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Brodmann, M

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Other titles: Selected papers of the international workshop, Guanajuato, Mexico. New York, NY

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ZORA URL: <https://doi.org/10.5167/uzh-21931>

Book Section

Originally published at:

Brodmann, M (2002). Cohomological invariants of coherent sheaves over projective schemes: a survey.

In: Lyubeznik, G. Local cohomology and its applications (Guanajuato, 1999). New York: Dekker, 91-120.

**COHOMOLOGICAL INVARIANTS OF COHERENT
SHEAVES OVER PROJECTIVE SCHEMES
– A SURVEY**

M. BRODMANN

ABSTRACT. We give a survey on certain results related to the cohomology of projective schemes with coefficients in coherent sheaves. In particular we present results on cohomological patterns, cohomological Hilbert functions and cohomological Hilbert polynomials. Bounding results for Castelnuovo-Mumford regularities, Severi coregularities and cohomological postulation numbers are discussed. Moreover, a number of open questions is presented.

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1. INTRODUCTION

Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous noetherian ring. Let $X := \text{Proj}(R)$ be the projective scheme induced by R . Let $i \in \mathbb{N}_0$. For a sheaf of \mathcal{O}_X -modules \mathcal{G} , the i -th Serre cohomology group $H^i(X, \mathcal{G})$ of X with coefficients in \mathcal{F} carries a natural structure of R_0 -module. If \mathcal{G} is coherent, it follows from Serre's finiteness theorem, that the R_0 -module $H^i(X, \mathcal{G})$ is finitely generated (cf. [42, III, Theorem 5.2], [72, §66, Théorème 1]).

Now, fix a coherent sheaf of \mathcal{O}_X -modules \mathcal{F} . For $n \in \mathbb{Z}$ consider the coherent sheaf of \mathcal{O}_X -modules $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}(1)^{\otimes n}$, e.g. the n -th twist of \mathcal{F} (with respect to the very ample sheaf $\mathcal{O}_X(1)$).

A considerable number of results in algebraic geometry can be expressed in the form of vanishing statements for some cohomology groups $H^i(X, \mathcal{F}(n))$. Correspondingly, there are several numerical cohomological invariants of the pair (X, \mathcal{F}) related to the vanishing and non-vanishing of the groups $H^i(X, \mathcal{F}(n))$.

Let us mention first the *cohomological dimension* of (X, \mathcal{F}) , which is given by

$$(1.1) \quad cd(X, \mathcal{F}) = cd(\mathcal{F}) := \sup\{i \in \mathbb{N}_0 \mid \exists n \in \mathbb{Z} : H^i(X, \mathcal{F}(n)) \neq 0\}.$$

(Throughout this paper we use the convention that $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.) It is important to notice that $cd(\mathcal{F}) < \infty$. In fact the invariant $cd(\mathcal{F})$ depends only on the topological behaviour of \mathcal{F} along the fibres of the natural morphism $\pi : X \rightarrow X_0 := \text{Spec}(R_0)$, (s. (2.3)), in accordance with the *Vanishing Theorem of Serre-Grothendieck* (cf. [72, §66, Théorème 1], [42, III, Theorem 2.7]).

Next, for each $i \in \mathbb{N}$, we may define the i -th *cohomological right vanishing order* of (X, \mathcal{F}) by

$$(1.2) \quad \mu_{X, \mathcal{F}}^i = \mu_{\mathcal{F}}^i := \sup\{n \in \mathbb{Z} \mid H^i(X, \mathcal{F}(n-1)) \neq 0\}.$$

By the *Vanishing Theorem of Castelnuovo-Serre* (cf. [72, §66, Théorème 2 (b)]) we have $\mu_{\mathcal{F}}^i < \infty$ for all $i \in \mathbb{N}$.

For $k \in \mathbb{N}$, the (*Castelnuovo-Mumford*) *regularity* of (X, \mathcal{F}) above level k is defined by

$$(1.3) \quad reg_k(X, \mathcal{F}) = reg_k(\mathcal{F}) := \sup\{\mu_{\mathcal{F}}^i + i \mid i > k\}.$$

As $\mu_{\mathcal{F}}^i < \infty$ for all $i > 0$ and as $cd(\mathcal{F}) < \infty$, we have $reg_k(\mathcal{F}) < \infty$. Moreover, $reg(\mathcal{F}) := reg_0(\mathcal{F})$ is the (*Castelnuovo-Mumford*) *regularity* as originally introduced in [65].

In 1893 G. Castelnuovo proved a result which, in our language, says that $reg_0(\mathcal{J}) \leq d-1$, where $\mathcal{J} \subseteq \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^3}$ is the sheaf of vanishing ideals of a smooth projective curve in $\mathbb{P}_{\mathbb{C}}^3$, (s. [25]). This is apparently the first bounding result for cohomological right vanishing orders and regularities. For this reason, Mumford did speak of Castelnuovo regularity and for the same reason we

associate the name of Castelnuovo to all vanishing or bounding results on the groups $H^i(X, \mathcal{F}(n))$ in the range $n \geq -i$.

The behaviour of the groups $H^i(X, \mathcal{F}(n))$ for $n \ll 0$ is more subtle than in the ‘‘Castelnuovo’’ range: in general, these groups do not vanish if $n \ll 0$. The problem of vanishing in this range essentially can be attacked only if i does not exceed the *cohomological finiteness dimension*

$$(1.4) \quad \begin{aligned} f(X, \mathcal{F}) &= f(\mathcal{F}) \\ &:= \inf\{i \in \mathbb{N}_0 \mid H^i(X, \mathcal{F}(n)) \neq 0 \text{ for infinitely many } n \leq 0\} \end{aligned}$$

of the pair (X, \mathcal{F}) . Under certain assumptions on R_0 , the finiteness dimension $f(\mathcal{F})$ depends only on the behaviour of \mathcal{F} along the fibers of the natural morphism $\pi : X \rightarrow X_0 = \text{Spec}(R_0)$, (s. (2.4) (iii)), in accordance with the *Vanishing Theorem of Severi-Enriques-Zariski-Serre* (cf. [72, §74, Théorème 2], [42, III, Theorem 7.6 (b)]).

Now, obviously for each $i < f(\mathcal{F})$, we may define the *left-vanishing order* of (X, \mathcal{F}) by

$$(1.5) \quad \nu_{X, \mathcal{F}}^i = \nu_{\mathcal{F}}^i := \inf\{n \in \mathbb{Z} \mid H^i(X, \mathcal{F}(n+1)) \neq 0\}$$

and thus have $\nu_{\mathcal{F}}^i > -\infty$ for all $i < f(\mathcal{F})$.

For $k \in \{0, 1, \dots, f(\mathcal{F}) - 1\}$ we define the (*Severi*-) *coregularity of (X, \mathcal{F}) at and below level k* by

$$(1.6) \quad \text{coreg}^k(X, \mathcal{F}) = \text{coreg}^k(\mathcal{F}) := \inf\{\nu_{\mathcal{F}}^i + i \mid i \leq k\}.$$

As $\nu_{\mathcal{F}}^i > -\infty$ for all $i < f(\mathcal{F})$ we have $\text{coreg}^k(\mathcal{F}) > -\infty$ for all $k < f(\mathcal{F})$. If $f(\mathcal{F}) > 0$ we call the number $\text{coreg}(\mathcal{F}) := \text{coreg}^{f(\mathcal{F})-1}(\mathcal{F})$ the (*Severi*-) *coregularity of (X, \mathcal{F})* .

In 1942 F. Severi proved a result which, in the language of sheaf cohomology, says that $f(\omega_X) > 1$ for the canonical sheaf ω_X of a smooth projective surface $X \subseteq \mathbb{P}_{\mathbb{C}}^3$, (s. [73]). In 1949, F. Enriques gave an attempt to generalize this to the case where $X \subseteq \mathbb{P}_{\mathbb{C}}^r$ is a smooth projective variety of dimension > 1 , (s. [29]). In 1952, Zariski proved that for an arbitrary invertible sheaf \mathcal{L} over a normal projective variety X of dimension > 1 , we have $f(\mathcal{L}) > 1$, (s. [82]). In 1955, J.P. Serre brought this result to its final form by showing that for an arbitrary coherent sheaf \mathcal{F} over a projective variety X , $f(\mathcal{F})$ equals the minimum of the depths of the stalks of \mathcal{F} at closed points $x \in X$, (cf. [72, §74, Théorème 2]).

Apparently, Severi was the first to prove a non-trivial vanishing result for some of the groups $H^i(X, \mathcal{F}(n))$ in the range $n \ll 0$. Therefore, we speak of Severi coregularity. Moreover, for the same reason, we associate the name of Severi to all vanishing or bounding results on the groups $H^i(X, \mathcal{F}(n))$ in the range $n \leq -i$, $i < f(\mathcal{F})$.

In order to study and to formulate vanishing results, it is convenient to introduce the so called *cohomological pattern*

$$(1.7) \quad P(X, \mathcal{F}) = P(\mathcal{F}) := \{(i, n) \in \mathbb{N}_0 \times \mathbb{Z} \mid H^i(X, \mathcal{F}(n)) \neq 0\}$$

of the pair (X, \mathcal{F}) . We shall discuss this concept in Section 2.

Besides the mere vanishing and non-vanishing of the cohomology groups $H^i(X, \mathcal{F}(n))$ one may also ask for the “size” of these R_0 -modules. A natural approach to this is to assume that the base ring R_0 is artinian: then the R_0 -modules $H^i(X, \mathcal{F}(n))$ are of finite length, so that this length may be used to measure their size.

So, assume now, that R_0 is artinian. For any $i \in \mathbb{N}_0$, any $n \in \mathbb{Z}$ and any coherent sheaf of \mathcal{O}_X -modules \mathcal{F} we consider the length

$$(1.8) \quad h_{X, \mathcal{F}}^i(n) = h_{\mathcal{F}}^i(n) := \text{length}_{R_0}(H^i(X, \mathcal{F}(n)))$$

of the R_0 -module $H^i(X, \mathcal{F}(n))$. For fixed $i \in \mathbb{N}_0$, the function

$$(1.9) \quad h_{X, \mathcal{F}}^i = h_{\mathcal{F}}^i : \mathbb{Z} \longrightarrow \mathbb{N}_0; \quad (n \longmapsto h_{\mathcal{F}}^i(n))$$

is called the i -th *cohomological Hilbert function* of (X, \mathcal{F}) .

Now, fix $k \in \mathbb{N}_0$. Then, there are two easy *vanishing constraints* for the numbers $h_{\mathcal{F}}^i(n)$ (cf. (2.4) (i), (ii)). The first of these constraints says:

$$(1.10) \quad \begin{aligned} & \text{If } h_{\mathcal{F}}^i(-i) = 0 \text{ for all } i > k, \\ & \text{then } h_{\mathcal{F}}^i(n) = 0 \text{ for all } i > k \text{ and all } n \geq -i. \end{aligned}$$

The second constraint says:

$$(1.11) \quad \begin{aligned} & \text{If } h_{\mathcal{F}}^i(-i) = 0 \text{ for all } i \leq k, \\ & \text{then } h_{\mathcal{F}}^i(n) = 0 \text{ for all } i \leq k \text{ and all } n \leq -i. \end{aligned}$$

Observe that the vanishing assumption in the constraint (1.11) implies in particular that $k < f(\mathcal{F})$.

In sections 3 and 4 we shall present a couple of bounding results, which extend the above constraints. We actually have to distinguish three types of such results.

The first type gives bounds on the invariant $\text{reg}_k(\mathcal{F})$ and on the numbers $h_{\mathcal{F}}^i(n)$ in the range $i > k$, $n \geq -i$ in terms of the *cohomology diagonal* $(h_{\mathcal{F}}^i(-i))_{i=k+1}^{\text{cd}(\mathcal{F})}$ of \mathcal{F} above level k . We refer to these bounds as (*diagonal*) *bounds of Castelnuovo type* – in accordance with our earlier convention.

The second type of result applies if $k < f(\mathcal{F})$ and bounds the invariant $\text{coreg}^k(\mathcal{F})$ and the numbers $h_{\mathcal{F}}^i(n)$ in the range $i \leq k$, $n \leq -i$ in terms of the *cohomology diagonal* $(h_{\mathcal{F}}^i(-i))_{i=0}^k$ of \mathcal{F} at and below level k . Here, we speak of (*diagonal*) *bounds of Severi type*.

The third type of result is referred to as bounds of *extended Severi type*. Their aim is to extend the bounds of Severi type beyond the situation in

which $k < f(\mathcal{F})$. This first of all needs some conceptual modification of the ideas underlying the previous type of bound.

The crucial point is to keep in mind that for each $i \in \mathbb{N}_0$ there is a (unique) polynomial

$$(1.12) \quad p_{X,\mathcal{F}}^i = p_{\mathcal{F}}^i \in \mathbb{Q}[\mathbf{x}] \text{ with } h_{\mathcal{F}}^i(n) = p_{\mathcal{F}}^i(n), \forall n \ll 0,$$

the i -th *cohomological Hilbert polynomial* of (X, \mathcal{F}) , (s. [21, (20.4.12)]). Moreover, $\deg(p_{\mathcal{F}}^i) \leq i$.

Now, we may define the i -th *cohomological deficiency function* of (X, \mathcal{F}) :

$$(1.13) \quad \Delta_{X,\mathcal{F}}^i = \Delta_{\mathcal{F}}^i : \mathbb{Z} \longrightarrow \mathbb{Z}, \quad (n \longmapsto \Delta_{\mathcal{F}}^i(n) := h_{\mathcal{F}}^i(n) - p_{\mathcal{F}}^i(n)),$$

and the i -th *cohomological postulation number* of (X, \mathcal{F}) :

$$(1.14) \quad \nu_{X,\mathcal{F}}^i = \nu_{\mathcal{F}}^i := \inf\{n \in \mathbb{Z} \mid \Delta_{\mathcal{F}}^i(n+1) \neq 0\}.$$

Then, clearly $\nu_{\mathcal{F}}^i > -\infty$ for all $i \in \mathbb{N}_0$. As $p_{\mathcal{F}}^i \equiv 0$ for all $i < f(\mathcal{F})$, we have $\Delta_{\mathcal{F}}^i = h_{\mathcal{F}}^i$ for all these i . Therefore, the concept of cohomological postulation number naturally extends the concept of cohomological left-vanishing order (cf. (1.5)) to the range $i \geq f(\mathcal{F})$. But now, it is clear, what bounds of extended Severi type should achieve: They should bound the invariants $\nu_{\mathcal{F}}^i$ and the numbers $\Delta_{\mathcal{F}}^i(n)$ in the range $n \leq -i$ for arbitrary values of i .

We shall discuss a result of this type, which gives a bound on the numbers $\nu_{\mathcal{F}}^i$ for $i \leq k$ in terms of the cohomology diagonal $(h_{\mathcal{F}}^i(-i))_{i=0}^k$ of \mathcal{F} at and below level k and the cohomological Hilbert polynomial $p_{\mathcal{F}}^k$ in section 3. In section 4 we consider the case in which the base ring R_0 is a field. In this particular situation we are able to bound the cohomological postulation numbers $\nu_{\mathcal{F}}^i$ in terms of the *full cohomology diagonal* $(h_{\mathcal{F}}^i(-i))_{i=0}^{cd(\mathcal{F})}$ of \mathcal{F} . As a consequence of this we obtain that there are only finitely many choices for each of the cohomological Hilbert functions $h_{\mathcal{F}}^i$ if the cohomology diagonal $(h_{\mathcal{F}}^i(-i))_{i=0}^{cd(\mathcal{F})}$ is fixed.

All the bounds mentioned so far, are *a priori bounds*, valid for arbitrary pairs (X, \mathcal{F}) (with appropriately chosen base ring R_0). Moreover, they use (part of) the cohomology diagonal as a bounding system. In section 5 we also shall consider bounds for the regularity which depend on the so called *Hilbert coefficients* and hold for *b-sheaves* in the sense of Kleiman [38, Exp. XIII]. In section 6 we shall briefly discuss a *few specific bounds*, e.g. bounds concerning special pairs (X, \mathcal{F}) . Our interest is focussed on the classical cases, in which X is a projective space over an algebraically closed field and \mathcal{F} is an algebraic vector bundle or a sheaf of ideals defining a projective variety. We also consider the case in which X is a projective variety and $\mathcal{F} = \mathcal{O}_X$ is its structure sheaf.

Most of the results we present, are originally formulated and proved in terms of *local cohomology* rather than in terms of sheaf cohomology. So, we briefly recall the link between these two concepts. To do so, let the base ring R_0 be

arbitrary noetherian. Then, the coherent sheaf \mathcal{F} is induced by some finitely generated graded R -module M (s. [42, II, Proposition 5.11]).

Now, let $R_+ := \bigoplus_{n>0} R_n \subseteq R$ be the irrelevant ideal and let D_{R_+} denote the R_+ -transform functor, e.g. the linear left exact functor on the category of R -modules given by $D_{R_+}(\bullet) := \lim_{\rightarrow} Hom_R((R_+)^n, \bullet)$.

For $i \in \mathbb{N}_0$, let $\mathcal{R}^i D_{R_+}$ denote the i -th right derived functor of D_{R_+} . Then, the R -modules $\mathcal{R}^i D_{R_+}(M)$ carry a natural grading (s. [21, (12.4.5)]). Moreover, there are isomorphisms of R_0 -modules

$$(1.15) \quad H^i(X, \mathcal{F}(n)) \cong \mathcal{R}^i D_{R_+}(M)_n, \quad (\forall i \in \mathbb{N}_0, \forall n \in \mathbb{Z})$$

in which \bullet_n denotes the formation of n -th graded parts (s. [21, (20.4.4)]). Now, keep in mind that the local cohomology modules $H_{R_+}^i(M)$ carry a natural grading (s. [21, (12.3.3)]) or consult one of [23] or [26] and that the natural exact sequence

$$0 \longrightarrow H_{R_+}^0(M) \longrightarrow M \longrightarrow D_{R_+}(M) \longrightarrow H_{R_+}^1(M) \longrightarrow 0$$

and the natural isomorphisms

$$\mathcal{R}^i D_{R_+}(M) \cong H_{R_+}^i(M), \quad (\forall i \in \mathbb{N})$$

respect these gradings (s. [21, (12.4.2), (12.4.5) (iii)]). So, altogether, for each $n \in \mathbb{Z}$ we obtain a short exact sequence of R_0 -modules

$$(1.16) \quad 0 \longrightarrow H_{R_+}^0(M)_n \longrightarrow M_n \longrightarrow H^0(X, \mathcal{F}(n)) \longrightarrow H_{R_+}^1(M)_n \longrightarrow 0$$

and isomorphisms of R_0 -modules

$$(1.17) \quad H^i(X, \mathcal{F}(n)) \cong H_{R_+}^{i+1}(M)_n, \quad (\forall i \in \mathbb{N}).$$

The relations given by (1.16) and (1.17) are a version of the so called *Serre-Grothendieck correspondence* (cf. [21, Chap. 20], [37]).

2. COHOMOLOGICAL PATTERNS

Let $X = Proj(R = \bigoplus_{n \geq 0} R_n)$ and \mathcal{F} be as in the introduction. In this section we shall discuss a few properties of the cohomological pattern $P(\mathcal{F})$ introduced in (1.7). Let us start with the following definition.

2.1. Definition. (i) Let $w \in \mathbb{N}_0$. A set $P \subseteq \mathbb{N}_0 \times \mathbb{Z}$ is called a *combinatorial pattern of width w* , if it satisfies the following five conditions:

- (π_1) $\exists m, n \in \mathbb{Z} : (0, m), (w, n) \in P$;
- (π_2) $(i, n) \in P \implies i \leq w$;
- (π_3) $(i, n) \in P \implies \exists j \leq i : (j, n + i - j + 1) \in P$;
- (π_4) $(i, n) \in P \implies \exists k \geq i : (k, n + i - k - 1) \in P$;
- (π_5) $i > 0 \implies (i, n) \notin P, \forall n \gg 0$.

The *width* of a combinatorial pattern P is denoted by $w(P)$.

- (ii) Let $i \in \mathbb{N}_0$. A combinatorial pattern P is said to be *tame at level i* , if either $(i, n) \in P$ for all $n \ll 0$, or $(i, n) \notin P$ for all $n \ll 0$.

Observe that any combinatorial pattern P is tame at level 0 and at level $w(P)$. A combinatorial pattern P is said to be *tame* if it is tame at any level $i \in \mathbb{N}_0$.

- (iii) A combinatorial pattern P is said to be *minimal* if there is no combinatorial pattern $Q \subset P$.

- (iv) Fix $w \in \mathbb{N}_0$ and let \mathbb{M}_w be the set of all monotonically decreasing functions $\mu : \mathbb{Z} \rightarrow \{0, \dots, w\}$ for which $0, w \in \mu(\mathbb{Z})$. For any $\mu \in \mathbb{M}_w$, let us introduce the “skew graph” of μ , e.g. the set

$$P[\mu] := \{(\mu(n); n - \mu(n)) \mid n \in \mathbb{Z}\} \subseteq \mathbb{N}_0.$$

It is easy to verify that $P[\mu]$ is a minimal combinatorial pattern of width w . •

In the following remark we summarize a few properties of combinatorial patterns. For a more detailed and complete presentation of the listed facts we refer to [13, Sec. 2].

- 2.2. Remark.** (i) Let $P \subseteq \mathbb{N}_0 \times \mathbb{Z}$ be a combinatorial pattern and let $(i, n) \in P$. Then, there is a integer $w \geq i$ and a function $\mu \in \mathbb{M}_w$ such that

$$(i, n) \in P[\mu] \subseteq P.$$

- (ii) In view of what we just remarked, it is easy to see, that the minimal combinatorial patterns of width w are precisely the patterns of the form $P[\mu]$ with $\mu \in \mathbb{M}_w$.

- (iii) On use of the previous observation one now proves immediately:

A set $P \subseteq \mathbb{N}_0 \times \mathbb{Z}$ is a tame combinatorial pattern if and only if it is the union of finitely many minimal combinatorial patterns. •

After these combinatorial preliminaries, let us turn back to cohomological patterns.

- 2.3. Proposition.** *In the notation of (1.1) and (1.7) we have:*

$$P(\mathcal{F}) \text{ is a combinatorial pattern of width } \\ cd(\mathcal{F}) = \sup\{\dim(\text{Supp}(\mathcal{F}) \cap \pi^{-1}(x_0)) \mid x_0 \in X_0\}.$$

Proof: s. [13, (3.5)]. □

- 2.4. Remarks.** (i) Observe that according to Proposition (2.3) the pattern $P(\mathcal{F})$ satisfies in particular the two conditions π_4 and π_3 , which correspond respectively to the following two *vanishing constraints* (in which $k \in \mathbb{N}_0$ and $r \in \mathbb{Z}$ are fixed):

$$(*)_1 \quad H^i(X, \mathcal{F}(r-i)) = 0, \quad \forall i > k \implies H^i(X, \mathcal{F}(n-i)) = 0, \quad \forall i > k, \quad \forall n \geq r$$

$$(*)_2 \quad H^i(X, \mathcal{F}(r-i)) = 0, \quad \forall i \leq k \implies H^i(X, \mathcal{F}(m-i)) = 0, \\ \forall i \leq k, \quad \forall m \leq r.$$

In the special case, where R_0 is a field, the constraint $(*)_1$ is found in Mumfords work [65]. For a proof of both constraints in the present form see [13, Sec. 3]. Obviously, if R_0 is artinian and $r = 0$, the constraints $(*)_1$ and $(*)_2$ respectively correspond to the constraints (1.10) and (1.11).

(ii) The constraints $(*)_1, (*)_2$ tell us respectively: If $H^i(X, \mathcal{F}(n))$ vanishes “along a diagonal above (resp. “at and below”) level k ” then it vanishes everywhere “to the right” (resp. “to the left”) of this diagonal.

In particular, these constraints give alternative descriptions of the concepts of regularity above level k and of coregularity at and below level k (s. (1.3), (1.5)), namely:

$$\begin{aligned} \text{reg}_k(\mathcal{F}) &= \inf\{r \in \mathbb{Z} \mid H^i(X, \mathcal{F}(r-i)) = 0, \forall i > k\}; \\ \text{coreg}^k(\mathcal{F}) &= \sup\{c \in \mathbb{Z} \mid H^i(X, \mathcal{F}(c-i)) \neq 0, \forall i \leq k\}, \quad (k < f(\mathcal{F})). \end{aligned}$$

(iii) Clearly, the cohomological finiteness dimension $f(\mathcal{F})$ of \mathcal{F} is the lowest level i at which there are infinitely many $(i, n) \in P(\mathcal{F})$ with $n < 0$. The invariant $f(\mathcal{F})$ is related to the local behaviour of \mathcal{F} along the fibres of the morphism $\pi : X \rightarrow X_0$ by the inequality

$$f(\mathcal{F}) \leq \inf\{\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\pi^{-1}(\pi(x)) \cap \overline{\{x\}}) \mid x \in X\} =: \delta(\mathcal{F}).$$

Moreover, if R_0 is a homomorphic image of a regular ring, equality holds (cf. [21, (20.4.20)]). •

Now, let us say a few words about the tameness of the pattern $P(\mathcal{F})$. First of all, let us mention the following result (cf. [13, (4.3)]).

2.5. Theorem. *If the base ring R_0 is semilocal and of dimension ≤ 1 , the cohomological pattern $P(\mathcal{F})$ of the coherent sheaf \mathcal{F} is tame.* □

2.6. Comment and Problem. We do not know, whether the conclusion of (2.5) holds without any restriction on the (noetherian) base ring R_0 . So, let us pose the following *tameness problem*:

Is $P(\mathcal{F})$ always tame? •

2.7. Remark. Obviously, one might try to answer the tameness problem at certain particular levels i . Indeed, for certain values of i the requested tameness is easily verified. So, by the axioms π_3 and π_4 and in view of (2.3) it is clear, that $P(\mathcal{F})$ is tame at level 0 and at all levels $i > cd(\mathcal{F})$. Moreover, $P(\mathcal{F})$ obviously is tame at any level $i < f(\mathcal{F})$. •

In view of the previous remark, $i = f(\mathcal{F})$ is the first level at which the tameness question is non-trivial. At this particular level $P(\mathcal{F})$ is indeed tame. Actually, we have the following result, which is an easy consequence of [13, (5.6)] and the Serre-Grothendieck Correspondence (1.16), (1.17).

2.8. Theorem. *The sets $Ass_{R_0}(H^{f(\mathcal{F})}(X, \mathcal{F}(n)))$ are asymptotically stable for $n \rightarrow -\infty$, e.g. there exists an integer n_0 such that*

$$Ass_{R_0}(H^{f(\mathcal{F})}(X, \mathcal{F}(n))) = Ass_{R_0}(H^{f(\mathcal{F})}(X, \mathcal{F}(n_0))) \quad \text{for all } n \leq n_0.$$

In particular $P(\mathcal{F})$ is tame at level $f(\mathcal{F})$. \square

2.9. Remark and Problems. (i) For arbitrary values of $i \in \mathbb{N}_0$, the sets $Ass_{R_0}(H^i(X, \mathcal{F}(n)))$ need not be asymptotically stable for $n \rightarrow -\infty$. Indeed, using a construction of Singh [75] one can easily write down a positively graded homogeneous noetherian domain $R = \bigoplus_{n \geq 0} R_n$ such that with $X = Proj(R)$ we have $cd(\mathcal{O}_X) = 2$ and

$$\# \left(\bigcup_{n < 0} Ass_{R_0}(H^2(X, \mathcal{O}_X(n))) \right) = \infty, \quad (\text{cf. [13, (5.7)]}).$$

(ii) On the other hand, we do not know an example for which the sets $Ass_{R_0}(H^i(X, \mathcal{F}(n)))$ are not ‘‘asymptotically increasing’’ for $n \rightarrow -\infty$. So, let us ask the following question:

Is there some $n_0 \in \mathbb{Z}$ such that for all $n \leq n_0$

$$Ass_{R_0}(H^i(X, \mathcal{F}(n-1))) \supseteq Ass_{R_0}(H^i(X, \mathcal{F}(n)))?$$

Clearly, an affirmative answer to this question implies that $P(\mathcal{F})$ is tame (at level i).

(iii) The base ring R_0 in the example mentioned in part (i) is not local, and in fact we do not know an example with local base ring R_0 for which the sets $Ass_{R_0}(H^i(X, \mathcal{F}(n)))$ are not asymptotically stable for $n \rightarrow -\infty$. So, let us pose the following problem:

Assume that R_0 is local. Is there some $n_0 \in \mathbb{Z}$ such that

$$Ass_{R_0}(H^i(X, \mathcal{F}(n))) = Ass_{R_0}(H^i(X, \mathcal{F}(n_0))) \quad \text{for all } n \leq n_0?$$

(iv) Let us also mention here the following fact, which is an easy consequence of [13, (5.5)] and the Serre-Grothendieck Correspondence (1.16), (1.17):

If $Ass_{R_0}(H^i(X, \mathcal{F}(n)))$ is asymptotically stable for $n \rightarrow -\infty$ and if \mathcal{F} is induced by the finitely generated graded R -module M , then $Ass_R(H_{R_+}^{i+1}(M))$ is a finite set.

In the special case, where (R_0, \mathfrak{m}_0) is local and $\mathfrak{m} := \mathfrak{m}_0 + R_+$ denotes the graded maximal ideal of R , the two sets $Ass_R(H_{R_+}^{i+1}(M))$ and

$Ass_{R_{\mathfrak{m}}}(H_{(R_+)_{R_{\mathfrak{m}}}}^{i+1}(M_{\mathfrak{m}}))$ are in natural bijection. So, the ‘‘stability problem’’ mentioned in part (iii) is closely related to the *finiteness problem* for the sets of associated primes $Ass_R(H_{\mathfrak{a}}^i(M))$ of the local cohomology modules $H_{\mathfrak{a}}^i(M)$ of a finitely generated module M over a noetherian (local) ring R with respect to some ideal $\mathfrak{a} \subseteq R$, (cf. [47]). This problem recently has found an affirmative answer for some particular values of i (cf. [14], [19] and also [12]) and is known to have a negative answer in general (s. [75]). In the special case, where $R = M$ is a local ring, there are some partial results (cf. [43]), but even in the case of a local Cohen-Macaulay ring R , the problem is

open. Only in the case where $R = M$ is regular local ring, there are complete (and further reaching) results (s. [49], [55], [56]). One might ask, whether the ideas developed in [74] could lead to some progress in the study of the “graded case” of the above finiteness problem. •

It is natural to ask, whether the pattern axioms (2.1) $(\pi_1 - \pi_r)$ (together with the requirement of tameness) are the only general constraints to which cohomological patterns are subject. More precisely: Is any tame combinatorial pattern the pattern of a pair (X, \mathcal{F}) ? This question has indeed an affirmative answer.

2.10. Theorem. *Let R_0 be a semilocal noetherian ring of dimension ≤ 1 and let $w \in \mathbb{N}_0$. Then, the tame combinatorial patterns of width $\leq w$ are precisely the combinatorial patterns of coherent sheaves of $\mathcal{O}_{\mathbb{P}_{R_0}^w}$ -modules.*

Proof: (s. [13, (4.8)]). □

2.11. Remark. (i) The crucial step in the proof of (2.10) is to show that for any field K and any minimal combinatorial pattern P of width w , there is a coherent sheaf \mathcal{F} of $\mathcal{O}_{\mathbb{P}_K^w}$ -modules with $P(\mathcal{F}) = P$. In view of the vanishing Theorem of Severi-Enriques-Zariski-Serre, such a \mathcal{F} must be an algebraic vector bundle. Indeed, we can choose the requested sheaf \mathcal{F} as an (indecomposable) algebraic vector bundle of rank $\leq w!$ (s. [13, (4.5)]). Here, let us ask the following problem:

What is the least possible rank of an algebraic vector bundle \mathcal{F} over \mathbb{P}_k^w for which $P(\mathcal{F}) = P$?

In [13, (4.5)] a candidate for \mathcal{F} is constructed as the direct image $\pi_*\mathcal{L}$ of a line bundle $\mathcal{L} = \otimes_{i=1}^w \pi_i^* \mathcal{O}_{\mathbb{P}_K^1}(r_i)$ over the w -fold Segre product $Y = \mathbb{P}_K^1 \times \cdots \times \mathbb{P}_K^1$ of a projective line \mathbb{P}_K^1 under a generic projection $\pi : Y \rightarrow \mathbb{P}_K^w$. (Here $\pi_i : Y \rightarrow \mathbb{P}_K^1$ denotes the projection to the i -th factor.) The twisting orders r_i are determined appropriately on use of the Künneth formulas (s. [77], [31]). To answer the above question one likewise has to use a different method of construction.

(ii) It is easy to write down examples of tame combinatorial patterns of width w which are realizable only by decomposable vector bundles (s. [13, (4.9)]) over \mathbb{P}_K^w . This makes arise the following question:

Is there a purely combinatorial characterization of those patterns, which may be realized by an indecomposable algebraic vector bundle over \mathbb{P}_K^w ? •

Let K be a field. By Theorem (2.10) (and by the Vanishing Theorem of Severi-Enriques-Zariski-Serre) the combinatorial patterns of algebraic vector bundles over \mathbb{P}_K^w are precisely the combinatorial patterns P of width w such that $(i, n) \notin P$ for all $i < w$ and all $n \ll 0$. Inspired by this observation one may be tempted to ask whether there is a purely combinatorial description of the patterns $P(\mathcal{O}_X)$ of a smooth projective variety $X \subseteq \mathbb{P}_K^r$ over an

algebraically closed field K (of characteristic 0). In order to say a few words about this question, we introduce some notions:

2.12. Definition and Remark. (i) A combinatorial pattern P is said to be *positive*, if $(0, 0) \in P$ and if $(i, n) \notin P$ for all $i < w(P)$ and all $n < 0$. A combinatorial pattern P is said to be *straight*, if $(0, 0) \in P$ and if $(i, n) \notin P$ whenever $i < w(P)$ and $n < 0$ or $0 < i < w(P)$ and $n > 0$. Straight patterns are positive.

(ii) If P is a combinatorial pattern of width w , we set

$$\alpha(P) := \sup\{n \in \mathbb{Z} \mid (w, n) \in P\}.$$

If P is positive, the axiom (2.1) (π_3) implies that $\alpha(P) \geq -w(P) - 1$, whereas axiom (π_4) gives

$$(i, 0) \notin P \quad \text{for all } i > w + \alpha(P) + 1. \quad \bullet$$

Now, we are ready to show:

2.13. Proposition. *Let P a straight combinatorial pattern of width $w > 0$ and let K be an algebraically closed field. Then, there is a smooth projective variety X over K such that $P(\mathcal{O}_X) = P$.*

Proof: (Induction on w) First, let $w = 1$, so that $\alpha(P) \geq -2$. If $\alpha(P) = -2$, choose $X = \mathbb{P}_K^1$. Otherwise, let $X \subseteq \mathbb{P}_K^2$ be a smooth curve of degree $\alpha(P) + 3$.

Now, let $w > 1$, $\alpha := \alpha(P)$ and $m = \max\{i \in \mathbb{N}_0 \mid (i, 0) \in P\}$. Then $\alpha \geq m - w - 1$ (s. (2.12) (ii)).

Assume first, that $\alpha = m - w - 1$. Then $m < w$ and $\alpha < 0$. If $m = 0$, choose $X = \mathbb{P}_K^w$. So, let $m > 0$. Then, there is a straight pattern Q with $w(Q) = m$ and $Q \cap (\mathbb{N}_0 \times \{0\}) = P \cap (\mathbb{N}_0 \times \{0\})$. So, by induction, there is a smooth projective non-degenerate variety $Y \subseteq \mathbb{P}_K^r$ with $P(\mathcal{O}_Y) = Q$. Let $Z := \mathbb{P}_K^{w-m}$. Then, the Segre product $X := Y \times Z \subseteq \mathbb{P}_K^{(r+1)(w-m+1)-1}$ is a smooth projective variety. As $P(\mathcal{O}_Z) = (\{0\} \times \mathbb{N}_0) \cup (\{w-m\} \times \mathbb{Z}_{\leq m-w-1})$, the Künneth formulas give for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$

$$\begin{aligned} H^i(X, \mathcal{O}_X(n)) &\cong H^i(Y, \mathcal{O}_Y(n)) \otimes_K H^0(Z, \mathcal{O}_Z(n)) \\ &\oplus H^{i-m-w}(Y, \mathcal{O}_Y(n)) \otimes_K H^{w-m}(Z, \mathcal{O}_Z(n)) \end{aligned}$$

(with the convention that $H^j \equiv 0$ for $j < 0$). Using again the shape of $P(\mathcal{O}_Y)$ and of $P(\mathcal{O}_Z)$ it follows that $P(\mathcal{O}_X) = P$.

Next, assume that $m - w - 1 < \alpha < 0$. Then $m \leq w - 1$. If $m = w - 1$, we have $\alpha = -1$. Moreover, there is a straight pattern Q with $w(Q) = w - 1$ and such that $Q \cap (\mathbb{N}_0 \times \{0\}) = P \cap (\mathbb{N}_0 \times \{0\})$. If $m < w - 1$, there is a straight pattern Q with $w(Q) = w - 1$, $\alpha(Q) = \alpha$ and $Q \cap (\mathbb{N}_0 \times \{0\}) = P \cap (\mathbb{N}_0 \times \{0\})$. By induction, there is a smooth non-degenerate projective variety $Y \subseteq \mathbb{P}_K^r$ such that $P(\mathcal{O}_Y) = Q$. Let $Z \subseteq \mathbb{P}_K^2$ be a smooth quadratic curve. Then, the Segre product $X := Y \times Z \subseteq \mathbb{P}_K^{3r+2}$ is a smooth projective variety. As

$P(\mathcal{O}_Z) = (\{0\} \times \mathbb{N}_0) \cup (\{1\} \times \mathbb{Z}_{<0})$, the Künneth formulas give for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$

$$\begin{aligned} H^i(X, \mathcal{O}_X(n)) &\cong H^i(Y, \mathcal{O}_Y(n)) \otimes_K H^0(Z, \mathcal{O}_Z(n)) \\ &\oplus H^{i-1}(Y, \mathcal{O}_Y(n)) \otimes_K H^1(Z, \mathcal{O}_Z(n)). \end{aligned}$$

Using once more the shape of $P(\mathcal{O}_Y)$ and of $P(\mathcal{O}_Z)$ it follows $P(\mathcal{O}_X) = P$. So, let $\alpha \geq 0$. Then, there is a straight pattern M with $\alpha(M) \in \{0, -1\}$, $w(M) = w - 1$ and such that $M \cap (\mathbb{N}_0 \times \{0\}) = P \cap (\{0, \dots, w-1\} \times \{0\})$. By induction, there is some smooth projective non-degenerate variety $T \subset \mathbb{P}_K^r$ such that $P(\mathcal{O}_T) = M$. Now, let $W \subseteq \mathbb{P}_K^9$ the third Veronesean of the projective plane, so that $P(\mathcal{O}_W) = (\{0\} \times \mathbb{N}_0) \cup (\{2\} \times \mathbb{Z}_{<0})$. Moreover, let $U := T \times W \subset \mathbb{P}_K^{10r+9}$ be the Segre product of T and W . Then, U is a smooth projective variety and the Künneth formulas give

$$\begin{aligned} H^i(U, \mathcal{O}_U(n)) &\cong H^i(T, \mathcal{O}_T(n)) \otimes_K H^0(W, \mathcal{O}_W(n)) \\ &\oplus H^{i-2}(T, \mathcal{O}_T(n)) \otimes_K H^2(W, \mathcal{O}_W(n)), \end{aligned}$$

for all $i, n \in \mathbb{Z}$. It follows, that $P(\mathcal{O}_U)$ is a straight pattern with $w(P(\mathcal{O}_U)) = w + 1$, $\alpha(P(\mathcal{O}_U)) = -1$ and such that

$$P(\mathcal{O}_U) \cap (\mathbb{N}_0 \times \{0\}) = P \cap (\{0, \dots, w-1\} \times \{0\}).$$

Now, let $H \subset \mathbb{P}_K^{10r+9}$ be a generic hypersurface of degree $2\alpha + 1$. Then, by Bertini, $V := U \cap H \subset \mathbb{P}_K^{10r+9}$ is a smooth projective variety (cf. [50]). Now, the exact sequences

$$\begin{aligned} H^i(U, \mathcal{O}_U(n - 2\alpha - 1)) &\longrightarrow H^i(U, \mathcal{O}_U(n)) \longrightarrow H^i(V, \mathcal{O}_V(n)) \\ &\longrightarrow H^{i+1}(U, \mathcal{O}_U(n - 2\alpha - 1)) \\ &\longrightarrow H^{i+1}(U, \mathcal{O}_U(n)) \end{aligned}$$

show, that

$$H^i(V, \mathcal{O}_V(n)) = 0 \quad \text{if} \quad \begin{cases} i > w \\ 0 < i < w, n \neq 0, 2\alpha + 1 \\ i = 0, n < 0 \end{cases}$$

$$H^i(V, \mathcal{O}_V(0)) \cong H^i(U, \mathcal{O}_U(0)) \quad \text{for} \quad 0 \leq i < w;$$

$$H^w(V, \mathcal{O}_V(2\alpha)) \neq 0; \quad H^w(V, \mathcal{O}_V(n)) = 0 \quad \text{for all} \quad n > 2\alpha.$$

Now, let $X \subseteq \mathbb{P}_K^{50r^2+105r+54}$ be the second Veronese transform of V . As $X \cong V$, X is smooth. It now follows easily from the above statements on the groups $H^i(V, \mathcal{O}_V(n))$ that $P(\mathcal{O}_X) = P$. \square

2.14. Remark and Problems. (i) In view of the Kodaira Vanishing Theorem (cf. [52]) the pattern $P(\mathcal{O}_X)$ of the structure sheaf of a smooth projective complex variety X is always positive. So one may be lead to ask:

Is each positive pattern the cohomological pattern of the

structure sheaf \mathcal{O}_X of a smooth projective variety X ?

(ii) The method of construction used in the proof of (2.14) realizes straight patterns by smooth projective varieties $X \subseteq \mathbb{P}_K^r$ of rather large degree. So, we ask:

What can be said about the minimal possible degree of a smooth projective variety $X \subseteq \mathbb{P}_K^r$ whose structure sheaf \mathcal{O}_X realizes a given straight pattern?

(iii) Let us also mention the relation of the above problems to the following *Non-Rigidity Theorem* of Evans-Griffiths (s. [30] and [63]): Given $r - c - 1$ graded modules L_1, \dots, L_{r-c-1} of finite length over the polynomial ring $K[\mathbf{x}_0, \dots, \mathbf{x}_r]$ over the algebraically closed field K , ($2 \leq c \leq r - 2$), there is a normal projective variety $X \subseteq \mathbb{P}_K^r$ of codimension c and an integer t such that there are graded isomorphisms $\bigoplus_{n \in \mathbb{Z}} H^i(X, \mathcal{O}_X(n)) \cong L_i(t)$ for $1 \leq i \leq r - c - 1$. •

3. COHOMOLOGICAL HILBERT FUNCTIONS

Let $X = Proj(R = \bigoplus_{n \geq 0} R_n) \xrightarrow{\pi} X_0 = Spec(R_0)$ and \mathcal{F} be as in the introduction, but assume from now on, that R_0 is artinian. We now shall present a few results on the cohomological Hilbert functions $h_{\mathcal{F}}^i$ introduced in (1.9). Let us start with a few preliminary remarks.

3.1. Remark. (i) As R_0 is artinian, we have

$$\sup\{\dim(Supp(\mathcal{F})) \cap \pi^{-1}(x_0) \mid x_0 \in X_0\} = \dim(Supp(\mathcal{F})) =: \dim(\mathcal{F}),$$

so that (cf. (2.3))

$$cd(\mathcal{F}) = \dim(\mathcal{F}).$$

(ii) Clearly, now the invariant

$$\delta(\mathcal{F}) = \inf\{\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \dim(\pi^{-1}(\pi(x)) \cap \overline{\{x\}}) \mid x \in X\}$$

of (2.4) (iii) takes the value

$$\inf\{\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \mid x \in X, \overline{\{x\}} = \{x\}\}.$$

As R_0 is artinian, its localizations are complete and hence homomorphic images of regular local rings, by Cohens Structure Theorem (s. [60]). By (2.4) (iii) it follows now easily that

$$f(\mathcal{F}) = \delta(\mathcal{F}) = \inf\{\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \mid x \in X, \overline{\{x\}} = \{x\}\}. \quad \bullet$$

Now, let us formulate a first bounding result – a diagonal bound of Castelnuovo type.

3.2. Theorem. *Let $d := \dim(\mathcal{F})$. Then*

$$a) \quad \text{reg}_k(\mathcal{F}) \leq \left(2 \sum_{i=k+1}^d \binom{d-k-1}{i-k-1} h_{\mathcal{F}}^i(-i)\right)^{2^{d-k-1}}, \quad (0 \leq k < d).$$

$$b) \quad h_{\mathcal{F}}^j(n) \leq \frac{1}{2} (2 \sum_{i=j}^d \binom{d-j}{i-j} h_{\mathcal{F}}^i(-i))^{2^{d-j}}, \quad (0 < j \leq d, n \geq -j).$$

Proof: See [16, Sec. 4]. □

Our next result gives a diagonal bound of Severi type.

3.3. Theorem. *Let $k < \delta(\mathcal{F})$. Then*

$$a) \quad \text{coreg}^k(\mathcal{F}) \geq - (2 \sum_{i=0}^k \binom{k}{i} h_{\mathcal{F}}^i(-i))^{2^k};$$

$$b) \quad h_{\mathcal{F}}^j(n) \leq \frac{1}{2} (2 \sum_{i=0}^j \binom{j}{i} h_{\mathcal{F}}^i(-i))^{2^j}, \quad (0 < j < k, n \leq -j).$$

Proof: See [16, Sec. 5]. □

3.4. Definition. (i) Let \mathcal{C} denote the class of all pairs (X, \mathcal{F}) in which $X = \text{Proj}(\oplus_{n \geq 0} R_n)$ is a projective scheme over an artinian base ring R_0 and \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules. Let $\mathcal{D} \subseteq \mathcal{C}$ be a subclass. Now, in the spirit of [16, (4.8)] and [21, (16.4.1)] we define a *numerical invariant on \mathcal{D}* to be an assignment $\mu : \mathcal{D} \rightarrow \mathbb{Z} \cup \{\pm\infty\}$. We say that the numerical invariant is *finite*, if $\mu(\mathcal{D}) \subseteq \mathbb{Z}$.

(ii) Let \mathcal{C}, \mathcal{D} be as in part (i) and let $\mu_1, \dots, \mu_s, \rho$ be numerical invariants on \mathcal{D} . We say that μ_1, \dots, μ_s form an *upper (resp. lower) bounding system* for ρ on \mathcal{D} , if the invariants μ_1, \dots, μ_s are finite and if there is a function $B : \mathbb{Z}^s \rightarrow \mathbb{Z}$ such that

$$\begin{aligned} \rho(U) &\leq B(\mu_1(U), \dots, \mu_s(U)) \\ \text{(resp. } \rho(U) &\geq B(\mu_1(U), \dots, \mu_s(U))) \text{ for all } U \in \mathcal{D}. \end{aligned}$$

(iii) In the situation of part (ii) we call B an *upper (resp. lower) bounding function for ρ in terms of μ_1, \dots, μ_s* . Instead of saying that μ_1, \dots, μ_s form an upper (resp. lower) bounding system for ρ on \mathcal{D} , we also say that ρ is bounded above (resp. below) by (or - in terms of) μ_1, \dots, μ_s on \mathcal{D} . We say that ρ is *polynomially bounded above (resp. below)* on \mathcal{D} by μ_1, \dots, μ_s if it is bounded by a polynomial bounding function B .

(iv) Let \mathcal{D}, \mathcal{C} and $\mu_1, \dots, \mu_s, \rho : \mathcal{D} \rightarrow \mathbb{Z}$ be as above. We say that μ_1, \dots, μ_s form a *minimal upper (resp. lower) bounding system for ρ on \mathcal{D}* if they form such a system, but if none of the s systems $\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_s$ does ($i = 1, \dots, s$). •

3.5. Lemma. *Let K be a field, let $d \in \mathbb{N}$, let $i \in \{0, \dots, d\}$ and let \mathcal{D} be the class of all pairs $(\mathbb{P}_K^d, \mathcal{E})$ for which \mathcal{E} is an indecomposable algebraic vector bundle over \mathbb{P}_K^d . Then*

a) *If $0 \leq k < i$, the invariants $h_{\bullet}^0(0), \dots, h_{\bullet}^{i-1}(-i+1), h_{\bullet}^{i+1}(-i-1), \dots, h_{\bullet}^d(-d)$ do not form an upper bounding system for the invariant $\text{reg}_k(\bullet)$ on the class \mathcal{D} .*

b) *If $i \leq k < d$, the invariants $h_{\bullet}^0(0), h_{\bullet}^{i-1}(-i+1), h_{\bullet}^{i+1}(-i-1), \dots, h_{\bullet}^d(-d)$ do not form a lower bounding system for the invariant $\text{coreg}^k(\bullet)$ on the class \mathcal{D} .*

Proof: Let $t \in \mathbb{N}$. It is easy to see, that there is a minimal combinatorial pattern $P_{i,t}$ of width d with the properties that $(i, -i - t), (i, -i + t) \in P_{i,t}$ and $(j, -j) \notin P_{i,t}$ for all $j \neq i$. By (2.10) and in view of (2.11) (i) there is an indecomposable algebraic vector bundle $\mathcal{E}_{i,t}$ over \mathbb{P}_K^d such that $P(\mathcal{E}_{i,t}) = P_{i,t}$. In particular we have $h_{\mathcal{E}_{i,t}}^j(-j) = 0$ for all $j \neq i$.

Moreover, if $0 \leq k < i$ we conclude from $(i, -i + t) \in P_{i,t}$ that $\text{reg}_k(\mathcal{E}_{i,t}) > t$. If $i \leq k < d$, we use $(i, -i - t) \in P_{i,t}$ to show that $\text{coreg}^k(\mathcal{E}_{i,t}) < -t$. This proves our claim. \square

3.6. Notation. For $r \in \mathbb{N}_0$, let $\mathcal{C}^{(r)}$ denote the class of all pairs $(X, \mathcal{F}) \in \mathcal{C}$ with $\dim(\mathcal{F}) = r$ and let ${}^{(r)}\mathcal{C}$ denote the class of all pairs $(X, \mathcal{F}) \in \mathcal{C}$ with $\delta(\mathcal{F}) > r$.

3.7. Corollary. Let $k \in \mathbb{N}_0$.

a) For each $d \geq k$, the invariant $\text{reg}_k(\bullet)$ is bounded above polynomially in terms of the invariants $h_{\bullet}^{k+1}(-k-1), h_{\bullet}^{k+2}(-k-2), \dots, h_{\bullet}^d(-d)$ on the class $\mathcal{C}^{(d)}$. Moreover, these latter invariants form a minimal bounding system for $\text{reg}_k(\bullet)$ on $\mathcal{C}^{(d)}$.

b) $\text{coreg}^k(\bullet)$ is bounded below polynomially in terms of the invariants $h_{\bullet}^0(0), h_{\bullet}^1(-1), \dots, h_{\bullet}^k(-k)$ on the class \mathcal{C} . Moreover, these latter invariants form a minimal bounding system for $\text{coreg}^k(\bullet)$ on ${}^{(k)}\mathcal{C}$.

Proof: Immediate by (3.2), (3.3) and (3.5). \square

3.8. Remarks and Problems. (i) The polynomial bounds of Theorems (3.2) and (3.3) are not the sharpest possible bounds. In [16] we give sharper bounds in terms of certain piecewise polynomial and recursively defined bounding functions

$$B_{(d+1)}^{(i)} : \mathbb{N}_0^{d+1-i} \times \mathbb{Z}_{\geq -i} \longrightarrow \mathbb{N}_0, \quad (0 < i \leq d)$$

$$C_{(d+1)}^{(i)} : \mathbb{N}_0^{d+1-i} \longrightarrow \mathbb{Z}_{\geq -i}$$

and

$$B_{(k+1)}^{(i)} : \mathbb{N}_0^{i+1} \times \mathbb{Z}_{\leq -i} \longrightarrow \mathbb{N}_0, \quad (0 \leq i < k)$$

$$C_{(k+1)}^{(i)} : \mathbb{N}_0^{i+1} \longrightarrow \mathbb{Z}_{\leq -i}$$

such that

$$h_{\mathcal{F}}^i(n) \leq \underline{B}_{(d+1)}^{(i)}(h_{\mathcal{F}}^i(-i), \dots, h_{\mathcal{F}}^d(-d); n), \quad \forall n \geq -i;$$

$$\forall (X, \mathcal{F}) \in \mathcal{C}^{(d)}$$

$$\text{reg}_{i-1}(\mathcal{F}) \leq \underline{C}_{(d+1)}^{(i)}(h_{\mathcal{F}}^i(-i), \dots, h_{\mathcal{F}}^d(-d)) + i;$$

and

$$h_{\mathcal{F}}^i(n) \leq B_{(k+1)}^{(i)}(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^i(-i); n), \quad \forall n \leq -i$$

$$\forall (X, \mathcal{F}) \in {}^{(k)}\mathcal{C}.$$

$$\text{coreg}^i(\mathcal{F}) \leq C_{(k+1)}^{(i)}(h_{\mathcal{F}}^0(0), \dots, h_{\mathcal{F}}^i(-i)) + i;$$

(ii) In spite of the sharper bounds mentioned in part (i), one may ask, whether the polynomial bounds of Theorem (3.2) and (3.3) reflect the *rate of growth* for the invariants $\text{reg}_k(\mathcal{F})$ (resp. $\text{coreg}^k(\mathcal{F})$) in terms of the corresponding diagonal values $h_{\mathcal{F}}^i(-i)$. More precisely, let $d, k \in \mathbb{N}_0$ with $k < d$. Then, we ask

Is there a sequence $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}^{(d)}$ and a constant $c > 0$ such that

$$\text{reg}_k(\mathcal{F}_n) \geq c \left(2 \sum_{i=k+1}^d \binom{d-k-1}{i-k-1} h_{\mathcal{F}_n}^i(-i) \right)^{2^{d-k-1}} \quad \text{for all } n \in \mathbb{N}?$$

Is there a sequence $(Y_n, \mathcal{G}_n)_{n \in \mathbb{N}} \subseteq {}^{(k)}\mathcal{C}$ and a constant $b > 0$ such that

$$\text{coreg}^k(\mathcal{G}_n) \leq -b \left(2 \sum_{i=1}^k \binom{k}{i} h_{\mathcal{G}_n}^i(-i) \right)^{2^k} \quad \text{for all } n \in \mathbb{N}?$$

(iii) In the special case where $k = 0$, diagonal bounds of Castelnuovo type similar to those in Theorem (3.2) are given in [21, Chap. 16]. There, the main tool is an algebraic version of the *Lemma of Mumford-Le Potier* (cf. [21, (16.1.4)]). This lemma applies only to the function $n \mapsto h_{\mathcal{F}}^1(n)$ in the range $n \geq 0$ and thus is not of great help to bound $\text{reg}_k(\mathcal{F})$ for $k > 0$ and to bound $\text{coreg}^k(\mathcal{F})$ at all. So, we use another idea to derive the bounding results mentioned under (3.2), (3.3) and (3.4): a generalized version of the *principle of linear systems of hyperplane sections*. This principle has been used to derive *a-priori* bounds of Castelnuovo type and of Severi type in the case where $R_0 = K$ is an algebraically closed field in [5], [7], [9], [18]. In [4] and [16] versions of this principle adopted to the case where R_0 is local (and artinian) are used. •

Now, let us say a few words on *a-priori bounds of extended Severi type*, e.g. bounds on the cohomological deficiency functions $n \mapsto \Delta_{\mathcal{F}}^i(n)$ (cf. (1.13)) and the cohomological postulation numbers $\nu_{\mathcal{F}}^i$ (cf. (1.14)) for $i \geq f(\mathcal{F})$, e.g. without any restriction on i .

3.9. Theorem. *Let $i \in \mathbb{N}_0$. Then, in the notation of (1.12-14):*

$$a) \nu_{\mathcal{F}}^i \geq - \left(2 \left(1 + \sum_{j=0}^i \binom{i}{j} (h_{\mathcal{F}}^j(-j) + |p_{\mathcal{F}}^i(-j)|) \right) \right)^{2^i} + 2 .$$

$$b) |\Delta_{\mathcal{F}}^i(n)| \leq \frac{1}{2} \left(2 \left(1 + \sum_{j=0}^i \binom{i}{j} (h_{\mathcal{F}}^j(-j) + |p_{\mathcal{F}}^i(-j)|) \right) \right)^{2^i}, \quad (n \leq -i).$$

Proof: See [17, Sec. 3]. □

3.10. Remarks and Problems. (i) Here again, the polynomial bounds of Theorem (3.9) are not the sharpest possible bounds. In [17] we give sharper bounds in terms of certain piecewise polynomial and recursively defined bounding functions.

(ii) Similar as in (3.8) (ii) one may ask here:

Does the polynomial bound (3.9) a) “reflect the rate of growth” of the invariant $\nu_{\mathcal{F}}^i$?

(iii) It is easy to see that the method of linear systems of general hyperplane sections (cf. (3.8) (iii)) cannot be used to derive bounds of extended Severi type. So, in order to prove (3.9), one has to apply a different method: the method of *sequences of admissible linear forms*. This method was introduced by Matteotti [61] and turns out to be successful even in greater generality than here.

(iv) It turns out, that the $2(i+1)$ numerical invariants $h_{\bullet}^0(0), \dots, h_{\bullet}^i(-i), p_{\bullet}^i(0), \dots, p_{\bullet}^i(-i)$ do not form a minimal lower bounding system for the invariant ν_{\bullet}^i on the class \mathcal{C} (s. (3.4) (i)). In fact, one has (s. [17, (4.12)]):

Let $i \in \mathbb{N}_0$ and let $k \in \{0, \dots, i-1\}$. then, the $2i$ invariants $h_{\bullet}^j(-j)$, ($j = 0, \dots, i$), and $p_{\bullet}^i(-j)$, ($j = 0, \dots, i-1; j \neq k$), form a lower bounding system for the invariant ν_{\bullet}^i on the class \mathcal{C} .

This observation gives rise to the following question:

For which sets $\mathbb{M} \subseteq \{0, \dots, i\}$ do the invariants $h_{\bullet}^j(-j)$, ($j = 0, \dots, i$), $p_{\bullet}^i(-l)$, ($l \in \mathbb{M}$) form a minimal lower bounding system for the invariant ν_{\bullet}^i on the class \mathcal{C} ? •

4. PURELY DIAGONAL BOUNDS

Keep the previous notations and hypotheses. We shall discuss in this section the role of the full cohomology diagonal $(h_{\mathcal{F}}^i(-i))_{i=0}^{cd(\mathcal{F})=\dim(\mathcal{F})}$ as a bounding system in the special case where the base ring R_0 is a field. For these considerations the case where $R = K[\underline{\mathbf{x}}] = K[\mathbf{x}_0, \dots, \mathbf{x}_r]$ is a polynomial ring over a field K is crucial. We begin with some prerequisites.

4.1. Definition and Remark. (i) Let M be a finitely generated and graded module over the positively graded homogeneous ring $R = \bigoplus_{n \geq 0} R_n$. Let $R_+ := \bigoplus_{n > 0} R_n \subset R$ be the irrelevant ideal of R and let $k \in \mathbb{N}_0$. For a graded R -module $T = \bigoplus_{n \in \mathbb{Z}} T_n$ let $\text{end}(T) := \sup\{n \in \mathbb{Z} \mid T_n \neq 0\}$ denote the *end* of T . We define the *regularity of M at and above level k* by (cf. [21, (15.2.9)])

$$\text{reg}^k(M) := \sup\{\text{end}(H_{R_+}^i(M)) + i \mid i \geq k\},$$

bearing in mind that the local cohomology modules $H_{R_+}^i(M)$ carry a natural grading. Then $\text{reg}(M) := \text{reg}^0(M)$ is the Castelnuovo-Mumford regularity of the module M , as it was introduced by Ooishi [68].

(ii) Keep the notations and hypotheses of part (i). Let $X = \text{Proj}(R)$ and let $\mathcal{F} := \tilde{M}$ be the sheaf of \mathcal{O}_X -modules induced by M . Then, on use of the Serre-Grothendieck Correspondence (1.16), (1.17) we have

$$\text{reg}_k(\mathcal{F}) = \text{reg}^{k+2}(M), \quad (\forall k \geq 0).$$

In particular we have (cf. [21, (20.2.4)])

$$\text{reg}(\mathcal{F}) = \text{reg}^2(M);$$

$$\text{reg}(M) = \max\{\text{reg}(\mathcal{F}), \text{end}(H_{R_+}^0(M)), \text{end}(H_{R_+}^1(M)) + 1\}.$$

(iii) Now, let $K[\mathbf{x}] = K[\mathbf{x}_0, \dots, \mathbf{x}_r]$ be a polynomial ring over the field K . Let M be a finitely generated and graded $K[\mathbf{x}]$ -module with a minimal graded free resolution of the form

$$\begin{aligned} 0 \longrightarrow \bigoplus_{i=1}^{b_p} K[\mathbf{x}](a_i^{(p)}) &\longrightarrow \bigoplus_{i=1}^{b_{p-1}} K[\mathbf{x}](a_i^{(p-1)}) \\ &\longrightarrow \dots \longrightarrow \bigoplus_{i=1}^{b_0} K[\mathbf{x}](a_i^{(0)}) \longrightarrow M \longrightarrow 0. \end{aligned}$$

Then, the well known *syzygetic characterization* of regularity (cf. [21, (15.3.7)], [27]) gives

$$\text{reg}(M) = \max\{-a_i^{(j)} - j \mid 0 \leq j \leq p, 1 \leq i \leq b_j\}. \quad \bullet$$

4.2. Definition. Let $R = \bigoplus_{n \geq 0} R_n$ be as in the introduction. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finitely generated and graded R -module. We define the *generating degree* $d(M)$ of M as the smallest integer d such that M is generated over R by homogeneous elements of degree $\leq d$. Thus

$$d(M) := \inf\{d \in \mathbb{Z} \mid M = \bigoplus_{n \leq d} M_n R\}. \quad \bullet$$

Now, we may formulate the following bounding result.

4.3. Theorem. *Let $r \in \mathbb{N}$. Then, there is a polynomial $P_r \in \mathbb{Q}[\mathbf{s}, \mathbf{t}]$ such that for each $s \in \mathbb{N}$, each field K , each polynomial ring $K[\mathbf{x}] = K[\mathbf{x}_0, \dots, \mathbf{x}_r]$ and for each graded submodule $M \subseteq K[\mathbf{x}]^{\oplus s}$*

$$\text{reg}(M) \leq P_r(s, d(M)).$$

Proof: See [15, Sec. 2]. □

4.4. Remark and Problems. (i) One way of obtaining a bounding polynomial $P_r \in \mathbb{Q}[\mathbf{s}, \mathbf{t}]$ as in (4.3) is to use the classical *Hentzelt-Hermann bound* for the generating degree of the kernel of a polynomial matrix, (s. [44], [45] but also [59], [71] and [76]). This bound says that the kernel $\text{Ker}(F) \subseteq K[\mathbf{x}]^{\oplus t}$ of a matrix $F = [f_{ij} \mid 1 \leq i \leq s, 1 \leq j \leq t] \in K^{s \times t}$ of size $s \times t$ with entries $f_{ij} \in K[\mathbf{x}]$ is generated by elements of degree $< (2s \deg(F))^{2^r}$, where $\deg(F) := \max\{\deg(f_{ij}) \mid 1 \leq i \leq s, 1 \leq j \leq t\}$. Starting from this bound one obtains

$$P_r(\mathbf{s}, \mathbf{t}) = (2\mathbf{st})^{2^{r(r+1)}} \in \mathbb{Q}[\mathbf{s}, \mathbf{t}]$$

as a possible choice for P_r .

(ii) A considerable smaller bounding polynomial P_r than in part (i) is obtained if one proceeds as in the thesis of A.F. Lashgari [53]: First, the *regularity criterion of Bayer-Stillman* [3] is extended from the case of graded ideals of $K[\underline{\mathbf{x}}]$ to the case of graded submodules of $K[\underline{\mathbf{x}}]^{\oplus s}$ for arbitrary $s \in \mathbb{N}$. Then adopting the arguments which furnish the *Bayer-Mumford regularity bound* for graded ideals in $K[\underline{\mathbf{x}}]$ (cf. [2]) one obtains

$$P_r(\mathbf{s}, \mathbf{t}) = \mathbf{s}^{e_r} (2\mathbf{t})^{r!} \in \mathbb{Q}[\mathbf{s}, \mathbf{t}]$$

as a possible candidate for P_r , where the exponents e_r are defined recursively by

$$e_0 := 1; \quad e_m = me_{m-1} + 1 \quad \text{for } m \in \mathbb{N}.$$

(iii) There is some evidence, that the bounding polynomial of part (ii) is for apart from giving sharp bounds either: Namely, for a graded ideal $\mathfrak{a} \subseteq K[\underline{\mathbf{x}}]$ we may apply theorem (4.3) with the bounding polynomial $P_r(\mathbf{s}, \mathbf{t})$ mentioned in part (ii) and obtain $\text{reg}(\mathfrak{a}) \leq (2d(\mathfrak{a}))^{r!}$. This is the regularity bound of Bayer-Mumford found in [2, (3.8)] and mentioned already in part (ii). It is still an open problem, whether in our special case one has the sharper estimate $\text{reg}(\mathfrak{a}) \leq (2d(\mathfrak{a}))^{2^{r-1}}$. For base fields K of characteristic 0, this inequality is in fact true (cf. [2, (3.7)], [32], [33]). On the other hand, examples due to E. Mayr and A. Meyer [62] show, that this latter bound is “fairly close” to be sharp. More precisely, for each $r \in \mathbb{N}$, there is a graded ideal $\mathfrak{a}_r \subseteq \mathbb{C}[\mathbf{x}_0, \dots, \mathbf{x}_r]$ such that $d(\mathfrak{a}_r) = 4$ and $\text{reg}(\mathfrak{a}_r) \geq 2^{2^{\lfloor (r-1)/10 \rfloor}} + 1$ (cf. [2, (3.11)]).

(iv) In view of the above observations it seems natural to ask, whether in Theorem (4.3) one might expect a mere “doubly exponential” bound on $\text{reg}(M)$. More precisely:

Are there constants $a, b > 0$ such that for each $r \in \mathbb{N}$, each $s \in \mathbb{N}$, each polynomial ring $K[\underline{\mathbf{x}}] = K[\underline{\mathbf{x}}_0, \dots, \underline{\mathbf{x}}_r]$ and each graded submodule $M \subseteq K[\underline{\mathbf{x}}]^{\oplus s}$ we have $\text{reg}(M) \leq (asd(M))^{b^r}$? •

Now, let $X = \text{Proj}(\oplus_{n \geq 0} R_n = R)$ and \mathcal{F} be as in the introduction. Let

$$\mathcal{F}^\vee := \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$

be the dual of \mathcal{F} . Our next result says that, over projective spaces, the “cohomology diagonal of a coherent sheaf bounds the regularity of the dual sheaf”.

4.5. Theorem. *Let $r \in \mathbb{N}_0$. Then, there is a polynomial $Q_r \in \mathbb{Q}[\mathbf{t}_0, \dots, \mathbf{t}_r]$ such that for each field K and each coherent sheaf of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules \mathcal{F} we have*

$$\text{reg}(\mathcal{F}^\vee) \leq Q_r(h_{\mathcal{F}}^0(0), h_{\mathcal{F}}^1(-1), \dots, h_{\mathcal{F}}^r(-r)).$$

Proof: See [15, Sec. 3]. □

4.6. Example and Problems. (i) For a locally free sheaf \mathcal{F} of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules (e.g. an algebraic vector bundle over \mathbb{P}_K^r) Serre-duality gives $h_{\mathcal{F}^\vee}^i(n) = h_{\mathcal{F}}^{r-i}(-n-r-1)$ for all $i \in \{0, \dots, r\}$ and all $n \in \mathbb{Z}$. Fix $j \in \{0, \dots, r-1\}$ and let $t \in \mathbb{N}_0$. Consider the minimal combinatorial pattern

$$P_{j,t} := (\{0\} \times \mathbb{N}) \cup (\{j\} \times \{k \mid -j-t \leq k \leq -j\}) \cup (\{r\} \times \mathbb{Z}_{<-t-r})$$

of width r . Then, there is an indecomposable algebraic vector bundle $\mathcal{F}_{j,t}$ of rank $\leq r!$ such that $P(\mathcal{F}_{j,t}) = P_{j,t}$ (cf. (2.11) (i)). Consequently, $P(\mathcal{F}_{j,t}^\vee) = (\{0\} \times \mathbb{Z}_{\geq t}) \cup (\{r-j\} \times \{k \mid j-r-1 \leq k \leq j-r-1+t\}) \cup (\{r\} \times \mathbb{Z}_{<r-1})$. In particular, we have $\text{reg}(\mathcal{F}_{j,t}) = 0$ and $h_{\mathcal{F}_{j,t}}^i(-i) = 0$ for all $i \neq j$ and $\text{reg}(\mathcal{F}_{j,t}^\vee) = t$. This shows at the same time that the regularity of a coherent sheaf of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules \mathcal{F} need not bound the regularity of \mathcal{F}^\vee and that the full cohomology diagonal is a minimal upper bounding system for the regularity on the class of all these sheaves.

(ii) In view of the previous observation one is lead to ask:

Which invariants of \mathcal{F} bound the regularity of \mathcal{F}^\vee ? •

As a consequence of Theorem (4.5) one now gets that the “*full cohomology diagonal bounds the cohomological postulation numbers*”. To formulate this result, let us introduce the following notation.

4.7. Notation. Let $\tilde{\mathcal{C}}$ denote the class of all pairs (X, \mathcal{F}) in which $X = \text{Proj}(\bigoplus_{n \geq 0} R_n)$ is a projective scheme over a field $R_0 = K$ and \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules. For $r \in \mathbb{N}_0$ let $\tilde{\mathcal{C}}^{(r)}$ denote the class of all pairs $(X, \mathcal{F}) \in \tilde{\mathcal{C}}$ with $\dim(\mathcal{F}) (= cd(\mathcal{F})) = r$. Observe that in the notation of (3.4) (i) and (3.6) we have $\tilde{\mathcal{C}} \subseteq \mathcal{C}$ and $\tilde{\mathcal{C}}^{(r)} \subseteq \mathcal{C}^{(r)}$. •

Now, we are ready to formulate the announced bounding result.

4.8. Theorem. *Let $r \in \mathbb{N}_0$. Then, there is a polynomial $T_r \in \mathbb{Q}[\mathbf{t}_0, \dots, \mathbf{t}_r]$ such that for each pair $(X, \mathcal{F}) \in \tilde{\mathcal{C}}^{(r)}$ and each $i \in \{0, \dots, r\}$ we have*

$$\nu_{\mathcal{F}}^i \geq T_r(h_{\mathcal{F}}^0(0), h_{\mathcal{F}}^1(-1), \dots, h_{\mathcal{F}}^r(-r)).$$

Proof: See [15, Sec. 4] or [53]. □

As an interesting consequence of the previous result we get that “*the full cohomology diagonal numerically bounds cohomology*”.

4.9. Corollary. *Let $i, r \in \mathbb{N}_0$ and let $h^0, \dots, h^r \in \mathbb{N}_0$. Then, there are only finitely many cohomological Hilbert functions $h_{\mathcal{F}}^i$ with $(X, \mathcal{F}) \in \tilde{\mathcal{C}}^{(r)}$ and $h_{\mathcal{F}}^j(-j) = h^j$ for $j = 0, \dots, r$.*

Proof: See [15, Sec. 5] or [53]. □

4.10. Remark and Problems. (i) For $r = 2$, the conclusions of (4.8) and (4.9) even hold on the class $\mathcal{C}^{(2)}$, e.g. over arbitrary artinian base rings R_0 (cf. [17, Sec. 4], (3.10) iv)).

(ii) The observation made in part (i) gives rise to the question:

Do the results (4.8) and (4.9) hold for arbitrary r on the class $\mathcal{C}^{(r)}$? •

5. b -SHEAVES AND HILBERT COEFFICIENTS

A fundamental issue of algebraic geometry is the relation between the so called Hilbert coefficients and the regularity of coherent sheaves – notably for sheaves of ideals or for invertible sheaves. Results of this type are of basic significance for the theory of Hilbert schemes and Picard schemes (cf. [34], [40], [57], [58], [38], [51], [65]).

5.1. Reminder and Remark. (i) Let $X = Proj(R = \bigoplus_{n \geq 0} R_n)$ and $\mathcal{F} \neq 0$ be as in the previous sections and assume in addition that the base ring R_0 is artinian. For $i \in \{0, \dots, cd(\mathcal{F}) = \dim(\mathcal{F})\}$ let $e_i(\mathcal{F}) \in \mathbb{Z}$ be the i -th *Hilbert coefficient* of \mathcal{F} , so that the *Hilbert-Serre polynomial* of \mathcal{F} is given by

$$\chi_{\mathcal{F}}(\mathbf{t}) = \sum_{i=0}^{\dim(\mathcal{F})} (-1)^i e_i(\mathcal{F}) \binom{\mathbf{t} + \dim(\mathcal{F}) - i}{\dim(\mathcal{F}) - i} \in \mathbb{Q}[\mathbf{t}].$$

Keep in mind that $e_0(\mathcal{F}) \in \mathbb{N}$ and that

$$\chi_{\mathcal{F}}(n) = \sum_{i=0}^{\dim(\mathcal{F})} (-1)^i h_{\mathcal{F}}^i(n) \quad \text{for all } n \in \mathbb{Z}.$$

Sometimes it is useful to introduce the invariants

$$a_i(\mathcal{F}) = (-1)^{\dim(\mathcal{F})-i} e_{\dim(\mathcal{F})-i}(\mathcal{F}), \quad (i = 0, \dots, \dim(\mathcal{F})),$$

so that

$$\chi_{\mathcal{F}}(\mathbf{t}) = \sum_{i=0}^{\dim(\mathcal{F})} \binom{\mathbf{t} + i}{i} a_i(\mathcal{F}).$$

(ii) Let K be an algebraically closed field and let $r \in \mathbb{N}$. A fundamental result of Mumford [65] says that the regularity $reg(\mathcal{J})$ of a coherent sheaf $\mathcal{J} \subseteq \mathcal{O}_{\mathbb{P}_K^r}$ of ideals is bounded polynomially in terms of the Hilbert coefficients $e_0(\mathcal{J}), \dots, e_r(\mathcal{J})$ of \mathcal{J} .

For each $m \in \mathbb{N}$ let $\mathcal{F}_m := \mathcal{O}_{\mathbb{P}_K^1}(-m) \oplus \mathcal{O}_{\mathbb{P}_K^1}(m)$, so that $e_0(\mathcal{F}_m) = 2$, $e_1(\mathcal{F}_1) = 0$ and $reg(\mathcal{F}_m) = m$. This example shows that, even on the class of fully decomposable algebraic vector bundles over \mathbb{P}_K^r , regularity is not bounded by the Hilbert coefficients. •

In spite of the failure just observed, there is an extension of Mumfords bound result mentioned in (5.1) (ii) to a larger class of sheaves. This extension is due to S. Kleiman (s. [38, Exp. XIII]) and based on the notion of \underline{b} -sheaf. We recall this notion for the readers convenience.

5.2. Definition and Remark. (i) Let $X = Proj(R = \bigoplus_{n \geq 0} R_n)$ and \mathcal{F} be as in the introduction. A sequence of global sections $f_1, \dots, f_r \in H^0(X, \mathcal{O}_X(1))$ is said to be \mathcal{F} -regular, if

$$H_i \cap Ass_X(\mathcal{F} \upharpoonright_{H_1 \cap \dots \cap H_{i-1}}) = \emptyset \text{ for } i = 1, \dots, r,$$

where $H_i \subseteq X$ is the subscheme defined by f_i . It is equivalent to say, that the natural homomorphisms of sheaves

$$f_i : \mathcal{F} \upharpoonright_{H_1 \cap \dots \cap H_{i-1}}(n) \longrightarrow \mathcal{F} \upharpoonright_{H_1 \cap \dots \cap H_{i-1}}(n+1)$$

are injective for all $i \in \{1, \dots, r\}$ and one (resp. all) $n \in \mathbb{Z}$.

If R_0 contains an infinite field, there are \mathcal{F} -regular sequences $f_1, \dots, f_r \in H^0(X, \mathcal{O}_X(1))$ of arbitrary length r .

(ii) Now, let $X = Proj(R = \bigoplus_{n \geq 0} R_n)$ and \mathcal{F} be as above and assume, that R_0 is in addition artinian. Let $r \in \mathbb{N}_0$ and let $\underline{b} = (b_0, \dots, b_r) \in \mathbb{N}_0^{r+1}$. Then, \mathcal{F} is said to be a \underline{b} -sheaf, if $\dim(\mathcal{F}) \leq r$ and

$$h_{\mathcal{F}}^0 \upharpoonright_{H_1 \cap \dots \cap H_i}(-1) \leq b_i \text{ for } i = 0, \dots, r,$$

where again $H_j \subseteq X$ is the subscheme defined by f_j . •

Then, *Kleimans Main Theorem on \underline{b} -sheaves* says

5.3. Theorem. (cf. [38, Exp. XIII, Théorème 1.11]) *Let $r \in \mathbb{N}_0$. Then, there is a polynomial $U_r \in \mathbb{Q}[\mathfrak{t}_0, \dots, \mathfrak{t}_r]$ such that for each algebraically closed field K , each projective scheme X over K and each coherent sheaf of \mathcal{O}_X -modules which is a $\underline{b} = (b_0, \dots, b_r)$ -sheaf of dimension r we have*

$$reg(\mathcal{F}) \leq U_r(b_0 - a_0(\mathcal{F}), \dots, b_r - a_r(\mathcal{F})). \quad \square$$

5.4. Remark and Problems. (i) Let X be a purely r -dimensional reduced projective scheme of degree d over an algebraically closed field K . Then, on use of Bertini, it follows easily that \mathcal{O}_X is a $(0, \dots, 0, d)$ -sheaf. As the property of being a \underline{b} -sheaf is inherited by coherent subsheaves (cf. [38, Exp. XIII, Prop. 1.6]) it follows from (5.3) that $reg(\mathcal{J}) \leq U_r(-a_0(\mathcal{J}), \dots, -a_{r-1}(\mathcal{J}), d - a_r(\mathcal{J}))$ for each coherent sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_X$. This clearly recovers the bounding result of Mumford mentioned in (5.1) (ii).

(ii) Let K be an algebraically closed field, let $m_1, \dots, m_t \in \mathbb{Z}$ and let $\mathcal{F} \subseteq \bigoplus_{i=1}^t \mathcal{O}_{\mathbb{P}_K^r}(m_i)$ be a coherent subsheaf. It follows easily from (5.3) that $reg(\mathcal{F})$ is polynomially bounded in terms of the Hilbert coefficients $e_i(\mathcal{F})$ of \mathcal{F} and the integers m_j . In [21, Chap. 17] we proved an algebraic version of this, in which K may be replaced by an arbitrary artinian ring.

(iii) Now, let $X = Proj(R = \bigoplus_{n \geq 0} R_n)$ and \mathcal{F} as in (5.1) (i). Assume that $f_1, \dots, f_r \in H^0(X, \mathcal{O}_X(1))$ form an \mathcal{F} -regular sequence. For $j \in \{0, \dots, r\}$

let $H_j \subseteq X$ denote the subscheme of X defined by f_j . Then, it follows easily by induction on k , that

$$h_{\mathcal{F}|_{H_1 \cap \dots \cap H_k}}^i(n) \leq \sum_{j=0}^k \binom{k}{j} h_{\mathcal{F}}^{i+j}(n-j); \quad (n \in \mathbb{Z}, 0 \leq k \leq r).$$

So, if $\dim(\mathcal{F}) \leq r$ and with $b_k := \sum_{j=0}^k \binom{k}{j} h_{\mathcal{F}}^j(-j)$, ($k = 0, \dots, r$) it follows that $\mathcal{F}(1)$ is a $\underline{b} = (b_0, \dots, b_r)$ -sheaf.

Thus, if R_0 contains an infinite field, for each coherent sheaf \mathcal{F} of \mathcal{O}_X -modules of dimension r with given cohomology diagonal the twisted sheaf $\mathcal{F}(1)$ is a \underline{b} -sheaf, where $\underline{b} = (b_0, \dots, b_r)$ is defined as above. It is easy to see that the converse of this does not hold:

If K is an algebraically closed field, one may choose a sequence $(\mathcal{J}_n)_{n \in \mathbb{N}}$ of coherent sheaves of ideals $\mathcal{J}_n \subseteq \mathcal{O}_{\mathbb{P}_K^r}$ such that $\lim_{n \rightarrow \infty} \text{reg}(\mathcal{J}_n) = \infty$. Then clearly the set of cohomology diagonals $\{(h_{\mathcal{J}_n(-1)}^i(-i))_{i=0}^r \mid n \in \mathbb{N}\}$ is unbounded by (3.2), whereas the sheaves $(\mathcal{J}_n(-1))(1) = \mathcal{J}_n$ are all $(0, \dots, 0, 1)$ -sheaves (cf. part (i)).

(iv) For $r \in \mathbb{N}_0$ let $\overline{\mathcal{C}}^{(r)}$ be the class of all pairs (X, \mathcal{F}) in which X is a projective scheme over an algebraically closed field and \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules with $\dim(\mathcal{F}) = r$.

The observations of part (iii) make us ask the following question:

Let $\underline{b} = (b_0, \dots, b_r) \in \mathbb{N}_0^{r+1}$, $(c_0, \dots, c_r) \in \mathbb{N}_0^{r+1}$ and let \mathcal{D} be the class of all pairs $(X, \mathcal{F}) \in \overline{\mathcal{C}}^{(r)}$ in which \mathcal{F} is a \underline{b} -sheaf and for which $|e_i(\mathcal{F})| \leq c_i$ for $i = 0, \dots, r$. Is the set of cohomology diagonals $\{(h_{\mathcal{F}(-1)}^i(-i))_{i=0}^r \mid (X, \mathcal{F}) \in \mathcal{D}\}$ finite?

Obviously, the best answer to this would be given by polynomial upper bounds for the numbers $h_{\mathcal{F}(-1)}^i(-i) = h_{\mathcal{F}}^i(-i-1)$ in terms of the numbers b_k and the Hilbert coefficients $e_j(\mathcal{F})$.

(v) In view of [38, Exp. XIII, Théorème (6.4), (6.7)] one also could ask the following modified version of the previous question:

Let $\underline{b}, \underline{c}$ be as in part (vi), let $m \in \mathbb{N}_0$ and let \mathcal{D}'_m resp. \mathcal{D}''_m be the classes of all pairs $(X, \mathcal{F}) \in \overline{\mathcal{C}}^{(r)}$ for which $\mathcal{F}(m)$ is generated by global sections and \mathcal{F} is a \underline{b} -sheaf resp. $|e_i(\mathcal{F})| \leq c_i$ for $i = 0, \dots, r$. Are the sets $\{(h_{\mathcal{F}(-1)}^i(-i))_{i=0}^r \mid (X, \mathcal{F}) \in \mathcal{D}'_m\}$ and $\{(h_{\mathcal{F}(-1)}^i(-i))_{i=0}^r \mid (X, \mathcal{F}) \in \mathcal{D}''_m\}$ finite? •

6. A FEW SPECIFIC BOUNDS

Now, we shall say a few words about the cohomological invariants of some specific classes of pairs (X, \mathcal{F}) . We start with pairs $(\mathbb{P}_K^r, \mathcal{J})$ with $r > 1$, where K is an algebraically closed field and $\mathcal{J} \subseteq \mathcal{O}_{\mathbb{P}_K^r}$ is the sheaf of

vanishing ideals of a projective variety $Y \subseteq \mathbb{P}_K^r$, e.g. a closed reduced irreducible subscheme. We always shall assume in this situation, that Y is non-degenerately embedded, e.g. that it is contained in no hyperplane $\mathbb{P}_K^{r-1} \subseteq \mathbb{P}_K^r$.

6.1. Reminder, Remark and Problem. (i) Keep the above hypotheses and notations. The function $h_{\mathcal{J}}^1 : \mathbb{Z} \rightarrow \mathbb{N}_0$ is called the *Hartshorne-Rao function* of the variety $Y \subseteq \mathbb{P}_K^r$. As Y is reduced and K is algebraically closed, we have $h_{\mathcal{J}}^1(n) = 0$ for all $n \leq 0$. Write $\bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}_K^r, \mathcal{O}_{\mathbb{P}_K^r}(n)) = K[\mathbf{x}_0, \dots, \mathbf{x}_r] = K[\underline{\mathbf{x}}]$, let $I := \bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}_K^r, \mathcal{J}(n)) \subseteq K[\underline{\mathbf{x}}]$ be the homogeneous vanishing ideal of Y and let $A := K[Y] := K[\underline{\mathbf{x}}]/I$ be the homogeneous coordinate ring of Y . As $\mathcal{J} = \tilde{I}$, the Serre-Grothendieck correspondence (1.16), (1.17) yields $h_{\mathcal{J}}^1(n) = \dim_K(H_{K[\underline{\mathbf{x}}]_+}^2(I)_n)$ for all $n \in \mathbb{Z}$. So, the graded short exact sequence $0 \rightarrow I \rightarrow K[\underline{\mathbf{x}}] \rightarrow A \rightarrow 0$ shows that $h_{\mathcal{J}}^1(n) = \dim_K(H_{A_+}^1(A)_n)$ for all $n \in \mathbb{Z}$.

(ii) Keep the previous notations and hypotheses, and let us briefly recall the geometric meaning of the numbers $h_{\mathcal{J}}^1(n)$. First of all, let us recall that $h_{\mathcal{J}}^1(1)$ is the *projection excess* of $Y \subseteq \mathbb{P}_K^r$, e.g. the largest number $e \in \mathbb{N}_0$ for which there is a non-degenerate projective variety $Y' \subseteq \mathbb{P}_K^{r+e}$ and a projection $\pi : \mathbb{P}_K^{r+e} \setminus \mathbb{P}_K^{e-1} \rightarrow \mathbb{P}_K^r$ from a linear subspace $\mathbb{P}_K^{e-1} \subseteq \mathbb{P}_K^{r+e}$ (with the convention that $\mathbb{P}_K^{-1} = \emptyset$) such that $\mathbb{P}_K^{e-1} \cap Y' = \emptyset$ and π induces an isomorphism $\pi| : Y' \xrightarrow{\cong} Y$.

For arbitrary $n \in \mathbb{N}$ let $Y^{(n)} \subseteq \mathbb{P}_K^{\binom{r+n}{r}-1}$ the n -th *Veronesean* of Y and let $\mathbb{P}_K^{t(n)} \subseteq \mathbb{P}_K^{\binom{r+n}{r}-1}$ be the linear subspace spanned by $Y^{(n)}$. Then $h_{\mathcal{J}}^1(n)$ is nothing else than the projection excess of $Y^{(n)} \subseteq \mathbb{P}_K^{t(n)}$.

So here, the Hartshorne-Rao function has a description in purely geometric terms.

(iii) The Hartshorne-Rao function of space curves $Y \subseteq \mathbb{P}_K^3$ has found much interest recently and here, very satisfactory results have been proved (cf. [64] for example). In [20] the structure of the Hartshorne-Rao module $H^1(A) \cong \bigoplus_{n \in \mathbb{Z}} H^1(\mathbb{P}_K^3, \mathcal{J}(n))$ of a curve $Y \subseteq \mathbb{P}_K^3$ of degree $r+2$ is completely described. The fact, that in this situation only 4 different Hartshorne-Rao functions occur if $r \geq 4$ allows a fairly good understanding of these curves. The ideas of [20] have been used in [10] to study surfaces $Y \subseteq \mathbb{P}_K^r$ of degree $r+1$. But here, we have not been able to describe all occurring Hartshorne-Rao functions. So, let us ask the following question:

Which are the possible Hartshorne-Rao functions of surfaces $Y \subseteq \mathbb{P}_K^r$ of degree $r+1$?

A related problem (open even in the surface case) is:

What is a sharp upper bound for the Hartshorne-Rao function of a projective variety $Y \subseteq \mathbb{P}_K^r$ of degree $\leq r - \dim(Y) + 3$? •

6.2. Remark and Problems. (i) Let K be an algebraically closed field and let $Y \subseteq \mathbb{P}_K^r$ be a closed subscheme, $\mathcal{J} \subseteq \mathcal{O}_{\mathbb{P}_K^r}$ its sheaf of vanishing ideals. The study of the regularity $\text{reg}(Y) := \text{reg}(\mathcal{J})$ is an extremely active field of algebraic geometry. One of the most appealing problems in this field is a conjecture of Eisenbud-Goto [27] saying that

$$\text{reg}(Y) \leq \text{deg}(Y) - \text{codim}(Y) + 1$$

whenever Y is reduced and irreducible. For smooth curves in $\mathbb{P}_\mathbb{C}^3$, the above estimate corresponds to Castelnuovo's classical result (cf. [25]). For curves $Y \subseteq \mathbb{P}_K^r$, the above inequality has been established by Gruson-Lazarsfeld-Peskine [39]. For smooth surfaces and threefolds $Y \subseteq \mathbb{P}_K^r$ in characteristic 0, the stated inequality has been shown by Lazarsfeld [54] and Pinkham [69] respectively by Ran [70].

(ii) Besides the proves of the above inequality a tremendous amount of upper bounds for the regularity of specific classes of projective schemes has been established (s. [46], [67], [78] for example). For investigations of the same kind in some modified context see also [80] or [81]. The regularity of projective schemes is of fundamental interest for computational algebraic geometry (s. [2] for example) and thus related to Gröbner base techniques (cf. [24], [26], [33], [46], [57], [81]).

(iii) For arbitrary 1-Buchsbaum surfaces in characteristic 0 the Eisenbud-Goto inequality of part (i) has been established in [22]. The essential problem was to establish this inequality for 1-Buchsbaum surfaces $Y \subseteq \mathbb{P}_K^r$ of degree $r+1$. In [10], the Eisenbud-Goto inequality has been established in arbitrary characteristic for a larger class of surfaces $Y \subseteq \mathbb{P}_K^r$ of degree $r+1$, but not for all of them. So we ask the following question – closely related to the problem in (6.1) (iii):

Do all surfaces (resp. all projective varieties) $Y \subseteq \mathbb{P}_K^r$ of degree $r+1$ (resp. of degree $\leq r - \dim(Y) + 3$) satisfy the Eisenbud-Goto inequality? •

6.3. Remark and Problems. (i) Elenywaig and Forster [28] have shown that all cohomological Hilbert functions $h_{\mathcal{E}}^i$ of an algebraic vector bundle \mathcal{E} over a complex projective space $\mathbb{P}_\mathbb{C}^r$ are bounded in terms of the first two Chern numbers $c_1(\mathcal{E}), c_2(\mathcal{E})$ and the span $\sigma(\mathcal{E})$ of the generic splitting type of \mathcal{E} . If \mathcal{E} is in addition semistable, the Grauert-Mulich theorem gives bounds on the functions $h_{\mathcal{E}}^i$ in terms of $c_1(\mathcal{E}), c_2(\mathcal{E})$ and $\text{rank}(\mathcal{E})$, thus generalizing a result of [41]. In [5] and [18] such bound of Elenywaig-Forster type are studied over algebraically closed base fields K of arbitrary characteristic by means of the method of linear systems of hyperplane sections.

(ii) In (2.11) we have asked already a few questions related to the cohomology of algebraic vector bundles over a projective space. Remember, that for a coherent sheaf \mathcal{F} of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules \mathcal{F} with a given cohomology diagonal only finitely many different cohomological Hilbert functions $h_{\mathcal{F}}^i$ may occur

(cf. (4.9)). This makes us ask, under which conditions on the cohomology diagonal all the occurring functions $h_{\mathcal{F}}^i$ with $i < r$ are “left vanishing”:

Is there a non-trivial criterion on the full cohomology diagonal $(h_{\mathcal{F}(-i)}^i)_{i=0}^r$ of a coherent sheaf of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules \mathcal{F} which induces that \mathcal{F} is a vector bundle? •

6.4. Remark and Problems. (i) One also can just study the cohomological Hilbert functions $h_{\mathcal{O}_X}^i$, where $X \subseteq \mathbb{P}_K^r$ is a projective variety (e.g. a closed reduced and irreducible subscheme of \mathbb{P}_K^r) over the algebraically closed field K . Bearing in mind the short exact sequence $0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_{\mathbb{P}_K^r} \rightarrow \mathcal{O}_X \rightarrow 0$ (in which $\mathcal{J} \subseteq \mathcal{O}_{\mathbb{P}_K^r}$ is the sheaf of vanishing ideals of X) this essentially comes up to study the cohomological Hilbert functions $h_{\mathcal{J}}^i$ for $i > 1$, e.g. different from the Hartshorne-Rao function. Using the method of linear systems of hyperplane sections, some bounding results on these functions are deduced in [5], [6] and [8].

(ii) Among the fundamental vanishing theorems for (local) cohomology groups one may distinguish results of Hartshorne-Lichtenbaum type (cf. [21, Chap. 8] and [48] notably) which concern cohomology defined with respect to open subsets of projective schemes and results of Kodaira type (cf. [35], [52], [66]) which are closely related to the vanishing of the numbers $h_{\mathcal{O}_X}^i(n)$ in the range $n < 0$, $i < \dim(X)$, where X is a projective variety over an algebraically closed field K . We focus only on the second type of result. Here, the situation considered in the geometric context actually is more general: One considers an ample invertible sheaf \mathcal{L} of \mathcal{O}_X -modules and studies the vanishing of the groups $H^i(X, \mathcal{L}^{\otimes n})$ for $i < \dim(X)$ and $n < 0$. Choosing $\mathcal{L} = \mathcal{O}_X(1)$ one obtains vanishing statements for the numbers $h_{\mathcal{O}_X}^i(n)$ in the requested range. Let us recall the main result of [52] which says that the above cohomology groups $H^i(X, \mathcal{L}^{\otimes n})$ vanish if X is smooth and $K = \mathbb{C}$. Let us also recall a vanishing result of Mumford [66], which says that the groups $H^1(X, \mathcal{L}^{\otimes n})$ vanish for all $n < 0$ if X is normal of dimension > 1 and if $\text{Char}(K) = 0$. In [66] it is also shown that this result fails if $\text{Char}(K) > 0$.

(iii) In positive characteristic one has at least some bounding results for the dimensions $h^1(X, \mathcal{L}^{\otimes n}) := \dim_K H^1(X, \mathcal{L}^{\otimes n})$. So, in [1] it is shown that with

$$e(X) := \sum_{p \in X, \{p\} = \overline{\{p\}}} \dim_K (H_{\mathfrak{m}_{X,p}}^1(\mathcal{O}_{X,p}))$$

we have

$$e(X) \leq h^1(X, \mathcal{L}^{\otimes n}) \leq \min\{e(X), h^1(X, \mathcal{O}_X) + n(h^0(X, \mathcal{L}) - 1)\}$$

for all $n < 0$, if X is of dimension > 1 and has only finitely many non-normal points. In [1], we actually consider a more general situation, with the aim to prove a *relative version* of the mentioned bounding result: The canonical morphism $X \rightarrow \text{Spec}(K)$ is replaced by a projective morphism $X \rightarrow X_0 = \text{Spec}(R_0)$ with geometrically normal fibres (cf. [36]), where R_0

is a domain essentially of finite type over a perfect field. We have not been able to prove similar bounds for higher cohomology groups. So let us ask the following question:

Are there upper bounds of the above type for the numbers $h^i(X, \mathcal{L}^{\otimes n})$ in the range $n < 0$, $i < \dim(X)$ if X is smooth?

(iv) One consequence of the bound mentioned in part (ii) is that $h^1(X, \mathcal{O}_X(n)) = e(X)$ for all non-degenerate surfaces $X \subseteq \mathbb{P}_K^r$ with sectional genus $< r$ and having only finitely many non-normal points. In [11] we have applied this fact in order to give a characteristic free approach to the cohomological behaviour of non-degenerate surfaces $X \subseteq \mathbb{P}^r$ of degree $\leq 2r - 2$ and having only finitely many non-normal points. This leads to a generalization of a corresponding result of Chern and Griffiths for smooth complex projective surfaces. It seems natural to ask whether these results may be generalized to amply polarized surfaces:

Let X be a projective surface with only finitely many non-normal points and let \mathcal{L} be an ample invertible sheaf with $\deg(\mathcal{L}) \leq 2h^0(X, \mathcal{L}) - 4$. Is it true that $h^1(X, \mathcal{L}^{\otimes n}) = e(X)$ for all $n < 0$? What is the nature of the morphism $X \rightarrow \mathbb{P}_K^{h^0(X, \mathcal{L})-1}$ induced by \mathcal{L} , if $\deg(\mathcal{L}) = 2h^0(X, \mathcal{L}) - 4$? •

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Markus Brodmann
Institute of Mathematics
University of Zurich
Winterthurerstrasse 190
CH – 8057 Zürich, Switzerland
e-mail: brodmann@math.unizh.ch