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van den Berg, M ; Bolthausen, E

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AREA VERSUS CAPACITY AND SOLIDIFICATION IN THE CRUSHED ICE MODEL

M. VAN DEN BERG AND E. BOLTHAUSEN

1. Introduction

Let K be a non-polar, compact set in Euclidean space $\mathbb{R}^m$ $(m = 2, 3, \dots)$ with boundary ∂K. Let $u : \mathbb{R}^m \setminus K \times [0, \infty) \to \mathbb{R}$ be the unique weak solution of

$$\Delta u = \frac{\partial u}{\partial t}, \qquad x \in \mathbb{R}^m \setminus K, \quad t > 0, \tag{1.1}$$

with initial condition

$$u(x; 0) = 0, \qquad x \in \mathbb{R}^m \setminus K, \tag{1.2}$$

and with boundary condition

$$u(x; t) = 1, \qquad x \in \partial K, \quad t > 0. \tag{1.3}$$

Then $u(x; t)$ represents the temperature at time t at a point $x \in \mathbb{R}^m \setminus K$ when $\mathbb{R}^m \setminus K$ has initial temperature 0 and ∂K is kept at temperature 1 for all time t. We define the heat content $E_K(t)$ by the total heat flow from K into $\mathbb{R}^m \setminus K$ up to time t:

$$E_K(t) = \int_{\mathbb{R}^m \setminus K} u(x; t)\, dx. \tag{1.4}$$

The asymptotic behaviour of $E_K(t)$ for large t was obtained by Spitzer [14]. For $m \geq 3$ he showed that

$$E_K(t) = C(K)t + o(t), \qquad t \to \infty, \tag{1.5}$$

where $C(K)$ is the Newtonian capacity of K. If $m = 2$ and K has positive logarithmic capacity then

$$E_K(t) = \frac{4\pi t}{\log t} + o\left(\frac{t}{\log t}\right), \qquad t \to \infty. \tag{1.6}$$

For refinements we refer to Spitzer's original paper and to two papers by Le Gall [6], [7].

It is an important consequence of the maximum principle that the asymptotic behaviour of the solution u of (1.1)–(1.3) for $t \to 0$ is locally computable in terms of the geometry of K. Exploiting this it was shown [2] that if ∂K is smooth then the heat content for $t \to 0$ is area dominated and

$$E_K(t) = 2\pi^{-1/2} A(\partial K) t^{1/2} + o(t^{1/2}), \qquad t \to 0, \tag{1.7}$$

where $A(\partial K)$ is the area of ∂K.

Results of Rauch and Taylor [11] suggest that (1.7) may not hold if K is the closure of an infinite union of disjoint closed balls with finite total area. In [1] we obtained conditions on the geometry of the small balls in K such that the heat content satisfies (1.7). In this paper we give some examples where these geometrical conditions are not satisfied, and where the asymptotic behaviour of the heat content is anomalous, i.e. different from (1.7).

Let $B(c;r)$ be the closed ball with centre c and radius r,

$$B(c;r) = \{\, x \in \mathbb{R}^m : |c - x| \leq r \,\}, \tag{1.8}$$

and let

$$B = \bigcup_{i=1}^{\infty} B(c_i; r_i), \tag{1.9}$$

where the balls in B are pairwise disjoint

$$B(c_i; r_i) \cap B(c_j, r_j) = \emptyset, \qquad i \neq j, \tag{1.10}$$

and are labeled such that

$$r_1 \geq r_2 \geq \dots. \tag{1.11}$$

Let K be the closure of B,

$$K = \bar{B}. \tag{1.12}$$

In general $K \setminus B$ may be non empty and may even have positive capacity, in which case the set will contribute to the heat content. In order to avoid this we consider the heat equation on a half space. Let

$$H^{\pm} = \{\, (x_1, \dots, x_m) \in \mathbb{R}^m : x_1 \gtrless 0 \,\} \tag{1.13}$$

be the open half-spaces, and denote by

$$N = \partial H^+ = \partial H^- \tag{1.14}$$

their common boundary. Throughout the paper we assume that the balls satisfy (1.10) and that

$$B \subset H^+, \qquad K \setminus B \subset N. \tag{1.15}$$

Let $u : H^+ \setminus B \times [0, \infty) \to \mathbb{R}$ be the solution of

$$\Delta u = \frac{\partial u}{\partial t}, \qquad x \in H^+ \setminus B, \quad t > 0, \tag{1.16}$$

with initial condition

$$u(x;0) = 0, \qquad x \in H^+ \setminus B, \tag{1.17}$$

and with the following boundary conditions. On ∂B we impose Dirichlet conditions,

$$u(x;t) = 1, \qquad x \in \partial B, \quad t > 0, \tag{1.18}$$

and on N we impose insulating Neumann boundary conditions,

$$\frac{\partial u}{\partial x_1}(x;t) = 0, \qquad x \in N, \quad t > 0. \tag{1.19}$$

We denote the total heat flow from B into $H^+ \setminus B$ up to time t by

$$\widetilde{E}_B(t) = \int_{H^+ \setminus B} u(x;t)\, dx. \tag{1.20}$$

In [1] we proved that (1.7) remains correct in this setting provided the contribution from the small balls in B is not too large.

Theorem 1.1. *Suppose B satisfies* (1.10), (1.11) *and* (1.15). *Suppose that if $m \geq 3$ and*

$$\sum_{i=1}^{\infty} r_i^{m-2} < \infty \tag{1.21}$$

or that if $m = 2$ and

$$\sum_{i=1}^{\infty} \left(\log \frac{2r_1}{r_i} \right)^{-1} < \infty, \tag{1.22}$$

then

$$\widetilde{E}_B(t) = 2\pi^{-1/2} A(\partial B) t^{1/2} + o(t^{1/2}), \qquad t \to 0, \tag{1.23}$$

where $A(\partial B)$ is the area of ∂B,

$$A(\partial B) = m\omega_m \sum_{i=1}^{\infty} r_i^{m-1}, \tag{1.24}$$

and where ω_m is the volume of a ball in $\mathbb{R}^m$ with radius 1. *If in addition there is a constant $\eta > 1$ such that*

$$B(c_i; \eta r_i) \cap B(c_j; \eta r_j) = \emptyset, \qquad i \neq j, \tag{1.25}$$

then for $m \geq 3$

$$\widetilde{E}_B(t) = 2\pi^{-1/2} A(\partial B) t^{1/2} + O(t), \qquad t \to 0, \tag{1.26}$$

and for $m = 2$

$$\widetilde{E}_B(t) = 2\pi^{-1/2} A(\partial B) t^{1/2} + O\left(t \log \frac{1}{t} \right), \qquad t \to 0. \tag{1.27}$$

The main result of this paper is the detailed analysis of an example where this condition is not satisfied. The example below was first introduced by Molchanov and Vainberg [9]. The model has two parameters $\alpha > 1$ and $\beta > 0$.

Let Q be a cuboid in H^+ such that one of the faces of Q is contained in N. Q is partitioned into layers $L_1, L_2, \ldots$ parallel to N, and such that L_1 is the toplayer in Q, and L_j is bounded by L_{j-1} and L_{j+1} for $j \geq 2$. The thickness of each layer L_j is $[j^\alpha]^{-1}$, $j = 1, 2, \ldots$. Each layer L_j is partitioned into $[j^\alpha]^{m-1}$ disjoint open cubes Q_{ij}, $i = 1, \ldots, [j^\alpha]^{m-1}$ of volume $[j^\alpha]^{-m}$.

Let c_{ij} be the center of cube Q_{ij}, and let

$$B_{\alpha,\beta} = \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{[j^\alpha]^{m-1}} B(c_{ij}; \varrho_j) \tag{1.28}$$

where $\{\varrho_j : j = 1, 2, \ldots\}$ is for $m \geq 3$ given by

$$\varrho_j = a j^{-\beta} \tag{1.29}$$

and is for $m = 2$ given by

$$\varrho_j = a e^{-j^\beta}. \tag{1.30}$$

It is easily seen that $B_{\alpha,\beta}$ satisfies (1.15) and (1.25) if

$$\beta \geq \alpha > 1 \qquad (m \geq 3), \tag{1.31}$$
$$\beta > 0, \quad \alpha > 1 \qquad (m = 2),$$

and if a is sufficiently small and positive.

The sum in (1.21) is finite if and only if

$$\beta > \frac{(m-1)\alpha+1}{m-2} \qquad (m \geq 3), \tag{1.32}$$

or

$$\beta > \alpha + 1 \qquad (m = 2). \tag{1.33}$$

Hence if $m \geq 3$ and α, β satisfy (1.31) and (1.32) then $\tilde{E}_{B_{\alpha,\beta}}(t)$ satisfies (1.26). Similarly if $m = 2$ and α, β satisfy (1.31) and (1.33) then $\tilde{E}_{B_{\alpha,\beta}}(t)$ satisfies (1.27).

Before we state the main results we introduce some further notation. For $m \geq 3$ we denote the Newtonian capacity of the ball $B(0;a)$ by

$$c_m(a) = C(B(0;a)) = 4\pi^{m/2}\left(\Gamma\left(\frac{m}{2}-1\right)\right)^{-1} a^{m-2}. \tag{1.34}$$

We denote by $\mu_m(a)$, $a < \frac{1}{2}$ the first eigenvalue of the Laplace operator on $L^2(Q_1 \setminus B(0;a))$ where Q_1 is a unit cube in $\mathbb{R}^m$ with center 0 and with Dirichlet boundary conditions on $\partial B(0;a)$ and with Neumann boundary conditions on ∂Q_1. In the remainder of this paper we put $B_{\alpha,\beta} = B$, and in the statements of the main results we will not repeat the conditions on α and on β made in (1.31).

Theorem 1.2. *Let $m \geq 3$. If*

$$\frac{(m-1)\alpha+1}{m-2} > \beta > \alpha + \frac{2}{m-2}, \tag{1.35}$$

then

$$\widetilde{E}_B(t) = 2\pi^{-1/2}(A(\partial B)+1)t^{1/2} + O(t^{(\alpha-1)/(m\alpha-(m-2)\beta)}). \tag{1.36}$$

If

$$\alpha + \frac{2}{m-2} > \beta > \alpha, \tag{1.37}$$

then

$$\widetilde{E}_B(t) = C_{\alpha,\beta,m} t^{(\alpha-1)/(m\alpha-(m-2)\beta)}(1+o(1)), \tag{1.38}$$

where

$$\begin{aligned} C_{\alpha,\beta,m} = (\alpha-1)^{-1}\Gamma\left(\frac{1+(m-1)\alpha-(m-2)\beta}{m\alpha-(m-2)\beta}\right) \\ \times c_m(a)^{(\alpha-1)/(m\alpha-(m-2)\beta)}. \end{aligned} \tag{1.39}$$

If

$$\beta = \frac{(m-1)\alpha+1}{m-2}, \tag{1.40}$$

then

$$\widetilde{E}_B(t) = 2\pi^{-1/2}(A(\partial B)+1)t^{1/2} + O\left(t\log\frac{1}{t}\right). \tag{1.41}$$

If $\beta = \alpha$ then

$$\begin{aligned} &\liminf_{t\to 0} t^{(1-\alpha)/(2\alpha)}\widetilde{E}_B(t) \\ &\geq (1-\omega_m a^m)(\alpha-1)^{-1}\Gamma\left(\frac{1+\alpha}{2\alpha}\right)\mu_m(a)^{(\alpha-1)/(2\alpha)}. \end{aligned} \tag{1.42}$$

Theorem 1.3. *Let $m = 2$. If*

$$\alpha + 1 > \beta > 2, \tag{1.43}$$

then

$$\widetilde{E}_B(t) = 2\pi^{-1/2}(A(\partial B) + 1)t^{1/2} + O(t^{(\alpha-1)/(2\alpha-\beta)}). \tag{1.44}$$

If

$$2 > \beta > 0, \tag{1.45}$$

then

$$\widetilde{E}_B(t) = D_{\alpha,\beta}\, t^{(\alpha-1)/(2\alpha-\beta)}(1 + o(1)), \tag{1.46}$$

where

$$D_{\alpha,\beta} = (\alpha - 1)^{-1}(2\pi)^{(\alpha-1)/(2\alpha-\beta)}\Gamma\left(\frac{1+\alpha-\beta}{2\alpha-\beta}\right). \tag{1.47}$$

If

$$\beta = \alpha + 1, \tag{1.48}$$

then

$$\widetilde{E}_B(t) = 2\pi^{-1/2}(A(\partial B) + 1)t^{1/2} + O\left(t \log \frac{1}{t}\right). \tag{1.49}$$

There are essentially two regimes. There is a middle regime (1.35) or (1.43) where the radii of the balls decrease less rapidly than in (1.21) or (1.22) respectively, but not as slowly as to destroy the leading $\sqrt{t}$ behaviour of Theorem 1.1. There is a bottom regime (1.37) or (1.45)where the radii of the balls decrease even more slowly, and where the small balls give rise to an anomalous behaviour of the heat content with an exponent strictly less than 1/2. The remaining regimes (1.40),(1.42) or (1.48) are intermediate regimes. Our methods of proof are too crude to reveal the precise leading asymptotic behaviour of the intermediate cases $\beta = \alpha + 2/(m-2)$, $(m \geq 3)$ or $\beta = 2$ $(m = 2)$. However, in both of these cases one obtains estimates which are comparable with $\sqrt{t}$.

The main tool in the proofs of these results is the probabilistic representation of the solution of (1.1)–(1.3) or of (1.16)–(1.19) respectively. Let $(B(s),\, s \geq 0 : \mathbb{P}_x,\, x \in \mathbb{R}^m)$ be a Brownian motion associated to Δ and let $(\widetilde{B}(s),\, s \geq 0 : \mathbb{P}_x,\, x \in H^+)$ be the reflected Brownian motion defined by

$$\widetilde{B}(s) = (|B_1(s)|, B_2(s), \dots, B_m(s)), \tag{1.50}$$

where $B_i(\cdot), i = 1, \dots, m$ are components of $B(\cdot)$. Let

$$T_K = \inf\{\, s \geq 0 : B(s) \in K \,\}. \tag{1.51}$$

The solution of (1.1)–(1.3) is given by

$$u(x;t) = \mathbb{P}_x[T_K \leq t]. \tag{1.52}$$

Similarly let

$$\widetilde{T}_B = \inf\{\, s \geq 0 : \widetilde{B}(s) \in B \,\}. \tag{1.53}$$

It follows from the reflection principle that

$$\widetilde{T}_B = T_{B \cup \widetilde{B}} \tag{1.54}$$

where $\tilde{B}$ is the set B reflected at the hyperplane N. By symmetry $\mathbb{P}_x[T_{B\cup\tilde{B}} \le t]$ satisfies the Neumann boundary conditions at N, and so

$$u(x;t) = \mathbb{P}_x[\widetilde{T}_B \le t] \tag{1.55}$$

is the solution of (1.16)–(1.19).

The interpretation of Theorems 1.2 and 1.3 is as follows. The heat flow from a ball with radius r is area dominated and given by (1.7) if $t \ll r^2$, and is capacity dominated and given by (1.5) (or (1.6) if $m = 2$) if $t \gg r^2$. The same holds true for all balls in B with the proviso that a ball in cube Q_{ij} in layer L_j will contribute at most $|Q_{ij}| = [j^\alpha]^{-m}$ to the heat content. The periodicity of the balls in layer L_j has in good approximation the effect of putting insulating Neumann boundary conditions on the boundaries of the cubes Q_{ij} in L_j.

Suppose $m \ge 3$ and

$$\beta \le \frac{(m-1)\alpha + 1}{m-2}. \tag{1.56}$$

The layers L_j which have temperature of order 1 are those for which

$$tc_m(\varrho_j) \ge [j^\alpha]^{-m}. \tag{1.57}$$

By (1.29), (1.34) and (1.57) it follows that the cubes in L_j have temperature of order 1 if $j \ge J$, where

$$J \asymp t^{-1/(m\alpha-(m-2)\beta)}. \tag{1.58}$$

The volume of all these layers with $j \ge J$ is of order

$$\sum_{j>J} [j^\alpha]^{-1} \asymp J^{1-\alpha} \asymp t^{(\alpha-1)/(m\alpha-(m-2)\beta)}. \tag{1.59}$$

We see that (1.59) dominates the heat flow of the balls in B for which $t \ll r^2$ if $(\alpha - 1)/(m\alpha - (m-2)\beta) < 1/2$ i.e. if (1.37) holds.

Suppose $m = 2$ and

$$\beta \le \alpha + 1. \tag{1.60}$$

The layers L_j which have temperature of order 1 are those for which

$$4\pi t\left(\log \frac{t}{\varrho_j^2}\right)^{-1} \ge [j^\alpha]^{-2}. \tag{1.61}$$

By (1.30) and (1.61) it follows that the cubes in L_j have temperature of order 1 if $j \ge J$, where

$$J \asymp t^{-1/(2\alpha-\beta)}. \tag{1.62}$$

The volume of these layers with $j \ge J$ is of order

$$J^{1-\alpha} \asymp t^{(\alpha-1)/(2\alpha-\beta)}. \tag{1.63}$$

We see that (1.63) dominates the heat flow of the balls in B for which $t \ll r^2$ if $(\alpha - 1)/(2\alpha - \beta) < 1/2$ i.e. if (1.45) holds.

So a reflecting Brownian motion starting in any of the layers $L_{J+1}, L_{J+2}, \dots$ has a probability of order 1 of hitting a ball in B before t. I.e. the Brownian motion is trapped. For $(\alpha-1)/(m\alpha-(m-2)\beta) > \frac{1}{2}$ (or $(\alpha-1)/(2\alpha-\beta) > \frac{1}{2}$ if $m = 2$) we have that the total volume of these trapping layers is $o(t^{1/2})$. However, we will show in the proof of Proposition 1.4 that trapping with probability 1 still occurs for all Brownian motions starting at $N \cap \partial Q$.

Proposition 1.4. *If $m \geq 3$ and β satisfies* (1.56) *or if $m = 2$ and β satisfies* (1.60) *then for all $x \in N \cap \partial Q$*

$$\mathbb{P}_x[\widetilde{T}_B = 0] = 1. \tag{1.64}$$

The effect of trapping at $N \cap \partial Q$ with probability 1 is the same as putting Dirichlet boundary 1 conditions on $N \cap \partial Q$. This is the cause of the additional $2\pi^{-1/2}t^{1/2}$ in both (1.36) and (1.44). A two-dimensional example where Brownian motion starting at $x \in H^+ \backslash B$ hits N with positive probability and is trapped with probability one was given by Gamelin and Lyons (Section 8 in [5]).

The phenomenon of trapping or solidification was discovered by Rauch and Taylor [11] in a different context. Consider n closed balls of equal radius r_n, evenly spaced in some region Ω of Euclidean space $\mathbb{R}^m$. The behaviour of nr_n^{m-2} as $n \to \infty$ $(m \geq 3)$ determines the efficiency of the heat flow from the n balls. If $nr_n^{m-2} \to 0$ then the heat flow becomes negligible and the obstacle set fades, while if $nr_n^{m-2} \to \infty$ the heat flow becomes infinitely efficient and the set Ω solidifies. A Brownian motion starting at $x \in \Omega$ will immediately hit the obstacle set. The relevant quantity for $m = 2$ is $n \log \frac{1}{r_n}$.

The obstacle set $B_{\alpha,\beta} = B$ is fixed and solidification occurs for $\alpha + 2/(m-2) > \beta \geq \alpha$ (or $2 > \beta > 0$ if $m = 2$) in a time dependent set, and with a probability of order 1. The non-trivial coefficients $C_{\alpha,\beta,m}$ and $D_{\alpha,\beta}$ in (1.38) and (1.46) respectively reflect the space average of this probability.

The proofs of these results are organized as follows. In Sections 2 and 3 we prove the lower and upper bounds of Theorems 1.2 and 1.3 respectively in the solidification regimes. In Section 4 we complete the proof of Theorems 1.2 and 1.3 in the intermediate regimes (1.35) and (1.43) respectively assuming the validity of Proposition 1.4. The proof of which is postponed until Section 5.

2. Lower bounds in the solidification regime

In this section we prove the lower bounds in Theorems 1.2 and 1.3 in the solidification regimes. The basic idea is a coarse graining argument of the cuboid Q. We partition Q as follows. Let $J_1 < J_2 < \cdots < J_k$ be positive integers, and let $M_0, \ldots, M_k$ be the collection of open, pairwise disjoint cuboids such that

$$\bigcup_{l=0}^{k} M_l \subset Q, \tag{2.1}$$

$$M_0 \supset \bigcup_{j=1}^{J_1} L_j, \tag{2.2}$$

$$M_l \supset \bigcup_{j=J_l+1}^{J_{l+1}} L_j, \qquad l = 1, \ldots, k-1, \tag{2.3}$$

$$M_k \supset \bigcup_{j>J_k} L_j. \tag{2.4}$$

Then

$$\begin{aligned}\widetilde{E}_B(t) &= \int_{H^+\setminus B} \mathbb{P}_x[\widetilde{T}_B \le t]\,dx \\ &\ge \sum_{l=0}^{k} \int_{M_l\setminus B} \mathbb{P}_x[\widetilde{T}_{B\cap M_l} \le t]\,dx \\ &\ge \sum_{l=0}^{k} \int_{M_l\setminus B} \mathbb{P}_x[T_{B\cap M_l} \le t]\,dx \\ &\ge \sum_{l=0}^{k} \int_{M_l\setminus B} \big(\mathbb{P}_x[T_{(B\cap M_l)\cup\partial M_l} \le t] - \mathbb{P}_x[T_{\partial M_l} \le t]\big)\,dx.\end{aligned} \tag{2.5}$$

The proof of the following can be found in [1].

Lemma 2.1. *Let K be a closed set in $\mathbb{R}^m$. Then for $t > 0$*

$$\mathbb{P}_x[T_K \le t]\,dx \le 2^{(2+m)/2} e^{-d(x,K)^2/(8t)}, \tag{2.6}$$

where

$$d(x,K) = \min\{\,|x-y| : y \in K\,\}. \tag{2.7}$$

By Lemma 2.1

$$\begin{aligned}\int_{M_l\setminus B} \mathbb{P}_x[T_{\partial M_l} \le t]\,dx &\le \int_{M_l} \mathbb{P}_x[T_{\partial M_l} \le t]\,dx \\ &\le 2^{(2+m)/2} \int_{M_l} e^{-d(x,\partial M_l)^2/(8t)}\,dx \\ &\le 2^{(2+m)/2} A(\partial M_l) \int_0^\infty e^{-r^2/(8t)}\,dr \\ &\le 2^{(2+m)/2} A(\partial Q)\cdot(2\pi t)^{1/2} \le k_1 t^{1/2},\end{aligned} \tag{2.8}$$

for some constant k_1 depending upon m only.

By (2.5) and (2.8)

$$\widetilde{E}_B(t) \ge \sum_{l=0}^{k} \int_{M_l\setminus B} \mathbb{P}_x[T_{(B\cap M_l)\cup\partial M_l} \le t]\,dx - (k+1)k_1 t^{1/2}. \tag{2.9}$$

Let $p_{M_l\setminus B}(x,y;t)$ denote the heat kernel on the open set $M_l \setminus B$ with Dirichlet boundary conditions on $(\partial B\cap M_l)\cup\partial M_l$. The spectrum of the corresponding Dirichlet Laplacian on $L^2(M_l \setminus B)$ is discrete and is denoted by $\lambda_{1,l} < \lambda_{2,l} \le \dots$ with a corresponding orthonormal set of eigenfunctions $\{\phi_{1,l}, \phi_{2,l}, \dots\}$. Then

$$p_{M_l\setminus B}(x,y;t) = \sum_{n=1}^{\infty} e^{-t\lambda_{n,l}} \phi_{n,l}(x)\phi_{n,l}(y). \tag{2.10}$$

By (2.10) and by Parseval's identity

$$\begin{aligned}\int_{M_l\setminus B}\mathbb{P}_x[T_{(B\cap M_l)\cup\partial M_l}\le t]\,dx &= |M_l\setminus B| - \int_{M_l\setminus B}\mathbb{P}_x[T_{(B\cap M_l)\cup\partial M_l}> t]\,dx\\ &= |M_l\setminus B| - \sum_{n=1}^{\infty} e^{-t\lambda_{n,l}}\Big(\int_{M_l\setminus B}\phi_{n,l}\Big)^2\\ &\ge |M_l\setminus B| - e^{-t\lambda_{1,l}}\sum_{n=1}^{\infty}\Big(\int_{M_l\setminus B}\phi_{n,l}\Big)^2\\ &= |M_l\setminus B|(1-e^{-t\lambda_{1,l}}).\end{aligned} \tag{2.11}$$

In Lemmas 2.2, 2.3 and 2.4 we obtain lower bounds for $\lambda_{1,l}$ in terms of J_l.

Lemma 2.2. *Let $m\ge 3$ and*

$$\alpha+\frac{2}{m-2}\ge\beta>\alpha. \tag{2.12}$$

Then for any $\varepsilon\in(0,1)$ there exists $J_1\in\mathbb{N}$ depending upon α, β, m, a and ε such that for all $l=1,\dots,k$

$$\lambda_{1,l}\ge c_m(a)(J_l+1)^{m\alpha-(m-2)\beta}(1-\varepsilon). \tag{2.13}$$

Proof. We use Dirichlet–Neumann bracketing [12] to obtain a lower bound. First we replace the Dirichlet boundary conditions on ∂M_l by Neumann boundary conditions. Secondly we insert additional Neumann boundary conditions on the boundaries of all cubes Q_{ij}, while retaining the Dirichlet boundary conditions on the balls. The bottom of the spectrum of the resulting mixed Laplace operator is, by scaling, equal to

$$[j^\alpha]^2\mu_m(aj^{-\beta}[j^\alpha]), \tag{2.14}$$

where $\mu_m(\delta)$ is the first eigenvalue of a unit cube Q_1 in $\mathbb{R}^m$ with Neumann boundary conditions on ∂Q_1 and with Dirichlet conditions on the boundary of the ball with radius δ centered at Q_1. Since $\beta>\alpha$, $j^{-\beta}[j^\alpha]\to 0$ as $j\to\infty$. It is a standard result ([4], see also [8] for a review) that

$$\mu_m(\delta)=c_m(1)\delta^{m-2}(1+o(1)),\qquad \delta\to 0. \tag{2.15}$$

Hence for $\varepsilon\in(0,1)$, there exists $J_1\in\mathbb{N}$ such that for all $j>J_1$

$$\mu_m(aj^{-\beta}[j^\alpha])\ge c_m(a)j^{-(m-2)\beta}j^{\alpha(m-2)}(1-\varepsilon), \tag{2.16}$$

and therefore the expression in (2.14) is bounded from below by

$$c_m(a)j^{m\alpha-(m-2)\beta}(1-\varepsilon). \tag{2.17}$$

By (2.12), $m\alpha-(m-2)\beta>0$. But $j\ge J_l+1\ge J_1+1$ for all $Q_{ij}\subset M_l$. Then (2.17) is bounded from below by the right hand side of (2.13). □

Lemma 2.3. *Let $\beta=\alpha$. Then for any $\varepsilon\in(0,1)$ there exists $J_1\in\mathbb{N}$ depending upon α, a and ε such that for all $l=1,\dots,k$*

$$\lambda_{1,l}\ge\mu_m(a)(J_l+1)^{2\alpha}(1-\varepsilon). \tag{2.18}$$

Proof. We follow the lines of the proof of Lemma 2.2 up to (2.14). Since $\beta = \alpha$ (2.14) becomes

$$[j^\alpha]^2 \mu_m(aj^{-\alpha}[j^\alpha]) \geq j^{2\alpha}(1-2j^{-\alpha})\mu_m(a(1-j^{-\alpha})). \tag{2.19}$$

For any $\varepsilon \in (0,1)$ there exists $J_1 \in \mathbb{N}$ such that for all $j > J_1$

$$2j^{-\alpha} < \frac{\varepsilon}{2}, \qquad \mu_m(a(1-j^{-\alpha})) > \mu_m(a)\left(1-\frac{\varepsilon}{2}\right). \tag{2.20}$$

Since $j \geq J_l + 1 \geq J_1 + 1$ for all Q_{ij} in M_l we arrive at (2.18) by (2.19) and (2.20). □

Lemma 2.4. *Let $m = 2$ and*

$$2 \geq \beta > 0. \tag{2.21}$$

Then for any $\varepsilon \in (0,1)$ there exists $J_1 \in \mathbb{N}$ depending on α, β, a and ε such that for all $l = 1,\dots,k$

$$\lambda_{1,l} \geq 2\pi(J_1+1)^{2\alpha-\beta}(1-\varepsilon). \tag{2.22}$$

Proof. Since the ball with Dirichlet boundary conditions in cube Q_{ij} has radius ae^{-j^β} we have that (2.14) is being replaced by

$$[j^\alpha]^2\mu_m(ae^{-j^\beta}[j^\alpha]). \tag{2.23}$$

For $m = 2$ it is well known [3] that

$$\mu_2(\delta) = 2\pi\left(\log\frac{1}{\delta}\right)^{-1}(1+o(1)), \qquad \delta \to 0. \tag{2.24}$$

Hence for $\varepsilon \in (0,1)$ there exists $J_1 \in \mathbb{N}$ such that for all $j > J_1$

$$\mu_2(ae^{-j^\beta}[j^\alpha]) \geq \mu_2(ae^{-j^\beta}) \geq 2\pi j^{-\beta}\left(1-\frac{\varepsilon}{2}\right), \tag{2.25}$$

$$[j^\alpha]^2 \geq j^{2\alpha}\left(1-\frac{\varepsilon}{2}\right). \tag{2.26}$$

By (2.21), $2\alpha - \beta \geq 2\alpha - 2 > 0$. Since $j \geq J_l + 1 \geq J_1 + 1$ for all $Q_{ij} \subset M_l$, we have that (2.23) is bounded from below by the right hand side of (2.22) by (2.25) and (2.26). □

In the following we choose $J_1,\dots,J_k$ such that each cuboid $M_1,\dots,M_k$ has approximately the same volume.

Lemma 2.5. *Let J_1 and k be positive integers such that*

$$\frac{J_1}{k} > 2(2^\alpha+1)(\alpha-1), \qquad J_1 \geq 2, \tag{2.27}$$

and let

$$J_l = \left[J_1\left(\frac{k+1}{k-l+2}\right)^{1/(\alpha-1)}\right], \qquad l = 2,\dots,k. \tag{2.28}$$

Then

$$J_1 < J_2 < \cdots < J_k, \tag{2.29}$$

and

$$|M_l| \geq (k+1)^{-1}(\alpha-1)^{-1}J_1^{1-\alpha} - (2^\alpha+1)J_1^{-\alpha}. \tag{2.30}$$

Proof. By (2.3) we have for $l = 2, \dots, k-1$

$$|M_l| \geq \sum_{j=J_l+1}^{J_{l+1}} [j^\alpha]^{-1} \geq \int_{J_l+1}^{J_{l+1}} [x^\alpha]^{-1}\, dx \geq (\alpha-1)^{-1}((J_l+1)^{1-\alpha} - J_{l+1}^{1-\alpha}). \tag{2.31}$$

By (2.28)

$$\begin{aligned}(J_l+1)^{1-\alpha} &\geq \left(J_1\left(\frac{k+1}{k-l+2}\right)^{1/(\alpha-1)} + 1\right)^{1-\alpha} \\ &\geq J_1^{1-\alpha}\left(\frac{k-l+2}{k+1}\right) - (\alpha-1)\left(\frac{k-l+2}{k+1}\right)^{\alpha/(\alpha-1)} J_1^{-\alpha} \\ &\geq J_1^{1-\alpha}\left(\frac{k-l+2}{k+1}\right) - (\alpha-1)J_1^{-\alpha},\end{aligned} \tag{2.32}$$

and for $J_1 \geq 2$

$$\begin{aligned}J_{l+1}^{1-\alpha} &\leq \left(J_1\left(\frac{k+1}{k-l+1}\right)^{1/(\alpha-1)} - 1\right)^{1-\alpha} \\ &\leq J_1^{1-\alpha}\left(\frac{k-l+1}{k+1}\right) + (\alpha-1)2^\alpha\left(\frac{k-l+1}{k+1}\right)^{\alpha/(\alpha-1)} J_1^{-\alpha} \\ &\leq J_1^{1-\alpha}\left(\frac{k-l+1}{k+1}\right) + (\alpha-1)2^\alpha J_1^{-\alpha}.\end{aligned} \tag{2.33}$$

Estimate (2.30) follows from (2.31)–(2.33). Finally note that $|M_l| > 0$ for all J_1 and k satisfying (2.27). Then each cuboid M_l contains at least one layer L_j. This implies (2.29). □

Proof of the lowerbound in (1.38) *for* $m \geq 3$ *and* β *satisfying*(1.37). By (2.9) and (2.11)

$$\begin{aligned}\widetilde{E}_B(t) &\geq \sum_{l=1}^{k} |M_l \setminus B|(1 - e^{-t\lambda_{1,l}}) - (k+1)k_1 t^{1/2} \\ &\geq \sum_{l=1}^{k} |M_l|(1 - e^{-t\lambda_{1,l}}) - \sum_{l=1}^{k} |M_l \cap B| - (k+1)k_1 t^{1/2}.\end{aligned} \tag{2.34}$$

By (1.37)

$$\sum_{l=1}^{k} |M_l \cap B| = \sum_{j=J_1+1}^{\infty} \omega_m a^m [j^\alpha]^{m-1} j^{-m\beta} \leq k_2 J_1^{1+(m-1)\alpha-m\beta}, \tag{2.35}$$

for some constant k_2 depending on α, β, a and m. By (2.34), (2.35), Lemma 2.2 and Lemma 2.5 we have for all $\varepsilon \in (0,1)$ and J_1 sufficiently large,

$$\begin{aligned}\widetilde{E}_B(t) \geq (k+1)^{-1}(\alpha-1)^{-1} J_1^{1-\alpha} &\sum_{l=1}^{k}\left(1 - e^{-(1-\varepsilon)tc_m(a)(J_l+1)^{m\alpha-(m-2)\beta}}\right) \\ &- (k+1)k_1 t^{1/2} - k_2 J_1^{1+(m-1)\alpha-m\beta} - k(1+2^\alpha)J_1^{-\alpha}.\end{aligned} \tag{2.36}$$

By (2.28)

$$\begin{aligned}&\sum_{l=1}^{k}\big(1-e^{-(1-\varepsilon)tc_m(a)(J_l+1)^{m\alpha-(m-2)\beta}}\big)\\&\geq\sum_{l=1}^{k}\big(1-e^{-(1-\varepsilon)tc_m(a)J_1^{m\alpha-(m-2)\beta}((k+1)/(k-l+2))^{(m\alpha-(m-2)\beta)/(\alpha-1)}}\big)\\&\geq\int_1^k dx\big(1-e^{-(1-\varepsilon)tc_m(a)J_1^{m\alpha-(m-2)\beta}((k+1)/(k-x+2))^{(m\alpha-(m-2)\beta)/(\alpha-1)}}\big)\\&=(k+1)\int_1^{(k+1)/2}dy\,y^{-2}\big(1-e^{-(1-\varepsilon)tc_m(a)J_1^{m\alpha-(m-2)\beta}y^{(m\alpha-(m-2)\beta)/(\alpha-1)}}\big),\end{aligned}\tag{2.37}$$

since the summand is monotonically increasing in l. An integration by parts gives

$$\begin{aligned}&\int_0^\infty\frac{dy}{y^2}\big(1-e^{-(1-\varepsilon)tc_m(a)J_1^{m\alpha-(m-2)\beta}y^{(m\alpha-(m-2)\beta)/(\alpha-1)}}\big)\\&\quad=\big((1-\varepsilon)tc_m(a)\big)^{\frac{\alpha-1}{m\alpha-(m-2)\beta}}J_1^{\alpha-1}\Gamma\bigg(\frac{1+(m-1)\alpha-(m-2)\beta}{m\alpha-(m-2)\beta}\bigg).\end{aligned}\tag{2.38}$$

Furthermore

$$\begin{aligned}&\int_{(k+1)/2}^\infty\frac{dy}{y^2}\big(1-e^{-(1-\varepsilon)tc_m(a)J_1^{m\alpha-(m-2)\beta}y^{(m\alpha-(m-2)\beta)/(\alpha-1)}}\big)\\&\quad\leq\int_{(k+1)/2}^\infty\frac{dy}{y^2}=\frac{2}{k+1}.\end{aligned}\tag{2.39}$$

Finally, since $m\alpha-(m-2)\beta>\alpha-1$

$$\begin{aligned}&\int_0^1\frac{dy}{y^2}\big(1-e^{-(1-\varepsilon)tc_m(a)J_1^{m\alpha-(m-2)\beta}y^{(m\alpha-(m-2)\beta)/(\alpha-1)}}\big)\\&\quad\leq(1-\varepsilon)tc_m(a)J_1^{m\alpha-(m-2)\beta}\int_0^1 dy\cdot y^{\frac{m\alpha-(m-2)\beta}{\alpha-1}-2}\\&\quad\leq k_3J_1^{m\alpha-(m-2)\beta}t,\end{aligned}\tag{2.40}$$

where k_3 depends on α, β, a and m. By (2.36)–(2.40) and (1.39)

$$\begin{aligned}\widetilde{E}_B(t)\geq{}&C_{\alpha,\beta,m}((1-\varepsilon)t)^{(\alpha-1)/(m\alpha-(m-2)\beta)}-(k+1)k_1t^{1/2}\\&-k_2J_1^{1+(m-1)\alpha-m\beta}-k(1+2^\alpha)J_1^{-\alpha}\\&-2(k+1)^{-1}(\alpha-1)^{-1}J_1^{1-\alpha}-k_3J_1^{1+(m-1)\alpha-(m-2)\beta}t.\end{aligned}\tag{2.41}$$

We make the following choices for J_1 and k:

$$J_1=[t^{-1/(m\alpha-(m-2)\beta)+\gamma}],\tag{2.42}$$

$$k=[t^{-1/4}J_1^{(1-\alpha)/2}],\tag{2.43}$$

where

$$0<\gamma<1/(m\alpha-(m-2)\beta).\tag{2.44}$$

Then, $J_1 \geq 2$ and J_1 satisfies the conditions in Lemma 2.2 for all t sufficiently small. Moreover,

$$\frac{J_1}{k} = t^{\frac{1}{4} - \frac{\alpha+1}{2(m\alpha-(m-2)\beta)} + \frac{(\alpha+1)\gamma}{2}}, \tag{2.45}$$

so that $J_1/k \to \infty$ as $t \to 0$ for

$$\gamma < \frac{(m-2)(\beta-\alpha)+2}{(\alpha+1)(m\alpha-(m-2)\beta)}. \tag{2.46}$$

We also have

$$kt^{1/2} \asymp k^{-1}J^{1-\alpha} \asymp t^{\frac{1}{4} + \frac{\alpha-1}{2(m\alpha-(m-2)\beta)} + \frac{(1-\alpha)\gamma}{2}}, \tag{2.47}$$

$$kJ_1^{-\alpha} \asymp t^{-\frac{1}{4}} J_1^{\frac{1-3\alpha}{2}} \asymp t^{-\frac{1}{4} + (\gamma - \frac{1}{m\alpha-(m-2)\beta})(\frac{1-3\alpha}{2})}, \tag{2.48}$$

$$J_1^{1+(m-1)\alpha-m\beta} \asymp t^{\frac{m\beta-(m-1)\alpha-1}{m\alpha-(m-2)\beta} + \gamma(1+(m-1)\alpha-m\beta)}, \tag{2.49}$$

$$tJ_1^{1+(m-1)\alpha-(m-2)\beta} \asymp t^{\frac{\alpha-1}{m\alpha-(m-2)\beta} + \gamma(1+(m-1)\alpha-(m-2)\beta)}. \tag{2.50}$$

It is easily seen that the terms in (2.47)–(2.50) are negligible compared to the first term in the right hand side of (2.41) if

$$\begin{aligned} &\gamma < \frac{(m-2)(\alpha-\beta)+2}{2(\alpha-1)(m\alpha-(m-2)\beta)}, \qquad \gamma < \frac{(m-2)(\beta-\alpha)+2}{2(3\alpha-1)(m\alpha-(m-2)\beta)}, \\ &\gamma < \frac{m(\beta-\alpha)}{(m\beta-(m-1)\alpha-1)(m\alpha-(m-2)\beta)}, \\ &\gamma(1+(m-1)\alpha-(m-2)\beta) > 0. \end{aligned} \tag{2.51}$$

Since β satisfies (1.37) the set of γ's satisfying (2.44), (2.46) and (2.51) is non-empty. Choosing such a γ gives

$$\liminf_{t\to 0} t^{(1-\alpha)/(m\alpha-(m-2)\beta)} \widetilde{E}_B(t) \geq (1-\varepsilon)^{(\alpha-1)/(m\alpha-(m-2)\beta)} C_{\alpha,\beta,m}. \tag{2.52}$$

Since $\varepsilon \in (0,1)$ was arbitrary we obtain

$$\liminf_{t\to 0} t^{(1-\alpha)/(m\alpha-(m-2)\beta)} \widetilde{E}_B(t) \geq C_{\alpha,\beta,m}. \tag{2.53}$$

□

Proof of the lower bound in (1.42) *for* $m \geq 3$ *and* $\beta = \alpha$. For $\beta = \alpha$ we have

$$|M_l \setminus B| = (1-\omega_m a^m)|M_l|. \tag{2.54}$$

By (2.9), (2.11), (2.54), Lemma 2.3 and Lemma 2.5 we have for all $\varepsilon \in (0,1)$ and J_1 sufficiently large

$$\begin{aligned} \widetilde{E}_B(t) \geq{}& (1-\omega_m a^m)(\alpha-1)^{-1} J_1^{1-\alpha}(k+1)^{-1} \sum_{l=1}^{k} \big(1 - e^{-(1-\varepsilon)t\mu_m(a)(J_l+1)^{2\alpha}}\big) \\ &- (k+1)k_1 t^{1/2} - k(1+2^\alpha)J_1^{-\alpha}. \end{aligned} \tag{2.55}$$

Following the steps from (2.37)–(2.40) we obtain

$$\begin{aligned}\widetilde{E}_B(t) \geq (1-\omega_m a^m)(\alpha-1)^{-1}\Gamma\left(\frac{\alpha-1}{2\alpha}\right)(\mu_m(a)(1-\varepsilon)t)^{(\alpha-1)/(2\alpha)} \\ -(k+1)k_1 t^{1/2} - k(1+2^\alpha)J_1^{-\alpha} \\ -2(\alpha-1)^{-1}(k+1)^{-1}J_1^{1-\alpha} - k_3 t J_1^{1+\alpha}.\end{aligned} \tag{2.56}$$

Replacing β by α in (2.42)–(2.48) and (2.50) we arrive at the corresponding first, second and fourth requirements for γ in (2.51) and for γ in (2.44), (2.46). This set of γ's is non-empty. We conclude that

$$\begin{aligned}\liminf_{t\to 0} t^{(1-\alpha)/(2\alpha)}\widetilde{E}_B(t) \\ \geq (1-\omega_m a^m)(\alpha-1)^{-1}\Gamma\left(\frac{1+\alpha}{2\alpha}\right)(\mu_m(a)(1-\varepsilon))^{(\alpha-1)/(2\alpha)}.\end{aligned} \tag{2.57}$$

Since $\varepsilon \in (0,1)$ was arbitrary we obtain (1.42). □

We note that the volume term (2.49) is absent in this case. Indeed, for $\beta = \alpha$ this term is of the same order as the leading term and had to be taken into account at an earlier stage (2.54). This accounts for the additional factor $(1-\omega_m a^m)$ in (1.42).

Proof of the lower bound in (1.46) *for* $m = 2$ *and* $2 > \beta > 0$. By (1.30)

$$\sum_{l=1}^{k} |M_l \cap B| = \sum_{j=J_1+1}^{\infty} \pi a^2 [j^\alpha] e^{-2j^\beta} \leq k_4 e^{-J_1^\beta}, \tag{2.58}$$

for some constant k_4 depending on α, β and a. By (2.9), (2.11), (2.58), Lemma 2.4 and Lemma 2.5 we have for all $\varepsilon \in (0,1)$ and J_1 sufficiently large

$$\begin{aligned}\widetilde{E}_B(t) \geq (\alpha-1)^{-1}J_1^{1-\alpha}(k+1)^{-1}\sum_{l=1}^{k}\left(1-e^{-2\pi(1-\varepsilon)t(J_l+1)^{2\alpha-\beta}}\right) \\ -(k+1)k_1 t^{1/2} - k(1+2^\alpha)J_1^{-\alpha} - k_4 e^{-J_1^\beta}.\end{aligned} \tag{2.59}$$

Following the steps from (2.37)–(2.40) we obtain

$$\begin{aligned}\widetilde{E}_B(t) \geq ((1-\varepsilon)t)^{(\alpha-1)/(2\alpha-\beta)}D_{\alpha,\beta} - (k+1)k_1 t^{1/2} - k(1+2^\alpha)J_1^{-\alpha} - k_4 e^{-J_1^\beta} \\ -2(\alpha-1)^{-1}(k+1)^{-1}J_1^{1-\alpha} - k_5 t J_1^{1+\alpha-\beta},\end{aligned} \tag{2.60}$$

for some constant k_5 depending on α, β. We choose

$$J_1 = [t^{-1/(2\alpha-\beta)+\gamma}], \tag{2.61}$$

$$k = [t^{-1/4}J_1^{(1-\alpha)/2}], \tag{2.62}$$

where

$$0 < \gamma < 1/(2\alpha-\beta). \tag{2.63}$$

Then, $J_1 \geq 2$ and J_1 satisfies the conditions in Lemma 2.4 for all t sufficiently small. By (2.61) and (2.62)

$$\frac{J_1}{k} \asymp t^{\frac{1}{4}-\frac{(1+\alpha)}{2(2\alpha-\beta)}+\frac{(1+\alpha)\gamma}{2}}. \tag{2.64}$$

Hence $\frac{J_1}{k} \to \infty$ as $t \to 0$ for

$$\gamma < \frac{\beta+2}{2(\alpha+1)(2\alpha-\beta)}. \tag{2.65}$$

We also have

$$kt^{1/2} \asymp k^{-1}J_1^{1-\alpha} \asymp t^{\frac{1}{4}+\frac{\alpha-1}{2(2\alpha-\beta)}+\frac{(1-\alpha)\gamma}{2}}, \tag{2.66}$$

$$kJ_1^{-\alpha} \asymp t^{-\frac{1}{4}}J_1^{(1-3\alpha)/2} \asymp t^{-\frac{1}{4}+\frac{3\alpha-1}{2(2\alpha-\beta)}+\frac{(1-3\alpha)\gamma}{2}}, \tag{2.67}$$

$$tJ_1^{1+\alpha-\beta} \asymp t^{(\alpha-1)/(2\alpha-\beta)+\gamma(1+\alpha-\beta)}. \tag{2.68}$$

It is easily seen that the terms in (2.66)–(2.68) are negligible compared to the first term in the right hand side of (2.60) if

$$\gamma < \frac{2-\beta}{2(\alpha-1)(2\alpha-\beta)}, \qquad \gamma < \frac{\beta+2}{2(3\alpha-1)(2\alpha-\beta)}, \qquad \gamma(1+\alpha-\beta) > 0. \tag{2.69}$$

The set of γ's satisfying (2.63), (2.65) and (2.69) is non-empty for $0 < \beta < 2$. Moreover the term in (2.58) is exponentially small. By taking first the limit $t \to 0$ and then $\varepsilon \to 0$ we obtain

$$\liminf_{t\to 0} t^{(1-\alpha)/(2\alpha-\beta)}\widetilde{E}_B(t) \geq D_{\alpha,\beta}. \tag{2.70}$$

□

3. Upper bounds in the solidification regime

In this section we prove the upper bounds in Theorem 1.2 and 1.3 in the solidification regimes. We denote by $\mathcal{B}_j$ the collection of balls in layer L_j, and we let ζ_j be the first coordinate of the centres of the balls in $\mathcal{B}_j$:

$$\zeta_j = \sum_{k=j+1}^{\infty}[j^\alpha]^{-1} + \frac{1}{2}[j^\alpha]^{-1}. \tag{3.1}$$

Furthermore we put

$$\xi(\alpha,\beta) = \begin{cases} m\alpha-(m-2)\beta, & m \geq 3, \\ 2\alpha-\beta, & m=2, \end{cases} \tag{3.2}$$

$$\tau = t^{(\alpha-1)/\xi(\alpha,\beta)}, \tag{3.3}$$

and for $s > 0$

$$Q(s) = \{\, x \in Q : x_1 < s\tau \,\}. \tag{3.4}$$

Lemma 3.1. *For $m \geq 2$*

$$\int_{H^+\setminus Q} \widetilde{\mathbb{P}}_x[\widetilde{T}_B \leq t]\,dx = o(\tau), \quad \tau \to 0, \tag{3.5}$$

and for any $\varepsilon > 0$ there exist $0 < A_0 < A$ such that

$$\int_{(Q(A_0)\cup Q(A)^c)\setminus B} \widetilde{\mathbb{P}}_x[\widetilde{T}_B \leq t]\,dx \leq \varepsilon\tau. \tag{3.6}$$

Proof. By the reflection principle and Lemma 2.1

$$\begin{aligned}\int_{H^+\setminus Q} \widetilde{\mathbb{P}}_x[\widetilde{T}_B \le t]\,dx &= \frac{1}{2}\int_{\mathbb{R}^m\setminus(Q\cup\widetilde{Q})} \mathbb{P}_x[T_{B\cup\widetilde{B}} \le t]\,dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^m\setminus(Q\cup\widetilde{Q})} \mathbb{P}_x[T_{Q\cup\widetilde{Q}} \le t]\,dx \\ &\le 2^{m/2}\int_{\mathbb{R}^m\setminus(Q\cup\widetilde{Q})} e^{-d(x,Q\cup\widetilde{Q})^2/(8t)}\,dx = O(t^{1/2}).\end{aligned} \tag{3.7}$$

But

$$t^{1/2} = \tau^{\xi(\alpha,\beta)/(2\alpha-2)} = O(\tau), \tag{3.8}$$

by (3.2) and (1.37) or (1.45) respectively. This proves (3.5).

To prove (3.6) we have first of all the trivial estimate

$$\int_{Q(A_0)\setminus B} \widetilde{\mathbb{P}}_x[\widetilde{T}_B \le t]\,dx \le |Q(A_0)\setminus B| \le |Q(A_0)| = A_0\tau. \tag{3.9}$$

We choose $A_0 = \varepsilon/2$. By the reflection principle and by subadditivity (see [1]) we have

$$\int_{Q(A)^c\setminus B} \widetilde{\mathbb{P}}_x[\widetilde{T}_B \le t]\,dx \le \sum_{\{j:\zeta_j\ge A\tau\}} [j^\alpha]^{m-1} E_{\varrho_j}(t), \tag{3.10}$$

where

$$E_a(t) = \int_{\mathbb{R}^m\setminus B(0;a)} \mathbb{P}_x[T_{B(0;a)} \le t]\,dx. \tag{3.11}$$

It follows from Lemma's 4.1, 4.2 in this paper (see also [1]) that there exist constants $C(m)$, $m \ge 2$ such that

$$E_a(t) \le \begin{cases} C(m)a^{m-1}t^{1/2}, & 0 < t \le 2a^2,\ m \ge 2, \\ C(m)a^{m-2}t, & t > 2a^2, \quad m \ge 3, \\ C(2)t(\log(t/a^2))^{-1}, & t > 2a^2, \quad m = 2. \end{cases} \tag{3.12}$$

Let

$$J_1 = \max\{\, j \in \mathbb{N} : \zeta_j \ge A\tau \,\}. \tag{3.13}$$

Then by (3.1) there exists $c_1 > 0$ such that

$$J_1 \le c_1(A\tau)^{1/(1-\alpha)} \tag{3.14}$$

for all τ sufficiently small.

We first consider the case $m \ge 3$. Then by (3.12)

$$\begin{aligned}&\sum_{\{j:\zeta_j\ge A\tau\}} [j^\alpha]^{m-1} E_{\varrho_j}(t) \\ &\quad\le C(m)\sum_{j=1}^{J_1}\{a^{m-1}j^{\alpha(m-1)-\beta(m-1)}t^{1/2} + a^{m-2}j^{\alpha(m-1)-\beta(m-2)}t\}.\end{aligned} \tag{3.15}$$

By (1.37), (3.3) and (3.13)

$$\sum_{j=1}^{J_1} j^{\alpha(m-1)-\beta(m-2)} t \le c_2 A^{\frac{1+\alpha(m-1)-\beta(m-2)}{1-\alpha}} \tau, \tag{3.16}$$

$$\sum_{j=1}^{J_1} j^{\alpha(m-1)-\beta(m-1)} t^{1/2} \le \begin{cases} c_3 A^{\frac{\beta(m-1)-\alpha(m-1)-1}{\alpha-1}} \tau^{\frac{\beta m-\alpha(m-2)-2}{2(\alpha-1)}}, \\ \qquad \alpha < \beta < \alpha + \frac{1}{m-1}, \\ c_4 \max\{1, \log(c_1 (A\tau)^{1/(1-\alpha)})\} \tau^{1+\frac{m}{2(\alpha-1)(m-1)}}, \\ \qquad \beta = \alpha + \frac{1}{m-1}, \\ c_5 \tau^{\frac{m\alpha-(m-2)\beta}{2\alpha-2}}, \\ \qquad \alpha + \frac{1}{m-1} < \beta < \alpha + \frac{2}{m-2}, \end{cases} \tag{3.17}$$

where $c_2, \ldots, c_5$ are constants depending on α, β and m respectively, but which are independent of A and of τ. It is straightforward to check that the right hand side of (3.17) is $o(\tau)$ for $\tau \to 0$, where the remainder is uniform for $A \ge A_0$. We now choose $A > A_0$ such that

$$c_2 A^{\frac{1+\alpha(m-1)-\beta(m-2)}{1-\alpha}} < \frac{\varepsilon}{4}. \tag{3.18}$$

By (3.16) and (3.17) we conclude that the right hand side of (3.15) is bounded by $\varepsilon\tau/2$ for all A satisfying (3.18) and all τ sufficiently small. This, together with (3.9) and the choice $A_0 = \varepsilon/2$, proves (3.6) for $m \ge 3$.

Finally we consider the case $m = 2$. By (3.12) and (1.29)

$$\begin{aligned} &\sum_{\{j:\zeta_j \ge A\tau\}} [j^\alpha] E_{\varrho_j}(t) \\ &\qquad \le \sum_{j=1}^{\infty} C(2) a j^\alpha e^{-j^\beta} t^{1/2} + \sum_{\{j:\zeta_j \ge A\tau, t > 2\varrho_j^2\}} C(2) t (\log(t/\varrho_j^2))^{-1}. \end{aligned} \tag{3.19}$$

The first term in the right hand side of (3.19) is $o(\tau)$ by (3.8). To estimate the second sum in the right hand side of (3.19) we put

$$J_2 = \min\{\, j \in \mathbb{N} : t > 2a^2 e^{-2j^\beta} \,\}. \tag{3.20}$$

Since

$$\log(t/\varrho_j^2) = \log(t/\varrho_{J_2}^2) + \log(\varrho_{J_2}^2/\varrho_j^2) \ge \log 2 + 2(j^\beta - J_2^\beta), \tag{3.21}$$

we have that the second sum in the right hand side of (3.19) is bounded by

$$\begin{aligned} &\frac{C(2)t}{\log 2} \sum_{j=J_2}^{2J_2} j^\alpha + \frac{C(2)t}{2} \sum_{\{j:j \ge 2J_2+1, j \le J_1\}} j^\alpha (j^\beta - J_2^\beta)^{-1} \\ &\qquad \le \frac{C(2)t}{\log 2} 2^\alpha J_2^{1+\alpha} + \frac{C(2)t}{2} \sum_{\{j:j \ge 2J_2+1, j \le J_1\}} j^{\alpha-\beta} (1 - 2^{-\beta})^{-1} \\ &\qquad \le \frac{C(2)t}{\log 2} 2^\alpha J_2^{1+\alpha} + \frac{C(2)t}{2} (1 - 2^{-\beta})^{-1} (1 + \alpha - \beta)^{-1} J_1^{1+\alpha-\beta}. \end{aligned} \tag{3.22}$$

The first term in the right hand side of (3.22) is $O(t(\log t^{-1})^{(1+\alpha)/\beta})$ which is $o(\tau)$ by (3.2), (3.3) and (3.20). The second term in (3.22) is bounded by

$$c_1^{1+\alpha-\beta}C(2)(2-2^{1-\beta})^{-1}(1+\alpha-\beta)^{-1}A^{\beta-\alpha-1}\tau. \tag{3.23}$$

We now choose $A > A_0$ such that the coefficient of τ in (3.23) is less than $\varepsilon/4$. It follows that (3.19) is bounded by $\varepsilon\tau/2$ for all τ sufficiently small. This, together with (3.6) and the choice $A_0 = \varepsilon/2$ completes the proof of Lemma 3.1. □

In order to do this, we split this part into M smaller slabs

$$Q_\ell \overset{\text{def}}{=} Q\left(A_0 + \frac{\ell}{M}(A-A_0)\right) \Big\backslash\, Q\left(A_0 + \frac{\ell-1}{M}(A-A_0)\right),\ \ell = 1,\dots,M, \tag{3.24}$$

and we split the integral accordingly into a sum of integrals. M in the end will be chosen large, such that the ensuing sum will be the Riemann sum approximation of the integral, defining the constants $C_{\alpha,\beta,m}$ and $D_{\alpha,\beta}$ in (1.39) and (1.47).

The midpoint of the balls in the j-th layer are in Q_ℓ if and only if

$$\left(A_0 + \frac{\ell-1}{M}(A-A_0)\right)\tau < \zeta_j \le \left(A_0 + \frac{\ell}{M}(A-A_0)\right)\tau.$$

We denote the minimal j such that ζ_j is in this interval by $j_{\min,\ell}(t)$, and $j_{\max,\ell}(t)$ is defined accordingly. These quantities of course depend also on A_0, A. Remark that for t small enough, balls $D \in \mathcal{B}_j$ with $j < j_{\min,\ell}(t) - 1$ or $j > j_{\max,\ell}(t)$ do not intersect Q_ℓ. We write $\mathcal{B}^{(\ell)}$ for the set of balls which do intersect Q_ℓ. (We tacitly require our inequalities to hold for t sufficiently small only). Define

$$\psi_{A_0,A,\ell,M}(t) = \frac{t^{-1/\xi(\alpha,\beta)}}{\left[(\alpha-1)\left(A_0 + \frac{\ell}{M}(A-A_0)\right)\right]^{1/(\alpha-1)}}.$$

Then for all $0 < A_0 < A$

$$\lim_{M\to\infty} \limsup_{t\to 0} \max_{\ell} \left| \frac{j_{\min,\ell}(t)}{\psi_{A_0,A,\ell,M}(t)} - 1 \right| = 0, \tag{3.25}$$

and similarly with $j_{\max,\ell}(t)$. This follows immediately from the definitions.

In the course of the proof, we need also slightly enlarged balls. For $D \in \mathcal{B}_j$ and $m \ge 3$ we choose D' with the same center as D such that the probability for a Brownian motion to ever hit D when starting outside D' is small. The be specific, we take the radius of D' as $j^{-\beta}\log j$. For $m = 2$ we choose the radius $\exp\left[-j^{\beta/2}\right]$ for $m = 2$. By choosing t small (which implies that j is large), we may assume that the D' are contained in R_D, the cube of the subdivision of Q with the same midpoint as D. In particular, the different D' do not overlap. We also choose $N \in \mathbb{N}$.

We sketch the basic strategy for proving our estimate. In the region we are interested in, the probability that the Brownian hits one of the balls up to time t is of order 1, but also bounded away from 1, at least if the starting point is not too close to one of the balls. Therefore, the probability that it hits one of the balls up to time t/N is small if N is large. If B is not hit up to time t/N, and the end point is still not close to one of the balls, e.g. is in no one of the D', then the Brownian motion gets the next small chance for a hit in the next time interval of length t/N, and so on. In this way, we get for the probability of a hit up to time t an exponential expression. This expression is essentially constant in our slabs Q_ℓ, and by choosing M large and summing, we obtain the desired Riemann sum approximation. For technical reasons, we also have to stay away from the boundary of whole box Q_ℓ, and for this reason, we choose a number η with

$$1/2 > \eta > \frac{\alpha - 1}{\xi(\alpha, \beta)}, \tag{3.26}$$

and define for $0 \le n \le N$

$$S_{n,\ell} = \left\{ x \in Q_\ell : d\left(x, Q_\ell^c\right) \ge \left(1 - \frac{n}{N}\right) t^\eta \right\},$$

$$B' = \bigcup_{D \in \mathcal{B}} D', \; S'_{n,\ell} = S_{n,\ell} \backslash B',$$

For the moment, we keep M, N, A_0, A, η fixed, and we therefore suppress them in the formulae. Define

$$\phi(n, \ell, t) = \inf\left\{ \mathbb{P}_x\left(T_B > t\right) : x \in S'_{n,\ell} \right\}. \tag{3.27}$$

Of course, this quantity depends also on N and on M.

The upper bound in Theorem 1.2 and Theorem 1.3 will follow from the following three lemmas.

Lemma 3.2.

$$\int_{Q_\ell \backslash B} \widetilde{\mathbb{P}}_x\left(T_B \le t\right) \, dx \le \left|Q_\ell\right| \left(1 - \phi(0, \ell, t)\right) + o(\tau)$$

where $o(\tau)$ refers to $t \to 0$, for fixed M, N, A_0, A.

Lemma 3.3.

$$\phi(0, \ell, t) \ge \left[\rho_{N,\ell}(t) - \varepsilon_N(t)\right]^N$$

where

$$\lim_{t \to 0} t^{-k} \varepsilon_N(t) = 0,$$

for all $k > 0$, and

$$\rho_{N,\ell}(t) = \inf_{y \in S'_{N-1,\ell}} \mathbb{P}_y\left(T_B > t/N, \, B(t/N) \notin B'\right). \tag{3.28}$$

Lemma 3.4.

$$\liminf_{t \to 0} \rho_{N,\ell}(t) \ge 1 - \frac{1}{N}\left(1 + \delta(A_0, A, M)\right) z(\ell),$$

where

$$z(\ell) = c_m a^{m-2} \left[(\alpha - 1) \left(A_0 + \frac{\ell}{M}(A - A_0) \right) \right]^{-\xi(\alpha,\beta)/(\alpha-1)} \tag{3.29}$$

for $m \ge 3$. For $m = 2$, $z(\ell)$ is defined in the same way with $c_m a^{m-2}$ replaced by 2π. Furthermore

$$\lim_{M \to \infty} \delta(A_0, A, M) = 0$$

for all $0 < A_0 < A$

Before proving these lemmas, we show that they imply the desired upper bounds in Theorem 1.2 and Theorem 1.3.

Using $|Q_\ell| = (A - A_0)\,\tau/M$, we get

$$\begin{aligned}\limsup_{t\to 0}\frac{1}{\tau}\int_{Q_\ell\setminus K}\widetilde{\mathbb{P}}_x(T_B\le t)\,dx &\le \frac{A-A_0}{M}(1-\liminf_{t\to 0}\phi(0,\ell,t))\\ &\le \frac{A-A_0}{M}(1-\liminf_{t\to 0}[\rho_{N,\ell}(t)-\varepsilon_N(t)]^N)\\ &\le \frac{A-A_0}{M}\left(1-\left(1-\frac{1}{N}(1+\delta(A_0,A,M))z(\ell)\right)^N\right),\end{aligned}$$

where the first inequality is by Lemma 3.2, the second by Lemma 3.3, and the third by Lemma 3.4. This holds for all $N\in\mathbb{N}$, and therefore

$$\begin{aligned}&\limsup_{t\to 0}\frac{1}{\tau}\int_{Q(A)\setminus(Q(A_0)\cup B)}\widetilde{\mathbb{P}}_x\left(T_B\le t\right)\,dx\\ &\quad\le \frac{A-A_0}{M}\sum_{\ell=1}^{M}\left(1-\exp\left[-\left(1+\delta\left(A_0,A,M\right)\right)\,z\left(\ell\right)\right]\right).\end{aligned}$$

Letting first $M\to\infty$ with A_0 and A still fixed, and then letting $A_0\to 0$, and $A\to\infty$, we obtain by Lemma 3.1

$$\limsup_{t\to 0}\frac{1}{\tau}\int_{Q\setminus B}\widetilde{\mathbb{P}}_x\left(T_B\le t\right)\,dx\le\int_0^\infty\left(1-\exp\left[-c_m a^{m-2}\left[(\alpha-1)\,x\right]^{-\frac{\xi(\alpha,\beta)}{(\alpha-1)}}\right]\right)\,dx$$

($c_m a^{m-2}$ replaced by 2π for $m=2$). By an elementary substitution, the right hand side is $C_{\alpha,\beta,m}$ for $m\ge 3$, and $D_{\alpha,\beta}$ for $m=2$. Together with the lower bounds proved in Section 2, this completes the proof of Theorem 1.2 and Theorem 1.3 in the solidification regime. □

Proof of Lemma 3.2. We have

$$\begin{aligned}\int_{Q_\ell\setminus B}\widetilde{\mathbb{P}}_x\left(T_B\le t\right)\,dx &= |Q_\ell\setminus B| - \int_{Q_\ell\setminus B}\widetilde{\mathbb{P}}_x\left(T_B> t\right)\,dx\\ &\le |Q_\ell| - \phi\left(0,\ell,t\right)\left|S'_{0,\ell}\right| + o\left(\tau\right),\end{aligned}$$

where the $o\left(\tau\right)$-correction comes from replacing $\widetilde{\mathbb{P}}_x$ by $\mathbb{P}_x$. As

$$\left|S'_{0,\ell}\right|\ge|Q_\ell|-Ct^\eta-\sum_{D\in\mathcal{B}^{(\ell)}}|D'|=|Q_\ell|-o\left(\tau\right),$$

we get

$$\int_{Q_\ell\setminus K}\widetilde{\mathbb{P}}_x\left(T_B\le t\right)\,dx\le|Q_\ell|\left(1-\phi\left(0,\ell,t\right)\right)+o\left(\tau\right). \tag{3.30}$$

□

Proof of Lemma 3.3. For $0 \leq n < N$ and $x \in S'_{n,\ell}$ we have

$$\begin{aligned}
\mathbb{P}_x(T_B > t) &= \int \mathbb{P}_x(T_B > t/N,\, B(t/N) \in dy)\, \mathbb{P}_y(T_B > t - t/N) \\
&\geq \int_{S'_{n+1,\ell}} \mathbb{P}_x(T_B > t/N,\, B(t/N) \in dy)\, \mathbb{P}_y(T_B > t - t/N) \\
&\geq \phi(n+1, \ell, t - t/N) \mathbb{P}_x\left(T_B > t/N,\, B(t/N) \in S'_{n+1,\ell}\right) \qquad (3.31)\\
&\geq \phi(n+1, \ell, t - t/N) \\
&\times \Big(\inf_{x \in S'_{N-1,\ell}} \mathbb{P}_x(T_B > t/N,\, B(t/N) \notin B') \\
&\qquad - \sup_{x \in S_{n,\ell}} \mathbb{P}_x(B(t/N) \notin S_{n+1,\ell}) \Big).
\end{aligned}$$

Let

$$\varepsilon_N(t) \stackrel{\text{def}}{=} \sup_{1 \leq n < N} \max_{1 \leq \ell \leq M} \sup_{x \in S_{n,\ell}} \mathbb{P}_x(B(t/N) \notin S_{n+1,\ell}).$$

By our choice $\eta < 1/2$, we have

$$\begin{aligned}
\varepsilon_N(t) &\leq C \left(\frac{t}{N}\right)^{-m/2} \exp\left[-\frac{1}{4}\left(\frac{t^\eta}{N}\right)^2 \left(\frac{t}{N}\right)^{-1}\right] \qquad (3.32)\\
&= C N^{m/2} t^{-m/2} \exp\left[-\frac{1}{4N} t^{2(\eta - 1/2)}\right],
\end{aligned}$$

which decays faster than polynomially in t when $t \to 0$. With $\rho_{N,\ell}(t)$ from (3.28) and (3.31), we get for $0 \leq n < N$

$$\phi(n, \ell, t) \geq \phi(n+1, \ell, t - t/N)\left(\rho_{N,\ell}(t) - \varepsilon_N(t)\right),$$

and iterating this we obtain

$$\begin{aligned}
\phi(0, \ell, t) &\geq \phi(N, \ell, 0)\left[\rho_{N,\ell}(t) - \varepsilon_N(t)\right]^N \qquad (3.33)\\
&= \left[\rho_{N,\ell}(t) - \varepsilon_N(t)\right]^N.
\end{aligned}$$

□

The proof of Lemma 3.4 requires an additional result (Lemma 3.5). To formulate this, we consider a single disc $D \in \mathcal{B}^{(\ell)}$, fixed for the moment. For $m \geq 3$ let ν me the equilibrium distribution on D. For $m = 2$ we use a slight modification by replacing the Brownian motion by a Brownian motion with killing rate 1, and then we write again ν for the equilibrium distribution, i.e. the unique Radon measure on D (concentrated on ∂D) satisfying $\int \overline{g}(x, y)\, \nu(dy) = 1$ on D, where $\overline{g}(x, y) = \int_0^\infty \overline{p}(x, y; s)\, ds$, where $\overline{p}(x, y; s) = p(x, y; s)$ for $m \geq 3$, and $\mathrm{e}^{-s} p(x, y; s)$ for $m = 2$. For $m \geq 3$, $\nu(\partial D)$ is the Newtonian capacity of $D : \nu(\partial D) = c_m(r(D)) = c_m r(D)^{m-2}$, where $c_m = c_m(1)$. For $m = 2$, if r is the radius of D, $\gamma(r) = \nu(\partial D)$ satisfies

$$\gamma(r) \int_0^\infty \mathrm{e}^{-s} \frac{1}{4\pi s} \mathrm{e}^{-r^2/4s} ds = 1.$$

¿From this relation, we obtain

$$\lim_{r\to 0} \frac{\gamma(r)\log(1/r)}{2\pi} = 1. \tag{3.34}$$

Lemma 3.5.

$$\lim_{t\to 0} \sup_{x\in S'_{N-1,\ell}} \int_0^{t/N} ds \sum_{D\in\mathcal{B}^{(\ell)}} \left| \int_{\partial D} \nu(dy)\, \overline{p}(x,y;s) - \frac{\nu(\partial D)}{|R_D|} \int_{R_D} \overline{p}(x,y;s)\, dy \right| = 0. \tag{3.35}$$

(Recall that $\mathcal{B}^{(\ell)}$ depends on t. Also R_D is the cube of our basic subdivision of Q with the same midpoint m_D as D).

Proof. We have to distinguish between balls which are close to x and those which are not. We first consider the balls D in B where $x \in S'_{N-1,\ell}$ and

$$|x - m_D| \leq \operatorname{diam}(R_D). \tag{3.36}$$

For such a ball, we use the following estimates in the case $m \geq 3$:

$$\begin{aligned} \int_0^{t/N} ds \int_{\partial D} \nu(dy)\; p(x,y,s) &\leq C \int_{\partial D} \nu(dy)\; |x-y|^{-m+2} \\ &\leq C(\log j)^{-m+2}\; j^{\beta(m-2)} \nu(\partial D) \\ &= o(1), \end{aligned}$$

and

$$\begin{aligned} \int_0^{t/N} ds \frac{r(D)^{m-2}}{|R_D|} \int_{R_D} dy\, p(x,y,s) &\leq \frac{r(D)^{m-2}}{|R_D|} \int_{R_D} dy\; |x-y|^{-m+2} \\ &\leq \frac{r(D)^{m-2}}{|R_D|}\, j^{-2\alpha} = o(1), \end{aligned}$$

since $\beta > \alpha$. For $m = 2$ we use the estimate

$$\int_0^\infty \mathrm{e}^{-s} p(x,y;s)\, ds \leq C \frac{1}{\log|x-y|}$$

for $|x-y| < 1$. Therefore

$$\int_0^{t/N} ds \int_{\partial D} \nu(dy)\; \overline{p}(x,y;s) \leq C j^{-\beta/2} = o(1),$$

and

$$\int_0^{t/N} ds \frac{\nu(\partial D)}{|R_D|} \int_{R_D} dy\, \overline{p}(x,y;s) = o(1).$$

For fixed x, the number of boxes which satisfy (3.36) is bounded. We are therefore left with estimating in the sum $\sum_{D\in\mathcal{B}^{(\ell)}}$ in (3.35) only those balls which satisfy $|x - m_D| > \operatorname{diam}(R_D)$. In this case, we have the elementary estimates $|x-y| \leq |x-m_D| \leq$

$2|x-y|$ and

$$\begin{aligned}\delta_D(x,y) &= |\overline{p}(x,y;s)-\overline{p}(x,m_D;s)| \le |p(x,y;s)-p(x,m_D;s)| \\ &= \frac{1}{(4\pi s)^{m/2}}|\mathrm{e}^{-|x-y|^2/(8s)}-\mathrm{e}^{-|x-m_D|^2/(8s)}|(\mathrm{e}^{-|x-y|^2/(8s)}+\mathrm{e}^{-|x-m_D|^2/(8s)}) \\ &\le \frac{1}{(4\pi s)^{m/2}\,4s}\,|y-m_D|\,(|x-y|+|x-m_D|)\,\mathrm{e}^{-|x-y|^2/(8s)} \\ &\le C\frac{1}{\sqrt{s}}\,\mathrm{diam}\,(R_D)\,p\,(x,y;4s)\end{aligned}$$

for $y \in R_D$. Using this, we get

$$\begin{aligned}&\int_0^{t/N} ds \sum_{\substack{D\in\mathcal{B}^{(\ell)} \\ d(x,R_D)>j^{-\alpha}}} \left|\int_{\partial D}\nu\,(dy)\,\overline{p}\,(x,y,s)-\frac{\nu\,(\partial D)}{|R_D|}\int_{R_D}\overline{p}\,(x,y,s)\,dy\right| \\ &\le \int_0^{t/N} ds \sum_{\substack{D\in\mathcal{B}^{(\ell)} \\ d(x,R_D)>j^{-\alpha}}} \left\{\int_{\partial D}\nu\,(dy)\ \delta_D\,(x,y)+\frac{\nu\,(\partial D)}{|R_D|}\int_{R_D}dy\,\delta_D\,(x,y)\right\} \\ &\le Cj_{\min,\ell}^{-\alpha}\int_0^{t/N}\frac{1}{\sqrt{s}}\sum_{D\in\mathcal{B}^{(\ell)}}\frac{\nu\,(\partial D)}{|R_D|}\int_{R_D}dy\,p\,(x,y;4s) \\ &\le Cj_{\min,\ell}^{-\alpha}\sqrt{\frac{t}{N}}j_{\max,\ell}^{\alpha m}\max_{D\in\mathcal{B}^{(\ell)}}\nu\,(\partial D)=o\,(1)\,,\end{aligned}$$

because $\beta>\alpha$. This implies the lemma. □

Proof of Lemma 3.4. If $x \in S'_{N-1,\ell}$, then

$$\begin{aligned}\mathbb{P}_x\,(T_B \le t/N,\ B\,(t/N)\notin B') &\le \sum_{D\in\mathcal{B}}\mathbb{P}_x\,(T_D\le t/N,\ B\,(t/N)\notin D') \\ &\le \sum_{D\in\mathcal{B}^{(\ell)}}\mathbb{P}_x\,(T_D\le t/N,\ B\,(t/N)\notin D')+o\,(\tau)\end{aligned}$$

For $m=2$, we replace the Brownian motion by a Brownian motion with killing at rate 1. The correction for this replacement is $O\,(t)=o\,(\tau)$ which can be neglected. We tacitly assume for the rest of the proof that for $m=2$, $B\,(\cdot)$ is such a Brownian motion with killing. Let λ_D be the last exit time of the Brownian motion from D.

$$\begin{aligned}\mathbb{P}_x\,(T_D\le t/N,\ B\,(t/N)\notin D') &\le \frac{\int_{D'^c}\mathbb{P}_x\,(T_D\le t/N,\ B\,(t/N)\in dy)\,\mathbb{P}_y\,(T_D=\infty)}{\inf_{y\notin D'}\mathbb{P}_y\,(T_D=\infty)} \\ &= \frac{\mathbb{P}_x\,(\lambda_D\le t/N,\ B\,(t/N)\notin D')}{\inf_{y\notin D'}\mathbb{P}_y\,(T_D=\infty)} \\ &\le \frac{\mathbb{P}_x\,(\lambda_D\le t/N)}{\inf_{y\notin D'}\mathbb{P}_y\,(T_D=\infty)},\end{aligned} \tag{3.37}$$

For $m\ge 3$, we have $r\,(D')\,/r\,(D)\to\infty$ for $D\in\mathcal{B}^{(\ell)}$, $t\to 0$, and therefore $\inf_{y\notin D'}P_y\,(T_D=\infty)=1-o\,(1)$. The same relation is easily checked for $m=2$

(because of the killing). From this we get

$$\begin{aligned} \mathbb{P}_x\left(T_D \leq t/N, B\left(t/N\right) \notin D'\right) &\leq \left(1+o\left(1\right)\right)\mathbb{P}_x\left(\lambda_D \leq t/N\right) \\ &= \left(1+o\left(1\right)\right)\int_0^{t/N}\int_{\partial D}\nu(dy)\overline{p}(x,y,s)ds, \end{aligned} \tag{3.38}$$

where $\nu(dy)$ is the equilibrium distribution on ∂D. From (3.34) we get for $m=2$

$$\nu\left(\partial B\right) = 2\pi j^{-\beta}\left(1+o\left(1\right)\right),$$

Using (3.37), and Lemma 3.5, we get

$$\begin{aligned} &\sum_{D\in\mathcal{B}^{(\ell)}} P_x\left(T_D \leq t/N, B\left(t/N\right) \notin D'\right) \\ &\leq \sum_{D\in\mathcal{B}^{(\ell)}} \frac{\nu\left(\partial D\right)}{\left|R_D\right|}\int_0^{t/N} ds\int_{R_D} dy\, p\left(x,y;s\right) + o\left(1\right). \end{aligned}$$

For $m \geq 3$, this is

$$\leq c_m a^{m-2} j_{\min,\ell}^{-\beta(m-2)} j_{\max,\ell}^{\alpha m}\frac{t}{N} + o\left(1\right),$$

and for $m=2$

$$\leq 2\pi j_{\min,\ell}^{-\beta} j_{\max,\ell}^{2\alpha}\frac{t}{N} + o\left(1\right),$$

According to (3.25), we have that for $m \geq 3$

$$j_{\min,\ell}^{-\beta(m-2)} j_{\max,\ell}^{\alpha m} \leq \frac{1}{t}\left[\left(\alpha-1\right)\left(A_0 + \frac{\ell}{M}\left(A-A_0\right)\right)\right]^{-\frac{\xi(\alpha,\beta)}{(\alpha-1)}}\left(1+\delta\left(A_0,A,M\right)\right),$$

where $\lim_{M\to\infty}\delta\left(A_0,A,M\right)=0$, for all A, A_0. For $m=2$, the left hand side of the above expression is replaced by $j_{\min,\ell}^{-\beta} j_{\max,\ell}^{2\alpha}$. Therefore,

$$\sum_{D\in\mathcal{B}^{(\ell)}} \mathbb{P}_x\left(T_D \leq t/N,\, B\left(t/N\right) \notin D'\right) \leq \frac{1}{N}\left(1+\delta\left(A_0,A,M\right)\right)z\left(\ell\right) + o\left(1\right),$$

where $z\left(\ell\right)$ was defined in (3.29), and therefore

$$\rho_{N,\ell}\left(t\right) \geq \inf_{y\in S'_{N-1,\ell}} P_y\left(B\left(t/N\right)\notin D'\right) - \frac{1}{N}\left(1+\delta\left(A_0,A,M\right)\right)\, z\left(\ell\right) - o\left(1\right).$$

Remark now, that for any N, L, M

$$\liminf_{t\to 0}\inf_{y\in S'_{N-1,\ell}} \mathbb{P}_y\left(B\left(t/N\right)\notin D'\right) = 1.$$

This easily follows from the fact that the variance of $B\left(t/N\right)$ is of order $\sqrt{t}$ which is much larger than the diameter of the cubes R_D, for $t\to 0$, and that by our assumptions, the volume of the balls is negligible compared with that of the R_D.

Therefore, the lemma is proved. □

4. Trapping at the Neumann boundary

In this section we prove Theorems 1.2 and 1.3 for the regimes (1.35), (1.40) and (1.43), (1.48) respectively. The lower bound will depend on Proposition 1.4 which we will prove in Section 5. To prove the upper bounds we need some results which were proved in [1].

Lemma 4.1. *Let $m \geq 2$. Then for all $t > 0$*

$$E_{B(c;r)}(t) \leq 2\pi^{-1/2} A(\partial B(c;r)) t^{1/2} + k_6 r^{m-2} t, \tag{4.1}$$

where k_6 depends upon m only.

Lemma 4.2. *Let $m = 2$. Then for $t \geq 2r^2$*

$$E_{B(c;r)}(t) \leq 4\pi e^2\, t\, (\log(t/r^2))^{-1}. \tag{4.2}$$

To prove the upper bounds we enclose the balls in layers $\bigcup_{j>J} L_j$ in a cuboid C_J whose boundary and interior will be put at temperature 1. We then use (4.1), (4.2) to obtain an upper bound for the heat content from the remaining balls in layers $L_1, \dots, L_J$. We then minimise over J.

For $J \in \mathbb{N}$ let

$$C_J = \Big\{ x \in H^+ : 0 < x_1 < \sum_{j>J} [j^\alpha]^{-1}, (x_2, \dots, x_m) \in N \cap \partial Q \Big\}. \tag{4.3}$$

So C_J contains the layers $L_{J+1}, L_{J+2}, \dots$. By subadditivity and monotonicity

$$\begin{aligned} \widetilde{E}_B(t) &\leq |C_J| + \int_{H^+ \setminus (B \cup C_J)} \mathbb{P}_x[\widetilde{T}_B \leq t]\, dx \\ &\leq |C_J| + \int_{H^+ \setminus C_J} \mathbb{P}_x[\widetilde{T}_{C_J} \leq t]\, dx + \int_{H^+ \setminus (B \setminus C_J)} \mathbb{P}_x[\widetilde{T}_{B \setminus C_J} \leq t]\, dx. \end{aligned} \tag{4.4}$$

Lemma 4.3. *Let K be a closed subset of H^+. Then for all $t > 0$*

$$\widetilde{E}_K(t) \leq E_K(t). \tag{4.5}$$

Proof. By (1.54), symmetry, subadditivity and monotonicity

$$\begin{aligned} \widetilde{E}_K(t) &= \frac{1}{2} \int_{\mathbb{R}^m \setminus (K \cup \widetilde{K})} \mathbb{P}_x[T_{K \cup \widetilde{K}} \leq t]\, dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^m \setminus (K \cup \widetilde{K})} (\mathbb{P}_x[T_K \leq t] + \mathbb{P}_x[T_{\widetilde{K}} \leq t])\, dx \\ &= \int_{\mathbb{R}^m \setminus (K \cup \widetilde{K})} \mathbb{P}_x[T_K \leq t]\, dx \\ &\leq \int_{\mathbb{R}^m \setminus K} \mathbb{P}_x[T_K \leq t]\, dx = E_K(t). \end{aligned} \tag{4.6}$$

□

Since B is a relatively closed subset of H^+, and C_J is an open subset of H^+, $B\setminus C_J$ is a closed subset of H^+. By Lemma 4.3 and subadditivity

$$\int_{H^+\setminus(B\setminus C_J)} \mathbb{P}_x[\widetilde{T}_{B\setminus C_J} \le t]\,dx \le \int_{\mathbb{R}^m\setminus(B\setminus C_J)} \mathbb{P}_x[T_{B\setminus C_J} \le t]\,dx \le \sum_{j=1}^{J} [j^\alpha]^{m-1} E_{B(0,\varrho_j)}(t). \tag{4.7}$$

Lemma 4.4. *Let $m \ge 2$. Then for all $t > 0$ and all $J \in \mathbb{N}$*

$$\int_{H^+\setminus C_J} \mathbb{P}_x[\widetilde{T}_{C_J} \le t]\,dx \le 2\pi^{-1/2}t^{1/2} + k_7 J^{1-\alpha}t^{1/2} + O(t). \tag{4.8}$$

Proof. Let $\widehat{Q}$ be the subset of H^+ with $(x_2,\dots,x_m) \in N \cap \partial Q$. By (1.54)

$$\begin{aligned}\int_{H^+\setminus C_J} &\mathbb{P}_x[\widetilde{T}_{C_J} \le t]\,dx \\ &\le \int_{H^+\setminus C_J} \mathbb{P}_x[T_{C_J\cup\widetilde{C}_J} \le t]\,dx \\ &\le \int_{(H^+\cap\widehat{Q})\setminus C_J} \mathbb{P}_x[T_{C_J\cup\widetilde{C}_J} \le t]\,dx + \int_{H^+\setminus\widehat{Q}} \mathbb{P}_x[T_{C_J\cup\widetilde{C}_J} \le t]\,dx.\end{aligned} \tag{4.9}$$

On the set $H^+\setminus\widehat{Q}$ we use Lemma 2.1 to obtain an upper bound. It is straightforward to show that

$$|\{\, x \in H^+ \setminus \widehat{Q} : d(x, C_J \cup \widetilde{C}_J) < r \,\}| \le 4^m(r + r^{m-1})\left(\sum_{j>J}[j^\alpha]^{-1} + r\right). \tag{4.10}$$

By Lemma 2.1 and (4.10)

$$\begin{aligned}\int_{H^+\setminus\widehat{Q}} &\mathbb{P}_x[T_{C_J\cup\widetilde{C}_J}]\,dx \\ &\le 2^{(3m-2)/2}t^{-1}\int_0^\infty e^{-r^2/(8t)}(r^2 + r^m)\left(\sum_{j>J}[j^\alpha]^{-1} + r\right)dr \\ &\le k_7 J^{1-\alpha}t^{1/2} + O(t),\end{aligned} \tag{4.11}$$

where k_7 depends on m and α only. For $x \in (H^+\cap\widehat{Q})\setminus C_J$ we use

$$\mathbb{P}_x[T_{C_J\cup\widetilde{C}_J} \le t] \le (\pi t)^{-1/2}\int_{d(x,C_J)}^\infty e^{-r^2/(4t)}\,dr, \tag{4.12}$$

to obtain that

$$\int_{(H^+\cap\widehat{Q})\setminus C_J} \mathbb{P}_x[T_{C_J\cup\widetilde{C}_J} \le t]\,dx \le 2\pi^{-1/2}A(N\cap\partial Q)t^{1/2} = 2\pi^{-1/2}t^{1/2}. \tag{4.13}$$

The Lemma follows from (4.9),(4.11) and (4.13). $\square$

Proof of the upper bound for $m \geq 3$. By Lemma 4.1 we have

$$\begin{aligned}\sum_{j=1}^{J}[j^{\alpha}]^{m-1}&E_{B(0;\varrho_j)}(t)\\ &\leq \sum_{j=1}^{J}[j^{\alpha}]^{m-1}\big(2\pi^{-1/2}A(\partial B(0;\varrho_j))t^{1/2}+k_6(aj^{-\beta})^{m-2}t\big)\\ &\leq 2\pi^{-1/2}A(\partial B)t^{1/2}+k_6a^{m-2}t\sum_{j=1}^{J}j^{(m-1)\alpha-(m-2)\beta}.\end{aligned} \tag{4.14}$$

Suppose β satisfies (1.35). Then for $J \to \infty$

$$t\sum_{j=1}^{J}j^{(m-1)\alpha-(m-2)\beta} \asymp tJ^{1+(m-1)\alpha-(m-2)\beta}, \tag{4.15}$$

$$|C_J| = \sum_{j>J}[j^{\alpha}]^{-1} \asymp J^{1-\alpha}. \tag{4.16}$$

We choose

$$J = \big[t^{-1/(m\alpha-(m-2)\beta)}\big]. \tag{4.17}$$

Then

$$J^{1-\alpha} \asymp tJ^{1+(m-1)\alpha-(m-2)\beta} = O\big(t^{(\alpha-1)/(m\alpha-(m-2)\beta)}\big), \tag{4.18}$$

$$J^{1-\alpha}t^{1/2} = o\big(t^{(\alpha-1)/(m\alpha-(m-2)\beta)}\big), \tag{4.19}$$

By (4.4), Lemma 4.4, (4.14)–(4.19) we obtain for β satisfying (1.35)

$$\widetilde{E}_B(t) \leq 2\pi^{-1/2}(A(\partial B)+1)t^{1/2}+O\big(t^{(\alpha-1)/(m\alpha-(m-2)\beta)}\big). \tag{4.20}$$

Suppose β satisfies (1.40). Then for $J \to \infty$

$$t\sum_{j=1}^{J}j^{(m-1)\alpha-(m-2)\beta} \asymp t\log J, \tag{4.21}$$

and

$$\widetilde{E}_B(t) \leq 2\pi^{-1/2}(A(\partial B)+1)t^{1/2}+O\bigg(t\log\frac{1}{t}\bigg), \tag{4.22}$$

by (4.4), Lemma 4.4, (4.14), (4.16)–(4.19) and (4.21).
Proof of the upper bound for $m = 2$. By Lemma 4.1, (4.7) and the fact that $\varrho_j \leq a$

$$\begin{aligned}\int_{H^+\setminus(B\setminus C_J)}\mathbb{P}_x[T_{B\setminus C_J}\leq t]\,dx \leq 2\pi^{-1/2}A(\partial B)t^{1/2}+k_6t\sum_{\{j:t<2a\varrho_j\}}[j^{\alpha}]\\ +4\pi e^2t\sum_{\{j\leq J,t\geq 2a\varrho_j\}}[j^{\alpha}](\log t/\varrho_j^2)^{-1}.\end{aligned} \tag{4.23}$$

By (1.30)

$$k_6t\sum_{\{j:t<2a\varrho_j\}}[j^{\alpha}] = O\Bigg(t\bigg(\log\frac{1}{t}\bigg)^{(1+\alpha)/\beta}\Bigg), \tag{4.24}$$

and

$$
\begin{aligned}
t \sum_{\{j\le J, t\ge 2a\varrho_j\}} [j^\alpha](\log t/\varrho_j^2)^{-1} &\le t \sum_{\{j\le J, t\ge 2a\varrho_j\}} [j^\alpha]\left(\log \frac{2a}{\varrho_j}\right)^{-1} \\
&\le t\sum_{j\le J} j^{\alpha-\beta}.
\end{aligned}
\tag{4.25}
$$

Suppose β satisfies (1.43). Then for $J \to \infty$

$$
t\sum_{j\le J} j^{\alpha-\beta} \asymp tJ^{1+\alpha-\beta}. \tag{4.26}
$$

We choose

$$
J = [t^{-1/(2\alpha-\beta)}]. \tag{4.27}
$$

By (4.4), Lemma 4.4, (4.16), (4.23)–(4.27) we obtain for β satisfying (1.43)

$$
\widetilde{E}_B(t) \le 2\pi^{-1/2}(A(\partial B)+1)t^{1/2} + O(t^{(\alpha-1)/(2\alpha-\beta)}). \tag{4.28}
$$

Suppose β satisfies (1.48). Then for $J \to \infty$

$$
t\sum_{j\le J} j^{\alpha-\beta} \asymp t\log J. \tag{4.29}
$$

By (4.4), Lemma 4.4, (4.16), (4.23)–(4.25), (4.27) and (4.29)

$$
\widetilde{E}_B(t) \le 2\pi^{-1/2}(A(\partial B)+1)t^{1/2} + O\left(t\log\frac{1}{t}\right). \tag{4.30}
$$

The upper bounds in the cases (1.35), (1.40) and (1.43),(1.48) follow from (4.22), (4.28) and (4.30) respectively. □

To prove the lower bounds in Theorems 1.2 and 1.3 for the regimes (1.35), (1.40) and (1.43), (1.48) we need the following result [1].

Lemma 4.5. *Let $m \ge 2$, and $2r > b > r$, $t > 0$. Then*

$$
\begin{aligned}
&\int_{\{x: r<x<b\}} \mathbb{P}_x[T_{B(0;r)} \le t]\,dx \\
&\qquad \ge 2\pi^{-1/2}A(\partial B(0;r))t^{1/2}(1-e^{-(b-r)^2/(4t)}) - k_8 r^{m-2}t,
\end{aligned}
\tag{4.31}
$$

where k_8 depends on m only.

Let $J \in \mathbb{N}$ be arbitrary. By Proposition 1.4

$$
\mathbb{P}_x[\widetilde{T}_B \le t] \ge \mathbb{P}_x[\widetilde{T}_{N\cap\partial Q} \le t]. \tag{4.32}
$$

Hence by positivity

$$
\begin{aligned}
\widetilde{E}_B(t) &\ge \int_{C_J\setminus B} \mathbb{P}_x[\widetilde{T}_B \le t]\,dx + \int_{H^+\setminus(C_J\cup B)} \mathbb{P}_x[\widetilde{T}_B \le t]\,dx \\
&\ge \int_{C_J\setminus B} \mathbb{P}_x[\widetilde{T}_{N\cap\partial Q} \le t]\,dx + \int_{H^+\setminus(C_J\cup B)} \mathbb{P}_x[\widetilde{T}_B \le t]\,dx.
\end{aligned}
\tag{4.33}
$$

Hence by symmetry and monotonicity

$$\begin{aligned}\int_{C_J\setminus B} \mathbb{P}_x[\widetilde{T}_{N\cap\partial Q} \le t]\,dx &\ge \int_{C_J} \mathbb{P}_x[\widetilde{T}_{N\cap\partial Q} \le t]\,dx - |B\cap C_J| \\ &\ge \frac{1}{2}\int_{\widehat{Q}\cup\widetilde{\widehat{Q}}} \mathbb{P}_x[T_{N\cap\partial Q} \le t]\,dx \\ &\quad - \frac{1}{2}\int_{(\widehat{Q}\cup\widetilde{\widehat{Q}})\setminus(C_J\cup\widetilde{C}_J)} \mathbb{P}_x[T_{N\cap\partial Q} \le t]\,dx \\ &\quad - \omega_m \sum_{j>J} [j^\alpha]^{m-1}\varrho_j^m.\end{aligned} \tag{4.34}$$

By Lemma 2.1 we obtain that the second term in the right hand side of (4.34) is bounded by

$$\begin{aligned}2^{(m+2)/2}\int_{r>\sum_{j>J}[j^\alpha]^{-1}} e^{-r^2/(8t)}\,dr &\le k_{10}t^{1/2}e^{-(\sum_{j>J}[j^\alpha]^{-1})^2/(8t)} \\ &\le k_{10}t^{1/2}e^{-k_{11}J^{2-2\alpha}t^{-1}},\end{aligned} \tag{4.35}$$

where k_{10} depends on m and k_{11} depends on α.

The first term in the right hand side of (4.34) equals

$$\begin{aligned}&\frac{1}{2}\int_{\widehat{Q}\cup\widetilde{\widehat{Q}}} \mathbb{P}_x[T_N \le t]\,dx - \frac{1}{2}\int_{\widehat{Q}\cup\widetilde{\widehat{Q}}} \mathbb{P}_x[T_{N\setminus(N\cap\partial Q)} \le t]\,dx \\ &\quad = 2\pi^{-1/2}t^{1/2} - \int_{\widehat{Q}} \mathbb{P}_x[T_{N\setminus(N\cap\partial Q)} \le t]\,dx \\ &\quad \ge 2\pi^{-1/2}t^{1/2} - 2^{(m+2)/2}\int_{\widehat{Q}} e^{-(d(x,N\setminus(N\cap\partial Q))^2/(8t)}\,dx,\end{aligned} \tag{4.36}$$

where we have used Lemma 2.1.

To estimate the integral in (4.36) we use the fact that $\widehat{Q} = (N\cap\partial Q)\times[0,\infty)$. Since N and $N\cap\partial Q$ are isometric with $\mathbb{R}^{m-1}$ and the unit cube in $\mathbb{R}^{m-1}$ respectively we have

$$|\{\, x\in\widehat{Q} : d(x, N\setminus(N\cap\partial Q)) < r\,\}| \le 2(m-1)r^2. \tag{4.37}$$

It follows that

$$\int_{\widehat{Q}} e^{-d(x,N\setminus(N\cap\partial Q))^2/(8t)}\,dx = O(t). \tag{4.38}$$

Since $B(=B_{\alpha,\beta})$ satisfies (1.15) and (1.25) with $\eta = 1/(2a)$, the collection of annuli

$$\{\, B(c_{ij};\varrho_j/(2a))\setminus B(c_{ij};\varrho_j) : j=1,\dots,J, i=1,\dots,[j^\alpha]^{m-1}\,\}, \tag{4.39}$$

is disjoint (apart from a set with measure 0). Hence

$$\begin{aligned}&\int_{H^+\setminus(C_J\cup B)} \mathbb{P}_x[\widetilde{T}_B \le t]dx \\ &\quad \ge \int_{H^+\setminus(C_J\cup B)} \mathbb{P}_x[T_B \le t]\,dx \\ &\quad \ge \sum_{j=1}^{J}[j^\alpha]^{m-1}\int_{B(0;\varrho_j/(2a))\setminus B(0;\varrho_j)} \mathbb{P}_x[T_{B(0;\varrho_j)} \le t]\,dx.\end{aligned} \tag{4.40}$$

We use Lemma 4.5 with $r = \varrho_j$ and

$$b = \varrho_j \min\left\{\frac{1}{2a}, 2\right\}, \tag{4.41}$$

and

$$e^{-(b-\varrho_j)^2/(4t)} \le (b-\varrho_j)^{-1} t^{1/2}. \tag{4.42}$$

It follows that

$$\begin{aligned} &\int_{H^+\setminus(C_J\cup B)} \mathbb{P}_x[\widetilde{T}_B \le t]\, dx \\ &\quad \ge 2\pi^{-1/2} A(\partial B) t^{1/2} - 2\pi^{-1/2} m\omega_m t^{1/2} \sum_{j>J} [j^\alpha]^{m-1} \varrho_j^{m-1} \\ &\quad\quad - \left(k_8 + 2\pi^{-1/2} m\omega_m \left(\min\left\{\frac{1}{2a}, 2\right\} - 1\right)^{-1}\right) t \sum_{j=1}^{J} [j^\alpha]^{m-1} \varrho_j^{m-2}. \end{aligned} \tag{4.43}$$

Proof of the lower bound for $m \ge 3$. Suppose β satisfies (1.35). Choose

$$J = [t^{-1/(2\beta)}]. \tag{4.44}$$

Then the last term in (4.34) becomes

$$\omega_m \sum_{j>J} [j^\alpha]^{m-1} \varrho_j^m \asymp J^{1+(m-1)\alpha - m\beta} \asymp t^{(m\beta-(m-1)\alpha-1)/(2\beta)}. \tag{4.45}$$

The second term in the right hand side of (4.43) has order

$$t^{1/2} \sum_{j>J} [j^\alpha]^{m-1} \varrho_j^{m-1} \asymp t^{1/2} J^{1+(m-1)\alpha-(m-1)\beta} \asymp t^{(m\beta-(m-1)\alpha-1)/(2\beta)}. \tag{4.46}$$

The third term in the right hand side of (4.43) has order

$$t \sum_{j=1}^{J} j^{(m-1)\alpha-(m-2)\beta} \asymp t J^{1+(m-1)\alpha-(m-2)\beta} \asymp t^{(m\beta-(m-1)\alpha-1)/(2\beta)}. \tag{4.47}$$

Finally, the right hand side of (4.35) is exponentially small in t since (1.35) implies $\beta > \alpha - 1$. Putting all the estimates together we obtain

$$\widetilde{E}_B(t) \ge 2\pi^{-1/2}(A(\partial B)+1)t^{1/2} + O(t^{(m\beta-(m-1)\alpha-1)/(2\beta)}). \tag{4.48}$$

Since β satisfies (1.35)

$$(m\beta-(m-1)\alpha-1)/(2\beta) > (\alpha-1)/(m\alpha-(m-2)\beta), \tag{4.49}$$

and so (4.48) completes, together with (4.20) the proof of (1.35), (1.36). □

Suppose β satisfies (1.40). The third term in the right hand side of (4.43) has order

$$t \sum_{j=1}^{J} j^{(m-1)\alpha-(m-2)\beta} \asymp t \log J \asymp t \log 1/t, \tag{4.50}$$

while (4.45), (4.46) are both $O(t)$ in this case. Hence for β satisfying (1.40)

$$\widetilde{E}_B(t) \ge 2\pi^{-1/2}(A(\partial B)+1)t^{1/2} + O\left(t \log \frac{1}{t}\right). \tag{4.51}$$

Then (1.40), (1.41) follows from (4.22) and (4.51). □

Proof of the lower bound for $m = 2$. Suppose β satisfies (1.43) or (1.48). The third term in the right hand side of (4.43) has order

$$\sum_{j>J} j^{\alpha} e^{-2j^{\beta}} = o(e^{-J^{\beta}}). \tag{4.52}$$

The second and third terms in the right hand side of (4.43) have order

$$t^{1/2} \sum_{j>J} j^{\alpha} e^{-j^{\beta}} = o(t^{1/2} e^{-J^{\beta}/2}), \tag{4.53}$$

$$t \sum_{j=1}^{J} [j^{\alpha}] \asymp t J^{1+\alpha}. \tag{4.54}$$

We choose J such that

$$e^{-J^{\beta}/2} \asymp t^{1/2}. \tag{4.55}$$

It follows that the right hand side of (4.35) is exponentially small. Putting all these estimates together we obtain

$$\widetilde{E}_B(t) \geq 2\pi^{-1/2}(A(\partial B) + 1)t^{1/2} + O\bigg(t\Big(\log \frac{1}{t}\Big)^{(1+\alpha)/\beta}\bigg), \tag{4.56}$$

and (1.43), (1.44) follows from (4.28), (4.56). Finally (1.48), (1.49) follows from (4.30) and (4.56). □

This completes the proof of Theorems 1.2 and 1.3.

5. Proof of Proposition 1.4

Throughout this section we assume (1.56) if $m \geq 3$ or (1.60) if $m = 2$. We introduce some further notation. Let $B_{\alpha,\beta} = B$ as before and let U_j be the union of all translates of the balls in layer L_j by vectors $[j^{\alpha}]^{-1}(0, k_2, \ldots, k_m)$, $k_i \in \mathbb{Z}$, and let $U = \underset{j}{\cup} U_j$. We denote the centers of the balls in layer L_j by Z_j. So these centers have first coordinate ζ_j. We denote $E_j = \{x \in H^+ : x_1 = \zeta_j\}$ for $j \in \mathbb{N}$. As before we let $\widetilde{B}(s)$, $s \geq 0$ be the reflected Brownian motion on $H^+ \cup N$. We assume that the Brownian motion is, as usual, defined on the set of continuous paths $\omega : [0, \infty) \to H^+ \cup N$. We write θ_t for the time shifts $\theta_t(\omega)_s = \omega_{s+t}$, and we write $\widetilde{\mathbb{P}}_x$ for the law of the reflected Brownian motion, starting in $x \in H^+ \cup N$. In order to prove Proposition 1.4 it evidently suffices to prove that

$$\widetilde{\mathbb{P}}_x[\widetilde{T}_U = 0] = 1, \quad x \in N. \tag{5.1}$$

We fix some arbitrary $J \in \mathbb{N}$ and define τ_J to be the first hitting time of E_J by $\widetilde{B}(s), s \geq 0$. As $\tau_J \to 0$ for $J \to \infty$ and $x \in N$ we see that in order to prove (5.1) it suffices to show that for any $J \in \mathbb{N}$

$$\widetilde{\mathbb{P}}_x[\widetilde{T}_U \leq \tau_J] = 1. \tag{5.2}$$

For the remainder of the proof we fix an arbitrary J, which will be suppressed in the notation. For the considerations below, we kill the reflected Brownian motion at τ_J. For $j = \mathbb{N}, j \geq 2$ we define

$$\sigma_j = \inf\{s \geq 0 : \widetilde{B}(s) \in E_{j-1} \cup E_{j+1}\} \tag{5.3}$$

Lemma 5.1. *Let $m \geq 3$. There exists j_0 depending on a, α, β and m such that for $j \geq j_0$ and $x \in E_j$*

$$\widetilde{\mathbb{P}}_x[\widetilde{T}_U \leq \sigma_j] \geq C(m,a) j^{-(m-2)(\beta-\alpha)}, \tag{5.4}$$

where $C(m,a)$ depends on m and on a only. Let $m = 2$. Then there exists j_0 depending on a, α and β such that for $j \geq j_0$ and $x = E_j$

$$\widetilde{\mathbb{P}}_x[\widetilde{T}_U \leq \sigma_j] \geq C(2,a) j^{-\beta}, \tag{5.5}$$

where $C(2,a)$ depends on a only.

Proof. If $x \in E_j$ then there exists a $y \in Z_j$ with $d(x,y) \leq 2^{-1}(m-1)^{1/2}[j^\alpha]^{-1}$. It suffices to prove that for $m \geq 3$

$$\widetilde{\mathbb{P}}_x[T_{B(y;\rho_j)} \leq \sigma_j] \geq C(m,a) j^{-(m-2)(\beta-\alpha)}, \tag{5.6}$$

and for $m = 2$ that

$$\widetilde{\mathbb{P}}_x[T_{B(y;\rho_j)} \leq \sigma_j] \geq C(2,a) j^{-\beta}. \tag{5.7}$$

First we will show that for $x \in E_j$ and $d(x,y) > \frac{1}{2}(j+1)^{-\alpha}$

$$\widetilde{\mathbb{P}}_x[T_{\partial B(y;\, 2^{-1}(j+1)^{-\alpha})} \leq \sigma_j] \geq C(m). \tag{5.8}$$

To see this we observe that there exists a cuboid with sides parallel to the coordinate planes, centered at x, and lying between E_{j-1} and E_{j+1}, and having the property that one face is entirely inside $B(y;\ 2^{-1}(j+1)^{-\alpha})$. Furthermore, the ratio of the side lengths of the cuboid are bounded independently of j and of α. So a Brownian motion starting at x has a positive probability, depending on these ratios only, to exit the cuboid at any specific face, in particular through the one inside $B(y;\ 2^{-1}(j+1)^{-\alpha})$. This implies (5.8).

By the strong Markov property, it therefore suffices to prove (5.6), (5.7) for $x \in B(y;\ 2^{-1}(j+1)^{-\alpha})$. Let $S = \partial B(y;\ 2(j+1)^{-\alpha}/3)$. If the Brownian motion starts at x, $x \in B(y;\ 2^{-1}(j+1)^{-\alpha})$ then it has to hit S before hitting $E_{j-1} \cup E_{j+1}$. Therefore, for $x \in B(y;\ 2^{-1}(j+1)^{-\alpha})$

$$\widetilde{\mathbb{P}}_x[T_{B(y;\rho_j)} < \sigma_j] \geq \mathbb{P}_x[T_{B(y;\rho_j)} < T_S]. \tag{5.9}$$

For $m \geq 3$ we have that (Proposition 1.5 on p.55 in [10]) the right hand side of (5.9) equals

$$\min\left\{1, \frac{(2(j+1)^{-\alpha}/3)^{2-m} - d(x,y)^{2-m}}{(2(j+1)^{-\alpha}/3)^{2-m} - {\rho_j}^{2-m}}\right\} \geq \min\left\{1, 2^{-1}a^{m-2}j^{(\alpha-\beta)(m-2)}\right\}, \tag{5.10}$$

which proves (5.6) for j sufficiently large. For $m = 2$ we have that(Proposition 4.8 on p.75 in [10]) the right hand side of (5.9) equals

$$\min\left\{1, \frac{\log(2(j+1)^{-\alpha}/3) - \log d(x,y)}{\log(2(j+1)^{-\alpha}/3) - \log \rho_j}\right\} \geq \min\left\{1, \frac{\log(4/3)}{\log(\frac{2}{3a}(j+1)^{-\alpha}e^{j\beta})}\right\}. \tag{5.11}$$

We now choose $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$

$$(j+1)^\alpha \geq \frac{e}{2a}. \tag{5.12}$$

Then for all $j \geq j_0$ the right hand side of (5.11) is bounded from below by $j^{-\beta}$. □

In the following we give some results for one-dimensional Brownian motion which will be used in the sequel.

Lemma 5.2. *Let* $(\beta(s),\ s \geq 0;\ \mathbb{P}_x,\ x \in \mathbb{R})$ *be a one-dimensional Brownian motion. Let* $a > 0,\ b > 0$ *and let* $\tau = \inf\{s \geq 0 : \beta(s) \notin (-a, b)\}$*. Then*

$$\mathbb{P}_0[\tau \geq t \mid \beta(\tau) = -a] \leq e^{-t/(a+b)^2}, \tag{5.13}$$

$$\mathbb{P}_0[\tau \geq t \mid \beta(\tau) = b] \leq e^{-t/(a+b)^2}, \tag{5.14}$$

for all

$$t \geq (a+b)^2. \tag{5.15}$$

Proof. It suffices to prove the lemma for the interval (0, 1), and with a Brownian motion starting at $x \in (0,1)$. By symmetry

$$\mathbb{P}_{1-x}\ [\tau \geq t \mid \beta(\tau) = 0]\ =\ \mathbb{P}_x\ [\tau \geq t \mid \beta(\tau) = 1], \tag{5.16}$$

where τ is the first exit time from $(0,1)$. By (5.16) it suffices to an upper bound for $\mathbb{P}_x[\tau \geq t \mid \beta(\tau) = 1]$. By Proposition 8.3 in [10] we have that

$$\begin{aligned} \mathbb{P}_x[\tau \geq t \mid \beta(\tau) = 1] &= \frac{1}{x}\int_0^1 p_{(0,1)}(x,y;t)y\ dy \\ &= \frac{2}{x}\sum_{k=1}^{\infty}\ e^{-t\pi^2k^2}\sin(\pi kx)\int_0^1 \sin(\pi ky)y\ dy \\ &= \frac{2}{x}\sum_{k=1}^{\infty} e^{-t\pi^2k^2}\sin(\pi kx)(\pi k)^{-1}(-1)^{k-1} \\ &\leq 2\sum_{k=1}^{\infty}\ e^{-t\pi^2k^2} \\ &\leq 2e^{-t}\sum_{k=1}^{\infty} e^{-t(\pi^2k^2-1)}. \end{aligned} \tag{5.17}$$

For all $t \geq 1$ we have that

$$2\sum_{k=1}^{\infty} e^{-t(\pi^2k^2-1)} \leq 2e\sum_{k=1}^{\infty} e^{-\pi^2k^2} < 1. \tag{5.18}$$

The lemma follows from (5.17) and (5.18). □

Let $\beta_s,\ s \geq 0$ be a standard Brownian motion and $\widetilde{\beta}_s = |\beta_s|$. By the Trotter Theorem, there exists a continuous local time for (β_s): There exists a mapping

$$\lambda : [a,\infty) \times \mathbb{R} \times \Omega \to [0,\infty)$$

which is jointly continuous in the first two arguments, and such for almost all $\omega \in \Omega$ one has for all Borel sets $A \subset \mathbb{R}$ and all $t > 0$

$$\int_0^t 1_A(\omega_s)\,ds = \int_A \lambda(t,x,\omega)\,dx.$$

The local time for $\widetilde{\beta}_s$ is then simply

$$\tilde{\lambda}(t,x) = \lambda(t,x) + \lambda(t,-x), \quad x \geq 0.$$

If $a > 0$, let $\widetilde{T}_a$ be the first hitting time of a by $(\widetilde{\beta}_s)$.

Lemma 5.3. *$\tilde{\lambda}(\widetilde{T}_a, ., \omega)$ is positive on $[0, a)$ for almost all ω.*

Proof. If T_a is the first hitting time of a by (β_s), we have $\widetilde{T}_a = T_a \wedge T_{-a}$. Therefore

$$\tilde{\lambda}(\widetilde{T}_a, x) \geq 1_{T_a < T_{-a}} \lambda(T_a, x) + 1_{T_{-a} < T_a} \lambda(T_{-a}, -x), \quad x \geq 0.$$

By the Ray-Knight Theorem, $(\lambda(T_a, a - x))_{0 \leq x \leq a}$ is the square of a two-dimensional Bessel process (see [13, Ch. XI, Theorem 2.2]). The claim follows since the two-dimensional Brownian motion is not point recurrent. □

As above, we write $\widetilde{B}(s),\ s \geq 0$ for reflected Brownian motion. By abuse of notation we write $\widetilde{\mathbb{P}}_N$ for its law when the starting point is in N, and $\widetilde{\mathbb{P}}_x$ when $x \in H^+$. For any $j \geq J$ we define the sequence of stopping times $\gamma_1^{(j)}, \gamma_2^{(j)}, \ldots$ of successive visits of reflected Brownian motion in E_j by,

$$\gamma_1^{(j)} = \inf\{s \geq 0 : \widetilde{B}(s) \in E_j\}, \tag{5.19}$$

which is finite $\widetilde{\mathbb{P}}_N$ almost surely, and

$$\sigma_k^{(j)} = \gamma_k^{(j)} + \sigma_j \circ \theta_{\gamma_k}(j), \tag{5.20}$$

$$\gamma_{k+1}^{(j)} = \sigma_k^{(j)} + \gamma_1^{(j)} \circ \theta_{\sigma_k}^{(j)}. \tag{5.21}$$

We note that if $j = J + 1$ we may have $\sigma_k^{(j)} = \infty$, even if $\gamma_k^{(j)} < \infty$, namely if the Brownian motion hits E_J, where it is killed. Furthermore, $\gamma_{k+1}^{(j)}$ can be infinity for all j and $k > 1$ under $\widetilde{\mathbb{P}}_N$. It is clear that $\widetilde{\mathbb{P}}_N$ almost surely at most finitely many $\gamma_k^{(j)}$ are finite. Define

$$n_j = \max\{k : \gamma_k^{(j)} < \infty\}. \tag{5.22}$$

Lemma 5.4. *For $m \geq 3$*

$$\sum_{j=J+2}^{\infty} n_j \ j^{(m-2)(\alpha-\beta)} = +\infty, \tag{5.23}$$

$\widetilde{\mathbb{P}}_N$ almost surely, and for $m = 2$

$$\sum_{j=J+2}^{\infty} n_j \ j^{-\beta} = +\infty, \tag{5.24}$$

$\widetilde{\mathbb{P}}_N$ almost surely.

Proof. Define

$$\tau_k^{(j)} = \sigma_k^{(j)} - \gamma_k^{(j)} \tag{5.25}$$

These random variables are finite on the set $\{\gamma_k^{(j)} < \infty\}$. Let Λ_j be the total time spent by the Brownian motion between the planes E_{j-1} and E_{j+1} before τ_J. Clearly $\Lambda_j < \infty$ almost surely. It is of course just the total time spent by a reflected one-dimensional Brownian motion $\widetilde{\beta}_s,\ s \geq 0$ in the interval $[\zeta_{j+1}, \zeta_{j-1}]$ before killing at ζ_J. Alternatively, it is the total time a standard one-dimensional Brownian motion spends in $[\zeta_{j+1}, \zeta_{j-1}] \cup [-\zeta_{j-1}, -\zeta_{j+1}]$ before killing at the boundary of the interval

$(-\zeta_J, \zeta_J)$. We write λ_J for the local time for the reflected Brownian motion, killed at reaching ζ_J. Then

$$\Lambda_j = \int_{\zeta_{j+1}}^{\zeta_{j-1}} \lambda_J\ (x) dx, \tag{5.26}$$

and

$$\Lambda_j \leq \sum_{k=1}^{n_{j-1}} \tau_k^{(j-1)} + \sum_{k=1}^{n_j} \tau_k^{(j)} + \sum_{k=1}^{n_{j+1}} \tau_k^{(j+1)}. \tag{5.27}$$

By the strong Markov property and Lemma 5.2 the right hand side of (5.27) is stochatically dominated by

$$\sum_{k=1}^{n_{j-1}} \tilde{\tau}_k^{(j-1)} + \sum_{k=1}^{n_j} \tilde{\tau}_k^{(j)} + \sum_{k=1}^{n_{j+1}} \tilde{\tau}_k^{(j+1)}, \tag{5.28}$$

where

$$\tilde{\tau}_k^{(j)} = \max\{\mu_j^2,\ \zeta_k^{(j)}\}\ ,\ \ k \geq 1, \tag{5.29}$$

$$\mu_j = \zeta_{j-1} - \zeta_{j+1} \quad , \tag{5.30}$$

and where $\zeta_k^{(j)}$ are independent exponentially distributed random variables with mean $(\zeta_{j-1} - \zeta_{j+1})^2$. Therefore the right hand side of (5.27) is stochastically dominated by a sum of $n_{j-1} + n_j + n_{j+1}$ independent random variables

$$\hat{\tau}_k = \max\{\mu_{j-1}^2\ ,\ \zeta_k^{(j-1)}\}, \tag{5.31}$$

where the $\zeta_k^{(j-1)}$ are independent exponentially distributed with mean $(\zeta_{j-2} - \zeta_j)^2$. Then for any $u > 0$

$$\begin{aligned} &\widetilde{\mathbb{P}}_N(\Lambda_j \geq 15uj^{-\alpha},\ n_{j-1} + n_j + n_{j+1} \leq uj^{\alpha}) \\ &\qquad \leq \widetilde{\mathbb{P}}_N\left(\sum_{k=1}^{[uj^\alpha]}\ \hat{\tau}_k \geq 15uj^{-\alpha}\right) \\ &\qquad \leq \widetilde{\mathbb{P}}_N\left(\sum_{k=1}^{[uj^\alpha]} \zeta_k^{(j-1)} \geq 15uj^{-\alpha} - [uj^\alpha](\zeta_{j-2} - \zeta_j)^2\right) \\ &\qquad = \widetilde{\mathbb{P}}_N\left(\sum_{k=1}^{[uj^\alpha]} X_k \geq 15uj^{-\alpha}(\zeta_{j-2} - \zeta_j)^{-2} - [uj^\alpha]\right) \\ &\qquad \leq \widetilde{\mathbb{P}}_N\left(\sum_{k=1}^{[uj^\alpha]} (X_k - 1) \geq 15uj^{-\alpha}(\zeta_{j-2} - \zeta_j)^{-2} - 2uj^\alpha\right), \end{aligned} \tag{5.32}$$

where the X_k are independent exponentially distributed random variables with mean 1. Since

$$(\zeta_{j-2} - \zeta_j)^2 = (\frac{1}{2}[(j-2)^\alpha]^{-1} + [(j-1)^\alpha]^{-1} + \frac{1}{2}[j^\alpha]^{-1})^{-2} \geq \frac{1}{5} j^{2\alpha} \tag{5.33}$$

for all j sufficiently large, we have that the right hand side of (5.32) is bounded from above by

$$\tilde{\mathbb{P}}_N\left(\sum_{k=1}^{[uj^\alpha]}(X_k-1)\geq uj^\alpha\right). \tag{5.34}$$

For any $\lambda\in(0,\frac{1}{2})$ we have by Tchebycheff's inequality and the independence of the X_k 's

$$\begin{aligned}\widetilde{\mathbb{P}}_N\left(\sum_{k=1}^{[uj^\alpha]}(X_k-1)\geq uj^\alpha\right)&\leq\exp\left[-\lambda uj^\alpha\right]\left\{E\left(\exp\left[\lambda\left(X_1-1\right)\right]\right)\right\}^{[uj^\alpha]}\\&\leq\exp\left[-\lambda uj^\alpha\right]\left\{\frac{\mathrm{e}^{-\lambda}}{1-\lambda}\right\}^{[uj^\alpha]}\\&\leq\exp\left[-\frac{uj^\alpha}{8}\right],\end{aligned}\tag{5.35}$$

in the last step choosing $\lambda=1/4$.

Define the event $A_{j,u}$ by

$$A_{j,u}=\left\{\Lambda_j\geq 15uj^{-\alpha},n_{j-1}+n_j+n_{j+1}\leq uj^\alpha\right\}. \tag{5.36}$$

Then by Borel-Cantelli, we conclude that

$$\widetilde{\mathbb{P}}_N\left(\limsup_{j\to\infty}A_{j,u}\right)=0 \tag{5.37}$$

for any $u>0$.

We are now coming back to the local time $\lambda_J(x)$. According to Lemma 5.3, $\lambda_J(x)$ is continuous and positive on $[0,\zeta_J)$, almost surely. Therefore, for any $\eta>0$, we may choose a $u>0$, such that

$$\widetilde{\mathbb{P}}_N\left(\inf_{x\in[0,\zeta_{J+1}]}\lambda_J(x)\geq 15u\right)\geq 1-\eta. \tag{5.38}$$

On the event $\left\{\inf_{x\in[0,\zeta_{J+1}]}\lambda_J(x)\geq 15u\right\}$, we have $\Lambda_j\geq 15uj^{-\alpha}$. Therefore, on

$$\left\{\inf_{x\in[0,\zeta_{J+1}]}\lambda_J(x)\geq 15u\right\}\cap\left\{\limsup_{j\to\infty}A_{j,u}\right\}^c, \tag{5.39}$$

we have $n_{j-1}+n_j+n_{j+1}>uj^\alpha$ for all large enough j, i.e. $\sum_{j=J+2}^\infty n_j\,j^{(m-2)(\alpha-\beta)}=\infty$ for $m\geq 3$, and $\sum_{j=J+2}^\infty n_j\,j^{-\beta}=\infty$ for $m=2$. Therefore

$$\widetilde{\mathbb{P}}_N\left(\sum_{j=J+2}^\infty n_j\,j^{(m-2)(\alpha-\beta)}=\infty\right)\geq 1-\eta,\ m\geq 3 \tag{5.40}$$

for any $\eta>0$, and similarly for $m=2$. This proves the claim. □

With the help of this lemma, the proof of the proposition can be finished in the following way. We introduce an additional cutoff at $K>J$. If the starting point of the Brownian motion is on E_j, then we define $\tau=\inf\{t\geq 0:\widetilde{B}(t)\in E_{j-1}\cup E_{j+1}\}$. For $j=K$, we set $\overline{\tau}=\tau$ if $\widetilde{B}(\tau)\in E_{K-1}$, $\overline{\tau}=\inf\{t\geq\tau:\widetilde{B}(t)\in E_K\}$ if $\widetilde{B}(\tau)\in E_{K+1}$. For

$j = J+1$, we define $\overline{\tau}{=}\tau$ if $\widetilde{B}(\tau) \in E_{J+2,}$ and $\overline{\tau}{=}\infty$ if $\widetilde{B}(\tau) \in E_J$. In all other cases, $\overline{\tau}{=}\tau$. Consider the successive visiting times $\sigma_1, \sigma_2, \ldots$ of the planes E_j, $J < j \le K$, recursively defined by

$$\sigma_1 = \sigma_1(K) = \inf\left\{t \ge 0 : \widetilde{B}(t) \in E_K\right\} < \infty. \tag{5.41}$$

If σ_k is defined and finite, and $\widetilde{B}(\sigma_k) \in E_j$, $J+2 \le j \le K-1$ then

$$\sigma_{k+1}{=}\sigma_k + \overline{\tau} \circ \theta_{\sigma_k}, \tag{5.42}$$

where θ is the usual shift operation. We set $X_k{=}j$ if $\sigma_k < \infty$ and $\widetilde{B}(\sigma_k) \in E_j$. If $\sigma_k = \infty$, we define $X_k{=}\infty$. It is clear that the sequence (X_k) forms a Markov chain itself. We claim now that for any $r \in \mathbb{N}$, and any sequence $j_1 = K, j_2, \ldots, j_r \in \{J+1, \ldots, K\}$ we have

$$\begin{aligned} &\widetilde{\mathbb{P}}_N\left(\widetilde{B}(s) \notin U,\, s \in [\sigma_1, \sigma_r],\, X_1 = j_1, X_2 = j_2, \ldots, X_r = j_r\right) \\ &\le \prod_{i=1}^{r-1}\left(1 - C j_i^{(m-2)(\alpha-\beta)}\right) \widetilde{\mathbb{P}}_N\left(X_1 = j_1, X_2 = j_2, \ldots, X_r = j_r\right),\ m \ge 3, \end{aligned} \tag{5.43}$$

for some constant C (which may depend on m and a only), and

$$\begin{aligned} &\widetilde{\mathbb{P}}_N\left(\widetilde{B}(s) \notin U,\, s \in [\sigma_1, \sigma_r],\, X_1 = j_1, X_2 = j_2, \ldots, X_r = j_r\right) \\ &\le \prod_{i=1}^{r-1}\left(1 - C j_i^{-\beta}\right) \widetilde{\mathbb{P}}_N\left(X_1 = j_1, X_2 = j_2, \ldots, X_r = j_r\right),\ m = 2. \end{aligned} \tag{5.44}$$

We give the proof for the case $m \ge 3$. The proof for $m = 2$ is similar. In order to prove (5.43), we use induction on r. For $r = 1$, there is nothing to prove. Assume, the estimate is proved for r, and we want to prove it for $r+1$. By the strong Markov property and Lemma 5.1, we have

$$\begin{aligned} &\widetilde{\mathbb{P}}_N\Big(\bigcap_{s\in[\sigma_1,\sigma_{r+1})}\{\widetilde{B}(s) \notin U\},\, X_{[1,r+1]} = (j_1, \ldots, j_{r+1})\Big) \\ &= E_N\Big(\widetilde{\mathbb{P}}_{\widetilde{B}(\sigma_r)}\big(\bigcap_{s\in[0,\overline{\tau})}\{\widetilde{B}(s) \notin U\},\, \widetilde{B}(\overline{\tau}) \in E_{j_{r+1}}\big);\, \bigcap_{s\in[\sigma_1,\sigma_r]}\{\widetilde{B}(s) \notin U\},\, X_{[1,r]} = (j_1, \ldots, j_r)\Big) \\ &\le E_N\Big(\widetilde{\mathbb{P}}_{\widetilde{B}(\sigma_r)}\big(\bigcap_{s\in[0,\tau)}\{\widetilde{B}(s) \notin U\},\, \widetilde{B}(\overline{\tau}) \in E_{j_{r+1}}\big);\, \bigcap_{s\in[\sigma_1,\sigma_r)}\{\widetilde{B}(s) \notin U\},\, X_{[1,r]} = (j_1, \ldots, j_r)\Big) \\ &\le (1 - C j_r^{(m-2)(\alpha-\beta)}) \widetilde{\mathbb{P}}_N\left(X_1 = j_{r+1} | X_0 = j_r\right) \widetilde{\mathbb{P}}_N\Big(\bigcap_{s\in[\sigma_1,\sigma_r]}\{\widetilde{B}(s) \notin U\},\, X_{[1,r]} = (j_1, \ldots, j_r)\Big) \\ &\le \prod_{i=1}^{r}\left(1 - C j_i^{(m-2)(\alpha-\beta)}\right) \widetilde{\mathbb{P}}_N\left(X_{[1,r+1]} = (j_1, j_2, \ldots, j_{r+1})\right). \end{aligned} \tag{5.45}$$

Therefore, (5.43) is proved.

Letting now $r \to \infty$ in this estimate, we get

$$\begin{aligned} \widetilde{\mathbb{P}}_N\left(\widetilde{B}(s) \notin U,\, s \in [\sigma_1(K), \infty)\right)) &\le E_N\left(\prod_{i=1}^{\infty}\left(1 - C X_i^{(m-2)(\alpha-\beta)}\right)\right) \\ &= E_N\left(\prod_{j=J+1}^{K}\left(1 - C j^{(m-2)(\alpha-\beta)}\right)^{n_j}\right). \end{aligned} \tag{5.46}$$

Now, we can let $K \to \infty$, obtaining

$$\widetilde{\mathbb{P}}_N \left(\widetilde{B}(s) \notin U, \, s \geq 0 \right) \leq E_N \left(\prod_{j=J+1}^{\infty} \left(1 - C j^{(m-2)(\alpha-\beta)} \right)^{n_j} \right) = 0, \tag{5.47}$$

the last equality by Lemma 5.4 The case $m = 2$ follows by a straightforward modification.

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M. van den Berg, School of Mathematics, University of Bristol,University Walk, Bristol BS8 ITW, United Kingdom

E. Bolthausen, Institut für Mathematik, Abt. Angewandte Mathematik, Universität Zürich, Winterthurerstrasse 190, CH–8057 Zürich, Switzerland