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Rassias, Michael Th ; Yang, Bicheng ; Raigorodskii, Andrei

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A new Hardy–Mulholland-type inequality with a mixed kernel

Michael Th. Rassias[1,2,3] · Bicheng Yang[4] · Andrei Raigorodskii[2,5,6,7]

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Abstract

By the use of weight coefficients and techniques of real analysis, we establish a new Hardy–Mulholland-type inequality with a mixed kernel and a best possible constant factor in terms of the hypergeometric function. Equivalent forms, an operator expression with the norm and reverses are also considered.

Keywords Hardy–Mulholland-type inequality · Weight coefficients · Equivalent form · Hypergeometric function · Operator · Reverse

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✉ Michael Th. Rassias
michail.rassias@math.uzh.ch; mthrassias@yahoo.com

Bicheng Yang
bcyang@gdei.edu.cn; bcyang818@163.com

Andrei Raigorodskii
mraigor@yandex.ru

1 Institute of Mathematics, University of Zurich, 8057 Zurich, Switzerland

2 Moscow Institute of Physics and Technology, Institutskiy per, d. 9, 141700 Dolgoprudny, Russia

3 Program in Interdisciplinary Studies, Institute for Advanced Study, 1 Einstein Dr, Princeton, NJ 08540, USA

4 Department of Mathematics, Guangdong University of Education, Guangzhou 510303, Guangdong, People's Republic of China

5 Moscow State University, 119991 Moscow, Russia

1 Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$,

$$\|a\|_p = \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} > 0, \quad \|b\|_q > 0,$$

then we have the following Hardy–Hilbert inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q. \tag{1}$$

We also have the following Mulholland inequality

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=2}^{\infty} \frac{a_m^p}{m^{1-p}} \right)^{\frac{1}{p}} \left(\sum_{n=2}^{\infty} \frac{b_n^q}{n^{1-q}} \right)^{\frac{1}{q}}, \tag{2}$$

with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [5]). The inequalities (1) and (2) are important in Mathematical Analysis and its various applications (cf. [5, 15, 35–38]).

If $\mu_i, \upsilon_j > 0 \, (i,j \in \mathbf{N} = \{1, 2, \ldots\})$,

$$U_m := \sum_{i=1}^{m} \mu_i, \quad V_n := \sum_{j=1}^{n} \upsilon_j \qquad (m, n \in \mathbf{N}), \tag{3}$$

then we have the following Hardy–Hilbert-type inequality (cf. Theorem 321 of [5], replacing $\mu_m^{1/q} a_m$ and $\upsilon_n^{1/p} b_n$ with a_m and b_n) :

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\sum_{m=1}^{\infty} \frac{a_m^p}{\mu_m^{p-1}} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{b_n^q}{\upsilon_n^{q-1}} \right)^{\frac{1}{q}}. \tag{4}$$

For $\mu_i = \upsilon_j = 1 \, (i,j \in \mathbf{N})$, inequality (4) reduces to (1).

In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [34] proved an extension of the integral analogous of (1) with the kernel $\frac{1}{(x+y)^{\lambda}}$ for $p = q = 2$. Recently, Yang [36] presented the following extensions of (1) and (2): If $\lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1 + \lambda_2 = \lambda, k_{\lambda}(x, y)$ is a finite non-negative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1) = \int_0^{\infty} k_{\lambda}(t, 1) t^{\lambda_1 - 1} \, \mathrm{d}t \in \mathbf{R}_+,$$

and $k_{\lambda}(x, y) x^{\lambda_1 - 1}$ $(k_{\lambda}(x, y) y^{\lambda_2 - 1})$ is decreasing with respect to $x > 0 \, (y > 0)$,

$$\phi(x) = x^{p(1-\lambda_1)-1}, \quad \psi(x) = x^{q(1-\lambda_2)-1},$$

then for $a_m, b_n \geq 0$,

$$a = \{a_m\}_{m=1}^{\infty} \in l_{p,\phi} = \left\{ a; \|a\|_{p,\phi} := \left(\sum_{m=1}^{\infty} \phi(m)|a_m|^p \right)^{\frac{1}{p}} < \infty \right\},$$

$b = \{b_n\}_{n=1}^{\infty} \in l_{q,\psi}$, $\|a\|_{p,\phi}, \|b\|_{q,\psi} > 0$, we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} k_{\lambda}(m,n) a_m b_n < k(\lambda_1) \|a\|_{p,\phi} \|b\|_{q,\psi}, \tag{5}$$

where the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for $\lambda = 1$, $\lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, $k_1(x,y) = \frac{1}{x+y}$, the inequality (5) reduces to (1). Some other new results including multidimensional Hilbert-type inequalities, Hardy–Mulholland-type inequalities and Hardy–Hilbert-type inequalities are established in [1–4, 6–10, 12, 14, 16–31, 33, 39–48].

In this paper, by the use of weight coefficients and techniques of real analysis, we prove a new Hardy–Mulholland-type inequality with the following mixed kernel

$$\frac{(\min\{\ln U_m, \ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}} \quad (0 \leq -2\alpha < \lambda \leq 2; m, n \in \mathbf{N} \backslash \{1\}).$$

and a best possible constant factor expressed in terms of the hypergeometric function. This inequality constitutes an extension of (2). Equivalent forms, operator expressions with the norm and reverses are considered as well.

2 An example and some lemmas

In the sequel, we consider that

$$p \neq 0, 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \lambda_1 + \lambda_2 = \lambda, \quad \mu_i, \upsilon_j > 0 \ (i,j \in \mathbf{N}),$$

with $\mu_1 = \upsilon_1 = 1$, U_m and V_n are defined in (3), $a_m, b_n \geq 0$,

$$\|a\|_{p,\Phi_{\lambda}} := \left(\sum_{m=2}^{\infty} \Phi_{\lambda}(m) a_m^p \right)^{\frac{1}{p}} \quad \text{and} \quad \|b\|_{q,\Psi_{\lambda}} := \left(\sum_{n=2}^{\infty} \Psi_{\lambda}(n) b_n^q \right)^{\frac{1}{q}},$$

where

$$\Phi_{\lambda}(m) := \frac{(\ln U_m)^{p(1-\lambda_1)-1}}{U_m^{1-p} \mu_m^{p-1}}, \quad \Psi_{\lambda}(n) := \frac{(\ln V_n)^{q(1-\lambda_2)-1}}{V_n^{1-q} \upsilon_n^{q-1}} \qquad (m, n \in \mathbf{N} \backslash \{1\}).$$

We introduce the following hypergeometric function (cf. [32]):

$$F(\alpha, \beta, \gamma, z) := \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\, dt, \tag{6}$$

where $\mathrm{Re}(\gamma) > \mathrm{Re}(\beta) > 0, |\arg(1-z)| < \pi; (1-zt)^{-\alpha} = 1$, for $z = 0$.

***Example* 2.1** For $-\alpha < \lambda_1, \lambda_2 \leq 1\ (-2\alpha < \lambda = \lambda_1 + \lambda_2 \leq 2)$, we set

$$k_\lambda(x,y) := \frac{(\min\{x,y\})^\alpha}{(x+y)^{\lambda+\alpha}} \quad ((x,y) \in \mathbf{R}_+^2).$$

(1) Since $\lambda + \alpha > -\alpha$, there exists a constant $L_\alpha = \max\{2^\alpha, 1\} > 0$, such that

$$\frac{1}{(t+1)^{\lambda+\alpha}} \leq (t+1)^\alpha \leq L_\alpha \quad (t \in (0,1)).$$

For $\lambda_1, \lambda_2 > -\alpha$, we get that

$$\begin{aligned} 0 < k_\alpha(\lambda_1) &:= \int_0^\infty k_\lambda(1,t) t^{\lambda_2 - 1}\,\mathrm{d}t = \int_0^\infty k_\lambda(t,1) t^{\lambda_1 - 1}\,\mathrm{d}t \\ &= \int_0^\infty \frac{(\min\{t,1\})^\alpha}{(t+1)^{\lambda+\alpha}} t^{\lambda_1 - 1}\,\mathrm{d}t = \int_0^1 \frac{t^{\lambda_1+\alpha-1} + t^{\lambda_2+\alpha-1}}{(t+1)^{\lambda+\alpha}}\,\mathrm{d}t \\ &\leq L_\alpha \int_0^1 (t^{\lambda_1+\alpha-1} + t^{\lambda_2+\alpha-1})\,\mathrm{d}t = L_\alpha\left(\frac{1}{\lambda_1+\alpha} + \frac{1}{\lambda_2+\alpha}\right) < \infty, \end{aligned} \tag{7}$$

and thus by (6) and (7), it follows that

$$k_\alpha(\lambda_1) = \sum_{i=1}^2 \frac{1}{\lambda_i + \alpha} F(\lambda+\alpha, \lambda_i+\alpha, \lambda_i+\alpha+1, -1) \in \mathbf{R}_+. \tag{8}$$

(i) For $\alpha = 0$, we obtain $k_\lambda(x,y) = \frac{1}{(x+y)^\lambda}\ (0 < \lambda \leq 2)$ and

$$k_0(\lambda_1) = \int_0^\infty \frac{t^{\lambda_1 - 1}\,\mathrm{d}t}{(t+1)^\lambda} = B(\lambda_1, \lambda_2) = \sum_{i=1}^2 \frac{1}{\lambda_i} F(\lambda, \lambda_i, \lambda_i + 1, -1); \tag{9}$$

(ii) for $-\alpha < \lambda + \alpha \leq 1\ (< 2 + \alpha)$, we derive that

$$\begin{aligned} k_\alpha(\lambda_1) &= \int_0^1 \frac{t^{\lambda_1+\alpha-1} + t^{\lambda_2+\alpha-1}}{(t+1)^{\lambda+\alpha}}\,\mathrm{d}t \\ &= \int_0^1 \sum_{k=0}^\infty \binom{-\lambda-\alpha}{k} t^k (t^{\lambda_1+\alpha-1} + t^{\lambda_2+\alpha-1})\,\mathrm{d}t \\ &= \int_0^1 \sum_{k=0}^\infty (-1)^k \binom{\lambda+\alpha+k-1}{k} t^k (t^{\lambda_1+\alpha-1} + t^{\lambda_2+\alpha-1})\,\mathrm{d}t \\ &= \int_0^1 \sum_{k=0}^\infty \binom{\lambda+\alpha+2k-1}{2k}\left(1 - \frac{\lambda+\alpha+2k}{2k+1} t\right) t^{2k} (t^{\lambda_1+\alpha-1} + t^{\lambda_2+\alpha-1})\,\mathrm{d}t. \end{aligned}$$

Since

$$1-\frac{\lambda+\alpha+2k}{2k+1}t\geq\frac{1-(\lambda+\alpha)+2k(1-t)}{2k+1}\geq 0,$$

in view of the Lebesgue term by term integration theorem (cf. [13]), we obtain that

$$\begin{aligned}k_\alpha(\lambda_1)&=\sum_{k=0}^{\infty}\int_0^1\binom{\lambda+\alpha+2k-1}{2k}\left(1-\frac{\lambda+\alpha+2k}{2k+1}t\right)t^{2k}(t^{\lambda_1-1}+t^{\lambda_2-1})\,\mathrm{d}t\\&=\sum_{k=0}^{\infty}\binom{-\lambda-\alpha}{k}\int_0^1 t^k(t^{\lambda_1+\alpha-1}+t^{\lambda_2+\alpha-1})\,\mathrm{d}t\\&=\sum_{k=0}^{\infty}\binom{-\lambda-\alpha}{k}\left(\frac{1}{k+\lambda_1+\alpha}+\frac{1}{k+\lambda_2+\alpha}\right).\end{aligned}\tag{10}$$

(2) Suppose that $\alpha\leq 0\,(\alpha>-1)$. We have

$$\lambda+\alpha>-\alpha\geq 0,\quad 0<\lambda_i+\alpha\leq\lambda_i\,(i=1,2).$$

For $\lambda_2\leq 1\,(\lambda_2+\alpha\leq\lambda_2\leq 1)$ and fixed $x>0$, we deduce that

$$k_\lambda(x,y)\frac{1}{y^{1-\lambda_2}}=\begin{cases}\dfrac{1}{(x+y)^{\lambda+\alpha}y^{1-(\lambda_2+\alpha)}}, & 0<y<x\\ \dfrac{x^\alpha}{(x+y)^{\lambda+\alpha}y^{1-\lambda_2}}, & y\geq x\end{cases}$$

is strictly decreasing with respect to $y>0$. Similarly, for $\lambda_1\leq 1$ and fixed $y>0$,

$$k_\lambda(x,y)\frac{1}{x^{1-\lambda_1}}$$

is strictly decreasing with respect to $x>0$.

Lemma 2.2 *If* $0\leq-\alpha<\lambda_1,\lambda_2\leq 1, k_\alpha(\lambda_1)$ *is defined as in* (7), *and we define the following weight coefficients*:

$$\omega(\lambda_2,m):=\sum_{n=2}^{\infty}\frac{(\min\{\ln U_m,\ln V_n\})^\alpha}{(\ln U_mV_n)^{\lambda+\alpha}}\frac{\upsilon_n\ln^{\lambda_1}U_m}{V_n\ln^{1-\lambda_2}V_n},\quad m\in\mathbf{N}\backslash\{1\},\tag{11}$$

$$\varpi(\lambda_1,n):=\sum_{m=2}^{\infty}\frac{(\min\{\ln U_m,\ln V_n\})^\alpha}{(\ln U_mV_n)^{\lambda+\alpha}}\frac{\mu_m\ln^{\lambda_2}V_n}{U_m\ln^{1-\lambda_1}U_m},\quad n\in\mathbf{N}\backslash\{1\},\tag{12}$$

then the following inequalities hold true:

$$\omega(\lambda_2,m)<k_\alpha(\lambda_1)(-\alpha<\lambda_2\leq 1,\lambda_1>-\alpha;m\in\mathbf{N}\backslash\{1\}),\tag{13}$$

$$\varpi(\lambda_1,n)<k_\alpha(\lambda_1)(-\alpha<\lambda_1\leq 1,\lambda_2>-\alpha;n\in\mathbf{N}\backslash\{1\}).\tag{14}$$

Proof In view of (3), we set

$$\mu(t) := \mu_m,\ t \in (m-1, m]\ (m \in \mathbf{N});\quad \upsilon(t) := \upsilon_n,\ t \in (n-1, n]\ (n \in \mathbf{N}),$$

and

$$U(x) := \int_0^x \mu(t)\,\mathrm{d}t\,(x \geq 0),\quad V(y) := \int_0^y \upsilon(t)\,\mathrm{d}t\,(y \geq 0). \tag{15}$$

Then it follows that

$$U(m) = U_m,\quad V(n) = V_n \qquad (m, n \in \mathbf{N}).$$

For $x \in (m-1, m)$,

$$U'(x) = \mu(x) = \mu_m \quad (m \in \mathbf{N});$$

for $y \in (n-1, n)$,

$$V'(y) = \upsilon(y) = \upsilon_n \quad (n \in \mathbf{N}).$$

Since $V(y)$ is strictly increasing in $(n-1, n](n \in \mathbf{N})$, and $-\alpha < \lambda_2 \leq 1$, $\lambda_1 > -\alpha$, in view of Example 2.1(2), we derive that

$$\begin{aligned}\omega(\lambda_2, m) &= \sum_{n=2}^{\infty} \int_{n-1}^{n} \frac{(\min\{\ln U_m, \ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}} \frac{\ln^{\lambda_1} U_m}{V_n \ln^{1-\lambda_2} V_n} V'(y)\,\mathrm{d}y \\ &< \sum_{n=2}^{\infty} \int_{n-1}^{n} \frac{(\min\{\ln U_m, \ln V(y)\})^{\alpha}}{(\ln U_m V(y))^{\lambda+\alpha}} \frac{\ln^{\lambda_1} U_m}{V(y) \ln^{1-\lambda_2} V(y)} V'(y)\,\mathrm{d}y \\ &= \int_{1}^{\infty} \frac{(\min\{\ln U_m, \ln V(y)\})^{\alpha}}{(\ln U_m V(y))^{\lambda+\alpha}} \frac{\ln^{\lambda_1} U_m}{V(y) \ln^{1-\lambda_2} V(y)} V'(y)\,\mathrm{d}y.\end{aligned}$$

Setting $t = \frac{\ln V(y)}{\ln U_m}$, we obtain that

$$\frac{V'(y)}{V(y)}\,\mathrm{d}y = \ln U_m\,\mathrm{d}t,\quad \ln V(1) = \ln \upsilon_1 = 0\ (\upsilon_1 = 1)$$

and

$$\begin{aligned}\omega(\lambda_2, m) &< \int_0^{\frac{\ln V(\infty)}{\ln U_m}} \frac{(\min\{1, t\})^{\alpha}}{(1+t)^{\lambda+\alpha}} t^{\lambda_2 - 1}\,\mathrm{d}t \\ &\leq \int_0^{\infty} \frac{(\min\{1, t\})^{\alpha}}{(1+t)^{\lambda+\alpha}} t^{\lambda_2 - 1}\,\mathrm{d}t = k_{\alpha}(\lambda_1).\end{aligned}$$

Hence, we obtain (13). Similarly, for $-\alpha < \lambda_1 \leq 1, \lambda_2 > -\alpha$, we have (14). □

Lemma 2.3 *If* $0 \leq -\alpha < \lambda_1, \lambda_2 \leq 1, k_{\alpha}(\lambda_1)$ *is defined in* (7), $U_{\infty} = V_{\infty} = \infty$, *there exist* $m_0, n_0 \in \mathbf{N}$, *such that* $\{\mu_m\}_{m=m_0}^{\infty}$ *and* $\{\upsilon_n\}_{n=n_0}^{\infty}$ *are decreasing, then*:

(i) *for* $m, n \in \mathbf{N}\backslash\{1\}$, *we have*

$$k_\alpha(\lambda_1)(1-\theta(\lambda_2,m))<\omega(\lambda_2,m) \quad (-\alpha<\lambda_2\leq 1, \lambda_1>-\alpha), \tag{16}$$

$$k_\alpha(\lambda_1)(1-\vartheta(\lambda_1,n))<\varpi(\lambda_1,n) \quad (-\alpha<\lambda_1\leq 1, \lambda_2>-\alpha), \tag{17}$$

where

$$\begin{aligned}\theta(\lambda_2,m) &:= \frac{1}{k_\alpha(\lambda_1)}\int_0^{\frac{\ln V_{n_0+1}}{\ln U_m}} \frac{(\min\{t,1\})^\alpha}{(t+1)^{\lambda+\alpha}} t^{\lambda_2-1}\,\mathrm{d}t \\ &= O\left(\frac{1}{\ln^{\alpha+\lambda_2} U_m}\right) \in (0,1),\end{aligned} \tag{18}$$

$$\begin{aligned}\vartheta(\lambda_1,n) &:= \frac{1}{k_\alpha(\lambda_1)}\int_0^{\frac{\ln U_{m_0+1}}{\ln V_n}} \frac{(\min\{t,1\})^\alpha}{(t+1)^{\lambda+\alpha}} t^{\lambda_1-1}\,\mathrm{d}t \\ &= O\left(\frac{1}{\ln^{\alpha+\lambda_1} V_n}\right) \in (0,1);\end{aligned} \tag{19}$$

(ii) *for any* $a>0$, *we have*

$$\sum_{m=2}^{\infty}\frac{\mu_m}{U_m \ln^{1+a} U_m} = \frac{1}{a}\left[\frac{1}{\ln^a U_{m_0+1}} + aO_1(1)\right], \tag{20}$$

$$\sum_{n=2}^{\infty}\frac{\upsilon_n}{V_n \ln^{1+a} V_n} = \frac{1}{a}\left[\frac{1}{\ln^a V_{n_0+1}} + aO_2(1)\right]. \tag{21}$$

Proof Since $\upsilon_n \geq \upsilon_{n+1}$ $(n \geq n_0)$, $-\alpha<\lambda_2\leq 1, \lambda_1>-\alpha$ and $V(\infty)=\infty$, by Example 2.1(2), we have

$$\begin{aligned}\omega(\lambda_2,m) &\geq \sum_{n=n_0+1}^{\infty}\frac{(\min\{\ln U_m, \ln V_n\})^\alpha}{(\ln U_m V_n)^{\lambda+\alpha}}\frac{\upsilon_{n+1}\ln^{\lambda_1} U_m}{V_n \ln^{1-\lambda_2} V_n} \\ &> \sum_{n=n_0+1}^{\infty}\int_n^{n+1}\frac{(\min\{\ln U_m, \ln V(y)\})^\alpha}{(\ln U_m V(y))^{\lambda+\alpha}}\frac{V'(y)\ln^{\lambda_1} U_m}{V(y)\ln^{1-\lambda_2} V(y)}\,\mathrm{d}y \\ &= \int_{n_0+1}^{\infty}\frac{(\min\{\ln U_m, \ln V(y)\})^\alpha}{(\ln U_m V(y))^{\lambda+\alpha}}\frac{V'(y)\ln^{\lambda_1} U_m}{V(y)\ln^{1-\lambda_2} V(y)}\,\mathrm{d}y \quad \left(t=\frac{\ln V(y)}{\ln U_m}\right) \\ &= \int_{\frac{\ln V_{n_0+1}}{\ln U_m}}^{\infty}\frac{(\min\{1,t\})^\alpha}{(1+t)^{\lambda+\alpha}} t^{\lambda_2-1}\,\mathrm{d}t = k_\alpha(\lambda_1)(1-\theta(\lambda_2,m)) > 0.\end{aligned}$$

For $U_m \geq V_{n_0+1}$ $(m \geq 2)$, by Example 2.1(1), we obtain that

$$
\begin{aligned}
0<\theta(\lambda_2,m) &= \frac{1}{k_\alpha(\lambda_1)}\int_0^{\frac{\ln V_{n_0+1}}{\ln U_m}} \frac{(\min\{1,t\})^\alpha}{(1+t)^{\lambda+\alpha}} t^{\lambda_2-1}\,\mathrm{d}t \\
&= \frac{1}{k_\alpha(\lambda_1)}\int_0^{\frac{\ln V_{n_0+1}}{\ln U_m}} \frac{t^\alpha}{(1+t)^{\lambda+\alpha}} t^{\lambda_2-1}\,\mathrm{d}t \\
&\leq \frac{L_\alpha}{k_\alpha(\lambda_1)}\int_0^{\frac{\ln V_{n_0+1}}{\ln U_m}} t^{\alpha+\lambda_2-1}\,\mathrm{d}t = \frac{L_\alpha}{(\alpha+\lambda_2)k_\alpha(\lambda_1)}\left(\frac{\ln V_{n_0+1}}{\ln U_m}\right)^{\alpha+\lambda_2},
\end{aligned}
$$

namely, (16) and (18) follow. Similarly, for $-\alpha<\lambda_1\leq 1, \lambda_2 > -\alpha$, we obtain (17) and (19).

For $a>0$, we deduce that

$$
\begin{aligned}
\sum_{m=2}^{\infty}\frac{\mu_m}{U_m \ln^{1+a} U_m} &= \sum_{m=2}^{m_0+1}\frac{\mu_m}{U_m \ln^{1+a} U_m} + \sum_{m=m_0+2}^{\infty}\frac{\mu_m}{U_m \ln^{1+a} U_m} \\
&< \sum_{m=2}^{m_0+1}\frac{\mu_m}{U_m \ln^{1+a} U_m} + \sum_{m=m_0+2}^{\infty}\int_{m-1}^{m}\frac{U'(x)}{U(x)\ln^{1+a} U(x)}\,\mathrm{d}x \\
&= \sum_{m=2}^{m_0+1}\frac{\mu_m}{U_m \ln^{1+a} U_m} + \int_{m_0+1}^{\infty}\frac{U'(x)}{U(x)\ln^{1+a} U(x)}\,\mathrm{d}x \\
&= \frac{1}{a}\left(\frac{1}{\ln^a U_{m_0+1}} + a\sum_{m=2}^{m_0+1}\frac{\mu_m}{U_m \ln^{1+a} U_m}\right),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{m=2}^{\infty}\frac{\mu_m}{U_m \ln^{1+a} U_m} &\geq \sum_{m=m_0+1}^{\infty}\frac{\mu_{m+1}}{U_m \ln^{1+a} U_m} > \sum_{m=m_0+1}^{\infty}\int_{m}^{m+1}\frac{U'(x)\,\mathrm{d}x}{U(x)\ln^{1+a} U(x)} \\
&= \int_{m_0+1}^{\infty}\frac{U'(x)\,\mathrm{d}x}{U(x)\ln^{1+a} U(x)} = \frac{1}{a\ln^a U_{m_0+1}}.
\end{aligned}
$$

Hence we obtain (20). Similarly, we derive (21). □

Lemma 2.4 *If* $0\leq -\alpha<\lambda_1,\lambda_2\leq 1, k_\alpha(\lambda_1)$ *is defined as in* (7), *then for*

$$0<\delta<\min\{\alpha+\lambda_1,\alpha+\lambda_2\},$$

we have

$$k_\alpha(\lambda_1\pm\delta) = k_\alpha(\lambda_1)+o(1)(\delta\to 0^+). \tag{22}$$

Proof For $0<\delta<\min\{\alpha+\lambda_1,\alpha+\lambda_2\}$, we get that

$$
\begin{aligned}
|k_\alpha(\lambda_1+\delta)-k_\alpha(\lambda_1)| &\le \int_0^\infty \frac{(\min\{t,1\})^\alpha t^{\lambda_1-1}|t^\delta-1|}{(t+1)^{\lambda+\alpha}}\,\mathrm{d}t \\
&= \int_0^1 \frac{t^{\alpha+\lambda_1-1}(1-t^\delta)}{(t+1)^{\lambda+\alpha}}\,\mathrm{d}t + \int_0^1 \frac{t^{\alpha+\lambda_2-1}(t^{-\delta}-1)}{(t+1)^{\lambda+\alpha}}\,\mathrm{d}t \\
&\le L_\alpha\left[\int_0^1 t^{\alpha+\lambda_1-1}(1-t^\delta)\,\mathrm{d}t + \int_0^1 t^{\alpha+\lambda_2-1}(t^{-\delta}-1)\,\mathrm{d}t\right] \\
&= L_\alpha\left[\frac{1}{\alpha+\lambda_1}-\frac{1}{\alpha+\lambda_1+\delta}+\frac{1}{\alpha+\lambda_2-\delta}-\frac{1}{\alpha+\lambda_2}\right] \\
&\to 0(\delta\to 0^+).
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&|k_\alpha(\lambda_1-\delta)-k_\alpha(\lambda_1)| \\
&\quad\le L_\alpha\left[\frac{1}{\alpha+\lambda_1-\delta}-\frac{1}{\alpha+\lambda_1}+\frac{1}{\alpha+\lambda_2}-\frac{1}{\alpha+\lambda_2+\delta}\right] \\
&\quad\to 0\,(\delta\to 0^+).
\end{aligned}
$$

Hence, we derive (22). □

3 Main results and operator expressions

We also set the following functions:

$$
\begin{aligned}
\widetilde{\Phi}_\lambda(m) &:= \omega(\lambda_2,m)\frac{(\ln U_m)^{p(1-\lambda_1)-1}}{U_m^{1-p}\mu_m^{p-1}}, \\
\widetilde{\Psi}_\lambda(n) &:= \varpi(\lambda_1,n)\frac{(\ln V_n)^{q(1-\lambda_2)-1}}{V_n^{1-q}\upsilon_n^{q-1}} \quad (m,n\in\mathbf{N}\backslash\{1\}).
\end{aligned} \tag{23}
$$

Theorem 3.1 *If* $0\le-\alpha<\lambda_1,\lambda_2\le 1$, *then*:

(i) *for* $p>1$, *we have the following equivalent inequalities*:

$$
I := \sum_{n=2}^\infty\sum_{m=2}^\infty \frac{(\min\{\ln U_m,\ln V_n\})^\alpha}{(\ln U_mV_n)^{\lambda+\alpha}}a_mb_n \le \|a\|_{p,\widetilde{\Phi}_\lambda}\|b\|_{q,\widetilde{\Psi}_\lambda}, \tag{24}
$$

$$
\begin{aligned}
J &:= \left\{\sum_{n=2}^\infty \frac{\upsilon_n\ln^{p\lambda_2-1}V_n}{(\varpi(\lambda_1,n))^{p-1}V_n}\left[\sum_{m=2}^\infty \frac{(\min\{\ln U_m,\ln V_n\})^\alpha}{(\ln U_mV_n)^{\lambda+\alpha}}a_m\right]^p\right\}^{\frac{1}{p}} \\
&\le \|a\|_{p,\widetilde{\Phi}_\lambda};
\end{aligned} \tag{25}
$$

(ii) *for* $0<p<1$ (*or* $p<0$), *we have the equivalent reverses of* (24) *and* (25).

Proof (i) By Hölder's inequality with weight (cf. [11]) and (12), we have

$$\left[\sum_{m=2}^{\infty}\frac{(\min\{\ln U_m,\ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}}a_m\right]^p$$
$$=\left[\sum_{m=2}^{\infty}\frac{(\min\{\ln U_m,\ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}}\right.$$
$$\left.\times\left(\frac{U_m^{1/q}(\ln U_m)^{(1-\lambda_1)/q}}{(\ln V_n)^{(1-\lambda_2)/p}\mu_m^{1/q}}a_m\right)\left(\frac{(\ln V_n)^{(1-\lambda_2)/p}\mu_m^{1/q}}{U_m^{1/q}(\ln U_m)^{(1-\lambda_1)/q}}\right)\right]^p$$
$$\leq\sum_{m=2}^{\infty}\frac{(\min\{\ln U_m,\ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}}\frac{U_m^{p-1}(\ln U_m)^{(1-\lambda_1)p/q}}{(\ln V_n)^{1-\lambda_2}\mu_m^{p/q}}a_m^p \tag{26}$$
$$\times\left[\sum_{m=2}^{\infty}\frac{(\min\{\ln U_m,\ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}}\frac{(\ln V_n)^{(1-\lambda_2)(q-1)}\mu_m}{U_m(\ln U_m)^{1-\lambda_1}}\right]^{p-1}$$
$$=\frac{(\varpi(\lambda_1,n))^{p-1}V_n}{(\ln V_n)^{p\lambda_2-1}\upsilon_n}$$
$$\times\sum_{m=2}^{\infty}\frac{(\min\{\ln U_m,\ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}}\frac{\upsilon_n U_m^{p-1}(\ln U_m)^{(1-\lambda_1)(p-1)}a_m^p}{V_n(\ln V_n)^{1-\lambda_2}\mu_m^{p-1}}.$$

Hence by (11), we deduce that

$$J\leq\left[\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{(\min\{\ln U_m,\ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}}\frac{\upsilon_n U_m^{p-1}(\ln U_m)^{(1-\lambda_1)(p-1)}}{V_n(\ln V_n)^{1-\lambda_2}\mu_m^{p-1}}a_m^p\right]^{\frac{1}{p}}$$
$$=\left[\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{(\min\{\ln U_m,\ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}}\frac{\upsilon_n(\ln U_m)^{\lambda_1}}{V_n(\ln V_n)^{1-\lambda_2}}\frac{(\ln U_m)^{p(1-\lambda_1)-1}}{U_m^{1-p}\mu_m^{p-1}}a_m^p\right]^{\frac{1}{p}} \tag{27}$$
$$=\left[\sum_{m=2}^{\infty}\omega(\lambda_2,m)\frac{(\ln U_m)^{p(1-\lambda_1)-1}}{U_m^{1-p}\mu_m^{p-1}}a_m^p\right]^{\frac{1}{p}},$$

and then (25) follows.

By Hölder's inequality (cf. [11]), we have

$$\begin{aligned} I=\sum_{n=2}^{\infty}&\left[\frac{(\ln V_n)^{\lambda_2-\frac{1}{p}}\upsilon_n^{1/p}}{(\varpi(\lambda_1,n))^{\frac{1}{q}}V_n^{\frac{1}{p}}}\sum_{m=2}^{\infty}\frac{(\min\{\ln U_m,\ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}}a_m\right]\\ &\times\left[(\varpi(\lambda_1,n))^{\frac{1}{q}}\frac{(\ln V_n)^{\frac{1}{p}-\lambda_2}}{V_n^{\frac{-1}{p}}\upsilon_n^{\frac{1}{p}}}b_n\right]\leq J\|b\|_{q,\widetilde{\Psi}_\lambda}. \end{aligned} \tag{28}$$

Then by (25), we derive (24).

On the other hand, assuming that (24) holds true, we set

$$b_n:=\frac{(\ln V_n)^{p\lambda_2-1}\upsilon_n}{(\varpi(\lambda_1,n))^{p-1}V_n}\left[\sum_{m=2}^{\infty}\frac{(\min\{\ln U_m,\ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}}a_m\right]^{p-1},\quad n\in\mathbf{N}\backslash\{1\}. \tag{29}$$

Then we obtain that $J^p = \|b\|^q_{q,\widetilde{\Psi}_\lambda}$. If $J = 0$, then (25) is trivially valid; if $J = \infty$, then by (27), (25) takes the form of equality ($= \infty$). Suppose that $0 < J < \infty$. By (24), it follows that

$$\|b\|^q_{q,\widetilde{\Psi}_\lambda} = J^p = I \leq \|a\|_{p,\widetilde{\Phi}_\lambda} \|b\|_{q,\widetilde{\Psi}_\lambda}, \tag{30}$$

$$\|b\|^{q-1}_{q,\widetilde{\Psi}_\lambda} = J \leq \|a\|_{p,\widetilde{\Phi}_\lambda}, \tag{31}$$

and then (25) follows, which is equivalent to (24).

(ii) For $0 < p < 1$ (or $p < 0$), by the reverse Hölder inequality with weight (cf. [11]) and (12), we obtain the reverse of (26) (or (26)). Then by (11), we derive the reverse of (27), and thus the reverse of (25) follows. By Hölder's inequality (cf. [11]), we obtain the reverse of (28) and then by the reverse of (25), we deduce the reverse of (24).

On the other hand, assuming that the reverse of (24) holds true, we set b_n as in (29). Then we obtain that $J^p = \|b\|^q_{q,\widetilde{\Psi}_\lambda}$. If $J = \infty$, then the reverse of (25) is trivially valid; if $J = 0$, then by the reverse of (27), (25) takes the form of equality ($= 0$). Suppose that $0 < J < \infty$. By the reverse of (24), it follows that the reverses of (30) and (31) are valid and then the reverse of (25) follows, which is equivalent to the reverse of (24). □

Theorem 3.2 *If* $0 \leq -\alpha < \lambda_1, \lambda_2 \leq 1, k_\alpha(\lambda_1)$ *is defined as in* (8), *there exist* $m_0, n_0 \in \mathbf{N}$, *such that* $\{\mu_m\}_{m=m_0}^\infty$ *and* $\{\upsilon_n\}_{n=n_0}^\infty$ *are decreasing*, $U_\infty = V_\infty = \infty$, *then for* $p > 1$, $\|a\|_{p,\Phi_\lambda} \in \mathbf{R}_+$ *and* $\|b\|_{q,\Psi_\lambda} \in \mathbf{R}_+$, *we have the following equivalent inequalities*:

$$\sum_{n=2}^\infty \sum_{m=2}^\infty \frac{(\min\{\ln U_m, \ln V_n\})^\alpha}{(\ln U_m V_n)^{\lambda+\alpha}} a_m b_n < k_\alpha(\lambda_1) \|a\|_{p,\Phi_\lambda} \|b\|_{q,\Psi_\lambda}, \tag{32}$$

$$\begin{aligned} J_1 := \left\{ \sum_{n=2}^\infty \frac{\upsilon_n \ln^{p\lambda_2 - 1} V_n}{V_n} \left[\sum_{m=2}^\infty \frac{(\min\{\ln U_m, \ln V_n\})^\alpha}{(\ln U_m V_n)^{\lambda+\alpha}} a_m \right]^p \right\}^{\frac{1}{p}} \\ < k_\alpha(\lambda_1) \|a\|_{p,\Phi_\lambda}, \end{aligned} \tag{33}$$

where the constant factor $k_\alpha(\lambda_1)$ *is the best possible.*

Proof Applying (13) and (14) in (24) and (25), since

$$(\omega(\lambda_2, m))^{\frac{1}{p}} < (k_\alpha(\lambda_1))^{\frac{1}{p}} \quad (p > 1), \qquad (\varpi(\lambda_1, n))^{\frac{1}{q}} < (k_\alpha(\lambda_1))^{\frac{1}{q}} \quad (q > 1),$$

and

$$\frac{1}{(k_\alpha(\lambda_1))^{p-1}} < \frac{1}{(\varpi(\lambda_1, n))^{p-1}} \quad (p > 1),$$

by simplification, we obtain the equivalent inequalities (32) and (33).

For $\varepsilon \in (0, q(\alpha+\lambda_2))$, we set $\widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}\ (>-\alpha)$, $\widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}\ (\in(-\alpha,1))$, and $\widetilde{a} = \{\widetilde{a}_m\}_{m=2}^{\infty}$, $\widetilde{b} = \{\widetilde{b}_n\}_{n=2}^{\infty}$,

$$\begin{aligned}\widetilde{a}_m &:= \frac{\mu_m}{U_m}\ln^{\widetilde{\lambda}_1-\varepsilon-1} U_m = \frac{\mu_m}{U_m}\ln^{\lambda_1-\frac{\varepsilon}{p}-1} U_m,\\ \widetilde{b}_n &= \frac{\upsilon_n}{V_n}\ln^{\widetilde{\lambda}_2-1} V_n = \frac{\upsilon_n}{V_n}\ln^{\lambda_2-\frac{\varepsilon}{q}-1} V_n.\end{aligned} \tag{34}$$

Then by (20), (21) and (19), we have

$$\begin{aligned}\|\widetilde{a}\|_{p,\Phi_\lambda}\|\widetilde{b}\|_{q,\Psi_\lambda} &= \left(\sum_{m=2}^{\infty}\frac{\mu_m}{U_m\ln^{1+\varepsilon} U_m}\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\frac{\upsilon_n}{V_n\ln^{1+\varepsilon} V_n}\right)^{\frac{1}{q}}\\ &= \frac{1}{\varepsilon}\left[\frac{1}{\ln^{\varepsilon} U_{m_0+1}} + \varepsilon O_1(1)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^{\varepsilon} V_{n_0+1}} + \varepsilon O_2(1)\right]^{\frac{1}{q}},\\ \widetilde{I} &:= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{(\min\{\ln U_m, \ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}}\widetilde{a}_m\widetilde{b}_n\\ &= \sum_{m=2}^{\infty}\left[\sum_{n=2}^{\infty}\frac{(\min\{\ln U_m, \ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}}\frac{\upsilon_n\ln^{\widetilde{\lambda}_1} U_m}{V_n\ln^{1-\widetilde{\lambda}_2} V_n}\right]\frac{\mu_m}{U_m\ln^{\varepsilon+1} U_m}\\ &= \sum_{m=2}^{\infty}\omega(\widetilde{\lambda}_2, m)\frac{\mu_m}{U_m\ln^{\varepsilon+1} U_m}\\ &\geq k_\alpha(\widetilde{\lambda}_1)\sum_{m=2}^{\infty}\left(1 - O\left(\frac{1}{\ln^{\widetilde{\lambda}_2+\alpha} U_m}\right)\right)\frac{\mu_m}{U_m\ln^{\varepsilon+1} U_m}\\ &= k_\alpha(\widetilde{\lambda}_1)\left[\sum_{m=2}^{\infty}\frac{\mu_m}{U_m\ln^{\varepsilon+1} U_m} - \sum_{m=2}^{\infty} O\left(\frac{\mu_m}{U_m(\ln U_m)^{(\frac{\varepsilon}{p}+\alpha+\lambda_2)+1}}\right)\right]\\ &= \frac{1}{\varepsilon}k_\alpha\left(\lambda_1 + \frac{\varepsilon}{q}\right)\left[\frac{1}{\ln^{\varepsilon} U_{m_0+1}} + \varepsilon(O_1(1) - O(1))\right].\end{aligned}$$

If there exists a positive constant $K \leq k_\alpha(\lambda_1)$, such that (32) is valid when replacing $k_\alpha(\lambda_1)$ by K, then in particular, we have $\varepsilon\widetilde{I} < \varepsilon K\|\widetilde{a}\|_{p,\Phi_\lambda}\|\widetilde{b}\|_{q,\Psi_\lambda}$, namely

$$\begin{aligned}&k_\alpha\left(\lambda_1 + \frac{\varepsilon}{q}\right)\left[\frac{1}{\ln^{\varepsilon} U_{m_0+1}} + \varepsilon(O_1(1) - O(1))\right]\\ &\quad < K\left[\frac{1}{\ln^{\varepsilon} U_{m_0+1}} + \varepsilon O_1(1)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^{\varepsilon} V_{n_0+1}} + \varepsilon O_2(1)\right]^{\frac{1}{q}}.\end{aligned} \tag{35}$$

In view of (22), it follows that $k_\alpha(\lambda_1) \leq K\ (\varepsilon \to 0^+)$. Hence, $K = k_\alpha(\lambda_1)$ is the best possible constant factor of (32).

Similarly to (28), we can still obtain that

$$I \leq J_1\|b\|_{q,\Psi_\lambda}. \tag{36}$$

Hence, we can prove that the constant factor $k_\alpha(\lambda_1)$ in (33) is the best possible. Otherwise, we would reach a contradiction by (36) that the constant factor in (32) is not the best possible.

For $p > 1$,

$$\Psi_\lambda^{1-p}(n) = \frac{\upsilon_n}{V_n}(\ln V_n)^{p\lambda_2 - 1} \quad (n \in \mathbf{N}\backslash\{1\}),$$

we define the following normed spaces:

$$l_{p,\Phi_\lambda} := \{a = \{a_m\}_{m=2}^\infty; \|a\|_{p,\Phi_\lambda} < \infty\},$$
$$l_{q,\Psi_\lambda} := \{b = \{b_n\}_{n=2}^\infty; \|b\|_{q,\Psi_\lambda} < \infty\},$$
$$l_{p,\Psi_\lambda^{1-p}} := \{c = \{c_n\}_{n=2}^\infty; \|c\|_{p,\Psi_\lambda^{1-p}} < \infty\}.$$

Assuming that $a = \{a_m\}_{m=2}^\infty \in l_{p,\Phi_\lambda}$, setting

$$c = \{c_n\}_{n=2}^\infty, \quad c_n := \sum_{m=2}^\infty \frac{(\min\{\ln U_m, \ln V_n\})^\alpha}{(\ln U_m V_n)^{\lambda+\alpha}} a_m, \ n \in \mathbf{N}\backslash\{1\},$$

we can rewrite (33) as follows:

$$\|c\|_{p,\Psi_\lambda^{1-p}} < k_\alpha(\lambda_1)\|a\|_{p,\Phi_\lambda} < \infty,$$

namely, $c \in l_{p,\Psi_\lambda^{1-p}}$.

Definition 3.3 Define a Hardy–Mulholland-type operator $T : l_{p,\Phi_\lambda} \to l_{p,\Psi_\lambda^{1-p}}$ as follows: For any $a = \{a_m\}_{m=2}^\infty \in l_{p,\Phi_\lambda}$, there exists a unique representation $Ta = c \in l_{p,\Psi_\lambda^{1-p}}$. Define the formal inner product of Ta and $b = \{b_n\}_{n=2}^\infty \in l_{q,\Psi_\lambda}$ as follows:

$$(Ta, b) := \sum_{n=2}^\infty \left[\sum_{m=2}^\infty \frac{(\min\{\ln U_m, \ln V_n\})^\alpha}{(\ln U_m V_n)^{\lambda+\alpha}} a_m\right] b_n. \tag{37}$$

Then we can rewrite (32) and (33) as bellow:

$$(Ta, b) < k_\alpha(\lambda_1)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{38}$$

$$\|Ta\|_{p,\Psi_\lambda^{1-p}} < k_\alpha(\lambda_1)\|a\|_{p,\Phi_\lambda}. \tag{39}$$

Define the norm of operator T as follows:

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\Phi_\lambda}} \frac{\|Ta\|_{p,\Psi_\lambda^{1-p}}}{\|a\|_{p,\Phi_\lambda}}.$$

Then by (39), we derive that $\|T\| \leq k_\alpha(\lambda_1)$. Since the constant factor in (39) is the best possible, we have

$$\|T\| = k_\alpha(\lambda_1) = \sum_{i=1}^{2} \frac{1}{\lambda_i + \alpha} F(\lambda + \alpha, \lambda_i + \alpha, \lambda_i + \alpha + 1, -1). \tag{40}$$

4 Some equivalent reverses

In the following, we also set

$$\begin{aligned} \widetilde{\Omega}_\lambda(m) &:= (1 - \theta(\lambda_2, m)) \frac{(\ln U_m)^{p(1-\lambda_1)-1}}{U_m^{1-p} \mu_m^{p-1}}, \\ \widetilde{\Upsilon}_\lambda(n) &:= (1 - \vartheta(\lambda_1, n)) \frac{(\ln V_n)^{q(1-\lambda_2)-1}}{V_n^{1-q} \upsilon_n^{q-1}} \quad (m, n \in \mathbf{N}\backslash\{1\}). \end{aligned} \tag{41}$$

For $0 < p < 1$ or $p < 0$, we still use $\|a\|_{p,\Phi_\lambda}$, $\|b\|_{q,\Psi_\lambda}$, $\|a\|_{p,\widetilde{\Omega}_\lambda}$ and $\|b\|_{q,\widetilde{\Upsilon}_\lambda}$ as the formal symbols.

Theorem 4.1 *With regard to the assumptions of Theorem 3.2, if* $0 < p < 1$, $\|a\|_{p,\Phi_\lambda} \in \mathbf{R}_+$ *and* $\|b\|_{q,\Psi_\lambda} \in \mathbf{R}_+$, *then we have the following equivalent inequalities with the best possible constant factor* $k_\alpha(\lambda_1)$:

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{(\min\{\ln U_m, \ln V_n\})^\alpha}{(\ln U_m V_n)^{\lambda+\alpha}} a_m b_n > k_\alpha(\lambda_1) \|a\|_{p,\widetilde{\Omega}_\lambda} \|b\|_{q,\Psi_\lambda}, \tag{42}$$

$$\left\{ \sum_{n=2}^{\infty} \frac{\upsilon_n \ln^{p\lambda_2 - 1} V_n}{V_n} \left[\sum_{m=2}^{\infty} \frac{(\min\{\ln U_m, \ln V_n\})^\alpha}{(\ln U_m V_n)^{\lambda+\alpha}} a_m \right]^p \right\}^{\frac{1}{p}} > k_\alpha(\lambda_1) \|a\|_{p,\widetilde{\Omega}_\lambda}. \tag{43}$$

Proof Using (16) and (14) in the reverses of (24) and (25), since

$$\begin{aligned} &(\omega(\lambda_2, m))^{\frac{1}{p}} > (k_\alpha(\lambda_1))^{\frac{1}{p}} (1 - \theta(\lambda_2, m))^{\frac{1}{p}} \quad (0 < p < 1), \\ &(\varpi(\lambda_1, n))^{\frac{1}{q}} > (k_\alpha(\lambda_1))^{\frac{1}{q}} \quad (q < 0), \end{aligned}$$

and

$$\frac{1}{(k_\alpha(\lambda_1))^{p-1}} > \frac{1}{(\varpi(\lambda_1,n))^{p-1}} \quad (0<p<1),$$

by simplification, we obtain the equivalent inequalities (42) and (43).

For $\varepsilon \in (0, p(\alpha+\lambda_1))$, we set

$$\begin{aligned}
&\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}\,(\in(-\alpha,1)), \quad \widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}\,(>-\alpha), \quad \text{and}\\
&\widetilde{a} = \{\widetilde{a}_m\}_{m=2}^{\infty}, \quad \widetilde{b} = \{\widetilde{b}_n\}_{n=2}^{\infty},\\
&\widetilde{a}_m := \frac{\mu_m}{U_m}\ln^{\widetilde{\lambda}_1-1} U_m = \frac{\mu_m}{U_m}\ln^{\lambda_1-\frac{\varepsilon}{p}-1} U_m,\\
&\widetilde{b}_n = \frac{\upsilon_n}{V_n}\ln^{\widetilde{\lambda}_2-\varepsilon-1} V_n = \frac{\upsilon_n}{V_n}\ln^{\lambda_2-\frac{\varepsilon}{q}-1} V_n.
\end{aligned}$$

Then by (20), (21) and (14), we obtain that

$$\begin{aligned}
&\|a\|_{p,\widetilde{\Omega}_\lambda}\|b\|_{q,\Psi_\lambda}\\
&= \left[\sum_{m=2}^{\infty}(1-\theta(\lambda_2,m))\frac{\mu_m}{U_m\ln^{1+\varepsilon}U_m}\right]^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\frac{\upsilon_n}{V_n\ln^{1+\varepsilon}V_n}\right)^{\frac{1}{q}}\\
&= \left(\sum_{m=2}^{\infty}\frac{\mu_m}{U_m\ln^{1+\varepsilon}U_m} - \sum_{m=2}^{\infty}O(\frac{\mu_m}{U_m\ln^{1+\alpha+\lambda_2+\varepsilon}U_m})\right)^{\frac{1}{p}}\\
&\quad\times\left(\sum_{n=2}^{\infty}\frac{\upsilon_n}{V_n\ln^{1+\varepsilon}V_n}\right)^{\frac{1}{q}}\\
&= \frac{1}{\varepsilon}\left[\frac{1}{\ln^{\varepsilon}U_{m_0+1}} + \varepsilon(O_1(1)-O(1))\right]^{\frac{1}{p}}\left[\frac{1}{\ln^{\varepsilon}V_{n_0+1}} + \varepsilon O_2(1)\right]^{\frac{1}{q}},\\
&\widetilde{I} := \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{(\min\{\ln U_m,\ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}}\widetilde{a}_m\widetilde{b}_n\\
&= \sum_{n=2}^{\infty}\left[\sum_{m=2}^{\infty}\frac{(\min\{\ln U_m,\ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}}\frac{\mu_m\ln^{\widetilde{\lambda}_2}V_n}{U_m\ln^{1-\widetilde{\lambda}_1}U_m}\right]\frac{\upsilon_n}{V_n\ln^{\varepsilon+1}V_n}\\
&= \sum_{n=2}^{\infty}\varpi(\widetilde{\lambda}_1,n)\frac{\upsilon_n}{V_n\ln^{\varepsilon+1}V_n} \le k_\alpha(\widetilde{\lambda}_1)\sum_{n=2}^{\infty}\frac{\upsilon_n}{V_n\ln^{\varepsilon+1}V_n}\\
&= \frac{1}{\varepsilon}k_\alpha\left(\lambda_1-\frac{\varepsilon}{p}\right)\left[\frac{1}{\ln^{\varepsilon}V_{n_0+1}} + \varepsilon O_2(1)\right].
\end{aligned}$$

If there exists a positive constant $K \ge k_\alpha(\lambda_1)$, such that (42) is valid when replacing $k_\alpha(\lambda_1)$ by K, then in particular, we have $\varepsilon\widetilde{I} > \varepsilon K\|\widetilde{a}\|_{p,\widetilde{\Omega}_\lambda}\|\widetilde{b}\|_{q,\Psi_\lambda}$, namely

$$k_{\alpha}\left(\lambda_1 - \frac{\varepsilon}{p}\right)\left[\frac{1}{\ln^{\varepsilon} V_{n_0+1}} + \varepsilon O_2(1)\right]$$
$$> K\left[\frac{1}{\ln^{\varepsilon} U_{m_0+1}} + \varepsilon(O_1(1) - O(1))\right]^{\frac{1}{p}}\left[\frac{1}{\ln^{\varepsilon} V_{n_0+1}} + \varepsilon O_2(1)\right]^{\frac{1}{q}}.$$

In view of (22), it follows that $k_{\alpha}(\lambda_1) \geq K\ (\varepsilon \to 0^+)$. Hence, $K = k_{\alpha}(\lambda_1)$ is the best possible constant factor of (42). The constant factor $k_{\alpha}(\lambda_1)$ in (43) is still the best possible. Otherwise, we would reach a contradiction by the reverse of (36) that the constant factor in (42) is not the best possible.

Theorem 4.2 *With regard to the assumptions of Theorem 3.2, if* $p<0$, $\|a\|_{p,\Phi_{\lambda}} \in \mathbf{R}_+$ *and* $\|b\|_{q,\Psi_{\lambda}} \in \mathbf{R}_+$, *then we have the following equivalent inequalities with the best possible constant factor* $B(\lambda_1, \lambda_2)$:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{(\min\{\ln U_m, \ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}} a_m b_n > k_{\alpha}(\lambda_1)\|a\|_{p,\Phi_{\lambda}}\|b\|_{q,\widetilde{\Upsilon}_{\lambda}}, \tag{44}$$

$$J_2 := \left\{\sum_{n=2}^{\infty}\frac{\upsilon_n \ln^{p\lambda_2-1} V_n}{(1-\vartheta(\lambda_1,n))^{p-1} V_n}\left[\sum_{m=2}^{\infty}\frac{(\min\{\ln U_m, \ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}} a_m\right]^p\right\}^{\frac{1}{p}} > k_{\alpha}(\lambda_1)\|a\|_{p,\Phi_{\lambda}}. \tag{45}$$

Proof Using (13) and (17) in the reverses of (24) and (25), since

$$(\omega(\lambda_2, m))^{\frac{1}{p}} > (k_{\alpha}(\lambda_1))^{\frac{1}{p}} \quad (p<0),$$
$$(\varpi(\lambda_1, n))^{\frac{1}{q}} > (k_{\alpha}(\lambda_1))^{\frac{1}{q}}(1-\vartheta(\lambda_1,n))^{\frac{1}{q}} \quad (0<q<1),$$

and

$$\left[\frac{1}{(k_{\alpha}(\lambda_1))^{p-1}(1-\vartheta(\lambda_1,n))^{p-1}}\right]^{\frac{1}{p}} > \left[\frac{1}{(\varpi(\lambda_1,n))^{p-1}}\right]^{\frac{1}{p}} \quad (p<0),$$

by simplification, we obtain equivalent inequalities (44) and (45).

For $\varepsilon \in (0, q(\alpha+\lambda_2))$, we set

$$\widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}\ (> -\alpha), \quad \widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}\ (\in(-\alpha, 1)), \quad \text{and} \quad \widetilde{a} = \{\widetilde{a}_m\}_{m=2}^{\infty}, \quad \widetilde{b} = \{\widetilde{b}_n\}_{n=2}^{\infty},$$
$$\widetilde{a}_m = \frac{\mu_m}{U_m}\ln^{\widetilde{\lambda_1-\varepsilon}-1} U_m = \frac{\mu_m}{U_m}\ln^{\lambda_1-\frac{\varepsilon}{p}-1} U_m,$$
$$\widetilde{b}_n = \frac{\upsilon_n}{V_n}\ln^{\widetilde{\lambda_2}-1} V_n = \frac{\upsilon_n}{V_n}\ln^{\lambda_2-\frac{\varepsilon}{q}-1} V_n.$$

Then by (20), (21) and (12), we have

$$\begin{aligned}
&\|\widetilde{a}\|_{p,\Phi_\lambda}\|\widetilde{b}\|_{q,\widetilde{q}_\lambda}\\
&= \left(\sum_{m=2}^{\infty}\frac{\mu_m}{U_m \ln^{\varepsilon+1} U_m}\right)^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}(1-\vartheta(\lambda_1,n))\frac{\upsilon_n}{V_n \ln^{\varepsilon+1} V_n}\right]^{\frac{1}{q}}\\
&= \left(\sum_{m=2}^{\infty}\frac{\mu_m}{U_m \ln^{\varepsilon+1} U_m}\right)^{\frac{1}{p}}\\
&\quad\times\left[\sum_{n=2}^{\infty}\frac{\upsilon_n}{V_n \ln^{\varepsilon+1} V_n}-\sum_{n=2}^{\infty}O\left(\frac{\upsilon_n}{V_n \ln^{1+(\alpha+\lambda_1+\varepsilon)} V_n}\right)\right]^{\frac{1}{q}}\\
&= \frac{1}{\varepsilon}\left[\frac{1}{\ln^{\varepsilon} U_{m_0+1}}+\varepsilon O_1(1)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^{\varepsilon} V_{n_0+1}}+\varepsilon(O_2(1)-O(1))\right]^{\frac{1}{q}},\\
\widetilde{I} &= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{(\min\{\ln U_m,\ln V_n\})^{\alpha}}{(\ln U_m V_n)^{\lambda+\alpha}}\widetilde{a}_m\widetilde{b}_n\\
&= \sum_{m=2}^{\infty}\left[\sum_{n=2}^{\infty}\frac{\ln^{\widetilde{\lambda}_1} U_m}{\ln^{\lambda}(U_m V_n)}\frac{\upsilon_n}{V_n}\ln^{\widetilde{\lambda}_2-1} V_n\right]\frac{\mu_m}{U_m \ln^{\varepsilon+1} U_m}\\
&= \sum_{m=2}^{\infty}\omega(\widetilde{\lambda}_2,m)\frac{\mu_m}{U_m \ln^{\varepsilon+1} U_m}\le k_\alpha(\widetilde{\lambda}_1)\sum_{n=2}^{\infty}\frac{\mu_m}{U_m \ln^{\varepsilon+1} U_m}\\
&= \frac{1}{\varepsilon}k_\alpha\left(\lambda_1+\frac{\varepsilon}{q}\right)\left[\frac{1}{\ln^{\varepsilon} U_{m_0+1}}+\varepsilon O_1(1)\right].
\end{aligned}$$

If there exists a positive constant $K \ge k_\alpha(\lambda_1)$, such that (44) is satisfied when replacing $k_\alpha(\lambda_1)$ by K, then in particular, we have $\varepsilon\widetilde{I} > \varepsilon K\|\widetilde{a}\|_{p,\Phi_\lambda}\|\widetilde{b}\|_{q,\widetilde{\Upsilon}_\lambda}$, namely

$$\begin{aligned}
k_\alpha\left(\lambda_1+\frac{\varepsilon}{q}\right)\left[\frac{1}{\ln^{\varepsilon} U_{m_0+1}}+\varepsilon O_1(1)\right] &> K\left[\frac{1}{\ln^{\varepsilon} U_{m_0+1}}+\varepsilon O_1(1)\right]^{\frac{1}{p}}\\
&\times\left[\frac{1}{\ln^{\varepsilon} V_{n_0+1}}+\varepsilon(O_2(1)-O(1))\right]^{\frac{1}{q}}.
\end{aligned}$$

In view of (22), it follows that $k_\alpha(\lambda_1) \ge K(\varepsilon \to 0^+)$. Hence, $K = k_\alpha(\lambda_1)$ is the best possible constant factor of (44). Similarly to the reverse of (28), we still obtain that

$$I \ge J_2\|b\|_{q,\widetilde{\Upsilon}_\lambda}. \tag{46}$$

Hence, the constant factor $k_\alpha(\lambda_1)$ in (45) is still the best possible. Otherwise, we would reach a contradiction by (46) that the constant factor in (44) is not the best possible. □

***Remark* 4.3** For $\alpha = 0$ in (32), by (9), we have

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{(\ln U_m V_n)^{\lambda}} < B(\lambda_1, \lambda_2)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}. \tag{47}$$

For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}, \mu_i = \upsilon_i = 1 (i \in \mathbf{N})$ in (47), we deduce (2). Hence, (47) is an extension of (2); so is (32).

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Affiliations

Michael Th. Rassias[1,2,3] · Bicheng Yang[4] · Andrei Raigorodskii[2,5,6,7]

Bicheng Yang
bcyang@gdei.edu.cn; bcyang818@163.com

Andrei Raigorodskii
mraigor@yandex.ru

[1] Institute of Mathematics, University of Zurich, 8057 Zurich, Switzerland

[2] Moscow Institute of Physics and Technology, Institutskiy per, d. 9, 141700 Dolgoprudny, Russia

[3] Program in Interdisciplinary Studies, Institute for Advanced Study, 1 Einstein Dr, Princeton, NJ 08540, USA

[4] Department of Mathematics, Guangdong University of Education, Guangzhou 510303, Guangdong, People's Republic of China

[5] Moscow State University, 119991 Moscow, Russia

[6] Buryat State University, Ulan-Ude, Russia

[7] Caucasus Mathematical Center, Adyghe State University, Maykop, Russia