Year: 2004

On some mixed boundary value problems with nonlocal diffusion

Chang, N ; Chipot, M

Originally published at:
Chang, N; Chipot, M (2004). On some mixed boundary value problems with nonlocal diffusion. Advances in Mathematical Sciences and Applications, 14(1):1-24.

ON SOME MIXED BOUNDARY VALUE PROBLEMS WITH NONLOCAL DIFFUSION

N.-H. CHANG & M. CHIPOT

Abstract. We study here a class of nonlinear nonlocal problems. First we consider the issue of existence and uniqueness for the parabolic setting. Then we study the asymptotic behaviour of the solution for large time. This leads us to introduce and investigate in details the associated stationary problem.

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^n$, $n \geq 1$ with a Lipschitz boundary Γ. We suppose that Γ is split into two measurable subsets Γ_D and $\Gamma_N = \Gamma \setminus \Gamma_D$. We denote by $a = a(\zeta)$ a function such that

$$\begin{cases} a \text{ is continous,} \\ \exists\, m, M \text{ such that } 0 < m \leq a(\zeta) \leq M \quad \forall\, \zeta \in \mathbb{R}. \end{cases} \tag{1.1}$$

We consider then the problem of finding $u = u(x,t)$ solution to

$$\begin{cases} u_t - a(\ell(u(t)))\Delta u + u = f \quad \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 \text{ on } \Gamma_D \times \mathbb{R}^+, \quad \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_N \times \mathbb{R}^+, \\ u(\cdot, 0) = u_0 \text{ in } \Omega. \end{cases} \tag{1.2}$$

In the above system Δ is the usual Laplace operator, ν is the unit outward normal to Γ, $\frac{\partial u}{\partial \nu}$ the outward normal derivative, ℓ is a linear form on $L^2(\Omega)$ so that

$$\ell(u(t)) = \int_\Omega g(x)u(x,t)\,dx, \quad g \in L^2(\Omega), \tag{1.3}$$

u_0 and f are some functions such that

$$f \in L^2(\Omega), \qquad u_0 \in L^2(\Omega). \tag{1.4}$$

This kind of model problem arises for instance in diffusion of bacteria: $u(x,t)$ is the density of population located at x at the time t, f is the density of bacteria supplied from outside, u_0 is the initial density of population, a is the diffusion rate (depending on $\ell(u(t))$), the lower order term u is the density of population eliminated by death at a constant rate taken for the sake of simplicity equal to 1. The nonlocal dependence of a on $\ell(u(t))$ includes for instance the case where the diffusion rate of the population depends on the entire population – take $g = 1$ in (1.3). We could have also imposed a nonlocal dependence on the death rate (see [2], [3]), however for simplicity we only consider here the case (1.2). Among the interesting issues of (1.2) is the study of the asymptotic behaviour of the density $u(x, t)$ as $t \to +\infty$. This is one of the focus of our paper. Note that the problem with no lower order term has been investigated in [6], [7], [8] – see also [15], but the technique there could not apply here. For further aspects of nonlocal problems see [9].

2. Existence and uniqueness of solution to (1.2)

First we consider the question of existence. We consider (1.2) in the slightly more general setting where

$$f \in L^2(0, T; L^2(\Omega)), \qquad u_0 \in L^2(\Omega), \tag{2.1}$$

(T is some arbitrary positive number). We define

$$V = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D \}. \tag{2.2}$$

Then we have:

Theorem 2.1. *Suppose* (1.1), (1.3), (2.1)*. There exists a function $u = u(x, t)$ solution to:*

$$\begin{cases} u \in L^2(0, T; V) \cap C([0, T], L^2(\Omega)), \quad u_t \in L^2(0, T; V'), \\ u(\cdot, 0) = u_0, \\ \dfrac{d}{dt}(u, v) + a(\ell(u(t))) \displaystyle\int_\Omega \nabla u \nabla v \, dx + (u, v) \\ \qquad = (f, v) \text{ in } \mathcal{D}'(0, T), \quad \forall v \in V. \end{cases} \tag{2.3}$$

Here $(\cdot, \cdot)$ is the usual scalar product on $L^2(\Omega)$, $\mathcal{D}'(0, T)$ is the space of distributions on $(0, T)$, V' is the dual of V.

Proof. The proof is based on the Schauder fixed point theorem. We refer the reader to [10], [1], [8], [4] for precise definitions of the functional spaces used here. For $w \in L^2(0, T; L^2(\Omega))$ the mapping

$$t \mapsto a(\ell(w(\cdot, t)))$$

is clearly measurable and thus belongs to $L^\infty(0, T)$. From a well known result of J.L. Lions (cf. [10], [4]) there exists a unique u solution to

$$\begin{cases} u \in L^2(0, T; V) \cap C([0, T], L^2(\Omega)), \quad u_t \in L^2(0, T; V'), \\ u(\cdot, 0) = u_0, \\ \dfrac{d}{dt}(u, v) + a(\ell(w(t))) \displaystyle\int_\Omega \nabla u \nabla v \, dx + (u, v) \\ \qquad = (f(t), v) \text{ in } \mathcal{D}'(0, T), \quad \forall v \in V. \end{cases} \tag{2.4}$$

($f(t)$ denotes the mapping $f(\cdot, t)$). Then the idea is to show that the map

$$w \mapsto u = R(w) \tag{2.5}$$

has a fixed point. For that taking $v = u$ in (2.4) we get

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + a(\ell(w(t))) \int_\Omega |\nabla u|^2 \, dx + |u|_2^2 = (f(t), u) \le |f(t)|_2 |u|_2 \tag{2.6}$$

($|\cdot|_2$ denotes the usual $L^2(\Omega)$-norm, $|\cdot|$ the usual euclidean norm).

From (1.1) and Young's inequality we derive

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + m\big||\nabla u|\big|_2^2 + |u|_2^2 \le \frac{1}{2}|f(t)|_2^2 + \frac{1}{2}|u|_2^2. \tag{2.7}$$

Hence

$$\frac{d}{dt}|u|_2^2 + m_0\big\{\big||\nabla u|\big|_2^2 + |u|_2^2\big\} \le |f(t)|_2^2 \tag{2.8}$$

where $m_0 = \min(2m, 1)$. Integrating on $(0,t)$ we obtain for $t \in [0,T]$

$$|u(t)|_2^2 + m_0|u|^2_{L^2(0,t;V)} \le |f|^2_{L^2(0,t;L^2(\Omega))} + |u_0|_2^2. \tag{2.9}$$

(We denote by $|\cdot|_{L^2(0,T;X)}$ the usual norm in $L^2(0,T;X)$, cf. [10], [1], the norm in V is the usual $H^1(\Omega)$-norm). Thus we have

$$|u|_{L^2(0,T;V)} \le C \tag{2.10}$$

where $C = \{(|f|^2_{L^2(0,T;L^2(\Omega))} + |u_0|_2^2)/m_0\}^{1/2}$ is a constant independent of w. Writing the last equation of (2.4) as

$$u_t - a(\ell(w(t)))\Delta u + u = f \quad \text{in } V', \tag{2.11}$$

since $f \in L^2(0,T;L^2(\Omega))$, $u \in L^2(0,T;V)$ then $u_t \in L^2(0,T;V')$ and it holds that

$$|u_t|_{L^2(0,T;V')} \le \widetilde{C} \tag{2.12}$$

for some constant $\widetilde{C}$ independent of w. We may then consider

$$B = \{\, v \in L^2(0,T;L^2(\Omega)) \mid |v|_{L^2(0,T;L^2(\Omega))} \le C \,\} \tag{2.13}$$

where C is the constant in (2.10). It is then clear that the mapping $w \mapsto u = R(w)$ maps B into itself. In order to apply the Schauder fixed point theorem we need to verify the following:

(i) $R(B)$ is relatively compact in B,

(ii) R is continuous in $L^2(0,T;L^2(\Omega))$.

(i) is obvious from (2.10), (2.12) (cf. [13]). The point (ii) follows the lines of [8] and is left to the reader. This completes the proof of the theorem. □

Regarding uniqueness, under the assumption of Theorem 2.1 we have:

Theorem 2.2. *Suppose that a is locally Lipschitz continuous – i.e.*

$$\forall\, z > 0\ \exists\, A_z > 0 \text{ such that } \quad |a(\zeta) - a(\zeta')| \le A_z|\zeta - \zeta'| \quad \forall\, \zeta, \zeta' \in [-z, z]. \tag{2.14}$$

Then, the problem (2.3) *admits a unique solution.*

Proof. Let u_1, u_2 be two solutions of (2.3) with the same initial data u_0. We have

$$\begin{aligned} &\frac{d}{dt}(u_1, v) + a(\ell(u_1(t)))\int_\Omega \nabla u_1 \nabla v\, dx + (u_1, v) \\ &\quad = \frac{d}{dt}(u_2, v) + a(\ell(u_2(t)))\int_\Omega \nabla u_2 \nabla v\, dx + (u_2, v) \quad \text{in } \mathcal{D}'(0,T),\ \forall\, v \in V. \end{aligned} \tag{2.15}$$

This leads to

$$
\begin{aligned}
&\frac{d}{dt}(u_1 - u_2, v) + a(\ell(u_1(t))) \int_\Omega \nabla(u_1 - u_2)\nabla v \, dx + (u_1 - u_2, v) \\
&\quad = \{a(\ell(u_2(t))) - a(\ell(u_1(t)))\} \int_\Omega \nabla u_2 \nabla v \, dx \quad \forall\, v \in V.
\end{aligned} \tag{2.16}
$$

Since $u_1, u_2 \in C([0,T], L^2(\Omega))$ we have for some $z > 0$

$$
\ell(u_i(t)) \in [-z, z], \quad i = 1, 2.
$$

Taking $v = u_1 - u_2$ in (2.16) we obtain easily

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_1 - u_2|_2^2 + a(\ell(u_1(t))) \int_\Omega |\nabla(u_1 - u_2)|^2 \, dx + |u_1 - u_2|_2^2 \\
&\quad \le A_z |\ell(u_2(t)) - \ell(u_1(t))| \, \big|\nabla u_2\big|_2 \big|\nabla(u_1 - u_2\big|_2 \\
&\quad \le A_z |g|_2 \big|\nabla u_2\big|_2 \big|\nabla(u_1 - u_2)\big|_2 |u_1 - u_2|_2
\end{aligned} \tag{2.17}
$$

(we used Cauchy–Schwarz inequality). Using (1.1) and the Young inequality

$$
ab \le \frac{m}{2}a^2 + \frac{1}{2m}b^2 \quad \forall\, a, b \ge 0
$$

it comes

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_1 - u_2|_2^2 + m\big|\nabla(u_1 - u_2)\big|_2^2 + |u_1 - u_2|_2^2 \\
&\quad \le \frac{m}{2}\big|\nabla(u_1 - u_2)\big|_2^2 + \frac{1}{2m}A_z^2 |g|_2^2 \big|\nabla u_2\big|_2^2 |u_1 - u_2|_2^2
\end{aligned} \tag{2.18}
$$

which leads to

$$
\frac{d}{dt}|u_1 - u_2|_2^2 \le C(t)|u_1 - u_2|_2^2 \tag{2.19}
$$

with

$$
C(t) = \frac{1}{m}A_z^2 |g|_2^2 \big|\nabla u_2\big|_2^2 \in L^1(0,T). \tag{2.20}
$$

Rewriting (2.19) as

$$
\frac{d}{dt}\big\{e^{-\int_0^t C(s)\, ds}|u_1 - u_2|_2^2\big\} \le 0
$$

this shows that $t \mapsto e^{-\int_0^t C(s)\, ds}|u_1 - u_2|_2^2$ is nonincreasing. Since for $t = 0$

$$
u_1(\cdot, 0) = u_2(\cdot, 0) = u_0
$$

this function vanishes at 0 and thus identically. This completes the proof of uniqueness. □

Remark 2.1. For $f = f(x) \in L^2(\Omega)$ we have

$$
f \in L^2(0, T; L^2(\Omega)) \tag{2.21}
$$

for any $T > 0$. Thus, in this case, under the assumptions of Theorem 2.2, for any $T > 0$ there exists a unique solution to (2.3). This is from now on what we will call a weak solution to (1.2) and this is the asymptotic behaviour of this solution that we will investigate.

3. Stationary solutions

To study the asymptotic behaviour of the solution obtained in the preceding section we start by investigating the corresponding stationary problem

$$\begin{cases} -a(\ell(u))\Delta u + u = f \quad \text{in } \Omega, \\ u = 0 \text{ on } \Gamma_D, \qquad \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_N, \end{cases} \tag{3.1}$$

with

$$\ell(u) = \int_\Omega g(x)u(x)\,dx \tag{3.2}$$

and $f, g \in L^2(\Omega)$. Recalling the definition

$$V = \{\, v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D \,\} \tag{3.3}$$

the problem (3.1) can be written in a weak form as

$$\begin{cases} u \in V, \\ a(\ell(u)) \int_\Omega \nabla u \nabla v\,dx + \int_\Omega uv\,dx = \int_\Omega fv\,dx \quad \forall\, v \in V. \end{cases} \tag{3.4}$$

The idea to solve (3.1), (3.4) is to rely on an algebraic equation in $\mathbb{R}$. For any $a > 0$ we consider $\varphi = \varphi_a$ the solution to

$$\begin{cases} -a\Delta\varphi_a + \varphi_a = f \quad \text{in } \Omega, \\ \varphi_a = 0 \text{ on } \Gamma_D, \quad \dfrac{\partial \varphi_a}{\partial \nu} = 0 \text{ on } \Gamma_N, \end{cases} \tag{3.5}$$

or in a weak form

$$\begin{cases} \varphi_a \in V, \\ a \int_\Omega \nabla\varphi_a \nabla v\,dx + \int_\Omega \varphi_a v\,dx = \int_\Omega fv\,dx \quad \forall\, v \in V. \end{cases} \tag{3.6}$$

(The existence of φ_a is a straightforward consequence of the Lax–Milgram theorem). Then, we have

Theorem 3.1. *The mapping*

$$u \mapsto \ell(u)$$

is a one-to-one mapping from the set of solutions of (3.1), (3.4) *to the set of solutions of the equation in* $\mathbb{R}$

$$\mu = \int_\Omega g\varphi_{a(\mu)}\,dx \tag{3.7}$$

– i.e. solving (3.7) *will provide all the solutions to* (3.4).

Proof. Suppose that u is solution to (3.4). Due to (3.6) we have

$$u = \varphi_{a(\ell(u))}. \tag{3.8}$$

Multiplying this equality by g and integrating over Ω we get

$$\ell(u) = \int_\Omega gu\,dx = \int_\Omega g\varphi_{a(\ell(u))}\,dx$$

which implies that $\ell(u)$ belongs to the set of solution of (3.7). Let now μ solve (3.7) and set

$$u = \varphi_{a(\mu)}. \tag{3.9}$$

Multiplying by g and integrating over Ω we obtain

$$\ell(u) = \int_\Omega gu\,dx = \int_\Omega g\varphi_{a(\mu)}\,dx = \mu.$$

Moreover, by (3.9)

$$u = \varphi_{a(\ell(u))}$$

and u is solution to (3.4). This shows that the map ℓ is onto. If $\ell(u_1) = \ell(u_2)$ where u_1, u_2 are solutions to (3.1), (3.4) one has of course $u_1 = u_2$ and the injectivity of ℓ follows. This completes the proof of the theorem. □

After having established the correspondence between the solutions to (3.1), (3.4) and (3.7) we are able to analyze more deeply the stationary system (3.1). For that – recall (3.7) – we define for $a > 0$

$$K(a) = \int_\Omega g(x)\varphi_a(x)\,dx. \tag{3.10}$$

The properties of K are given by the following lemma:

Lemma 3.1. *We have:*

$$K \text{ is continuous on } (0, +\infty), \tag{3.11}$$

$$\lim_{a\to 0} K(a) = \int_\Omega g(x)f(x)\,dx, \tag{3.12}$$

$$\lim_{a\to +\infty} K(a) = 0 \quad \text{if} \quad |\Gamma_D| \neq 0, \tag{3.13}$$

$$\lim_{a\to +\infty} K(a) = |\Omega| \fint_\Omega g\,dx \fint_\Omega f\,dx \quad \text{if} \quad |\Gamma_D| = 0.$$

($|\Gamma_D|$ denotes the measure area of Γ_D. $\fint_\Omega h\,dx$ is the average of h on Ω given by $\fint_\Omega h\,dx = \frac{1}{|\Omega|}\int_\Omega h(x)\,dx$, $|\Omega|$ is the measure of Ω).

Proof: (i) *Proof of* (3.11). By (3.10) we have for $a_1, a_2 \in (0, +\infty)$

$$|K(a_1) - K(a_2)| = \left|\int_\Omega g(\varphi_{a_1} - \varphi_{a_2})\,dx\right| \le |g|_2 |\varphi_{a_1} - \varphi_{a_2}|_2. \tag{3.14}$$

Let us then get a priori bounds for the solution $\varphi = \varphi_a$ of (3.6). Choosing in (3.6)

$$v = \varphi_a$$

we obtain

$$a\big||\nabla\varphi_a|\big|_2^2 + |\varphi_a|_2^2 = (f, \varphi_a) \le |f|_2|\varphi_a|_2. \tag{3.15}$$

Thus, we get, neglecting the first term above:

$$|\varphi_a|_2^2 \le |f|_2|\varphi_a|_2$$

and it comes

$$|\varphi_a|_2 \le |f|_2. \tag{3.16}$$

Reporting in (3.15) this leads also to

$$\big||\nabla\varphi_a|\big|_2 \le |f|_2/\sqrt{a}. \tag{3.17}$$

Let us come back to the proof of the continuity of K. We have by (3.6)

$$\begin{aligned}
a_1\int_\Omega \nabla\varphi_{a_1}\nabla v\,dx + \int_\Omega \varphi_{a_1} v\,dx &= \int_\Omega f v\,dx \quad \forall\, v\in V,\\
a_2\int_\Omega \nabla\varphi_{a_2}\nabla v\,dx + \int_\Omega \varphi_{a_2} v\,dx &= \int_\Omega f v\,dx \quad \forall\, v\in V,
\end{aligned}$$

and by subtraction

$$\begin{aligned}
a_1\int_\Omega \nabla(\varphi_{a_1}-\varphi_{a_2})\nabla v\,dx &+ \int_\Omega (\varphi_{a_1}-\varphi_{a_2}) v\,dx\\
&= (a_2-a_1)\int_\Omega \nabla\varphi_{a_2}\nabla v\,dx \qquad \forall\, v\in V.
\end{aligned} \tag{3.18}$$

Taking $v=\varphi_{a_1}-\varphi_{a_2}$ we derive

$$a_1\big||\nabla(\varphi_{a_1}-\varphi_{a_2})|\big|_2^2 + |\varphi_{a_1}-\varphi_{a_2}|_2^2 \le |a_2-a_1|\,\big||\nabla\varphi_{a_2}|\big|_2\big||\nabla(\varphi_{a_1}-\varphi_{a_2}|\big|_2.$$

Using (3.17) we obtain

$$a_1\big||\nabla(\varphi_{a_1}-\varphi_{a_2})|\big|_2^2 + |\varphi_{a_1}-\varphi_{a_2}|_2^2 \le |a_2-a_1|\,|f|_2\big||\nabla(\varphi_{a_1}-\varphi_{a_2})|\big|/\sqrt{a_2}. \tag{3.19}$$

The second term above being nonnegative this leads to

$$\big||\nabla(\varphi_{a_1}-\varphi_{a_2})|\big|_2 \le |a_2-a_1|\,|f|_2/a_1\sqrt{a_2} \tag{3.20}$$

and thus going back to (3.19) we obtain

$$|\varphi_{a_1}-\varphi_{a_2}|_2^2 \le (a_2-a_1)^2|f|_2^2/a_1a_2. \tag{3.21}$$

By (3.14) it comes

$$|K(a_1)-K(a_2)| \le |g|_2|f|_2|a_1-a_2|/\sqrt{a_1a_2}$$

and the continuity (local Lipschitz continuity) of K follows.

(ii) *Proof of* (3.12). Since by (3.16) φ_a is bounded in $L^2(\Omega)$ independently of a, we have for some "subsequence" of a and some φ_0

$$\varphi_a \rightharpoonup \varphi_0 \quad \text{in } L^2(\Omega) \text{ when } a\to 0. \tag{3.22}$$

Taking $v\in\mathcal{D}(\Omega)$ – the space of C^∞-functions with compact support in Ω – in (3.6) we get after integration by parts

$$a(\varphi_a,\Delta v) + (\varphi_a, v) = (f,v). \tag{3.23}$$

Letting $a\to 0$ this leads to

$$(\varphi_0, v) = (f,v) \quad \forall\, v\in\mathcal{D}(\Omega), \tag{3.24}$$

i.e. to $\varphi_0=f$. Since this is true for every "subsequence" of $a\to 0$ we have shown that

$$\varphi_a \rightharpoonup f \quad \text{in } L^2(\Omega) \text{ when } a\to 0. \tag{3.25}$$

(3.12) follows by passing to the limit in the definition of K.

(iii) *Proof of* (3.13). First if $|\Gamma_D| \neq 0$ then

$$\big||\nabla v|\big|_2 \tag{3.26}$$

is a norm on V equivalent to the $H^1(\Omega) =$ norm (see for instance [5]). Thus from (3.17) we deduce that when $a \to +\infty$

$$\varphi_a \to 0 \quad \text{in } H^1(\Omega) \tag{3.27}$$

and the first part of (3.13) follows. Next if $|\Gamma_D| = 0$ – i.e. when $V = H^1(\Omega)$ – from (3.16) we derive that for a "subsequence" of $a \to +\infty$ and some φ_∞ it holds that

$$\varphi_a \rightharpoonup \varphi_\infty \quad \text{in } L^2(\Omega) \text{ when } a \to +\infty. \tag{3.28}$$

This implies that for any $i = 1, \ldots, n$ we have

$$\frac{\partial \varphi_a}{\partial x_i} \to \frac{\partial \varphi_\infty}{\partial x_i} \quad \text{in } \mathcal{D}'(\Omega). \tag{3.29}$$

Now from (3.17) we have

$$\frac{\partial \varphi_a}{\partial x_i} \to 0 \quad \text{in } L^2(\Omega)$$

for every i and by uniqueness of the limit in $\mathcal{D}'(\Omega)$ we derive that

$$\frac{\partial \varphi_\infty}{\partial x_i} = 0 \quad \forall\, i = 1, \ldots, n$$

that is to say φ_∞ is a constant. Taking now $v = 1$ in (3.6) we get

$$\int_\Omega \varphi_a \, dx = \int_\Omega f \, dx \tag{3.30}$$

and letting the "subsequence" a go to $+\infty$ we obtain

$$\varphi_\infty |\Omega| = \int_\Omega f \, dx \quad \Leftrightarrow \quad \varphi_\infty = \fint_\Omega f \, dx.$$

Since the limit is independent of the subsequence considered we have shown that

$$\varphi_a \rightharpoonup \fint_\Omega f \, dx \quad \text{in } L^2(\Omega) \text{ when } a \to +\infty.$$

Passing to the limit in the definition of $K(a)$, this completes the proof of the lemma. □

As a consequence we can prove the following:

Theorem 3.2. *Suppose that*

$$a \text{ is a continuous function from } \mathbb{R} \text{ into } (0, +\infty). \tag{3.31}$$

Then if $f, g \in L^2(\Omega)$ there exists a solution to (3.1), (3.4).

Proof. It follows from Theorem 3.1 that the problem reduces to solve the equation

$$\mu = K(a(\mu)). \tag{3.32}$$

Due to Lemma 3.1 the function $\mu \mapsto K(a(\mu))$ is continuous and bounded. Then it holds that

$$\lim_{\mu \to -\infty} \mu - K(a(\mu))) = -\infty, \qquad \lim_{\mu \to +\infty} \mu - K(a(\mu)) = +\infty \tag{3.33}$$

and clearly one can find μ such that (3.32) holds. This completes the proof of the theorem. $\square$

We now would like to investigate the number of solutions to (3.1), (3.4). For that we will use the following lemma:

Lemma 3.2. *Let f be a function such that*

$$f \in H^1(\Omega), \qquad f \geq 0, \tag{3.34}$$

$$\int_\Omega \nabla f \nabla v \, dx \geq 0 \quad \forall v \in V, \quad v \geq 0, \tag{3.35}$$

then the mapping $a \mapsto \varphi_a$ is nonincreasing, i.e.

$$a_1 \geq a_2 \quad \Rightarrow \quad \varphi_{a_1} \leq \varphi_{a_2} \quad a.e. \ in \ \Omega. \tag{3.36}$$

If in addition

$$\Delta f \not\equiv 0 \ in \ \Omega \quad or \quad f \not\equiv 0 \ on \ \Gamma_D \quad or \quad \frac{\partial f}{\partial \nu} \not\equiv 0 \ on \ \Gamma_N, \tag{3.37}$$

then the mapping is decreasing in the sense that

$$a_1 > a_2 \quad \Rightarrow \quad \varphi_{a_1} < \varphi_{a_i} \quad a.e. \ in \ \Omega. \tag{3.38}$$

Proof. Let us first establish (3.36). We claim first that for any $a \in (0, +\infty)$ it holds that

$$(\varphi_a - f)^+ = 0. \tag{3.39}$$

Indeed from (3.6) we obtain

$$a \int_\Omega \nabla(\varphi_a - f) \nabla v \, dx + \int_\Omega (\varphi_a - f) v \, dx = -a \int_\Omega \nabla f \nabla v \, dx \quad \forall v \in V. \tag{3.40}$$

Since $f \geq 0$ we have – see [12], [4]

$$(\varphi_a - f)^+ \in V$$

and by (3.40), (3.35) we get taking $v = (\varphi_a - f)^+$

$$a \int_\Omega |\nabla(\varphi_a - f)^+|^2 \, dx + \int_\Omega (\varphi_a - f)^{+2} \, dx = -a \int_\Omega \nabla f \nabla (\varphi_a - f)^+ \leq 0$$

which leads to (3.39). Next, using (3.6) again we have

$$a \int_\Omega \nabla \varphi_a \nabla v \, dx = -\int_\Omega (\varphi_a - f) v \, dx \quad \forall v \in V. \tag{3.41}$$

By (3.39) it follows that it holds that

$$a \int_\Omega \nabla \varphi_a \nabla v \, dx = \int_\Omega (\varphi_a - f)^- v \, dx \geq 0 \quad \forall v \in V, \ v \geq 0. \tag{3.42}$$

We then go back to (3.18) we have for $a_1 > a_2$

$$\begin{aligned} a_1 \int_\Omega \nabla(\varphi_{a_1} - \varphi_{a_2}) \nabla v \, dx + \int_\Omega (\varphi_{a_1} - \varphi_{a_2}) v \, dx \\ = (a_2 - a_1) \int_\Omega \nabla \varphi_{a_2} \nabla v \, dx \quad \forall v \in V. \end{aligned} \tag{3.43}$$

Taking $v = (\varphi_{a_1} - \varphi_{a_2})^+$ by (3.42) we derive

$$a_1 \int_\Omega |\nabla(\varphi_{a_1} - \varphi_{a_2})^+|^2 \, dx + \int_\Omega (\varphi_{a_1} - \varphi_{a_2})^{+2} \, dx \le 0$$

that is to say

$$(\varphi_{a_1} - \varphi_{a_2})^+ = 0 \quad \Leftrightarrow \quad \varphi_{a_1} \le \varphi_{a_2}. \tag{3.44}$$

This completes the proof of the first part of the theorem. For the second part we notice that by (3.43) it holds that – take $v \in \mathcal{D}(\Omega)$:

$$-a_1 \Delta(\varphi_{a_1} - \varphi_{a_2}) + \varphi_{a_1} - \varphi_{a_2} = (a_1 - a_2)\Delta\varphi_{a_1} \quad \text{in } \mathcal{D}'(\Omega). \tag{3.45}$$

Now, from (3.6) we have also if we take $a = a_1$

$$-a_1 \Delta\varphi_{a_1} = -(\varphi_{a_1} - f) \tag{3.46}$$

and by (3.39)

$$\Delta\varphi_{a_1} = -(\varphi_{a_1} - f)^- / a_1 \in H^1(\Omega)$$

and

$$\Delta\varphi_{a_1} \le 0.$$

If $\Delta\varphi_{a_1} \not\equiv 0$ by (3.45) and the maximum principle we have

$$\varphi_{a_1} < \varphi_{a_2} \quad \text{a.e. in } \Omega$$

and we are done (see remark below). But if

$$\Delta\varphi_{a_1} \equiv 0$$

from (3.46) we derive

$$f = \varphi_{a_1}$$

and thus

$$\Delta f = 0 \text{ in } \Omega, \quad f = 0 \text{ on } \Gamma_D, \quad \frac{\partial f}{\partial \nu} = 0 \text{ on } \Gamma_N$$

which contradicts (3.37). This completes the proof of the lemma. □

Remark 3.1. The function $u = \varphi_{a_1} - \varphi_{a_2}$ satisfies – see (3.45)

$$u \le 0, \quad -a_1 \Delta u + u = h \in L^2(\Omega), \quad h \le 0. \tag{3.47}$$

Then, we claim that if

$$h \le 0, \qquad h \not\equiv 0 \tag{3.48}$$

it holds that

$$u < 0 \quad \text{a.e. in } \Omega. \tag{3.49}$$

(This is what we used above). If $u \in W^{2,n}_{\text{loc}}(\Omega)$ the result follows from Theorem 9.6 of [11]. If one assumes only $h \in L^2(\Omega)$ the result seems to be not available in the literature. To convince the reader that it holds also in this case we can proceed as follows. If $h \in L^2(\Omega)$, $h \le 0$ it is possible to find a sequence of simple functions h_k such that

$$h \le h_k \le 0, \qquad h_k \searrow h \quad \text{a.e. } x \in \Omega. \tag{3.50}$$

Of course since $h \not\equiv 0$ we can assume $h_k \not\equiv 0$. Thus if we introduce u_k the solution to

$$\begin{cases} u_k \in V, \\ a_1 \int_\Omega \nabla u_k \nabla v \, dx + \int_\Omega u_k v \, dx = \int_\Omega h_k v \, dx \quad \forall \, v \in V, \end{cases} \tag{3.51}$$

since $h_k \in L^p(\Omega)$, $p > n$ it holds that $u_k \in W^{2,n}_{\text{loc}}(\Omega)$ and thus (cf. Theorem 9.6 of [11])

$$u_k < 0 \quad \text{a.e. in } \Omega.$$

Since u is solution to

$$\begin{cases} u \in V, \\ a_1 \int_\Omega \nabla u \nabla v \, dx + \int_\Omega u v \, dx = \int_\Omega h v \, dx \quad \forall \, v \in V, \end{cases} \tag{3.52}$$

taking $v = (u - u_k)^+$ in (3.51), (3.52) one derives easily that

$$u \leq u_k < 0 \quad \text{a.e. in } \Omega$$

which completes the proof.

As a consequence of Lemma 3.2 we have:

Lemma 3.3. *Let f be a function satisfying* (3.5)–(3.7). *Suppose that*

$$g \geq 0, \qquad g \not\equiv 0 \tag{3.53}$$

then it holds that

$$K \text{ is decreasing on } (0, +\infty). \tag{3.54}$$

Proof. This follows immediately from (3.38), (3.53) and the definition of K. □

Remark 3.2. Somehow (3.54) cannot hold under the simple assumption that

$$f, g \geq 0, \qquad f, g \not\equiv 0, \tag{3.55}$$

(cf. [14], [15]). Indeed, suppose for instance that (3.55) holds and that f, g have disjoint supports. Then, by Lemma 3.1 we have

$$\lim_{a \to 0} K(a) = \int_\Omega g(x) f(x) \, dx = 0, \qquad \lim_{a \to +\infty} K(a) \geq 0, \quad K > 0$$

and thus K cannot be decreasing. By a perturbation argument since K depends continuously of f and g, K could fail also to be decreasing for $f, g > 0$.

Remark 3.3. For $a \in (0, +\infty)$ let us denote by ψ_a the weak solution to

$$\begin{cases} -a\Delta\psi_a + \psi_a = g \text{ in } \Omega, \\ \psi_a = 0 \text{ on } \Gamma_D, \quad \dfrac{\partial \psi_a}{\partial \nu} = 0 \text{ on } \Gamma_N, \end{cases} \tag{3.56}$$

i.e.

$$\begin{cases} \psi_a \in V, \\ a \int_\Omega \nabla \psi_a \nabla v \, dx + \int_\Omega \psi_a v \, dx = \int_\Omega g v \, dx \quad \forall \, v \in V. \end{cases} \tag{3.57}$$

It holds that

$$K(a) = \int_\Omega g\varphi_a \, dx = \int_\Omega f\psi_a \, dx. \tag{3.58}$$

Indeed taking $v = \psi_a$ in (3.6) and $v = \varphi_a$ in (3.57) it comes

$$a \int_\Omega \nabla\psi_a \nabla\varphi_a \, dx + \int_\Omega \psi_a\varphi_a \, dx = \int_\Omega g\varphi_a \, dx = \int_\Omega f\psi_a \, dx \tag{3.59}$$

which is (3.59). Then, f and g are playing a symmetric rôle and assuming that

$$f \geq 0, \qquad f \equiv 0 \tag{3.60}$$

$$\text{(3.34)–(3.36) holds for } g \text{ in place of } f, \tag{3.61}$$

we have that

$$K \text{ is decreasing.} \tag{3.62}$$

This is indeed a consequence of (3.58) since by (3.61), $a \mapsto \psi_a$ is decreasing. In particular if $|\Gamma_D| \neq 0$ then K is decreasing for

$$f = 1, \quad g \geq 0, \quad g \not\equiv 0, \quad \text{or} \quad g = 1, \quad f \geq 0, \quad f \not\equiv 0. \tag{3.63}$$

(3.34), (3.35), (3.37) holds for $f = 1$ if $|\Gamma_D| \neq 0$).

Let us see what can happen in the case where K is decreasing regarding the solutions to (3.1). Suppose for example that $|\Gamma_D| \neq 0$ so that K is a one-to-one mapping from $(0, +\infty)$ into $(\int_\Omega f0 \, dx, g)$ – see Lemma 3.1. Denote by K^{-1} the inverse of K. Then the equation (3.7) can be written – see (3.32) –

$$\mu = K(a(\mu)) \quad \Leftrightarrow \quad a(\mu) = K^{-1}(\mu) \tag{3.64}$$

and (3.1) can have then a unique solution, several solutions, a continuum of solutions depending on the number of solutions to (3.64) – see Theorem 3.1. The different situations are explained in the figures below.

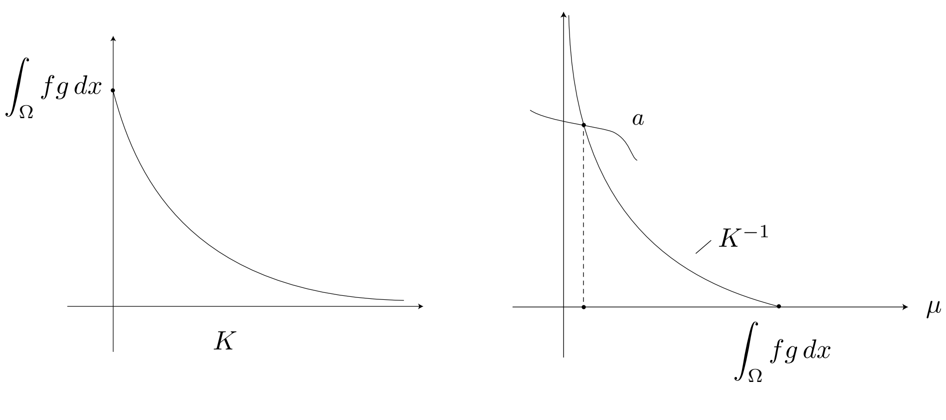


FIGURE 3.1. A case with a single solution

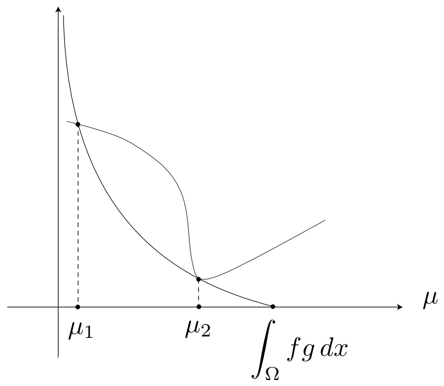


FIGURE 3.2. A case with two solutions

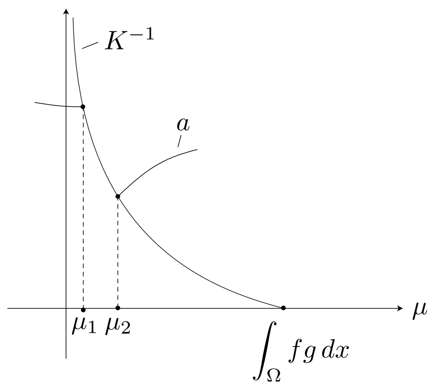


FIGURE 3.3. A case with a continuum of solutions

4. Asymptotic behaviour

In this section we consider $u = u(x,t)$ the weak solution to (1.2) – see Remark 2.1. We would like to study the asymptotic behaviour of $u(x,t)$ when $t \to +\infty$. Of course the general situation is very complicated and we have to restrict ourselves to special cases.

4.1. **A simple example.** Let us consider $u = u(x,t)$ solutions to

$$\begin{cases} u_t - a\Big(\displaystyle\int_\Omega u(x,t)\,dx\Big)\Delta u + u = f \quad \text{in } \Omega \times \mathbb{R}^+, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma \times \mathbb{R}^+, \\ u(\cdot,0) = u_0, \end{cases} \tag{4.1}$$

i.e. u is solution to (1.2) with $\Gamma_D = \emptyset$, $\Gamma = \Gamma_N$, $g \equiv 1$. Assuming that a is locally Lipschitz continuous it follows from Theorems 2.1, 2.2 that a unique weak solution to (4.1) does exist. A stationary solution associated to (4.1) is a function u_∞ such that

$$\begin{cases} -a\Big(\displaystyle\int_\Omega u_\infty(x)\,dx\Big)\Delta u_\infty + u_\infty = f \quad \text{in } \Omega, \\ \dfrac{\partial u_\infty}{\partial \nu} = 0 \quad \text{on } \Gamma. \end{cases} \tag{4.2}$$

Since $\Gamma_D = \emptyset$ the condition (3.37) fails for $f, g = 1$. However, it is easy to show that (4.2) admits a unique solution. Considering for instance (4.2) under its weak formulation

$$\begin{cases} u_\infty \in V = H^1(\Omega), \\ a\Big(\int_\Omega u_\infty(x)\,dx\Big)\int_\Omega \nabla u_\infty \nabla v\,dx \\ \qquad + \int_\Omega u_\infty v\,dx = \int_\Omega f v\,dx \quad \forall\, v \in H^1(\Omega), \end{cases} \tag{4.3}$$

we get by taking $v = 1$

$$\int_\Omega u_\infty(x)\,dx = \int_\Omega f\,dx$$

and u_∞ is the unique solution to

$$\begin{cases} -a\Big(\int_\Omega f(x)\,dx\Big)\Delta u_\infty + u_\infty = f \quad \text{in } \Omega, \\ \dfrac{\partial u_\infty}{\partial \nu} = 0 \quad \text{on } \Gamma. \end{cases} \tag{4.4}$$

Similarly taking $v = 1$ in the weak formulation of (4.1) – or integrating the equation (4.1) on Ω – we obtain:

$$\begin{aligned} &\frac{d}{dt}\Big\{\int_\Omega u(x,t)\,dx\Big\} + \int_\Omega u(x,t)\,dx = \int_\Omega f(x)\,dx \\ \iff \quad &\frac{d}{dt}\Big\{e^t\int_\Omega u(x,t)\,dx\Big\} = e^t\int_\Omega f(x)\,dx. \end{aligned} \tag{4.5}$$

Integrating in t between 0 and t it comes

$$\int_\Omega u(x,t)\,dx = e^{-t}\int_\Omega u_0(x)\,dx + (1 - e^{-t})\int_\Omega f(x)\,dx. \tag{4.6}$$

Thus when $t \to +\infty$ it holds that

$$\int_\Omega u(x,t)\,dx \to \int_\Omega f(x)\,dx. \tag{4.7}$$

Since a is continuous it follows that

$$a\Big(\int_\Omega u(x,t)\,dx\Big) \to a\Big(\int_\Omega f(x)\,dx\Big) \tag{4.8}$$

and it is then relatively easy to show that

$$u(\cdot,t) \to u_\infty \quad \text{in } L^2(\Omega) \tag{4.9}$$

when $t \to +\infty$. We refer the reader to [8] or [4] for a proof.

4.2. **A case with a single equilibrium.** In this section we suppose that we are under the conditions of Theorems 2.1, 2.2 and we denote by $u = u(x,t)$ the unique weak solution to (1.2) – see Remark 2.1. Moreover, let us assume that (3.34), (3.35), (3.37) holds in such a way that K is a decreasing function. Finally, let us suppose that there exists a unique μ_∞ solution to

$$a(\mu_\infty) = K^{-1}(\mu_\infty) \tag{4.10}$$

i.e. there exists a unique stationary solution to (1.2) given by u_∞ where u_∞ satisfies

$$\begin{cases} -a(\mu_\infty)\Delta u_\infty + u_\infty = f \quad \text{in } \Omega, \\ u_\infty = 0 \text{ on } \Gamma_D, \quad \dfrac{\partial u_\infty}{\partial \nu} = 0 \text{ on } \Gamma_N. \end{cases} \tag{4.11}$$

We would like to show that in these conditions, when $t \to +\infty$ it holds that

$$u(\cdot, t) \to u_\infty \quad \text{in } L^2(\Omega). \tag{4.12}$$

For that let us set

$$\ell_0 = \liminf_{t\to+\infty} \ell(u(\cdot,t)), \qquad L_0 = \limsup_{t\to+\infty} \ell(u(\cdot,t)), \tag{4.13}$$

$$m_0 = \operatorname*{Inf}_{[\ell_0,L_0]} a(\xi), \qquad M_0 = \operatorname*{Sup}_{[\ell_0,L_0]} a(\xi). \tag{4.14}$$

Then, we have the following lemma.

Lemma 4.1. *Under the above conditions it holds that*

$$\ell(\varphi_{M_0}) \le \ell_0 \le L_0 \le \ell(\varphi_{m_0}). \tag{4.15}$$

Proof. Recall that for $a > 0$, φ_a denotes the solution to (3.6) – i.e. to

$$\begin{cases} \varphi_a \in V, \\ a\displaystyle\int_\Omega \nabla\varphi_a \nabla v \, dx + \int_\Omega \varphi_a v \, dx = \int_\Omega f v \, dx \quad \forall\, v \in V. \end{cases} \tag{4.16}$$

By the definition of ℓ_0 and L_0 have

$$\ell_0 = \operatorname*{Sup}_{t_0} \inf_{t\ge t_0} \ell(u(\cdot,t)), \qquad L_0 = \operatorname*{Inf}_{t_0} \operatorname*{Sup}_{t\ge t_0} \ell(u(\cdot,t)). \tag{4.17}$$

Let $\varepsilon > 0$ be fixed. By the above definitions for $t_0 = t_0(\varepsilon)$ large enough we have

$$\ell_0 - \varepsilon \le \ell(u(\cdot,t)) \le L_0 + \varepsilon \quad \forall\, t \ge t_0. \tag{4.18}$$

This implies if we set $u(\cdot,t) = u(t)$

$$m_0 - \delta(\varepsilon) \le a(\ell(u(t))) \le M_0 + \delta(\varepsilon) \quad \forall\, t \ge t_0 \tag{4.19}$$

for some $\delta = \delta(\varepsilon)$ such that

$$\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0. \tag{4.20}$$

Let us prove the first inequality of (4.15). From the weak formulation of (1.2) and (4.16) written for $a = M_0 + \delta$ we derive

$$\begin{aligned} &\frac{d}{dt}(u - \varphi_{M_0+\delta}, v) + a(\ell(u(t))) \int_\Omega \nabla(u - \varphi_{M_0+\delta})\nabla v \, dx + (u - \varphi_{M_0+\delta}, v) \\ &\qquad = \{M_0 + \delta - a(\ell(u(t)))\} \int_\Omega \nabla\varphi_{M_0+\delta} \nabla v \, dx \quad \forall\, v \in V, \end{aligned} \tag{4.21}$$

(this equality holding for instance in $\mathcal{D}'(0,+\infty)$.) Taking

$$v = -(u - \varphi_{M_0+\delta})^- \tag{4.22}$$

it comes

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|(u-\varphi_{M_0+\delta})^-|_2^2 + a(\ell(u(t)))\big|\nabla(u-\varphi_{M_0+\delta})^-\big|_2^2 + |(u-\varphi_{M_0+\delta})^-|_2^2 \\ = \{a(\ell(u(t))) - (M_0+\delta)\}\int_\Omega \nabla\varphi_{M_0+\delta}\nabla(u-\varphi_{M_0+\delta})^-\,dx.\end{aligned} \tag{4.23}$$

For $t \geq t_0$ we have by (4.19) and (3.42)

$$a(\ell(u(t))) - (M_0+\delta) \leq 0, \qquad \int_\Omega \nabla\varphi_{M_0+\delta}\nabla(u-\varphi_{M_0+\delta})^-\,dx \geq 0.$$

It follows that for $t \geq t_0$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|(u-\varphi_{M_0+\delta})^-|_2^2 + a(\ell(u(t))\big|\nabla(u-\varphi_{M_0+\delta})^-\big|_2^2 \\ + |(u-\varphi_{M_0+\delta})^-|_2^2 \leq 0\end{aligned} \tag{4.24}$$

and thus

$$\begin{aligned} &\frac{d}{dt}|(u-\varphi_{M_0+\delta})^-|_2^2 + 2|(u-\varphi_{M_0+\delta})^-|_2^2 \leq 0 \\ \iff \quad &\frac{d}{dt}\{e^{2t}|(u-\varphi_{M_0+\delta})^-|_2^2\} \leq 0.\end{aligned} \tag{4.25}$$

From this we derive that

$$\begin{aligned} &e^{2t}|(u-\varphi_{M_0+\delta})^-|_2^2 \leq e^{2t_0}|(u(t_0)-\varphi_{M_0+\delta})^-|_2^2 \\ \Longrightarrow \quad &|(u(t)-\varphi_{M_0+\delta})^-|_2 \leq e^{-(t-t_0)}|(u(t_0)-\varphi_{M_0+\delta})^-|_2, \quad t \geq t_0.\end{aligned}$$

Thus, we have when $t \to +\infty$,

$$(u(t)-\varphi_{M_0+\delta})^- \to 0 \quad \text{in } L^2(\Omega). \tag{4.26}$$

From

$$u-\varphi_{M_0+\delta} = (u-\varphi_{M_0+\delta})^+ - (u-\varphi_{M_0+\delta})^- \geq -(u-\varphi_{M_0+\delta})^-$$

we derive since $g \geq 0$, $g \in L^2(\Omega)$

$$\ell(u-\varphi_{M_0+\delta}) \geq -\ell((u-\varphi_{M_0+\delta})^-) \geq -\varepsilon \tag{4.27}$$

for t large enough. Thus for $t \geq t_1 = t_1(\varepsilon)$ we have

$$\ell(u(t)) \geq \ell(\varphi_{M_0+\delta}) - \varepsilon. \tag{4.28}$$

Passing to the limit this implies

$$\ell_0 = \liminf_{t\to+\infty} \ell(u(t)) \geq \ell(\varphi_{M_0+\delta}) - \varepsilon. \tag{4.29}$$

Letting now $\varepsilon \to 0$ the first inequality of (4.15) follows. The second inequality of (4.15) is clear. The third can be obtained as above and for the sake of completeness we outline briefly the proof. From the weak formulation of (1.2) and (4.16) written for $a = m_0 - \delta$ (we suppose ε small enough in such a way that $m_0 > \delta$) we derive as in (4.21):

$$\begin{aligned}\frac{d}{dt}(u-\varphi_{m_0-\delta}, v) + a(\ell(u(t)))\int_\Omega \nabla(u-\varphi_{m_0-\delta})\nabla v\,dx + (u-\varphi_{m_0-\delta}, v) \\ = \{(m_0-\delta) - a(\ell(u(t)))\}\int_\Omega \nabla\varphi_{m_0-\delta}\nabla v\,dx \quad \forall\, v \in V, \text{ in } \mathcal{D}'(0,+\infty).\end{aligned} \tag{4.30}$$

Taking $v = (u - \varphi_{m_0-\delta})^+$ and using (4.19), (3.42) we get for $t \geq t_0$

$$\frac{1}{2}\frac{d}{dt}|(u - \varphi_{m_0-\delta})^+|_2^2 + |(u - \varphi_{m_0-\delta})^+|_2^2 \leq 0$$

which implies as in (4.26) that

$$(u - \varphi_{m_0-\delta})^+ \to 0 \quad \text{in } L^2(\Omega) \tag{4.31}$$

as $t \to +\infty$. Thus arguing as in (4.27) we obtain

$$\ell(u - \varphi_{m_0-\delta}) \leq \ell((u - \varphi_{m_0-\delta})^+) \leq \varepsilon$$

for t large enough and thus

$$L_0 = \limsup_{t\to+\infty} \ell(u(t)) \leq \ell(\varphi_{m_0-\delta}) + \varepsilon.$$

Letting $\varepsilon \to 0$ the third inequality in (4.15) follows. Note that we have used here the continuity of the mapping $a \mapsto \varphi_a$ which results from (3.21). This completes the proof of the lemma. □

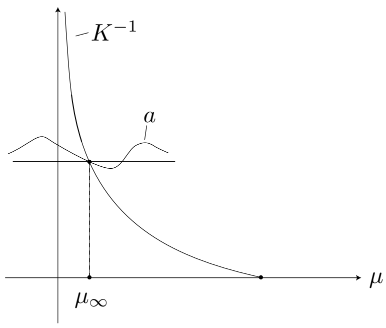
In addition to the above assumptions we are going to assume that

$$a(\mu) \geq a(\mu_\infty) \quad \forall\, \mu \leq \mu_\infty \qquad \text{or} \qquad a(\mu) \leq a(\mu_\infty) \quad \forall\, \mu \geq \mu_\infty, \tag{4.32}$$

that is to say that we are in one of the cases of the figures below.

□

Figure 4



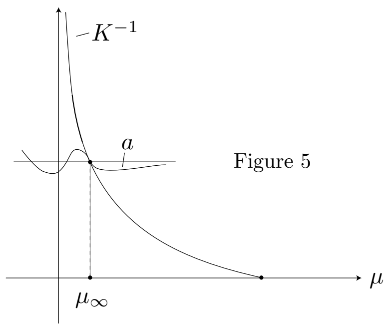


Figure 5

Remark 4.1. Note that (4.32) needs only to be satisfied on the domain of definition of K^{-1}.

Then we can prove the following:

Theorem 4.1. *Under the above assumptions we have when* $t \to +\infty$

$$u(\cdot, t) \to u_\infty \quad \textit{in } L^2(\Omega) \tag{4.33}$$

where u_∞ *is the unique solution to* (4.11).

Proof. If we can show that

$$\lim_{t\to+\infty} \ell(u(t)) = \mu_\infty \tag{4.34}$$

we will have

$$\lim_{t\to+\infty} a(\ell(u(t))) = a(\mu_\infty) \tag{4.35}$$

and the result will follow – see [8], [4]. To prove (4.34) it is enough to show that

$$\ell_0 = L_0 = \mu_\infty. \tag{4.36}$$

Let us first show that

$$\ell_0 = L_0. \tag{4.37}$$

If not we have

$$\ell_0 < L_0. \tag{4.38}$$

Case 1. $\mu_\infty \leq \ell_0 < L_0$
Denote by m_1 a point of $[\ell_0, L_0]$ such that

$$m_0 = \operatorname*{Inf}_{[\ell_0,L_0]} a(\xi) = a(m_1). \tag{4.39}$$

If $m_1 = \mu_\infty$, by (4.15) we have

$$L_0 \leq \ell(\varphi_{m_0}) = \ell(\varphi_{a(m_1)}) = K(a(m_1)) = \mu_\infty \tag{4.40}$$

which is impossible (recall that $K(a) = \ell(\varphi_a)$). If $m_1 > \mu_\infty$, by (4.15) we have now

$$L_0 \leq \ell(\varphi_{m_0}) = \ell(\varphi_{a(m_1)}) = K(a(m_1)) < m_1 \tag{4.41}$$

which contradicts the definition of m_1. In the above, the last inequality follows of the uniqueness of the solution to

$$\mu_\infty = K(a(\mu_\infty)).$$

Indeed due to this uniqueness we have

$$\mu > K(a(\mu)) \quad \forall\, \mu > \mu_\infty, \qquad \mu < K(a(\mu)) \quad \forall\, \mu < \mu_\infty. \tag{4.42}$$

Thus, the case 1 above is impossible.

Case 2. $\ell < L_0 \leq \mu_\infty$
Let us denote by M_1 a point of $[\ell_0, L_0]$ such that

$$M_0 = \operatorname*{Sup}_{[\ell_0,L_0]} a(\xi) = a(M_1). \tag{4.43}$$

If $M_1 = \mu_\infty$, by (4.15) it holds that

$$\ell_0 \geq \ell(\varphi_{M_0}) = \ell(\varphi_{a(M_1)}) = K(a(M_1)) = \mu_\infty \tag{4.44}$$

which is impossible. If $M_1 < \mu_\infty$ then

$$\ell_0 \geq \ell(\varphi_{M_0}) = \ell(\varphi_{a(M_1)}) = K(a(M_1)) > M_1 \tag{4.45}$$

(by (4.39)). This is also impossible and this case 2 cannot occur.

Case 3. $\ell_0 < \mu_\infty < L_0$
Define m_1 and M_1 as before. Suppose for instance that we are in the case of Figure 4 i.e.

$$a(\mu) \geq a(\mu_\infty) \quad \forall\, \mu \leq \mu_\infty. \tag{4.46}$$

Then without loss of generality we can assume

$$m_1 \in [\mu_\infty, L_0].$$

If $m_1 = \mu_\infty$ we have (4.40) and a contradiction. If $m_1 > \mu_\infty$ we have (4.41) and another impossibility. In the case of Figure 5 one argues similarly using M_1. Thus we have established the impossibility of (4.38) and we have

$$\ell_0 = L_0. \tag{4.47}$$

In this case (4.15) becomes

$$\ell_0 = L_0 \le \ell(\varphi_{a(\ell_0)}) = K(a(\ell_0)) \tag{4.48}$$

and thus

$$\ell_0 = L_0 = \mu_\infty.$$

This completes the proof of the theorem. □

Remark 4.2. If we drop the assumption (4.32) and consider an arbitrary a, the analysis of the cases 1 and 2 together with (4.48) shows that we have in this case

$$\ell_0 < \mu_\infty < L_0 \qquad \text{or} \qquad \ell_0 = L_0 = \mu_\infty. \tag{4.49}$$

4.3. **The case of two equilibria.** In this section we assume that we are in the case of Figure 3.2. In particular we will assume

$$f \text{ satisfies } (3.34), (3.35), (3.37), \tag{4.50}$$

$$g > 0 \quad \text{in } \Omega, \tag{4.51}$$

$$a(\mu_i) = K^{-1}(\mu_i) \quad i = 1, 2, \qquad a(\mu) > K^{-1}(\mu) \quad \forall\, \mu \in (\mu_1, \mu_2), \tag{4.52}$$

$$a(\mu_2) \le a(\mu) \le a(\mu_1) \quad \forall\, \mu \in (\mu_1, \mu_2). \tag{4.53}$$

Let us denote by $u_i = \varphi_{a(\mu_i)}$, $i = 1, 2$ the solution to

$$\begin{cases} -a(\ell(u_i))\Delta u_i + u_i = f \text{ in } \Omega, \\ u_i = 0 \text{ on } \Gamma_D, \quad \dfrac{\partial u_i}{\partial \nu} = 0 \text{ on } \Gamma_N. \end{cases} \tag{4.54}$$

Due to (3.38) we have

$$u_1 < u_2. \tag{4.55}$$

We consider then u solution to (1.2) with u_0 such that

$$u_1 \le u_0 \le u_2, \quad u_0 \ne u_2. \tag{4.56}$$

We have:

Proposition 4.1. *Let u be the weak solution to* (1.2). *Under the above assumptions it holds that*

$$u_1 \le u(\cdot, t) \le u_2 \quad \forall\, t. \tag{4.57}$$

Proof. Let us denote by E the set

$$E = \{\, t \mid \ell(u(s)) \in [\mu_1, \mu_2] \ \forall\, s \le t \,\}. \tag{4.58}$$

By (4.56), E contains 0 (recall that $g \ge 0$, $\ell(u_i) = \mu_i$). Set

$$t^* = \operatorname{Sup}\{\, t \mid t \in E \,\}. \tag{4.59}$$

By continuity of the mapping $t \mapsto u(t)$ in $L^2(\Omega)$, $t \mapsto \ell(u(t))$ is continuous and

$$\ell(u(t^*)) \in [\mu_1, \mu_2], \quad t^* \in E. \tag{4.60}$$

We claim next that

$$u_1 \leq u(t) \leq u_2 \quad \forall\, t \in [0, t^*]. \tag{4.61}$$

Suppose that we want to prove the left hand side inequality. Using the weak formulation of (1.2) and (4.54) we have in $\mathcal{D}'(0, t^*)$

$$\begin{aligned} &\frac{d}{dt}(u - u_1, v) + a(\ell(u(t))) \int_\Omega \nabla(u - u_1)\nabla v \, dx + (u - u_1, v) \\ &\qquad = \{a(\mu_1) - a(\ell(u(t)))\} \int_\Omega \nabla u_1 \nabla v \, dx \quad \forall\, v \in V. \end{aligned} \tag{4.62}$$

Due to (4.53), (4.60), (3.42) we have on $(0, t^*)$

$$\begin{aligned} &a(\mu_1) - a(\ell(u(t))) \geq 0, \\ &\int_\Omega \nabla u_1 \nabla v \, dx \geq 0 \quad \forall\, v \in V, \ v \geq 0. \end{aligned}$$

Taking $v = -(u - u_1)^-$ in (4.62) we get easily

$$\frac{1}{2}\frac{d}{dt}|(u - u_1)^-|_2^2 + |(u - u_1)^-|_2^2 \leq 0. \tag{4.63}$$

Since $(u - u_1)^-(0) = (u_0 - u_1)^- = 0$ it follows from the Gronwall inequality – see for instance (4.25) – that $(u - u_1)^- = 0$ and thus $u \geq u_1$. The right hand side of the inequality (4.61) can be proven in the same way. This established (4.61). Next, by definition of t^*, if $t^* < +\infty$ we have

$$\ell(u(t^*)) = \ell(u_1) \quad \text{or} \quad \ell(u_2).$$

By (4.51), (4.61) this implies

$$u(t^*) = u_1 \quad \text{or} \quad u_2$$

and by the uniqueness of the solution to (1.2) this equality remains valid for further time which contradicts the definition of t^*. We have thus $t^* = +\infty$ and (4.61) gives (4.57). This completes the proof of the proposition. □

Remark 4.3. Since we did not use the inequality in (4.52) Proposition 4.1 remains valid in the case of Figure 3.3. The assumption (4.51) could be relaxed using the dynamical system theory – see [8].

We can now show

Theorem 4.2. *Assume that we are under the conditions of Proposition 4.1 with in addition*

$$f = g. \tag{4.64}$$

If u is the weak solution to (1.2) *with u_0 satisfying* (4.56)*, when $t \to +\infty$, it holds that*

$$u(\cdot, t) \to u, \quad \text{in } L^2(\Omega). \tag{4.65}$$

We first exhibit a Lyapunov function for the problem. Indeed we have:

Lemma 4.2. $|u(t)|_2^2$ *is a Lyapunov function – i.e. decreases with time.*

Proof. The weak formulation of (1.2) reads in $\mathcal{D}'(0,+\infty)$

$$\langle u_t, v\rangle - a(\ell(u(t)))\langle \Delta u, v\rangle + (u,v) = (f,v) \quad \forall\, v \in V \tag{4.66}$$

($\langle\cdot\rangle$ is the duality bracket between V' and V). Taking $v = \varphi_{a(\ell(u(t))} = \varphi_a$ we get

$$\begin{aligned} & \langle u_t, \varphi_a\rangle - a\langle \Delta u, \varphi_a\rangle + (u,\varphi_a) = (f,\varphi_a) = (g,\varphi_a) \\ \iff \quad & \langle u_t, \varphi_a\rangle + (u, -a\Delta\varphi_a + \varphi_a) = (g,\varphi_a) = K(a) \\ \iff \quad & \langle u_t, \varphi_{a(\ell(u(t)))}\rangle = K(a(\ell(u(t))) - \ell(u(t)). \end{aligned}$$

Since $\ell(u(t)) \in [\mu_1,\mu_2]$, by (4.52) it holds that

$$a(\ell(u(t))) \geq K^{-1}(\ell(u(t))) \quad \Leftrightarrow \quad K(a(\ell(u(t))) \leq \ell(u(t))$$

and thus

$$\langle u_t, \varphi_{a(\ell(u(t)))}\rangle \leq 0. \tag{4.67}$$

Combining (3.6) written for $a = a(\ell(u(t)))$ and (4.66) we obtain in $\mathcal{D}'(0,+\infty)$:

$$\langle u_t, v\rangle + a(\ell(u(t))) \int_\Omega \nabla(u-\varphi_a)\nabla v\, dx + (u-\varphi_a, v) = 0 \quad \forall\, v \in V. \tag{4.68}$$

Choosing in the above equality $v = u - \varphi_a = u - \varphi_{a(\ell(u(t)))}$ we get by (4.67)

$$\begin{aligned} \langle u_t, u\rangle &= \langle u_t, \varphi_a\rangle - a(\ell(u(t))) \int_\Omega |\nabla(u-\varphi_a)|^2\, dx - |u-\varphi_a|_2^2 \\ &\leq -a(\ell(u(t)))\big|\big|\nabla(u-\varphi_a)\big|\big|_2^2 - |u-\varphi_a|_2^2 \end{aligned} \tag{4.69}$$

i.e.

$$\langle u_t, u\rangle = \frac{1}{2}\frac{d}{dt}|u(t)|_2^2 \leq 0.$$

This completes the proof of the lemma. □

Proof of Theorem 4.2. Note that (4.69) holds in the distributional sense in $\mathcal{D}'(0,+\infty)$ or a.e. t. Then we claim that

$$\operatorname*{limsupess}_{t\to+\infty}\langle u_t, u\rangle = \operatorname*{Inf}_{t_0} \operatorname*{essup}_{t\geq t_0}\langle u_t, u\rangle(t) = 0. \tag{4.70}$$

Indeed – due to (4.69) – if it is not true, for t_0 large enough it holds that

$$\operatorname*{essup}_{t\geq t_0}\langle u_t, u\rangle \leq -\alpha < 0$$

for some positive α. Thus we have

$$\frac{d}{dt}|u(t)|_2^2 \leq -2\alpha \quad \text{a.e. } t \geq t_0.$$

Integrating between t_0 and t we obtain

$$|u(t)|_2^2 \leq -2\alpha(t-t_0) + |u_0|_2^2 \tag{4.71}$$

(recall that u is continuous). But then (4.71) is impossible for t large enough (the $L^2(\Omega)$-norm of u is bounded since $|u(t)|_2$ is nonincreasing). This shows (4.70). Going back to (4.69) it follows that

$$0 = \operatorname*{limsupess}_{t\to+\infty}\langle u_t, u\rangle \leq \operatorname*{limsupess}_{t\to+\infty}\{-a(\ell(u(t)))\big|\big|\nabla(u-\varphi_a)\big|\big|_2^2 - |u-\varphi_a|_2^2\} \leq 0. \tag{4.72}$$

It follows – recall that u is continuous – that we have

$$\liminf_{t\to+\infty} |u(t) - \varphi_{a(\ell(u(t)))}|_2^2 = 0. \tag{4.73}$$

Thus, there exists a sequence $t_n \to +\infty$ such that

$$u(t_n) - \varphi_{a(\ell(u(t_n)))} \to 0 \tag{4.74}$$

as $t_n \to +\infty$. Since $u(t_n)$ is bounded in $L^2(\Omega)$ we can extract from t_n a subsequence, for simplicity still labelled t_n, such that for some u_∞

$$u(t_n) \rightharpoonup u_\infty \quad \text{in } L^2(\Omega). \tag{4.75}$$

The set

$$C = \{\, v \in L^2(\Omega) \mid u_1(x) \le v(x) \le u_2(x) \text{ a.e. } x \in \Omega \,\} \tag{4.76}$$

is closed and convex in $L^2(\Omega)$. It is also weakly closed and by Proposition 4.1 and (4.75) we derive

$$u_1 \le u_\infty \le u_2 \quad \text{in } \Omega. \tag{4.77}$$

Moreover, from (4.74) we get

$$u_\infty = \varphi_{a(\ell(u_\infty))} \tag{4.78}$$

that is to say u_∞ is a stationary point and by (4.77)

$$u_\infty = u_1 \quad \text{or} \quad u_2.$$

Since $|u(t)|_2^2$ is nonincreasing – recall that $u_i \ge 0$ – by (4.56) we can only have

$$u_\infty = u_1.$$

Since the weak limit of any subsequence is unique we have shown that

$$u(t_n) \rightharpoonup u_1 \quad \text{in } L^2(\Omega). \tag{4.79}$$

Next consider another sequence $t'_n \to +\infty$ such that

$$u(t'_n) \rightharpoonup v_\infty \quad \text{in } L^2(\Omega).$$

Since $u(t'_n) \in C$ – cf. Proposition 4.1 – we also have $v_\infty \in C$ and in particular

$$v_\infty \ge u_1.$$

From (4.74), (4.79) we deduce that

$$u(t_n) \to u_1 \quad \text{in } L^2(\Omega).$$

Since $|u(t)|_2^2$ is nonincreasing it admits a limit when $t \to +\infty$ and this limit can only be $|u_1|_2^2$. Thus, passing to the limit in the inequality

$$|u(t'_n)|_2^2 - (u(t'_n), u_1) = (u(t'_n), u(t'_n) - u_1) \ge 0$$

we get

$$|u_1|_2^2 - (v_\infty, u_1) = (u_1 - v_\infty, u_1) \ge 0.$$

Since $u_1 > 0$, $v_\infty - u_1 \ge 0$ this clearly imposes

$$v_\infty = u_1.$$

Thus, every subsequence of $u(t)$ converging weakly towards u_1, we have as $t \to +\infty$

$$u(t) \rightharpoonup u_1 \quad \text{in } L^2(\Omega).$$

The strong convergence follows from the fact that

$$|u(t)|_2 \to |u_1|_2.$$

This completes the proof of the theorem. □

Remark 4.4. Some other cases can be treated. For instance in the case of Figure 3.3 and $f = g$ one can show for u_0 satisfying $u_1 \leq u_0 \leq u_2$ that $u(\cdot, t)$ converges toward an equilibrium when $t \to +\infty$. One can consider also cases where a is below K^{-1}. To keep the paper short we postpone these issues to forthcoming works – cf. [3].

Acknowledgements: This work was done partly when the first author was visiting the University of Zürich and the second the Academia Sinica. We thank both institutions for their hospitality. The authors would also like to acknowledge the support of the Swiss National Science Foundation under the contract # 20-67618.02.

References

[1] T. Cazenave, A. Haraux: *An Introduction to Semilinear Evolution Equations,* Oxford Science Publications (1998).
[2] N.-H. Chang, M. Chipot: *On some model diffusion problems with a nonlocal lower order term,* (to appear).
[3] N.-H. Chang, M. Chipot: *Nonlinear nonlocal evolution problems,* (in preparation).
[4] M. Chipot: *Elements of Nonlinear Analysis,* Birkhäuser Advanced Text (2000).
[5] M. Chipot: *ℓ goes to plus infinity,* Birkhäuser Advanced Text (2001)
[6] M. Chipot, B. Lovat: *Existence and uniqueness results for a class of nonlocal elliptic and parabolic problems,* DCDIS, Series A, 8 (2001), # 1, p. 35–51.
[7] M. Chipot, B. Lovat: *On the asymptotic behaviour of some nonlocal problems,* Positivity, 3 (1999), p. 65–81.
[8] M. Chipot, L. Molinet: *Asymptotic behaviour of some nonlocal diffusion problems,* Applicable Analysis, 80 (2001), p. 273–315.
[9] M. Chipot, J.F. Rodrigues: *On a class of nonlocal nonlinear elliptic problems,* Math. Mod. and Num. Anal., 26 (1992), p. 447–468.
[10] R. Dautray, J.L. Lions: *Mathematical analysis and numerical methods for science and technology,* Springer Verlag (1990).
[11] D. Gilbarg, N.S. Trudinger: *Elliptic Partial Differential Equations of Second Order,* Springer Verlag, New York (1983).
[12] D. Kinderlehrer, G. Stampacchia: *An introduction to variational inequalities and their applications,* Acad. Press (1980).
[13] J.L. Lions: *Quelques méthodes de résolution des problèmes aux limites non linéaires,* Dunod-Gauthier-Villards (1969).
[14] M. Siegwart: *Private communication.*
[15] M. Siegwart: *Thesis,* University of Zürich (to appear).

N.-H. Chang, Academia Sinica, Institute of Mathematics, Taipei 11529 Taiwan
E-mail address: `nhchang@math.sinica.edu.tw`

M. Chipot, Institut für Mathematik, Abt. Angewandte Mathematik, Universität Zürich, Winterthurerstrasse 190, CH–8057 Zürich, Switzerland
E-mail address: `chipot@amath.unizh.ch`