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A DIAGONAL BOUND FOR COHOMOLOGICAL POSTULATION NUMBERS OF PROJECTIVE SCHEMES

M. BRODMANN AND A.F. LASHGARI

ABSTRACT. Let X be a projective scheme over a field K and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules. We show that the cohomological postulation numbers $\nu_{\mathcal{F}}^i$ of \mathcal{F} , e.g. the ultimate places at which the cohomological Hilbert functions $n \mapsto \dim_K (H^i(X, \mathcal{F}(n))) =: h_{\mathcal{F}}^i(n)$ start to be polynomial for $n \ll 0$, are (polynomially) bounded in terms of the cohomology diagonal $(h_{\mathcal{F}}^i(-i))_{i=0}^{\dim(\mathcal{F})}$ of \mathcal{F} . As a consequence we obtain that there are only finitely many different cohomological Hilbert functions $h_{\mathcal{F}}^i$ if \mathcal{F} runs through all coherent sheaves of \mathcal{O}_X -modules with fixed cohomology diagonal. In order to prove these results we extend the regularity bound of Bayer-Mumford [1] from graded ideals to graded modules. Moreover we prove that the Castelnuovo-Mumford regularity of the dual \mathcal{F}^\vee of a coherent sheaf of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules \mathcal{F} is (polynomially) bounded in terms of the cohomology diagonal of \mathcal{F} .

1. INTRODUCTION

Let X be a projective scheme over a field K with twisting sheaf $\mathcal{O}_X(1)$ and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules. For $i \in \mathbb{N}_0$, the i -th *cohomological Hilbert function of $(X$ with respect to) \mathcal{F}* is defined as the function

$$(1.1) \quad h_{\mathcal{F}}^i : \mathbb{Z} \rightarrow \mathbb{N}_0, \quad n \mapsto h_{\mathcal{F}}^i(n) := \dim_K (H^i(X, \mathcal{F}(n))),$$

where $H^i(X, \mathcal{F}(n))$ denotes the i -th cohomology group of X with coefficients in the n -th twist $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)^{\otimes n}$ of \mathcal{F} .

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It is well known that the function $n \mapsto h_{\mathcal{F}}^i(n)$ is polynomial for all $n \ll 0$. The corresponding polynomial

$$(1.2) \quad p_{\mathcal{F}}^i \in \mathbb{Q}[\mathbf{x}] \quad \text{with} \quad p_{\mathcal{F}}^i(n) = h_{\mathcal{F}}^i(n), \quad \forall n \ll 0$$

is called the i -th *cohomological Hilbert polynomial of $(X$ with respect to \mathcal{F}* and is of degree $\leq i$ (cf [5, 20.4.14]).

Now, for each $i \in \mathbb{N}_0$, we may define the i -th *cohomological postulation number (of X with respect to \mathcal{F})* by

$$(1.3) \quad \nu_{\mathcal{F}}^i := \inf\{n \in \mathbb{Z} \mid h_{\mathcal{F}}^i(n) \neq p_{\mathcal{F}}^i(n)\} - 1$$

with the usual convention that $\inf \emptyset = \infty$. The basic aim of the present paper is to establish the following bounding result (cf Theorem 4.6)

For each $r \in \mathbb{N}_0$ there is a polynomial $M_r \in \mathbb{Q}[\mathbf{u}_0, \dots, \mathbf{u}_r]$ such that for each $i \in \mathbb{N}_0$, each field K , each projective scheme X over K and each coherent sheaf of \mathcal{O}_X -modules \mathcal{F} with $\dim(\mathcal{F}) \leq r$ we have

$$\nu_{\mathcal{F}}^i \geq M_r(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r)).$$

This extends a result of Matteotti [7], who has calculated bounding functions for the numbers $\nu_{\mathcal{F}}^i$ in terms of the "cohomology diagonal $(h_{\mathcal{F}}^j(-j))_{j \leq i}$ at and below level i " and in terms of the corresponding cohomological Hilbert polynomials $(p_{\mathcal{F}}^j)_{j \leq i}$. So, what we shall prove is that one may bound the numbers $\nu_{\mathcal{F}}^i$ without knowing the polynomials $p_{\mathcal{F}}^i$, only in terms of the "full" cohomology diagonal.

As a consequence of the mentioned bounding result we shall prove the following finiteness result (cf Theorem 5.4)

Let $r \in \mathbb{N}_0$ and let $h_0, \dots, h_r \in \mathbb{N}_0$. Let (X, \mathcal{F}) run through all pairs in which X is a projective scheme over some field and \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules such that

$$\dim \mathcal{F} \leq r \quad \text{and} \quad h_{\mathcal{F}}^j(-j) \leq h_j \quad \text{for} \quad j = 0, \dots, r.$$

Then only finitely many different cohomological Hilbert functions $h_{\mathcal{F}}^i$ may occur.

Keep the previous notation and hypothesis and let $k \in \mathbb{N}_0$. Then, the *Castelnuovo-Mumford regularity of \mathcal{F} above level k* is defined by (cf [4, 1.11])

$$(1.4) \quad \text{reg}_k(\mathcal{F}) := \inf\{t \in \mathbb{Z} \mid h_{\mathcal{F}}^i(n-i) = 0, \quad \forall n \geq t, \forall i > k\},$$

so that

$$(1.5) \quad \text{reg}_0(\mathcal{F}) =: \text{reg}(\mathcal{F})$$

is the usual *Castelnuovo-Mumford regularity* of \mathcal{F} .

One of the main steps towards the bounding result mentioned above is to show that the regularity of the dual $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_K^r}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}_K^r})$ of a coherent sheaf \mathcal{F} over a projective space \mathbb{P}_K^r is bounded in terms of the cohomology diagonal of \mathcal{F} . More precisely (cf Theorem 3.8)

For each $r \in \mathbb{N}_0$ there is a polynomial $L_r \in \mathbb{Q}[\mathbf{u}_0, \dots, \mathbf{u}_r]$ such that for each field K and each coherent sheaf of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules \mathcal{F} we have

$$\text{reg}(\mathcal{F}^\vee) \leq L_r(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r)).$$

One important ingredient in order to prove the preceding result is the following bounding result which relates the Castelnuovo-Mumford regularity $\text{reg}(M)$ and the generating degree $d(M)$ (s. 2.2 B) resp. 2.1 C) for the definitions) of a graded submodule M of a graded free module over a polynomial ring (cf Theorem 2.6).

For each $r \in \mathbb{N}_0$ there is a polynomial $F_r \in \mathbb{Q}[\mathbf{s}, \mathbf{t}]$ such that for each $s \in \mathbb{N}$, for each field K , for each polynomial ring $K[\mathbf{x}] = K[\mathbf{x}_0, \dots, \mathbf{x}_r]$ and for each graded submodule $M \subseteq K[\mathbf{x}]^{\oplus s}$ we have

$$\text{reg}(M) \leq F_r(s, d(M)).$$

In fact, polynomial regularity bounds of the above type may be deduced by classical results on syzygies, (cf [3, Sec. 4]). We include a proof of the above bounding result mainly because of our choice of the bounding polynomial F_r : namely, if $s = 1$ our bound coincides with the regularity bound of Bayer-Mumford (s. [1, 2.3]). The classical syzygetic method furnishes much weaker bounds.

The results mentioned above, partly are modified versions of the main results of the thesis of the second author [6] and have been announced without proofs in [3].

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2. A REGULARITY BOUND OF BAYER-MUMFORD TYPE

Let $K[\mathbf{x}] := K[\mathbf{x}_0, \dots, \mathbf{x}_r]$ be a polynomial ring over a field K . The principal aim of this section is to extend the regularity bound of Bayer-Mumford [1] for graded ideals $\mathfrak{a} \subseteq K[\mathbf{x}]$ to graded submodules $M \subseteq K[\mathbf{x}]^{\oplus s}$ for all $s \in \mathbb{N}$. We begin with some preliminaries on graded rings and modules.

2.1. Definition and Remark. A) Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous noetherian ring (so that $R = R_0[R_1]$) and let $R_+ := \bigoplus_{n > 0} R_n$ denote the irrelevant ideal of R . If T is a graded R -module and $n \in \mathbb{Z}$, we denote by T_n the n -th homogeneous part of T , so that $T = \bigoplus_{n \in \mathbb{Z}} T_n$. Using this notation we define the *beginning* and the *end* of T respectively by

$$\begin{aligned} \text{beg}(T) &:= \inf\{n \in \mathbb{Z} \mid T_n \neq 0\}, \\ \text{end}(T) &:= \sup\{n \in \mathbb{Z} \mid T_n \neq 0\}, \end{aligned}$$

with the usual convention that \inf and \sup are formed in $\mathbb{Z} \cup \{\pm\infty\}$ with $\inf \emptyset = \infty$, $\sup \emptyset = -\infty$.

B) Let R and T be as above. For $m \in \mathbb{Z}$ we define the m -th *left-* resp. *right-truncation* of T as the R_0 -submodules

$$T_{\geq m} := \bigoplus_{n \geq m} T_n, \quad T_{\leq m} := \bigoplus_{n \leq m} T_n.$$

Clearly, $T_{\geq m}$ is a graded submodule of T .

C) Let R and T be as above. The *generating degree* of T is defined by

$$d(T) := \inf\{m \in \mathbb{Z} \mid T = (T_{\leq m})R\}.$$

D) Let R and T be as above. Let R'_0 be a noetherian R_0 -algebra. Then $R' := R'_0 \otimes_{R_0} R = \bigoplus_{n \geq 0} R'_0 \otimes_{R_0} R_n$ carries a natural grading which turns it into a homogeneous noetherian ring with irrelevant ideal $R'_+ = R_+ R'$. Moreover $T' = R' \otimes_R T = R'_0 \otimes_{R_0} T = \bigoplus_{n \in \mathbb{Z}} R'_0 \otimes_{R_0} T_n$ becomes a graded R' -module.

If R'_0 is faithfully flat over R_0 , then R' is faithfully flat over R and moreover

$$\text{beg}(T') = \text{beg}(T), \quad \text{end}(T') = \text{end}(T), \quad d(T') = d(T). \quad \bullet$$

2.2. Reminder and Remark. A) Let $R = \bigoplus_{n \geq 0} R_n$ be as in 2.1 and let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a graded R -module. Let $i \in \mathbb{N}_0$. Then, the i -th *local cohomology module* $H_{R_+}^i(M)$ of M with respect to the irrelevant ideal $R_+ \subseteq R$ carries a natural grading (cf [5, Chap. 12]).

B) Let R and M be as in part A) and assume in addition, that M is finitely generated. Let $i \in \mathbb{N}_0$ and let $n \in \mathbb{Z}$. Then, the n -th homogeneous part $H_{R_+}^i(M)_n$ of $H_{R_+}^i(M)$ is a finitely generated R_0 -module and vanishes if n is sufficiently large. This allows to define the

(Castelnuovo-Mumford) regularity of M at and above level k by

$$\text{reg}^k(M) := \sup\{\text{end}(H_{R_+}^i(M)) + i \mid i \geq k\} \in \mathbb{Z} \cup \{-\infty\}$$

for each $k \in \mathbb{Z}_0$ (cf [5, (15.2.9)]). Then

$$\text{reg}(M) := \text{reg}^0(M)$$

is the usual (Castelnuovo-Mumford) regularity of M .

C) Keep the above notations and hypotheses. Assume in addition that R'_0 is a faithfully flat R_0 -algebra. Then $R'_0 \otimes_{R_0} M$ is a finitely generated graded module over $R'_0 \otimes_{R_0} R$ and the graded flat base change theorem gives rise to natural isomorphisms of R'_0 -modules

$$H_{(R'_0 \otimes_{R_0} R)_+}^i(R'_0 \otimes_{R_0} M)_n \cong R'_0 \otimes_{R_0} H_{R_+}^i(M)_n$$

for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ (cf [5, (15.2.2)]).

In particular we have

$$\text{reg}^k(R'_0 \otimes_{R_0} M) = \text{reg}^k(M)$$

for all $k \in \mathbb{Z}_0$.

D) Assume now, that $R = K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \dots, \mathbf{x}_r]$ is a polynomial ring over the field K . Let M be a finitely generated and graded R -module and let

$$0 \rightarrow F_p \rightarrow F_{p-1} \rightarrow \dots \rightarrow F_0 \rightarrow M \rightarrow 0$$

be a minimal graded free resolution of M . Then we have the following well known "syzygetic characterization" of regularity (cf [5, (15.3.7)])

$$\text{reg}(M) = \max\{d(F_j) - j \mid 0 \leq j \leq p\}. \quad \bullet$$

2.3. Reminder and Remark. A) Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous noetherian ring and let M be a finitely generated and graded R -module. A homogeneous element $f \in R_n$ of R is said to be (R_+) -filter-regular with respect to M , if it is a non-zero divisor with respect to the module $M/H_{R_+}^0(M)$.

B) It is easy to see that $f \in R_n$ is filter-regular with respect to M if and only if the annihilator $(0 \underset{M}{:} f)$ of f in M is contained in $H_{R_+}^0(M)$. So, $f \in R_n$ is filter-regular with respect to M iff $e := \text{end}(0 \underset{M}{:} f) < \infty$. Moreover, if this is the case, we have $e \leq \text{end}(H_{R_+}^0(M))$. If in addition $n > 0$, we have $e = \text{end}(H_{R_+}^0(M))$.

C) If R_0 is an infinite field and if $R_1 \neq 0$, then there is an element $f \in R_1 \setminus \{0\}$ which is filter-regular with respect to M (cf [5, 15.1.4]). •

2.4. Lemma. *Let $K[\underline{x}] = K[\mathbf{x}_0, \dots, \mathbf{x}_r]$ be a polynomial ring over the field K , let U be a finitely generated and graded $K[\underline{x}]$ -module, let $m \in \mathbb{Z}$ and let $M, N \subseteq U$ be two graded submodules such that $d(M), d(N) \leq m$ and $\text{reg}(M + N) < m$. Then $d(M \cap N) \leq m$.*

Proof: There are graded epimorphisms $\pi : F \rightarrow M, \varrho : G \rightarrow N$ in which F and G are graded free $K[\underline{x}]$ -modules of finite rank with $d(F), d(G) \leq m$. In particular we have $\text{reg}(F \oplus G) \leq m$. So, the graded short exact sequence

$$0 \rightarrow \text{Ker}(\pi + \varrho) \rightarrow F \oplus G \xrightarrow{\pi + \varrho} M + N \rightarrow 0$$

yields $\text{reg}(\text{Ker}(\pi + \varrho)) \leq m$ (cf [5, (15.2.15) (i)]), thus $d(\text{Ker}(\pi + \varrho)) \leq m$. The commutative diagram

$$\begin{array}{ccc} M \oplus N & \xrightarrow{\sigma := id_M + id_N} & M + N \\ \uparrow \pi \oplus \varrho & & \uparrow \pi + \varrho \\ F \oplus G & \xlongequal{\quad} & F \oplus G \end{array}$$

shows that $(\pi \oplus \varrho)(\text{Ker}(\pi + \varrho)) = \text{Ker}(\sigma)$, hence $d(\text{Ker}(\sigma)) \leq m$. In view of the graded isomorphism $M \cap N \cong \text{Ker}(\sigma)$ we get our claim. ■

2.5. Definition and Remark. A) We define a sequence of polynomials $(F_r)_{r \in \mathbb{N}_0} \subseteq \mathbb{Q}[\mathbf{s}, \mathbf{t}]$ as follows:

$$F_0(\mathbf{s}, \mathbf{t}) := \mathbf{t};$$

$$F_r(\mathbf{s}, \mathbf{t}) := F_{r-1}(\mathbf{s}, \mathbf{t}) + \mathbf{s} \binom{F_{r-1}(\mathbf{s}, \mathbf{t}) + r}{r}, \quad \forall r \in \mathbb{N}.$$

We call F_r the r -th *Bayer-Mumford* polynomial.

B) We define a sequence of integers $(e_r)_{r \in \mathbb{N}_0}$ by

$$e_0 := 0; \quad e_r := r e_{r-1} + 1, \quad \forall r \in \mathbb{N}.$$

It follows easily by induction, that

$$t \leq F_r(s, t) \leq F_r(s', t') \text{ if } 0 \leq s \leq s', \quad 0 \leq t \leq t';$$

$$F_r(s, t) < s^{e_r} (2t)^{r!} \text{ if } s, t \in \mathbb{N}. \quad \bullet$$

Now, we are ready to formulate and to prove the main result of this section.

2.6. Theorem. *Let $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \dots, \mathbf{x}_r]$ be a polynomial ring over the field K . Let $s, d \in \mathbb{N}$ and let $M \subseteq K[\underline{\mathbf{x}}]^{\oplus s}$ be a non-zero graded submodule with $d(M) \leq d$. Then*

$$\text{reg}(M) \leq F_r(s, d).$$

Proof: In view of 2.1 D) and 2.2 C) we may replace K by any of its extension fields and thus assume that K is infinite.

We proceed by induction on r . If $r = 0$, M is a graded free $K[\mathbf{x}_0]$ -module of finite rank, so that $\text{reg}(M) \leq d = F_0(s, d)$, (cf 2.2 D)).

So, let $r > 0$. We write $R := K[\underline{\mathbf{x}}]$ and $T := R^{\oplus s}/M$. As $\text{reg}(R^{\oplus s}) = 0$ and in view of the graded short exact sequence $0 \rightarrow M \rightarrow R^{\oplus s} \rightarrow T \rightarrow 0$ it suffices to show that $\text{reg}(T) \leq F_r(s, d) - 1$ (cf [5, (15.2.15) (i)]).

By 2.3 C) there is an element $f \in R_1 \setminus \{0\}$ which is filter-regular with respect to T . After a linear change of coordinates we may assume that $f = \mathbf{x}_r$.

Now, let $\overline{R} := R/\mathbf{x}_r R = K[\mathbf{x}_0, \dots, \mathbf{x}_{r-1}]$. Then, the graded \overline{R} -module $\overline{M} := (M + \mathbf{x}_r R^{\oplus s})/\mathbf{x}_r R^{\oplus s} \subseteq \overline{R}^{\oplus s}$ satisfies $d(\overline{M}) \leq d$. So, by induction we have $\text{reg}(\overline{M}) \leq F_{r-1}(s, d)$. By the base ring independence of local cohomology, this inequality remains valid if we consider \overline{M} as an R -module. As $\text{reg}(\mathbf{x}_r R^{\oplus s}) = 1$, the graded short exact sequence $0 \rightarrow \mathbf{x}_r R^{\oplus s} \rightarrow (M + \mathbf{x}_r R^{\oplus s}) \rightarrow \overline{M} \rightarrow 0$ therefore gives (cf [5, (15.2.15) (iii)])

$$(1) \quad \text{reg}(M + \mathbf{x}_r R^{\oplus s}) \leq F_{r-1}(s, d).$$

As $\text{reg}(R^{\oplus s}) = 0$, the graded short exact sequence $0 \rightarrow (M + \mathbf{x}_r R^{\oplus s}) \rightarrow R^{\oplus s} \rightarrow T/\mathbf{x}_r T \rightarrow 0$ gives $\text{reg}(T/\mathbf{x}_r T) \leq \text{reg}(M + \mathbf{x}_r R^{\oplus s}) - 1$ and hence

$$(2) \quad \text{reg}(T/\mathbf{x}_r T) \leq F_{r-1}(s, d) - 1.$$

By [5, (18.3.11)] we also have $\text{reg}^1(T) \leq \text{reg}(T/\mathbf{x}_r T)$, so that $\text{reg}^1(T) \leq F_{r-1}(s, d) - 1$, hence $\text{reg}^1(T) \leq F_r(s, d) - 1$. It therefore remains to show that $\text{end}(H_{R_+}^0(T)) \leq F_r(s, d) - 1$.

Applying cohomology to the graded short exact sequence $0 \rightarrow T/(0 \underset{T}{:} \mathbf{x}_r) \xrightarrow{\mathbf{x}_r} T(1) \rightarrow (T/\mathbf{x}_r T)(1) \rightarrow 0$ we get exact sequences

$$0 \rightarrow H_{R_+}^0 \left(T/(0 \underset{T}{:} \mathbf{x}_r) \right)_n \rightarrow H_{R_+}^0(T)_{n+1} \rightarrow H_{R_+}^0(T/\mathbf{x}_r T)_{n+1}.$$

In view of the inequality (2) we thus get isomorphisms

$$(3) \quad H_{R_+}^0 \left(T/(0 \underset{T}{:} \mathbf{x}_r) \right)_n \cong H_{R_+}^0(T)_{n+1}, \quad \forall n \geq F_{r-1}(s, d) - 1.$$

If we apply cohomology to the graded short exact sequence $0 \rightarrow (0 \underset{T}{:} \mathbf{x}_r) \rightarrow T \rightarrow T/(0 \underset{T}{:} \mathbf{x}_r) \rightarrow 0$ and keep in mind that $(0 \underset{T}{:} \mathbf{x}_r) \subseteq H_{R_+}^0(T)$

(cf 2.3 B)) we get exact sequences $0 \rightarrow (0 \underset{T}{:} \mathbf{x}_r)_n \rightarrow H_{R_+}^0(T)_n \rightarrow H_{R_+}^0\left(T/(0 \underset{T}{:} \mathbf{x}_r)\right)_n \rightarrow 0$ for all $n \in \mathbb{Z}$. So, in view of the isomorphism (3) we obtain short exact sequences

$$(4) \quad 0 \rightarrow (0 \underset{T}{:} \mathbf{x}_r)_n \rightarrow H_{R_+}^0(T)_n \xrightarrow{\pi_n} H_{R_+}^0(T)_{n+1} \rightarrow 0, \quad \forall n \geq F_{r-1}(s, d).$$

If we apply Lemma 2.4 to the submodules $M, \mathbf{x}_r R^{\oplus s} \subseteq R^{\oplus s}$ and keep in mind that $d(M), d(\mathbf{x}_r R^{\oplus s}) \leq d \leq F_{r-1}(s, d)$ and $\text{reg}(M + \mathbf{x}_r R^{\oplus s}) \leq F_{r-1}(s, d)$, (see (1)), we get $d(M \cap \mathbf{x}_r R^{\oplus s}) \leq F_{r-1}(s, d) + 1$. As $M \cap \mathbf{x}_r R^{\oplus s} = \mathbf{x}_r(M \underset{R^{\oplus s}}{:} \mathbf{x}_r)$, we obtain $d(M \underset{R^{\oplus s}}{:} \mathbf{x}_r) \leq F_{r-1}(s, d)$. But this means that $d(0 \underset{T}{:} \mathbf{x}_r) \leq F_{r-1}(s, d)$. So, if the epimorphism π_n in (4) is injective for some $n \geq F_{r-1}(s, d)$, the map π_m is an isomorphism for all $m \geq n$, hence $H_{R_+}^0(T)_m = 0$ for all $m \geq n$. So, in the range $n \geq F_{r-1}(s, d)$, the function $n \mapsto \dim_K(H_{R_+}^0(T)_n)$ is strictly decreasing until it reaches the value 0. But this implies $\text{end}(H_{R_+}^0(T)) \leq F_{r-1}(s, d) + \dim_K(H_{R_+}^0(T)_{F_{r-1}(s, d)}) - 1$. As $\dim_K(T_{F_{r-1}(s, d)}) \leq \dim_K((R^{\oplus s})_{F_{r-1}(s, d)}) = s \binom{F_{r-1}(s, d) + r}{r}$ and $H_{R_+}^0(T)_{F_{r-1}(s, d)} \subseteq T_{F_{r-1}(s, d)}$ it follows

$$\text{end}(H_{R_+}^0(T)) \leq F_{r-1}(s, d) + s \binom{F_{r-1}(s, d) + r}{r} - 1 = F_r(s, d) - 1,$$

and this concludes our proof. \blacksquare

2.7. Corollary. *Let $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \dots, \mathbf{x}_r]$ be a polynomial ring over the field K , let $s \in \mathbb{N}$ and let $M \subseteq K[\underline{\mathbf{x}}]^{\oplus s}$ be a graded submodule. Then,*

$$\text{reg}(M) \leq s^{e_r} (2d(M))^{r!}.$$

Proof: As $\text{reg}(0) = -\infty$, we may assume that $M \neq 0$. If $d(M) = 0$, by Nakayama, there is a graded isomorphism $M \cong K[\underline{\mathbf{x}}]^{\oplus u}$ with some $u \in \{1, \dots, s\}$, so that $\text{reg}(M) = 0$. Therefore we may assume that $d(M) > 0$. Then, we conclude by Theorem 2.6 and the estimate at the end of 2.5 B). \blacksquare

2.8. Remark. A) For $s = 1$, Corollary 2.7 gives the regularity bound of Bayer-Mumford [1].

C) In [6, (2.1)] it is shown that under the hypothesis of Corollary 2.7 one has $\text{reg}(M) \leq (2d(M))^{s^r r!}$. There, a different approach is used: First, the regularity criterion of Bayer-Stillman [2] is extended to graded submodules of free modules (cf [6, (1.10)]). Then, this extended criterion is used to prove the mentioned bound. Actually a slight modification of the proof of [6, (2.1)] gives the bound of Corollary 2.7. \bullet

3. REGULARITY OF DUAL SHEAVES

Let $r \in \mathbb{N}$, let K be a field and let \mathcal{F} be a coherent sheaf of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules. The aim of this section is to show that the regularity $reg(\mathcal{F}^\vee)$ of the dual sheaf $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_K^r}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}_K^r})$ is bounded in terms of the full cohomology diagonal $(h_{\mathcal{F}}^i(-i))_{i=0}^r$ of \mathcal{F} by a universal polynomial. We first give a few preliminaries.

3.1. Reminder and Remark. A) Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous noetherian ring and let $X = Proj(R)$ be the projective scheme defined by R . Let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules, and let M be a finitely generated graded R -module such that $\mathcal{F} = \tilde{M}$, e.g. \mathcal{F} is the sheaf of \mathcal{O}_X -modules induced by M . Then, the *Serre-Grothendieck correspondence* (cf [5, (20.4.4)]) yields an exact sequence of graded R -modules

$$(1) \quad 0 \rightarrow H_{R_+}^0(M) \rightarrow M \rightarrow \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{F}(n)) \rightarrow H_{R_+}^1(M) \rightarrow 0$$

and isomorphisms of graded R -modules

$$(2) \quad H_{R_+}^{i+1}(M) \cong \bigoplus_{n \in \mathbb{Z}} H^i(X, \mathcal{F}(n)), \quad (\forall i \in \mathbb{N}).$$

For the regularity of \mathcal{F} (cf (1.5)) we thus get

$$reg(\mathcal{F}) = reg^2(M).$$

B) Keep the previous notations and hypotheses and consider the graded R -module $T(\mathcal{F}) := \bigoplus_{n \geq 0} H^0(X, \mathcal{F}(n))$. Then, the exact sequence (1) of

A) shows that $T(\mathcal{F})_n = M_n$ for all $n \gg 0$, so that $T(\mathcal{F})$ is finitely generated and $T(\mathcal{F})^\sim \cong \mathcal{F}$. Applying the sequence (1) with $T(\mathcal{F})$ instead of M we now see that

$$H_{R_+}^0(T(\mathcal{F})) = 0, \quad \text{end}(H_{R_+}^1(T(\mathcal{F}))) < 0$$

and hence, if $\mathcal{F} \neq 0$:

$$\max\{0, reg(\mathcal{F})\} = reg(T(\mathcal{F})).$$

C) Keep the above notations and hypotheses and let $f \in R_1$. Then, f is filter-regular with respect to M if and only if it is a non-zero divisor with respect to $T(\mathcal{F})$ or - equivalently - if the homomorphism of sheaves $f : \mathcal{F} \rightarrow \mathcal{F}(1)$ is injective. In this situation, we also say that f is *regular with respect to \mathcal{F}* . If this is the case - with $Y := Proj(R/fR)$ - we have

exact sequences

$$(3) \quad \begin{array}{ccccccc} H^i(X, \mathcal{F}(n)) & \rightarrow & H^i(X, \mathcal{F}(n+1)) & \rightarrow & H^i(Y, \mathcal{F}|_Y(n+1)) & \rightarrow \\ H^{i+1}(X, \mathcal{F}(n)) & \rightarrow & H^{i+1}(X, \mathcal{F}(n+1)) & \rightarrow & H^{i+1}(Y, \mathcal{F}|_Y(n+1)) & \rightarrow \end{array}$$

in which $\mathcal{F}|_Y$ denotes the restriction of \mathcal{F} to Y .

D) Keep the previous hypothesis and notations and assume in addition, that $R_0 = K$ is a field. Let K' be an extension field of K . Then $X' = Proj(K' \otimes_K R)$ is a projective scheme over K' , $\mathcal{F}' := (K' \otimes_K M)^\sim$ is a coherent sheaf of $\mathcal{O}_{X'}$ -modules and

$$H^i(X', \mathcal{F}'(n)) \cong K' \otimes_K H^i(X, \mathcal{F}(n))$$

for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{N}$ (cf 2.2 C), 3.1 A). So in the notation of (1.1) we have

$$h_{\mathcal{F}'}^i(n) = h_{\mathcal{F}}^i(n) \text{ for all } i \in \mathbb{N}_0 \text{ and all } n \in \mathbb{Z}.$$

E) Keep the above notation and hypotheses. Let $\bullet^* = {}^* Hom_R(\bullet, R)$ denote the functor of taking duals in the category of graded R -modules. Then $(M^*)^\sim \cong (M^\sim)^\vee$ and hence $(T(\mathcal{F})^*)^\sim \cong \mathcal{F}^\vee$. \bullet

3.2. Definition and Remark. A) For $r \in \mathbb{N}_0$ we introduce the polynomials $G_r \in \mathbb{Q}[\mathbf{u}_0, \dots, \mathbf{u}_r, \mathbf{v}]$ defined by

$$\begin{aligned} G_0(\mathbf{u}_0; \mathbf{v}) &:= \mathbf{u}_0 \\ G_r(\mathbf{u}_0, \dots, \mathbf{u}_r; v) &:= \mathbf{u}_0 + \sum_{w=1}^v G_{r-1}(\mathbf{u}_0 + \mathbf{u}_1, \dots, \mathbf{u}_{r-1} + \mathbf{u}_r; w) \end{aligned}$$

for all $v \in \mathbb{N}_0$ and all $r > 0$.

B) Moreover, for each $r \in \mathbb{N}$ we consider the polynomial $H_r \in \mathbb{Q}[\mathbf{u}_1, \dots, \mathbf{u}_r]$ defined by

$$H_r(\mathbf{u}_1, \dots, \mathbf{u}_r) := \left(2 \sum_{j=1}^r \binom{r-1}{j-1} \mathbf{u}_j \right)^{2^{r-1}}.$$

Finally, let $H_0 := 0$.

C) Let $u_0, \dots, u_r, v, u'_0, \dots, u'_r, v' \in \mathbb{N}_0$ such that $u'_j \leq u_j$ for all $j \in \{0, \dots, r\}$ and $v \leq v'$. Then

$$\begin{aligned} 0 &\leq G_r(u_0, \dots, u_r, v) \leq G_r(u'_0, \dots, u'_r, v'), \\ 0 &\leq H_r(u_1, \dots, u_r) \leq H_r(u'_1, \dots, u'_r). \end{aligned} \quad \bullet$$

3.3. Lemma. *Let $r \in \mathbb{N}_0$, let K be a field and let \mathcal{F} be a coherent sheaf of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules. Then:*

- a) $h_{\mathcal{F}}^0(n) \leq G_r(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r); n)$ for all $n \geq 0$;
 b) $\text{reg}(\mathcal{F}) \leq \text{reg}(T(\mathcal{F})) \leq H_r(h_{\mathcal{F}}^1(-1), \dots, h_{\mathcal{F}}^r(-r)).$

Proof: For $r = 0$, both statements are obvious. So let $r > 0$.

"a)": In view of 3.1 D) we may assume that K is infinite. We write $\mathbb{P}_K^r = \text{Proj}(K[\underline{\mathbf{x}}])$, with a polynomial ring $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \dots, \mathbf{x}_r]$ and choose a finitely generated graded $K[\underline{\mathbf{x}}]$ -module M with $\tilde{M} = \mathcal{F}$. Then, there is an element $f \in K[\underline{\mathbf{x}}]_1 \setminus \{0\}$ which is filter-regular with respect to M , hence regular with respect to \mathcal{F} (cf 2.3 C), 3.1 C). We may assume that $f = \mathbf{x}_r$ and write $\mathbb{P}_K^{r-1} = \text{Proj}(K[\underline{\mathbf{x}}]/(\mathbf{x}_r))$ and $\mathcal{G} := \mathcal{F} \upharpoonright_{\mathbb{P}_K^{r-1}}$. In view of the sequences (3) of 3.1 C) we thus get the inequalities

$$h_{\mathcal{F}}^0(n) \leq h_{\mathcal{F}}^0(0) + \sum_{m=1}^n h_{\mathcal{G}}^0(m), \quad \text{for all } n \in \mathbb{N}_0;$$

$$h_{\mathcal{G}}^i(-i) \leq h_{\mathcal{F}}^i(-i) + h_{\mathcal{F}}^{i+1}(-(i+1)), \quad \text{for all } i \in \mathbb{N}_0.$$

By induction and in view of the monotony statement of 3.2 C), we now get

$$\begin{aligned} h_{\mathcal{F}}^0(n) &\leq h_{\mathcal{F}}^0(0) \\ &+ \sum_{m=1}^n G_{r-1}(h_{\mathcal{F}}^0(0) + h_{\mathcal{F}}^1(-1), \dots, h_{\mathcal{F}}^{r-1}(-(r-1)) + h_{\mathcal{F}}^r(-r); m) \\ &= G_r(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r); n). \end{aligned}$$

"b)": See [4, Rem. 6] and observe 3.2 C). ■

3.4. Definition and Remark. A) Let $R = \bigoplus_{n \geq 0} R_n$ be a noetherian, homogeneous ring such that $R_0 = K$ is a field. If T is a finitely generated and graded R -module, we use $\mu(T)$ to denote the minimal number of homogeneous elements needed to generate T .

B) Keep the notations and hypothesis of part A) and assume in addition that there is an element $f \in R_1$ which is a non-zero divisor with respect to T . Then

$$\mu(T) \leq \dim_K(T_n) \text{ for all } n \geq d(T). \quad \bullet$$

3.5. Notation. For $r \in \mathbb{N}_0$, let us introduce the polynomial

$$U_r := \binom{H_r(\mathbf{u}_1, \dots, \mathbf{u}_r) + r + 1}{r} G_r(\mathbf{u}_0, \dots, \mathbf{u}_r; H_r(\mathbf{u}_1, \dots, \mathbf{u}_r)) \\ \in \mathbb{Q}[\mathbf{u}_0, \dots, \mathbf{u}_r],$$

where G_r and H_r are defined according to 3.2 A), B). •

3.6. Lemma. *Let $r \in \mathbb{N}_0$, let K be a field, let $\mathcal{F} \neq 0$ be a coherent sheaf of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules and let $F_1 \rightarrow F_0 \xrightarrow{\pi} T(\mathcal{F}) \rightarrow 0$ be a minimal graded free presentation of the module $T(\mathcal{F})$ over the polynomial ring $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \dots, \mathbf{x}_r]$. Then*

- a) $rank(F_0) \leq G_r(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r); H_r(h_{\mathcal{F}}^1(-1), \dots, h_{\mathcal{F}}^r(-r)))$;
- b) $rank(F_1) \leq U_r(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r))$.

Proof: Let K' be an arbitrary extension field of K and consider the coherent sheaf of $\mathcal{O}_{\mathbb{P}_{K'}^r}$ -modules $\mathcal{F}' := \left(K' \otimes_K T(\mathcal{F}) \right) \tilde{\sim}$. Then the graded isomorphism $T(\mathcal{F}') \cong K' \otimes_K T(\mathcal{F})$ and the equalities $h_{\mathcal{F}'}^i(-i) = h_{\mathcal{F}}^i(-i)$ (cf 3.1 D)) allow to replace K and \mathcal{F} by K' and \mathcal{F}' . So, we may assume that K is infinite. Thus, there is an element $f \in K[\underline{\mathbf{x}}]_1 \setminus \{0\}$ which is filter-regular with respect to $T(\mathcal{F})$. As $H_{K[\underline{\mathbf{x}}]_+}^0(T(\mathcal{F})) = 0$ (cf 3.1 B)), f is a non-zero divisor with respect to $T(\mathcal{F})$. As $0 \leq d(T(\mathcal{F})) \leq reg(T(\mathcal{F})) \leq H_r(h_{\mathcal{F}}^1(-1), \dots, h_{\mathcal{F}}^r(-r))$ (cf 2.2 D), Lemma 3.3 b)) we conclude by 3.4 B) that $rank(F_0) = \mu(T(\mathcal{F})) \leq dim_K \left(T(\mathcal{F})_{H_r(h_{\mathcal{F}}^1(-1), \dots, h_{\mathcal{F}}^r(-r))} \right)$. So, by Lemma 3.3 a) and the definition of $T(\mathcal{F})$ we obtain:

$$(4) \quad rank(F_0) \leq G_r(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r); H_r(h_{\mathcal{F}}^1(-1), \dots, h_{\mathcal{F}}^r(-r))).$$

This proves in particular statement a). In view of 2.2 D) and Lemma 3.3 b) we have $d(Ker(\pi)) = d(F_1) \leq reg(T(\mathcal{F})) + 1 \leq H + 1$, where

$$(5) \quad H := H_r(h_{\mathcal{F}}^1(-1), \dots, h_{\mathcal{F}}^r(-r)).$$

As $Ker(\pi) \subseteq F_0$ is torsion free, 3.4 B) now gives $rank(F_1) = \mu(Ker(\pi)) \leq dim(Ker(\pi)_{H+1}) \leq dim_K((F_0)_{H+1})$. As $beg(F_0) = beg(T(\mathcal{F})) = 0$, we have $dim_K((F_0)_{H+1}) \leq rank(F_0) \binom{H+r+1}{r}$, thus $rank(F_1) \leq rank(F_0) \binom{H+r+1}{r}$. In view of (4) and (5) this proves statement b). ■

3.7. Definition and Remark. A) Let $r \in \mathbb{N}_0$. Using the notation of 2.5 A), 3.2 B) and 3.5 we define the polynomial

$$L_r := F_r(U_r(\mathbf{u}_0, \dots, \mathbf{u}_r), H_r(\mathbf{u}_1, \dots, \mathbf{u}_r) + 1) \in \mathbb{Q}[\mathbf{u}_0, \dots, \mathbf{u}_r].$$

B) By the monotony statements of 3.2 C) and 2.5 B) it follows

$$0 \leq L_r(u_0, \dots, u_r) \leq L_r(u'_0, \dots, u'_r)$$

for all $u_0, \dots, u_r, u'_0, \dots, u'_r \in \mathbb{N}_0$ with $u_j \leq u'_j$ for $j = 0, \dots, r$. \bullet

Now, we are ready to formulate and to prove the main result of the present section.

3.8. Theorem. *Let $r \in \mathbb{N}_0$, let K be a field and let \mathcal{F} be a coherent sheaf of $\mathcal{O}_{\mathbb{P}^r_K}$ -modules. Then, for the dual $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^r_K}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^r_K})$ of \mathcal{F} we have*

$$\text{reg}(\mathcal{F}^\vee) \leq L_r(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r)).$$

Proof: We may assume that $\mathcal{F} \neq 0$. Let $\underline{h} := (h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r))$ and consider a minimal graded free presentation $F_1 \rightarrow F_0 \rightarrow T(\mathcal{F}) \rightarrow 0$ of the graded module $T(\mathcal{F})$ (cf 3.1 B)) over the polynomial ring $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \dots, \mathbf{x}_r]$.

Applying the functor $\bullet^* = {}^* \text{Hom}_{K[\underline{\mathbf{x}}]}(\bullet, K[\underline{\mathbf{x}}])$ of graded duals to the above presentation, we get a graded exact sequence

$$(6) \quad 0 \rightarrow T(\mathcal{F})^* \rightarrow F_0^* \rightarrow Q \rightarrow 0,$$

in which Q is a graded submodule of F_1^* . As $-\text{beg}(F_1^*) = d(F_1) \leq \text{reg}(T(\mathcal{F})) + 1 \leq H_r(\underline{h}) + 1$, (cf 2.2 D), Lemma 3.3 b)), we have a graded embedding $F_1^* \hookrightarrow K[\underline{\mathbf{x}}]^{\oplus \text{rank}(F_1)}(H_r(\underline{h}) + 1)$. By Lemma 3.6 we know that $\text{rank}(F_1) \leq U_r(\underline{h})$. So $Q(-H_r(\underline{h}) - 1)$ becomes a graded submodule of $K[\underline{\mathbf{x}}]^{\oplus U_r(\underline{h})}$. As $d(Q) \leq d(F_0^*) = -\text{beg}(F_0) = 0$, we have $d(Q(-H_r(\underline{h}) - 1)) \leq H_r(\underline{h}) + 1$. So, by Theorem 2.6 we obtain $\text{reg}(Q(-H_r(\underline{h}) - 1)) \leq F_r(U_r(\underline{h}), H_r(\underline{h}) + 1)$ and hence

$$\text{reg}(Q) \leq F_r(U_r(\underline{h}), H_r(\underline{h}) + 1) - H_r(\underline{h}) - 1 \leq L_r(\underline{h}) - 1.$$

As $\text{reg}(F_0^*) = d(F_0^*) = -\text{beg}(F_0) = 0$, the exact sequence (6) yields the estimate $\text{reg}(T(\mathcal{F}^*)) \leq \max\{0, \text{reg}(Q) + 1\} \leq L_r(\underline{h})$. As $(T(\mathcal{F})^*)^\sim = \mathcal{F}^\vee$, (cf 3.1 E)), this proves our claim in view of 3.1 B). \blacksquare

3.9. Corollary. *Keep the notations and hypotheses of Theorem 3.8 and let $h_0, \dots, h_r \in \mathbb{N}_0$ be such that $h_{\mathcal{F}}^i(-i) \leq h_i$ for $i = 0, \dots, r$. Then*

$$\text{reg}(\mathcal{F}^\vee) \leq \text{reg}(T(\mathcal{F}^\vee)) \leq L_r(h_0, \dots, h_r).$$

Proof: As $\text{reg}(T(\mathcal{F}^\vee)) = \max\{0, \text{reg}(\mathcal{F}^\vee)\}$, (cf 3.1 B)) our statement follows from Theorem 3.8 and the inequalities of 3.7 B). \blacksquare

4. BOUNDING COHOMOLOGICAL POSTULATION NUMBERS

Let $r \in \mathbb{N}_0$ and let X be a projective scheme over the field K and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules such that $\dim(\mathcal{F})$ ($:= \dim(\text{Supp}(\mathcal{F}))$) $\leq r$. We shall prove that the cohomological postulation numbers $\nu_{\mathcal{F}}^i$ (see (1.3)) are bounded in terms of the cohomology diagonal $(h_{\mathcal{F}}^i(-i))_{i=0}^r$ by a universal polynomial.

We begin with the following auxiliary result, in which the notation introduced in 3.1 B) and 3.2 is used.

4.1. Lemma. *Let $r \in \mathbb{N}$, let K be a field, let \mathcal{F} be a coherent sheaf of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules. Let F be a graded free module of finite rank over the polynomial ring $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \dots, \mathbf{x}_r]$ and let $\pi : F \twoheadrightarrow T(\mathcal{F})$ be a minimal graded epimorphism. Let $\mathcal{G} := \text{Ker}(\pi)^\sim$ be the sheaf of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules induced by $\text{Ker}(\pi)$. Then:*

- a) $T(\mathcal{G}(1)) = T(\mathcal{G})(1) \cong \text{Ker}(\pi)(1)$;
 - b) $h_{\mathcal{G}(1)}^0(0) \leq r h_{\mathcal{F}}^0(0) + G_r(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r); 1)$;
 - c) $h_{\mathcal{G}(1)}^0(n) = 0$ for all $n < 0$;
 - d) if $r > 1$, then $h_{\mathcal{G}(1)}^1(n) = 0$ for all $n < 0$ and moreover $h_{\mathcal{G}(1)}^i(n) = h_{\mathcal{F}}^{i-1}(n+1)$ for $1 < i < r$ and all $n \in \mathbb{Z}$;
 - e) $h_{\mathcal{G}(1)}^r(-r) \leq h_{\mathcal{F}}^{r-1}(-r-1) + \binom{H+r}{r} G_r(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r); H)$,
- where $H := H_r(h_{\mathcal{F}}^1(-1), \dots, h_{\mathcal{F}}^r(-r))$.

Proof: Consider the exact sequence of sheaves of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules

$$(1) \quad 0 \rightarrow \mathcal{G} \rightarrow \tilde{F} \xrightarrow{\tilde{\pi}} \mathcal{F} \rightarrow 0.$$

"a)": Applying cohomology to (1) and keeping in mind that $r > 0$ and $\text{beg}(F) \geq 0$, we get a commutative diagram of graded $K[\underline{\mathbf{x}}]$ -modules with first exact row

$$\begin{array}{ccccccc} 0 & \longrightarrow & T(\mathcal{G}) & \longrightarrow & T(\tilde{F}) & \longrightarrow & T(\mathcal{F}) \\ & & & & \cong \uparrow & & \parallel \\ & & & & F & \xrightarrow{\pi} & T(\mathcal{F}) \end{array}$$

So, there is a graded isomorphism $T(\mathcal{G}) \cong \text{Ker}(\pi)$. As π is minimal, we have $\text{Ker}(\pi)_0 = 0$ and hence $T(\mathcal{G})_0 = 0$. It follows $T(\mathcal{G}(1)) = T(\mathcal{G})(1)_{\geq 0} = T(\mathcal{G})(1) \cong \text{Ker}(\pi)(1)$.

"b)": As usual we may assume that K is infinite. In view of statement a) we have $h_{\mathcal{G}(1)}^0(0) = \dim_K(\text{Ker}(\pi)_1) \leq \dim(F_1) = \dim_K(K[\underline{\mathbf{x}}]_1 F_0) + \dim_K(F_1/K[\underline{\mathbf{x}}]_1 F_0)$. As π is minimal, $\dim_K(F_0) = h_{\mathcal{F}}^0(0)$, hence

$\dim_K(K[\underline{\mathbf{x}}]_1 F_0) \leq (r+1)\dim_K(F_0) = (r+1)h_{\mathcal{F}}^0(0)$. Also, by the minimality of π we have $\dim_K(F_1/K[\underline{\mathbf{x}}]_1 F_0) = \dim_K(T(\mathcal{F})_1/K[\underline{\mathbf{x}}]_1 T(\mathcal{F})_0) = h_{\mathcal{F}}^0(1) - \dim_K(K[\underline{\mathbf{x}}]_1 H^0(X, \mathcal{F}))$. As K is infinite, there is an element $f \in K[\underline{\mathbf{x}}]_1$ which is filter-regular with respect to $T(\mathcal{F})$ (cf 2.3 C)) and the resulting monomorphism $f : H^0(X, \mathcal{F}) \hookrightarrow H^0(X, \mathcal{F}(1))$ (cf 3.1 C)) implies that $\dim_K(K[\underline{\mathbf{x}}]_1 H^0(X, \mathcal{F})) \geq h_{\mathcal{F}}^0(0)$. Altogether we obtain $h_{\mathcal{G}(1)}^0(0) \leq r h_{\mathcal{F}}^0(0) + h_{\mathcal{F}}^0(1)$ and the stated inequality follows by Lemma 3.3 a).

"c)": Clear from statement a) which gives $h_{\mathcal{G}}^0(0) = 0$.

"d)": Observe that $h_{\mathcal{G}}^0(n) = 0$ for all $n \leq 0$ (by c)), that $h_{\mathcal{F}}^0(0) = h_{\mathcal{F}}^0(0)$ by the minimality of π and that $h_{\mathcal{F}}^j(m) = 0$ for all $i \in \{1, \dots, r-1\}$ and all $m \in \mathbb{Z}$. Then apply cohomology to (1).

"e)": If we apply cohomology to (1) we obtain

$$h_{\mathcal{G}(1)}^r(-r) \leq h_{\mathcal{F}}^{r-1}(-(r-1)) + h_{\mathcal{F}}^r(-r+1).$$

There is a graded exact sequence

$$0 \rightarrow K[\underline{\mathbf{x}}]^{\oplus \text{rank}(F)}(-d(F)) \rightarrow F \rightarrow N \rightarrow 0$$

with $\dim(N) \leq r$. Passing to induced sheaves and then to cohomology, we get $h_{\mathcal{F}}^r(-r+1) \leq h_{\mathcal{O}_{\mathbb{P}^r_K}}^r(-d(F)-r+1)\text{rank}(F) = \binom{d(F)+r-2}{r}\text{rank}(F)$. As $\text{rank}(F) \leq G_r(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r), H)$ (cf Lemma 3.6 a)) and $d(F) = d(T(\mathcal{F})) \leq \text{reg}(T(\mathcal{F})) \leq H$ (cf 2.2 D) and Lemma 3.3 b)) we get our claim. \blacksquare

In our next Lemma, we use the notation of (1.3) and of 3.7.

4.2. Lemma. *Let $r \in \mathbb{N}_0$ let K be a field and let \mathcal{F} be a coherent sheaf of $\mathcal{O}_{\mathbb{P}^r_K}$ -modules. Then $\nu_{\mathcal{F}}^r \geq -r-1 - L_r(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r))$.*

Proof: Let $n \in \mathbb{Z}$. By duality we have $\text{Hom}_K(H^r(\mathbb{P}^r_K, \mathcal{F}(n)), K) \cong H^0(\mathbb{P}^r, \mathcal{F}^\vee(-n-r-1))$, hence $h_{\mathcal{F}}^r(n) = h_{\mathcal{F}^\vee}^0(-n-r-1)$. So, if $-n-r-1 \geq \text{reg}(\mathcal{F}^\vee)$, we have $h_{\mathcal{F}}^r(n) = \chi_{\mathcal{F}^\vee}(-n-r-1)$. As $n \mapsto \chi_{\mathcal{F}^\vee}(-n-r-1)$ is a polynomial function, it follows that $\nu_{\mathcal{F}}^r \geq -r-1 - \text{reg}(\mathcal{F}^\vee)$ and our claim results from Theorem 3.8. \blacksquare

4.3. Lemma. *Let K be a field, let $r \in \mathbb{N}_0$, let \mathcal{F} be a coherent sheaf of $\mathcal{O}_{\mathbb{P}^r_K}$ -modules and let $e := \sum_{x \in \mathbb{P}^r_K, x \text{ closed}} \text{length}_{\mathcal{O}_{\mathbb{P}^r_K, x}}(H_{\mathfrak{m}_{\mathbb{P}^r_K, x}}^0(\mathcal{F}_x))$. Then,*

$$e \leq h_{\mathcal{F}}^0(n-1) \leq \max\{e, h_{\mathcal{F}}^0(n) - 1\},$$

for each $n \in \mathbb{Z}$.

Proof: Let $\mathcal{H} \subseteq \mathcal{F}$ be the (unique) maximal coherent subsheaf with finite support. Then \mathcal{H} is of length e and $\mathcal{G} := \mathcal{F}/\mathcal{H}$ has no closed associated points. Therefore $h_{\mathcal{H}}^0(m) = e$ and $h_{\mathcal{H}}^1(m) = 0$, so that $h_{\mathcal{F}}^0(m) = e + h_{\mathcal{G}}^0(m)$ for all $m \in \mathbb{Z}$. As $h_{\mathcal{G}}^0(n-1) \leq \max\{0, h_{\mathcal{G}}^0(n) - 1\}$ (cf [4, 5.3]), we get our claim. \blacksquare

Now, in order to prove and to formulate the main result of this section, let us define one more class of bounding polynomials.

4.4. Definition and Remark. A) Let $r \in \mathbb{N}_0$. Then, for $p = 0, \dots, r$ we define polynomials $M_{r,p} \in \mathbb{Q}[\mathbf{u}_0, \dots, \mathbf{u}_r]$ by

$$\begin{aligned} M_{r,0} &:= -\mathbf{u}_0, \text{ and for } 1 \leq p \leq r \\ M_{r,p} &:= -r - L_r(\mathbf{u}_0, \dots, \mathbf{u}_r) + M_{r,p-1}(V, 0, \mathbf{u}_1, \dots, \mathbf{u}_{r-2}, W), \end{aligned}$$

where

$$\begin{aligned} V &:= r\mathbf{u}_0 + G_r(\mathbf{u}_0, \dots, \mathbf{u}_r; 1) \\ W &:= \mathbf{u}_{r-1} + \binom{H_r(\mathbf{u}_1, \dots, \mathbf{u}_r) + r}{r} G_r(\mathbf{u}_0, \dots, \mathbf{u}_r; H_r(\mathbf{u}_1, \dots, \mathbf{u}_r)), \end{aligned}$$

(and where G_r, H_r, L_r are defined as in 3.2 A), B) and 3.7 A) respectively). Finally, we set

$$M_r := M_{r,r}.$$

B) In view of the monotony statements of 3.2 C) and 3.7 B) it follows that

$$-u_0 \geq M_{r,p}(u_0, \dots, u_r) \geq M_{r,p}(u'_0, \dots, u'_r)$$

for all $u_0, \dots, u_r, u'_0, \dots, u'_r \in \mathbb{N}_0$ with $u_j \leq u'_j$ for $j = 0, \dots, r$ and for all $p \in \{0, \dots, r\}$. \bullet

In the following lemma, $pd_R(M)$ is used to denote the projective dimension of an R -module M .

4.5. Lemma. *Let $r \in \mathbb{N}$, let $p \in \{0, \dots, r\}$, let K be a field and let \mathcal{F} be a coherent sheaf of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules with $pd_{K[\underline{\mathbf{x}}]}(T(\mathcal{F})) \leq p$, where $K[\mathbf{x}_0, \dots, \mathbf{x}_r] = K[\underline{\mathbf{x}}]$ is a polynomial ring. Then*

$$\nu_{\mathcal{F}}^i \geq M_{r,p}(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r)) \text{ for } i = 0, \dots, r.$$

Proof: (Induction on p) Let $p = 0$. Then $T(\mathcal{F})$ is a graded free module with $beg(T(\mathcal{F})) \geq 0$. As $\mathcal{F} = T(\mathcal{F})^\sim$ we have $h_{\mathcal{F}}^i(n) = 0$ for all $i \neq 0, r$ and all $n \in \mathbb{Z}$. But this implies that $\nu_{\mathcal{F}}^i \geq \nu_{\mathcal{F}}^0$ for all $i \in \mathbb{N}_0$. In view of Lemma 4.3 we have $\nu_{\mathcal{F}}^0 \geq -h_{\mathcal{F}}^0(0) = M_{r,0}(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r))$.

So, let $p > 0$, let $\pi : F \rightarrow T(\mathcal{F})$ be a minimal graded epimorphism from a graded free $K[\underline{\mathbf{x}}]$ -module F and let $\mathcal{G} := Ker(\pi)^\sim$. By statement a) of Lemma 4.1 we have $pd_{K[\underline{\mathbf{x}}]}(T(\mathcal{G}(1))) < p$. So, by induction $\nu_{\mathcal{G}(1)}^i \geq M_{r,p-1}(h_{\mathcal{G}(1)}^0(0), \dots, h_{\mathcal{G}(1)}^r(-r))$ for all $i \in \mathbb{N}_0$. By statements b), c), d)

and e) of Lemma 4.1 and in the notation of 4.4 we thus get $\nu_{\mathcal{F}}^{i-1} = \nu_{\mathcal{G}(1)}^i \geq M$ for $i = 2, \dots, r-1$, where

$$M := M_{r,p-1}(V(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r)), 0, h_{\mathcal{F}}^1(-1), \dots, h_{\mathcal{F}}^{r-2}(-(r-2)), \\ W(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r))).$$

By Lemma 4.3, the definition of V and the lefthand side inequality of 4.4 B) we have $\nu_{\mathcal{F}}^0 \geq -h_{\mathcal{F}}^0(0) \geq -V(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r)) \geq M$. So $\nu_{\mathcal{F}}^i \geq M_{r,p}(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r))$ for $i = 0, \dots, r-2$. By Lemma 4.2 we also have $\nu_{\mathcal{F}}^r \geq -r-1 - L_r(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r)) \geq -r - L_r(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r)) + M = M_{r,p}(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r))$. As the function $n \mapsto \sum_{i=0}^r (-1)^i h_{\mathcal{F}}^i(n) = \chi_{\mathcal{F}}(n)$ is polynomial, we get $\nu_{\mathcal{F}}^{r-1} \geq M_{r,p}(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r))$. This concludes our proof. \blacksquare

4.6. Theorem. *Let $r \in \mathbb{N}_0$, let X be a projective scheme over the field K and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules such that $\dim(\mathcal{F}) \leq r$. Then $\nu_{\mathcal{F}}^i \geq M_r(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r))$ for all $i \in \mathbb{N}_0$.*

Proof: As usual, we may assume that K is infinite. Let $\mathcal{J} \subseteq \mathcal{O}_X$ be the annihilator sheaf of \mathcal{F} and let $Y \subseteq X$ be the closed subscheme defined by \mathcal{J} . Then $h_{\mathcal{F}}^i(n) = h_{\mathcal{F}|_Y}^i(n)$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$. This allows to replace X by Y and hence to assume that $\dim(X) = \dim(\mathcal{F}) \leq r$. As K is infinite we thus find a finite morphism $\varphi : X \rightarrow \mathbb{P}_K^r$ induced by global sections of $\mathcal{O}_X(1)$. It follows that $h_{\mathcal{F}}^i(n) = h_{\varphi_*\mathcal{F}}^i(n)$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{N}$. This allows to replace \mathcal{F} by $\varphi_*\mathcal{F}$ and hence to assume that $X = \mathbb{P}_K^n = Proj(K[\underline{\mathbf{x}}])$, where $K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \dots, \mathbf{x}_r]$ is a polynomial ring. As $H_{K[\underline{\mathbf{x}}]_+}^0(T(\mathcal{F})) = 0$ (cf 3.1 B)), we have $pd_{K[\underline{\mathbf{x}}]}(T(\mathcal{F})) \leq r$. So, Lemma 4.5 gives that $\nu_{\mathcal{F}}^i \geq M_{r,r}(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^r(-r))$ for all $i \in \mathbb{N}_0$. As $M_r = M_{r,r}$, this proves our claim. \blacksquare

4.7. Corollary. *Keep the notations and hypotheses of Theorem 4.6 and let $h_0, \dots, h_r \in \mathbb{N}_0$ be such that $h_{\mathcal{F}}^i(-i) \leq h_i$ for $i = 0, \dots, r$. Then $\nu_{\mathcal{F}}^i \geq M_r(h_0, \dots, h_r)$ for all $i \in \mathbb{N}_0$.*

Proof: Clear from Theorem 4.6 and the monotony property of 4.4 B). \blacksquare

5. A FINITENESS RESULT

Again, let $r \in \mathbb{N}_0$, let X be a projective scheme over a field K and let $h_0, \dots, h_r \in \mathbb{N}_0$. In this section we shall prove that there are only finitely many possible cohomological Hilbert functions $h_{\mathcal{F}}^i : \mathbb{Z} \rightarrow \mathbb{N}_0$ if \mathcal{F} runs through all coherent sheaves of \mathcal{O}_X -modules with $\dim(\mathcal{F}) \leq r$ and satisfying $h_{\mathcal{F}}^i(-i) = h_i$ for $i = 0, \dots, r$. So, what we prove is that

the cohomology diagonal $(h_{\mathcal{F}}^i(-i))_{i=0}^{\dim \mathcal{F}}$ of a coherent sheaf \mathcal{F} of \mathcal{O}_X -modules bounds the number of possible cohomological Hilbert functions $h_{\mathcal{F}}^i$ of \mathcal{F} . First, we prove the following result, in which the polynomials $H_k \in \mathbb{Q}[\mathbf{u}_1, \dots, \mathbf{u}_r]$ and $G_k \in \mathbb{Q}[\mathbf{u}_0, \dots, \mathbf{u}_k]$ are defined according to 3.2.

5.1. Lemma. *Let $r \in \mathbb{N}_0$, let K be a field, let X be a projective scheme over K . Let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules with $\dim \mathcal{F} \leq r$. Then:*

- a) $h_{\mathcal{F}}^i(n) \leq \frac{1}{2}H_{r-i+1}(h_{\mathcal{F}}^i(-i), \dots, h_{\mathcal{F}}^r(-r))$ for $1 \leq i \leq r$ and for all $n \geq -i$.
- b) $h_{\mathcal{F}}^i(n) \leq G_i(h_{\mathcal{F}}^i(-i), \dots, h_{\mathcal{F}}^0(0); -n - i)$ for $0 \leq i \leq r$ and for all $n \leq -i$.

Proof: "a)" See [4, Rem. 6].

"b)" As usual, we may assume that K is infinite. Let $X = Proj(R)$, where $R = \bigoplus_{n \geq 0} R_n$ is a homogeneous noetherian ring with $R_0 = K$.

Then, there is an element $f \in R_1$ which is filter-regular with respect to $T(\mathcal{F})$ and hence regular with respect to \mathcal{F} . The induced monomorphisms $H^0(X, \mathcal{F}(n)) \xrightarrow{f} H^0(X, \mathcal{F}(n+1))$ prove the case $i = 0$. So, let $i > 0$ and set $Y = Proj(R/fR)$ and $\mathcal{G} := \mathcal{F}|_Y$. Then, $\dim(\mathcal{G}) < r$ and by induction we have $h_{\mathcal{G}}^{i-1}(m) \leq G_{i-1}(h_{\mathcal{G}}^{i-1}(-(i-1)), \dots, h_{\mathcal{G}}^0(0); -m - i + 1)$, ($\forall m \leq -i + 1$). Moreover the sequences (3) of 3.1 C) show that $h_{\mathcal{F}}^i(n) \leq h_{\mathcal{F}}^i(-i) + \sum_{m=n+1}^{-i} h_{\mathcal{G}}^{i-1}(m)$ for all $n \leq -i$ and $h_{\mathcal{G}}^j(-j) \leq h_{\mathcal{F}}^j(-j) + h_{\mathcal{F}}^{j+1}(-(j+1))$ for all $j \leq i - 1$. In view of the monotony property of G_{i-1} and the definition of G_i (cf 3.2), we get our claim. \blacksquare

5.2. Notation. Let $r \in \mathbb{N}_0$ and let $\underline{h} = (h_0, \dots, h_r) \in \mathbb{N}_0^{r+1}$. By $\overline{\mathcal{C}}_{\leq \underline{h}}^{(r)}$ we denote the class of all pairs (X, \mathcal{F}) in which X is a projective scheme over some field K and in which \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules such that

$$\dim(\mathcal{F}) \leq r \text{ and } h_{\mathcal{F}}^i(-i) \leq h_i \text{ for } i = 0, \dots, r. \quad \bullet$$

Now, we have the following finiteness result for cohomological Hilbert polynomials and characteristic polynomials. (We use the symbol $\#$ to denote cardinality.)

5.3. Proposition. *Let $r \in \mathbb{N}_0$ and let $\underline{h} = (h_0, \dots, h_r) \in \mathbb{N}_0^{r+1}$. Then*

- a) $\forall i \in \{0, \dots, r\} : \# \left\{ p_{\mathcal{F}}^i \mid (X, \mathcal{F}) \in \overline{\mathcal{C}}_{\leq \underline{h}}^{(r)} \right\} < \infty$.
- b) $\# \left\{ \chi_{\mathcal{F}} \mid (X, \mathcal{F}) \in \overline{\mathcal{C}}_{\leq \underline{h}}^{(r)} \right\} < \infty$.

Proof: "a)": Fix $i \in \{0, \dots, r\}$. Let $\mathcal{C} := \overline{\mathcal{C}}_{\leq \underline{h}}^{(r)}$. We define in the notation of 4.4 and 3.2

$$N_{r,i}(\underline{h}) := \prod_{n=M_r(\underline{h})-i}^{M_r(\underline{h})} (G_i(h_i, \dots, h_0; -n-i) + 1).$$

By 4.7 we have $\nu_{\mathcal{F}}^i \geq M_r(\underline{h})$ whenever $(X, \mathcal{F}) \in \mathcal{C}$. So, by 5.1 b) and the monotony properties of G_i we obtain $0 \leq p_{\mathcal{F}}^i(n) \leq G_i(h_i, \dots, h_0; -n-i)$ for all $(X, \mathcal{F}) \in \mathcal{C}$ and all $n \leq M_r(\underline{h})$. As $p_{\mathcal{F}}^i$ is a polynomial of degree $\leq i$, it is determined by the values $p_{\mathcal{F}}^i(n)$ with $M_r(\underline{h}) - i \leq n \leq M_r(\underline{h})$. So, at most $N_{r,i}(\underline{h})$ different cohomological Hilbert polynomials $p_{\mathcal{F}}^i$ occur, if (X, \mathcal{F}) runs through \mathcal{C} .

"b)": Follows from a) as $\chi_{\mathcal{F}} = \sum_{i=0}^r (-1)^i p_{\mathcal{F}}^i$ whenever \mathcal{F} is a coherent sheaf of dimension $\leq r$ over a projective scheme over a field. ■

5.4. Theorem. *Let $0 \leq i \leq r$ and let $\underline{h} = (h_0, \dots, h_r) \in \mathbb{N}_0^{r+1}$. Then*

$$\# \left\{ h_{\mathcal{F}}^i \mid (X, \mathcal{F}) \in \overline{\mathcal{C}}_{\leq \underline{h}}^{(r)} \right\} < \infty.$$

Proof: Let $\mathcal{C} := \overline{\mathcal{C}}_{\leq \underline{h}}^{(r)}$. Assume first that $i > 0$. Let

$$\begin{aligned} \underline{h}_{\geq k} &:= (h_k, \dots, h_r), (k = 1, \dots, r), \\ S_{r,i}(\underline{h}) &:= \prod_{n=M_r(\underline{h})+1}^{-i} (G_i(h_i, \dots, h_0; -n+i) + 1), \\ T_{r,i}(\underline{h}) &:= \left(\frac{1}{2} H_{r-i+1}(\underline{h}_{\geq i}) + 1 \right)^{H_r(\underline{h}_{\geq 1})-1}. \end{aligned}$$

By Lemma 5.1 and the monotony properties of G_i and of H_{r-i+1} we see that at most $S_{r,i}(\underline{h})T_{r,i}(\underline{h})$ different functions

$$h_{\mathcal{F}}^i \upharpoonright: [M_r(\underline{h}) + 1, H_r(\underline{h}_{\geq 1}) - 1] \rightarrow \mathbb{N}_0$$

may occur if (X, \mathcal{F}) runs through \mathcal{C} .

For each pair $(X, \mathcal{F}) \in \mathcal{C}$ we have $\nu_{\mathcal{F}}^i \geq M_r(\underline{h})$ and $\text{reg}(\mathcal{F}) \leq H_r(\underline{h}_{\geq 1})$, (s. 4.7 and 3.3 b)), so that $h_{\mathcal{F}}^i(n) = p_{\mathcal{F}}^i(n)$ for all $n \leq M_r(\underline{h})$ and $h_{\mathcal{F}}^i(n) = 0$ for all $n \geq H_r(\underline{h}_{\geq 1})$. By Proposition 5.3 it follows that $\{h_{\mathcal{F}}^i \mid (X, \mathcal{F}) \in \mathcal{C}\}$ is a finite set.

As $\sum_{i=0}^r (-1)^i h_{\mathcal{F}}^i(n) = \chi_{\mathcal{F}}(n)$ for all $n \in \mathbb{Z}$, $h_{\mathcal{F}}^0(n) = \chi_{\mathcal{F}}(n)$ for all $n \geq H_r(\underline{h}_{\geq 1})$ and $h_{\mathcal{F}}^0(n) = p_{\mathcal{F}}^0(n)$ for all $n \leq M_r(\underline{h})$, the finiteness of the set $\{h_{\mathcal{F}}^0 \mid (X, \mathcal{F}) \in \mathcal{C}\}$ follows by Proposition 5.3 and the finiteness of the sets $\{h_{\mathcal{F}}^i \mid (X, \mathcal{F}) \in \mathcal{C}\}$ for $i > 0$. ■

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