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BILIPSCHITZ EMBEDDINGS OF NEGATIVE SECTIONAL CURVATURE IN PRODUCTS OF WARPED PRODUCT MANIFOLDS

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ABSTRACT. Generalizing results due to Brady and Farb (1998) we prove the existence of bilipschitz embedded manifolds of negative sectional curvature in Riemannian products of certain types of warped products.

1. INTRODUCTION

In [BrFa] the authors proved that the product $X := H^{m_1} \times ... \times H^{m_k}$ of hyperbolic spaces H^{m_i} admits an embedding of $Y := H^{m_1+...+m_k-k+1}$ in X that is quasi-isometric in the sense that the Riemannian distance functions d_X and d_Y on X and Y are related via

$$d_X|_Y \ \leq d_Y \leq \ \alpha \cdot d_X|_Y \ + \ \beta,$$

with constants $\alpha, \beta \in \mathbb{R}^+$, $\alpha > 1$.

It turns out that a variation of their method of proof yields the existence of Riemannian submanifolds (Y, g_Y) of negative sectional curvature within a wide class of Riemannian products $(X, g_X) = \mathcal{M}_1 \times \mathcal{M}_2 \times ... \times \mathcal{M}_k$ of warped products $\mathcal{M}_i$. It will be shown that in certain cases those submanifolds Y are embedded, such that the Riemannian distance functions d_X on X and d_Y on Y are Lipschitz related. More precisely there exists a $1 < \alpha \in \mathbb{R}^+$, such that

$$d_X|_Y \ \leq d_Y \leq \ \alpha \cdot d_X|_Y.$$

In order for the canonical embedding (Y, g_Y) that we are going to consider be of negative sectional curvature, there are certain conditions on the base space $(\mathcal{B}, g_{\mathcal{B}})$, the fibres $(\mathcal{F}_i, g_{\mathcal{F}_i})$, as well as the warping functions f_i that have to be satisfied. To be precise, the following holds.

Theorem 1. *Let $(\mathcal{B}, g_{\mathcal{B}})$ be either a one dimensional Riemannian manifold or a Riemannian manifold of arbitrary dimension $n(\mathcal{B})$ with negative sectional curvature. Further, let $f_i : \mathcal{B} \longrightarrow \mathbb{R}^+$, $i = 1, 2, ..., k$, be strictly convex functions on $(\mathcal{B}, g_{\mathcal{B}})$ without minimum that satisfy the relations*

$$grad\, f_i\ (f_j)\ >\ 0 \qquad\qquad \forall i, j\ \in\ \{1, 2, ..., k\},$$

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and $(\mathcal{F}_i, g_{\mathcal{F}_i})$, $i = 1, 2, ..., k$, *are either one dimensional Riemannian manifolds or Riemannian manifolds of nonpositive sectional curvatures.*

Let $g_{\mathcal{M}_i} := \pi_i^* g_{\mathcal{B}} + (f_i \circ \pi_i)^2 \eta_i^* g_{\mathcal{F}_i}$ *be the warped product metrics on* $\mathcal{M}_i := \mathcal{B} \times \mathcal{F}_i$, *where* π_i *and* η_i *denote the canonical projections from* $\mathcal{M}_i$ *to* $\mathcal{B}$ *and* $\mathcal{F}_i$ *respectively and define the Riemannian manifold* (X, g_X) *and the manifold* Y *via*

$$\begin{array}{lllllllllll} X & := & & (\mathcal{M}_1, g_{\mathcal{M}_1}) & \times & (\mathcal{M}_2, g_{\mathcal{M}_2}) & \times & .... & \times & (\mathcal{M}_k, g_{\mathcal{M}_k}) & \textit{and} \\ Y & := & \mathcal{B} \quad \times & \mathcal{F}_1 & \times & \mathcal{F}_2 & \times & .... & \times & \mathcal{F}_k & . \end{array}$$

The embedding $i : Y \longrightarrow X$

$$i : (t^m, x_1^{1_l}, x_2^{2_l}, ..., x_k^{k_l}) \longrightarrow \Big((t^m, x_1^{1_l}), (t^m, x_2^{2_l}), ..., (t^m, x_k^{k_l}) \Big)$$

defines a metric $g_Y := i^* g_X$ *on* Y *that makes* (Y, g_Y) *a Riemannian manifold with negative sectional curvature.*

Furthermore, in the case that the warping functions f_i are all the same, (Y, g_Y) turns out to be bilipschitz embedded in (X, g_X).

Theorem 2. *For* (X, g_X) *and* (Y, g_Y) *as above with* $f_i = f_j =: f \;\forall\, i, j = 1, 2, ..., k$, *the Riemannian distance functions* d_X *and* d_Y *on* (X, g_X) *and* (Y, g_Y) *are related via*

$$d_X|_Y \overset{i)}{\leq} d_Y \overset{ii)}{\leq} \Big(2 \cdot \sqrt{k} \; + \; k \Big) \cdot d_X|_Y .$$

2. Proofs of the theorems

We are going to denote the various canonical projections as follows:

$$\begin{array}{ll} \sigma_j : X \longrightarrow \mathcal{M}_j, & \eta^Y : Y \longrightarrow \mathcal{F}_1 \times .... \times \mathcal{F}_k, \\ \pi_j : \mathcal{M}_j \longrightarrow \mathcal{B}, & \pi \;\; : Y \longrightarrow \mathcal{B}, \\ \eta_j : \mathcal{M}_j \longrightarrow \mathcal{F}_j, & \eta_j^Y : Y \longrightarrow \mathcal{F}_j. \end{array}$$

2.1. **Proof of Theorem 1.**

Proof. A straightforward calculation proves

Proposition 1. *The metric* g_Y *above is given through*

$$g_Y \quad = \quad \pi^*(k \cdot g_{\mathcal{B}}) \; + \; \sum_{i=1}^{k} (f_i \circ \pi)^2 (\eta_i^Y)^* g_{\mathcal{F}_i}. \tag{1}$$

Another standard calculation gives

Lemma 1. *Let* h *be a smooth, strictly convex function on* $(\mathcal{B}, g_{\mathcal{B}})$ *that satisfies*

$$grad\, f \; (h) \quad > \quad 0$$

for another smooth function f *on* $\mathcal{B}$. *Then the lift* $\tilde{h}$ *of* h *on* $\mathcal{B} \times_f \mathcal{F}$ *is strictly convex, too.*

Recall the sectional curvature's equation for a warped product $(\mathcal{N}, g_{\mathcal{N}}) := \mathcal{B} \times_f \mathcal{F}$ ([BiO'N]): Let Π be the nondegenerate plane in the tangent space $T_{(p,q)}\mathcal{N}$ of $\mathcal{N}$ at $(p, q) \in \mathcal{N}$ that is spanned by the two orthonormal vectors $\tilde{x} + \tilde{v}$ and $\tilde{y} + \tilde{w}$ with $\tilde{x}, \tilde{y}, \tilde{v}, \tilde{w}$ being lifts of vectors $x, y \in T_p\mathcal{B}$ and $v, w \in (T_q\mathcal{F})$, i.e. $g_{\mathcal{N}}(\tilde{x} + \tilde{v}, \tilde{x} + \tilde{v}) =$

$1 = g_{\mathcal{N}}(\tilde{y}+\tilde{w},\tilde{y}+\tilde{w})$, $g_{\mathcal{N}}(\tilde{x}+\tilde{v},\tilde{y}+\tilde{w}) = 0$. The sectional curvature $K(\Pi)$ of Π is then

$$\begin{aligned}(2)\ \ K(\Pi) \ \ = \ \ & K^{\mathcal{B}}(x,y)\ \ ||\tilde{x}\wedge\tilde{y}||^2 \\ & -\Big\{\frac{H^f(y,y)}{\tilde{f}g_{\mathcal{B}}(y,y)}\, g_{\mathcal{B}}(y,y)\, g_{\mathcal{N}}(\tilde{v},\tilde{v}) \ + \ \frac{H^f(x,x)}{\tilde{f}g_{\mathcal{B}}(x,x)}\, g_{\mathcal{B}}(x,x)\, g_{\mathcal{N}}(\tilde{w},\tilde{w}) \\ & \quad - 2\,\frac{H^f(x,y)}{\tilde{f}g_{\mathcal{B}}(x,y)}\, g_{\mathcal{B}}(x,y)\, g_{\mathcal{N}}(\tilde{v},\tilde{w})\Big\}\frac{||\tilde{x}\wedge\tilde{w}\ -\ \tilde{y}\wedge\tilde{v}||^2}{||\tilde{x}\wedge\tilde{w}\ -\ \tilde{y}\wedge\tilde{v}||^2} \\ & + \Big\{\frac{1}{\tilde{f}^2}K^{\mathcal{F}}(v,w)\ -\ \frac{g_{\mathcal{N}}(grad(\tilde{f}),grad(\tilde{f}))}{\tilde{f}^2}\Big\}\ ||\tilde{v}\wedge\tilde{w}||^2,\end{aligned}$$

where $K^{\mathcal{B}}$ and $K^{\mathcal{F}}$ denote the sectional curvatures on $\mathcal{B}$ and $\mathcal{F}$ and the norm is defined by $||c\wedge d||^2 = G_{\mathcal{N}}(c\wedge d, c\wedge d)$, with

$$G_{\mathcal{N}}(c\wedge d, c\wedge d) \ := \ g_{\mathcal{N}}(c,c)g_{\mathcal{N}}(d,d) \ - \ g_{\mathcal{N}}(c,d)^2.$$

Note that the particular form of equation (2) is chosen most appropriately for our purposes. The terms that are not defined for particular choices of $\tilde{x},\tilde{y},\tilde{v}$ and $\tilde{w}$ vanish in those cases.

On $Y_l := \mathcal{B}\times\mathcal{F}_1\times ...\times\mathcal{F}_l$ we define

$$g_Y^l \ := \ {\pi_{Y_l}}^*(k\cdot g_{\mathcal{B}}) \ + \ \sum_{i=1}^{l}(f_i\circ\pi_{Y_l})^2(\eta_i^{Y_l})^* g_{\mathcal{F}_i}$$

with π_{Y_l} and $\eta_i^{Y_l}$ on Y_l analogous to π and η_i^Y on Y, and conclude as follows: The f_i are strictly convex on $(\mathcal{B}, g_{\mathcal{B}})$ and thus of course on $(\mathcal{B}, k\cdot g_{\mathcal{B}})$ as well. Furthermore f_1 has no minimum and therefore is strictly convex on $(\mathcal{B}\times\mathcal{F}_1, g_Y^1)$.

Applying Lemma 1 successively shows that f_i is strictly convex on $(\mathcal{B}\times\mathcal{F}_1\times ...\times\mathcal{F}_{i-1}, g_Y^{i-1})$. Now the $(\mathcal{F}_j, g_{\mathcal{F}_j})$ have nonpositive sectional curvatures and thus the validity of the theorem follows from Proposition 1, equation (2) and equations (9) and (10) successively applied to $(\mathcal{B}\times\mathcal{F}_1\times ...\times\mathcal{F}_{i-1}, g_Y^{i-1})$. □

From equations (2), (8), (9) and (10) we further conclude:

Corollary 1. *Let* $K^{\mathcal{B}} \leq -\delta^{\mathcal{B}} < 0$, *and assume that for* $f_j : \mathcal{B} \longrightarrow \mathbb{R}$, $j = 1,2,...,k$, *there exist* $\delta^{f_j}, \delta^{ij} \in \mathbb{R}$ *such that*

$$\frac{H^{f_j}(x,x)}{f_j g_{\mathcal{B}}(x,x)} \geq \delta^{f_j} > 0 \qquad\qquad \forall\, x\in\ \mathcal{TB},\ j = 1,2,...,k,\ \textit{and}$$

$$\frac{grad^{\mathcal{B}} f_i(f_j)}{f_i\ f_j} \geq \delta^{ij} > 0 \qquad\qquad \forall\, i,j = 1,2,...,k,$$

where $grad^{\mathcal{B}} f_i$ *is the gradient of* f_i *in* $(\mathcal{B}, g_{\mathcal{B}})$. *Then the sectional curvature* K^Y *of* (Y, g_Y) *is bounded by*

$$K^Y \ \leq \ \frac{1}{k^2}\max_{i,j=1,2,...,k}\Big\{ - k\delta^{\mathcal{B}}, -(\delta^{f_j})^2, -(\delta^{ij})^2, -k\delta^{ii}, -k^2\Big\} \ < \ 0.$$

Proof. Write $grad^{\tilde{\mathcal{B}}} f_1$ for the gradient of f_1 relative to $(\mathcal{B}, k\cdot g_{\mathcal{B}})$ and $H^{\tilde{f}_j}$ for the Hessian of the lift $\tilde{f}_j$ of f_j to (Y_j, g_Y^j).

Consider (Y, g_Y) as the following warped product:

$$(Y, g_Y) \ = \ \Big(...\Big(\Big(\tilde{\mathcal{B}} \times_{f_1} \mathcal{F}_1\Big) \times_{f_2} \mathcal{F}_2\Big) \times_{f_3} ... \times_{f_{k-1}} \mathcal{F}_{k-1}\Big) \times_{f_k} \mathcal{F}_k,$$

where $\tilde{\mathcal{B}}$ denotes the Riemannian manifold $(\mathcal{B}, kg_{\mathcal{B}})$ and $\mathcal{F}_j$ is short for $(\mathcal{F}_j, g_{\mathcal{F}_j}),\ j = 1, ..., k$.

For any f_j one has

$$\frac{H^{\tilde{f}_j}(z,z)}{\tilde{f}_j g_Y^j(z,z)} \ \geq \ \frac{1}{k} \min_{l=1,2,\ldots,j} \Big\{\delta^{f_j}, \delta^{lj}\Big\}. \tag{3}$$

In order to see this let $Z_1 = \tilde{X} + \tilde{V}_1$ be a vector of unit length in (Y_1, g_Y^1) and calculate

$$\begin{aligned}
\frac{H^{\tilde{f}_j}(\tilde{X}+\tilde{V}_1, \tilde{X}+\tilde{V}_1)}{\tilde{f}_j} \ &= \ \frac{H^{\tilde{f}_j}(\tilde{X}, \tilde{X})}{\tilde{f}_j} \ + \ \frac{H^{\tilde{f}_j}(\tilde{V}_1, \tilde{V}_1)}{\tilde{f}_j} \\
&= \ \frac{H^{f_j}(X,X)}{f_j} \ + \ g_Y^1(\tilde{V}_1, \tilde{V}_1)\frac{grad^{\tilde{\mathcal{B}}}\tilde{f}_1(\tilde{f}_j)}{\tilde{f}_1\tilde{f}_j} \\
&\geq \ k \cdot g_{\mathcal{B}}(X,X)\ \frac{\delta^{f_j}}{k} \ + \ g_Y^1(\tilde{V}_1, \tilde{V}_1)\ \frac{\delta^{12}}{k} \\
&\geq \ \frac{1}{k}\ \min\Big\{\delta^{f_j}, \delta^{1j}\Big\}. \qquad (4)
\end{aligned}$$

Now let $Z_2 = \tilde{X} + \tilde{V}_1 + \tilde{V}_2 \in Y_2$ be a vector of unit (Y_2, g_Y^2)-length. The same calculation as in (4) shows that

$$\frac{H^{\tilde{f}_j}(\tilde{X}+\tilde{V}_1+\tilde{V}_2, \tilde{X}+\tilde{V}_1+\tilde{V}_2)}{\tilde{f}_j} \ \geq \ \frac{1}{k}\ \min\{\delta^{f_j}, \delta^{1,j}, \delta^{2j}\}.$$

It is obvious that the required inequality (3) follows by applying this argument successively to the (Y_l, g_Y^l), $l = 3, ..., k$.

Now apply equation (2) to the warped products (Y_j, g_Y^j) and notice that one has

$$||\tilde{x}_j \wedge \tilde{y}_j||^2 \ + \ ||\tilde{x}_j \wedge \tilde{w}_j \ - \ \tilde{y}_j \wedge \tilde{v}_j||^2 \ + \ ||\tilde{v}_j \wedge \tilde{w}_j||^2 \ = \ 1: \tag{5}$$

$$\begin{aligned}
K^{Y_1}(\Pi) \quad = \quad & K^{\tilde{\mathcal{B}}}(x_1, y_1)||\tilde{x}_1 \wedge \tilde{y}_1||^2 \\
& - \frac{B(\tilde{x}_1 \wedge \tilde{w}_1 - \tilde{y}_1 \wedge \tilde{v}_1, \tilde{x}_1 \wedge \tilde{w}_1 - \tilde{y}_1 \wedge \tilde{v}_1)}{G(\tilde{x}_1 \wedge \tilde{w}_1 - \tilde{y}_1 \wedge \tilde{v}_1, \tilde{x}_1 \wedge \tilde{w}_1 - \tilde{y}_1 \wedge \tilde{v}_1)}||\tilde{x}_1 \wedge \tilde{w}_1 - \tilde{y}_1 \wedge \tilde{v}_1||^2 \\
& - \Big\{\frac{1}{f_1^2}K^{\mathcal{F}_1}(v_1, w_1) \ - \ \frac{g_{Y_1}(grad\tilde{f}_1, grad\tilde{f}_1))}{\tilde{f}_1^2}\Big\}||\tilde{v}_1 \wedge \tilde{w}_1||^2 \\
& \leq - \frac{1}{k}\delta^{\mathcal{B}}||\tilde{x}_1 \wedge \tilde{y}_1||^2 \\
& - \min\Big\{1, \frac{1}{k^2}(\delta^{f_1})^2, \frac{1}{k^2}(\delta^{11})^2\Big\}||\tilde{x}_1 \wedge \tilde{w}_1 - \tilde{y}_1 \wedge \tilde{v}_1||^2 \\
& -\frac{1}{k}\delta^{11}||\tilde{v}_1 \wedge \tilde{w}_1||^2 \\
& \leq \frac{1}{k^2}\max\Big\{ - k\delta^{\mathcal{B}}, -(\delta^{f_1})^2, -(\delta^{11})^2, -k\delta^{11}, -k^2\Big\},
\end{aligned}$$

where the first inequality is due to equations (3), (5), (8) and (9). The corollary follows by applying the same arguments successively to the warped products $(Y_l, g_Y^l)\times_{f_{l+1}}\mathcal{F}_{l+1}$, $l = 1, ..., k-1$. $\square$

Arguments analogous to those above finally yield a proof for

Corollary 2. *Let $K^{\mathcal{B}} \geq -\epsilon^{\mathcal{B}} < 0$, $f_j : \mathcal{B} \longrightarrow \mathbb{R}$, $j = 1, 2, ..., k$, be such that there exist $\epsilon^{f_j}, \epsilon^{ij} \in \mathbb{R}$ with*

$$\frac{H^{f_j}(x,x)}{f_j g_{\mathcal{B}}(x,x)} \leq \epsilon^{f_j} > 0 \qquad \forall\, x \in \mathcal{TB},\ j = 1, 2, ..., k,\ \textit{and}$$

$$\frac{grad^{\mathcal{B}} f_i(f_j)}{f_i\ f_j} \leq \epsilon^{ij} > 0 \qquad \forall\, i, j = 1, 2, ..., k,$$

and assume that $\frac{K^{\mathcal{F}_j}}{{f_j}^2}$ is bounded below by $-\epsilon^j$, $j = 1, 2, ..., k$. Then the sectional curvature K^Y of (Y, g_Y) is bounded by

$$K^Y \ \geq\ \frac{1}{k^2} \min_{i,j=1,2,\ldots,k} \Big\{ -k\epsilon^{\mathcal{B}}, -(\epsilon^{f_j})^2, -(\epsilon^{ij})^2, -k\epsilon^i, k\epsilon^{ii}, -k^2 \Big\} \ <\ 0\,.$$

2.2. **Proof of Theorem 2.**

Proof. 1) The inequality $i)$ holds trivially, since (Y, g_Y) is a Riemannian submanifold of (X, g_X).

2) In order to show the second inequality consider an arbitrary differentiable curve $c : [t_p, t_q] \longrightarrow X$ connecting $i(p) \in X$ with $i(q) \in X$. The idea is now to construct a curve $\tilde{c} : [\alpha, \omega] \longrightarrow Y$ in Y that connects $p \in Y$ with $q \in Y$, whose Riemannian length $L_{(Y,g_Y)}(\tilde{c})$ in (Y, g_Y) is bounded by a constant times the Riemannian length $L_{(X,g_X)}(c)$ of c in (X, g_X).

Therefore we consider the projections $c_j := \sigma_j \circ c$ of c to $\mathcal{M}_j$, that connect $p_j := \sigma_j(i(p))$ with $q_j := \sigma_j(i(q))$. The $\pi_j \circ c_j$ are continuous, thus the set $K := \bigcup_j\ (\pi_j \circ c_j)([t_p, t_q]) \subset \mathcal{B}$ is compact. Let $b \in [t_p, t_q]$ be a parameter such that the warping function f takes its minimum in K at $(\pi_{j_0} \circ c_{j_0})(b)$ for some index $j_0 \in \{1, 2, ..., k\}$ (compare Figure 1).

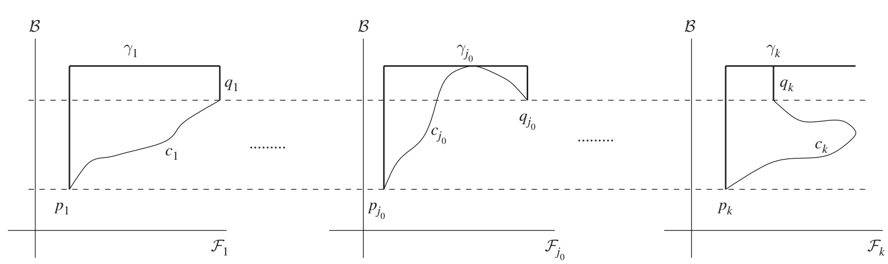

FIGURE 1. This figure shows possible projections of γ to the factors $\mathcal{M}_i$. Take the base space to be the set of real numbers and assume the warping function to be monotonously increasing. In this case the maximum of f is attained at the point of c with largest projection to $\mathbb{R}$, as the figure suggests.

The continuous and piecewise differentiable curve γ in Y that we are going to follow from p to q consists of three differentiable segments v_1, λ, and v_2 as follows:

- v_1 has constant projection $\eta^Y \circ v_1$ to $\mathcal{F}_1 \times \mathcal{F}_2 \times ... \times \mathcal{F}_k$ that is given through $\eta^Y \circ v_1 \equiv (\eta_1(p_1), ..., \eta_k(p_k))$, while its projection to the base $\mathcal{B}$ is $\pi \circ v_1 = (\pi_{j_0} \circ c_{j_0})|_{[t_p,b]}$.
- λ is the curve keeping its projection to the base constant: $(\pi \circ \lambda) \equiv (\pi_{j_0} \circ c_{j_0})(b)$, while varying along $\mathcal{F}_1 \times \mathcal{F}_2 \times ... \times \mathcal{F}_k$ with $\eta^Y \circ \lambda = (\eta_1 \circ c_1, ..., \eta_k \circ c_k)$.
- v_2 again has constant projection $\eta^Y \circ v_2$ to $\mathcal{F}_1 \times \mathcal{F}_2 \times ... \times \mathcal{F}_k$, that is, $\eta^Y \circ v_2 \equiv (\eta_1(q_1), ..., \eta_k(q_k))$. Its projection to the base $\mathcal{B}$ is $\pi \circ v_2 = (\pi_{j_0} \circ c_{j_0})|_{[b,t_q]}$.

The length of $\gamma := v_2 * \lambda * v_1$ is the sum

$$L_{(Y,g_Y)}(\gamma) \;=\; L_{(Y,g_Y)}(v_1) \;+\; L_{(Y,g_Y)}(\lambda) \;+\; L_{(Y,g_Y)}(v_2).$$

From (1) it directly follows that

$$\begin{aligned} L_{(Y,g_Y)}(v_m) &= \sqrt{k} \cdot L_{(\mathcal{B},g_{\mathcal{B}})}((\pi_{j_0} \circ c_{j_0})|_{I_m}) \\ &\leq \sqrt{k} \cdot L_{(\mathcal{B},g_{\mathcal{B}})}(\pi_{j_0} \circ c_{j_0}) \\ &\leq \sqrt{k} \cdot L_{(X,g_X)}(c), \end{aligned} \tag{6}$$

where $I_1 = [t_p, b]$ and $I_2 = [b, t_q]$.

Again with (1) and the particular choice of b it is

$$\begin{aligned} L_{(Y,g_Y)}(\lambda) &= \int_{t_p}^{t_q} \sqrt{g_Y(\lambda'(\tau), \lambda'(\tau))}\, d\tau \\ &= \int_{t_p}^{t_q} \sqrt{\sum_{j=1}^{k} f^2\Big((\pi_{j_0} \circ c_{j_0})(b)\Big)\; g_{\mathcal{F}_j}\Big((\eta_j \circ c_j)'(\tau), (\eta_j \circ c_j)'(\tau)\Big)}\; d\tau \\ &\leq \sum_{j=1}^{k} \int_{t_p}^{t_q} \sqrt{f^2\Big((\pi_{j_0} \circ c_{j_0})(b)\Big)\; g_{\mathcal{F}_j}\Big((\eta_j \circ c_j)'(\tau), (\eta_j \circ c_j)'(\tau)\Big)}\; d\tau \\ &\leq \sum_{j=1}^{k} \int_{t_p}^{t_q} \sqrt{f^2\Big((\pi_j \circ c_j)(\tau)\Big)\; g_{\mathcal{F}_j}\Big((\eta_j \circ c_j)'(\tau), (\eta_j \circ c_j)'(\tau)\Big)}\; d\tau \\ &= \sum_{j=1}^{k} L_{(\mathcal{M}_j, g_{\mathcal{M}_j})}(c_j) \\ &\leq k \cdot \max_{j=1,...,k} \Big\{ L_{(\mathcal{M}_j, g_{\mathcal{M}_j})}(c_j) \Big\} \;\leq\; k \cdot L_{(X,g_X)}(c)\,. \end{aligned} \tag{7}$$

Thus with (6) and (7) we can conclude that for an arbitrary curve c in X connecting two points $i(p), i(q) \in i(Y) \subset X$ there exists a curve γ in Y connecting p and q with

$$L_{(Y,g_Y)}(\gamma) \;\leq\; \Big(2\sqrt{k} \;+\; k\Big) L_{(X,g_X)}(c).$$

Thus the required result follows by the definition of the Riemannian length functions. □

Considering different warping functions f_i on $\mathcal{B}$, the proof above fails in general. However, taking the f_i to be Lipschitz related to each other, the same result may be achieved. A more interesting further generalization is obtained if one takes the base $\mathcal{B}$ to be the set of real numbers $\mathbb{R}$. If the f_i are all monotonous increasing (decreasing respectively) d_X and d_Y once again turn out to be Lipschitz related.

Finally note that the embeddings Y considered above are not quasiconvex in X in general. If one of the factors $\mathcal{F}_i$ of $\mathcal{M}_i$ does not lie quasiconvex in $\mathcal{M}_i$, then it is easy to see that Y is not quasiconvex in X either. For the cases of products of hyperbolic spaces, see [BrFa].

Appendix A.

Consider the warped product $(\mathcal{N}, g_{\mathcal{N}}) := \mathcal{B}\times_f\mathcal{F}$ at $n := (p,q) \in \mathcal{B}\times\mathcal{F}$. The two positive definite symmetric bilinear forms $f^2 g_{\mathcal{F}} : T_q\mathcal{F}\times T_q\mathcal{F} \longrightarrow \mathbb{R}$ and $\frac{H^f}{f}$: $T_p\mathcal{B}\times T_p\mathcal{B} \longrightarrow \mathbb{R}$ define a positive definite symmetric bilinear form on the direct sum $T_p\mathcal{B} + T_q\mathcal{F} = T_{(p,q)}\mathcal{N}$ via

$$b(\tilde{x}+\tilde{v},\tilde{x}+\tilde{v}) \;:=\; g_{\mathcal{N}}(\tilde{v},\tilde{v}) \;+\; \frac{H^f(x,x)}{f g_{\mathcal{B}}(x,x)} g_{\mathcal{B}}(x,x).$$

If $\frac{H^f}{f}$ is strictly positive, then so is b. In order to see this, note that for every unit vector $\tilde{x}+\tilde{v}$ in $T_{(p,q)}\mathcal{N}$ one has

$$\begin{aligned} b(\tilde{x}+\tilde{v},\tilde{x}+\tilde{v}) &= g_{\mathcal{N}}(\tilde{v},\tilde{v}) \;+\; \frac{H^f(x,x)}{f g_{\mathcal{B}}(x,x)} g_{\mathcal{B}}(x,x) \\ &\geq \min\left\{1, \frac{H^f(x,x)}{f g_{\mathcal{B}}(x,x)}\right\} \Big(g_{\mathcal{N}}(\tilde{v},\tilde{v}) \;+\; g_{\mathcal{B}}(x,x)\Big) \\ &= \min\left\{1, \frac{H^f(x,x)}{f g_{\mathcal{B}}(x,x)}\right\} =: \epsilon. \end{aligned} \tag{8}$$

Denote the extensions of b and $g_{\mathcal{N}}$ to bivectors by B and $G_{\mathcal{N}}$ and define $\hat{b} : T_{(p,q)}\mathcal{N} \longrightarrow T_{(p,q)}\mathcal{N}$ via

$$b(c,d) \;=:\; g_{\mathcal{N}}\Big(\hat{b}(c), d\Big) \qquad\qquad \forall\; c,d \in T_{(p,q)}\mathcal{N}$$

and $\hat{B} : \Lambda^2_{(p,q)}\mathcal{N} \longrightarrow \Lambda^2_{(p,q)}\mathcal{N}$ analogously. Observe that with $\hat{b}$ being symmetric relative to $g_{\mathcal{N}}$, $\hat{B}$ is symmetric relative to $G_{\mathcal{N}}$. Furthermore for an orthonormal basis $\{u_1,...,u_n\}$ of eigenvectors of $\hat{b}$ with eigenvalues λ_i, $\{u_i\wedge u_j \,|\, i<j, j=1,...,n\}$ is an orthonormal basis of eigenvectors of $\hat{B}$ with eigenvalues $\lambda_i\lambda_j$. Thus with $\hat{b}$ being bounded by ϵ $\hat{B}$ turns out to be bounded by ϵ^2:

$$\frac{B(c\wedge d, c\wedge d)}{G_{\mathcal{N}}(c\wedge d, c\wedge d)} \;=\; \frac{G_{\mathcal{N}}(\tilde{B}(c\wedge d), c\wedge d)}{G_{\mathcal{N}}(c\wedge d, c\wedge d)} \;\geq\; \epsilon^2. \tag{9}$$

Note that with exactly the same arguments as above one proves that an upper bound of $\frac{H^f}{f}$ yields an upper bound for $\frac{B}{G_{\mathcal{N}}}$.

Finally note that for $\tilde{x}, \tilde{y}, \tilde{v}, \tilde{w}$ as in subsection 2.1

$$\begin{aligned} & B(\tilde{x} \wedge \tilde{w} - \tilde{y} \wedge \tilde{v}, \tilde{x} \wedge \tilde{w} - \tilde{y} \wedge \tilde{v}) \\ = \; & B(\tilde{x} \wedge \tilde{w}, \tilde{x} \wedge \tilde{w}) \; + \; B(\tilde{y} \wedge \tilde{v}, \tilde{y} \wedge \tilde{v}) \; - \; 2\, B(\tilde{x} \wedge \tilde{w}, \tilde{y} \wedge \tilde{v}) \\ = \; & \Big\{ \frac{H^{\tilde{f}}(y,y)}{\tilde{f} g_{\mathcal{B}}(y,y)}\, g_{\mathcal{B}}(y,y)\; g_{\mathcal{N}}(\tilde{v},\tilde{v}) \; + \; \frac{H^{\tilde{f}}(x,x)}{\tilde{f} g_{\mathcal{B}}(x,x)}\, g_{\mathcal{B}}(x,x)\; g_{\mathcal{N}}(\tilde{w},\tilde{w}) \\ & - 2\, \frac{H^{\tilde{f}}(x,y)}{\tilde{f} g_{\mathcal{B}}(x,y)}\, g_{\mathcal{B}}(x,y)\; g_{\mathcal{N}}(\tilde{v},\tilde{w}) \Big\} \end{aligned} \tag{10}$$

and

$$\begin{aligned} & G_{\mathcal{N}}(\tilde{x} \wedge \tilde{w} - \tilde{y} \wedge \tilde{v}, \tilde{x} \wedge \tilde{w} - \tilde{y} \wedge \tilde{v}) \\ = \; & G_{\mathcal{N}}(\tilde{x} \wedge \tilde{w}, \tilde{x} \wedge \tilde{w}) \; + \; G_{\mathcal{N}}(\tilde{y} \wedge \tilde{v}, \tilde{y} \wedge \tilde{v}) \; - \; 2\, G_{\mathcal{N}}(\tilde{x} \wedge \tilde{w}, \tilde{y} \wedge \tilde{v}) \\ = \; & g_{\mathcal{B}}(x,x) g_{\mathcal{N}}(\tilde{w},\tilde{w}) \; + g_{\mathcal{B}}(y,y) g_{\mathcal{N}}(\tilde{v},\tilde{v}) \; - \; 2\, g_{\mathcal{B}}(x,y) g_{\mathcal{N}}(\tilde{v},\tilde{w}). \end{aligned} \tag{11}$$

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