iota}$ and $l > \underline{l} + \iota$, we have under the circumstances of (D3) :

$$f^i z \notin D_1^{l-\iota}, \quad i = 0, 1, \dots, k,$$

¹⁷In this statement we will use the unit circle model for \mathbb{T} rather than \mathbb{R}/\mathbb{Z} .

¹⁸We prepare a pair of collars for each l to make the statements robust under perturbations.

where $D_1^{l-\iota} \Subset \Omega \setminus \bar{S}$ is a disk containing $D^{l-\iota}$ with a definite $\text{mod}(D_1^{l-\iota} \setminus D^{l-\iota})$; in particular, $D^l \cap D_1^{l-\iota} = \emptyset$;

(D5) Moreover, under the above circumstances,

$$f^i z \in \text{Comp}_0(A^{l-\iota}), \quad i = 0, 1, \dots, k,$$

and $A^{l-\iota} \Subset \text{Comp}_0(A^{l-2\iota})$.

All the bounds and constants depend only on (\bar{N}, K, ϵ) .

Proof. Let us consider the circle pairs renormalization $R_{cp}^l f : X_-^l \cup X_+^l \rightarrow Y^l$. For Y^l we will also use notation Y_0^l .

Any dynamical tile $I_k^l \in \mathcal{I}^l$ (3.4) is compactly contained in the topological disk Y_k^l obtained by pulling Y^l back by the conformal landing map, the complex extension of $L_l : I_k^l \rightarrow I_0^l$. Complex *a priori* bounds imply that I_k^l is contained well inside Y_k^l . Hence each Y_k^l contains a round disk $\Delta_\epsilon(I_k^l)$ based on the $(1 + \epsilon)$ -scaled interval I_k^l , where $\epsilon > 0$ depends only on *a priori* bounds. The union of these disks is an annulus $\mathcal{D}^l \supset \mathbb{D}$ whose boundary lies on dynamical depth l . Moreover, for $k \neq 0$, these disks lie well inside $\text{Dom}^h f$, since $Y_k^l \subset \text{Dom}^h f$.

Obviously, there is $\bar{\iota} = \bar{\iota}(\bar{N}, K, \epsilon)$ such that for any \underline{l} , we can select collars $A^l \Subset A_1^l \Subset \mathcal{D}^l \setminus \mathbb{D}$ with the following properties:

- (i) They satisfy property (A1);
 - (ii) Every point $z \in A_1^l$ lies in the middle of some half-disk $\Delta_\epsilon(I_k^l) \setminus \bar{\mathbb{D}}$;
 - (iii) Every $z \in \partial^i A^l \cup \partial^i A_1^l$ lies on depth $d(z)$ with $0 < d(z) - (l + \underline{l}) < \bar{\iota}$.
- Since f is quasiregular, there is $\bar{\iota} = \bar{\iota}(\bar{N}, K, \epsilon)$ such that

$$d(f(z)) \geq d(z) - \bar{\iota}, \quad z \in \text{Dom}^h f$$

Together with (iii), this implies that if \underline{l} is selected sufficiently big ($\underline{l} > 2\bar{\iota}$), then property (A2) is satisfied as well: no point can jump over the collar A_0^l .

Let us view the topological half-disk $Y^l \setminus \bar{\mathbb{D}}$ as the hyperbolic plane, and let $D^l = D^l(R)$ be the hyperbolic disk of radius R in Y^l centered at α^l . By the Koebe Distortion Theorem, these disks satisfy property (D1) with constants depending on R (or better to say, on an upper bound for R).

For R big enough (depending only on (\bar{N}, K, ϵ)) they also satisfy (D2). Indeed, since f is quasiregular, any sufficiently small disk $\mathbb{D}(c_0, r)$ contains a comparable disk $\mathbb{D}(\zeta, ar) \subset f^{-1}(\mathbb{D}) \setminus \mathbb{D}$. Since the domains Y^l have a bounded shape around c_0 , while the disks $D^l(R)$ closely approximate $Y^l \setminus \bar{\mathbb{D}}$ (uniformly in l), we conclude that for R big enough,

$$D^l(R) \supset \mathbb{D}(\zeta, ar/2) \text{ and } \text{area } D^l(R) \asymp \text{area } \mathbb{D}(\zeta, ar/2),$$

which yields the first part of (D2).

The second part of (D2) follows from Proposition 3.2 that implies that there is a point $\zeta \in f^{-1}(\mathbb{T})$ lying in the middle of Y^l . For R big enough, it lies in the middle of $D^l(R)$ as well.

If $z \in A^l$ then by Property (ii), z lies in the middle of some half-disk $Y_k^l \setminus \bar{\mathbb{D}}$. By the Koebe Distortion Theorem, under the landing map $L_l : Y_k^l \rightarrow Y^l$, it lands in the middle of $Y^l \setminus \bar{\mathbb{D}}$. Hence for R big enough $L_l(z)$ lies in the middle of $D^l(R)$ as well, which establishes property (D3).

Since the whole orbit $\{f^i z\}_{i=0}^k$ lies on depth $\geq l - O(1)$, it is separated from $D^{l-\iota}$ and from $A^{l-\iota}$, as long as ι is sufficiently big. Similarly, since $A^{l-\iota}$ lies on

depth $l - \iota$, it is separated from $A^{l-2\iota}$ for ι big enough. These remarks prove (D4) and (D5). \square

We say that the trapping disk $D = D^l$ is *centered* at α^l , or that depth $D = l$.

3.7. Cylinder circle renormalization.

3.7.1. Real definition. There is a different approach to the circle renormalization that avoids using circle pairs. The quotient of \mathbb{R} by the lift of f is a circle \mathbb{T}' , and the first return map to the fundamental interval $[0, f(0)]$ descends to a critical circle map of \mathbb{T}' . Identifying \mathbb{T}' with \mathbb{T} by means of an orientation preserving analytic diffeomorphism we obtain the renormalization $R_{\text{cyl}}f$ of f (defined up to an orientation preserving analytic conjugacy). The rotation number of $R_{\text{cyl}}f$ is equal to $-1/\theta \pmod{\mathbb{Z}}$.

This leads to the modified Gauss map $G_* : \theta \mapsto -1/\theta \pmod{\mathbb{Z}}$ accompanied by the modified continued fraction expansion

$$\theta = \frac{1}{N_1 - \frac{1}{N_2 - \dots}} \equiv [N_1, N_2, \dots]_*, \quad N \geq 2.$$

We will use the same notation for the rational approximands in this expansion, $p_m/q_m = [N_1, \dots, N_m]_*$. Of course, notion of “bounded type” is independent of which expansion we use.

The rotation numbers $\theta_N = [N, N, N, \dots]_*$ with equal entries¹⁹ $N \geq 3$ are called of *stationary type* (with respect to the modified expansion) The most familiar of these is the golden mean $\theta_3 = (3 - \sqrt{5})/2$.

3.7.2. Complexification. Let us start with a topological lemma:

Lemma 3.17. *For a butterfly map F (3.7), there exists an arc γ connecting the fixed point $\alpha \in X_+$ to β_+ in such a way that α is the only common point of γ and $f(\gamma)$. Moreover, the triangle bounded by γ , $f(\gamma)$ and the arc of $[\phi_+(\beta_+) \in \mathbb{R}$ is κ -qc equivalent (by a global map $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$) to the half-strip*

$$(3.9) \quad \{z : \text{Im } z \geq 0, 0 \leq \text{Re } z \leq 1\} \cup \{\infty\} \subset \hat{\mathbb{C}},$$

with κ depending only on the qc geometry of the pair of domains (Y, X_+) .

Proof. Let $\phi \equiv \phi_+$, $X \equiv X_+$, $\beta_+ \equiv \beta$, $J := [\phi(\beta), \beta]$. The pullback $J' := \phi^{-1}(J)$ is a smooth subarc of ∂X_+ touching J at β with angle $\pi/3$. Hence $J \cup J'$ is a quasia rc. Pulling it further, we obtain a sequence of smooth arcs $J^n := \phi^{-n}(J) \subset \bar{X}$, $n = 0, 1, \dots$, one touching the previous at angle $\pi/3$ and shrinking at a geometric rate. Their union $\bigcup J^n$ is a quasia rc converging to the fixed point α . Adding α to it, we obtain a closed quasia rc $\Gamma = [\alpha, \phi(\beta)]$ such that $f(\Gamma) = [\alpha, \phi(\beta)]$ is a longer quasia rc. Moreover, the dilatation of Γ depends only on the geometry of the pair (Y, X) (by compactness of the corresponding maps ϕ).

The map ϕ on X can be globally linearized by a κ -qc homeomorphism $\psi : (\mathbb{C}, X) \rightarrow (\mathbb{C}, \psi(X))$ which is conformal on X , $\psi(\phi(z)) = \lambda\psi(z)$, $z \in X$, with κ depending only on the geometry of (Y, X) . It can be further conjugate to the doubling map $T : z \mapsto 2z$ by a qc homeomorphism $h : \mathbb{C} \rightarrow \mathbb{C}$ that straightens the quasia rc Γ to the unit interval $[0, 1]$. In this model, we can let $\tilde{\gamma} \equiv h(\psi(\gamma))$ be a segment of a circle passing through 0 and 1 sufficiently close to \mathbb{R} so that it fits to

¹⁹Note that $\theta_2 = 1$.

the domain $h(\psi(X))$. Moreover, the triangle bounded by $\tilde{\gamma}$, $2 \cdot \tilde{\gamma}$ and $[1, 2]$ is κ -qc equivalent to the half-strip (3.9), with κ depending only on the geometry of the pair (Y, X) . \square

For m sufficiently big, the cylinder renormalizations $R_{\text{cyl}}^m f$ we have described above can be complexified as follows, see Yampolsky [Ya2]. Let us consider a periodic point α^m , $m \geq \underline{L}$, from Corollary 3.15. Then there is a \mathbb{T} -symmetric arc γ_m connecting α^m to the symmetric point²⁰ $1/\bar{\alpha}^m$ in such a way that $f^{qm}(\gamma_m)$ does not intersect γ_m . Let us consider the fundamental region $\Delta^m = \Delta^m(f)$ bounded by these two arcs.

Lemma 3.18. *Let $f \in \text{Cir}(\bar{N}, K, \epsilon)$. Then the regions Δ^m are κ -qc equivalent to the strip $0 \leq \text{Re } z \leq 1$, with κ depending only on (\bar{N}, K, ϵ) .*

Let us now identify the boundary components of Δ^m by means of f^{qm} . We obtain a cylinder Cyl^m which is conformally equivalent to the standard bi-infinite cylinder \mathbb{C}/\mathbb{Z} . The first return map to Δ^m descends to a holomorphic map on Cyl^m near the circle, and then can be transferred to $\exp(\mathbb{C}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = (\mathbb{C}^*, \mathbb{T})$. This is the *cylinder renormalization* of a holomorphic circle map (well defined up to affine conjugacy).

3.8. Quasiconformal conjugacy.

Theorem 3.19. *Two quasiconformal circle maps,*

$$f : \text{Dom}^h f \rightarrow Y \text{ and } \tilde{f} : \text{Dom}^h \tilde{f} \rightarrow \tilde{Y}, \text{ of class } \text{Cir}(\bar{N}, K, \epsilon)$$

with the same rotation number are L -qc conjugate, with $L = L(\bar{N}, K, \epsilon)$.

Proof. It is an application of *Sullivan's Pullback Argument*, see [MvS]. By Theorem 3.9, there is a quasiconformal map $h : \mathbb{C} \rightarrow \mathbb{C}$ conjugating f and \tilde{f} on the unit circle (with dilatation depending only on \bar{N}). Using the complex bounds (Theorem 3.11) this map can be adjusted so that it is equivariant on the boundary of the butterfly, with dilatation depending only on (K, ϵ) .

We can now start lifting the map h under the dynamics to make it equivariant on bigger and bigger parts of Ω_f^h . Since f is conformal on Ω_f^h , these lifts preserve the dilatation of h . By compactness of the space of normalized L -qc maps, we can pass to a limit and produce the desired conjugacy. \square

4. SIEGEL MAPS AND THEIR PERTURBATIONS

4.1. Douady-Ghys surgery.

4.1.1. *Blaschke model for Siegel polynomials.* Let us consider a quadratic polynomial

$$(4.1) \quad \mathbf{f}_\theta : z \mapsto e^{2\pi i \theta} z + z^2, \quad \theta \in \mathbb{R}/\mathbb{Z}.$$

When the rotation number θ has bounded type, it is linearizable near the origin, and thus has a Siegel disk $\mathbf{S} \equiv \mathbf{S}_{P_\theta} \equiv \mathbf{S}_\theta$. Here we will briefly describe the Blaschke model for this quadratic map due to Douady and Ghys. It is based on a surgery that turns an appropriate Blaschke product into \mathbf{f}_θ .

²⁰Here we describe it in terms of the unit circle \mathbb{T} in \mathbb{C} .

Consider a family of Blaschke products

$$B_\alpha(z) = e^{2\pi i \alpha} z^2 \frac{z-3}{1-3z}.$$

It induces a family of critical circle maps on the unit circle \mathbb{T} . Adjusting the parameter α one can make the rotation number of B_λ assume an arbitrary value, so it can be made equal to the rotation number θ from (4.1).

Assume θ is of bounded type. Then by Theorem 3.9, $B_\lambda : \mathbb{T} \rightarrow \mathbb{T}$ is quasimetrically conjugate to the pure rotation T_θ . We can use this conjugacy to glue the Blaschke product on $\mathbb{C} \setminus \mathbb{D}$ to the rotation of \mathbb{D} . This produces a degree two quasiregular map F of a quasiconformal sphere. Moreover, F preserves the conformal structure obtained by spreading around the standard structure on the disk \mathbb{D} . By the Measurable Riemann Mapping Theorem, F is quasiconformally conjugate to some quadratic polynomial $z \mapsto \lambda z + z^2$. Since this quadratic polynomial has an invariant Siegel disk with rotation number θ , it coincides with \mathbf{f}_θ .

4.2. Expansion. Let us endow the complement $\mathbb{C} \setminus \bar{\mathbf{S}}$ of a Siegel disk $\mathbf{S} = \mathbf{S}_\theta$ of bounded type with the hyperbolic metric $\|\cdot\|_{\text{hyp}}$. A standard application of the Schwarz Lemma shows that the map $\mathbf{f} = \mathbf{f}_\theta$ is expanding in this metric,

$$\|D\mathbf{f}(z)\|_{\text{hyp}} > 1, \quad \text{if } z, \mathbf{f}(z) \in \mathbb{C} \setminus \bar{\mathbf{S}}.$$

Indeed, the map $\mathbf{f} : \mathbb{C} \setminus f^{-1}(\bar{\mathbf{S}}) \rightarrow \mathbb{C} \setminus \bar{\mathbf{S}}$ is a covering and hence a hyperbolic isometry. By the Schwarz Lemma, the embedding

$$(4.2) \quad i : \mathbb{C} \setminus \mathbf{f}^{-1}(\bar{\mathbf{S}}) \rightarrow \mathbb{C} \setminus \bar{\mathbf{S}}$$

is a hyperbolic contraction. Hence $\mathbf{f} \circ i^{-1} : \mathbb{C} \setminus \bar{\mathbf{S}} \rightarrow \mathbb{C} \setminus \bar{\mathbf{S}}$ is expanding on its domain of definition (i.e., on $\mathbb{C} \setminus \mathbf{f}^{-1}(\bar{\mathbf{S}})$).

Using the Blaschke model, McMullen showed that the expansion is uniform near the critical point:

Lemma 4.1 ([McM2]). *Let $\mathbf{f} = \mathbf{f}_\theta$ be a Siegel quadratic polynomial of type bounded by \bar{N} , and let $C > 0$. Then there exists $\rho = \rho(\bar{N}, C) > 1$ such that*

$$\|D\mathbf{f}(z)\|_{\text{hyp}} > \rho \quad \text{if } z, \mathbf{f}(z) \in \mathbb{C} \setminus \bar{\mathbf{S}}, \text{ and } |z - c_0| \leq C \text{ dist}(z, \mathbf{S}),$$

where the *dist* stands for the Euclidean one.

Proof. From the above argument we see that $\|D\mathbf{f}(z)\|_{\text{hyp}} = \|Di^{-1}\|_{\text{hyp}}$, where i is embedding (4.2). The latter is bounded in terms of the hyperbolic distance from z to $P^{-1}\bar{\mathbf{S}}$ (in $\mathbb{C} \setminus \bar{\mathbf{S}}$). For the Blaschke model, this hyperbolic distance is bounded in terms of C . The Blaschke model is K -qc equivalent to \mathbf{f} where K is bounded in terms of N . The conclusion follows. \square

Let us now consider a perturbation $\tilde{\mathbf{f}} = \mathbf{f}_{\tilde{\theta}}$ (not necessarily with real $\tilde{\theta}$) of the Siegel polynomial $\mathbf{f} = \mathbf{f}_\theta$. Let $\tilde{\mathbf{O}}$ be the postcritical set of $\tilde{\mathbf{f}}$. Endow its complement $\mathbb{C} \setminus \tilde{\mathbf{O}}$ with the hyperbolic metric $\|\cdot\|_{\text{hyp}}$. Then the map $\tilde{\mathbf{f}}$ is expanding with respect to this metric (for the same reason as the Siegel map \mathbf{f}). In fact, it is also uniformly expanding near the critical point:

Lemma 4.2. *Let the type of θ be bounded by \bar{N} , and let $C > 0$. Then there exists $\rho = \rho(\bar{N}, C) > 1$ such that for any compact set $K \Subset \mathbb{C} \setminus \bar{\mathbf{S}}$ there exists $\delta > 0$ with the*

following property. Let $|\tilde{\theta} - \theta| < \delta$, and assume $\tilde{\mathbf{O}}$ is contained in the δ -neighborhood of the Siegel disk \mathbf{S} . Then for any point $z \in K \setminus \tilde{\mathbf{f}}^{-1}(\tilde{\mathbf{O}})$ such that

$$(4.3) \quad |z - \mathbf{c}_0| \leq C \operatorname{dist}(z, \bar{\mathbf{S}}),$$

we have:

$$\|D\tilde{\mathbf{f}}(z)\|_{\text{hyp}} \geq \rho.$$

Proof. As the proof of Lemma 4.1 shows, the expansion factor ρ is bounded from below in terms of the hyperbolic distance from z to $\tilde{\mathbf{f}}^{-1}(\tilde{\mathbf{O}})$ in $\mathbb{C} \setminus \tilde{\mathbf{O}}$.

Let $U = U_\delta$ be the δ -neighborhood of $\bar{\mathbf{S}}$. For δ small enough, \bar{U} is disjoint from K . Then the hyperbolic metrics on $\mathbb{C} \setminus \bar{\mathbf{S}}$ and on $\mathbb{C} \setminus \bar{U}$ restricted to K are comparable (and in fact, close for δ small).

By assumption, the postcritical set $\tilde{\mathbf{O}}$ is contained in U . By the Schwarz Lemma, the hyperbolic metric $\|\cdot\|_{\text{hyp}}$ on $\mathbb{C} \setminus \tilde{\mathbf{O}}$ restricted to K is bounded by the hyperbolic metric on $\mathbb{C} \setminus \bar{U}$. Altogether, for δ sufficiently small we conclude:

$$\|\cdot\|_{\text{hyp}} \leq C_1 \|\cdot\|_{\text{hyp}} \quad \text{on } K,$$

with the constant C_1 depending only on N (in fact, C_1 can be taken arbitrary close to 1 for δ small).

Since the dynamics of \mathbf{f} on $\partial\mathbf{S}$ is minimal, the set $\tilde{\mathbf{O}}$ makes an ϵ -net for $\partial\mathbf{S}$ provided δ is small enough. Hence $\tilde{\mathbf{f}}^{-1}(\tilde{\mathbf{O}})$ makes an $O(\epsilon)$ -net for $\mathbf{f}^{-1}(\mathbf{S})$. As we know (see the proof of Lemma 4.1), condition (4.3) implies that the hyperbolic distance from z to $\mathbf{f}^{-1}(\mathbf{S})$ in $\mathbb{C} \setminus \bar{\mathbf{S}}$ is bounded. It follows that the hyperbolic distance from z to $\tilde{\mathbf{f}}^{-1}(\tilde{\mathbf{O}})$ in $\mathbb{C} \setminus \tilde{\mathbf{O}}$ is bounded as well. \square

4.3. Siegel maps.

4.3.1. *Definition.* A Siegel map $f : (\Omega, 0) \rightarrow (\mathbb{C}, 0)$, i.e., a holomorphic map on a Jordan disk $\Omega \equiv \Omega_f = \operatorname{Dom} f$ with the following properties:

- S1. f has a Siegel disk $S = S_f$ (centered at 0) which is a *quasidisk compactly contained in Ω* .
- S2. f has a non-degenerate critical point $c_0 \in \partial S$; we let $c_n = f^n c_0$;
- S3. The domain $\Omega_f^h = \{z \in \Omega \setminus \bar{S} : fz \in \Omega \setminus \bar{S}\}$ is obtained from the annulus $\Omega \setminus \bar{S}$ by removing a topological triangle

$$\mathcal{T} = \mathcal{T}_f := (\Omega \setminus \bar{\mathbb{D}}) \cup \{c_0\}$$

with a vertex at c_0 and the opposite side on the boundary of Ω ;

- S4. $f : \Omega_f^h \rightarrow \mathbb{C}$ is an immersion, and $f : \mathcal{T} \rightarrow S \cup \{c_1\}$ is an embedding.

We let $\operatorname{Dom}^h f = \Omega_f^h \cup \bar{S}$.

Remark 4.1. Note that Siegel maps are holomorphic by definition, so in this case superscript “h” is taken only by analogy with the circle case.

Given $\bar{N} \in \mathbb{N}$ and $\mu > 0$, let $\operatorname{Sieg}(\bar{N}, \mu, K)$ stand for the space of Siegel maps $f : \Omega \rightarrow \mathbb{C}$ of type bounded by \bar{N} and such that $\operatorname{mod}(\Omega \setminus S_f) \geq \mu$ and ∂S_f is a K -quasicircle. (If irrelevant, some of these parameters can be skipped in the notation.)

We will later use notation $\operatorname{Sieg}_\theta(\mu, K) \equiv \operatorname{Sieg}_N(\mu, K)$ for the class of Siegel maps $f \in \operatorname{Sieg}(\mu, K)$ with stationary rotation number $\theta = \theta_N$ and such that $\operatorname{mod}(\Omega \setminus S_f) \geq \mu$.

4.3.2. *Circle model for Siegel maps.* By performing the Douady-Ghys surgery on an arbitrary *analytic* critical circle map g of bounded type (not only on the Blaschke map), we can produce plenty of Siegel maps. However, to produce all of them, we need to allow *quasicritical* circle maps.

Proposition 4.3. *Any Siegel map $f : (\Omega, 0) \rightarrow (\mathbb{C}, 0)$ of bounded type can be obtained by performing the Douady-Ghys surgery on a quasicritical circle map.*

Proof. Let $\psi_+ : \mathbb{C} \setminus S \rightarrow \mathbb{C} \setminus \mathbb{D}$ be the uniformization of the complement of S normalized so that $\psi_+(c_0) = 1$. Since S is a quasidisk, it extends to a global quasiconformal map $\psi_+ : (\mathbb{C}, S) \rightarrow (\mathbb{C}, \mathbb{D})$. Then

$$g := \psi_+ \circ f \circ \psi_+^{-1} : (\mathbb{C}, \mathbb{D}) \rightarrow (\mathbb{C}, \mathbb{D})$$

is a global quasiregular map in a neighborhood of $\bar{\mathbb{D}}$ which is a holomorphic immersion on $\psi_+(\Omega^h)$. Applying the Schwarz Reflection Principle, we obtain a quasiregular map g near \mathbb{T} that restricts to a homeomorphism $\mathbb{T} \rightarrow \mathbb{T}$. Moreover, it is a holomorphic immersion on $\text{Dom}^h g$, and hence is real analytic on $\mathbb{T} \setminus \{1\}$. At the critical point $c_0 = 1$, it has local degree 3. Moreover, properties (S3) and (S4) of f readily translate to properties (Q3) and (Q4) of g . Thus, g is a quasicritical circle map.

On the other hand, the uniformization $\psi_- : \bar{S} \rightarrow \bar{\mathbb{D}}$ conjugates f to the rotation T_θ (and extends to a global qc map). Hence f is the quasiconformal welding between g and T_θ . \square

4.4. **Circle \rightsquigarrow Siegel transfer.** By means of the Douady-Ghys surgery, we can transfer the objects defined above for quasicritical circle maps to their Siegel counterparts. Somewhat abusing notation, we will usually keep the same notation for the transferred objects.

4.4.1. *Dynamical scales.* For any $f \in \text{Sieg}(\bar{N}, \mu, K)$, we can transfer the circle dynamical tilings (3.4) to the boundary of the Siegel disk S . Since the surgery is quasisymmetric, these *Siegel dynamical tilings* \mathcal{I}^m have bounded geometry as well (depending only on (\bar{N}, μ, K)), which gives us for any $z \in \partial S$ a notion of the *dynamical scales* near z .²¹

4.4.2. *Siegel butterfly renormalization.* Since any Siegel map f of bounded type is conjugate on the boundary of S to a quasicritical circle map, we can immediately define the *Siegel pairs* renormalizations $R_{Sp}f$ on ∂S . The complexification of this notion, a *Siegel butterfly*

$$(4.4) \quad R_{Sp}^m : X_+^m \cup X_-^m \rightarrow Y^m,$$

corresponds, via the surgery, to the external part of the circle butterfly. Theorem 3.11 implies:

Theorem 4.4. *Let $f \in \text{Sieg}(\bar{N}, \mu, K)$ be a Siegel map of bounded type. Then there exists an \underline{l} depending only on (\bar{N}, μ, H) such that for all $m \geq \underline{l} - 1$ the renormalizations $R_{Sp}^m f$ on ∂S can be extended to Siegel butterflies*

$$R_{Sp}^m f : X_-^m \cup X_+^m \rightarrow Y^m$$

²¹ with the constant C_0 from (3.5) replaced with an analogous constant $C_0 = C_0(\bar{N}, \mu, K)$ controlling the geometry of the tilings for Siegel maps.

with $Y^{\underline{l}-1} \ni Y^{\underline{l}} \ni \dots$ such that the Y^m are quasircles of bounded shape and

$$\text{dist}(\partial Y^m \setminus \partial S, Y^{m+1}) \asymp \text{dist}(\partial Y^m \setminus \partial S, X_{\pm}^m) \asymp \text{diam } Y^m.$$

All constants and bounds depend on (\bar{N}, μ, K) only.

As in the circle case, these *a priori* bounds lead to external expansion:

Corollary 4.5. *Under the circumstance of Theorem 4.4, the renormalizations $f_m := R_{S_p}^m f$ are expanding in the hyperbolic metric of Y^m . Moreover,*

$$\|Df_m(z)\|_{\text{hyp}} \geq \rho > 1$$

with ρ depending only on (\bar{N}, K, ϵ) and a lower bound on $\text{dist}(z, \bar{S}) / \text{dist}(z, c_0)$.

Corollary 4.6. *Let $f \in \text{Cir}(\bar{N}, \mu, K)$ be a Siegel map. Then there exist $a > 0$ and $\rho > 1$ depending only on (\bar{N}, μ, K) such that if $z \in Y^m \cap \text{Dom}^h f^n$ and $f^n z \in Y^{m-k}$ for some $n \in \mathbb{N}$, $0 < k < m$ (with $m - k > \underline{l}$), then*

$$\|Df^n(z)\|_{\text{hyp}} \geq a\rho^k,$$

where the norm is measured in the hyperbolic metric of $\mathbb{C} \setminus \bar{S}$.

4.4.3. *Periodic points α^l .* Proposition 3.15 implies:

Corollary 4.7. *For any Siegel map $f \in \text{Sieg}(\bar{N}, \mu, K)$, there exists $\underline{l} = \underline{l}(\bar{N}, \mu)$ such that for any $l \geq \underline{l} - 1$, f has a repelling periodic point α_l of period q_l in the l -th dynamical scale near the critical point c_0 .*

Remark 4.2. If $\mathbf{f} = \mathbf{f}_\theta$ is a Siegel quadratic polynomial with rotation number of bounded type, then the periodic point α^l was born in the *parabolic explosion* from the parabolic approximand $\mathbf{f}_{p_\kappa/q_\kappa}$. It can be characterized as the landing point of a ray with rotation number p_κ/q_κ .

4.4.4. *External collars of A^l and trapping disks D^l .* Let us now transfer, by means of the surgery, the collars and trapping disks from the circle plane to the Siegel plane. It is a direct consequence of Lemma 3.16 and quasisymmetry of quasiconformal maps.

Proposition 4.8. *For any Siegel map $f \in \text{Sieg}(\bar{N}, \mu, K)$ and any $l \geq \underline{l} - 1$, there exists a pair of smooth annuli (collars) $A_0^l \Subset A^l$ surrounding the Siegel disk $S = S_f$ in $\text{Dom } f \setminus \bar{S}$, and a smooth quasidisk $D^l \Subset \text{Dom } f \setminus \bar{S}$ containing α^l with the following properties:*

(A1) For any $z \in \partial^\circ A_0^l$, $\text{dist}(z, \partial^\circ A^l) \asymp \text{dist}(z, \partial^\circ S)$, and similarly for the inner boundaries $\partial^i A_0^l$ and $\partial^i A^l$;

(A2) It is impossible to “jump over the collar”:

$$\text{If } z \in \text{Comp}_0(\mathbb{C} \setminus A_0^l) \text{ while } f(z) \notin \text{Comp}_0(\mathbb{C} \setminus A_0^l) \text{ then } f(z) \in A_0^l;$$

(D1) The disk D^l has a bounded shape around α^l and it has the hyperbolic diameter of order 1 in $\mathbb{C} \setminus \bar{S}$;

(D2) A definite portion of D^l is contained in $f^{-1}(S) \setminus S$; moreover,

$$\text{there is a point } \beta \in f^{-1}(\partial S) \setminus \bar{S} \text{ that lies in the middle of } D^l;$$

(D3) If $z \in A^l$ then there exists a moment $k < q_{l+1}$ such that $f^k z$ lies in the middle of D^l ;

(D4) There exists $\underline{l} = \underline{l}(\bar{N}, \mu, K)$ such that for any $\iota > \underline{l}$ and $l > \underline{l} + \iota$, we have under the circumstances of (D3) :

$$f^i z \notin D_1^{l-\iota}, \quad i = 0, 1, \dots, k,$$

where $D_1^{l-\iota} \Subset \text{Dom } f \setminus \bar{S}$ is a disk containing $D^{l-\iota}$ with a definite $\text{mod}(D_1^{l-\iota} \setminus D^{l-\iota})$; in particular, $D^l \cap D_1^{l-\iota} = \emptyset$.

(D5) Moreover, under the above circumstances,

$$f^i z \in \text{Comp}_0(\mathbb{C} \setminus A^{l-\iota}), \quad i = 0, 1, \dots, k,$$

and $A^{l-\iota} \Subset \text{Comp}_0 \mathbb{C} \setminus (A^{l-2\iota})$.

All bounds and constants depend only on (\bar{N}, μ, K) .

4.5. Siegel cylinder renormalization.

4.5.1. *Definition.* Using the circle model, we can extend Yampolsky's construction of the *cylinder renormalization* R_S [Ya3] to all Siegel maps $f \in S_\theta$ of bounded type. Let g be the quasircritical circle map corresponding to f through the surgery. Let us transfer the arc used for the m th cylinder renormalization of g (see §3.7.2) to an arc δ_m connecting the periodic point α^m of f from Corollary 4.7 to the boundary of S_f . By continuing along the internal ray of S_f , extend δ_m to an arc γ_m connecting α^m to the Siegel fixed point 0. Then $f^{q_m}(\gamma_m)$ does not intersect γ_m , and these two arcs bound a *fundamental crescent* \mathcal{C}^m for f^{q_m} . Now we can proceed with the construction as in the circle case: identifying the boundary arcs of \mathcal{C}^m , we produce a map of the standard cylinder \mathbb{C}/\mathbb{Z} whose upper end corresponds to the Siegel fixed point. To recover this point back, let us map \mathbb{C}/\mathbb{Z} onto \mathbb{C}^* by means of e^{iz} . We obtain a Siegel map with rotation number $-1/\theta \pmod{1}$.

The following statement is a Siegel counterpart of Lemma 3.18 that follow from the latter by surgery.

Lemma 4.9. *Let f be a Siegel map of class $\text{Sieg}(\bar{N}, \mu, K)$. For any $m \geq \underline{l} - 1$, the fundamental crescent \mathcal{C}^m is κ -qc equivalent to the quadrilateral composed by attaching the half-strip (3.9) (corresponding to $\mathcal{C}^m \setminus S$) to a triangle with angle $2\pi/q$ at 0 (corresponding to $\mathcal{C}^m \cap \bar{S}$). The dilatation κ depends only on (\bar{N}, μ, K) .*

As in the circle case, let $\pi_m = \pi_m^f$ stand for the change of variable projecting the original dynamical plane to the renormalized one: it starts in the fundamental crescent \mathcal{C}^m and then is spread around by means of the dynamics.

5. INOU-SHISHIKURA CLASS

5.1. **Parabolic Renormalization.** Here we will briefly outline the Parabolic Renormalization Theory that provides us with a good control of bifurcations of parabolic maps. It was laid down in the work by Douady and Sentenac (see [DH1, D4]), Lavaurs [La], and Shishikura [Sh1], which can be consulted for details.

5.1.1. *Parabolic Piuseaut germs and their transit maps.* For $q \in \mathbb{N}$ and a small neighborhood U of 0, let $\mathcal{G}_0(U)$ be the space of parabolic germs near 0 given by Piuseaut series

$$(5.1) \quad f : z \mapsto z + z^2 + \sum_{k \in \mathbb{N}} a_k z^{2+k/q},$$

(continuous up to the boundary). By definition, it is isomorphic to the space of holomorphic germs

$$\hat{f} : \zeta \mapsto \zeta^q + \zeta^{2q} + \sum_{k \in \mathbb{N}} a_k \zeta^{2q+k}$$

on the neighborhood \hat{U} , the full preimage of U under the power change of variable $z = \zeta^q$. The latter space is endowed with uniform topology, which is inherited by $\mathcal{G}_0(U)$.

Let us consider the principal branch of f (which is real on \mathbb{R}_+) in the slit plane $\mathbb{C} \setminus (i\mathbb{R}_-)$. It is endowed with the following structure:

(C1) An *attracting petal* $\mathcal{P}^a \equiv \mathcal{P}^a(f)$, which is an open piecewise smooth Jordan disk with the following properties:

- \mathcal{P}^a is \mathbb{R} -symmetric and $\mathcal{P}^a \cap \mathbb{R} = (-\delta, 0)$ for some $\delta > 0$;
- \mathcal{P}^a touches the origin at a certain angle α which can be selected arbitrary in the range $(0, \pi)$. To be definite, we let $\alpha = \pi/2$;
- f univalently maps \mathcal{P}^a into itself, $f(\partial\mathcal{P}^a) \cap \partial\mathcal{P}^a = \{0\}$, and $f^n(z) \rightarrow 0$ as $n \rightarrow +\infty$ uniformly on \mathcal{P}^a .

Along with the attracting petal, there is a *repelling petal* $\mathcal{P}^r \equiv \mathcal{P}^r(f)$ containing an interval $(0, \delta)$ with some $\delta > 0$ that can be defined as the attracting petal for f^{-1} .

(C2) *The horn map* $H \equiv H_f : \mathcal{P}^r \rightarrow \mathcal{P}^a$. For any angle $\theta > 0$, there exist $\epsilon > 0$ and $n \in \mathbb{N}$ such that for any point $z \in \mathcal{P}^r$ with $|z| < \epsilon$ and $\arg z > \theta$ (where $\arg z$ is the principal value of the argument) we have $f^n z \in \mathcal{P}^a$. Moreover, ϵ and n can be selected the same for all maps $\tilde{f} \in \mathcal{G}_0(U)$ near f .

(C3) The attracting and repelling *Fatou coordinates*²²

$$\phi^a \equiv \phi_f^a : \mathcal{P}^a \rightarrow \{\operatorname{Re} z > 0\}, \quad \phi^r \equiv \phi_f^r : \mathcal{P}^r \rightarrow \{\operatorname{Re} z < 0\}.$$

that conformally conjugate f and f^{-1} to the translations $z \mapsto z+1$ and $z \mapsto z-1$ respectively. The Fatou coordinates are defined up to translation, so they are uniquely determined by normalization that specifies which points $c^{a/r} \equiv c_f^{a/r} \in \mathcal{P}^{a/r}(f)$ correspond to $\pm 1 \in \mathbb{C}$. Moreover, if the base points $c_f^{a/r}$ depend holomorphically on f then so do the normalized Fatou coordinates.

(C4) An *attracting fundamental crescent* $\mathcal{C}^a \equiv \mathcal{C}^a(f)$. It is the strip $\{1/2 \leq \operatorname{Re} z \leq 3/2\}$ properly embedded into the attracting petal \mathcal{P}^a such that $\partial\mathcal{C}^a \cap \partial\mathcal{P}^a = \{0\}$ and $f(\mathcal{C}^a) \cap \mathcal{C}^a$ is a boundary component of \mathcal{C}^a . To be definite, we will use the following choice:

$$\mathcal{C}^a \equiv \mathcal{C}^a(f) = \{z \in \mathcal{P}^a : 3/4 \leq \operatorname{Re} \phi^a(z) \leq 7/4\}.$$

Since the Fatou coordinate depends holomorphically on f , the crescent $\mathcal{C}^a(f)$ moves holomorphically with f .

Similarly, one can define the *repelling fundamental crescent*

$$\mathcal{C}^r \equiv \mathcal{C}^r(f) = \{z \in \mathcal{P}^r : -1/4 \leq \operatorname{Re} \phi^r(z) \leq -5/4\}.$$

(C5) The *Écalle-Voronin cylinders* $\operatorname{Cyl}^{a/r} \equiv \operatorname{Cyl}^{a/r}(f)$, which are the quotients of the petals $\mathcal{P}^{a/r}$ by the dynamics. They can be obtained by identifying the

²²To make sure that the Fatou coordinates are isomorphisms onto the corresponding half-planes requires a special choice of the petals, which will also be assumed in what follows.

boundary components of the corresponding fundamental crescents $\mathcal{C}^{a/r}$ by means of $z \sim f(z)$. The normalized Fatou coordinates induce isomorphisms of the pointed cylinders $(\text{Cyl}^{a/r}, c^{a/r})$ to the standard cylinder $(\mathbb{C}/\mathbb{Z}, 0)$, and in what follows, we will freely identify the cylinders with the standard model.

(C6) A complex one parameter family of *transit* isomorphisms

$$(5.2) \quad I_\lambda : \text{Cyl}^a \approx \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}/\mathbb{Z} \approx \text{Cyl}^r, \quad z \mapsto z + \lambda, \quad \lambda \in \mathbb{C}/\mathbb{Z}.$$

For any $\lambda \in \mathbb{D}_{1/4}$, the isomorphism I_λ lifts to translation

$$\{3/4 \leq \text{Re } z \leq 7/2\} \rightarrow \{\text{Re } z < 0\}, \quad z \mapsto z - 2 + \lambda,$$

which induces, by means of the Fatou coordinates $\phi_f^{a/r}$, an embedding

$$(5.3) \quad I_{f,\lambda} : \mathcal{C}^a(f) \rightarrow \mathcal{P}^r(f).$$

Holomorphic dependence of the Fatou coordinates on f implies that these embeddings depend nicely on the parameters:

Lemma 5.1. *Assume the base points $c_f^{a/r} \in \mathcal{P}^{a/r}(f)$ are selected holomorphically in f over some neighborhood $\mathcal{U}_0 \subset \mathcal{G}_0(U)$. Then the family of transit maps (5.3) depends holomorphically on $(f, \lambda) \in \mathcal{U}_0 \times \mathbb{D}_{1/4}$.*

The horn map $H \equiv H_f$ from (C2) also descends to the cylinders, and we will keep the same notation, $H : \text{Cyl}^r \rightarrow \text{Cyl}^a$, for the quotient.

(C7) *Parabolic renormalization* $R_{\text{par}}f$. Composing the transit maps with the horn map, we obtain a one-parameter family of return maps

$$(5.4) \quad I_\lambda \circ H_f : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}/\mathbb{Z}$$

defined near the ends of the repelling cylinder $\text{Cyl}^r \approx \mathbb{C}/\mathbb{Z}$. By means of²³

$$\text{Exp} : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}^*, \quad \text{Exp}(z) = -(4/27)e^{-2\pi iz},$$

we can identify the cylinder \mathbb{C}/\mathbb{Z} with \mathbb{C}^* so that its upper end corresponds to 0 and the boundary of the fundamental crescents $\mathcal{C}^{a/r}$ correspond to the ray $i\mathbb{R}_-$. Then family of return maps (5.4) becomes a one-parameter family $g_{f,\lambda}$ of conformal germs near 0.

Moreover, there is a unique choice of the transit parameter λ that makes the map $g_{f,\lambda}$ parabolic, with multiplier 1 at 0. This map $g_{f,\lambda}$ is called the *parabolic renormalization* $R_{\text{par}}f$ of f .

5.1.2. *Transit maps for perturbations and their geometric limits.* Let us now consider the space $\mathcal{G}(U)$ of Piuiseau germs (continuous up to the boundary)

$$(5.5) \quad f : z \mapsto e^{2\pi i\gamma}(z + z^2) + \sum_{k \in \mathbb{N}} a_k z^{2+k/q}$$

on U . We will refer to $\gamma \in \mathbb{C}/\mathbb{Z}$ as the *complex rotation number* of 0.

Let $\mathcal{U}_0 \subset \mathcal{G}_0(U)$ be a neighborhood of a parabolic map f_0 . Let us consider a neighborhood \mathcal{U} in $\mathcal{G}(U)$ consisting of maps $f = e^{2\pi i\gamma} \tilde{f}$, where $\tilde{f} \in \mathcal{U}_0$ and $|\arg \gamma| < \pi/4$.

If \mathcal{U} is sufficiently small then any map $f \in \mathcal{U} \setminus \mathcal{U}_0$ has a second fixed point $\beta = \beta_f$ near 0, and there exist crescent-shaped domains bounded by (closed) arcs

²³This special normalization of the exponential map is chosen to make it consistent with the one used by Inou and Shishikura, see below.

$\omega^{a/r} = \omega_f^{a/r}$ connecting 0 to β and their respective images $f^{\pm 1}(\omega^{a/r})$. Moreover, all four arcs, $\omega^{a/r}$ and $f^{\pm a}(\omega^{a/r})$ are pairwise disjoint (except for the endpoints). The domain $\mathcal{P} = \mathcal{P}(f)$ bounded by the arcs ω^a and ω^r will be referred as the *petal* for f .²⁴

As in the parabolic case, the perturbed map can be linearized on its petal. The linearizing coordinate

$$\phi \equiv \phi_f : \mathcal{P} \rightarrow \mathcal{C}, \quad \phi(fz) = \phi(z) + 1, \quad z \in \mathcal{P} \cap f^{-1}(\mathcal{P})$$

is called the *Fatou-Douady coordinate* (or *perturbed Fatou coordinate*). It is defined uniquely up to translation, so it can be normalized by prescribing a point $c^a \in \mathcal{P}$ corresponding to 1, or a point c^r corresponding to -1 . The normalized Douady coordinate depends holomorphically on $f \in \mathcal{U} \setminus \mathcal{U}_0$.

The petal \mathcal{P} can be selected so that $\phi(\mathcal{P})$ is a vertical strip $\{A < \operatorname{Re} z < B\}$ with big $B - A$, and in what follows we will assume such a choice. Let us define the *attracting and repelling fundamental crescents* as

$$\begin{aligned} \mathcal{C}^a &\equiv \mathcal{C}^a(f) = \{A + 3/4 \leq \operatorname{Re} \phi(z) \leq A + 7/4\}, \\ \mathcal{C}^r &\equiv \mathcal{C}^r(f) = \{B - 5/4 \leq \operatorname{Re} \phi(z) \leq B - 1/4\}. \end{aligned}$$

If a point c_f^a is selected in $\mathcal{C}(f)$ holomorphic in $f \in \mathcal{U}$ (including parabolic maps $f \in \mathcal{U}_0$), then the linearizing coordinate ϕ_f^a depends holomorphically, and hence continuously, on $f \in \mathcal{U}$. Thus, if $f_n \rightarrow f$ then for any compact set $K \subset \mathcal{P}^a(f)$, the $\phi_{f_n}^a$ are eventually well defined on K , and $\phi_{f_n}^a \rightarrow \phi_f^a$ uniformly on K .

A similar discussion applies to the repelling fundamental crescents.

The quotients of the petals $\mathcal{P}^{a/r}$ by the dynamics provide us with a pair of *Douady cylinders* $\operatorname{Cyl}^{a/r} = \operatorname{Cyl}^{a/r}(f)$. They can be obtained by identifying the boundary arcs of the crescents $\mathcal{C}^{a/r}$ by means of $z \sim f(z)$. As in the purely parabolic case, the Fatou-Douady coordinate ϕ induces an isomorphism between the cylinders $\operatorname{Cyl}^{a/r}$ and the standard cylinder \mathbb{C}/\mathbb{Z} , and we will freely identify the cylinders with the standard model.

Let us consider the transit map $T \equiv T_f : \mathcal{C}^a \rightarrow \mathcal{C}^r$, i.e., $Tz = f^j z$ where $f^k z \in \mathcal{P}$, $k = 0, 1, \dots, j$, and $f^j z \in \mathcal{C}^r$. It is usually discontinuous, but it induces a conformal isomorphism between the cylinders,²⁵

$$(5.6) \quad I_f : \operatorname{Cyl}^a \approx \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}/\mathbb{Z} \approx \operatorname{Cyl}^r, \quad z \mapsto z + \lambda, \quad \lambda = \lambda(f) \in \mathbb{C}/\mathbb{Z}.$$

Theorem 5.2. *Assume the base points $c_f^{a/r} \in \mathcal{C}^{a/r}(f)$ are selected holomorphically in f over some neighborhood $\mathcal{U} \subset \mathcal{G}(U)$. Let (Λ_f, c_f^r) be the lift of $(\mathbb{D}_{1/8}, 0) \subset (\mathbb{C}/\mathbb{Z}, 0)$ to $\mathcal{P}(f)$ (by means of the Fatou-Douady coordinate). Then for every sufficiently big j , there exist a holomorphic embedding*

$$\Phi_j : \mathcal{U}_0 \times \bar{\mathbb{D}}_{1/8} \rightarrow \mathcal{U}, \quad (\tilde{f}, \lambda) \mapsto e^{2\pi i \gamma_j} \tilde{f},$$

where $\gamma_{j, \tilde{f}} : \bar{\mathbb{D}}_{1/8} \rightarrow \mathbb{C}$ is a conformal embedding such that:

- $f = \Phi_j(\tilde{f}, \lambda)$ for some $(\tilde{f}, \lambda) \in \mathcal{U}_0 \times \bar{\mathbb{D}}_{1/8}$ iff

$$f^k(c^a) \in \mathcal{P}, \quad k = 0, 1, \dots, j, \quad f^j(c^a) \in \Lambda_f, \quad \text{and} \quad \lambda(f) = \lambda.$$

²⁴ What happens is that the attracting and repelling petals of a parabolic map “merge” under perturbation to form \mathcal{P} .

²⁵ Notice an essential difference with the parabolic case: in that case, there is a one-parameter family of isomorphisms between the cylinders, all on equal footing, while in the perturbed case, (5.6) is a *preferred* isomorphism induced by the dynamics.

- For $(\tilde{f}, \lambda) \in \mathcal{U}_0 \times \bar{\mathbb{D}}_{1/8}$ and $\epsilon > 0$, let $f = \Phi_j(\tilde{f}, \lambda)$ and

$$\mathcal{C}_\epsilon^a(f) = \{z : 1/2 - \epsilon < \operatorname{Re} \phi_f^a(z) < 3/2 + \epsilon\},$$

Then the transit maps $f^j : \mathcal{C}_\epsilon^a(f) \rightarrow \mathcal{P}(f)$ converge to the parabolic transit map $I_{\tilde{f}, \lambda} : \mathcal{C}^a(\tilde{f}) \rightarrow \mathcal{P}(\tilde{f})$ uniformly on compact subsets of $\mathcal{C}^a(\tilde{f})$, and uniformly over the tube $\mathcal{U}_0 \times \bar{\mathbb{D}}_{1/8}$.²⁶

- $\operatorname{diam} \operatorname{Im} \gamma_{j, \tilde{f}} \asymp j^{-2}$.

The images $Q^j = \operatorname{Im} \Phi_j$ will be called *parabolic tubes*. They are endowed with the *horizontal foliation* whose leaves $\mathcal{L}^j(\lambda) \approx \mathcal{U}_0$, $\lambda \in \bar{\mathbb{D}}_{1/8}$, correspond to the same transit parameter $\lambda \in \bar{\mathbb{D}}_{1/8}$.

The horn map from (C2) is robust under a perturbation $f = e^{2\pi i \gamma} \tilde{f}$ (5.5). The perturbed horn map $H \equiv H_f : \mathcal{P} \rightarrow \mathcal{P}$ is defined for $z \in \mathcal{P}$ with $|z| < \epsilon$ and $0 < \theta < \arg z < \pi/2$. It induces the cylinder horn map $\operatorname{Cyl}^r \rightarrow \operatorname{Cyl}^a$ near the upper²⁷ end of the Douady cylinders. We will use the same notation $H \equiv H_f$ for this map.

Composing it with the transit map $I_f : \operatorname{Cyl}^a \rightarrow \operatorname{Cyl}^r$, we obtain the return map $I_f \circ H_f : \operatorname{Cyl}_f^r \rightarrow \operatorname{Cyl}_f^r$ near the upper end of the cylinders. Viewed in the exp-coordinate, it becomes a germ $g_f : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$. Its rotation number is given by the (modified) complex Gauss map $G_*(\gamma) = -1/\gamma \pmod{\mathbb{Z}}$. If $G_*(\gamma)$ is small then this return map is close to the parabolic renormalization of \tilde{f} . It is called the *almost parabolic renormalization* of f . We will keep the same notation R_{par} for this operator.

5.1.3. *Case of rotation number p/q .* Let us now consider a holomorphic parabolic germ

$$(5.7) \quad f : \zeta \mapsto e^{2\pi i p/q} \zeta + \zeta^2 + \dots$$

with rotation number p/q . Assume it is non-degenerate, i.e., it has q petals (rather than a multiple of q petals). Then the q -th iterate f^q has a form

$$f^q : \zeta \mapsto \zeta + a_{q+1} \zeta^{q+1} + \dots, \quad \text{with } a_{q+1} \neq 0.$$

Performing a power change of variable $z = c\zeta^q$, we bring f^q to Puiseux form (5.1).

Let us now perturb the parabolic map f to

$$(5.8) \quad f_\epsilon : \zeta \mapsto e^{2\pi i(p/q+\epsilon)} \zeta + \zeta^2 + \dots$$

The q -th iterate f_ϵ^q has non-vanishing terms $a_k z^k$ with $1 < k < q+1$, but these terms can be killed by a conformal change of variable. Performing further a power change of variable $z = c\zeta^q$, we bring f_ϵ to Puiseux form (5.5). As all the above coordinate changes depend holomorphically on f , this allows us to apply the above theory to the space of germs (5.7).

²⁶Under these circumstances, the pair $(\tilde{f}_\infty, I_\lambda)$ is called the *geometric limit* of the sequence $\{f_j\}$.

²⁷The assumption that $\arg \gamma > \theta$ breaks the symmetry between the ends as it ensures that the points within a compact set of \mathcal{C}_f^a escape through the upper end of \mathcal{C}_f^r .

5.2. Inou-Shishikura class. Inou and Shishikura [IS] have constructed a class \mathcal{IS}_0 of maps with the following properties:

(P1) Any map $f \in \mathcal{IS}_0$ is holomorphic on some quasidisk Ω_f containing 0, and has a form $P_0 \circ \phi^{-1}$ where P_0 is the restriction of $z \mapsto z(1+z)^2$ to some domain Ω_0 , and $\phi : \Omega_0 \rightarrow \mathbb{C}$ is an appropriately normalized univalent map that admit a global qc extension to \mathbb{C} ;

(P2) 0 is the parabolic fixed point of any $f \in \mathcal{IS}_0$;

(P3) Any $f \in \mathcal{IS}_0$ has a single quadratic critical point $c_0 = c_0(f)$; moreover, the orbit of c_0 does not escape Ω_f , and $f^n(c_0) \rightarrow 0$ as $n \rightarrow \infty$;

(P4) The class is endowed with the *Bers-Teichmüller topology* and complex structure inherited from the space of Schwarzian derivatives $S\phi$; they make it isomorphic to the Universal Teichmüller Space;

(P5) The class is also endowed with *weak topology* induced by the compact-open topology on the space of univalent functions $\phi : \Omega_0 \rightarrow \mathbb{C}$; the weak completion $\overline{\mathcal{IS}}_0$ is compact;

(P6) The parabolic renormalization R acts from $\overline{\mathcal{IS}}_0$ to \mathcal{IS}_0 ; its restriction to \mathcal{IS}_0 is a compact holomorphic operator;

(P7) The parabolic renormalization of the quadratic map $z \mapsto z+z^2$ has a restriction in \mathcal{IS} .

For $\theta \in \mathbb{R}/\mathbb{Z}$, define the class \mathcal{IS}_θ as $e^{2\pi i\theta} \cdot \mathcal{IS}_0$, and let $\mathcal{IS} = \bigcup_{\theta} \mathcal{IS}_\theta$. (Notation $\overline{\mathcal{IS}}_\theta$ and $\overline{\mathcal{IS}}$ has a similar meaning.) Property (P6) is robust under perturbation:

Theorem 5.3 ([IS]). *If θ is sufficiently small then the almost parabolic renormalization R_{par} induces an operator $R_{\mathcal{IS}} : \overline{\mathcal{IS}}_\theta \rightarrow \mathcal{IS}_{-1/\theta}$ that restricts to a compact holomorphic operator $R_{\mathcal{IS}} : \mathcal{IS}_\theta \rightarrow \mathcal{IS}_{-1/\theta}$.*

We will call this operator $R_{\mathcal{IS}}$ (and in this section we will often abbreviate it, without saying, to R).

Corollary 5.4. *There exists \underline{N} such that if $\theta = [N_1, N_2, \dots, N_m, \dots]_*$ with $N_i > \underline{N}$, $i = 1, \dots, m$, then any map $\bar{f} \in \overline{\mathcal{IS}}_\theta$ is m times renormalizable under $R_{\mathcal{IS}}$. Hence it is infinitely renormalizable if θ is irrational.*

We say that a rotation number $\theta \in \mathbb{R}/\mathbb{Z}$ (rational or irrational) has *high type* if all $N_i > \underline{N}$ with \underline{N} as above. Let $\mathcal{IS}(\underline{N})$ stand for the union of the spaces \mathcal{IS}_θ over all θ of high type. For $\theta = [N, N, \dots]_*$ of high stationary type ($N > \underline{N}$) we will also use notation $\mathcal{IS}_N \equiv \mathcal{IS}_\theta$. Similar notation will be used for the weak completion $\overline{\mathcal{IS}}$.

5.3. Postcritical set. Inou and Shishikura have deduced from the above results

Proposition 5.5 ([IS]). *For any map $f \in \overline{\mathcal{IS}}(\underline{N})$, the critical point is non-escaping (i.e., $f^n(c_0) \in \Omega_f$, $n = 0, 1, \dots$) and stays away from the boundary of $\text{Dom } f$. Thus, the postcritical set \mathcal{O}_f is compactly contained in Ω_f (uniformly over $\overline{\mathcal{IS}}$). In the parabolic case we have: $f^n(c_0) \rightarrow 0$ as $n \rightarrow \infty$. In general, $\text{orb } c_0$ is non-periodic.*

Proof. (Sketch.) The mere fact that the IS renormalization Rf is well defined implies that the first N_1 iterates of the critical point stay in Ω_f (where N_1 is the first entry of the rotation number). Existence of all the renormalizations imply that

the whole critical orbit stays in Ω_f . Uniform bounds on the postcritical set follow from compactness of $\overline{\mathcal{I}\mathcal{S}}$.

In the parabolic case, the map is finitely renormalizable and its last renormalization falls to the class $\mathcal{I}\mathcal{S}_0$. Property (P3) implies that $f^n(c_0) \rightarrow 0$ as $n \rightarrow \infty$. In the irrational case, f is infinitely renormalizable and all the renormalizations $R^m f$ are small perturbations of parabolic maps of class $\mathcal{I}\mathcal{S}_0$. Hence $R^m f(c_0) \neq c_0$. On the other hand, if c_0 was periodic, then it would be the fixed point for some renormalization. \square

5.4. Renormalization Telescope. In this section we will collect some technical results, essentially contained in the work of Buff & Cheritat [BC] and Cheraghi [Ch].

Given a map $f \in \overline{\mathcal{I}\mathcal{S}_\theta}$ and a topological sector \mathfrak{S} centered at 0, a *principal branch of the first return map to \mathfrak{S}* is an iterate $f^l : V \rightarrow \mathfrak{S}$, where V is a relatively open subset of \mathfrak{S} with $0 \in \partial V$ such that for any $z \in V$, $f^l(z)$ is the first return of orb z to \mathfrak{S} .

The following statement provides us with a convenient domain of definition for the renormalization change of variable:

Lemma 5.6 ([Ch], §2). *For any map $f \in \overline{\mathcal{I}\mathcal{S}_\theta}$ with $\theta = [N_1, N_2, \dots]_*$ sufficiently small, there exists a smooth sector $\mathfrak{S} = \mathfrak{S}_f$ attached to the fixed point 0 with the following properties:*

(0) *It has angle θ at 0;*

(i) *There exists a bounded $s = s_f$ such that $f^s(\mathfrak{S})$ is a sector containing the critical value c_1 of f . In an appropriate Fatou coordinate,²⁸ the latter sector becomes the half-strip*

$$(5.9) \quad \{3/4 \leq \operatorname{Re} z \leq 7/4, \operatorname{Im} z \geq -2\}.$$

(ii) *There exists a well defined change of variable $\pi = \pi_f : \mathfrak{S} \rightarrow \mathbb{C}$ which is univalent on \mathfrak{S} and $\sim z^{1/\theta}$ as $z \rightarrow 0$ (uniformly over the class). Moreover, $\pi(\mathfrak{S}) \supset \mathfrak{S}_{Rf}$, and the boundary of $\pi(\mathfrak{S})$ touches the boundary of \mathfrak{S}_{Rf} at a single point, the fixed point 0.*

(iii) *The change of variable is equivariant: it conjugates two principal branches of first return map to \mathfrak{S} and Rf on its full domain.²⁹*

(iv) *For some k independent of f , the union*

$$\Omega_f^1 = \bigcup_{n=0}^{N_1+s-k} f^n(\mathfrak{S})$$

is a neighborhood of 0 compactly containing $\{c_n\}_{n=0}^{N_1+s-k}$.

(v) *The sectors \mathfrak{S}_f depend continuously on $f \in \overline{\mathcal{I}\mathcal{S}}(N)$.*

For $t \geq 2$, let $\Delta = \Delta_f(t)$ be the subset of the sector \mathfrak{S}_f corresponding to the box

$$\{3/4 \leq \operatorname{Re} z \leq 7/4, -2 \leq \operatorname{Im} z \leq t\}$$

in the Fatou coordinate (compare (5.9)).

²⁸This coordinate is normalized so that the critical value is placed at 1.

²⁹There is a precise formula for the return times in terms of the arithmetic of θ , see Lemma 2.2 in [Ch].

Lemma 5.7. *Under the circumstances of Lemma 5.6, for t sufficiently big, the image $\pi_f(\Delta_f(t))$ compactly contains $\Delta_{Rf}(t)$, with a definite space in between. Moreover, the domain $\Delta_f(t)$ depends continuously on f .*

Proof. The last statement follows from item (v) of Lemma 5.6 and continuous dependence of the Fatou coordinate of f . Together with the weak compactness of the Inou-Shishikura class $\overline{\mathcal{IS}}$ and item (ii) of the lemma, this implies that the change of variable π_f on S_f is uniformly comparable with $z \mapsto z^{1/\theta}$. This map is attracting near 0, so the “bottom” of Δ_f (corresponding to $\{\text{Im } z = t\}$ in the Fatou coordinate) goes even closer to 0. Together with item (ii) of the Lemma, this implies that $\pi_f(\Delta_f(t))$ compactly contains $\Delta_{Rf}(t)$. Using weak compactness of $\overline{\mathcal{IS}}$ once again, we conclude that there is a definite space in between. \square

From now on, t will be fixed, and will not appear in notation.

If f is m times IS-renormalizable then we can compose the above changes of variable, to obtain a map

$$\pi_f^m = \pi_{R^{m-1}f} \circ \cdots \circ \pi_f,$$

well defined and univalent on a sector \mathfrak{S}_f^m attached to 0. Spreading these sectors around by the iterates of f , we obtain a neighborhood of 0:

$$(5.10) \quad \Omega_f^m = \bigcup_{n=0}^{r_m} f^n(\mathfrak{S}_f^m),$$

where r_m is an appropriate time expressed in terms of the arithmetic of θ , and $f^n|_{\mathfrak{S}_f^m}$ is at most 2-to-1 for $n \leq r_m$ (note that these maps are not branched coverings over their images). Moreover, the iterate $f^{s_m-1}|_{\mathfrak{S}_f^m}$ (whose image $\mathfrak{S}^m(c_0) \equiv \mathfrak{S}_f^m(c_0)$ contains the critical point c_0) is univalent. We let

$$(5.11) \quad \Pi_m \equiv \Pi_f^m = \pi_m \circ f^{-(s_m-1)} : \mathfrak{S}^m(c_0) \rightarrow \mathbb{C},$$

where $f^{-(s_m-1)}|_{\mathfrak{S}^m(c_0)}$ is the branch of the inverse map with image \mathfrak{S}^m .

The domain Ω_f^m can be inductively obtained from Ω_{Rf}^{m-1} by lifting the latter by an appropriate inverse branch of π_f , and then applying $O(N_1)$ number of iterates of f to “close up the gaps”. (See §2.2 of [Ch]) for a detailed description).

Property (P3) and Lemma 5.6 (iv) imply:

Lemma 5.8. *Let f be an m times IS-renormalizable map such that $R^m f$ is a parabolic map with multiplier 1. Then the postcritical set \mathcal{O}_f is trapped inside Ω_f^m .*

Let us also consider the lifts Δ_f^m of the domains $\Delta_{R^m f}$ under π_f^m . We let

$$\mathcal{N}_f^m = \bigcup_{n=0}^{r_m} f^n(\Delta_f^m),$$

where the times r_m are the same as in (5.10). Moreover, f^{s_m-1} maps Δ^m univalently and with bounded distortion onto its image $\Delta^m(c_0) \equiv \Delta^m(c_0)$ containing the critical point c_0 . Thus, change of variable Π_m (5.11) restricted to $\Delta^m(c_0)$,

$$(5.12) \quad \Pi_m : \Delta^m(c_0) \rightarrow \mathbb{C},$$

is a univalent map with bounded distortion. Notice also that by compactness of $\overline{\mathcal{IS}}$ and continuous dependence of Δ_g^m on $g = R^m f$, the image of the restricted map Π_m contains a definite neighborhood of the critical point.

Like the Ω_f^m , the sets \mathcal{N}_f^m can be inductively constructed by lifting and spreading. We call these sets *necklaces*.

Lemma 5.7 implies:

Corollary 5.9. *The image $\pi_f^m(\Delta_f^{m-1})$ compactly contains $\Delta_{R^m f}$, with a definite space in between. There exist $\rho = \rho(\bar{N}) > 1$ such that $\text{diam } \Delta_f^m = O(\rho^{-m})$. Moreover, for each m , the domain Δ_f^m depend continuously on f .*

Corollary 5.10. *Let $f \in \overline{\mathcal{IS}}_\theta$ be a map of IS class with irrational rotation number. Then the critical point is recurrent.*

Proof. Indeed, the critical point returns to all the domains $\Delta^m f$, and these domains shrink. \square

5.5. Siegel disks. The next statement shows that maps $f \in \mathcal{IS}_\theta$ with θ of high bounded type are Siegel maps:

Proposition 5.11 ([Ya3]). *Let $f \in \overline{\mathcal{IS}}_\theta$, where θ is a rotation number of high type bounded by some \bar{N} . Then f is a Siegel map: its Siegel disk S_f is a quasidisk compactly contained in Ω_f , and $\partial S_f \ni c_0$. Moreover, $f|_{\partial S_f}$ is quasisymmetrically conjugate to $\mathbf{f}_\theta|_{\partial \mathbf{S}_\theta}$.*

Proof. By replacing f with its IS renormalization $Rf \in \mathcal{IS}$, we can assume that $f \in \mathcal{IS}$ (see Property (P6)).

By §4.1.1, we know that the assertion is valid for the quadratic map \mathbf{f}_θ and hence for its renormalization $\mathbf{g} := R(\mathbf{f}_\theta) \in \mathcal{IS}_\theta$. Since \mathcal{IS}_θ is isomorphic to the Universal Teichmüller Space, any other map $f \in \mathcal{IS}_\theta$ can be connected to \mathbf{g} by a holomorphic Beltrami path f_λ , $\lambda \in \mathbb{D}$.

Let $c_0(\lambda)$ be the critical point of f_λ , and let $c_n(\lambda) = f^n(c_0(\lambda))$, $n \in \mathbb{N}$. By Proposition 5.5, all points $c_n(\lambda)$ are well defined, and then, they depend holomorphically on λ . Moreover, they do not collide: $c_n(\lambda) \neq c_m(\lambda)$ for $n \neq m$ (by Proposition 5.5 and Corollary 5.10). Hence, they form a holomorphic motion over \mathbb{D} .

By the λ -lemma, this motion extends to the postcritical set \mathbf{O} of \mathbf{g} , and provides us with a family of quasisymmetric homeomorphisms $h_\lambda : \mathcal{O} \rightarrow \mathcal{O}_\lambda$, $\lambda \in \mathbb{D}$, where \mathcal{O}_λ is the postcritical set for f_λ . It follows that \mathcal{O}_λ is a quasicircle for any $\lambda \in \mathbb{D}$, in particular, for the original map f .

Let D be a quasidisk bounded by \mathcal{O}_f . Then the family of iterates f^n is normal on D , so $D \subset S_f$. On the other hand, as the Siegel disk S_f does not contain preimages of c_0 , which are dense in $\partial D = \mathcal{O}_f$, S_f is contained in D . \square

5.6. IS Renormalization fixed point. Now the whole theory of Siegel maps developed in §4 (external tilings, periodic points, trapping disks, renormalization fixed points, etc.) is applicable to any class \mathcal{IS}_N , $N > \underline{N}$.

Theorem 5.12 ([IS]). *Let $\theta = \theta_N$ be a stationary rotation number of high type. Then the IS renormalization R has a unique hyperbolic fixed point $f_\infty \in \mathcal{IS}_N$. The unstable manifold $\mathcal{W}^u(f_\infty)$ is a complex curve that can be parametrized by the complex rotation number ranging over a neighborhood of $[0, \theta]$. Moreover, $R^n f \rightarrow f_\infty$ exponentially fast for any Siegel map $f \in \mathcal{I}_N$.*

Corollary 5.13. *Under the circumstances of the above lemma, let us consider a holomorphic family $\mathcal{F} \subset \mathcal{B}_\circ$ passing through a Siegel map $f_\circ \in \mathcal{IS}_N$ transversally*

to \mathcal{IS}_N . Then the sequence of the IS renormalizations $R^n(\mathcal{F})$, $n = 0, 1, \dots$, is precompact, and in fact, it converges to the unstable manifold $\mathcal{W}^u(f_\infty)$.

5.7. Perturbations of Siegel maps. The above control of one renormalization, together with existence of the hyperbolic renormalization fixed point, provides us with a good control of perturbations of Siegel disks of stationary type:

Lemma 5.14. *Let f_\circ be a Siegel map of Inou-Shishikura class with stationary rotation number $\theta_\circ = [N, N, \dots]_*$, and let $\mathcal{F} = \{f_\lambda\}$ be a holomorphic family through $f_\circ = f_{\lambda_\circ}$ transverse to \mathcal{IS}_N . Then for any rotation number $\theta \in \mathbb{R}/\mathbb{Z}$, there exists a map $f_\lambda \in \mathcal{F}$ such that the renormalization $R^m f_\lambda$ with the same combinatorics as $R^m f_\circ$ is well defined and has rotation number θ . Moreover, the set Ω_f^m is contained in $O(\rho^{-m})$ -neighborhood of S_\circ , where $\rho = \rho(N) > 1$.*

Proof. Existence of $f = f_\lambda$ follows from the Renormalization Theorem 5.12. Moreover, the renormalizations of f_λ shadow those of f_\circ :

$$(5.13) \quad \text{dist}(R^n f, R^n f_\circ) \leq C|\theta - \theta_\circ| \rho_0^{-(m-n)}, \quad n = 0, 1, \dots, m,$$

where $\rho_0 = \rho_0(N) > 1$.

Let us now apply the lifting and spreading procedure to control the necklaces, and hence the Ω^m -domains. Assume we have already constructed a necklace $\mathcal{N}_{R^n f}^{m-n}$ which is confined to a δ -neighborhood of $S_{R^n f_\circ}$. By Corollary 5.9, under sufficiently many further lifts, it will shrink by a big factor. Spreading this pullback around by a bounded number of iterates of $R^{m-n-k} f$, the necklace can be pulled farther away from S_\circ by exponentially small (in $m-n$) distance, see (5.13). These two mechanisms imply the desired. \square

Together with Lemma 5.8, this leads us to the following important conclusion:

Corollary 5.15 ([BC]). *Under the circumstances of Lemma 5.14, assume the map $R^m f_\lambda$ is parabolic with multiplier 1. Then the postcritical set \mathcal{O}_λ of f_λ is contained in the $O(\rho^{-m})$ -neighborhood of the Siegel disk S_\circ .*

6. CONSTRUCTION OF AN EXAMPLE

6.1. Outline. Let us start with a rough description of our example. Take a big $l \in \mathbb{N}$, a bigger $\kappa \in \mathbb{N}$, and an even much bigger $m \in \mathbb{N}$. Begin with a Siegel quadratic polynomial

$$\mathbf{f} = \mathbf{f}_\theta : z \mapsto e^{2\pi i \theta} z + z^2$$

with a stationary rotation number of high type, and consider its cylinder renormalization $f = R_S^{m-\kappa} \mathbf{f}$. It is a Siegel map of Inou-Shishikura class.

Moreover, f has a distinguished repelling periodic point $\alpha = \alpha^l$ of period q_l (that approximates the dynamics on ∂S_f in scale l). Perturb f to a parabolic approximand \tilde{f} with rotation number p_κ/q_κ . Then α gets perturbed to a periodic point $\tilde{\alpha}$ with the same period.

Furthermore, using the theory of parabolic bifurcation, one can perturb \tilde{f} to a Misiurewicz map f_{Mis} for which $\tilde{\alpha}$ becomes a postcritical point α_{Mis} . Since α_{Mis} can be approximated with precritical points, f_{Mis} can be further perturbed to a superattracting map f_\circ .

The last map can be anti-renormalized to obtain a superattracting quadratic polynomial \mathbf{f}_\circ such that $f_\circ = R_S^{m-\kappa} \mathbf{f}_\circ$. This quadratic polynomial determines

a renormalization combinatorics. The unique infinitely renormalizable quadratic polynomial f_* with this combinatorics is desired.

Our construction depends on six large integer parameters N, l, κ, t , and m, j , selected consecutively as listed, where the last two play somewhat different role than the first four. Once we select one of the first four parameters, we assume, sometimes without saying, that all the rest depends on this choice. A statement *For any consecutively selected* $(N, l, \kappa) > (\underline{N}, \underline{l}, \underline{\kappa}) \dots$ (or *For any consecutively selected sufficiently big* $(N, l, \kappa) \dots$) will mean

$$\exists \underline{N} \quad \forall N > \underline{N} \quad \exists \underline{l} = \underline{l}(N) \quad \forall l > \underline{l} \quad \exists \underline{\kappa} = \underline{\kappa}(N, l) \quad \forall \kappa > \underline{\kappa} \dots$$

We will also assume that the choice $\underline{l}(N)$ is made *monotonically increasing* in N , the choice of $\underline{\kappa}(N, l)$ is monotonically increasing in each variable, and similarly for any other parameter in question.

Let us now supply the details.

6.2. Perturbed periodic points and trapping disks.

6.2.1. *General perturbations.* Recall that $\text{Sieg}(\bar{N}, \mu, K)$ stands for the space of Siegel maps $f : (\Omega, 0) \rightarrow (\mathbb{C}, 0)$ introduced in §4.3.1.

When we perturb f below, we will use the uniform metric on Ω .

Lemma 6.1. *There exist natural numbers³⁰ \underline{l} and ι depending on (\bar{N}, μ, K) such that for any $l \geq \underline{l}$, there exists a $\delta_0 = \delta_0(\bar{N}, \mu, K, l) > 0$ with the following property. For any $\delta < \delta_0$, if a holomorphic map $\tilde{f} : \Omega \rightarrow \mathbb{C}$ is δ -close to a Siegel map $f : \Omega \rightarrow \mathbb{C}$ of class $\text{Sieg}(\bar{N}, \mu, K)$ then*

- (i) *There exists a periodic point $\tilde{\alpha}^l$ of period q_l which is a perturbation³¹ of the α^l ;*
- (ii) *There exists a collar³² A^l in $\Omega \setminus \bar{S}_f$ such that: it is impossible to jump over it under \tilde{f} :*

$$\text{If } z \in \text{Comp}_0(\mathbb{C} \setminus A^l), \quad \tilde{f}(z) \notin \text{Comp}_0(\mathbb{C} \setminus A^l), \quad \text{then } \tilde{f}(z) \in A^l;$$

- (iii) *There exists a trapping quasidisk $D^l \Subset \Omega \setminus \bar{S}_f$ with bounded shape around $\tilde{\alpha}^l$ whose hyperbolic diameter in $\Omega \setminus \bar{S}_f$ is of order 1; moreover,*

$$D^l \cap D^{l+\iota} = \emptyset;$$

- (iv) *A definite part of the disk D^l is contained in $\tilde{f}^{-1}(S_f) \setminus \bar{S}_f$; moreover, there is a point $\tilde{\beta} \in \tilde{f}^{-1}(\partial S_f) \setminus \bar{S}_f$ that lies in the middle of D^l ;*

- (v) *If $z \in A^l$ then at some moment $k < q_{l+1}$, $f^k z$ lands in the middle of D^l , while*

$$f^i z \in \text{Comp}_0(A^{l-2\iota}) \setminus D^{l-\iota}, \quad i = 0, 1, \dots, k.$$

All geometric bounds depend only on N , μ , and K .

Proof. The properties of Proposition 4.8 are manifestly robust under perturbations, keeping the same collars A^l and trapping disks D^l . (The auxiliary collars A_0^l and disks D_1^l , as well as the collars $A^{l-2\iota}$ in the last statement, were designed to secure robustness.) \square

³⁰In the polynomial case, we can let $\underline{l} = 1$.

³¹meaning that $\tilde{\alpha}^l$ is $\epsilon(\delta)$ -near α^l where $\epsilon(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

³²Objects associated with \tilde{f} are usually marked with “tilde”, but it can be skipped if the object is independent of \tilde{f} , e.g., $\tilde{A}^l \equiv A^l$, $\tilde{D}^l \equiv D^l$, etc.

As before, we say that the trapping disk $D = D^l$ is centered at α^l , or that depth $D = l$.

6.2.2. Expansion. For a perturbation \tilde{f} of a Siegel map f , we will use notation $R_{S_p}^l \tilde{f} : X_+^l \cup X_-^l \rightarrow \mathbb{C}$ for the corresponding perturbation of the butterfly renormalization $R_{S_p}^l f$.

Away from the Siegel disk, Corollary 4.5 is robust under perturbations:

Lemma 6.2. *Let $f : \Omega \rightarrow \mathbb{C}$ be a Siegel map of class $\text{Sieg}(\bar{N}, \mu, K)$. For any $\epsilon > 0$ there exists $\delta = \delta(\bar{N}, \mu, K; \epsilon) > 0$ with the following property. Let $\tilde{f} : \Omega \rightarrow \mathbb{C}$ be a holomorphic map which is δ -close to f , and let $z \in X_+^l \cup X_-^l$ be a point with the property that $R^l f(z) \in Y^l$ and $\text{dist}(R^l f(z), \bar{S}) \geq \epsilon$. Then*

$$\|D(R^l f)(z)\|_{\text{hyp}} \geq \rho > 1$$

with ρ depending only on (\bar{N}, μ, K) and ϵ .

In turn, it implies a perturbed version of Corollary 4.6:

Corollary 6.3. *Let $f : \Omega \rightarrow \mathbb{C}$ be a Siegel map of class $\text{Sieg}(\bar{N}, \mu, K)$. For any $\epsilon > 0$, there exist $a > 0$, $\rho > 1$, and δ with the following property. For any holomorphic map $\tilde{f} : \Omega \rightarrow \mathbb{C}$ which is δ -close to f , if $z \in Y^m$, $f^n z \in Y^{m-k}$ for some $n \in \mathbb{N}$, $0 < k < m$ (with $m - k > l$), while*

$$\text{dist}(f^i z, \bar{S}) \geq \epsilon, \quad i = 0, 1, \dots, n,$$

then

$$\|Df^n(z)\|_{\text{hyp}} \geq a\rho^k,$$

where the norm is measured in the hyperbolic metric of $\mathbb{C} \setminus \bar{S}$.

6.2.3. Cylinder renormalization of polynomial maps. To make the exposition more transparent, we will focus on the stationary case when $\theta = \theta_N$ is a stationary rotation number with $N > \underline{N}$. Let $\mathbf{f} = \mathbf{f}_\theta : z \mapsto e^{2\pi i \theta} z + z^2$ be the corresponding Siegel quadratic polynomial, and let $\tilde{\mathbf{f}} = \mathbf{f}_{\tilde{\theta}}$ be its polynomial perturbation (where $\tilde{\theta}$ is not necessarily real). By the Inou-Shishikura theory, all cylinder renormalization of \mathbf{f} are well defined and belong to the IS class:

$$(6.1) \quad f_i = R_S^i(\mathbf{f}) \in \mathcal{IS}_\theta, \quad i = 1, 2, \dots$$

Moreover, for any n , if $\tilde{\theta}$ is sufficiently close to θ , then the same is true for the first n cylinder renormalizations of $\tilde{\mathbf{f}}$. In this case, we let

$$(6.2) \quad \tilde{f}_i = R_S^i(\tilde{\mathbf{f}}) \in \mathcal{IS}_{G_*^i \tilde{\theta}}, \quad i = 1, 2, \dots, n,$$

where $G_* : \gamma \mapsto -1/\gamma \pmod{\mathbb{Z}}$ is the modified and complexified Gauss map.

Theorem 5.12 and its Corollary 5.13 provide us with a good control of the maps \tilde{f}_i :

Lemma 6.4. *There exist positive μ, K, ϵ_0, C , and $\rho > 1$ depending only on N such that:*

- $f_i \in \text{Sieg}(N, \mu, K)$, $i = 0, 1, \dots$;
- For any $\gamma \in \mathbb{C}$ which is ϵ_0 -close to θ and any $n \in \mathbb{N}$, there exists a unique $\tilde{\theta}$ such that the cylinder renormalizations \tilde{f}_i , $i = 0, 1, \dots, n$, are well defined, and \tilde{f}_n has complex rotation number γ ;

•

$$\text{dist}(f_i, \tilde{f}_i) \leq C \text{dist}(f_n, \tilde{f}_n) \rho^{-(n-i)}, \quad i = 0, 1, \dots, n.$$

• The Siegel maps f_i converge to the Siegel renormalization fixed point f_∞ , while the nearby maps \tilde{f}_i converge to a map \tilde{f}_∞ in the unstable manifold $\mathcal{W}_S(f_\infty)$.

6.2.4. *Parabolic approximand $\tilde{\mathbf{f}}$.* We will now specialize a perturbation $\tilde{\mathbf{f}} = \mathbf{f}_{\tilde{\theta}}$ of the Siegel polynomial $\mathbf{f} = \mathbf{f}_\theta$. Take two natural numbers $\kappa < m$. Let $\tilde{\theta} = p_m/q_m$ be the (modified) continued fraction approximand to θ , so that $\tilde{\mathbf{f}}$ is the parabolic quadratic polynomial with rotation number p_m/q_m at 0. It is m times cylinder renormalizable with all the renormalizations $\tilde{f}_i = R_S^j \tilde{\mathbf{f}}$, $i \geq 1$, in the IS class. Moreover, f_i is parabolic with rotation number p_{m-i}/q_{m-i} at 0. We will consider the maps

$$(6.3) \quad f_{m-\kappa} = R_S^{m-\kappa}(\mathbf{f}) \in \mathcal{IS}_\theta, \quad \tilde{f}_{m-\kappa} = R_S^{m-\kappa}(\tilde{\mathbf{f}}) \in \mathcal{IS}_{p_\kappa/q_\kappa},$$

and their limits f_∞ and \tilde{f}_∞ . To simplify notation, we will often skip the subscript $m - \kappa \in \bar{\mathbb{N}}$ letting

$$f \equiv f_{m-\kappa}, \quad \tilde{f} \equiv \tilde{f}_{m-\kappa}, \quad m \in \bar{\mathbb{N}}_\kappa.$$

By Lemma 6.4, \tilde{f} is δ -close to $f : \Omega \rightarrow \mathbb{C}$ for κ big enough, so Lemma 6.1 is applicable, providing us with the trapping discs D^l and the collars A^l .

6.2.5. *Transit from $\tilde{\mathcal{C}}^r$ to $\tilde{\alpha}^l$.* For the parabolic map $\tilde{f} = \tilde{f}_{m-\kappa}$, we let:

- $\tilde{\mathcal{C}}^r$ be its the repelling crescent;
- $\tilde{\Delta}^\kappa$ be the domain of the renormalization change of variable $\tilde{\pi}_\kappa$, see §5.4;

Lemma 6.5. *For any consecutively selected N and l , there exists $\underline{\kappa}$ such that for any natural $m > \kappa > \underline{\kappa}$, the parabolic map $\tilde{f} = \tilde{f}_{m-\kappa}$ has the following property. There exists $\bar{s} = \bar{s}(N, l, \kappa)$ and a point $\tilde{a} \in \tilde{\mathcal{C}}^r \cap \tilde{\Delta}^\kappa$ such that $\tilde{f}^s(\tilde{a}) \in \tilde{D}^l$ for some $s \leq \bar{s}$, and this happens before the orbit of \tilde{a} passes through the collar $A^{l-\iota}$, where $\iota = \iota(N)$. Moreover, the projection $\tilde{\pi}_\kappa(\tilde{a})$ lies in the middle of the repelling crescent $\mathcal{C}^r(\tilde{f}_m)$, with a constant depending on κ but independent of N and l .*

Proof. The range $\tilde{\pi}_\kappa(\tilde{\Delta}^\kappa)$ contains an annulus $\{\epsilon < |z| < r\}$ with a definite r and a small³³ ϵ , slit along the straight ray $i\mathbb{R}_-$. Moreover,

- The ray does not intersect the repelling crescent $\mathcal{C}^r(\tilde{f}_m)$ (since the crescent is contained in the \mathbb{R}_+ -symmetric wedge of size $\pi/2$);
- ϵ is so small that the truncated crescent

$$\mathcal{C}_{\text{tr}}^r(\tilde{f}_m) := \mathcal{C}^r(\tilde{f}_m) \cap \mathbb{D}_\epsilon$$

is contained in an attracting crescent of \tilde{f}_m (by property (C2) of §5.1.1 and compactness of $\bar{\mathcal{I}}S_0$).

The above truncated crescent lifts under π_κ to a truncated crescent $\tilde{\mathcal{C}}_{\text{tr}}^r$ for \tilde{f} . The latter contains a point \tilde{a} that escapes the domain Ω . (For otherwise, the union of the repelling and attracting petals would form a neighborhood of 0 on which the family of iterates, $\{\tilde{f}^n\}_{n=0}^\infty$, would be well defined and normal.) By Lemma 6.1, this forces orb \tilde{a} to pass through the trapping disk D^l at some moment s before it passes through the collar $A^{l-\iota}$ with $\iota = \iota(N)$.

³³ How small it is depends on the truncation level t defining the domains Δ^κ , see §5.4.

If we fix κ , then we obtain a compact family of maps $\tilde{f} \in \bar{\mathcal{I}}S_{p_\kappa/q_\kappa}$, and the fundamental crescent $\tilde{\mathcal{C}}^r$ can be selected in a locally continuous way. This allows us to make a locally continuous choice of $\tilde{a}_{\tilde{f}}$, which, by compactness, makes the escaping time s bounded and puts \tilde{a} in the middle of $\tilde{\mathcal{C}}_{\text{tr}}^r$. Since $\tilde{\pi}_\kappa$ has a bounded distortion on $\tilde{\Delta}^\kappa$, this puts $\tilde{\pi}_\kappa(\tilde{a})$ in the middle of $\mathcal{C}_{\text{tr}}^r(f_m)$. \square

6.2.6. Pullback of D .

Lemma 6.6. *For any consecutively selected N and l , there exists $\underline{\kappa}$ such that for any natural $m > \kappa > \underline{\kappa}$ the parabolic map $\tilde{f}_{m-\kappa}$ has the properties of Lemma 6.5, and the trapping disc $D = D^l$ can be univalently and with bounded distortion pulled back to \tilde{a} along the orbit $\{\tilde{f}^i \tilde{a}\}_{i=0}^s$. Moreover, the whole pullback $\{\tilde{D}_{-k}\}_{k=0}^s$ is contained in $\text{Comp}_0(\mathbb{C} \setminus A^{l-\iota})$ for some $\iota = \iota(N)$, while the last domain \tilde{D}_{-s} is contained in the repelling crescent $\tilde{\mathcal{C}}^r$.*

Proof. By Proposition 5.15, for κ big enough, the postcritical set $\tilde{\mathcal{O}}$ of \tilde{f} stays close to $S = S_f$. Since D is contained well inside $\Omega \setminus \tilde{S}$, it is also contained well inside $\Omega \setminus \tilde{\mathcal{O}}$. So it has a bounded hyperbolic diameter in $\Omega \setminus \tilde{\mathcal{O}}$.

Let us consider the parabolic map $\tilde{f}_m = R_S^m \tilde{\mathbf{f}} = R_S^\kappa(\tilde{f}_{m-\kappa})$ with multiplier 1 at the origin. By Lemma 6.5, there is an escaping point \tilde{a} in $\tilde{\Delta}^\kappa$ such that $\tilde{\pi}_\kappa(\tilde{a})$ lies in the middle of the repelling crescent $\mathcal{C}^r(f_m)$, while $\tilde{a}_s \equiv \tilde{f}^s(\tilde{a})$ lands in $D = D^l$.

Corollary 4.6 implies that for any $\epsilon > 0$, if κ is sufficiently big, there is $k \leq s$ such that

- (i) \tilde{a}_{s-k} is ϵ -close to the critical point \tilde{c}_0 ;
- (ii) D can be univalently pulled back along the orbit $\{\tilde{a}_{s-n}\}_{n=0}^k$; let \tilde{D}_{-n} denote the corresponding disks;
- (iii) The hyperbolic diameter of \tilde{D}_{-k} in $\Omega \setminus \tilde{\mathcal{O}}$ is less than ϵ .

The last property allows us to enlarge \tilde{D}_{-k} to a disk $\tilde{D}'_{-k} \Subset \Omega \setminus \tilde{\mathcal{O}}$ such that

$$(6.4) \quad \text{mod}(\tilde{D}'_{-k} \setminus \tilde{D}_{-k}) > \mu, \quad \text{diam}_{\text{hyp}} \tilde{D}'_{-k} < \epsilon,$$

where $\mu = \mu(\epsilon) \rightarrow \infty$ as $\epsilon \rightarrow 0$.

Properties (i) and (iii) ensure that \tilde{a} lies in the range of the renormalization change of variable $\tilde{\pi}_{m-\kappa}$, so it can be lifted to a point $\tilde{\mathbf{a}}$ in the domain $\tilde{\Delta}^{m-\kappa}$ of $\tilde{\pi}_{m-\kappa}$. By equivariance of $\tilde{\pi}_{m-\kappa}$, there exist moments \mathbf{s} and $\mathbf{s} - \mathbf{k}$ such that points $\tilde{\mathbf{a}}_{\mathbf{s}}$ and $\tilde{\mathbf{a}}_{\mathbf{s}-\mathbf{k}}$ belong to $\tilde{\Delta}^{m-\kappa}$ and project by $\tilde{\pi}_{m-\kappa}$ to \tilde{a}_s and \tilde{a}_{s-k} respectively.

Moreover, the disks $\tilde{D} \ni \tilde{a}_s$ and $\tilde{D}'_{-k} \supset \tilde{D}_{-k} \ni \tilde{a}_{s-k}$ lift by $\tilde{\pi}_{m-\kappa}$ to disks $\tilde{D} \ni \tilde{\mathbf{a}}_{\mathbf{s}}$ and $\tilde{D}'_{-\mathbf{k}} \supset \tilde{D}_{-\mathbf{k}} \ni \tilde{\mathbf{a}}_{-\mathbf{k}}$ in $\mathbb{C} \setminus \tilde{\mathcal{O}}$. Since \tilde{f} is a global polynomial map, the disks $\tilde{D}'_{-\mathbf{k}} \supset \tilde{D}_{-\mathbf{k}}$ can be further pulled back to disks $\tilde{D}'_{-\mathbf{s}} \supset \tilde{D}_{-\mathbf{s}} \ni \tilde{\mathbf{a}}_{-\mathbf{s}}$ in $\mathbb{C} \setminus \tilde{\mathcal{O}}$.

As we know (see §4.2), this pullback contracts the hyperbolic diameter in $\mathbb{C} \setminus \tilde{\mathcal{O}}$. Since $\tilde{D}'_{-\mathbf{k}}$ has a small hyperbolic diameter, see (6.4), so does $\tilde{D}'_{-\mathbf{s}}$. Hence it has a small Euclidean diameter compared with $\text{dist}(\tilde{\mathbf{c}}_{-\mathbf{s}}, \tilde{\mathbf{c}}_{q_{m-\mathbf{s}}})$, where $\tilde{\mathbf{c}}_{-\mathbf{s}}$ is the center of $\tilde{\Delta}^m$. On the other hand, $\text{diam} \tilde{\Delta}^m$ is comparable with the latter distance, and we conclude that $\tilde{D}'_{-\mathbf{s}} \subset \tilde{\Delta}^m$.

We can now apply to $\tilde{D}'_{-\mathbf{s}} \supset \tilde{D}_{-\mathbf{s}}$ the renormalization change of variable $\tilde{\pi}_{m-\kappa}$ to obtain disks $\tilde{D}'_{-s} \supset \tilde{D}_{-s} \ni \tilde{a}$ in $\tilde{\Delta}^\kappa \setminus \tilde{\mathcal{O}}$ which are univalent pullbacks of the disks

$\tilde{D}'_{-k} \ni \tilde{D}_{-k}$. Moreover, the change of variable $\tilde{\pi}_\kappa$ is well defined on these disks, and

$$\text{mod}(\tilde{\pi}_\kappa(\tilde{D}'_{-s}) \setminus \tilde{\pi}_\kappa(\tilde{D}_{-s})) = \text{mod}(\tilde{D}'_{-s} \setminus \tilde{D}_{-s}) = \text{mod}(\tilde{D}'_{-k} \setminus \tilde{D}_{-k}) \geq \mu,$$

with a big μ , see (6.4). Hence the hyperbolic diameter of $\tilde{\pi}_\kappa(\tilde{D}_{-s})$ in $\Omega \setminus \tilde{\mathcal{O}}(\tilde{f}_m)$ is small. Since $\tilde{\pi}_\kappa(\tilde{a})$ lies in the middle of the repelling crescent of \tilde{f}_m , the disk $\tilde{\pi}_\kappa(\tilde{D}_{-s})$ lies inside the crescent. \square

Passing to the limit as $m \rightarrow \infty$ (using Lemma 6.4), we conclude:

Lemma 6.7. *There exists $\underline{\kappa}$ such that for any natural $\kappa > \underline{\kappa}$ the map*

$$\tilde{f}_\infty = \lim_{m \rightarrow \infty} \tilde{f}_{m-\kappa} \in \mathcal{W}_S^u(f_\infty)$$

has the properties listed in Lemma 6.6.

6.3. Various connections. By a *connection* between two points, z and ζ , we mean a trajectory passing from a small neighborhood of z to a small neighborhood of ζ .

6.3.1. Connection between \tilde{c}_0 and 0. Property (P3) of the Inou-Shishikura class (§5.2) and compactness of the space $\text{Sieg}(N, \mu, K)$ (with $\mu = \mu(N)$ and $K = K(N)$) as in Lemma 6.4) imply:

Lemma 6.8. *There exists an $\bar{n} = \bar{n}(N, \kappa)$ such that for any parabolic map $\tilde{f} = \tilde{f}_{m-\kappa}$, $m \in \bar{\mathbb{N}}_\kappa$, we have: $\tilde{f}^n(\tilde{c}_0) \in \tilde{\mathcal{C}}^a$ for some $n \leq \bar{n}$.*

6.3.2. Connection between $\tilde{\alpha}^l$ and \tilde{c}_0 . Let us now make a connection between the periodic point α^l and the critical point c_0 :

Lemma 6.9. *For any (N, μ, K) there exists \underline{l} with the following property. For any natural $l > \underline{l}$ and any $\rho > 0$ there exists \underline{t} such that for any $t > \underline{t}$, any Siegel map $f \in \text{Sieg}(N, \mu, K)$ has a t -precritical point c_{-t} in the ρ -neighborhood of the periodic point α^l . Moreover, the orbit $\{c_n\}_{n=-t}^0$ is contained well inside $\Omega^{l-\nu}$ with ν depending only on (N, μ, K) . In particular, all these properties are valid uniformly for the maps $f_{m-\kappa}$, $m \in \bar{\mathbb{N}}_\kappa$.*

Proof. Let $\epsilon = \sigma \cdot \text{dist}(\alpha^l, c_0)$ with a small $\sigma \in (0, 1)$, and let W be the ϵ^2 -neighborhood of the critical value c_1 . Any point $z \in W \cap \partial S$, except c_1 itself, has a preimage $z_{-1} \notin \bar{S}$. Let k be the first moment when the backward orbit $\{c_{-n}\}$ of c_0 (along ∂S) lands in W . Then $k = k(N, \mu, K; l)$ and $\text{dist}(c_1, c_{-k}) \asymp \epsilon^2$ (with a constant depending on (N, μ, K) only).

The point c_{-k} has a preimage $c_{-k-1} \notin \bar{S}$ such that

$$\text{dist}(c_{-k-1}, c_0) \asymp \text{dist}(c_{-k-1}, \bar{S}) \asymp \epsilon.$$

It follows that if σ is sufficiently small then $c_{-k-1} \in Y^l$ and the hyperbolic distance $d := \text{dist}_{\text{hyp}}(c_{-k-1}, \alpha^l)$ in Y^l is bounded. (Here Y^l corresponds through the surgery to the range of the holomorphic circle pair from Theorem 3.11).

Let $D \ni c_{-k-1}$ be the hyperbolic disk in Y^l of radius $2d$ centered at α^l . By the Schwarz Lemma, $f^{-q_i}(D)$ is a subset of D of bounded hyperbolic diameter (where $f^{-l_{q_i}}$ is the inverse branch fixing α^l). A few more (of order $-\log \rho$) pullbacks of c_{-k-1} by f^{-q_i} will bring our point to the ρ -neighborhood of α^l .

Since this backward orbit stays in D , it is trapped inside $\text{Comp}_0(\mathbb{C} \setminus A^{l-\iota})$ with $\iota = \iota(N)$. Since points of $\partial A^{l-\iota}$ lie on depth $l - \iota$, while those of $\partial \Omega^{l-\nu}$ lie on depth $l - \nu$, we see that $A^{l-\iota}$ is contained well inside $\Omega^{l-\nu}$ for ν big enough (depending on (N, μ, K) only). The conclusion follows. \square

The above connection is robust:

Corollary 6.10. *For any (N, μ, K) there exists \underline{l} with the following property. For any natural $l > \underline{l}$ and any $\rho > 0$ there exist \underline{t} and $\delta_0 > 0$ such that for any $\delta < \delta_0$ and any natural $t > \underline{t}$, the following holds. If a map \tilde{f} is δ -close to a Siegel map $f \in \text{Sieg}(N, \mu, K)$ then it has a t -precritical point \tilde{c}_{-t} in the ρ -neighborhood of the periodic point $\tilde{\alpha}^l$. Moreover, the orbit $\{\tilde{c}_n\}_{n=-t}^0$ is contained in $\tilde{\Omega}^{l-\nu}$ with $\nu = \nu(N, \mu, K)$. In particular, these properties are valid for any parabolic map $\tilde{f}_{m-\kappa}$, $m \in \tilde{\mathbb{N}}$.*

6.3.3. *Connection between 0 and $\tilde{\alpha}^l$.*

Lemma 6.11. *For any consecutively selected sufficiently big N, l, κ and any $\rho > 0$, there exist \underline{t} and \bar{s} such that for any natural $t > \underline{t}$ and some $s \leq \bar{s}$, the following holds. For any parabolic map $\tilde{f} = \tilde{f}_{m-\kappa}$, $m \in \tilde{\mathbb{N}}$, there exists a precritical point \tilde{c}_{-s-t} lying in the middle of the repelling crescent \tilde{C}^r such that $\tilde{f}^s(\tilde{c}_{-s-t}) = \tilde{c}_{-t}$ where \tilde{c}_{-t} is ρ -close to the periodic point $\tilde{\alpha}^l$, and the orbit $\{\tilde{f}^i(\tilde{c}_{-s-t})\}_{i=0}^s$ is contained in $\tilde{\Omega}^{l-\nu}$ with some $\nu = \nu(N)$.*

Proof. By Lemma 6.4, for κ sufficiently big, $\tilde{f}_{m-\kappa}$ is close to $f_{m-\kappa}$, uniformly in $m \in \tilde{\mathbb{N}}$. Hence we can apply

- Lemma 6.1 to find a trapping disk $D \equiv D^l$ around α^l ;
- Lemma 6.5 to find \bar{s} and a point $\tilde{a} \in \tilde{C}^r$ such that $\tilde{f}^s \tilde{a} \in D$ for some $s \leq \bar{s}$;
- Corollary 6.10 to find, for any $t > \underline{t}$, a precritical point $\tilde{c}_{-t} \in D$ which is ρ -close to α^l .

By Lemma 6.6, the trapping disk D can be univalently pulled back to the point \tilde{a} . Moreover, this pullback is contained in $\tilde{\Omega}^{l-\nu}$ for some $\nu = \nu(N)$, and the last domain \tilde{D}_{-s} is compactly contained in the repelling crescent \tilde{C}^r . The corresponding pullback of the precritical point $\tilde{c}_{-t} \in D$ gives us the desired point \tilde{c}_{-s-t} . \square

6.3.4. *Transit from the repelling crescent to the attracting one.* Combining the last lemma with Lemma 6.8 and Corollary 6.10, we obtain:

Corollary 6.12. *For any consecutively selected sufficiently big N, l, κ , and for any $\rho > 0$, there exist \underline{t} , \bar{n} and \bar{s} with the following properties. For any $m \in \tilde{\mathbb{N}}_\kappa$ and any $t > \underline{t}$, there exist $n \leq \bar{n}$ and $s \leq \bar{s}$ such that the parabolic map $\tilde{f} = \tilde{f}_{m-\kappa}$ has a precritical point $\tilde{c}_{-s-t} \in \tilde{C}^r$ and a postcritical point $\tilde{c}_n \in \tilde{C}^a$ such that the whole orbit $\{\tilde{c}_k\}_{k=-s-t}^n$ is trapped in $\tilde{\Omega}^{l-\nu}$ with some $\nu = \nu(N)$.*

Recall that $\tilde{\mathbf{f}} = \mathbf{f}_{p_m/q_m}$ is the parabolic quadratic polynomial with rotation number p_m/q_m , and that $\tilde{\mathbf{C}}^{a/r}$ stand for the attracting and repelling crescents for $\tilde{\mathbf{f}}$. As $\tilde{f}_{m-\kappa} = R_S^{m-\kappa} \mathbf{f}$, we obtain:

Corollary 6.13. *The points \tilde{c}_{-s-t} and \tilde{c}_n from Corollary 6.12 lift to a precritical point $\tilde{\mathbf{c}}_{-s-t} \in \tilde{\mathbf{C}}^r$ and a postcritical point $\tilde{\mathbf{c}}_n \in \tilde{\mathbf{C}}^a$ for the parabolic polynomial $\tilde{\mathbf{f}}$ such that the whole orbit $\{\tilde{\mathbf{c}}_k\}_{k=-s-t}^n$ is trapped in $\tilde{\Omega}^{m-\kappa+l-\nu}$ with $\nu = \nu(N)$.*

6.4. **Quadratic-like Renormalization.**

6.4.1. *Superattracting parameter.* Let us now perturb the parabolic map $\tilde{f} \equiv \tilde{f}_{m-\kappa}$, $m \in \bar{\mathbb{N}}_\kappa$, to a superattracting map $f_\circ \equiv f_{m-\kappa, j; \circ}$, $j \in \mathbb{N}$, that will determine the desired renormalization combinatorics. Its superattracting cycle³⁴ $\{c_k^\circ\}_{k=0}^{p-1}$ follows the following route:

- first, it passes from the critical point c_0° to a postcritical point c_n° in the attracting crescent C_\circ^a (where n comes from Lemma 6.8);
- then it goes through the parabolic gate to a precritical point c_{-t-s}° in the repelling crescent C_\circ^r (where s and t come from Lemmas 6.5 and 6.9);
- then it penetrates trough the boundary of the virtual Siegel disk S_f , approaches a periodic point α_\circ just missing it to land at c_{-t}° ;
- and finally, it returns to c_0° .

Here is a formal statement:

Lemma 6.14. *Let $\theta = \theta_N$ be a stationary rotation number of high type $N > \underline{N}$, and let $l > \underline{l}$ be a level selected in Lemma 6.1. For any $\delta > 0$, there exists $\underline{\kappa} = \underline{\kappa}(N, l; \delta)$ such that for any $\kappa > \underline{\kappa}$, some $n < \bar{n}(\kappa)$, $s < \bar{s}(\kappa)$, and any $t > \underline{t}(\underline{\kappa})$ and $m \geq \kappa$, there exists a superattracting map*

$$f_\circ \equiv f_{m-\kappa, j; \circ} = R_S^{m-\kappa}(\mathbf{f}_\circ) : \Omega \rightarrow \mathbb{C}$$

δ -close to the parabolic map \tilde{f} (6.3), with a superattracting cycle of period $p = n + j + s + t$, such that near the critical point c_0° we have

$$f_\circ^p = f_\circ^{s+t} \circ I_\circ \circ f_\circ^n,$$

where $I_\circ : C_\circ^a \rightarrow C_\circ^r$ is a transit map between the crescents of f_\circ . Moreover, the whole cycle of c_0° is contained in $\text{Comp}_0(\mathbb{C} \setminus A_\circ^{l-\iota})$ with some $\iota = \iota(N)$.

The same properties hold for the limit map

$$f_{\infty, j; \circ} = \lim_{m \rightarrow \infty} f_{m-\kappa, j; \circ}$$

in the unstable manifold of the renormalization fixed point (compare Lemma 6.7).

Proof. Let us consider postcritical point $\tilde{\mathbf{c}}_n \in \tilde{\mathbf{C}}^a$ and a precritical point $\tilde{\mathbf{c}}_{-s-t} \in \tilde{\mathbf{C}}^r$ from Corollary 6.13. Let $\mathbf{I} : \text{Cyl}^a \rightarrow \text{Cyl}^r$ be the isomorphism between the cylinders such that $\mathbf{I}(\tilde{\mathbf{c}}_n) = \tilde{\mathbf{c}}_{-s-t}$. By the Parabolic Bifurcation Theory (Theorem 5.2) for any j sufficiently big, $\tilde{\mathbf{f}}$ can be perturbed to a superattracting polynomial map $\mathbf{f}_\circ \equiv \mathbf{f}_{j; \circ}$ for which

$$\mathbf{f}_\circ^j(\mathbf{c}_n^\circ) = \mathbf{c}_{-s-t}^\circ.$$

Let $f_\circ = R_S^{m-\kappa}(\mathbf{f}_\circ)$ for $m \in \mathbb{N}_\kappa$. The desired properties for these maps, and their limit is $m \rightarrow \infty$, are evident. \square

6.4.2. *Quadratic-like families for parabolic maps.* Similarly, we can construct the whole quadratic-like family with the desired renormalization combinatorics:

Lemma 6.15. *For any consecutively selected natural $(N, l, \kappa, t) > (\underline{N}, \underline{l}, \underline{\kappa}, \underline{t})$, and any $m \in \bar{\mathbb{N}}_\kappa$, any parabolic map $\tilde{f}_{m-\kappa}$ admits a family of transit maps $I_\lambda : \tilde{\text{Cyl}}^a \rightarrow \tilde{\text{Cyl}}^r$, $\lambda \in \Lambda$, with the following properties. There is a family of disks $\tilde{U}_\lambda \subset \tilde{V}$ around \tilde{c}_0 and moments $(n, s) \leq (\bar{n}, \bar{s})$ (from Lemmas 6.8 and 6.5) such that:*

³⁴We will mark the objects related to f_\circ with a subscript or superscript “ \circ ”. On the other hand, for the (pre-/post-) critical points c_k , we skip (here and below) subscripts indicating their dependence on various parameter’s: m, j , etc.

(0) V is a quasidisk with bounded dilatation and definite size depending only on (N, l, κ) ,³⁵

(i) The maps

$$(6.5) \quad \tilde{F}_\lambda = \tilde{f}^{s+t} \circ I_\lambda \circ \tilde{f}^n : \tilde{U}_\lambda \rightarrow \tilde{V}$$

form a proper unfolded quadratic-like family over Λ ;

(ii) The closures of all intermediate disks,

$$\tilde{f}^k(\tilde{U}_\lambda), \quad k = 0, 1, \dots, n, \quad \text{and} \quad \tilde{f}^k \circ I_\lambda \circ \tilde{f}^n(\tilde{U}_\lambda), \quad k = 0, 1, \dots, s+t-1,$$

that appear in composition (6.5) are pairwise disjoint;

(iii)

$$\bar{\mu}(N, l, \kappa, t) \geq \text{mod}(V \setminus \tilde{U}_\lambda) \geq \underline{\mu}(N, l, \kappa, t) \rightarrow \infty \text{ as } t \rightarrow \infty \text{ with } N, l, \kappa \text{ fixed.}$$

(iv) In case of connected Julia set $J(\tilde{F}_\lambda)$ (i.e., when λ belongs to the Mandelbrot set $\mathcal{M}'_{N, l, \kappa, t, m}$ of q -l family (6.5)), the disk \tilde{U}_λ is an $L(N, l, \kappa, t)$ -quasidisk with

$$\text{area } \tilde{U}_\lambda \geq c(N, l, \kappa, t) > 0.$$

All constants and bounds are independent of m .

Proof. In the \tilde{f} -plane, select a disk $\tilde{V} \ni \tilde{c}_0$, and let \tilde{V}_{-i} , $i = 0, 1, \dots, t$, be its pullback to \tilde{c}_{-t} . Let us show that if \tilde{V} is small enough, depending on N, l , and κ but independently of t , then the closures of these disks are pairwise disjoint. Consider a linearization domain W around the periodic point \tilde{a}_l (so, f^{q_l} maps W univalently onto $f^{q_l}(W) \ni W$). Note that its size depends on N and l only. It takes a bounded number of iterates ($\leq t_0(N, l, \kappa)$) for the pullback in question to get trapped in W . By adjusting W and selecting \tilde{V} sufficiently small, we ensure that the first t_0 pullbacks \tilde{V}_{-i} , $i < t_0$, are pairwise disjoint and disjoint from W , while $\tilde{V}_{-t_0} \Subset W$. Then the further pullback $\tilde{V}_{-i} \subset W$, $t_0 \leq i \leq t$, will stay pairwise disjoint and disjoint from the first ones. So, independently of t , the whole pullback \tilde{V}_{-i} , $i = 0, 1, \dots, t$, will consist of pairwise disjoint domains. Moreover,

$$(6.6) \quad \text{diam } \tilde{V}_{-t} \rightarrow 0 \text{ as } t \rightarrow \infty \text{ with } (N, l, \kappa) \text{ fixed.}$$

Let us now pull \tilde{V}_{-t} further to c_{-t-s} . The number s of iterates is bounded by $\bar{s}(N, l, \kappa)$ from Lemma 6.5, so for t sufficiently big, (6.6) ensures that these pullbacks stay small and pairwise disjoint. Since c_{-t-s} lies in the middle of the repelling crescent $\tilde{\mathcal{C}}^r$, the final domain \tilde{V}_{-t} is trapped well inside \mathcal{C}^r . Hence it projects to a disk compactly contained in the repelling cylinder $\text{Cyl}^r \approx \mathbb{C}/\mathbb{Z}$. (We will keep the same notation for it.)

Consider a parameter domain $\Lambda \subset \mathbb{C}/\mathbb{Z}$ such that $I_\lambda(\tilde{c}_n) \in \tilde{V}_{-t-s}$ for any transit parameter λ in Λ (in fact, under our normalizations and notational conventions, $\Lambda = \tilde{V}_{-t-s}$). Pull \tilde{V}_{-t-s} further back by this transit map, and then further back to \tilde{c}_0 by the iterates of \tilde{f} . Call the corresponding pullbacks $\tilde{V}_{\lambda, -t-s-l-i}$, $i \leq n$. By Lemma 6.8, it involves at most $\bar{n}(N, l, \kappa)$ iterates. Hence all these pullbacks have a small diameter and stay pairwise disjoint and disjoint from the initial pullbacks, which proves assertion (ii). It also follows that the disc $\tilde{U}_\lambda := \tilde{V}_{\lambda, -t-s-l-n}$ is trapped well inside \tilde{V} , which implies that the maps F_λ defined by (6.5) are quadratic-like.

³⁵In fact, for given (N, l, κ) , the disk itself can be selected independently of t ; for m sufficiently big, it can be selected independently of m either.

Since the transit map $I_\lambda : \mathcal{C}_\lambda^a \rightarrow \mathcal{C}^r$ depends holomorphically on $\lambda \in \Lambda$, these q-l maps form a quadratic-like family. For the same reason, the domains $\tilde{V}_{\lambda, -t-s-I}$, and hence their further pullbacks $\tilde{V}_{-s-t-I-i}$, move holomorphically with λ , so our family is equipped (see §2.1.1). For $\lambda \in \partial\Lambda$, we have $I_\lambda(c_n) \in \partial\tilde{V}_{-t-s}$, and hence $F_\lambda(c_0) \in \partial\tilde{V}$. Thus, our q-l family is proper. Finally, as λ goes once around $\partial\Lambda$ then $I_\lambda(c_n)$ goes once around \tilde{V}_{-t-s} (recall that with our normalizations, $\Lambda = \tilde{V}_{-t-s}$). So, our q-l family is unfolded. This completes the proof of (i).

The upper estimate in item (iii) and item (iv) follow from the property that the total number of \tilde{f} -iterates involved in our construction is bounded in terms of (N, l, κ) and t , while the transit maps I_λ , $\lambda \in \bar{\Lambda}$, form a compact family. Hence the size of U_λ is definite in terms of (N, l, κ) and t .

On the other hand, as $t \rightarrow \infty$ with (N, l, κ) fixed, (6.6) implies that $\text{diam } U_\lambda \rightarrow 0$. This yields the lower bound in item (iii). \square

For notational convenience, let us shift the m -parameter:

$$\mathbf{m} = m - \kappa \in \bar{\mathbb{N}} = \{1, 2, \dots, \infty\},$$

and let $\mathcal{F}_\mathbf{m} = R_S^\mathbf{m} \mathcal{F}$, where \mathcal{F} is the quadratic family (\mathbf{f}_γ) . By Theorem 5.12, these are holomorphic curves converging to the unstable manifold $\mathcal{F}_\infty = \mathcal{W}^u(f_\infty)$ for the Siegel renormalization.

6.4.3. *Quadratic-like families for parabolic perturbations.* Perturbing our parabolic maps $\tilde{f}_\mathbf{m}$ within the families $\mathcal{F}_\mathbf{m}$, we can construct genuinely renormalizable maps:

Lemma 6.16. *Under the circumstances of Lemma 6.15, for any $\mathbf{m} \in \bar{\mathbb{N}}_0$ and $j > \underline{j}(N, l, \kappa, t)$, there exists a holomorphic subfamily $\mathcal{F}_{\mathbf{m}, j} = (f_{\mathbf{m}, j; \lambda})$ of $\mathcal{F}_\mathbf{m}$ parametrized by some domain $\Lambda_{\mathbf{m}, j}$ with the following properties:*

(i) *Each family $\mathcal{F}_{\mathbf{m}, j}$ gives rise to a primitive proper unfolded q-l family*

$$F_{\mathbf{m}, j; \lambda} = f_{\mathbf{m}, j; \lambda}^{\mathbf{p}} : U_{\mathbf{m}, j; \lambda} \rightarrow V_\mathbf{m}, \quad \lambda \in \Lambda_{\mathbf{m}, j},$$

with period $\mathbf{p} = n + j + s + t$;

(i) *As $\mathbf{m} \rightarrow \infty$, the families $\mathcal{F}_{\mathbf{m}, j}$ converge, uniformly in \mathbf{m} , to the families $\mathcal{F}_{\infty, j}$ in $\mathcal{F}_\infty = \mathcal{W}_S^u(f_\infty)$;*

(iii)

$$\bar{\mu}(N, l, \kappa, t) \geq \text{mod}(V_\mathbf{m} \setminus U_{\mathbf{m}, j; \lambda}) \geq \underline{\mu}(N, l, \kappa, t) \rightarrow \infty \text{ as } t \rightarrow \infty \text{ with } N, l, \kappa \text{ fixed.}$$

(iv) *In case of connected Julia set $J(F_\lambda)$ (i.e., when λ belongs to the corresponding little Mandelbrot set $\mathcal{M}'_{N, l, \kappa, t, m, j}$), the disks U_λ^j are $L(N, l, \kappa, t)$ -quasidisks with*

$$\text{area } U_{\mathbf{m}, j; \lambda} \geq c(N, l, \kappa, t) > 0.$$

All geometric constants and bounds are independent of \mathbf{m} and j .

Proof. Throughout this argument, (N, l, κ, t) will be fixed, and dependences on them will not be mentioned. Parameters m and j will be free.

By Corollary 5.13, the families $\mathcal{F}_\mathbf{m}$ stay within a compact collection of families crossing the Siegel class $\{f \in \mathcal{IS} : f'(0) = e^{2\pi i \theta}\}$ transversally at points $f_\mathbf{m} = R_S^\mathbf{m} \mathbf{f}_\theta$. In fact, they converge, as $\mathbf{m} \rightarrow \infty$, to the unstable manifold $\mathcal{W}^u(f_\infty) \equiv \mathcal{F}_\infty$ of the Siegel fixed point. Moreover, the parabolic maps

$$\tilde{f}_\mathbf{m} = R_S^\mathbf{m}(\mathbf{f}_{p_\mathbf{m}/q_\mathbf{m}}) = f_{m; p_\kappa/q_\kappa} \in \mathcal{F}_\mathbf{m},$$

converge to $\tilde{f}_\infty \in \mathcal{F}_\infty$. This allows us to apply the Parabolic Bifurcation Theory in a uniform way to the families \mathcal{F}_m near the maps \tilde{f}_m .

Let us start with the limiting parabolic map $\tilde{f} = \tilde{f}_\infty$. Let $V \ni c_0$ be the disk selected for this map in Lemma 6.15, and let $\tilde{V}_{-s-t} \ni c_{-s-t}$ be its pullback constructed in that lemma. It is compactly contained in the repelling crescent $\tilde{\mathcal{C}}^r$, and hence it is compactly contained in some smooth disk $\Lambda \Subset \tilde{\mathcal{C}}^r$.

There is a neighborhood $\Upsilon \subset \mathcal{F}_\infty$ of \tilde{f} such that for any map $f_\gamma \equiv f_{\infty, \gamma} \in \Upsilon$, the pullback $V_{-s-t}^\gamma \ni c_{-s-t}$ of V under f_γ^{s+t} is compactly contained in Λ as well (uniformly over $f_\gamma \in \Upsilon$). Moreover, since the disks \tilde{V}_{-s-t}^γ are univalent pullbacks of a fixed disk V by a holomorphic family of maps f_γ^{s+t} , they move holomorphically in γ ; let

$$h_\gamma : \tilde{V}_{-s-t} \rightarrow V_{-s-t}^\gamma$$

be this holomorphic motion (based at \tilde{f}).

By Theorem 5.2, for any sufficiently big j , there exists a holomorphic function $\gamma = \gamma_j(\lambda)$ on Λ such that the transit maps $I_\gamma^j : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}/\mathbb{Z}$ induced by f_γ^j , have the following properties:

- $I_\gamma^j(0) = \lambda$ (recall that the uniformizations of the Douady cylinders $\text{Cyl}^{a/r}$ by \mathbb{C}/\mathbb{Z} are selected so that $c_n \in \text{Cyl}^a$ and $c_{-s-t} \in \text{Cyl}^r$ correspond to $0 \in \mathbb{C}/\mathbb{Z}$);
- As $j \rightarrow \infty$, the transit maps $I_{\gamma(\lambda)}^j$ converge uniformly on compact sets of \mathbb{C}/\mathbb{Z} and uniformly in $\lambda \in \Lambda$ to the transit map $I_\lambda : z \mapsto z + \lambda$ between the Ecalle-Voronin cylinders for the parabolic map \tilde{f} .

By the Argument Principle³⁶ for any $z \in \partial\tilde{V}_{-s-t}$ there exists a unique $\lambda \in \Lambda$ such that

$$h_\gamma(z) = I_\gamma^j(0), \quad \text{with } \gamma = \gamma_j(\lambda),$$

and these λ 's go around a Jordan curve $\Gamma^j \Subset \Lambda$. This implies that each quadratic-like family

$$(6.7) \quad F_{j;\lambda} = f_\gamma^{s+t} \circ I_\gamma^j \circ f_\gamma^n : U_{j;\gamma} \rightarrow \tilde{V}$$

is proper and unfolded over the disk bounded by $\Gamma^j \Subset \Lambda$, where $\gamma = \gamma_j(\lambda)$ and $U_{j;\gamma} \ni c_0$ is the pullback of V_{-t-s}^γ by $I_\gamma^j \circ f_\gamma^n$. We obtain assertions (i) and (ii) for $m = \infty$.

Assertions (iii) and (iv) for $m = \infty$ follow from the corresponding assertions of Lemma 6.15 since the quadratic-like families (6.7) are small perturbations (for big j) of the family \tilde{F}_λ (6.5).

For each finite m , we can apply the same argument to the family \mathcal{F}_m , which provides us with quadratic-like families $F_{m,j;\lambda}$ with desired properties, except that the geometric constants and bounds may depend on m . To make them uniform, we can apply a perturbative argument near \mathcal{F}_∞ . Namely, let us start with the same disk $V \ni c_0$ as for $\tilde{f} \equiv \tilde{f}_\infty$, and pull it back by $f_{m;\gamma}^{s+t}$. We obtain a holomorphically moving family of disks $V_{-s-t}^{m;\gamma} \Subset \mathcal{C}^r(f_{m;\gamma})$ which is a small perturbations of the above family (V_{-s-t}^γ) for the f_γ . In particular, for m big enough, all these disks are uniformly compactly contained in the domain Λ used for $m = \infty$.

Moreover, by Theorem 5.2, as $m, j \rightarrow \infty$, the transit maps $I_\gamma^{m,j}$, with $\gamma = \gamma_{m,j}(\lambda)$, associated with $f_{m;\gamma}$, converge to I_λ . It follows that for m and j sufficiently

³⁶This is an occasion of the standard Phase-Parameter relation.

big, the quadratic-like families $(F_{\mathbf{m},j;\lambda})$ are small perturbations of the family (\tilde{F}_λ) from Lemma 6.15. The uniformity of the geometric bounds follows. \square

6.4.4. *Renormalizations in the quadratic family.* Lifting the above renormalization to the quadratic family by means of change of variable $\Pi_{m-\kappa}$ (5.12) we obtain:

Corollary 6.17. *Let $\underline{N}, \underline{l}, \underline{\kappa}$, and \underline{t} be as above. Then for any natural $(N, l, \kappa, t) > (\underline{N}, \underline{l}, \underline{\kappa}, \underline{t})$, there exist \underline{m} and \underline{j} with the following properties. For each natural $(m, j) > (\underline{m}, \underline{j})$, consider the holomorphic family $\mathcal{F}_{m,j} = (\mathbf{f}_{m,j;\lambda})$ of quadratic polynomials such that*

$$f_{m-\kappa,j;\lambda} = R_S^{\mathbf{m}}(\mathbf{f}_{m-\kappa,j;\lambda}),$$

where $(f_{m-\kappa,j;\lambda})$ is the family from Lemma 6.16, Then:

(i) Each family $\mathcal{F}_{m,j}$ admits a primitive proper unfolded q -l renormalization

$$\mathbf{F}_{m,j;\lambda} = \mathbf{f}_{m,j;\lambda}^{\mathbf{p}} : \mathbf{U}_{m,j;\lambda} \rightarrow \mathbf{V}_m, \quad \lambda \in \Lambda_{m,j};$$

(ii)

$\mu(N, l, \kappa, t) \geq \text{mod}(\mathbf{V}_m \setminus \mathbf{U}_{m,j;\lambda}) \geq \underline{\mu}(N, l, \kappa, t) \rightarrow \infty$ as $t \rightarrow \infty$ with N, l, κ fixed.

(iii) In case of connected Julia set $\mathbf{J} \equiv J(\mathbf{F}_\lambda)$ (i.e., when λ belongs to the corresponding little Mandelbrot set $\mathcal{M}'_{N,l,\kappa,t,m,j}$), the disks $\mathbf{U}_{m,j;\lambda}$ are $L(N, l, \kappa, t)$ -quasidisks with

$$\text{area } \mathbf{U}_{m,j;\lambda} \geq c \text{ area } \mathbf{V}_m, \quad \text{where } c = c(N, l, \kappa, t) > 0.$$

All geometric constants and bounds are independent of \mathbf{m} and j .

The little Mandelbrot copies $\mathcal{M}' = \mathcal{M}'_{N,l,\kappa,t,m,j} \subset \mathcal{M}$ generated by these quadratic-like families determine the desired renormalization combinatorics. Below, a map \mathbf{f}_λ will be called *renormalizable* if it is DH renormalizable with these combinatorics (and similarly, for the Siegel map f_λ).

6.5. **A priori bounds.** Along with lower thresholds $(\underline{N}, \underline{l}, \underline{\kappa})$ let us select some upper bounds $(\bar{N}, \bar{l}, \bar{\kappa}) > (\underline{N}, \underline{l}, \underline{\kappa})$ satisfying the following requirements:

$$\bar{N} > \underline{N}, \quad \bar{l} > \underline{l} = \underline{l}(\bar{N}), \quad \bar{\kappa} > \underline{\kappa} = \underline{\kappa}(\bar{N}, \bar{l}).$$

Let $\mathbf{f}_* : \mathbf{U} \rightarrow \mathbf{V}$ be an infinitely renormalizable quadratic polynomial with bounded combinatorics $(\mathcal{M}^i)_{i=0}^\infty$, where $\mathcal{M}^i = \mathcal{M}'_{N_i, l_i, \kappa_i, t_i, m_i, j_i}$ are the little Mandelbrot copies constructed above with

$$(6.8) \quad (\underline{N}, \underline{l}, \underline{\kappa}) < (N_i, l_i, \kappa_i) \leq (\bar{N}, \bar{l}, \bar{\kappa})$$

(while the bounds on t_i, m_i and j_i are not yet specified³⁷).

Proposition 6.18. *For any sequence (N_i, l_i, κ_i) satisfying (6.8) there exists \underline{t} such that if*

$$t_i > \underline{t}, \quad i = 0, 1, \dots,$$

then the quadratic polynomial \mathbf{f}_* has a priori bounds $\nu(\bar{N}, \bar{l}, \bar{\kappa}) > 0$ independent of (t_i, m_i, j_i) .

³⁷In fact, in this section one can consider maps \mathbf{f}_* with unbounded t_i, m_i, j_i

Proof. If g is a quadratic-like map with $\text{mod } g > \mu$ then it is K -qc conjugate to a quadratic polynomial \mathbf{f}_θ , where $K = K(\mu) \searrow 1$ as $\mu \rightarrow \infty$. Hence, if g is DH renormalizable with any combinatorics $\mathcal{M}' = \mathcal{M}'_{N,l,\kappa,t,m,j}$ under consideration, then its renormalization Rg has modulus at least $K^{-1}\underline{\mu}$, where $\underline{\mu} = \underline{\mu}(N, l, \kappa, t)$ is from Corollary 6.17, and $K = K(\underline{\mu})$.

Let us select ν so that $K(\nu) < 2$ and then \underline{t} so that $\underline{\mu}(N, l, \kappa, t) > 2\nu$ for any $t > \underline{t}$ (which is possible by Corollary 6.17). Then for any quadratic-like map g with $\text{mod } g > \nu$ which is renormalizable with combinatorics \mathcal{M}' , we have $\text{mod } Rg > \nu$ as well.

It follows that ν gives a *a priori bound* for any quadratic-like map g with $\text{mod } g > \nu$ which is infinitely renormalizable with combinatorics (\mathcal{M}^t) . \square

6.6. Landing probability. Let $f_* = R_S^{m-\kappa} \mathbf{f}_*$, and let $Rf_* : U_* \rightarrow V_*$ be its DH pre-renormalization (with the combinatorics constructed in §6.4.1).

The next lemma shows that there is a definite probability of landing in the renormalization domain U_* of the map f_* .

Lemma 6.19. *Let \underline{l} and ι be as in Lemma 6.1. Let $l > \underline{l} + \iota$ and let $D_* = D_*^{l-\iota}$ be the trapping disk for f_* constructed in that lemma. Then D_* contains domains $U' \subset V'$ of comparable (with D_*) size (with constants depending on N, l, κ , and t) which are mapped respectively to U_* and V_* under some iterate of f_* . Moreover, D_* is contained well inside $\text{Dom } f_* \setminus \mathcal{O}_*$ (with a lower bound depending on N only), where $\mathcal{O}_* = \mathcal{O}_{f_*}$ is the postcritical set for f_* .*

Proof. Recall that f_* is a small perturbation of the Siegel map f whose Siegel disk is called $S = S_f$. Let S' be the component of $f^{-1}(S)$ which is different from S . The trapping disk $D^{l-\iota}$ for f contains in the middle some point of $\partial S'$. If f_* is sufficiently close to f the $D_* = D_*^{l-\iota}$ contains in the middle some point of $f_*^{-1}(\partial S)$. Hence $f_*(D_*)$ contains in the middle some point of ∂S .

The renormalization range V_* can be selected at a much deeper (but still depending only on N, l, κ , and t) dynamical scale than $f_*(D_*)$. Then $f_*(D_*)$ contains many (in fact, we need only one) univalent and bounded distortion pullbacks of V_* under the Siegel map f . Moreover, these pullbacks have size comparable with $\text{diam } f_*(D_*)$. Selecting f_* sufficiently close to f , we ensure the same property for f_* . Then D_* also contains a comparable pullback of V_* . The corresponding pullback of U_* has a comparable size as well (all in terms of N, l, κ , and t).

The last assertion follows from the property that the postcritical set \mathcal{O}_* lies well inside A_*^{l-1} while D_* lies outside A_*^{l-1} . \square

We call the disk $D = D_*^{l-\iota}$ (and similar disks that appear below) a *safe trapping disk* since it can be “safely” pulled back, with a bounded distortion (depending on N only), along any orbit landing in it. As before, we say that D is *centered* at $\alpha^{l-\iota}$, or that depth $D = l - \iota$.

Lifting this disk by the renormalization change of variable $\Pi_{m-\kappa}$ (5.12), we obtain:

Corollary 6.20. *The quadratic polynomial \mathbf{f}_* has a safe trapping disk $\mathbf{D} := \mathbf{D}_*^{m-\kappa+l-\iota}$ that contains domains $\mathbf{U}' \subset \mathbf{V}'$ of comparable (with \mathbf{D}) size which are mapped respectively to \mathbf{U}_* and \mathbf{V}_* under some iterate of \mathbf{f}_* . The constant depends on N, l, κ , and t but is independent of m .*

We will refer to the above disk D as the *base safe trapping disk*.

Spreading the disks $U' \subset V'$ around by the landing map, we obtain:

Corollary 6.21. *For any point z whose orbit passes through the safe trapping disk D under the iterates of f_* , there exist quasidisks $U(z) \subset V(z)$ with bounded dilatation whose size is comparable with $\text{dist}(z, V(z))$, and such that*

$$f_*^n(U(z)) = U_*, \quad f_*^n(V(z)) = V_* \quad \text{for some } n = n(z).$$

All constants and bounds depend on N, l, κ and t , but not on m .

We are now ready to show the map f_* has a definite landing probability η .

Proposition 6.22. *For the polynomial f_* , the landing probability η is bounded from below in terms of N, l, κ , and t , uniformly in m .*

Proof. It is known that almost all point of the Julia set $J_* = J(f_*)$ land in U_* [L1], so it is sufficient to deal with the Fatou set. Since the Siegel disk $S = S_f$ occupies certain area, it is sufficient to check that a definite portion of points $z \in S \setminus J_*$ land in U_* . But any point $z \in S \setminus J_*$ on its way from S to ∞ must pass through the base safe trapping disk D . Then Lemma 6.21 provides us with a domain $U(z)$ of points landing in U_* that occupies a definite portion of some neighborhood of z . The conclusion follows. \square

6.7. Escaping probability ξ .

6.7.1. *Porosity.* Let us start with a general measure-theoretic lemma asserting that if a set X has density less than $1 - \epsilon$ in many scales then it has small area.

By a *gap* in X of radius r we mean a round disk of radius r disjoint from X .

Lemma 6.23. *For any $\rho \in (0, 1)$, $C > 0$ and $\epsilon > 0$ there exist $\sigma \in (0, 1)$ and $C_1 > 0$ with the following property. Assume that a measurable set $X \subset \mathbb{D}_r$ has the property that for any $z \in X$ there are n disks $\mathbb{D}(z, r_k)$ with radii*

$$C^{-1}\rho^{l_k} \leq r_k/r \leq C\rho^{l_k}, \quad l_k \in \mathbb{N}, \quad l_1 < l_2 < \dots < l_n,$$

containing gaps in X of radii ϵr_k . Then $\text{area } X \leq C_1 \sigma^n r^2$.

Proof. Since the assertion is scaling invariant, we can assume without loss of generality that $r = 1$. We can also assume that X is compact, and we can work with squares instead of disks. Using the first scale l_1 for points of X , we can subdivide the unit square \mathbb{Q} into dyadic squares Q_i^1 (of varying scales) such that each Q_i^1 contains a comparable dyadic square B_i^1 (of relative scale depending on ϵ) disjoint from X . Let $\mathbb{Q}^1 \supset X$ be the union of $Q_i^1 \setminus B_i^1$. Then

$$\text{area } \mathbb{Q}_1 \leq \sigma_0 \text{ area } \mathbb{Q},$$

where $\sigma_0 \in (0, 1)$ is roughly equal to $1 - \epsilon^2$.

Then we can subdivide each Q_i^1 into squares of size B_i^1 and repeat the construction with all non-empty squares of this subdivision (using a deeper scale l_j with a sufficiently big but bounded j). It will produce a set $\mathbb{Q}_2 \supset X$ such that

$$\text{area } \mathbb{Q}_2 \leq \sigma_0 \text{ area } \mathbb{Q}_1.$$

We can repeat this procedure roughly n/j times, which implies the desired. \square

6.7.2. *Landing branches.* Let us consider a safe trapping disk $\mathbf{D} = \mathbf{D}^1$ for \mathbf{f}_* , centered at the periodic point α_1 . By definition, it has a bounded hyperbolic³⁸ diameter of order 1 in $\mathbb{C} \setminus \mathbf{O}_*$:

$$(6.9) \quad d^{-1} \leq \text{diam}_{\text{hyp}} \mathbf{D} \leq d \quad \text{with } d = d(N).$$

For instance, \mathbf{D} can be the base trapping disk of depth $\mathbf{l} = m - \kappa + l - \iota$ from Corollary 6.20, but we will also consider much more shallow disks.

For any point z , let

$$0 \leq r_1(z) < \dots < r_n(z) < \dots$$

be all *landing times* of orb z at \mathbf{D} , i.e. the moments for which $\mathbf{f}_*^{r_n}(z) \in \mathbf{D}$ listed consecutively (this list can be infinite, finite, or empty). Let $T^n : \text{Dom } T^n \rightarrow \mathbf{D}$ be the corresponding *landing maps*, i.e., for a point $z \in \text{Dom } T^n$, the landing moment $r_n(z)$ is well defined and $T^n(z) = \mathbf{f}_*^{r_n}(z)$. Let $P^n(z) \ni z$ be the pullback of \mathbf{D} along the orbit $\{\mathbf{f}_*^i(z)\}_{i=0}^{r_n}$. Since $\mathbf{D} \Subset \mathbb{C} \setminus \mathbf{O}_*$, the maps

$$(6.10) \quad \mathbf{f}_*^{r_n} : P^n(z) \rightarrow \mathbf{D}$$

are univalent. We will refer to these maps as the *landing branches*.

For a domain $P = P^n(z)$, we will also use notation r_P for the landing time $r_n(z)$ (which is independent of $z \in P$), and will use notation $T_P = \mathbf{f}_*^{r_P}$ for the corresponding landing branch $P \rightarrow \mathbf{D}$.

Let $\mathcal{P}(\mathbf{D})$ be the family of all domains $P = P^n(z)$.

Lemma 6.24. • *The landing branches $T_P : P \rightarrow \mathbf{D}$, $P \in \mathcal{P}(\mathbf{D})$, have uniformly bounded distortion; the domains $P \in \mathcal{P}(\mathbf{D})$ have a bounded shape and are well inside $\mathbb{C} \setminus \mathbf{O}_*$ (with bounds and constants depending only on \bar{N});*

• *Each domain $P \in \mathcal{P}(\mathbf{D})$ contains a pullback of \mathbf{V}_* of comparable size (with the constant depending only on the parameters $\bar{N}, \underline{L}, \underline{\kappa}, \underline{\iota}$).*

Proof. The first assertion follows from the property that \mathbf{D} is well inside $\mathbb{C} \setminus \mathbf{O}_*$ and the Koebe Distortion Theorem. Together with Corollary 6.20, it implies the second assertion. \square

Along with \mathbf{D} , let us consider another trapping disk \mathbf{D}' (which is allowed to coincide with \mathbf{D}). Let $\mathcal{P}_{\mathbf{D}'}(\mathbf{D})$ be the family of all the domains $P = P^n(z) \in \mathcal{P}(\mathbf{D})$ intersecting \mathbf{D}' .

Lemma 6.25. *For any domain $P \in \mathcal{P}_{\mathbf{D}'}(\mathbf{D})$,*

$$\text{diam } P \leq C_0 \text{diam } \mathbf{D}' \quad \text{with } C_0 = C_0(\bar{N}),$$

where $\text{diam} \equiv \text{diam}_{\text{Euc}}$ stands for the Euclidean diameter;

Proof. By Lemma 4.1, the inverse branch $T_P^{-1} : \mathbf{D} \rightarrow P$ is a hyperbolic contraction. Hence $\text{diam}_{\text{hyp}} P \leq \text{diam}_{\text{hyp}} \mathbf{D} \leq d$. Since $P \cap \mathbf{D}' \neq \emptyset$ and $\text{diam}_{\text{hyp}} \mathbf{D} \leq d$ as well, we have:

$$(6.11) \quad \text{diam}_{\text{hyp}}(\mathbf{D} \cup P) \leq 2d.$$

It follows that the conformal factor $\rho(z)$ between the hyperbolic and Euclidean metrics has a bounded oscillation on $\mathbf{D}' \cup P$:

$$\sup_{z \in \mathbf{D}' \cup P} \rho(z) \leq C \inf_{z \in \mathbf{D}' \cup P} \rho(z), \quad C = C(N).$$

³⁸Below, “hyperbolic” will always refer to the hyperbolic metric in $\mathbb{C} \setminus \mathbf{O}_*$.

Hence

$$(6.12) \quad \frac{\text{diam}_{\text{Euc}} P}{\text{diam}_{\text{Euc}} \mathbf{D}'} \leq C \frac{\text{diam}_{\text{hyp}} P}{\text{diam}_{\text{hyp}} \mathbf{D}'} \leq Cd^2.$$

□

The following lemma shows that pullbacks of trapping disks to some point z lie in different scales:

Lemma 6.26. *For any $\sigma \in (0, 1)$, there exists $\nu = \nu(N, \sigma) \in \mathbb{N}$ with the following property. Let \mathbf{D}_i , $i = 1, \dots, \nu$, be safe trapping disks, not necessarily distinct. Consider a point z landing at the \mathbf{D}_i at moments r_i , where $0 \leq r_1 < \dots < r_\nu$, and let $P^\nu \ni z$ be the corresponding pullback of the \mathbf{D}_i . Then*

$$\text{diam } P^\nu < \sigma \text{diam } P^1.$$

Proof. Let $P \equiv P^\nu$, and let $P_i := \mathbf{f}_*^{r_i}(P)$, $i = 1, \dots, \nu$. Then $P_i \cap \mathbf{D}_i \neq \emptyset$. By property (6.11),

$$(6.13) \quad \text{diam}_{\text{hyp}} \mathbf{D}_i \cup P_i \leq 2d,$$

which implies (4.3) for all $z \in P_i$. It allows us to apply Lemma 4.2 and to conclude that all the maps $\mathbf{f}_*^{r_{i+1}-r_i} : P_i \rightarrow P_{i+1}$ are hyperbolic expansions by some factor $\lambda = \lambda(N) > 1$. Hence the map $\mathbf{f}_*^{r_\nu-r_1} : P_1 \rightarrow P_\nu$ (which is the same as $\mathbf{f}_*^{r_1}(P) \rightarrow \mathbf{D}_\nu$) is a hyperbolic expansion by $\lambda^{\nu-1}$. Hence

$$\text{diam}_{\text{hyp}}(\mathbf{f}_*(P)) \leq \lambda^{-\nu+1} \text{diam}_{\text{hyp}} \mathbf{D}_\nu \leq d \lambda^{-\nu+1}.$$

On the other hand, $\text{diam}_{\text{hyp}}(\mathbf{f}_*(P^1)) \equiv \text{diam}_{\text{hyp}} \mathbf{D}_1 \geq d^{-1}$, so

$$\text{diam}_{\text{hyp}}(\mathbf{f}_*(P)) \leq d^2 \lambda^{-\nu+1} \text{diam}_{\text{hyp}}(\mathbf{f}_*(P^1)).$$

Property (6.13) with $i = 1$ allows us to switch in the last estimate from the hyperbolic diameters to the Euclidean ones (like in (6.12)) and then to apply the Koebe Distortion Theorem to the map $\mathbf{f}_*^{r_1}$ on $P \cup P^1$. The conclusion follows. □

6.7.3. Truncated Poincaré series. Let us now fix a safe trapping disk \mathbf{D} (in applications, it will be the base trapping disk), and let $\mathcal{P} := \mathcal{P}_{\mathbf{D}}(\mathbf{D})$. Of course, a domain $P \in \mathcal{P}$ can admit several representations as $P^n(z)$. Let

$$\chi(P) = \max\{n : \exists z \in P \text{ such that } P = P^n(z)\}.$$

Let \mathcal{P}^n be the family of domains $P \in \mathcal{P}$ with $\chi(P) \leq n$. We also let

$$\mathbb{P} = \bigcup_{\mathcal{P}} P, \quad \mathbb{P}^n = \bigcup_{\mathcal{P}^n} P$$

Lemma 6.27. *There exists $C = C(N)$ such that*

$$\sum_{\mathcal{P}^n} \text{area } P \leq Cn \text{area } \mathbf{D}.$$

Proof. Note that the family \mathcal{P}^n has the intersection multiplicity at most n . Indeed, if some point z is contained in k sets P_i of this family then $P_i = P^{n_i}(z)$ with $n_i = n_i(z) \leq n$. But since the n_i are pairwise distinct, $\max n_i \geq k$.

Hence

$$(6.14) \quad \sum_{\mathcal{P}^n} \text{area } P \leq n \text{area } \mathbb{P}^n \leq n \text{area } \mathbb{P}.$$

By Lemma 6.25 (i), \mathbb{P} is contained in a Euclidean neighborhood of \mathbf{D} of size $\leq C_0 \text{diam } \mathbf{D}$. Since \mathbf{D} has a bounded shape, $\text{area } \mathbb{P} \leq C \text{area } \mathbf{D}$, with $C = C(N)$. Together with (6.14), this implies the desired. \square

Let us consider the following *truncated Poincaré series*: for $\zeta \in \mathbf{D}$, let

$$\phi_n(\zeta) = \sum_{P \in \mathcal{P}^n} \frac{1}{|DT_P(\zeta_P)|^2}, \quad \text{where } \zeta_P \in P \text{ and } T_P(\zeta_P) = \zeta.$$

Lemma 6.28. *We have $\phi_n(\zeta) \leq Cn$, where $C = C(\bar{N})$.*

Proof. We have:

$$\int_{\mathbf{D}} \phi_n(\zeta) d \text{area}(\zeta) = \sum_{\mathcal{P}^n} \text{area } P \leq Cn \text{area } \mathbf{D},$$

where the last estimate is the content of Lemma 6.27. But since the branches $T_P : P \rightarrow \mathbf{D}$ have a bounded distortion, $\phi_n(\zeta) \asymp \phi_n(\zeta')$ for any $\zeta, \zeta' \in \mathbf{D}$ (with constants depending only on N). The conclusion follows. \square

6.7.4. *Probability of few returns to the base.* Let us start with an observation that for m big enough, our quadratic polynomial \mathbf{f}_* has plenty of safe trapping disks:

Lemma 6.29. *For any natural $\tau \in \mathbb{N}$, there exists $\underline{m} = \underline{m}(N, l, \kappa, t, \tau)$ such that for any $m > \underline{m}$, the polynomial \mathbf{f}_* has at least τ safe trapping disks \mathbf{D}_i satisfying properties of Lemma 6.1. Moreover, these trapping disks are pairwise disjoint and disjoint from the base safe trapping disk $\mathbf{D} = \mathbf{D}^{m-\kappa+l-l}$.*

Proof. By Lemma 6.4, our polynomial \mathbf{f}_* is ϵ_m -close to the Siegel polynomial \mathbf{f} , where $\epsilon_m \rightarrow 0$ as $m \rightarrow \infty$ (keeping the other parameters, N, l, κ and t , frozen). Hence for m big enough, Lemma 6.1 (applied directly to \mathbf{f}_*) supplies us with arbitrary many safe trapping disks \mathbf{D}_i . \square

From now on, \mathbf{D} will stand for the base trapping disk. Recall that \mathbf{J}_* is the Julia set of \mathbf{f}_* . Let Z be the set of points $z \in \mathbf{D} \setminus \mathbf{J}_*$ that under the iterates of \mathbf{f}_* never return back to \mathbf{D} . The following lemma shows that for m sufficiently big, it is difficult to escape from \mathbf{D} :

Lemma 6.30. *For any natural $\tau \in \mathbb{N}$, there exists $\underline{m} = \underline{m}(N, l, \kappa, t, \tau)$ such that for any $m > \underline{m}$,*

$$\text{area } Z \leq C\sigma^\tau \text{area } \mathbf{D},$$

with $\sigma \in (0, 1)$ and $C > 0$ depending only on N .

Proof. Let $z \in Z$. If m is sufficiently big then on its way from \mathbf{D} to ∞ , the orbit of z must visit τ safe trapping disks \mathbf{D}_i from Lemma 6.29 at some moments $r_1 < r_2 < \dots < r_\tau$. By Lemma 6.1, definite parts W_i of these trapping disks are contained in $\mathbf{f}_*^{-1}(\mathbf{S})$. Since $\text{orb } z$ never returns back to \mathbf{D} , it cannot visit the Siegel disk $\mathbf{S} = S_{\mathbf{f}}$, and hence it cannot land in the domains W_i either.

Since each disk \mathbf{D}_j is safe, it can be univalently and with bounded distortion pulled back to z . The corresponding pullback of W_i creates a gap of definite size in Z near z . By Lemma 6.26, these gaps lie in $\asymp \tau$ different scales. Lemma 6.23 completes the proof. \square

Let

$$Z_n = \bigcup_{P \in \mathcal{P}^n} T_P^{-1}(Z),$$

where Z is from Lemma 6.30. Notice that points of Z_n escape \mathbf{D} forever after at most n returns.

Lemma 6.31. *For any natural $\tau \in \mathbb{N}$, there exists $\underline{m} = \underline{m}(N, l, \kappa, t, \tau)$ such that for any $m > \underline{m}$,*

$$\text{area } Z_n \leq C n \sigma^\tau \text{ area } \mathbf{D},$$

where $\sigma \in (0, 1)$ and $C > 0$ depend only on N .

Proof. Since

$$\text{area } Z_n = \int_Z \phi_n(\zeta) d \text{area}(\zeta),$$

the conclusion follows from Lemma 6.28 and Corollary 6.30. \square

6.7.5. *Many returns to the base.* Let

$$\mathbb{S}^n = \bigcup_{\chi(P) > n} P = \bigcup_{\mathcal{P} \setminus \mathcal{P}^n} P.$$

Lemma 6.32. *There exist $C > 0$ and $\sigma \in (0, 1)$ depending on $N, \underline{l}, \underline{\kappa}$, and \underline{t} such that for any $n \in \mathbb{N}$ the area of the set of points of \mathbb{S}^n that never land in \mathbf{V} is at most $C \sigma^n \text{ area } \mathbf{D}$.*

Proof. Take a point $\zeta \in \mathbb{S}^n$. It belongs to some domain $P \in \mathcal{P}$ with $\chi(P) > n$. Then P contains a point z that lands in \mathbf{D} at least n times, and $P^n(z) = P$. By Lemma 6.26, the nest

$$P^1(z) \supset P^2(z) \supset \cdots \supset P^n(z) = P$$

represents $\asymp n$ different scales. By Lemma 6.24, each of these domains contains a pullback of \mathbf{V} of comparable size. Now the desired follows from Lemma 6.23. \square

6.7.6. *Escaping probability.* We are finally ready to show that the escaping probability ξ for \mathbf{f}_* can be made arbitrary small by selecting m sufficiently big (while keeping the previously selected parameters, N, l, κ , and t , unchanged).

Proposition 6.33. *For any $\epsilon > 0$ there exists \underline{m} such that $\xi < \epsilon$ for any $m > \underline{m}$.*

Proof. Let Y be the set of points in \mathbf{D} that never land in \mathbf{V}_* . We will show first that for m sufficiently big,

$$(6.15) \quad \text{area } Y < \epsilon \text{ area } \mathbf{D}.$$

For any $n \in \mathbb{N}$, let us cover Y by three sets:

$$Y_0 = Y \cap J(\mathbf{f}_*), \quad Y_1^n = Y \cap \mathbb{S}^n, \quad Y_2^n = Y \setminus (Y_0 \cup Y_1^n).$$

It is known that almost all point of $J(\mathbf{f}_*)$ land in \mathbf{V} [L1], so $\text{area } Y_0 = 0$,
By Lemma 6.32,

$$\text{area } Y_1^n \leq C \sigma^n \text{ area } \mathbf{D} < (\epsilon/2) \text{ area } \mathbf{D}.$$

as long as n is sufficiently big.

Let us take now any point $z \in Y_2^n$. Then

$$\chi(z) := \max\{\chi(P) : P \in \mathcal{P}, P \ni z\} \leq n,$$

and orb z returns back to \mathbf{D} at most n times. Let $k \leq n$ be the number of returns, and let $P := P^k(z)$. Since $P \ni z$, we have $P \in \mathcal{P}^{\chi(z)} \subset \mathcal{P}^n$. Moreover, under the return map $T_P : P \rightarrow \mathbf{D}$, the point z must land in Z since it will never come back to \mathbf{D} again. Hence $z \in Z_n$. Thus $Y_2^n \subset Z_n$. Applying Lemma 6.31, we see that $\text{area } Y_2^n < (\epsilon/2) \text{area } \mathbf{D}$ for m sufficiently big, and estimate (6.15) follows.

To pass from (6.15) to an estimate of ξ , we need to transfer the density estimate for Y to the fundamental annulus $\mathbf{V}_* \setminus \mathbf{U}_*$. Let \mathcal{Y} be the set of points in $\mathbf{V}_* \setminus \mathbf{U}_*$ that never land in \mathbf{V}_* . Again, since almost all points of $J(\mathbf{f}_*)$ land in \mathbf{V}_* , it is sufficient to deal with the Fatou set $\mathcal{Y} \setminus \mathbf{J}_*$. Any point $z \in \mathcal{Y} \setminus \mathbf{J}_*$ eventually lands in the “middle” of the base trapping disk \mathbf{D} . Pulling \mathbf{D} back to z , we obtain a domain $Q(z)$ of bounded shape in which the set $\mathcal{Y} \cap Q(z)$ (the pullback of Y) has density $\leq C\epsilon$. Applying the Besikovich Covering Lemma (see [Ma]), we conclude that \mathcal{Y} has density $\leq C'\epsilon$ in $\mathbf{V}_* \setminus \mathbf{U}_*$. \square

6.8. Positive area.

Theorem 6.34. *For any stationary rotation number $\theta = \theta_N$ of high type (i.e., $N > \underline{N}$), there exist $\underline{l}, \underline{\kappa}$, \underline{t} and \underline{m} , \underline{j} with the following property. If κ, l, t, m, j are larger than the corresponding underlined parameters, then the Feigenbaum polynomial \mathbf{f}_* with stationary combinatorics $\mathcal{M}'_{N,l,\kappa,t,m,j}$ has the Julia set of positive area.*

Proof. By Proposition 6.22, the map \mathbf{f}_* has *a priori* bounds depending only on N, l, κ , and t .

By Proposition 6.22, it has a definite *landing parameter* η depending on the same four parameters only.

By Proposition 6.33, it has an arbitrary small *escaping parameter* ξ as long as m, j are sufficiently big (with frozen N, l, κ , and t).

Now the Black Hole Criterion (Theorem 2.3) implies the desired. \square

7. APPENDIX: FURTHER COMMENTS AND OPEN PROBLEMS

7.1. Probabilistically balanced maps. There is an interesting approach to creating balanced (in some stronger sense) maps by variation of a continuous parameter (we thank Jean-Christophe Yoccoz for this suggestion). Consider a renormalization horseshoe associated to a pair of renormalization combinatorics, such that one of the fixed points is lean and the other is a black hole. For each $0 \leq p \leq 1$, let μ_p be the Bernoulli measure on the horseshoe giving probability p to the “Lean” combinatorics and $1 - p$ to the “Black hole” one. Then for each p , the limit

$$c_p = \lim \frac{1}{n} \log \frac{\eta_n}{\xi_n}$$

should exist μ_p -a.e. and be independent of a particular μ_p -typical combinatorics. Moreover, the dependence $p \mapsto c_p$ is conceivably continuous, and since $c_0 < 0 < c_1$, we must have $c_{p_*} = 0$ for some $0 < p_* < 1$ (justification of all those facts would depend on a suitable extension of the analysis of [AL1]). Let us call a μ_{p_*} -typical Feigenbaum map *probabilistically balanced*. (They are “better balanced” than generic *topologically balanced* examples constructed in [AL1].) The geometry of the probabilistically balanced Julia sets would be a good approximation to

the geometry of (perhaps, non-existing) balanced Julia sets with periodic combinatorics.³⁹

7.2. Computer experiments. After identifying theoretically the main dynamical phenomena which should lead to Black hole behavior, we have attempted an informal numeric investigation of a particularly simple sequence of renormalization combinatorics displaying them. Consider the quadratic map p_c with a golden mean Siegel disk, with rotation number $[1, 1, 1, \dots]$, and let p_m/q_m be the sequence of rational approximands ($p_m = q_{m-1}$ being the Fibonacci sequence). Visual inspection of the (p_m/q_m) -limb reveals a pair of largest primitive Mandelbrot copies with period $q_m + q_{m-2}$. Choosing one of them, we explore in detail the parameter z_m in this copy for which the first renormalization has a golden mean Siegel disk. This parameter is very close to the actual Feigenbaum parameter with this stationary combinatorics, and considerably easier to determine numerically.

In parameter space, one sees that $\frac{z_{2m-1}-c}{z_{2m+1}-c} \rightarrow \beta = \frac{7+3\sqrt{5}}{2}$. Moreover, centering the Mandelbrot copies at the superattracting parameter and rescaling by β^m shows manifest convergence of the copies in the Hausdorff topology.

In the dynamical plane, one sees that $p_{z_{2m+1}}^{q_{2m+1}+q_{2m-1}}$ restricts to a quadratic-like map $g_{2m+1} : U_{2m+1} \rightarrow V_{2m+1}$, where V_{2m+1} is a disk of radius $\sqrt{38}|w_{2m+1}|$ and w_{2m+1} is the center of the Siegel disk for g_{2m+1} . Moreover, $\frac{w_{2m-1}}{w_{2m+1}}$ converges to some real constant greater than 1, and up to rescaling by $|w_{2m+1}|^{-1}$, g_{2m+1} is seen to converge. The proportion of $p_{z_{2m+1}}$ -orbits starting in the original Siegel disk of p_c that eventually land in V_{2m+1} is clearly seen to approach 1 (so that $\eta(2m+1)$ is bounded from below), while $\xi(2m+1)$ appears to decay exponentially. Julia sets of positive area might already emerge then for period 2207 ($\xi \approx 0.0622$), and more likely for period 15127 ($\xi \approx 0.0215$).⁴⁰

All those observations would be justified by the existence of an hyperbolic Siegel renormalization fixed point associated to the golden mean, with one-dimensional unstable manifold containing (up to straightening) the Mandelbrot copies we explore. While the existence of a hyperbolic Siegel renormalization fixed point was established by McMullen and Yampolsky [McM4, Ya3] (and plays a central role in our argument), the current techniques do not go so far as to prove its hyperbolicity in the particular case of the golden mean. But even this would not be enough: then one needs to show that the unstable manifold is large enough to contain those particular Mandelbrot copies that we want, which looks like a hard problem.

7.3. More Julia sets of positive area? It remains an open problem whether Julia sets of positive area may exist for real quadratic maps. Any such example would have to be infinitely renormalizable, and would imply their existence already in the class of real Feigenbaum quadratic maps with periodic combinatorics. A natural candidate would be the “original” Feigenbaum map corresponding to the period doubling bifurcation, since fixed points with high (essential) period are known to

³⁹Note however that μ_p -a.e. Feigenbaum Julia set has full hyperbolic dimension for every $0 < p < 1$ (see Lemma 7.2 and Theorem 8.1 of [AL1]), and while $c_p > 0$ should imply positive area, $c_p < 0$ would not imply Hausdorff dimension less than 2.

⁴⁰Those estimates are valid for the quadratic map and not for the renormalization fixed point, so there is still some extra distortion to consider. Heuristically (ignoring distortion), ξ should be small compared to the relative area of the filled Julia set with a Siegel disk, which near the fixed point is around 0.06.

be Lean. However, in the doubling case numerical experiments still suggest that the η_m decay, so this case appears to be Lean as well. It is thus plausible that all real quadratic Feigenbaum Julia sets are Lean. However, resolving this problem one way or another may depend on computer assistance.

In the higher degree case, the situation is even less conclusive. In this case, there is even a chance of existence of a non-renormalizable unicritical polynomial with positive area Julia set (and even real): see an attempt to prove it by Nowicki and van Strien for the Fibonacci map of high degree, see [B]).

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