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Gauge theoretical equivariant Gromov-Witten invariants and the full Seiberg-Witten invariants of ruled surfaces

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Abstract

Let F be a differentiable manifold endowed with an almost Kähler structure (J,ω), α a J-holomorphic action of a compact Lie group $\hat{K}$ on F, and K a closed normal subgroup of $\hat{K}$ which leaves ω invariant.

The purpose of this article is to introduce gauge theoretical invariants for such triples (F,α,K). The invariants are associated with moduli spaces of solutions of a certain vortex type equation on a Riemann surface Σ.

Our main results concern the special case of the triple

$$(\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^{r_0}),\alpha_{\mathrm{can}},U(r))\ ,$$

where α_{can} denotes the canonical action of $\hat{K}=U(r)\times U(r_0)$ on $\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^{r_0})$. We give a complex geometric interpretation of the corresponding moduli spaces of solutions in terms of gauge theoretical quot spaces, and compute the invariants explicitely in the case $r=1$.

Proving a comparison theorem for virtual fundamental classes, we show that the full Seiberg-Witten invariants of ruled surfaces, as defined in [OT2], can be identified with certain gauge theoretical Gromov-Witten invariants of the triple $(\mathrm{Hom}(\mathbb{C},\mathbb{C}^{r_0}),\alpha_{\mathrm{can}},U(1))$. We find the following formula for the full Seiberg-Witten invariant of a ruled surface over a Riemann surface of genus g:

$$SW^{-\mathrm{sign}\langle c,[F]\rangle}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})=0\ ,$$

$$SW^{\mathrm{sign}\langle c,[F]\rangle}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(l)=\mathrm{sign}\langle c,[F]\rangle\left\langle\sum_{i\geq\max(0,g-\frac{w_c}{2})}^{g}\frac{\Theta_c^i}{i!}\wedge l\ ,\ l_{\mathcal{O}_1}\right\rangle\ ,$$

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where $[F]$ denotes the class of a fibre. The computation of the invariants in the general case $r > 1$ should lead to a generalized Vafa-Intriligator formula for "twisted" Gromov-Witten invariants associated with sections in Grassmann bundles.

1 Introduction

1.1 The general set up

Let F be a differentiable manifold, ω a symplectic form on F, and J a compatible almost complex structure. Let α be a J-holomorphic action of a compact Lie group $\hat{K}$ on F, and let K be a closed normal subgroup of $\hat{K}$ which leaves ω invariant.

Put $K_0 := \hat{K}/K$, and let π be the projection of $\hat{K}$ onto this quotient. We fix an invariant inner product on the Lie algebra $\hat{\mathfrak{k}}$ of $\hat{K}$ and denote by $\mathrm{pr}_{\mathfrak{k}} : \hat{\mathfrak{k}} \to \mathfrak{k}$ the orthogonal projection onto the Lie algebra of K.

The topological data of our moduli problem are a K_0-bundle P_0 on a compact oriented differentiable 2-manifold Σ, and an equivalence class $\mathfrak{c}$ of pairs $(\lambda, \hat{h})$ consisting of a π-morphism $\hat{P} \xrightarrow{\lambda} P_0$ and a homotopy class $\hat{h}$ of sections in the associated bundle $E := \hat{P} \times_{\hat{K}} F$. Two pairs $(\lambda, \hat{h})$, $(\lambda', \hat{h}')$ are equivalent if there exists an isomorphism $\hat{P} \to \hat{P}'$ *over* P_0 which maps $\hat{h}$ onto $\hat{h}'$.

The pair $(P_0, \mathfrak{c})$ should be regarded as the *discrete parameter* on which our moduli problem depends. It plays the same role as the data of a $SU(2)$- or a $PU(2)$-bundle in Donaldson theory, or the data of an equivalence class of *Spin*^{c}-structures in Seiberg-Witten theory.

For every representant $(\hat{P} \xrightarrow{\lambda} P_0, \hat{h})$ of $\mathfrak{c}$ we denote by $\Gamma^\lambda(\mathfrak{c}) \subset \Gamma(\Sigma, E)$ the union of all homotopy classes $\hat{h}'$ of sections in E for which $(\lambda, \hat{h}') \in \mathfrak{c}$. This set $\Gamma^\lambda(\mathfrak{c})$ is the union of the homotopy classes in the orbit of $\hat{h}$ with respect to the action of the group $\pi_0(\mathrm{Aut}_{P_0}(\hat{P}))$ on the set $\pi_0(\Gamma(\Sigma, E))$. In other words, $\Gamma^\lambda(\mathfrak{c})$ is the saturation of $\hat{h}$ with respect to the $\mathrm{Aut}_{P_0}(\hat{P})$-action on the space of sections.

Now fix a representant $(\hat{P} \xrightarrow{\lambda} P_0, \hat{h})$ of $\mathfrak{c}$. Let μ be a $\hat{K}$-equivariant moment map for the restricted K-action $\alpha|_K$ on F, let g be a metric on Σ, and let A_0 be a connection on P_0.

The triple $\mathfrak{p} := (\mu, g, A_0)$ is the *continuous parameter* on which our moduli problem depends. It plays the role of the Riemannian metric on the base manifold in Donaldson theory [DK], or the role of the pair (Riemannian metric, self-dual form) in Seiberg-Witten theory [W2], [OT1], [OT2].

The orthogonal projection $\mathrm{pr}_{\mathfrak{k}}$ induces a bundle projection which we denote by the same symbol

$$\mathrm{pr}_{\mathfrak{k}} : \hat{P} \times_{\mathrm{ad}} \hat{\mathfrak{k}} \longrightarrow \hat{P} \times_{\mathrm{ad}} \mathfrak{k} \ .$$

Since $\hat{K}$ acts J-holomorphically, a connection $\hat{A}$ in $\hat{P}$ defines an almost holomorphic structure $J_{\hat{A}}$ in the associated bundle E; $J_{\hat{A}}$ agrees with J on the vertical tangent bundle $T_{E/\Sigma}$ of E and it agrees with the holomorphic structure J_g defined by g on Σ on the $\hat{A}$-horizontal distribution of E.

Our *gauge group* is

$$\mathcal{G} = \mathrm{Aut}_{P_0}(\hat{P}) \simeq \Gamma(\Sigma, \hat{P} \times_{\mathrm{Ad}} K) ,$$

and it acts from the right on our *configuration space*

$$\mathcal{A} := \mathcal{A}_{A_0}(\hat{P}, \lambda) \times \Gamma^{\lambda}(\mathfrak{c}) .$$

Here $\mathcal{A}_{A_0}(\hat{P}, \lambda)$ is the affine space of connections $\hat{A}$ in $\hat{P}$ which project onto A_0 via λ.

For a pair $(\hat{A}, \varphi) \in \mathcal{A}$ we consider the equations

$$\left\{ \begin{array}{ccc} \varphi & \text{is} & J_{\hat{A}} \text{ holomorphic} \\ \mathrm{pr}_{\mathfrak{k}} \Lambda F_{\hat{A}} + \mu(\varphi) & = & 0 \ . \end{array} \right. \tag{$V_{\mathfrak{p}}$}$$

These vortex type equations are obviously gauge invariant. The first condition of $(V_{\mathfrak{p}})$ can be rewritten as

$$\bar{\partial}_{\hat{A}} \varphi = 0 ,$$

where $\bar{\partial}_{\hat{A}} \varphi \in \Gamma(\Sigma, \Lambda^{0,1}(\varphi^*(T_{E/\Sigma}))$ is the $(0,1)$-component of the derivative $d_{\hat{A}} \varphi \in \Gamma(\Sigma, \Lambda^1(\varphi^*(T_{E/\Sigma}))$.

In the particular case where $K = \hat{K}$, these equations were independently found and studied in [Mu1], [CGS], and [G].

Denote by $\mathcal{M} = \mathcal{M}_{\mathfrak{p}}(\lambda, \hat{h})$ the moduli space of solutions of the equations $(V_{\mathfrak{p}})$ modulo gauge equivalence.

Let $\mathcal{A}^*$ be the open subspace of $\mathcal{A}$ consisting of irreducible pairs, i. e. of pairs with trivial stabilizer, and denote by $\mathcal{M}^*$ the moduli space of irreducible solutions; $\mathcal{M}^*$ can be regarded as a subspace of the infinite dimensional quotient $\mathcal{B}^* := {}^{\mathcal{A}^*}\!/_{\mathcal{G}}$ of irreducible pairs. The space $\mathcal{B}^*$ becomes a Banach manifold after suitable Sobolev completions. The parameters $\mathfrak{p}$ for which $\mathcal{M}^* \neq \mathcal{M}$ are called *bad parameters*, and the set of bad parameters is called the *bad locus* or the *wall*.

Our purpose is to define invariants for triples (F, α, K) by evaluating certain tautological cohomology classes on the virtual fundamental class of moduli spaces $\mathcal{M}$ corresponding to good parameters, provided these spaces are compact (or have a canonical compactification) and possess a canonical virtual fundamental class. The invariants will depend on the choice of the discrete parameter $(P_0, \mathfrak{c})$, and a chamber C, i. e. a component of the complement of the bad locus in the space of continuous parameters.

Some general ideas for the construction of Gromov-Witten type invariants associated with moduli spaces of solutions of vortex-type equations have been outlined in [CGS]; in [Mu2] such invariants are rigorously defined in the special case that F is compact Kähler and $K = \hat{K} = S^1$. Note that our program is fundamentally different: Our main construction begins with an important new idea, the parameter symmetry group $K_0 := \hat{K}/K$. This group, whose introduction was motivated by our previous work on Seiberg-Witten theory, leads to an essentially new set up and plays a crucial role in the following. Without it none of our main results could even be formulated.

Our first aim is to construct tautological cohomology classes on the infinite dimensional quotient $\mathcal{B}^*$.

Note first that any section $\varphi \in \Gamma(\Sigma, E)$ can be regarded as a $\hat{K}$-equivariant map $\hat{P} \to F$.

Therefore one gets a $\hat{K}$-equivariant evaluation map

$$\mathrm{ev} : \mathcal{A}^* \times \hat{P} \to F \ ,$$

which is obviously $\mathcal{G}$-invariant. Let $\hat{\mathcal{P}} := \mathcal{A}^* \times_{\mathcal{G}} \hat{P}$ be the universal $\hat{K}$-bundle over $\mathcal{B}^* \times \Sigma$. The evaluation map descends to a $\hat{K}$-equivariant map

$$\Phi : \mathcal{A}^* \times_{\mathcal{G}} \hat{P} \to F$$

which can be regarded as the universal section in the associated universal F-bundle $\mathcal{A}^* \times_{\mathcal{G}} E$. Let

$$\Phi^* : H^*_{\hat{K}}(F, \mathbb{Z}) \to H^*(\mathcal{B}^* \times \Sigma, \mathbb{Z}) \ .$$

be the map induced by Φ in $\hat{K}$-equivariant cohomology. Using the same idea as in Donaldson theory, we define for every $c \in H^*_{\hat{K}}(F, \mathbb{Z})$ and $\beta \in H_*(\Sigma)$ the element $\delta^c(\beta) \in H^*(\mathcal{B}^*, \mathbb{Z})$ by

$$\delta^c(\beta) := \Phi^*(c)/\beta \ .$$

Recall that one has natural morphisms

$$H^*(BK_0, \mathbb{Z}) \xrightarrow{\lambda^*} H^*(B\hat{K}, \mathbb{Z}) \xrightarrow{\eta^*} H^*_{\hat{K}}(F, \mathbb{Z})$$

which are induced by the natural maps

$$E\hat{K} \times_{\hat{K}} F \xrightarrow{\eta} B\hat{K} \xrightarrow{\lambda} BK_0 \ .$$

Let $\hat{\kappa} : \Sigma \to B\hat{K}$ be a classifying map for the bundle $\hat{P}$, and let $\kappa_0 := \lambda \circ \hat{\kappa}$ be the corresponding classifying map for P_0.

Denote by $\hat{h}^*$ the morphism $H^*_{\hat{K}}(F, \mathbb{Z}) \to H^*(\Sigma, \mathbb{Z})$ defined by $\hat{h}$.

Proposition 1.1 *The assigment $(c, \beta) \mapsto \delta^c(\beta)$ has the following properties:*
1. It is linear in both arguments.

2. For any homogeneous elements $c \in H^*_{\hat{K}}(F,\mathbb{Z})$, $\beta \in H_*(\Sigma,\mathbb{Z})$ of the same degree, one has

$$\delta^c(\beta) = \langle \hat{h}^*(c), \beta\rangle \cdot 1_{H^0(\mathcal{B}^*,\mathbb{Z})} \ .$$

3. For any homogeneous elements c, $c' \in H^*_{\hat{K}}(F,\mathbb{Z})$, one has

$$\delta^{c\cup c'}([*]) = \delta^c([*]) \cup \delta^{c'}([*]) \ .$$

4. For any homogeneous elements c, $c' \in H^*_{\hat{K}}(F,\mathbb{Z})$, $\beta \in H_1(\Sigma,\mathbb{Z})$ one has

$$\delta^{c\cup c'}(\beta) = (-1)^{\mathrm{deg}c'} \ \delta^c(\beta) \cup \delta^{c'}([*]) + \delta^c([*]) \cup \delta^{c'}(\beta) \ .$$

5. Let $(\beta_i)_{1\leq i\leq 2g(\Sigma)}$ be a basis of $H_1(\Sigma,\mathbb{Z})$. Then for any homogeneous elements c, $c' \in H^*_{\hat{K}}(F,\mathbb{Z})$ one has

$$\delta^{c\cup c'}([\Sigma]) = \delta^c([\Sigma]) \cup \delta^{c'}([*]) + \delta^c([*]) \cup \delta^{c'}([\Sigma]) - (-1)^{\mathrm{deg}c'} \sum_{i,j=1}^{2g(\Sigma)} \delta^c(\beta_i) \cup \delta^{c'}(\beta_j)(\beta_i \cdot \beta_j) \ .$$

6. For every $c_0 \in H^*(BK_0,\mathbb{Z})$ one has

$$\delta^{c\cup(\eta^*\lambda^* c_0)}(\beta) = \delta^c(\kappa_0^*(c_0) \cap \beta) \ .$$

The properties 1. – 5. follow from general properties of the slant product, whereas the last property follows from the natural identification

$$\hat{\mathcal{P}} \times_{\hat{K}} K_0 \simeq \mathcal{B}^* \times P_0 \ .$$

To every pair of *homogeneous* elements $c \in H^*_{\hat{K}}(F,\mathbb{Z})$, $\beta \in H_*(\Sigma,\mathbb{Z})$ satisfying $\mathrm{deg}c \geq \mathrm{deg}\beta$ we associate the symbol $\begin{pmatrix} c \\ \beta \end{pmatrix}$, considered as an element of degree $\mathrm{deg}c - \mathrm{deg}\beta$.

Let $\mathbb{A} = \mathbb{A}(F,\alpha,K,\mathfrak{c})$ be the graded-commutative graded $\mathbb{Z}$-algebra which is generated by the symbols $\begin{pmatrix} c \\ \beta \end{pmatrix}$, subject to the relations which correspond to the properties 1. – 6. in the proposition above. This algebra depends only on the homotopy type of our topological data.

The assignment $\begin{pmatrix} c \\ \beta \end{pmatrix} \mapsto \delta^c(\beta)$ defines a morphism of graded-commutative $\mathbb{Z}$-algebras $\delta : \mathbb{A} \to H^*(\mathcal{B}^*,\mathbb{Z})$.

Now fix a discrete parameter $(P_0,\mathfrak{c})$ and choose a representant $(\hat{P} \xrightarrow{\lambda} P_0, \hat{h})$ of $\mathfrak{c}$ as above. Choose a continuous parameter $\mathfrak{p}$ not on the wall. When the moduli space $\mathcal{M}_{\mathfrak{p}}(\lambda,\hat{h})^*$ is compact and possesses a virtual fundamental class $[\mathcal{M}_{\mathfrak{p}}(\lambda,\hat{h})^*]^{vir}$, then this class defines an invariant

$$GGW_{\mathfrak{p}}^{(P_0,\mathfrak{c})}(F,\alpha,K) : \mathbb{A}(F,\alpha,K,\mathfrak{c}) \longrightarrow \mathbb{Z} \ ,$$

given by

$$GGW_{\mathfrak{p}}^{(P_0,\mathfrak{c})}(F,\alpha,K)(a) := \langle \delta(a), [\mathcal{M}_{\mathfrak{p}}(\lambda,\hat{h})^*]^{vir}\rangle \ .$$

The 6 properties listed in the proposition above show that:

Remark 1.2 *Let $\mathfrak{G}$ be a set of homogeneous generators of $H^*_{\hat{K}}(F,\mathbb{Z})$, regarded as a graded $H^*(BK_0,\mathbb{Z})$-algebra. Then $\mathbb{A}$ is generated as a graded $\mathbb{Z}$-algebra by elements of the form $\begin{pmatrix} c \\ \beta \end{pmatrix}$ with $c \in \mathfrak{G}$, $\beta \in H_*(\Sigma,\mathbb{Z})$, and* $\mathrm{deg}c > \mathrm{deg}\beta$.

Suppose for example that we are in the simple situation where $\hat{K}$ splits as $\hat{K} = U(r) \times K_0$ and F is contractible. In this case the graded algebra

$$H^*_{\hat{K}}(F,\mathbb{Z}) = H^*(B\hat{K},\mathbb{Z}) = H^*(BU(r),\mathbb{Z}) \otimes H^*(BK_0,\mathbb{Z})$$

is generated as a $H^*(BK_0,\mathbb{Z})$-algebra by the universal Chern classes $c_i \in H^*(BU(r),\mathbb{Z})$, $1 \leq i \leq r$, and one has a natural identification

$$\mathbb{A} \simeq \mathbb{Z}[u_1,\dots,u_r,v_2,\dots,v_r] \otimes \Lambda^* \left[\bigoplus_{i=1}^{r} H_1(X,\mathbb{Z})_i\right] \ . \qquad (I)$$

Here $u_i = \begin{pmatrix} c_i \\ [*] \end{pmatrix}$, $v_i = \begin{pmatrix} c_i \\ [\Sigma] \end{pmatrix}$ have degree $2i$ and $2i-2$ respectively, whereas

$$H_1(\Sigma,\mathbb{Z})_i := \left\{ \begin{pmatrix} c_i \\ \beta \end{pmatrix} \middle| \ \beta \in H_1(\Sigma,\mathbb{Z}) \right\}$$

is a copy of $H_1(\Sigma,\mathbb{Z})$ whose elements are homogenous of degree $2i-1$.

Note also that in the case $\hat{K} = K \times K_0$, $\hat{P}$ splits as the fibre product of an K-bundle P and P_0, and the gauge group $\mathcal{G}$ can be identified with $\mathrm{Aut}(P)$.

Similarly, the universal $\hat{K}$-bundle $\hat{\mathcal{P}}$ over $\mathcal{B}^* \times \Sigma$ splits as the fibre product of the universal K-bundle $\mathcal{P} := \mathcal{A}^* \times_{\mathcal{G}} P$ with the K_0-bundle $\mathrm{pr}^*_{\Sigma}(P_0)$.

If $K = U(r)$, one can use this bundle to give a geometric interpretation of the images via δ of the classes u_i, v_i, $\begin{pmatrix} c_i \\ \beta \end{pmatrix} \in H^i_1(\Sigma,\mathbb{Z})$ defined above:

$$\delta(u_i) = c_i(\mathcal{P})/[*] \ , \ \delta(v_i) = c_i(\mathcal{P})/[\Sigma] \ , \ \delta\begin{pmatrix} c_i \\ \beta \end{pmatrix} = c_i(\mathcal{P})/\beta \ .$$

In the special case $r = 1$, one just gets

$$\mathbb{A} \simeq \mathbb{Z}[u] \otimes \Lambda^*(H_1(\Sigma,\mathbb{Z})) \ .$$

This shows that when $\hat{K} = S^1 \times K_0$ and F is contractible, the gauge theoretical Gromov-Witten invariants can be described by an inhomogeneous form $GGW_{\mathfrak{p}}^{(P_0,\mathfrak{c})}(F,\alpha,S^1) \in \Lambda^*(H^1(\Sigma,\mathbb{Z}))$ setting

$$GGW_{\mathfrak{p}}^{(P_0,\mathfrak{c})}(F,\alpha,S^1)(l) := \left\langle \delta(\sum_{j\geq 0} u^j \cup l) \ , \ [\mathcal{M}_{\mathfrak{p}}(\lambda,\hat{h})^*]^{vir} \right\rangle \ .$$

for any $l \in \Lambda^*(H_1(\Sigma, \mathbb{Z}))$. Here $(\lambda, \hat{h})$ represents $\mathfrak{c}$ and $\mathfrak{p}$ is a good continuous parameter.

1.2 Special cases

Twisted Gromov-Witten invariants: This is the special case $K = \{1\}$.

Here the gauge group $\mathcal{G}$ is trivial, the moduli space $\mathcal{M}$ is the space of J_{A_0}-holomorphic sections of the bundle E, and giving $\mathfrak{c}$ is equivalent to fixing a homotopy class h_0 of sections in $P_0 \times_{K_0} F$. The resulting invariants, when defined, should be regarded as twisted Gromov-Witten invariants, because we have replaced the space of F-valued maps on Σ in the definition of the standard Gromov-Witten invariants ([Gr] [LiT], [R]), by the space of sections in a F-bundle $P_0 \times_{K_0} F$. These invariants are associated with the almost Kähler manifold F, the K_0-action, and they depend on the discrete parameter (P_0, h_0) and the continuous parameter A_0. The invariants are defined on a graded algebra $\mathbb{A}(F, \alpha, h_0)$ obtained by applying the construction above in this special case.

Note that even in the particular case when the bundle P_0 is trivial, varying the parameter connection A_0 provides interesting deformations of the usual Gromov-Witten moduli spaces. In some situations one can prove a transversality result with respect to the parameter A_0 and then compute the standard Gromov-Witten invariants using a general parameter.

Equivariant symplectic quotients: This is the special case where the K-action on $\mu^{-1}(0)$ is free and μ is a submersion around $\mu^{-1}(0)$. In this case our data define a symplectic factorization problem, and one has a symplectic quotient $F_\mu := {\mu^{-1}(0)}/_{K}$ with an induced compatible almost complex structure and an induced almost holomorphic K_0-action α_μ. When $K_0 \neq \{1\}$, the system (F, α, K, μ) should be called *symplectic factorization problem with additional symmetry*, since the symplectic manifold F was endowed with a larger symmetry than the Hamiltonian symmetry used in performing the symplectic factorization.

For any homotopy class h_0 of sections in $P_0 \times_{K_0} F_\mu$ one can consider the twisted Gromov-Witten invariants of the pair (F_μ, α_μ) corresponding to the parameters (P_0, h_0) and A_0. One can associate to h_0 a class $\mathfrak{c}(h_0) = [\lambda, \hat{h}]$ as follows. We choose a section $\varphi_0 \in h_0$ regarded as a K_0-equivariant map $P_0 \to F_\mu$, put $\hat{P} := P_0 \times_{F_\mu} \mu^{-1}(0)$ endowed with the natural $\hat{K}$- action and the obvious morphism $\hat{P} \xrightarrow{\lambda} P_0$, and let $\hat{h}$ be the class of the section defined by the $\hat{K}$ -equivariant map $(p_0, f) \mapsto f$.

It is then an interesting and natural problem to compare the twisted Gromov-Witten invariants of the pair (F_μ, α_μ) with the gauge theoretical Gromov-Witten invariants of the initial triple (F, α, K) via the natural morphism

$$\mathbb{A}(F, \alpha, K, \mathfrak{c}(h_0)) \to \mathbb{A}(F_\mu, \alpha_\mu, h_0) \ .$$

In the non-twisted case $K_0 = \{1\}$, this problem was treated in [G], [CGS].

1.3 Main results

In section 2 we study the moduli spaces $\mathcal{M}_t(E, E_0, A_0)$ of solutions of the vortex type equations over Riemann surfaces (Σ, g), associated with the triple

$$(\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{r_0}), \alpha_{\mathrm{can}}, U(r))$$

and the moment map $\mu_t(f) = \frac{i}{2} f^* \circ f - it\mathrm{id}$, $t \in \mathbb{R}$.

In section 2.1 we introduce the gauge theoretical quot space $GQuot^E_{\mathcal{E}_0}$ of a holomorphic bundle $\mathcal{E}_0$ on a general compact complex manifold X. The space $GQuot^E_{\mathcal{E}_0}$ parametrizes the quotients of $\mathcal{E}_0$ with locally free kernels of fixed $\mathcal{C}^\infty$-type E, and can be identified with the corresponding analytical quot space when X is a curve. We prove a transversality result (Proposition 2.4) which states that, when X is a curve, $GQuot^E_{\mathcal{E}_0}$ is smooth and has the expected dimension for an open dense set of holomorphic structures $\mathcal{E}_0$ in a fixed $\mathcal{C}^\infty$-bundle E_0.

In section 2.2 we use the Kobayashi-Hitchin correspondence for the vortex equation [B] over a Riemann surface (Σ, g), to identify the irreducible part $\mathcal{M}^*_t(E, E_0, A_0)$ of $\mathcal{M}_t(E, E_0, A_0)$ with the gauge theoretical moduli space of $\frac{Vol_g(\Sigma)}{2\pi} t$ - stable pairs. The latter can be identified with a gauge theoretical quot space when t is sufficiently large (Corollary 2.8, Proposition 2.9).

In section 2.3 we prove transversality and compactness results for the moduli spaces $\mathcal{M}_t(E, E_0, A_0)$.

In section 3 we introduce formally our gauge theoretical Gromov-Witten invariants for the triple $(\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{r_0}), \alpha_{\mathrm{can}}, U(r))$ and prove an explicit formula in the abelian case $r = 1$.

We define the invariants using Brussee's formalism of virtual fundamental classes associated with Fredholm sections [Br] applied to the sections cutting out the moduli spaces $\mathcal{M}^*_t(E, E_0, A_0)$. The comparison Theorem 3.2 states that one can alternatively use the virtual fundamental class of the corresponding moduli space of stable pairs. This provides a complex geometric interpretation of our invariants. The results of section 2, and a complex geometric description of the abelian quot spaces as complete intersections in projective bundles, enables us to compute explicitely the full invariant in the abelian case $r = 1$:

Theorem: *Put $v = \chi(\mathrm{Hom}(L, E_0)) - (1 - g(\Sigma))$. The Gromov-Witten invariant $GGW^{(E_0, \mathfrak{c}_d)}_{\mathfrak{p}}(\mathrm{Hom}(\mathbb{C}, \mathbb{C}^{r_0}), \alpha_{\mathrm{can}}, U(1)) \in \Lambda^*(H^1(\Sigma, \mathbb{Z}))$ is given by the formula*

$$GGW^{(E_0, \mathfrak{c}_d)}_{\mathfrak{p}}(\mathrm{Hom}(\mathbb{C}, \mathbb{C}^{r_0}), \alpha_{\mathrm{can}}, U(1))(l) = \left\langle \sum_{i \geq \max(0, g(\Sigma) - v)}^{g(\Sigma)} \frac{(r_0 \Theta)^i}{i!} \wedge l \ , \ l_{\mathcal{O}_1} \right\rangle ,$$

for any $l \in \Lambda^*(H_1(\Sigma, \mathbb{Z}))$. Here Θ is the class in $\Lambda^2(H_1(\Sigma, \mathbb{Z}))$ given by the intersection form on Σ, and $l_{\mathcal{O}_1}$ is the generator of $\Lambda^{2g(\Sigma)}(H^1(\Sigma, \mathbb{Z}))$ defined by *the complex orientation $\mathcal{O}_1$ of $H^1(\Sigma, \mathbb{R})$.*

As an application we give in section 3.4 an explicit formula for the number of points in certain abelian quot spaces of expected dimension 0. This answers a classical problem in Algebraic Geometry. A generalisation of this result to the case $r > 1$ requires a wall-crossing formula for the non-abelian invariants.

The main result of section 4 is a natural identification of the full Seiberg-Witten invariants of ruled surfaces with certain abelian gauge theoretical Gromov-Witten invariants.

This result is a direct consequence of two important comparison theorems: The standard description of the effective divisors on a ruled surface $X := \mathbb{P}(\mathcal{V}_0) \xrightarrow{\pi} \Sigma$ over a curve, identifies the Hilbert schemes of effective divisors on X with certain quot schemes associated with symmetric powers of the 2-bundle $\mathcal{V}_0$ over Σ. In section 4.1 we show that, if one replaces the Hilbert schemes and the quot schemes by their gauge theoretical analoga *GDou*, *GQuot*, one has

Theorem: *For every $\mathcal{C}^\infty$ - line bundle L on Σ, there is a canonical isomorphism of complex spaces*

$$GDou(\pi^*(L) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}_0)}(n)) \simeq GQuot^{L^\vee}_{S^n(\mathcal{V}_0)}$$

which maps the virtual fundamental class $[GDou(\pi^*(L) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}_0)}(n))]^{vir}$ to the virtual fundamental class $[GQuot^{L^\vee}_{S^n(\mathcal{V}_0)}]^{vir}$.

On the other hand, the gauge theoretical Douady space on the left can be identified with the moduli space of monopoles on X which corresponds to the $\pi^*(L) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}_0)}(n)$-twisted canonical $Spin^c$-structure. In section 4.2, we show that this identification respects virtual fundamental classes too. Combining all these results we see that the full Seiberg-Witten invariant of the ruled surface X as defined in [OT2] can be identified with a corresponding gauge theoretical Gromov-Witten invariant for the triple $(\mathrm{Hom}(\mathbb{C}, \mathbb{C}^{n+1}), \alpha_{\mathrm{can}}, S^1)$. Using the explicit formula proven in section 3, one gets an independent check of the universal wall-crossing formula for the full Seiberg-Witten invariant in the case $b_+ = 1$.

2 Moduli spaces associated with the triple $(\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{r_0}), \alpha_{\mathrm{can}}, U(r))$

Because of the very technical compactification problem, we will not introduce our gauge theoretical invariants formally in the general framework described in section 1.2. Instead we specialize to the case $K = U(r)$, $\hat{K} = U(r) \times U(r_0)$, and $F = \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{r_0})$ endowed with the natural left $\hat{K}$-action. The K-action on F has the following family of moment maps,

$$\mu_t(f) = \frac{i}{2} f^* \circ f - it\mathrm{id},\ t \in \mathbb{R}\ ,$$

which are all $\hat{K}$-equivariant. Since F is contractible, one has only one homotopy class of sections in any fixed F-bundle.

Hence in this case our topological data reduce to the data of a differentiable Hermitian bundle E_0 of rank r_0 and a class of differentiable Hermitian bundles E of rank r. Therefore, when we fix the bundle E_0, the set of equivalence classes of pairs $(\lambda, \hat{h})$ as above can be identified with $\mathbb{Z}$ via the map $E \mapsto \deg(E)$. We will denote the class corresponding to an integer d by $\mathfrak{c}_d$.

Our moduli problem becomes now:

Let A_0 be a fixed Hermitian connection in E_0 and let $\mathcal{E}_0$ be the associated holomorphic bundle. Classify all pairs (A, φ) consisting of a Hermitian connection A in E and a (A, A_0)-holomorphic morphism $\varphi : E \to E_0$ such that the following vortex type equation is satisfied:

$$i\Lambda F_A - \frac{1}{2}\varphi^* \circ \varphi = -t\mathrm{id}_E \ .$$

Our first purpose is to show that, in a suitable chamber, the moduli space of solutions of this equation can be identified with a certain space of quotients of the holomorphic bundle $\mathcal{E}_0$. This remark will allow us later to describe the invariants explicitely in the abelian case $r = 1$.

2.1 Gauge theoretical quot spaces

Let $\mathcal{E}_0$ be a holomorphic bundle of rank r_0 on a compact connected complex manifold X of dimension n, and fix a differentiable vector bundle E of rank r on X. There is a simple gauge theoretical way to construct a complex space $GQuot^E_{\mathcal{E}_0}$ parametrizing equivalence classes of pairs $(\mathcal{E}, \varphi)$ consisting of a holomorphic bundle $\mathcal{E}$ of $\mathcal{C}^\infty$-type E and a sheaf monomorphism $\varphi : \mathcal{E} \hookrightarrow \mathcal{E}_0$; in other words, $GQuot^E_{\mathcal{E}_0}$ parametrizes the quotients of $\mathcal{E}_0$ with locally free kernel of fixed $\mathcal{C}^\infty$-type E.

Denote by E_0 the underlying differentiable bundle of $\mathcal{E}_0$ and by $\bar{\partial}_0$ the corresponding Dolbeault operator. Let $\bar{\mathcal{A}}(E)$ be the space of semiconnections in E and put $\bar{\mathcal{A}} := \bar{\mathcal{A}}(E) \times A^0\mathrm{Hom}(E, E_0)$. Let $\mathcal{G}^{\mathbb{C}} := \Gamma(X, GL(E))$ be the complex gauge group of the bundle E. A pair $(\delta, \varphi) \in \bar{\mathcal{A}}$ will be called:

- *simple* if its stabilizer with respect the natural action of $\mathcal{G}^{\mathbb{C}}$ is trivial,

- *integrable* if $\delta^2 = 0$ and $\bar{\partial}_{\delta, \bar{\partial}_0}\varphi = 0$.

We denote by $\bar{\mathcal{A}}^{simple}$ the open subspace of simple pairs, and by $\bar{\mathcal{B}}^{simple}$ its $\mathcal{G}^{\mathbb{C}}$-quotient; $\bar{\mathcal{B}}^{simple}$ becomes a possibly non-Hausdorff Banach manifold after suitable Sobolev completions.

Using similar methods as in [LO] one can construct a finite dimensional – but possibly non Hausdorff – complex subspace $\mathcal{M}^{simple}(E, \mathcal{E}_0)$ parametrizing the $\mathcal{G}^{\mathbb{C}}$-orbits of simple integrable pairs. This construction has been carried out in [Su].

It is easy to see – using Aronszajin's theorem [A]– that any pair (δ, φ), such that $\varphi_x : E_x \to E_{0,x}$ is injective in at least one point $x \in X$, is simple. Put

$$\bar{\mathcal{A}}^{inj} := \{(\delta, \varphi) \in \bar{\mathcal{A}}^{simple} \mid \exists\, x \in X \text{ with } \varphi_x \text{ injective}\} ,$$

and $\bar{\mathcal{B}}^{inj} := {\bar{\mathcal{A}}^{inj}}/{\mathcal{G}^{\mathbb{C}}}$.

Proposition 2.1 *After sufficiently high Sobolev completions, the open subspace $\bar{\mathcal{B}}^{inj}$ of $\bar{\mathcal{B}}^{simple}$ becomes an open Hausdorff submanifold of $\bar{\mathcal{B}}^{simple}$.*

Proof: Use the subscript $(\ )_k$ to denote Sobolev L^2_k-completions. The Sobolev index is chosen sufficiently large, such that L^2_k becomes an L^2_l module for any $l \geq k$. Let (δ_1, φ_1), $(\delta_1, \varphi_1) \in \bar{\mathcal{A}}^{inj}_k$ two pairs whose orbits $[\delta_1, \varphi_1]$, $[\delta_1, \varphi_1]$ cannot be separated. Then there exists a sequence of pairs $(\delta^n_1, \varphi^n_1) \in \bar{\mathcal{A}}^{inj}_k$ and a sequence of gauge transformations $g_n \in \mathcal{G}^{\mathbb{C}}_{k+1}$ such that

$$(\delta^n_1, \varphi^n_1) \to (\delta_1, \varphi_1) \ ,\ (\delta^n_1, \varphi^n_1) \cdot g_n \to (\delta_2, \varphi_2) \ .$$

Put $(\delta^n_2, \varphi^n_2) := (\delta^n_1, \varphi^n_1) \cdot g_n$. With this notation, one has

$$g_n \circ \delta^n_2 = \delta^n_1 \circ g_n \ ,\ \varphi^n_2 = \varphi^n_1 \circ g_n \ . \tag{1}$$

The first relation can be rewritten as

$$\delta^n_{12}(g_n) = 0 \ , \tag{2}$$

where δ^n_{12} is the semiconnection $\delta^n_1 \otimes (\delta^n_2)^\vee$ induced by δ^n_1, δ^n_2 in $\mathrm{End}(E)$. Put

$$f_n := \frac{1}{\parallel g_n \parallel_k} g_n \ .$$

Since the sequence (f_n) is bounded in L^2_k, we may suppose, passing to a subsequence if necessary, that (f_n) converges *weakly* in L^2_k to an element $f_{12} \in A^0(End(E))_k$. Now use (2) and the fact that δ^n_{12} converges to $\delta_{12} := \delta_1 \otimes (\delta_2)^\vee$ in L^2_k to see that $(\delta_{12}(f_n))$ converges strongly to 0 in L^2_k. This implies, by standard elliptic estimates, that (f_n) is bounded in L^2_{k+1}. Therefore, passing again to a subsequence if necessary, we may suppose that the convergence of (f_n) to f_{12} is *strong* in L^2_k; the limit must fulfill

$$\delta_{12}(f_{12}) = 0 \ , \parallel f_{12} \parallel_k = 1 \ . \tag{3}$$

The second relation in (1) implies

$$\frac{1}{\parallel g_n \parallel_k} = \frac{\parallel \varphi^n_1 \circ f_n \parallel_k}{\parallel \varphi^n_2 \parallel_k} \ .$$

The right hand term converges to $c_{12} := \frac{\|\varphi_2 \circ f\|_k}{\|\varphi_1\|_k}$. Taking $n \to \infty$ in (1), we get

$$\varphi_1 \circ f_{12} = c_{12}\ \varphi_2\ . \tag{4}$$

Similarly, we get a Sobolev endomorphism $f_{21} \in A^0(End(E))_k$ and a constant $c_{21} \in \mathbb{R}$ satisfying

$$\delta_{21}(f_{21}) = 0\ ,\ \| f_{21} \|_k = 1\ ,\ \varphi_2 \circ f_{21} = c_{21}\ \varphi_1\ . \tag{5}$$

Put $f_1 := f_{12} \circ f_{21}$, $f_2 := f_{21} \circ f_{12}$. Using (3) and (5) we find

$$\delta_{11}(f_1) = \delta_{22}(f_2) = 0\ ,\ \varphi_1 \circ f_1 = c_{12}c_{21}f_1\ ,\ \varphi_2 \circ f_2 = c_{12}c_{21}f_2\ . \tag{6}$$

Suppose that $c_{12} = 0$ or $c_{21} = 0$. Then by (4) or (5), f_{12} (respectively f_{21}) would vanish on the non-empty open set where φ_1 (respectively φ_2) is injective. But f_{12} (respectively f_{21}) is a solution of the Laplace equation $\delta_{12}^*\delta_{12}(u) = 0$ (respectively $\delta_{21}^*\delta_{21}(u) = 0$), where the Laplace operator on the left has the same symbol as the usual Dolbeault Laplace operator. By Aronszajin's identity theorem, this would imply $f_{12} = 0$ (or $f_{21} = 0$), which contradicts (3) or (5).

Therefore, we must have $c_{12} \neq 0$ and $c_{21} \neq 0$. Now formula (6) shows that $f_1 = c_{12}c_{21}\ \mathrm{id}_E$ on the open set where φ_1 is injective hence, by Aronszajin's theorem again, $f_1 = c_{12}c_{21}\ \mathrm{id}_E$ everywhere. This implies that the endomorphism $g_{12} := \frac{1}{c_{12}} f_{12}$ is a bundle isomorphism. Moreover, by (3) and (4), g_{12} satisfies

$$\delta_1 \circ g_{12} - g_{12} \circ \delta_2 = 0\ ,\ \varphi_1 \circ g_{12} = \varphi_2\ ,$$

so that the pairs (δ_i, φ_i) are gauge equivalent and $[\delta_1, \varphi_1] = [\delta_2, \varphi_2]$. ■

Note that an *integrable* pair (δ, φ) is in $\bar{\mathcal{A}}^{inj}$ if and only if φ defines an *injective sheaf homomorphism*.

Definition 2.2 *The gauge theoretical quot space $GQuot^E_{\mathcal{E}_0}$ of quotients of $\mathcal{E}_0$ with locally free kernels of $\mathcal{C}^\infty$-type E is defined as the open subspace*

$$GQuot^E_{\mathcal{E}_0} := \mathcal{M}^{simple}(E, \mathcal{E}_0) \cap \bar{\mathcal{B}}^{inj}$$

of $\mathcal{M}^{simple}(E, \mathcal{E}_0)$.

Note that $GQuot^E_{\mathcal{E}_0}$ is a Haudorff complex space of finite dimension.

For a holomorphic bundle $\mathcal{E}_0$ on a compact complex manifold, denote by $Quot^E_{\mathcal{E}_0}$ the complex analytic quot space parametrizing coherent quotients of $\mathcal{E}_0$ with locally free kernel of $\mathcal{C}^\infty$-type E [Dou].

When $\mathcal{E}_0$ is a bundle on an algebraic complex manifold endowed with an ample line bundle, denote by $Quot^P_{\mathcal{E}_0}$ the Grothendieck quot scheme over $\mathbb{C}$, parametrizing algebraic coherent quotients of $\mathcal{E}_0$ with Hilbert polynomial P. With these notations, one has:

Remark 2.3
1. Our gauge theoretical quot space $GQuot^E_{\mathcal{E}_0}$ *can be identified with the complex analytic quot space* $Quot^E_{\mathcal{E}_0}$.

This identification is an isomorphism of complex spaces, but a rigorous proof of this fact is very difficult [LL].

2. If $\mathcal{E}_0$ *is a bundle on an algebraic complex manifold endowed with an ample line bundle, then* $Quot^E_{\mathcal{E}_0}$ *can be identified with the underlying complex space of the open subscheme of* $Quot^{P_{\mathcal{E}_0}-P_E}_{\mathcal{E}_0}$ *consisting of coherent quotients of* $\mathcal{E}_0$ *with locally free kernel of* $\mathcal{C}^\infty$*-type* E *[S].*

3. When X *is a complex curve endowed with an ample line bundle of degree 1, the* $\mathcal{C}^\infty$*-type of a vector bundle on* X *is determined by its Hilbert polynomial. Furthermore, since torsion free sheaves on curves are locally free,* $GQuot^E_{\mathcal{E}_0}$ *parametrizes in this case* _all_ *quotients of* $\mathcal{E}_0$ *with Hilbert polynomial* $P_{\mathcal{E}_0} - P_E$. *In other words, on curves one has natural identifications of complex spaces*

$$GQuot^E_{\mathcal{E}_0} \simeq Quot^E_{\mathcal{E}_0} \simeq Quot^{P_{\mathcal{E}_0}-P_E}_{\mathcal{E}_0} \ .$$

Note that, on curves the first part of the integrability condition is automatically satisfied. Put $d_0 := \deg(E_0)$, $d := \deg(E)$. A simple transversality argument shows that

Proposition 2.4 *Let* X *is a curve, and put* $v(r_0, r, d, d_0) := \chi(E^\vee \otimes E_0) - \chi(E^\vee \otimes E)$. *The gauge theoretical quot space* $GQuot^E_{\mathcal{E}_0}$ *is smooth and has the expected dimension* $v(r_0, r, d, d_0)$ *for a dense open set of holomorphic structures* $\mathcal{E}_0$ *in* E_0.

Proof: We identify the space of holomorphic structures in a bundle over a curve with the affine space of semiconnection in the usual way.

Let $GQuot^E_{E_0} \subset \bar{\mathcal{B}}^{inj} \times \bar{\mathcal{A}}(E_0)$ be the parametrized gauge theoretical moduli space, i. e. the moduli space of solutions $([\delta, \varphi], \bar\partial_0)$ of the equation

$$\bar\partial_{\delta, \bar\partial_0} \varphi = 0 \ .$$

The space $GQuot^E_{\mathcal{E}_0}$ is the fibre of the projection $GQuot^E_{E_0} \to \bar{\mathcal{A}}(E_0)$ over the semiconnection $\bar\partial_0 \in \bar{\mathcal{A}}(E_0)$ corresponding to $\mathcal{E}_0$.

We denote by $f : \bar{\mathcal{A}}^{inj} \times \bar{\mathcal{A}}(E_0) \to A^{0,1}\mathrm{Hom}(E, E_0)$ the map $(\delta, \varphi, \bar\partial_0) \mapsto \bar\partial_{\delta, \bar\partial_0} \varphi$, and by $\bar f$ the induced section in the bundle

$$[\bar{\mathcal{A}}^{inj} \times \bar{\mathcal{A}}(E_0)] \times_{\mathcal{G}^{\mathbb{C}}} A^{0,1}\mathrm{Hom}(E, E_0)$$

over $\bar{\mathcal{B}}^{inj} \times \bar{\mathcal{A}}(E_0)$. We will show that (after suitable Sobolev completions) $\bar f$ is regular in every point of its vanishing locus, hence $GQuot^E_{E_0}$ becomes a smooth submanifold of $\bar{\mathcal{B}}^{inj} \times \bar{\mathcal{A}}(E_0)$. Equivalently, we will show that f is a submersion in every point of its vanishing locus $Z(f)$:

Let $\xi = (\delta, \varphi, \bar\partial_0) \in Z(f)$ and let $\beta \in A^{0,1}\mathrm{Hom}(E, E_0)$ be L^2-orthogonal to $\mathrm{im} d_\xi(f)$. One has

$$\frac{\partial}{\partial(\bar\partial_0)} f(\delta, \varphi, \bar\partial_0)(\alpha) = \alpha \circ \varphi,\ \ \alpha \in A^{0,1}\mathrm{End}(E_0)\ .$$

Note that the map $\mathrm{End}(\mathbb{C}^{r_0}) \to \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{r_0})$ given by $\Psi \mapsto \Psi \circ \Phi$ is surjective when Φ is injective. Therefore the image of $\frac{\partial}{\partial(\bar\partial_0)} f(\delta, \varphi, \bar\partial_0)$ contains the space $\Gamma_0(U, \Lambda^{0,1}\mathrm{Hom}(E, E_0))$ of $(0,1)$-forms with compact support contained in U, for every open set $U \subset \Sigma$ on which φ is a bundle monomorphism. This shows that β vanishes on U as a distribution, hence as a Sobolev section as well.

But β must also be orthogonal to the image of $\frac{\partial}{\partial(\delta,\varphi)}$, which is the first differential of the elliptic complex associated with the solution (δ, φ) and parameter $\bar\partial_0$. This means that β is a solution of an elliptic system with scalar symbol, so that another application of Aronszajin's identity theorem gives $\beta = 0$.

Since the projection $GQuot^E_{E_0} \to \bar{\mathcal{A}}(E_0)$ is proper, the set of regular values is open. By Sard's theorem – which applies since the projection $GQuot^E_{E_0} \to \bar{\mathcal{A}}(E_0)$ is a smooth Fredholm map defined on a Hausdorff manifold with countable basis [Sm] – the set of regular values is also dense. ■

A stronger form of this result refers to the embedding of quot spaces

$$GQuot^E_{\mathcal{F}_0} \hookrightarrow GQuot^E_{\mathcal{E}_0}$$

induced by a sheaf monomorphism $\psi : \mathcal{F}_0 \to \mathcal{E}_0$ with torsion quotient. The map ψ defines a bundle isomorphism over the complement of the finite set $S = \mathrm{sup}({}^{\mathcal{E}_0}/_{\mathcal{F}_0})$. Let U be a small neighbourhood of S. Any holomorphic structure $\mathfrak{E}_0$ in E_0 which coincides with $\mathcal{E}_0$ on U defines a holomorphic structure $\psi^*\mathfrak{E}_0$ on F_0 which coincides with $\mathcal{F}_0$ on U. We denote by $\bar{\mathcal{A}}_{U,\mathcal{E}_0}(E_0)$ the space of holomorphic structures in E_0 which coincide with $\mathcal{E}_0$ on U.

Using Aronszajin and Sard theorems again, one can prove the following *simultaneous regularity* result

Proposition 2.5 *Let X be a curve. The gauge theoretical quot spaces $GQuot^E_{\mathfrak{E}_0}$ and $GQuot^E_{\psi^*\mathfrak{E}_0}$ are smooth and have the expected dimensions for a dense open set of holomorphic structures $\mathfrak{E}_0 \in \bar{\mathcal{A}}_{U,\mathcal{E}_0}(E_0)$.*

2.2 Moduli spaces of vortices and stable holomorphic pairs of type $(E, \mathcal{E}_0)$

Consider again the case $K = U(r)$, $\hat K = U(r) \times U(r_0)$, $F = \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{r_0})$, and let (X, g) be a compact n-dimensional Kähler manifold.

Let E_0 be a fixed Hermitian bundle of rank r_0 on X endowed with a fixed integrable Hermitian connection A_0, and denote by $\mathcal{E}_0$ the corresponding holomorphic bundle. We also fix a Hermitian bundle E of rank r and denote by d its degree

$$\deg(E) = \langle c_1(E) \cup [\omega_g^{n-1}], [X] \rangle \ .$$

Our original gauge theoretical problem can be extended to this more general setting: For a given real number t, classify all pairs (A, φ) consisting of an *integrable* Hermitian connection in E and a (A, A_0)-holomorphic morphism $\varphi \in A^0\mathrm{Hom}(E, E_0)$ such that the following vortex type equation is satisfied:

$$i\Lambda F_A - \frac{1}{2}\varphi^* \circ \varphi = -t\mathrm{id}_E \ . \qquad (V_t^{A_0})$$

We denote by $\mathcal{M}_t(E, E_0, A_0)$ the moduli space cut out by the equation $(V_t^{A_0})$ and the integrability equations $F_A^{02} = 0$, $\bar\partial_{A,A_0}\varphi = 0$. The space $\mathcal{M}_t(E, E_0, A_0)$ is a subspace of the quotient

$$\mathcal{B}(E, E_0) = {\mathcal{A}(E) \times A^0\mathrm{Hom}(E, E_0)}\big/_{\mathrm{Aut}(E)} \ .$$

The irreducible part $\mathcal{M}_t^*(E, E_0, A_0) \subset \mathcal{M}_t(E, E_0, A_0)$ is the open subspace consisting of orbits with trivial stabilizer; this is a real analytic finite dimensional subspace of the free quotient

$$\mathcal{B}^*(E, E_0) = {[\mathcal{A}(E) \times A^0\mathrm{Hom}(E, E_0)]^*}\big/_{\mathrm{Aut}(E)} \ ,$$

which becomes a Banach manifold after suitable Sobolev completions.

The (slope) stability concept which corresponds to this gauge theoretical problem is well-known [B], [HL]:

Let τ be a real constant with $\deg(E)/\mathrm{rk}(E) > -\tau$. A pair $(\mathcal{E}, \varphi)$ consisting of a holomorphic bundle of $\mathcal{C}^\infty$-type E and a holomorphic sheaf morphism $\mathcal{E} \xrightarrow{\varphi} \mathcal{E}_0$ is τ-(semi)stable[1] if for every nontrival subsheaf $\mathcal{F} \subset \mathcal{E}$ one has

$$\begin{array}{cccl} \mu(\mathcal{E}/\mathcal{F}) & (\geq) & -\tau & \text{if } \mathrm{rk}(\mathcal{F}) < r, \\ \mu(\mathcal{F}) & (\leq) & -\tau & \text{if } \mathcal{F} \subset \ker(\varphi) \ . \end{array}$$

Definition 2.6 *The gauge theoretical moduli space $\mathcal{M}_\tau^{st}(E, \mathcal{E}_0)$ of τ-stable pairs of type $(E, \mathcal{E}_0)$ is the open subspace of $\mathcal{M}^{simple}(E, \mathcal{E}_0)$ consisting of τ-stable pairs.*

Note that it is not at all obvious that – even when (X, g) is a projective curve – the moduli space $\mathcal{M}_\tau^{st}(E, \mathcal{E}_0)$ can be identified with the underlying complex space of the corresponding quasi-projective moduli space of stable pairs of [HL]. For a proof of this fact we refer to [LL].

[1] The τ stability introduced here corresponds to the slope δ_1-stability of [HL] for $\delta_1 = \deg(\mathcal{E}) + \tau\mathrm{rk}(\mathcal{E})$

Theorem 2.7 *(Kobayashi-Hitchin correspondence for the equations $(V_t^{A_0})$) The moduli space $\mathcal{M}_t^*(E,E_0,A_0)$ of irreducible solutions of the equation $(V_t^{A_0})$ can be identified with the gauge theoretical moduli space $\mathcal{M}_\tau^{st}(E,\mathcal{E}_0)$ of τ-stable pairs where*

$$\tau = \frac{(n-1)!\mathrm{Vol}_g(X)}{2\pi} t \ .$$

Proof: The set theoretical identification follows from the work of Bradlow [B] and Lin [Lin]. The identification as real analytic spaces follows as in [LT] and [OT1].

Corollary 2.8 *Suppose $r=1$. Then one has $\mathcal{M}_t(E,E_0,A_0)=\mathcal{M}_t^*(E,E_0,A_0)$ for $t\neq -\frac{2\pi}{(n-1)!\mathrm{Vol}_g(X)}\frac{\deg(E)}{\mathrm{rk}(E)}$ and*

$$\mathcal{M}_t(E,E_0,A_0)=\left\{\begin{array}{ccc} \emptyset & \text{if} & t< -\frac{2\pi}{(n-1)!\mathrm{Vol}_g(X)}\frac{\deg(E)}{\mathrm{rk}(E)} \\ GQuot^E_{\mathcal{E}_0} & \text{if} & t> -\frac{2\pi}{(n-1)!\mathrm{Vol}_g(X)}\frac{\deg(E)}{\mathrm{rk}(E)} \end{array}\right. .$$

Proof: Indeed, integrating the equation $(V_t^{A_0})$ over X one finds

$$t> -\frac{2\pi}{(n-1)!\mathrm{Vol}_g(X)}\frac{\deg(E)}{\mathrm{rk}(E)}$$

when $(V_t^{A_0})$ has solutions with non-vanishing φ-component. Conversely, if $t> -\frac{2\pi}{(n-1)!\mathrm{Vol}_g(X)}\frac{\deg(E)}{\mathrm{rk}(E)}$, any solution (A,φ) must have $\varphi\neq 0$, so it must be irreducible. Using the theorem we get an isomorphism

$$\mathcal{M}_t(E,E_0,A_0)\simeq \mathcal{M}_\tau^{st}(E,\mathcal{E}_0)$$

where $\tau> -\frac{\deg(E)}{\mathrm{rk}(E)}$. Since any non-trivial morphism defined on a holomorphic line bundle is generically injective, we see that $\mathcal{M}_\tau^{st}(E,\mathcal{E}_0)=GQuot^E_{\mathcal{E}_0}$ if $\tau> -\frac{\deg(E)}{\mathrm{rk}(E)}$ and $r=1$. ■

In the non-abelian case, one has the following generalization of Corollary 2.8:

Proposition 2.9 *There exists a constant $c(\mathcal{E}_0,E)$ such that, for all $\tau\geq c(\mathcal{E}_0,E)$ the following holds:*
i) For every τ-semistable pair $(\mathcal{E},\varphi)$, φ is injective.
ii) Every pair $(\mathcal{E},\varphi)$ with φ injective is τ-stable.
iii) There is a natural isomorphism $\mathcal{M}_\tau^{st}(E,\mathcal{E}_0)=GQuot^E_{\mathcal{E}_0}$.
For all sufficiently large $t\in\mathbb{R}$ one has $\mathcal{M}_t(E,E_0,A_0)=\mathcal{M}_t^*(E,E_0,A_0)$ and a natural identification

$$\mathcal{M}_t(E,E_0,A_0)=GQuot^E_{\mathcal{E}_0}.$$

Proof:

i) Note first that, if $\ker(\varphi) \neq 0$, the second inequality of the stability condition for $\mathcal{F} = \ker(\varphi)$ implies

$$\deg(\mathrm{im}(\varphi)) \geq d + \tau \mathrm{rk}(\ker(\varphi)) \ .$$

But $\mathrm{im}(\varphi)$ is a non-trivial subsheaf of the fixed bundle $\mathcal{E}_0$, so one has an estimate of the form

$$\deg(\mathrm{im}(\varphi)) \leq C(\mathcal{E}_0) \ ,$$

where $C(\mathcal{E}_0) = \sup_{\mathcal{G} \subset \mathcal{E}_0} \deg(\mathcal{G})$ [Ko]. Therefore, as soon as

$$\tau > c(\mathcal{E}_0, E) := \max_{1 \leq i \leq r-1} \left[\frac{C(\mathcal{E}_0) - d}{i} \right] \ ,$$

any τ-semistable pair $(\mathcal{E}, \varphi)$ has an injective φ.

ii) Suppose now that φ is injective. The second part of the stability condition becomes empty, hence we only have to show that

$$\deg(\mathcal{F}) < d + \tau(r - \mathrm{rk}(\mathcal{F}))$$

for all subsheaves $\mathcal{F}$ of $\mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < r$. But if τ is larger than $c(\mathcal{E}_0, E)$, it follows that $d + \tau(r - s) > C(\mathcal{E}_0)$ for all $0 < s < r$. The inequality above is now automatically satisfied, since $\mathcal{F}$ can be regarded as a subsheaf of $\mathcal{E}_0$ via φ.

iii) This follows directly from i), ii) and Definition 2.6.

The last statement follows from *iii*) and the fact that any solution with generically injective φ-component is irreducible. ■

Corollary 2.8 shows that in the abelian case the moduli space $\mathcal{M}_t^*(E, E_0, A_0)$ is either empty or can be identified with a quot space.

In the non-abelian case, the space of parameters (t, g, A_0) has a chamber structure which can be very complicated. The wall in this parameter space consists of those points (t, g, A_0) for which reducible solutions (A, φ) appear in the moduli space $\mathcal{M}_t(E, E_0, A_0)$. Note that a solution (A, φ) is reducible if and only if either $\varphi = 0$, or A is reducible and φ vanishes on an A-parallel summand of E. When the parameter (t, g, A_0) crosses the wall, the corresponding moduli space changes by a "generalized flip" [Th1], [Th2], [OST].

Let $\mathbb{E} := \mathcal{A}^* \times_{\mathcal{G}} E$ be the universal complex bundle over $\mathcal{B}^* \times \Sigma$ associated with E. This bundle is the dual of the vector bundle $\mathcal{P} \times_{U(r)} \mathbb{C}^r$, where $\mathcal{P}$ is the universal K-bundle introduced in section 1.1. In order to compute the gauge theoretical Gromov-Witten invariants we will need an explicit description of the restriction of this bundle to $\mathcal{M}_t^*(E, E_0, A_0) \times \Sigma$. The following proposition provides a complex geometric interpretation of this bundle via the isomorphism given by Corollary 2.8, Proposition 2.9.

Proposition 2.10 *Suppose that t is large enough so that the Kobayashi-Hitchin correspondence defines an isomorphism* $\mathcal{M}_t^*(E, E_0, A_0) \simeq GQuot^E_{\mathcal{E}_0}$. *Via this isomorphism the restriction of the universal bundle* $\mathbb{E}$ *to* $\mathcal{M}_t^*(E, E_0, A_0) \times \Sigma$ *can be identified with the kernel of the universal quotient* $p_\Sigma^*(\mathcal{E}_0) \to \mathcal{Q}$ *over* $GQuot^E_{\mathcal{E}_0} \times \Sigma$.

2.3 Transversality and compactness for moduli spaces of vortices

We first prove a simple regularity result for moduli spaces of vortices over curves.

Proposition 2.11 *Let X be a curve.*

i) The moduli space $\mathcal{M}_t^*(E, E_0, A_0)$ *is smooth of expected dimension in every point* $[A, \varphi]$ *with* φ *generically surjective.*

ii) There is a dense second category set $\mathcal{C} \subset \mathcal{A}(E_0)$ *such that, for every* $A_0 \in \mathcal{C}$ *and every* $t \in \mathbb{R}$, *the open part* $\mathcal{M}_t(E, E_0, A_0)^{inj} \subset \mathcal{M}_t^*(E, E_0, A_0)$, *consisting of classes of pairs with generically injective* φ*-component, is smooth of expected dimension.*

Proof:

i) Since the Kobayashi-Hitchin correspondence is an isomorphism of real analytic spaces, it suffices to study the regularity of the moduli space $\mathcal{M}_\tau^{st}(E, \mathcal{E}_0)$ in the point $[\bar\partial_A, \varphi] \in \mathcal{M}_\tau^{st}(E, \mathcal{E}_0)$ which corresponds to $[A, \varphi]$. The first differential

$$D^1_{\bar\partial_A, \varphi} : A^1\mathrm{End}(E) \times A^0\mathrm{Hom}(E, E_0) \to A^{0,1}\mathrm{Hom}(E, E_0)$$

in the elliptic complex associated with the τ-stable pair $(\bar\partial_A, \varphi)$ is given by

$$D^1_{\bar\partial_A, \varphi}(\alpha, \phi) = \bar\partial_{A, A_0}\phi - \varphi \circ \alpha \ .$$

It suffices to see that, after suitable Sobolev completions, the first order differential operator $D^1_{\bar\partial_A, \varphi}$ is surjective. Let $\beta \in A^{0,1}\mathrm{Hom}(E, E_0)$ be L^2-orthogonal to $\mathrm{im}(D^1_{\bar\partial_A, \varphi})$. Note that the linear map $\mathrm{End}(\mathbb{C}^r) \to \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{r_0})$ given by $\Psi \mapsto \Phi \circ \Psi$ is surjective when Φ is surjective. Therefore, as in the proof of Proposition 2.4, we find that β vanishes as distribution, hence as a Sobolev section as well, on the open set where φ is surjective. But since β solves an elliptic second order system with scalar symbol, it follows that $\beta = 0$.

ii) Note that $\mathcal{M}_t(E, E_0, A_0)^{inj}$ can be identified via the Kobayashi-Hitchin correpondence with an open subspace of $GQuot^E_{\mathcal{E}_0}$. Therefore the statement follows from Proposition 2.4. ■

Theorem 2.12 *Let* (X, g) *be a compact Kähler manifold of dimension n, E and* E_0 *Hermitian bundles on X of ranks r and* r_0 *respectively. Suppose that* *either $n = 1$ or $r = 1$. Then the moduli spaces $\mathcal{M}_t(E, E_0, A_0)$ are compact for every $t \in \mathbb{R}$ and for every integrable Hermitian connection $A_0 \in \mathcal{A}(E_0)$.*

In particular, the moduli space $GQuot^E_{\mathcal{E}_0}$ is compact if X is a curve or $\mathrm{rk}(E) = 1$.

Proof: The Hermite-Einstein type equation

$$i\Lambda F_A - \frac{1}{2}\varphi^* \circ \varphi = -t\mathrm{id}_E$$

implies

$$\mu(E) - \frac{(n-1)!}{4\pi r} \parallel \varphi \parallel^2 = -\frac{(n-1)!Vol_g(X)}{2\pi} t \ .$$

The Weitzenböck formula for holomorphic sections in the holomorphic Hermitian bundle $E^\vee \otimes E_0$ with Chern connection $B := A^\vee \otimes A_0$ yields

$$i\Lambda\bar{\partial}\partial(\varphi,\varphi) = (i\Lambda F_B(\varphi),\varphi) - |\partial_B\varphi|^2 \leq ((i\Lambda F_{A_0}) \circ \varphi - \varphi \circ (i\Lambda F_A), \varphi) =$$
$$= (i\Lambda F_{A_0}(\varphi),\varphi) - (i\Lambda F_A, \varphi^* \circ \varphi) =$$
$$(i\Lambda F_{A_0}(\varphi),\varphi) + t|\varphi|^2 - \frac{1}{2}|\varphi^* \circ \varphi|^2 \ .$$

Notice that $|\varphi^* \circ \varphi|^2 \geq \frac{1}{r}|\varphi|^4$. Let x_0 be a point where the supremum of the function $|\varphi|^2$ is attained, and let $\lambda_M^{A_0}$ be the supremum of the highest eigenvalues of the Hermitian bundle endomorphism $i\Lambda F_{A_0}$. By the maximum principle we get

$$0 \leq [i\Lambda\bar{\partial}\partial|\varphi|^2]_{x_0} \leq (\lambda_M^{A_0} + t)|\varphi(x_0)|^2 - \frac{1}{2r}|\varphi(x_0)|^4 \ .$$

Therefore we have the following apriori $\mathcal{C}^0$-bound for the second component of a solution of $(V_t^{A_0})$:

$$\sup_X |\varphi|^2 \leq \max(0, 2r(\lambda_M^{A_0} + t)) \ .$$

Now, if $r = 1$, one can bring A in Coulomb gauge with respect to a fixed connection A^0 in E by a gauge transformation g_A. Moreover, one can choose g_A so that the projection of $g_A(A) - A^0$ on the kernel of the operator

$$d^+ + d^* : iA^1(X) \longrightarrow i[(A^{0,2}(X) + A^{2,0}(X) + A^{0,0}(X)) \cap A^2(X)]$$

(which coincides with the harmonic space $i\mathbb{H}^1(X)$ *in the Kählerian case*) belongs to a fixed fundamental domain D of the lattice $iH^1(X,\mathbb{Z})$. Now standard bootstrapping arguments apply as in the case of the abelian monopole equations [KM].

If X is a curve, then the contraction operator Λ is an isomorphism, so one gets an apriori L^∞-bound for the curvature of the connection component. The result follows now from Uhlenbeck's compactness theorems for connections with L^p-bound on the curvature [U]. ■

Corollary 2.13 *Let X be a projective manifold endowed with an ample line bundle H, and let P_L be the Hilbert polynomial of a locally free sheaf L of rank 1 with respect to H. Then the analytic quot space $Quot_{\mathcal{E}_0}^{P_{\mathcal{E}_0}-P_L}$ is compact.*

Proof: Indeed, by Remark 2.3, the gauge theoretical quot space $GQuot^L_{\mathcal{E}_0}$ is an open supspace of the underlying analytic space of $Quot_{\mathcal{E}_0}^{P_{\mathcal{E}_0}-P_L}$. But any torsion free sheaf on X with Hilbert polynomial P_L is a line bundle of $\mathcal{C}^\infty$-type L, so that the open embedding $GQuot^L_{\mathcal{E}_0} \hookrightarrow Quot_{\mathcal{E}_0}^{P_{\mathcal{E}_0}-P_L}$ is surjective. ■

3 The definition of the invariants and an explicit formula in the abelian case

3.1 Virtual fundamental classes for Fredholm sections and the definition of the invariants

We explain briefly – following [Br] – the definition and the basic properties of virtual fundamental classes of vanishing loci of Fredholm sections. For simplicty we discuss only the compact case.

Let E be a Banach bundle over the Banach manifold B, and let σ be a Fredholm section of index d in E with compact vanishing locus $Z(\sigma)$. Fix a trivialization θ of the real line bundle $\det(\mathrm{Index}(\mathrm{D}\sigma))$ in a neighbourhood of $Z(\sigma)$. One can associate with these data a Čech homology class $[Z(\sigma)]^{vir}_\theta \in \check{H}_d(Z(\sigma),\mathbb{Z})$ in the following way:

Notice first that one can choose a finite rank subbundle $F \subset E|_U$ of the restriction of E to a sufficiently small neighbourhood U of $Z(\sigma)$ in B such that $D_x\sigma + F_x = E_x$ for every $x \in Z(\sigma)$. This shows that the induced section $\bar\sigma$ in the quotient bundle $E|_U/F$ is regular in the points of $Z(\sigma)$, hence it is also regular on a neighbourhood $V \subset U$ of $Z(\sigma)$ in B. Put $M := Z(\bar\sigma|_V)$. Then M is a smooth closed submanifold of V of dimension $m := d + \mathrm{rk}F$.

Denote by $\mathrm{or}(M)$ the orientation sheaf of M, and let $[M] \in \check{H}^{\mathrm{cl}}_m(M, \mathrm{or}(M))$ be the fundamental class of M in Čech homology with closed supports.

Notice that the restriction $\sigma|_M$ takes values in the subbundle $F|_M$ of $E|_M$, and that the real line bundle $\Lambda^{\max}(T_M)^\vee \otimes \Lambda^{\max}(F|_M)$ can be identified with $\det(\mathrm{Index}(\mathrm{D}\sigma))|_M$; therefore it comes with a natural trivialization induced by $\theta|_M$. Let $e(F|_M, \sigma|_M) \in \check{H}^{\mathrm{rk}F}(M, M \setminus Z(s), \mathrm{or}(F|_M))$ be the localized Euler class of $(F|_M, \sigma|_M)$.

Using the trivialization of $\Lambda^{\max}(T_M)^\vee \otimes \Lambda^{\max}(F|_M)$, the virtual fundamental class is defined as

$$[Z(\sigma)]^{vir}_\theta := e(F|_M, \sigma|_M) \cap [M] \in \check{H}_d(Z(\sigma), \mathbb{Z}).$$

In this definition the cap product pairs Čech cohomology and Čech homology with closed supports ([Br], Lemma 13).

The homology class obtained in this way is well-defined, i. e. it does not depend on the choice of the subbundle F and the submanifold M used in the definition. Note also, that when $Z(\sigma)$ is locally contractible, e. g. when $Z(\sigma)$ is locally homeomorphic to a real analytic set, then its Čech homology can be identified with its singular homology. In this case one gets a well defined virtual fundamental class

$$[Z(\sigma)]^{vir}_{\theta} \in H_d(Z(\sigma), \mathbb{Z}) \ .$$

Remark 3.1 *When the section σ is regular in every point of its vanishing locus, then $Z(\sigma)$ is either empty or a smooth manifold of dimension d which comes with a natural orientation induced by θ. In this case, $[Z(\sigma)]^{vir}_{\theta}$ coincides with the usual fundamental class $[Z(\sigma)]_{\theta}$ of this oriented manifold.*

We will omitt the index θ when there is a natural choice of a trivialization, e. g. when B is a complex Banach manifold, E is a holomorphic Banach bundle, and σ is holomorphic.

The virtual fundamental class has the following two fundamental properties which will play an important role in this section ([Br], Proposition 14):

Associativity Property: *Let E be a Banach vector bundle on a Banach manifold B, and let σ be a Fredholm section in E with compact vanishing locus $Z(\sigma)$. Let*

$$0 \to E' \to E \to E'' \to 0 \qquad (e)$$

be an exact sequence of bundles and suppose that the section $\sigma'' \in \Gamma(B, E'')$ induced by σ is regular [2] in the points of its vanishing locus $B'' := Z(\sigma'')$. Let $\sigma' \in \Gamma(B'', E'|_{B''})$ be the section in $E'|_{B''}$ defined by σ. The inclusion $i : B'' \subset B$ induces:

1. An homeomorphism $Z(\sigma') \simeq Z(\sigma)$ which is an isomorphism of real (complex) analytic spaces if B, E, E', E'', σ and the morphisms in the exact sequence (e) are real (complex) analytic.

2. An identification of virtual fundamental classes $[Z(\sigma')]^{vir} \simeq [Z(\sigma)]^{vir}$.

Note that one has a well defined map $\check{H}_*(Z(\sigma)) \to H_*(B)$ induced by the composition $Z(\sigma) \hookrightarrow M \to B$ and the identification $\check{H}_*(M) = H_*(M)$. With this remark, we can state

Homotopy Invariance: *Let $(\sigma_t)_{t\in[0,1]}$ be a smooth 1-parameter family of sections in E such that the vanishing locus of the induced section in the bundle $\mathrm{pr}_B^*(E)$ over $B \times [0, 1]$ is compact. Then the images of $[Z(\sigma_0)]^{vir}$ and $[Z(\sigma_1)]^{vir}$ in $H_d(B)$ coincide.*

[2] A section in a Banach bundle with fibre Λ is regular in a point x of its vanishing locus if the associated Λ-valued map with respect to a trivialization around x is a submersion in x. In the non-Fredholm case, this condition is stronger than the surjectivity of the intrinsic derivative, but it is equivalent to this condition if the base manifold is a Hilbert manifold.

Now we can introduce our gauge theoretical Gromov-Witten invariants for the triple $(\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^{r_0}),\alpha_{\mathrm{can}},U(r))$. Let Σ be a compact oriented 2-manifold, and let E, E_0 be Hermitian bundles on Σ of ranks r, r_0 and degrees d, d_0 respectively. Choose a continuous parameter $\mathfrak{p}=(t,g,A_0)$ as in section 2.

The moduli space $\mathcal{M}_t^*(E,E_0,A_0)$ can be regarded as the vanishing locus of a Fredholm section $v_t^{A_0}$ in the vector bundle

$$\mathcal{A}^*\times_{\mathcal{G}}[A^{0,1}\mathrm{Hom}(E,E_0)\oplus A^0\mathrm{Herm}(E)]$$

over $\mathcal{B}^*$. The section $v_t^{A_0}$ is defined by the $\mathcal{G}$-equivariant map

$$(A,\varphi)\mapsto(\bar\partial_{A,A_0}\varphi\ ,\ i\Lambda F_A-\frac{1}{2}\varphi^*\circ\varphi+t\mathrm{id}_E)\ .$$

Moreover, this moduli space is compact for good parameters (t,g,A_0) by Theorem 2.12. We trivialize the determinant line bundle $\det(\mathrm{Index}Dv_t^{A_0})$ in the following way:

The kernel (cokernel) of the intrinsic derivative of $Dv_t^{A_0}$ in a solution $[A,\varphi]$ can be identified with the harmonic space $\mathbb{H}^1$ ($\mathbb{H}^2$) of the elliptic deformation complex associated with this solution. But the Kobayashi-Hitchin correspondence identifies these harmonic spaces with the corresponding harmonic spaces of the elliptic complex associated with the simple holomorphic pair $(\bar\partial_A,\varphi)$. We orient the kernel and the cokernel of the intrinsic derivative using the complex orientations of the latter harmonic spaces.

Following the general formalism described in section 1.1, we put

$$GGW_{\mathfrak{p}}^{(E_0,\mathfrak{c}_d)}(\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^{r_0}),\alpha_{\mathrm{can}},U(r))(a):=\langle\delta(a),[\mathcal{M}_t^*(E,E_0,A_0)]^{vir}\rangle$$

for any good continuous parameter $\mathfrak{p}=(t,g,A_0)$ and any element

$$a\in\mathbb{A}(F,\alpha,K,\mathfrak{c})=\mathbb{Z}[u_1,\dots,u_r,v_2,\dots,v_r]\otimes\Lambda^*\left[\bigoplus_{i=1}^r H_1(\Sigma,\mathbb{Z})_i\right]\ .$$

We have seen that, on curves, the moduli space $\mathcal{M}^{simple}(E,\mathcal{E}_0)$ of simple pairs of type $(E,\mathcal{E}_0)$ can be regarded as the vanishing locus of a Fredholm section $\bar v^{\mathcal{E}_0}$ of complex index $\chi(\mathrm{Hom}(E,E_0))-\chi(\mathrm{End}(E))$ in the Banach bundle

$$\bar{\mathcal{A}}^{simple}\times_{\mathcal{G}^{\mathbb{C}}}A^{0,1}\mathrm{Hom}(E,E_0)$$

over the non-Hausdorff Banach manifold $\bar{\mathcal{B}}^{simple}$. However, since $\mathcal{M}^{simple}(E,\mathcal{E}_0)$ is in general non-Hausdorff, one cannot endow it with a virtual fundamental class. On the other hand, by Theorem 2.7, the open subspace $\mathcal{M}_\tau^{stable}(E,\mathcal{E}_0)$ of τ-stable pairs of type $(E,\mathcal{E}_0)$ is always Hausdorff, and it is also compact if the corresponding parameter $\mathfrak{p}=(\frac{2\pi}{Vol_g(\Sigma)}\tau,g,A_0)$ is good. The following proposition shows that for good parameters $\mathfrak{p}$, $\mathcal{M}_\tau^{stable}(E,\mathcal{E}_0)$ can be endowed with a virtual fundamental class, and that the isomorphism given by Theorem 2.7 maps $[\mathcal{M}_t^*(E,E_0,A_0)]^{vir}$ onto $[\mathcal{M}_\tau^{stable}(E,\mathcal{E}_0)]^{vir}$. Therefore one can use the complex geometric virtual fundamental classes $[\mathcal{M}_\tau^{stable}(E,\mathcal{E}_0)]^{vir}$ to compute the gauge theoretical Gromow-Witten invariants.

This proposition is a particular case of a more general principle which states that *the Kobayashi-Hitchin type correspondence associated to a complex geometric moduli problem of "Fredholm type" respects virtual fundamental classes.* The proof below can be adapted to the general case.

Theorem 3.2 *Let $\mathfrak{p}=(t,g,A_0)$ be a good parameter, let $\mathcal{E}_0$ be the holomorphic bundle defined by $\bar\partial_{A_0}$ in E_0, and put $\tau:=\frac{Vol_g(\Sigma)}{2\pi}t$. Then*
i) $\mathcal{M}_\tau^{stable}(E,\mathcal{E}_0)$ is compact and has a Hausdorff neighbourhood in $\bar{\mathcal{B}}^{simple}$; it comes with a virtual fundamental class induced by the restriction of $\bar v^{\mathcal{E}_0}$ to such a neighbourhood.
ii) The isomorphism given by the Kobayashi-Hitchin correspondence maps the virtual fundamental class $[\mathcal{M}_t^*(E,E_0,A_0)]^{vir}$ onto $[GQuot^E_{\mathcal{E}_0}]^{vir}$.

Proof:

We apply the Associativity Property of the virtual fundamantal classes to the following exact sequence of Banach bundles over $\mathcal{B}^*$:

$$0\to\mathcal{A}^*\times_{\mathcal{G}}A^{0,1}\mathrm{Hom}(E,E_0)\to\mathcal{A}^*\times_{\mathcal{G}}[A^{0,1}\mathrm{Hom}(E,E_0)\oplus A^0\mathrm{Herm}(E)]\to$$
$$\to\mathcal{A}^*\times_{\mathcal{G}}A^0\mathrm{Herm}(E)\to 0\ .$$

The section $(v_t^{A_0})''$ in $\mathcal{A}^*\times_{\mathcal{G}}A^0\mathrm{Herm}(E)$ induced by $v_t^{A_0}$ is given by the $\mathcal{G}$-equivariant map

$$(A,\varphi)\mapsto i\Lambda F_A-\frac{1}{2}\varphi^*\circ\varphi+t\mathrm{id}_E\ .$$

Using the fact that this map comes from a formal moment map, one can prove ([LT] ch 4, [OT1]) that:
1. $(v_t^{A_0})''$ is regular around $Z(v_t^{A_0})$,
2. the natural map $\rho:Z((v_t^{A_0})'')\to\bar{\mathcal{B}}^{simple}$ given by $[A,\varphi]\mapsto[\bar\partial_A,\varphi]$ induces a bijection

$$Z(v_t^{A_0})=\mathcal{M}_t^*(E,E_0,A_0)\overset{\simeq}{\to}\mathcal{M}_\tau^{stable}(E,\mathcal{E}_0)\ ,$$

and is étale around $Z(v_t^{A_0})$. Since, by Theorem 2.12, $\mathcal{M}_t^*(E,E_0,A_0)$ is compact for a good parameter $\mathfrak{p}=(t,g,A_0)$, it follows that $\mathcal{M}_\tau^{stable}(E,\mathcal{E}_0)$ is compact, and that ρ maps a sufficiently small neighbourhood $\mathcal{V}$ of $\mathcal{M}_t^*(E,E_0,A_0)$ in $Z((v_t^{A_0})'')$ isomorphically onto a neighborhood $\mathcal{U}$ of $\mathcal{M}_\tau^{stable}(E,\mathcal{E}_0)$ in $\bar{\mathcal{B}}^{simple}$. This neighbourhood must be Hausdorff, because $\mathcal{B}^*$ is Hausdorff.

Via the natural identification

$$\rho^*(\bar{\mathcal{A}}^{simple}\times_{\mathcal{G}^{\mathbb{C}}}A^{0,1}\mathrm{Hom}(E,E_0))=[\mathcal{A}^*\times_{\mathcal{G}}A^{0,1}\mathrm{Hom}(E,E_0)]_{Z((v_t^{A_0})'')}\ ,$$

the section $(v_t^{A_0})'$ induced by $v_t^{A_0}$ in $[\mathcal{A}^* \times_{\mathcal{G}} A^{0,1}\mathrm{Hom}(E,E_0)]_{Z((v_t^{A_0})'')}$ corresponds via this identification to the section $\bar{v}^{\mathcal{E}_0}$. This shows that:

1. The vanishing locus of $\bar{v}^{\mathcal{E}_0}|_{\mathcal{U}}$ is exactly $\mathcal{M}_\tau^{stable}(E,\mathcal{E}_0)$, so we can define $[\mathcal{M}_\tau^{stable}(E,\mathcal{E}_0)]^{vir}$ as the virtual fundamental class defined by $\bar{v}^{\mathcal{E}_0}|_{\mathcal{U}}$.

2. The map ρ maps $[Z((v_t^{A_0})'|_{\mathcal{V}})]^{vir}$ onto $[\mathcal{M}_\tau^{stable}(E,\mathcal{E}_0)]^{vir}$.

The statement follows now directly from the Associativity Property. ■

In the non-abelian case one has a very complicated chamber structure and the invariants jump when the continuous parameter $\mathfrak{p}$ crosses the wall. Proving a wall crossing formula for these jumps is an important but very difficult problem.

In the abelian case the situation is much simpler: Recall that in this case the invariants are determined by an inhomogenous form

$$GGW_{\mathfrak{p}}^{(E_0,\mathfrak{c}_d)}(\mathrm{Hom}(\mathbb{C},\mathbb{C}^{r_0}),\alpha_{\mathrm{can}},S^1) \in \Lambda^*(H^1(\Sigma,\mathbb{Z})) \ .$$

Corollary 2.8 yields the following:

Proposition 3.3 *For any fixed topological data (E_0,d), there are exactly <u>two chambers</u> in the space of parameters. The "interesting chamber" C^+ – in which the moduli space can be non-empty – is defined by the inequality*

$$t > -\frac{2\pi}{\mathrm{Vol}_g(X)}\deg(E) \ .$$

For any parameter $\mathfrak{p}=(t,g,A_0)$ in this chamber, the corresponding gauge theoretical Gromov-Witten moduli space coincides with the gauge theoretical quot space $GQuot_{\mathcal{E}_0}^L$, where L is a line bundle of degree d on Σ and $\mathcal{E}_0$ is the holomorphic structure in E_0 associated with the connection A_0.

The wall is defined by the equation $t=-\frac{2\pi}{\mathrm{Vol}_g(X)}\deg(E)$, which does not involve the third parameter A_0, hence one cannot cross the wall by varying only A_0.

In order to compute the invariants in the "interesting chamber" C^+, we need an explicit description of the abelian quot spaces. We will see that these quot spaces can be described as subspaces of a projective bundle over a component of $\mathrm{Pic}(\Sigma)$, defined as the intersection of finitely many divisors representing the Chern class of the relative hyperplane line bundle.

3.2 Quot spaces in the abelian case

We begin with the following known results [Gh]:

Let $\mathcal{F}_0$ be vector bundle on a curve Σ, and let m be a sufficiently negative integer such that $H^1(\mathcal{M}^\vee \otimes \mathcal{F}_0)=0$ for all $\mathcal{M}\in \mathrm{Pic}^m(\Sigma)$.

Denote by $\mathfrak{P}$ a Poincaré line bundle on $\mathrm{Pic}^m(\Sigma)\times\Sigma$, by $\mathcal{V}$ the locally free sheaf $\left[[\mathrm{pr}_{\mathrm{Pic}^m(\Sigma)}]_*\mathrm{Hom}(\mathfrak{P},\mathrm{pr}_\Sigma^*(\mathcal{F}_0))\right]^\vee$ on $\mathrm{Pic}^m(\Sigma)$, and by $\mathbb{P}(\mathcal{V})$ its projectivization in Grothendieck's sense. Applying the projection formula to the projective morphism $p:\mathbb{P}(\mathcal{V})\times\Sigma\to\mathrm{Pic}^m(\Sigma)\times\Sigma$, we get

$$p_*(\mathrm{Hom}(p^*(\mathfrak{P})(-1),\mathrm{pr}_\Sigma^*(\mathcal{F}_0))=$$

$$=\mathrm{Hom}(\mathfrak{P},\mathrm{pr}_\Sigma^*(\mathcal{F}_0))\otimes[\mathrm{pr}_{\mathrm{Pic}^m(\Sigma)}]^*\left[[\mathrm{pr}_{\mathrm{Pic}^m(\Sigma)}]_*[\mathrm{Hom}(\mathfrak{P},\mathrm{pr}_\Sigma^*(\mathcal{F}_0)]\right]^\vee\ ,$$

hence on $\mathbb{P}(\mathcal{V})\times\Sigma$ there is a canonical monomorphism

$$\mathrm{pr}^*_{\mathrm{Pic}^m(\Sigma)\times\Sigma}(\mathfrak{P})(-1)\xrightarrow{\nu}\mathrm{pr}_\Sigma^*(\mathcal{F}_0)\ .$$

Let M be a differentiable line bundle of degree m.

Proposition 3.4 *Choose m sufficiently negative such that $H^1(\mathcal{M}^\vee\otimes\mathcal{F}_0)=0$ for all $\mathcal{M}\in\mathrm{Pic}^m(\Sigma)$. Then the quotient* $\mathrm{pr}_\Sigma^*(\mathcal{F}_0)/_{\mathrm{im}(\nu)}$ *is flat over $\mathbb{P}(\mathcal{V})$, and the associated morphism $\mathbb{P}(\mathcal{V})\to Quot^M_{\mathcal{F}_0}$ is an isomorphism.*

An epimorphism $\mathcal{F}_0\xrightarrow{\alpha}\mathcal{O}_x$, where $x\in\Sigma$ is a simple point, induces an epimorphism $\mathrm{pr}_\Sigma^*(\mathcal{F}_0)\xrightarrow{\tilde\alpha}\mathcal{O}_{\mathbb{P}(\mathcal{V})\times\{x\}}$ on $\mathbb{P}(\mathcal{V})\times\Sigma$. The composition $\tilde\alpha\circ\nu$ can be regarded, by adjunction, as a morphism

$$\mathrm{pr}^*_{\mathrm{Pic}^m(\Sigma)\times\{x\}}(\mathfrak{P}|_{\mathrm{Pic}^m(\Sigma)\times\{x\}})(-1)\to\mathcal{O}_{\mathbb{P}(\mathcal{V})\times\{x\}}\ ,$$

hence as a section σ_α in the line bundle $\mathrm{pr}^*_{\mathrm{Pic}^m(\Sigma)}(\mathfrak{P}_x)^\vee(1)$ over the projective bundle $\mathbb{P}(\mathcal{V})\simeq\mathbb{P}(\mathcal{V})\times\{x\}$. Here $\mathfrak{P}_x$ is the line bundle on $\mathrm{Pic}^m(\Sigma)$ corresponding to $\mathfrak{P}|_{\mathrm{Pic}^m(\Sigma)\times\{x\}}$.

Proposition 3.5 *Let Z_0 be a finite set of simple points in Σ, and consider for each $x\in Z_0$ an epimorphism $\alpha_x:\mathcal{F}_0\to\mathcal{O}_x$. Put $\alpha=\bigoplus_{x\in Z_0}\alpha_x:\mathcal{F}_0\to\bigoplus_{x\in Z_0}\mathcal{O}_x$, $Z:=\bigcap_{x\in Z_0}Z(\sigma_{\alpha_x})$, and let p_Σ be the projection $Z\times\Sigma\to\Sigma$. Then , for all m sufficiently negative, the quotient* $p_\Sigma^*(\ker\alpha)/_{\mathrm{im}(\nu|_{Z\times\Sigma})}$ *is flat over Z, and the induced morphism $Z\to Quot^M_{\ker\alpha}$ is an isomorphism.*

Let now L be differentiable line bundle of degree d and $\mathcal{E}_0$ a holomorphic bundle of rank r_0 and degree d_0 on Σ.

Let H be an ample line bundle on Σ and $n\in\mathbb{N}$ sufficiently large such that $\mathcal{E}_0^\vee\otimes H^{\otimes n}$ is globally generated. Then the cokernel of a generic morphism

$$\mathcal{O}_\Sigma^{\oplus r_0}\to\mathcal{E}_0^\vee\otimes H^{\otimes n}$$

has the form $\bigoplus_{i=1}^{k} \mathcal{O}_{x_i}$ with $k := -d_0 + r_0\mathrm{deg}(H)$ distinct *simple* points $x_i \in \Sigma$. Dualizing, one gets an exact sequence

$$0 \to \mathcal{E}_0 \otimes H^{\otimes -n} \to \mathcal{O}_\Sigma^{\oplus r_0} \xrightarrow{\rho} \bigoplus_{i=1}^{k} \mathcal{O}_{x_i} \to 0 \ . \qquad (*)$$

The i-th component $\rho_i : \mathcal{O}_\Sigma^{\oplus r_0} \to \mathcal{O}_{x_i}$ of ρ is defined by a non-trivial linear form $\rho^i : \mathbb{C}^{r_0} \to \mathbb{C}$.

Note also that one has a natural isomorphism

$$Quot_{\mathcal{E}_0}^{L} \simeq Quot_{\mathcal{E}_0 \otimes H^{\otimes -n}}^{L \otimes H^{\otimes -n}}$$

which identifies the corresponding virtual fundamental classes. Thus we can replace L by $L' := L \otimes H^{\otimes -n}$ and $\mathcal{E}_0$ by $\mathcal{E}_0' := \mathcal{E}_0 \otimes H^{\otimes -n}$.

The exact sequence $(*)$ shows now that, at least as a set, $Quot_{\mathcal{E}_0'}^{L'}$ can be identified with the subspace of $Quot_{\mathcal{O}_\Sigma^{\oplus r_0}}^{L'}$ consisting of quotients

$$0 \to \mathcal{L}' \xrightarrow{\varphi} \mathcal{O}_\Sigma^{\oplus r_0} \to {\mathcal{O}_\Sigma^{\oplus r_0}}\big/{\varphi(\mathcal{L}')} \to 0$$

of the free sheaf $\mathcal{O}_\Sigma^{\oplus r_0}$ with $\rho^i(\varphi(x_i)) = 0$. By Proposition 3.5 applied to $\mathcal{F}_0 = \mathcal{O}_\Sigma^{\oplus r_0}$ and $M = L'$, this identification is also an isomorphism of complex spaces, and we have

Corollary 3.6 *Let $\mathfrak{P}$ be a Poincaré line bundle over $\mathrm{Pic}^{d'}(\Sigma) \times \Sigma$, with $d' := \mathrm{deg}(L') = d - n\mathrm{deg}(H)$. If n is sufficiently large, then $Quot_{\mathcal{E}_0'}^{L'}$ can be identified with the analytic subspace Z of the projective bundle*

$$P := \mathbb{P}([(\mathrm{pr}_{\mathrm{Pic}^{d'}(\Sigma)})_*(\mathfrak{P}^\vee)^{\oplus r_0})]^\vee)$$

over $\mathrm{Pic}^{d'}(\Sigma)$ which is cut out by the sections $\sigma_{\rho_i} \in \Gamma(P, \mathrm{pr}_{\mathrm{Pic}^{d'}(\Sigma)}^*(\mathfrak{P}_{x_i})^\vee(1))$. The kernel of the universal quotient over $Z \times \Sigma$ is the restriction of the line bundle $\mathrm{pr}_{\mathrm{Pic}^{d'}(\Sigma)\times\Sigma}^*(\mathfrak{P})(-1)$ to $Z \times \Sigma$.

Consider the embedding $j : Quot_{\mathcal{E}_0}^{L} \hookrightarrow P$ given by the identification $Quot_{\mathcal{E}_0}^{L} = Quot_{\mathcal{E}_0'}^{L'}$ and Corollary 3.5. Denote by π the projection of the projective bundle P onto its basis $\mathrm{Pic}^{d'}(\Sigma)$, and let $\iota : \mathrm{Pic}^{d'}(\Sigma) \to \mathrm{Pic}^{-d'}(\Sigma)$ be the natural identification given by $\mathcal{M} \mapsto \mathcal{M}^{-1}$.

Lemma 3.7 *Via the isomorphism $\mathcal{M}_t^*(E, E_0, A_0) \simeq Quot_{\mathcal{E}_0}^{L}$ defined by the Kobayashi-Hitchin correspondence one has*

$$\delta(u)|_{\mathcal{M}_t^*(E,E_0,A_0)} = j^*[c_1(\pi^*(\mathfrak{P}_x^\vee)(1))] \ , \ \delta\begin{pmatrix} c_1 \\ \beta \end{pmatrix}\Bigg|_{\mathcal{M}_t^*(E,E_0,A_0)} = j^*[(\iota\circ\pi)^*(\beta)] \ .$$

In the second formula we used the natural identification

$$H_1(\Sigma,\mathbb{Z}) = H^1(\mathrm{Pic}^{-d'}(\Sigma),\mathbb{Z}) \ .$$

Proof: By Proposition 2.12 and Corollary 3.6 we find that the universal bundle $\mathbb{E}$ over $\mathcal{M}_t^*(E,E_0,A_0)\times\Sigma \simeq Quot^L_{\mathcal{E}_0}\times\Sigma$ is given by

$$\mathbb{E}\simeq j^*[\mathrm{pr}^*_{\mathrm{Pic}^{d'}(\Sigma)\times\Sigma}(\mathfrak{P})(-1)] \ .$$

Therefore, the restriction to $\mathcal{M}_t^*(E,E_0,A_0)\times\Sigma$ of the line bundle associated with the universal $U(1)$-bundle $\mathcal{P}$ (see section 1.1) is $j^*[\mathrm{pr}^*_{\mathrm{Pic}^{d'}(\Sigma)\times\Sigma}(\mathfrak{P}^\vee)(1)]$. The formulae given in section 1.1 give

$$\delta(u) = j^*\left[c_1(\mathrm{pr}^*_{\mathrm{Pic}^{d'}(\Sigma)\times\Sigma}(\mathfrak{P}^\vee)(1))/[x]\right] = j^*[c_1(\pi^*(\mathfrak{P}^\vee_x)(1))] \ ,$$

$$\delta\begin{pmatrix} c_1\\ \beta\end{pmatrix} = j^*\left[c_1(\mathrm{pr}^*_{\mathrm{Pic}^{d'}(\Sigma)\times\Sigma}(\mathfrak{P}^\vee)(1))/\beta\right] = j^*\left[c_1(\mathrm{pr}^*_{\mathrm{Pic}^{d'}(\Sigma)\times\Sigma}(\mathfrak{P}^\vee))/\beta\right] = $$
$$= j^*(\pi^*(c_1(\mathfrak{P}^\vee)/\beta)) \ .$$

To get the second equality we used the fact that $c_1(\mathcal{O}_P(1))$ has type $(2,0)$ with respect to the Künneth decomposition of $H^*(P\times\Sigma,\mathbb{Z})$. Note now that the line bundle $\mathfrak{P}_1 := (\iota\times\mathrm{id}_\Sigma)^*(\mathfrak{P}^\vee)$ is a Poincaré line bundle on $\mathrm{Pic}^{-d'}(\Sigma)\times\Sigma$. We get

$$\delta\begin{pmatrix} c_1\\ \beta\end{pmatrix} = j^*((\iota\circ\pi)^*(c_1(\mathfrak{P}_1)/\beta)) \ .$$

But the assigment $\beta\mapsto c_1(\mathfrak{P}_1)/\beta$ gives the standard identification $H_1(\Sigma,\mathbb{Z})\simeq H^1(\mathrm{Pic}^{-d'}(\Sigma),\mathbb{Z})$. ■

3.3 The explicit formula

We can now prove the following

Theorem 3.8 *Put* $g := g(\Sigma)$, $v = v(r_0,1,d,d_0) := \chi(\mathrm{Hom}(L,E_0)) - (1-g)$. *Let* $l_{\mathcal{O}_1}$ *be the generator of* $\Lambda^{2g}(H^1(\Sigma,\mathbb{Z}))$ *defined by the complex orientation* $\mathcal{O}_1$ *of* $H^1(\Sigma,\mathbb{R})$. *The Gromov-Witten invariant*

$$GGW^{(E_0,\mathfrak{c}_d)}_{\mathfrak{p}}(\mathrm{Hom}(\mathbb{C},\mathbb{C}^{r_0}),\alpha_{\mathrm{can}},S^1)\in\Lambda^*(H^1(\Sigma,\mathbb{Z}))$$

is given by the formula

$$GGW^{(E_0,\mathfrak{c}_d)}_{\mathfrak{p}}(\mathrm{Hom}(\mathbb{C},\mathbb{C}^{r_0}),\alpha_{\mathrm{can}},S^1)(l) = \left\langle \sum_{i\geq\max(0,g-v)}^{g}\frac{(r_0\Theta)^i}{i!}\wedge l \ , \ l_{\mathcal{O}_1}\right\rangle$$

for any $\mathfrak{p}$ *in the interesting chamber* C^+.

Proof: In the following computation we make the identifications

$$GQuot^L_{\mathcal{E}_0} \simeq Quot^L_{\mathcal{E}_0} \simeq Quot^{L'}_{\mathcal{E}'_0}$$

using the notations from above. By the homotopy invariance of the virtual class, we can use a general holomorphic structure $\mathcal{E}_0$. For such a structure $\mathcal{E}_0$, the quot space $Quot^L_{\mathcal{E}_0}$ is smooth and has the expected dimension v by Proposition 2.4.

Since the codimension of $Quot^L_{\mathcal{E}_0} = Quot^{L'}_{\mathcal{E}'_0}$ in P is k and this subspace is smooth, it follows from Corollary 3.6, shows that the section $\sigma := \bigoplus_{i=1}^{k} \sigma_{\rho_i}$ is regular along its vanishing locus. We can normalize the Poincaré line bundle $\mathfrak{P}$ such that the line bundles $\mathfrak{P}_x$, $x \in \Sigma$ are topologically trivial. Then Corollary 3.6 shows that the fundamental class $[Quot^{L'}_{\mathcal{E}'_0}] \in H_v(P,\mathbb{Z})$ is Poincaré dual to $c_1(\mathcal{O}_P(1))^k$.

By Lemma 3.7 we see that our problem reduces to the computation of the direct image of the homology classes $PD(c_1(\mathcal{O}_P(1))^i|_{Quot^L_{\mathcal{E}_0}})$ via the push-forward morphism

$$H_*(Quot^L_{\mathcal{E}_0},\mathbb{Z}) \xrightarrow{(\iota\circ\pi\circ j)_*} H_*(\mathrm{Pic}^{-d'}(\Sigma),\mathbb{Z}) \ .$$

Using the same arguments as in [OT2], the direct image of $PD(c_1(\mathcal{O}_P(1))^i|_{Quot^L_{\mathcal{E}_0}})$ in $H_*(\mathrm{Pic}^{-d'}(\Sigma),\mathbb{Z})$ can be identified with the Segre class s_{k+i} of the vector bundle $\mathrm{pr}^*_{\mathrm{Pic}^{-d'}(\Sigma)}(\mathfrak{P}_1)^{\oplus r_0}$ over $\mathrm{Pic}^{-d'}(\Sigma)$. The Chern classes of this bundle can be determined by applying the Grothendieck-Riemann-Roch theorem to the projection $\mathrm{Pic}^{d'}(\Sigma)\times\Sigma \to \mathrm{Pic}^{d'}(\Sigma)$. ■

3.4 Application: Counting quotients

We give a purely complex geometric application of our computation. Suppose that we have chosen the integers r, r_0, d, d_0 such that the expected dimension $v(r_0,r,d,d_0) = \chi(\mathrm{Hom}(E,E_0)) - \chi(\mathrm{End}(E))$ of the corresponding quot spaces is 0. Suppose also that for a particular bundle $\mathcal{E}_0$ of rank r_0 and degree d_0 the quot space $Quot^E_{\mathcal{E}_0}$ has dimension 0. We do *not* require that it is smooth. The problem is to estimate the number of points of such a 0-dimensional quot space.

Using the results above one can easily prove the following result.

Proposition 3.9 *Suppose* $v(r_0,r,d,d_0) = \dim(Quot^E_{\mathcal{E}_0}) = 0$. *Then*
i) The length of the 0-dimensional complex space $Quot^E_{\mathcal{E}_0}$ *is an invariant which does not depend on* $\mathcal{E}_0$, *but only on the integers* r, r_0, d, d_0.
ii) When $r = 1$, *this invariant is* r_0^g, *and the set* $Quot^E_{\mathcal{E}_0}$ *has at most* r_0^g *elements.*

Proof: The virtual fundamental class of a complex space Z which is cut out by a holomorphic section of index 0 with finite vanishing locus is just

$$\sum_{z\in Z} \dim(\mathcal{O}_{Z,z})[z] \in H_0(Z,\mathbb{Z}) \ .$$

This follows easily from the definition of the virtual fundamental class. It suffices now to use the Homotopy Invariance of virtual fundamental classes. The second statement follows directly from Theorem 3.7. ■

We close this subsection with the following remarks:

1. The quot spaces $Quot^E_{\mathcal{E}_0}$ *cannot* be regarded as fibres of a flat family as the holomorphic structure $\mathcal{E}_0$ in E_0 varies; the first statement is therefore not a consequence of the invariance of the length of zero dimensional spaces under deformations.

2. The main ingredient used in our proof is the fact that the quot spaces can be defined as vanishing loci of Fredholm sections and that the virtual fundamental class of the vanishing locus of such a section is invariant under *continuous deformations.*

3. It is more difficult to get the result above with purely complex geometric methods. In the particular case $r_0 = 2$ the inequality in ii) was obtained with such methods by Lange [L]. In the smooth case, the equality of ii) was proven by Oxbury [Oxb].

4. It is an interesting problem to compute the invariant introduced in i) also in the non-abelian case, i. e. for $r > 1$.

This reduces to the computation of the gauge theoretical Gromov-Witten invariant for $(\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{r_0}), \alpha_{\mathrm{can}}, U(r))$ in the chamber which corresponds to $t \gg 0$. The main difficulty is that, in the non-abelian case, there are many chambers, not just two.

5. More generally, consider spaces of holomorphic sections in a Grassmann bundle $\mathbb{G}_r(\mathcal{E}_0)$ over Σ.

The quot spaces $GQuot^E_{\mathcal{E}_0}$ are natural compactifications of these spaces to which the tautological cohomology classes extend naturally. These quot spaces map surjectively onto the Uhlenbeck compactifications of the spaces of sections in $\mathbb{G}_r(\mathcal{E}_0)$. It should therefore be possible to compare our non-abelian invariants with the twisted Gromov-Witten invariants (section 1.2) associated with sections in Grassmann bundles, and this should lead to an interesting generalization of the Vafa-Intriligator formula [BDW] [W1].

4 Gauge theoretical Gromov-Witten invariants and the full Seiberg-Witten invariants of ruled surfaces

4.1 Douady spaces of ruled surfaces and quot spaces on curves

Let $\mathcal{V}_0$ be a holomorphic bundle of rang 2 on a curve Σ and let $X = \mathbb{P}(\mathcal{V}_0)^3$ be the corresponding ruled surface; we denote by $\pi : X \to \Sigma$ the projection map. Let M a line bundle of Chern class $m := df + ns$ on X, where f is the Poincaré dual of a fibre and $s = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{V}_0)}(1))$.

An elementary computation shows that for every holomorphic structure $\mathcal{M}$ in M one has $\pi_*(\mathcal{M}) \simeq S^n(\mathcal{V}_0) \otimes \mathcal{L}$, where $\mathcal{L}$ is a holomorphic line bundle of degree d on the base curve Σ. Moreover, the assignement

$$\mathcal{L} \mapsto \mathcal{M} := \pi^*(\mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}_0)}(n)$$

defines an isomorphism $\mathrm{Pic}^d(\Sigma) \to \mathrm{Pic}^{df+ns}(X)$. Let $\mathcal{H}ilb(m)$ stand for the Hilbert scheme of effective divisors on X representing the homology class $PD(m)$ Poincaré dual to m. The family of identifications $H^0(X, \mathcal{M}) = H^0(\Sigma, \pi_*(\mathcal{M}))$ for $\mathcal{M} \in \mathrm{Pic}^m(X)$ gives rise to an isomorphism of schemes over $\mathbb{C}$

$$\mathcal{H}ilb(m) \simeq Quot^P_{S^n(\mathcal{V}_0)} \ , \qquad (I)$$

where P is the Hilbert polynomial $P = P_{S^n(\mathcal{V}_0)} - P_{\mathcal{L}^\vee}$ [Ha]. Notice that both moduli spaces in (I) have *complex analytic* as well as *gauge theoretical* versions. In complex analytic geometry one defines Douady spaces $Dou(m)$ of effective divisors representing $PD(m)$ and, more generally, complex analytic quot spaces. These analytic objects are isomorphic to the underlying complex spaces of the corresponding algebraic geometric objects, as explained in section 2.1.

The gauge theoretical quot spaces have been introduced in section 2. The gauge theoretical Douady space is defined as follows:

Let M de a differentiable line bundle on X. The gauge theoretical Douady space $GDou(M)$ is the space of equivalence classes of simple pairs $(\mathfrak{d}, \mathfrak{f})$, consisting of a holomorphic structure $\mathfrak{d}$ in M and a non-trivial $\mathfrak{d}$-holomorphic section in M. By the results of [LL] one has a natural identification of complex spaces $GDou(M) = Dou(c_1(M))$.

As explained in section 2, the complex analytic quot space $Quot^P_{S^n(\mathcal{V}_0)}$ can be identified with the gauge theoretical quot space $GQuot^{L^\vee}_{S^n(\mathcal{V}_0)}$, where L is a fixed smooth bundle of degree d on Σ.

We will need a gauge theoretical version of the isomorphism (I) which allows us to compare the virtual fundamental classes of the corresponding gauge theoretical complex spaces. We begin by defining the virtual fundamental class of $GDou(M)$:

Let $\bar{\mathcal{A}}_X := \bar{\mathcal{A}}(M) \times A^0(M)$, let $\bar{\mathcal{A}}^{inj}_X$ be the open subset of pairs whose section component does not vanish identically, and put

$$\bar{\mathcal{B}}^{inj}_X = {\bar{\mathcal{A}}^{inj}_X}\Big/{\mathcal{G}^{\mathbb{C}}_X} \ .$$

[3] We use here the Grothendieck convention for the projectivization of a bundle.

Over the Hausdorff Banach manifold $\bar{\mathcal{B}}_X^{inj}$ consider the bundles

$$E^i := \bar{\mathcal{A}}_X^{inj} \times_{\mathcal{G}_X^{\mathbb{C}}} [A_X^{0,i} \oplus A^{0,i-1}(M)] \ ,$$

and the bundle morphisms $D^i : E^i \to E^{i+1}$ given by

$$D^i_{(\mathfrak{d},\mathfrak{f})}(u,v) = (\bar{\partial} u, -u\mathfrak{f} - \mathfrak{d} v) \ .$$

Let E be the bundle

$$E := \ker(D^2) \subset E^2,$$

whose fibre in a point $[\mathfrak{d},\mathfrak{f}] \in \bar{\mathcal{B}}_X^{inj}$ is

$$E_{(\mathfrak{d},\mathfrak{f})} = \{(u,v) \in A_X^{0,2} \times A^{0,1}(M) |\ \mathfrak{d} v + u\mathfrak{f} = 0\} \ .$$

The fact that $\ker(D^2)$ is a subbundle of E^2 is crucial for our construction. It follows from the following

Remark 4.1 *After suitable Sobolev completions, the morphism $D^2 : E^2 \to E^3$ is a bundle epimorphism on $\bar{\mathcal{B}}^{inj}$.*

The proof uses a similar argument as the proof of Proposition 2.4.

The moduli space $GDou(M)$ is the vanishing locus of the Fredholm section $\mathfrak{s}$ in E, given by the $\mathcal{G}_X^{\mathbb{C}}$-equivariant map

$$(\mathfrak{d},\mathfrak{f}) \mapsto (-F_{\mathfrak{d}}^{02}, \mathfrak{d}(\mathfrak{f})) \ .$$

The index of this section is $w(m) = \chi(M) - \chi(\mathcal{O}_X)$.

Note that the section in E^2 defined by the same formula as $\mathfrak{s}$ is *not* Fredholm.

Definition 4.2 *The virtual fundmental class of $GDou(M)$ is defined as the virtual fundamental class $[GDou(M)]^{vir} \in H_{w(m)}(GDou(M),\mathbb{Z})$ associated with the pair $(E,\mathfrak{s})$.*

Now let V_0 be the underlying differentiable bundle of $\mathcal{V}_0$, H the underlying differentiable bundle of $\mathcal{O}_{\mathbb{P}(\mathcal{V}_0)}(1)$, and let $\mathfrak{h}$ be the corresponding semiconnection in H. Fix a smooth line bundle L of degree d on Σ.

Similarly as above let $\bar{\mathcal{A}}_\Sigma := \bar{\mathcal{A}}(L) \times A^0(L \otimes S^n(V_0))$, let $\bar{\mathcal{A}}_\Sigma^{inj}$ be the open subset of pairs whose section component is nondegenerate on a non-empty open set, and put

$$\bar{\mathcal{B}}_\Sigma^{inj} = {\bar{\mathcal{A}}_\Sigma^{inj}}\Big/_{\mathcal{G}_\Sigma^{\mathbb{C}}}$$

as in section 2.

Over the Hausdorff Banach manifold $\bar{\mathcal{B}}_{\Sigma}^{inj}$ consider the bundles F^i defined by

$$F^i := \bar{\mathcal{A}}_{\Sigma}^{inj} \times_{\mathcal{G}_{\Sigma}^{\mathbb{C}}} [A_{\Sigma}^{0,i} \oplus A^{0,i-1}(L \otimes S^n(V_0))] \ .$$

Put $F := F^2$. With this notation, the moduli space $GQuot_{S^n(\mathcal{V}_0)}^{L^\vee}$, which was introduced in section 2, is the vanishing locus of the Fredholm section s in F given by the equivariant map

$$(\delta, \varphi) \mapsto \bar{\partial}_{\delta,\delta_0} \varphi \ .$$

Here $\bar{\partial}_{\delta,\delta_0}$ stands for the semiconnection in $L \otimes S^n(V_0)$ associated with δ and the semiconnection δ_0 in V_0 corresponding to the holomorphic structure $\mathcal{V}_0$. The virtual fundamental class $[GQuot_{S^n(\mathcal{V}_0)}^{L^\vee}]^{vir}$ was defined as the virtual fundamental class associated with (F, s).

We can now state the main result of this section:

Theorem 4.3 *Let L be a differentiable on Σ, and put $M := \pi^*(L) \otimes H^{\otimes n}$. There is a canonical isomorphism of complex spaces*

$$GDou(M) \simeq GQuot_{S^n(\mathcal{V}_0)}^{L^\vee}$$

which maps the virtual fundamental class $[GDou(M)]^{vir}$ onto the virtual fundamental class $[GQuot_{S^n(\mathcal{V}_0)}^{L^\vee}]^{vir}$.

Remark 4.4 *One can prove a similar identification of moduli spaces for much more general fibrations ; in general, however, this identification will <u>not</u> respect the virtual fundamental classes.*

Proof: (of Theorem 4.3)

To every pair (δ, φ) consisting of a semiconnection in L and a section $\varphi \in A^0(L \otimes S^n(V_0)) = A^0\mathrm{Hom}(L^\vee, S^n(V_0))$ we associate the pair

$$(\tilde{\delta}, \tilde{\varphi}) \in \bar{\mathcal{A}}(M) \times A^0(M)$$

defined by

$$\tilde{\delta} := \pi^*(\delta) \otimes \mathfrak{h}^n \ ,$$

$$\tilde{\varphi}([e]) := \varphi(e) \text{ for } e \in V_0^\vee \setminus \{0- \text{ section}\} \ .$$

We will prove – using the Associativity Property of virtual fundamental classes – that the assignment $(\delta, \varphi) \to (\tilde{\delta}, \tilde{\varphi})$ induces an embedding $\bar{\mathcal{B}}_{\Sigma}^{inj} \hookrightarrow \bar{\mathcal{B}}_{X}^{inj}$ which maps the virtual fundemental class $[GDou(M)]^{vir}$ onto the virtual fundemental class $[GQuot_{S^n(\mathcal{V}_0))}^{L^\vee}]^{vir}$:

Consider the vertical subbundle $T_{X/\Sigma}$ of T_X, and define E' to be the linear subspace of E whose fibre in a point $[\mathfrak{d}, \mathfrak{f}]$ is

$$E'_{(\mathfrak{d},\mathfrak{f})} := \{(u,v) \in E_{(\mathfrak{d},\mathfrak{f})} | \ u = 0, v|_{T_{X/\Sigma}} = 0\} =$$

$$= \{(u,v) \in A^{0,2}_X \times A^{0,1}(M)|\ u=0,\ v|_{T_{X/\Sigma}}=0,\ \mathfrak{d}v=0\}\ .$$

The last two conditions mean that v defines a section in the line bundle $M \otimes \pi^*(\Lambda^{0,1}_\Sigma)$ which is $\mathfrak{d}$-holomorphic on the fibres.

Claim: After suitable Sobolev completions, E' is a subbundle of E.

Proof: We will omitt Sobolev indices to save on notations. The proof uses the fact that the fibres of π are projective lines in an essential way. Let $\mathfrak{d}$ be an arbitrary semiconnection in M. Since in the line bundle $H^{\otimes n}$ over $\mathbb{P}^1$ all semiconnections are gauge equivalent, there exists a gauge transformation $g_{\mathfrak{d}} \in \mathcal{G}^{\mathbb{C}}_X$, unique modulo $\mathcal{G}^{\mathbb{C}}_\Sigma$, such that $g_{\mathfrak{d}} \cdot \mathfrak{d}$ and $\mathfrak{h}^{\otimes n}$ coincide on the fibres. Moreover, one can choose $g_{\mathfrak{d}}$ to depend smoothly on $\mathfrak{d}$. The gauge transformation $g_{\mathfrak{d}}$ identifies the space $E'_{(\mathfrak{d},\mathfrak{f})}$ with the fixed space

$$E'_0 = \{\alpha \in A^0(M \otimes \pi^*(\Lambda^{0,1}))|\ \alpha|_{\rm fibres} \text{ is } \mathfrak{h}^{\otimes n}\text{−holomorphic}\} \simeq A^{0,1}_\Sigma(\Sigma, L \otimes S^n(V_0)).$$

The union

$$\coprod_{(\mathfrak{d},\mathfrak{f}) \in \bar{\mathcal{A}}^{inj}_X} E'_{(\mathfrak{d},\mathfrak{f})}$$

becomes therefore a trivial subbundle of $\bar{\mathcal{A}}^{inj}_X \times [A^{0,2}_X \times A^{0,1}(M)]$. Since it is gauge invariant, this subbundle descends to a subbundle E' of E, as required.

■

Now consider the exact sequence

$$0 \to E' \to E \to E'' \to 0$$

and let $\mathfrak{s}''$ be the section in E'' induced by $\mathfrak{s}$.

Claim: The section $\mathfrak{s}''$ is regular in every point of its vanishing locus.

Proof: We have to show that

$$E_{(\mathfrak{d},\mathfrak{f})} = \mathrm{im}(D^1_{(\mathfrak{d},\mathfrak{f})}) + E'_{(\mathfrak{d},\mathfrak{f})}$$

for every pair $(\mathfrak{d},\mathfrak{f})$ of the form $(\mathfrak{d},\mathfrak{f}) = (\tilde{\delta},\tilde{\varphi})$. In other words, we must prove that, for such pairs $(\mathfrak{d},\mathfrak{f})$, the natural map

$$E'_{(\mathfrak{d},\mathfrak{f})} \longrightarrow {E_{(\mathfrak{d},\mathfrak{f})}}\big/{\mathrm{im}(D^1_{(\mathfrak{d},\mathfrak{f})})} = H^2_{(\mathfrak{d},\mathfrak{f})}$$

is surjective. Here $H^2_{(\mathfrak{d},\mathfrak{f})}$ denotes the second cohomology group associated with the elliptic deformation complex of the holomorphic pair $(\mathfrak{d},\mathfrak{f})$. But it is easy to see that the image of this map coincides with the image of the pull-back map

$H^2_{(\delta,\varphi)} \to H^2_{(\mathfrak{d},\mathfrak{f})}$. In order to check the surjectivity of this map, put $\mathcal{M} := (M, \mathfrak{d})$, $\mathcal{L} := (L, \delta)$ and consider the following morphism of long exact cohomology sequences:

$$\begin{array}{ccccccc}
H^1(X,\mathcal{M}) & \to & H^2_{(\mathfrak{d},\mathfrak{f})} & \to & H^2(X,\mathcal{O}_X) & \to & H^2(X,\mathcal{M}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^1(\Sigma,\mathcal{L}\otimes S^n(\mathcal{V}_0)) & \to & H^2_{(\delta,\varphi)} & \to & 0 & \to & 0
\end{array}$$

One has $H^2(X,\mathcal{O}_X) = 0$, and the first vertical map is an epimorphism for all $n \geq 0$, since $H^0(\Sigma, R^1\pi_*(\mathcal{M})) = H^0(\Sigma, \mathcal{L} \otimes R^1\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{V}_0)}(n))) = 0$. The case $n < 0$ is not interesting, since in this case $\bar{\mathcal{B}}^{inj}_X = \bar{\mathcal{B}}^{inj}_\Sigma = \emptyset$.

This shows that the natural morphism of elliptic complexes $F^* \to E^*$ induces an epimorphism $H^2_{(\delta,\varphi)} \to H^2_{(\mathfrak{d},\mathfrak{f})}$ as desired.

Claim: One has natural identifications

$$Z(\mathfrak{s}'') = \bar{\mathcal{B}}^{inj}_\Sigma \ , \ F = E'|_{Z(\mathfrak{s}'')} \ , \ s = \mathfrak{s}' \ .$$

Proof: Indeed, when $(\mathfrak{d},\mathfrak{f}) \in Z(\mathfrak{s}'')$, then $\mathfrak{d}$ is integrable and $\mathfrak{d}\mathfrak{f}$ vanishes on the vertical tangent space. Applying the gauge transformation $g_{\mathfrak{d}}$ if neccesary, we may assume that $\mathfrak{d}$ coincides with $\mathfrak{h}$ on the fibres. We fix a semiconnection δ^0 in L. Since the difference $\mathfrak{d} - \pi^*(\delta^0) \otimes \mathfrak{h}^{\otimes n} \in A^{0,1}_X$ vanishes on the vertical tangent space and is $\bar\partial$-closed, it must be the pull-back of a $(0,1)$-form α on Σ. But this implies that $\mathfrak{d} = \pi^*(\delta^0 + \alpha) \otimes \mathfrak{h}^{\otimes n} = \tilde{\delta}_\alpha$, where $\delta_\alpha := \delta^0 + \alpha$. Similarly, the condition $\mathfrak{d}\mathfrak{f}|_{T_{X/\Sigma}} = 0$ implies that $\mathfrak{f}$ is $\mathfrak{h}^{\otimes n}$-holomorphic on the fibres, hence it has the form $\tilde{\varphi}$, where φ is a section of $L \otimes S^n(V_0)$.

Theorem 4.3 follows now directly from the Associativity Property of virtual fundmental classes. ■

4.2 Comparison of virtual fundamental classes of Seiberg-Witten moduli spaces and Douady spaces

Let (X, g) be a Kähler surface and let K_X be the differentiable line bundle underlying the canonical bundle of X . Every Hermitian line bundle M on X defines a $Spin^c$-structure

$$\gamma_M : \Lambda^1_X \to \mathbb{R}SU(\Lambda^0(M) \oplus \Lambda^{0,2}(M), \Lambda^{0,1}(M)) \ ,$$

obtained by tensoring the canonical $Spin^c$-structure with M. The determinant bundle of this $Spin^c$-structure is $M^{\otimes 2} \otimes K_X^{-1}$. The assignment $[M] \mapsto [\gamma_M]$ induces a bijective correspondence between the group of isomorphism classes of Hermitian line bundles, which can be identified with $H^2(X,\mathbb{Z})$, and the set of equivalence classes of $Spin^c$-structures on X.

Let $\beta \in A^{1,1}_{\mathbb{R}}$ be a closed form. The Kobayashi-Hitchin correspondence for the Seiberg-Witten monopole equations [OT1], [OT2] states that the moduli space $\mathcal{W}^{\gamma_M}_{X,\beta}$ of solutions $(A,\Psi) \in \mathcal{A}(L) \times [A^0(M) \oplus A^{0,2}(M)]$ of the twisted monopole equations

$$\left\{ \begin{array}{lll} \not{D}^{\gamma_M}_A \Psi & = & 0 \\ \gamma_M \left(F^+_A + 2\pi i \beta^+ \right) & = & 2(\Psi\bar{\Psi})_0, \end{array} \right. \qquad (SW^{\gamma_M}_\beta)$$

can be identified with the gauge theoretical Douady space $GDou(M)$ (respectively $GDou(K_X \otimes M^{-1})$) when $\langle (2c_1(M) - c_1(K_X) - [\beta]) \cup [\omega_g], [X] \rangle < 0$ (respectively > 0).

The fact that this identification is an isomorphism of real analytic spaces was proved in [Lu]. One has the following stronger result:

Theorem 4.5 *The Kobayashi-Hitchin correspondence for the Seiberg-Witten equations induces an isomorphism which maps the virtual fundamental class* $[\mathcal{W}^{\gamma_M}_{X,\beta}]^{vir}$*, computed with respect to the complex orientation data, onto the virtual fundamental class* $[GDou(M)]^{vir}$ *(respectively onto* $(-1)^{\chi(M)}[GDou(K_X \otimes M^{-1})]^{vir}$*) when* $\langle (2c_1(M) - c_1(K_X) - [\beta]) \cup [\omega_g], [X] \rangle < 0$ *(respectively* > 0*).*

Proof: Let C_0 be the standard connection induced by the Levi-Civita connection in the line bundle K_X^{-1}. Using the substitutions $A := C_0 \otimes B^{\otimes 2}$ with $B \in \mathcal{A}(M)$ and $\Psi =: \varphi + \alpha \in A^0(M) \oplus A^{0,2}(M)$, the configuration space of unknowns becomes $\mathcal{A} = \mathcal{A}(M) \times [A^0(M) \oplus A^{0,2}(M)]$, and a pair $(B, \varphi + \alpha)$ solves the twisted monopole equation $(SW^{\gamma_M}_\beta)$ iff

$$\begin{array}{ll} -F^{02}_A \ + \ \alpha \otimes \bar{\varphi} & = 0 \\ \bar{\partial}_B(\varphi) \ - \ i\Lambda \partial_B(\alpha) & = 0 \\ i\Lambda_g(F_A + 2\pi i \beta) + (\varphi\bar{\varphi} - *(\alpha \wedge \bar{\alpha})) & = 0 \ . \end{array}$$

We denote by $\mathcal{A}^*$ the open subspace of $\mathcal{A}$ with non-trivial spinor component, and by $\mathcal{B}^*$ its quotient ${}^{\mathcal{A}^*}/_{\mathcal{G}}$ by the gauge group $\mathcal{G} = \mathcal{C}^\infty(X, S^1)$.

Let $e^i(B, \varphi, \alpha)$, $i = 1, \ldots, 3$ stand for the map of $\mathcal{A}$ defined by the left hand term of the i-th equation above. This map induces a section ε^i in a certain bundle H^i over $\mathcal{B}^*$ which is associated with the principal $\mathcal{G}$-bundle $\mathcal{A}^* \to \mathcal{B}^*$.

The Seiberg-Witten moduli space $\mathcal{W}^{\gamma_M}_\beta$ is the analytic subspace of $\mathcal{B}^*$ cut out by the Fredholm section $\varepsilon = (\varepsilon^1, \varepsilon^2, \varepsilon^3)$ in the bundle $H := \oplus H^i$, and the virtual fundamental class $[\mathcal{W}^{\gamma_M}_\beta]^{vir}$ is by definition the virtual fundamental class in the sense of Brussee [Br], associated with this section and the complex orientation data.

We define a bundle morphism $q : H \to H^2 := \mathcal{A}^* \times_{\mathcal{G}} A^{0,2}(M)$ by

$$q_{(B,\varphi,\alpha)}(x^1, x^2, x^3) = \bar{\partial}_B x^2 + \frac{1}{2} x^1 \varphi \ .$$

One easily checks that

$$q\circ\varepsilon(B,\varphi,\alpha)=(\frac{1}{2}|\varphi|^2+\bar\partial_B\bar\partial_B^*)\alpha\ .$$

Suppose now that $\langle(2c_1(M)-c_1(K_X)-[\beta])\cup[\omega_g],[X]\rangle<0$. Integrating the third equation over X, one sees that any solution of the equations has a nontrivial φ-component. The space $\mathcal{A}^{SW}$ of solutions is therefore contained in the open subspace $\mathcal{A}^\circ$ consisting of triples (B,φ,α) with $\varphi\neq 0$. But the operator $(\frac{1}{2}|\varphi|^2+\bar\partial_B\bar\partial_B^*)$ is invertible for $\varphi\neq 0$. It follows that on $\mathcal{B}^\circ:=\mathcal{A}^\circ/\mathcal{G}$ the section $\varepsilon'':=q\circ\varepsilon$ is regular around its vanishing locus $Z(\varepsilon'')$, and that the submanifold $Z(\varepsilon'')\subset\mathcal{B}^\circ$ is just the submanifold cut out by the equation $\alpha=0$.

One checks that q is a bundle epimorphism on $\mathcal{B}^\circ$. Set $H':=\ker q$. The Associativity Property shows now that the virtual fundamental class $[\mathcal{W}_\beta^{\gamma_M}]^{vir}$ can be identified with the virtual fundamental class associated with the Fredholm section

$$\varepsilon':=\varepsilon|_{Z(\varepsilon'')}\in\Gamma(Z(\varepsilon''),H'|_{Z(\varepsilon'')})\ .$$

In other words, the virtual fundamental class of the Seiberg-Witten moduli space can be identified with the virtual fundamental class of the moduli space $\mathcal{V}_{(\frac{s}{2}-\pi\Lambda_g\beta)}(M)$ of $(\frac{s}{2}-\pi\Lambda_g\beta)$-*vortices* in M [OT1]. Recall that $\mathcal{V}_{(\frac{s}{2}-\pi\Lambda_g\beta)}(M)$ is defined as the space of equivalence classes of pairs $(B,\varphi)\in\mathcal{A}(M)\times[A^0(M)\setminus\{0\}]$ satisfying the equations

$$\begin{aligned}&(-F_A^{02},\bar\partial_B\varphi) &&=0\\ &i\Lambda_gF_B+\tfrac{1}{2}\varphi\bar\varphi+(\tfrac{s}{2}-\pi\Lambda_g\beta) &&=0\ .\end{aligned}$$

Here the first equation is considered as taking values in the subspace

$$G^1_{B,\varphi}:=\{(u,v)\in A^{0,2}_X\oplus A^{0,1}(M)|\ \bar\partial_Bv+u\varphi=0\}\ .$$

More precisely, let $\mathcal{C}^*$ be the quotient $\mathcal{C}^*:={\mathcal{A}(M)\times[A^0(M)\setminus\{0\}]}/{\mathcal{G}}$, and let G^1 be the subbundle of the associated bundle

$$[\mathcal{A}(M)\times[A^0(M)\setminus\{0\}]]\times_{\mathcal{G}}[A^{0,2}_X\oplus A^{0,1}(M)]$$

over $\mathcal{C}^*$, whose fibre in $[B,\varphi]$ is $G^1_{B,\varphi}$. Let G^2 be the trivial bundle $\mathcal{C}^*\times A^0(X)$ and $G:=G^1\oplus G^2$. The left hand terms of the equations above define sections g^i in the bundles G^i, and the section $g=(g^1,g^2)$ is Fredholm.

So far we have shown that the virtual fundamental class of the Seiberg-Witten moduli space can be identified with the virtual fundamental class of the moduli space $\mathcal{V}_{(\frac{s}{2}-\pi\Lambda_g\beta)}(M)$ associated with the Fredholm section g and the complex orientations.

To complete the proof, we have to identify the virtual fundamental class $[\mathcal{V}_{(\frac{s}{2}-\pi\Lambda_g\beta)}(M)]^{vir}$ with the virtual fundamental class $[GDou(M)]^{vir}$ of the corresponding gauge theoretical Douady space. This is again an application of the general principle which states the Kobayashi-Hitchin-type correspondence between moduli spaces associated with Fredholm problems respects virtual fundamental classes. We proceed as in the proof of Theorem 3.2:

Consider the exact sequence

$$0 \longrightarrow G^1 \longrightarrow G \xrightarrow{\pi} G^2 \longrightarrow 0$$

of bundles over $\mathcal{C}^*$. The section $g^2 = \pi \circ g$ comes from a formal moment map, so one can show:
1. g^2 is regular around $Z(g)$,
2. the natural map $\rho : Z(g^2) \to \bar{\mathcal{B}}^{inj}$ induces a bijection

$$Z(g) = \mathcal{V}_{(\frac{s}{2} - \pi\Lambda_g\beta)}(M) \xrightarrow{\simeq} GDou(M) \ ,$$

and is étale around $Z(g)$.
Using the notations of section 4.1, one obtains a natural identification

$$\rho^*(E) = G^1|_{Z(g^2)} \ ,$$

and $g^1|_{Z(g^2)}$ corresponds via this identification to the section $\mathfrak{s}$ which defines the virtual fundamental class $[GDou(M)]^{vir}$.

The result follows now by applying again the Associativity Property of virtual fundamental classes. ■

Recall from [OT2] that with any compact oriented 4-manifold X with $b_+ = 1$ one can associate a full Seiberg-Witten invariant $SW^{\pm}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}) \in \Lambda^* H^1(X,\mathbb{Z})$ which depends on an equivalence class $\mathfrak{c}$ of $Spin^c$-structures, an orientation $\mathcal{O}_1$ of $H^1(X,\mathbb{R})$, and a component $\mathbf{H}_0$ of the hyperquadric $\mathbf{H}$ of $H^2(X,\mathbb{R})$ defined by the equation $x \cdot x = 1$.

By the homotopy invariance of the virtual fundamantal classes, one has

Remark 4.6 *The Seiberg-Witten invariants defined in [OT2] using the generic regularity of Seiberg-Witten moduli spaces with respect to Witten's perturbation [W2], coincide with the Seiberg-Witten invariants defined in [Br] using the virtual fundamental classes of these moduli spaces.*

Combining Theorems 2.8, 3.2, 3.8, 4.3, 4.5 we obtain

Corollary 4.7 *Consider a ruled surface $X = \mathbb{P}(\mathcal{V}_0)$ over the Riemann surface Σ of genus g, and a class $\mathfrak{c}$ of $Spin^c$-strucures on X. Let c be the Chern class of the determinant line bundle of $\mathfrak{c}$ and let $w_c := \frac{1}{4}(c^2 - 3\sigma(X) - 2e(X))$ be the index of $\mathfrak{c}$. Denote by $[F]$ the class of a fibre of X over Σ, by $\Theta_c \in \Lambda^2(H_1(X,\mathbb{Z})) = \Lambda^2(H^1(X,\mathbb{Z}))^\vee$ the element defined by*

$$\Theta_c(a,b) := \frac{1}{2}\langle c \cup a \cup b, [X]\rangle \ ,$$

and let $l_{\mathcal{O}_1}$ be the generator of $\Lambda^{2g}(H^1(X,\mathbb{Z}))$ corresponding to $\mathcal{O}_1$.

The full Seiberg-Witten invariant of X corresponding to $\mathfrak{c}$, the complex orientation $\mathcal{O}_1$ of the cohomology space $H^1(X,\mathbb{R})$, and the component $\mathbf{H}_0$ of $\mathbf{H}$ which contains the Kähler cone, is given by

$$SW^{\pm}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}) = 0$$

if $\langle c,[F]\rangle = 0$; when $\langle c,[F]\rangle \neq 0$, it is given by

$$SW^{-\mathrm{sign}\langle c,[F]\rangle}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}) = 0 \ ,$$

$$SW^{\mathrm{sign}\langle c,[F]\rangle}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(l) = \mathrm{sign}\langle c,[F]\rangle \left\langle \sum_{i\geq \max(0,g-\frac{w_c}{2})}^{g} \frac{\Theta_c^i}{i!}\wedge l \ , \ l_{\mathcal{O}_1} \right\rangle \ .$$

Remarks:

1. This result cannot be obtained directly using the Kobayashi-Hitchin correspondence for the Seiberg-Witten equations, because the Douady spaces of divisors on ruled surfaces are in general oversized, non-reduced, and they can contain components of different dimensions. Moreover, it is not clear at all whether one can achieve regularity by varying the holomorphic structure $\mathcal{V}_0$ in V_0. This shows that the quot spaces of the form $Quot^{L^\vee}_{S^n(\mathcal{V}_0)}$ are very special within the class of quot spaces $Quot^{L^\vee}_{\mathcal{E}_0}$ with $\mathcal{E}_0$ $\mathcal{C}^\infty$- equivalent to $S^n(\mathcal{V}_0)$. The theory of gauge theoretical Gromov-Witten invariants and the comparison Theorem 4.3 show that one can however compute the full Seiberg-Witten invariant of X using a quot space $Quot^{L^\vee}_{\mathcal{E}_0}$ with $\mathcal{E}_0$ a general holomorphic bundle $\mathcal{C}^\infty$-equivalent to $S^n(\mathcal{V}_0)$, although such a quot space cannot be identified with a space of divisors of X.

2. The result provides an independent check of the universal wall-crossing formula for the full Seiberg-Witten invariant, proven in [OT2]. Note however that, in the formula given in [OT2], the sign in front of u_c, which corresponds to Θ_c above, is wrong. The error was pointed out to us by Markus Dürr, who also checked the corrected formula for a large class of elliptic surfaces [Dü].

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