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Sharp energy estimates for finite element  
approximations of non-convex problems

by

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# SHARP ENERGY ESTIMATES FOR FINITE ELEMENT APPROXIMATIONS OF NON-CONVEX PROBLEMS

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## 1. Introduction

The goal of this note is to expose in a simple situation the key arguments which allow one to prove sharp energy estimates in the numerical analysis of problems with multiple-well energy in the calculus of variations. Let us recall that such problems arise naturally for instance in materials science. We refer the reader for this matter to [1], [5], [6], [7], [10]. Note also that our technique borrows widely from [8], [9] and can be extended to more general situations - see [3].

So, we will assume that we are in dimension 2 and  $\Omega$  will denote the square  $(0, 1) \times (0, 1)$  with boundary  $\partial\Omega$ . Setting

$$W_0^{1,\infty}(\Omega) = \{v : \Omega \rightarrow \mathbf{R} \mid v \text{ is Lipschitz continuous, } v = 0 \text{ on } \partial\Omega\}$$

we would like to consider the problem

$$\inf_{W_0^{1,\infty}(\Omega)} \int_{\Omega} W(\nabla u) \, dx \, dy = \inf_{W_0^{1,\infty}(\Omega)} \int_{\Omega} |u_x| + ||u_y| - 1| \, dx \, dy \quad (1.1)$$

where we have set

$$\nabla u = (u_x, u_y) \quad , \quad W(\xi_1, \xi_2) = |\xi_1| + ||\xi_2| - 1|. \quad (1.2)$$

More precisely we would like to address a discrete version of this problem. We denote by  $\tau_h$  a regular (see [4]) family of triangulations of  $\Omega$ , with mesh

size  $h$ , i.e.

$$h = \max\{\text{diam}(K) \mid K \in \tau_h\}.$$

Then, if  $P_1$  denotes the set of polynomials of degree one, we set

$$V_0^h = \{v : \Omega \rightarrow \mathbf{R} \mid v \text{ continuous, } v|_k \in P_1 \ \forall k \in \tau_h, v = 0 \text{ on } \partial\Omega\}$$

and we consider the discretized analogue of (1.1) :

$$\inf_{V_0^h} \int_{\Omega} W(\nabla u) \, dx \, dy. \quad (1.3)$$

The energy density  $W$  is a nonnegative function that vanishes only on the two wells

$$\omega_+ = (0, 1) \quad , \quad \omega_- = (0, -1). \quad (1.4)$$

They play, of course, a crucial role in the minimization process. It is easy to establish that (1.1) does not admit a minimizer but that (1.3) does (see for instance [2]) thus the minimizers of (1.3) provide a minimizing sequence to (1.1). What we would like to establish here is:

**Theorem 1** *There exists a constant  $C$  independent of  $h \ll 1$  such that*

$$\inf_{V_0^h} \int_{\Omega} |u_x| + ||u_y| - 1| \, dx \, dy \leq Ch^{\frac{1}{2}}. \quad (1.5)$$

*Moreover, this estimate is sharp in the sense that there exists a family of triangulation  $\tau_h$ , a constant  $c$  independent of  $h$  such that*

$$\inf_{V_0^h} \int_{\Omega} |u_x| + ||u_y| - 1| \, dx \, dy \geq ch^{\frac{1}{2}}. \quad (1.6)$$

The rest of the paper will be devoted to the proof of this theorem.

## 2. Proof of Theorem 1

### 2.1. THE PROOF OF (1.5)

Let us denote by  $\alpha \in (0, 1)$  a real number that we will fix later on. Define

$$u(x, y) = \begin{cases} y & \text{if } y \in [0, h^\alpha], \\ 2h^\alpha - y & \text{if } y \in [h^\alpha, 2h^\alpha]. \end{cases} \quad (2.7)$$

Assume that  $u$  is extended periodically - with period  $2h^\alpha$  in the  $y$  direction - on the whole  $\mathbf{R}^2$ . Clearly one has

$$\nabla u = \omega_+ \quad \text{or} \quad \omega_- \quad \text{a.e. in } \Omega. \quad (2.8)$$

Since  $u$  does not vanish on the boundary of  $\Omega$  one sets

$$\hat{u} = u \wedge \text{dist}(x, \partial\Omega)$$

and then

$$u_h = \text{the interpolate of } \hat{u} \text{ on } \tau_h, \quad (2.9)$$

i.e.  $u_h$  denotes the function of  $V_0^h$  that agrees with  $\hat{u}$  at the nodes of the triangulation  $\tau_h$ . The two functions that are involved in the definition of  $\hat{u}$  have a gradient that is uniformly bounded. Since the triangulation  $\tau_h$  is regular, the gradient of  $u_h$  is uniformly bounded independently of  $h$  (see [4]). It then follows from (2.8) that one has

$$\int_{\Omega} W(\nabla u_h) dx dy = \int_{\{u_h \neq u\}} W(\nabla u_h) dx dy \leq C |\{u_h \neq u\}| \quad (2.10)$$

for some constant  $C$ , where  $|\{u_h \neq u\}|$  denotes the measure of the set where  $u_h$  is distinct from  $u$ . First one remarks that

$$0 \leq u \leq h^\alpha$$

(see (2.7)) and thus for  $\text{dist}(x, \partial\Omega) > h^\alpha$  one has  $\hat{u} = u$ . So, for  $\text{dist}(x, \partial\Omega) \geq h^\alpha + h$  the interpolate of  $\hat{u}$  will be equal to the interpolate of  $u$ . Now, the interpolate of  $u$  is  $u$  itself except on a strip of size  $2h$  around each of the lines  $y = kh^\alpha, k \in \mathbf{N}$ . Collecting this information one has clearly

$$|\{u_h \neq u\}| \leq 4(h^\alpha + h) + N 2h \leq 8h^\alpha + (N + 1)2h \quad (2.11)$$

where  $N$  is the number of strips cutting  $\Omega$ . Note that for  $h < 1$  and  $\alpha \in (0, 1)$  one has  $h < h^\alpha$ . Now, one has clearly

$$(N + 1)h^\alpha \leq 1$$

so that by (2.10), (2.11) one gets

$$\int_{\Omega} W(\nabla u_h) dx dy \leq 8C(h^\alpha + h^{1-\alpha}).$$

Taking  $\alpha = \frac{1}{2}$  leads to (1.5).

## 2.2. PROOF OF (1.6)

First remark that if we claim that (1.5) is sharp there are some particular triangulations - related to the wells - for which (1.5) could be improved. Indeed, choose for instance the triangulation of the (figure 1) and set

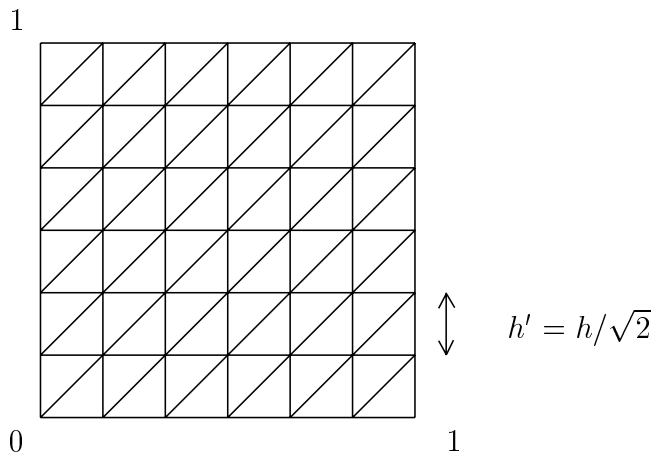


Figure 1.

$$u(x, y) = \begin{cases} y & \text{on } (0, h') \\ 2h' - y & \text{on } (h', 2h'). \end{cases}$$

Extending  $u$  periodically with  $2h'$ , and setting

$$u_h = \text{the interpolate of } u \wedge \text{dist}(x, \partial\Omega)$$

one clearly has

$$\inf_{V_0^h} \int_{\Omega} W(\nabla u) dx dy \leq \int_{\Omega} W(\nabla u_h) dx dy \leq ch \ll ch^{\frac{1}{2}}$$

for  $h$  small. However, this estimate depends on the triangulation as we are about to see. Indeed consider now a family of triangulations as on (figure 2). For  $x_0 = kh/2$ ,  $k \in \mathbf{N}$  we are going to consider the strip  $(x_0, x_0 + \frac{h}{2}) \times (0, 1)$ . Let  $u \in V_0^h$  and  $0 < \delta < \frac{1}{3}$ . Recall that  $\nabla u$  is constant on each of the triangles of  $\tau_h$ . Then for a given  $u$  let us adopt the following definition:

**Definition 1** Let  $T \in \tau_h$  with one side at least of length  $h$ . We will say that  $T$  is of

$$\begin{array}{ll} \text{type +} & \text{if } |u_x| \leq \delta, \quad |u_y - 1| \leq \delta, \\ \text{type -} & \text{if } |u_x| \leq \delta, \quad |u_y + 1| \leq \delta, \\ \text{type 0} & \text{else.} \end{array}$$

In other words  $T$  is of type + or - if  $\nabla u$  is close to  $\omega_+$  or  $\omega_-$  respectively. Then on the strip  $(x_0, x_0 + \frac{h}{2}) \times (0, 1)$  we have triangles of various types. We denote by  $N_0 = N_0(x_0)$  the number of triangles of type 0. We say that we have a change of phase along the line  $x_0$  when two triangles change from type + to type - (see figure 3). Then we can prove,

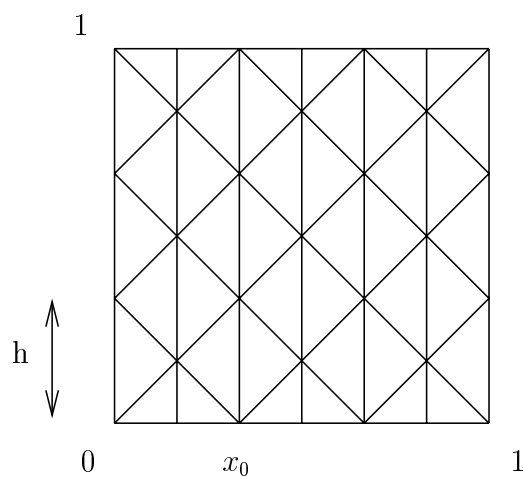


Figure 2.

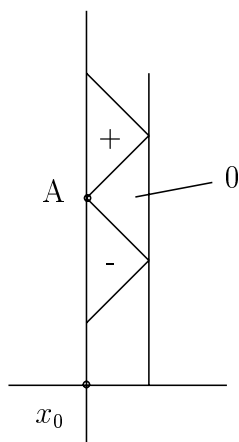


Figure 3.

**Lemma 1** *At each change of phase along the line  $x = x_0$  the triangle having only one vertex on  $x = x_0$  is of type 0, (see figure 3).*

**Proof.** Let us denote by  $(u_x^+, u_y^+)$ ,  $(u_x^-, u_y^-)$  the gradients of  $u$  in the triangle of type + and of type -, respectively. If  $A = (x_0, y_0)$  one has

$$u\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}\right) = u(x_0, y_0) + u_x^+ \frac{h}{2} + u_y^+ \frac{h}{2}$$



$$u\left(x_0 + \frac{h}{2}, y_0 - \frac{h}{2}\right) = u(x_0, y_0) + u_x^- \frac{h}{2} - u_y^- \frac{h}{2}.$$

So, in the triangle whose only vertex on  $x = x_0$  is  $A$  one has

$$u_y = \frac{1}{2}(u_x^+ - u_x^-) + \frac{1}{2}(u_y^+ + u_y^-).$$

Since  $u_x^+, u_x^- \in [-\delta, \delta]$ ,  $u_y^+ \in [1 - \delta, 1 + \delta]$ ,  $u_y^- \in [-1 - \delta, -1 + \delta]$  one deduces that  $|u_y| \leq 2\delta$  so that

$$|u_y \pm 1| \geq 1 - 2\delta > \delta.$$

Since  $0 < \delta < \frac{1}{3}$ . The considered triangle is thus of type 0.

**Lemma 2** *Let  $u$  be a piecewise  $\mathbf{C}^1$  function on  $(a, b)$  such that  $u'$  has a constant sign and  $|u'| \geq C$ . Then one has*

$$\int_a^b |u(z)| dz \geq C \left( \frac{(b-a)^2}{4} \right) \quad (2.12)$$

**Proof.** If  $u$  does not vanish on  $[a, b]$  then clearly

$$\int_a^b |u(z)| dz \geq \min \left( \int_a^b |u(z) - u(a)| dz, \int_a^b |u(z) - u(b)| dz \right)$$

So, considering possibly  $u - u(a)$  or  $u - u(b)$  instead of  $u$  one can assume that  $u$  vanishes at some point  $\xi$ . Since  $u'$  has a constant sign one has

$$|u(z)| = \left| u(\xi) + \int_{\xi}^z u'(z') dz' \right| = \int_{[\xi, z]} |u'(z)| dz \geq C|\xi - z|.$$

Integration in  $z$  on  $(a, b)$  leads to (2.12).

**Lemma 3** *Assume that  $(N_0 + 1)h \leq \frac{1}{2}$  then one has*

$$\int_0^1 |u(x_0, y)| dy \geq \frac{1 - \delta}{16(N_0 + 1)}. \quad (2.13)$$

**Proof.** Consider a maximal chain of triangles of type  $+$  or  $-$ . In other words consider  $(a_i, b_i)$  such that one has a change of phase or a “boundary point” at  $(x_0, a_i)$ ,  $(x_0, b_i)$  and all the sides of triangles of the strip  $(x_0, x_0 + \frac{h}{2}) \times (0, 1)$  located on  $x = x_0$  between  $a_i$  and  $b_i$  belong to triangles of the same type  $+$  or  $-$ . (A “boundary point” could be a vertex of one triangle of the boundary

with all its sides strictly smaller than  $h$ .) Since  $u_y$  has a constant sign and  $|u_y| \geq 1 - \delta$  one deduces from Lemma (2) that

$$\int_{a_i}^{b_i} |u(x_0, y)| dy \geq (1 - \delta) \frac{(b_i - a_i)^2}{4}. \quad (2.14)$$

Denote by  $N_1$  the number of maximal chains as above, i.e.  $i = 1, \dots, N_1$ . Then from (2.14) one deduces

$$\sum_{i=1}^{N_1} \int_{a_i}^{b_i} |u(x_0, y)| dy \geq \frac{(1 - \delta)N_1}{4} \sum_{i=1}^{N_1} \frac{(b_i - a_i)^2}{N_1}$$

and by a convexity argument

$$\int_0^1 |u(x_0, y)| dy \geq \sum_{i=1}^{N_1} \int_{a_i}^{b_i} |u(x_0, y)| dy \geq \frac{(1 - \delta)N_1}{4} \left( \sum_{i=1}^{N_1} \frac{b_i - a_i}{N_1} \right)^2. \quad (2.15)$$

Due to Lemma 1 the sides of triangles on  $x = x_0$  which do not belong to a chain belong to a triangle of type 0 or a small triangle of the boundary. So, one has

$$\sum_{i=1}^{N_1} b_i - a_i \geq 1 - (N_0 + 1)h \geq \frac{1}{2}$$

and from (2.15) we deduce that

$$\int_0^1 |u(x_0, y)| dy \geq \frac{(1 - \delta)}{16N_1}.$$

The result follows from the fact that by Lemma 1,  $N_1 \leq N_0 + 1$ .

**Lemma 4** *Let  $u \in V_0^h$  then one has*

$$\int_0^1 |u(x_0, y)| dy \leq \int_{\Omega} W(\nabla u) dx dy. \quad (2.16)$$

**Proof.** One has

$$u(x_0, y) = u(0, y) + \int_0^{x_0} u_x(\xi, y) d\xi = \int_0^{x_0} u_x(\xi, y) dy.$$

Thus

$$|u(x_0, y)| \leq \int_0^1 |u_x(\xi, y)| d\xi \leq \int_0^1 W(\nabla u(\xi, y)) d\xi$$

and the result follows by integration in  $y$ .

**Lemma 5** (*Estimate of the number of changes of phase*). Let  $u \in V_0^h$  such that

$$\int_{\Omega} W(\nabla u) \, dx \, dy \leq Ch^{\frac{1}{2}} \quad (2.17)$$

where  $C$  is a constant. If  $h \leq 64C^2$  then one has

$$N_0 + 1 \geq \frac{1 - \delta}{16C} h^{-\frac{1}{2}}. \quad (2.18)$$

(Recall that  $N_0$  is the number of triangles of type 0 in the strip  $(x_0, x_0 + \frac{h}{2}) \times (0, 1)$ .)

**Proof.** If (2.18) does not hold one has

$$(N_0 + 1)h < \frac{(1 - \delta)h^{\frac{1}{2}}}{16C} < \frac{h^{\frac{1}{2}}}{16C} \leq \frac{1}{2}.$$

Thus combining (2.13), (2.16) and (2.17) one obtains

$$\frac{1 - \delta}{16(N_0 + 1)} \leq \int_0^1 |u(x_0, y)| \, dy \leq Ch^{\frac{1}{2}}$$

which leads to (2.18).

**End of the proof of Theorem 1**

Knowing that (1.5) holds consider a  $u \in V_0^h$  such that

$$\int_{\Omega} W(\nabla u) \, dx \, dy \leq Ch^{\frac{1}{2}} \quad (2.19)$$

On a triangle of type 0 for such an  $u$  one has

$$|u_x| \geq \delta \text{ or } |u_y - 1| \geq \delta,$$

i.e.  $W(\nabla u) \geq \delta$ . Thus if  $S_{x_0} = (x_0, x_0 + \frac{h}{2}) \times (0, 1)$

$$\int_{S_{x_0}} W(\nabla u) \, dx \, dy \geq N_0 \delta \frac{h^2}{2}.$$

Assuming that  $h$  is small - more precisely  $\frac{(1-\delta)h^{-\frac{1}{2}}}{32C} > 1$  - one deduces from (2.18)

$$N_0 \geq \frac{1 - \delta}{32C} h^{-\frac{1}{2}}$$

which leads to

$$\int_{S_{x_0}} W(\nabla u) \, dx \, dy \geq \frac{(1-\delta)\delta h^{\frac{3}{2}}}{64C}.$$

Since the number of stripes  $S_{x_0}$  is  $\frac{2}{h}$  one deduces that

$$\int_{\Omega} W(\nabla u) \, dx \, dy \geq \frac{(1-\delta)\delta h^{\frac{1}{2}}}{32C}$$

which completes the proof.

**Remark.** The exponent  $1/2$  appearing in (1.6), (1.5) is related to the linear growth of  $W$ . Other growth conditions would produce other exponents in (1.6), (1.5). Our argument extend also to general polygonal domains and more general functionals. For all this we refer to [3].

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