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Local cohomology: an algebraic introduction with geometric applications

Brodmann, M ; Sharp, R

Abstract: This book provides a careful and detailed algebraic introduction to Grothendieck's local cohomology theory, and provides many illustrations of applications of the theory in commutative algebra and in the geometry of quasi-affine and quasi-projective varieties. Topics covered include Castelnuovo–Mumford regularity, the Fulton–Hansen connectedness theorem for projective varieties, and connections between local cohomology and both reductions of ideals and sheaf cohomology. It is designed for graduate students who have some experience of basic commutative algebra and homological algebra, and also for experts in commutative algebra and algebraic geometry.

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Local Cohomology: an algebraic introduction with geometric applications

M. P. Brodmann
Professor of Mathematics, Universität Zürich

R. Y. Sharp
Professor of Pure Mathematics, University of Sheffield

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1

The local cohomology functors

The main objective of this chapter is to introduce the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$ (throughout the book, $\mathfrak{a}$ always denotes an ideal in a (non-trivial) commutative Noetherian ring R) and its right derived functors $H^i_{\mathfrak{a}}$ $(i \geq 0)$, referred to as the local cohomology functors with respect to $\mathfrak{a}$. We shall see that $\Gamma_{\mathfrak{a}}$ is naturally equivalent to the functor $\varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(R/\mathfrak{a}^n, \,\bullet\,)$ and, indeed, that $H^i_{\mathfrak{a}}$ is naturally equivalent to the functor $\varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, \,\bullet\,)$ for each $i \geq 0$; moreover, as $\Gamma_{\mathfrak{a}}$ turns out to be left exact, the functors $\Gamma_{\mathfrak{a}}$ and $H^0_{\mathfrak{a}}$ are naturally equivalent.

This chapter also serves notice that our approach is based on fundamental techniques of homological commutative algebra, such as ones based on connected sequences of functors (see [52, pp. 212-214]): readers familiar with such ideas, and with the local cohomology functors, might like to just glance through this chapter and to move rapidly on to Chapter 2.

1.1 Torsion functors

1.1.1 Definition. For each R-module M, set $\Gamma_{\mathfrak{a}}(M) = \bigcup_{n\in\mathbb{N}}(0 :_M \mathfrak{a}^n)$, the set of elements of M which are annihilated by some power of $\mathfrak{a}$. Note that $\Gamma_{\mathfrak{a}}(M)$ is a submodule of M. For a homomorphism $f : M \longrightarrow N$ of R-modules, we have $f(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(N)$, and so there is a mapping $\Gamma_{\mathfrak{a}}(f) : \Gamma_{\mathfrak{a}}(M) \longrightarrow \Gamma_{\mathfrak{a}}(N)$ which agrees with f on each element of $\Gamma_{\mathfrak{a}}(M)$.

It is clear that, if $g : M \rightarrow N$ and $h : N \rightarrow L$ are further homomorphisms of R-modules and $r \in R$, then $\Gamma_{\mathfrak{a}}(h \circ f) = \Gamma_{\mathfrak{a}}(h) \circ \Gamma_{\mathfrak{a}}(f)$, $\Gamma_{\mathfrak{a}}(f+g) = \Gamma_{\mathfrak{a}}(f) + \Gamma_{\mathfrak{a}}(g)$, $\Gamma_{\mathfrak{a}}(rf) = r\Gamma_{\mathfrak{a}}(f)$ and $\Gamma_{\mathfrak{a}}(\mathrm{Id}_M) = \mathrm{Id}_{\Gamma_{\mathfrak{a}}(M)}$. Thus, with these assignments, $\Gamma_{\mathfrak{a}}$ becomes a covariant, R-linear functor from $\mathscr{C}(R)$ to itself. (We say that a functor $T : \mathscr{C}(R) \longrightarrow \mathscr{C}(R)$ is *R-linear* precisely when it is additive and $T(rf) = rT(f)$

for all $r \in R$ and all homomorphisms f of R-modules.) We call $\Gamma_\mathfrak{a}$ the *$\mathfrak{a}$-torsion functor.*

1.1.2 ♯Exercise. Let $\mathfrak{b}$ be a second ideal of R. Show that

$$\Gamma_\mathfrak{a}(\Gamma_\mathfrak{b}(M)) = \Gamma_{\mathfrak{a}+\mathfrak{b}}(M)$$

for each R-module M.

1.1.3 ♯Exercise. Let $\mathfrak{b}$ be a second ideal of R. Show that $\Gamma_\mathfrak{a} = \Gamma_\mathfrak{b}$ if and only if $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.

(The notation ♯, attached to some exercises, is explained in the section of 'Notation and conventions' following the Preface.)

1.1.4 Exercise. Suppose that the ideal $\mathfrak{b}$ of R is a reduction of $\mathfrak{a}$, that is, $\mathfrak{b} \subseteq \mathfrak{a}$ and there exists $s \in \mathbb{N}$ such that $\mathfrak{b}\mathfrak{a}^s = \mathfrak{a}^{s+1}$. Show that $\Gamma_\mathfrak{a} = \Gamma_\mathfrak{b}$.

1.1.5 Exercise. For a prime number p, find $\Gamma_{p\mathbb{Z}}(\mathbb{Q}/\mathbb{Z})$.

1.1.6 Lemma. *The $\mathfrak{a}$-torsion functor $\Gamma_\mathfrak{a} : \mathscr{C}(R) \longrightarrow \mathscr{C}(R)$ is left exact.*

Proof. Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be an exact sequence of R-modules and R-homomorphisms. We must show that

$$\begin{tikzcd} 0 \arrow[r] & \Gamma_\mathfrak{a}(L) \arrow[r, "\Gamma_\mathfrak{a}(f)"] & \Gamma_\mathfrak{a}(M) \arrow[r, "\Gamma_\mathfrak{a}(g)"] & \Gamma_\mathfrak{a}(N) \end{tikzcd}$$

is still exact. It is clear that $\Gamma_\mathfrak{a}(f)$ is a monomorphism and it follows immediately from 1.1.1 that $\Gamma_\mathfrak{a}(g) \circ \Gamma_\mathfrak{a}(f) = 0$, so that

$$\operatorname{Im}(\Gamma_\mathfrak{a}(f)) \subseteq \operatorname{Ker}(\Gamma_\mathfrak{a}(g)).$$

To prove the reverse inclusion, let $m \in \operatorname{Ker}(\Gamma_\mathfrak{a}(g))$. Thus $m \in \Gamma_\mathfrak{a}(M)$, so that there exists $n \in \mathbb{N}$ such that $\mathfrak{a}^n m = 0$, and $g(m) = 0$. Now there exists $l \in L$ such that $f(l) = m$, and our proof will be complete if we show that $l \in \Gamma_\mathfrak{a}(L)$. To achieve this, note that, for each $r \in \mathfrak{a}^n$, we have $f(rl) = rf(l) = rm = 0$, so that $rl = 0$ because f is a monomorphism. Hence $\mathfrak{a}^n l = 0$. □

The result of Lemma 1.1.6 will become transparent to many readers once we have covered a little more theory, and related the $\mathfrak{a}$-torsion functor $\Gamma_\mathfrak{a}$ to a functor defined in terms of direct limits of 'Hom' modules. However, before we proceed in that direction, we are going to introduce, at this early stage, the fundamental definition of the local cohomology modules of an R-module M with respect to $\mathfrak{a}$.

1.2 Local cohomology modules

1.2.1 Definitions. For $i \in \mathbb{N}_0$, the i-th right derived functor of $\Gamma_{\mathfrak{a}}$ is denoted by $H^i_{\mathfrak{a}}$ and will be referred to as the *i-th local cohomology functor with respect to* $\mathfrak{a}$.

For an R-module M, we shall refer to $H^i_{\mathfrak{a}}(M)$, that is, the result of applying the functor $H^i_{\mathfrak{a}}$ to M, as the *i-th local cohomology module of M with respect to* $\mathfrak{a}$, and to $\Gamma_{\mathfrak{a}}(M)$ as the *$\mathfrak{a}$-torsion submodule of* M. We shall say that M is *$\mathfrak{a}$-torsion-free* precisely when $\Gamma_{\mathfrak{a}}(M) = 0$, and that M is *$\mathfrak{a}$-torsion* precisely when $\Gamma_{\mathfrak{a}}(M) = M$, that is, if and only if each element of M is annihilated by some power of $\mathfrak{a}$.

It is probably appropriate for us to stress the implications of the above definition at this point, and list some basic properties of the local cohomology modules.

1.2.2 Properties of local cohomology modules. Let M be an arbitrary R-module.

(i) To calculate $H^i_{\mathfrak{a}}(M)$, one proceeds as follows. Take an injective resolution

$$I^{\bullet} : 0 \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \longrightarrow \cdots \longrightarrow I^i \xrightarrow{d^i} I^{i+1} \longrightarrow \cdots$$

of M, so that there is an R-homomorphism $\alpha : M \longrightarrow I^0$ such that the sequence

$$0 \longrightarrow M \xrightarrow{\alpha} I^0 \xrightarrow{d^0} I^1 \longrightarrow \cdots \longrightarrow I^i \xrightarrow{d^i} I^{i+1} \longrightarrow \cdots$$

is exact. Apply the functor $\Gamma_{\mathfrak{a}}$ to the complex $I^{\bullet}$ to obtain

$$0 \xrightarrow{\Gamma_{\mathfrak{a}}(d^{-1})} \Gamma_{\mathfrak{a}}(I^0) \longrightarrow \cdots \longrightarrow \Gamma_{\mathfrak{a}}(I^i) \xrightarrow{\Gamma_{\mathfrak{a}}(d^i)} \Gamma_{\mathfrak{a}}(I^{i+1}) \longrightarrow \cdots$$

and take the i-th cohomology module of this complex; the result,

$$\operatorname{Ker}(\Gamma_{\mathfrak{a}}(d^i)) / \operatorname{Im}(\Gamma_{\mathfrak{a}}(d^{i-1})),$$

which, by a standard fact of homological algebra, is independent (up to R-isomorphism) of the choice of injective resolution $I^{\bullet}$ of M, is $H^i_{\mathfrak{a}}(M)$.

(ii) Since $\Gamma_{\mathfrak{a}}$ is covariant and R-linear, it is automatic that each local cohomology functor $H^i_{\mathfrak{a}}$ ($i \in \mathbb{N}_0$) is again covariant and R-linear.

(iii) Since $\Gamma_{\mathfrak{a}}$ is left exact, $H^0_{\mathfrak{a}}$ is naturally equivalent to $\Gamma_{\mathfrak{a}}$. Thus, loosely, we can use this natural equivalence to identify these two functors.

(iv) The reader should be aware of the long exact sequence of local cohomology modules which results from a short exact sequence of R-modules and R-homomorphisms, and so we spell out the details here.

Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be an exact sequence of R-modules and R-homomorphisms. Then, for each $i \in \mathbb{N}_0$, there is a connecting homomorphism

$H^i_{\mathfrak{a}}(N) \to H^{i+1}_{\mathfrak{a}}(L)$, and these connecting homomorphisms make the resulting long sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0_{\mathfrak{a}}(L) \arrow[r, "H^0_{\mathfrak{a}}(f)"] & H^0_{\mathfrak{a}}(M) \arrow[r, "H^0_{\mathfrak{a}}(g)"] & H^0_{\mathfrak{a}}(N) \\
{} \arrow[r] & H^1_{\mathfrak{a}}(L) \arrow[r, "H^1_{\mathfrak{a}}(f)"] & H^1_{\mathfrak{a}}(M) \arrow[r, "H^1_{\mathfrak{a}}(g)"] & H^1_{\mathfrak{a}}(N) \\
{} \arrow[r] & \cdots & & \cdots \\
{} \arrow[r] & H^i_{\mathfrak{a}}(L) \arrow[r, "H^i_{\mathfrak{a}}(f)"] & H^i_{\mathfrak{a}}(M) \arrow[r, "H^i_{\mathfrak{a}}(g)"] & H^i_{\mathfrak{a}}(N) \\
{} \arrow[r] & H^{i+1}_{\mathfrak{a}}(L) \arrow[r] & \cdots &
\end{tikzcd}$$

exact. The reader should also be aware of the 'natural' or 'functorial' properties of these long exact sequences: if

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\lambda"] & M \arrow[r, "g"] \arrow[d, "\mu"] & N \arrow[r] \arrow[d, "\nu"] & 0 \\
0 \arrow[r] & L' \arrow[r, "f'"] & M' \arrow[r, "g'"] & N' \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of R-modules and R-homomorphisms with exact rows, then, for each $i \in \mathbb{N}_0$, we not only have a commutative diagram

$$\begin{tikzcd}
H^i_{\mathfrak{a}}(L) \arrow[r, "H^i_{\mathfrak{a}}(f)"] \arrow[d, "H^i_{\mathfrak{a}}(\lambda)"] & H^i_{\mathfrak{a}}(M) \arrow[r, "H^i_{\mathfrak{a}}(g)"] \arrow[d, "H^i_{\mathfrak{a}}(\mu)"] & H^i_{\mathfrak{a}}(N) \arrow[d, "H^i_{\mathfrak{a}}(\nu)"] \\
H^i_{\mathfrak{a}}(L') \arrow[r, "H^i_{\mathfrak{a}}(f')"] & H^i_{\mathfrak{a}}(M') \arrow[r, "H^i_{\mathfrak{a}}(g')"] & H^i_{\mathfrak{a}}(N')
\end{tikzcd}$$

(simply because $H^i_{\mathfrak{a}}$ is a functor!), but we also have a commutative diagram

$$\begin{tikzcd}
H^i_{\mathfrak{a}}(N) \arrow[r] \arrow[d, "H^i_{\mathfrak{a}}(\nu)"] & H^{i+1}_{\mathfrak{a}}(L) \arrow[d, "H^{i+1}_{\mathfrak{a}}(\lambda)"] \\
H^i_{\mathfrak{a}}(N') \arrow[r] & H^{i+1}_{\mathfrak{a}}(L')
\end{tikzcd}$$

in which the horizontal maps are the appropriate connecting homomorphisms.

The following remark will be used frequently in applications. It is an easy consequence of Exercise 1.1.3 and the definition of local cohomology functors in 1.2.1.

1.2.3 Remark. Let $\mathfrak{b}$ be a second ideal of R such that $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$. Then $H^i_{\mathfrak{a}} = H^i_{\mathfrak{b}}$ for all $i \in \mathbb{N}_0$, so that $H^i_{\mathfrak{a}}(M) = H^i_{\mathfrak{b}}(M)$ for each R-module M and all $i \in \mathbb{N}_0$.

The next four exercises might help the reader to consolidate the properties of local cohomology modules listed in 1.2.2. The first three of these exercises (for which non-trivial results from commutative algebra about injective dimension over the relevant rings are very helpful) give a tiny foretaste of results about the vanishing of local cohomology modules which are central to the subject, and which will feature prominently later in the book.

1.2.4 Exercise. Show that for every Abelian group (that is, $\mathbb{Z}$-module) G and for every $a \in \mathbb{Z}$, we have $H^i_{\mathbb{Z}a}(G) = 0$ for all $i \geq 2$.

1.2.5 Exercise. Suppose that $(R, \mathfrak{m})$ is a regular local ring of dimension d. Show that, for each R-module M, we have $H^i_{\mathfrak{a}}(M) = 0$ for all $i > d$.

1.2.6 Exercise. Suppose that $(R, \mathfrak{m})$ is a Gorenstein local ring (see [35, p. 142]) of dimension d. Show that, for each finitely generated R-module M of finite projective dimension, we have $H^i_{\mathfrak{a}}(M) = 0$ for all $i > d$. (Here is a hint: use the fact [35, Theorem 18.1] that the injective dimension of R as an R-module is d, and then use induction on the projective dimension of M.)

The next exercise investigates the behaviour of local cohomology modules under fraction formation: its results show that, speaking loosely, the local cohomology functors 'commute' with fraction formation. This is a fundamental fact in the subject; however, we shall actually derive it as an immediate consequence of a more general result in Chapter 4 concerning the behaviour of local cohomology under flat base change (and we shall not make use of it until it has been proved in Chapter 4). Nevertheless, even at this early stage, its proof should not present much difficulty for a reader familiar with the fact (proved in 10.1.13) that, if I is an injective R-module and S is a multiplicatively closed subset of R, then $S^{-1}I$ is an injective $S^{-1}R$-module.

1.2.7 Exercise. Let M be an R-module and let S be a multiplicatively closed subset of R. Show that $S^{-1}(\Gamma_{\mathfrak{a}}(M)) = \Gamma_{\mathfrak{a}S^{-1}R}(S^{-1}M)$, and that, for all $i \in \mathbb{N}_0$, there is an isomorphism of $S^{-1}R$-modules

$$S^{-1}(H^i_{\mathfrak{a}}(M)) \cong H^i_{\mathfrak{a}S^{-1}R}(S^{-1}M).$$

It is now time for us to relate the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$ to a functor defined in terms of direct limits of 'Hom' modules. Fundamental to the discussion is the natural isomorphism, for an R-module M and $n \in \mathbb{N}$,

$$\phi := \phi_{\mathfrak{a}^n,M} : \operatorname{Hom}_R(R/\mathfrak{a}^n, M) \xrightarrow{\cong} (0 :_M \mathfrak{a}^n)$$

for which $\phi(f) = f(1 + \mathfrak{a}^n)$ for all $f \in \operatorname{Hom}_R(R/\mathfrak{a}^n, M)$. In fact, we are going to put the various $\phi_{\mathfrak{a}^n,M}$ $(n \in \mathbb{N})$ together to obtain a natural isomorphism $\varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(R/\mathfrak{a}^n, M) \xrightarrow{\cong} \Gamma_{\mathfrak{a}}(M)$, but before we do this it might be helpful to the reader if we give some general considerations about functors and direct limits, as the principles involved will be used numerous times in this book.

1.2.8 Remarks. Let $(\Lambda, \leq)$ be a (non-empty) directed partially ordered set, and suppose that we are given an inverse system of R-modules $(W_\alpha)_{\alpha\in\Lambda}$ over Λ, with constituent R-homomorphisms $h^\alpha_\beta : W_\alpha \to W_\beta$ (for each $(\alpha, \beta) \in \Lambda \times \Lambda$ with $\alpha \geq \beta$). Let $T : \mathscr{C}(R) \times \mathscr{C}(R) \to \mathscr{C}(R)$ be an R-linear functor of two variables which is contravariant in the first variable and covariant in the second. (A functor $U : \mathscr{C}(R) \times \mathscr{C}(R) \longrightarrow \mathscr{C}(R)$ is said to be *R-linear* precisely when it is additive and $U(rf, g) = rU(f, g) = U(f, rg)$ for all $r \in R$ and all homomorphisms f, g of R-modules.) We show now how these data give rise to a covariant, R-linear functor

$$\varinjlim_{\alpha\in\Lambda} T(W_\alpha, \bullet) : \mathscr{C}(R) \longrightarrow \mathscr{C}(R).$$

Let M, N be R-modules and let $f : M \longrightarrow N$ be an R-homomorphism. For $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$, the homomorphism $h^\alpha_\beta : W_\alpha \longrightarrow W_\beta$ induces an R-homomorphism

$$T(h^\alpha_\beta, M) : T(W_\beta, M) \longrightarrow T(W_\alpha, M),$$

and the fact that T is a functor ensures that the $T(h^\alpha_\beta, M)$ turn the family $(T(W_\alpha, M))_{\alpha\in\Lambda}$ into a direct system of R-modules and R-homomorphisms over Λ. We may therefore form $\varinjlim_{\alpha\in\Lambda} T(W_\alpha, M)$. Moreover, again for $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$, we have a commutative diagram

$$\begin{tikzcd}
T(W_\beta, M) \arrow[r, "T(h^\alpha_\beta{,}M)"] \arrow[d, "T(W_\beta{,}f)"] & T(W_\alpha, M) \arrow[d, "T(W_\alpha{,}f)"] \\
T(W_\beta, N) \arrow[r, "T(h^\alpha_\beta{,}N)"] & T(W_\alpha, N)
\end{tikzcd};$$

therefore the $T(W_\alpha, f)$ $(\alpha \in \Lambda)$ constitute a morphism of direct systems and so induce an R-homomorphism

$$\varinjlim_{\alpha\in\Lambda} T(W_\alpha, f) : \varinjlim_{\alpha\in\Lambda} T(W_\alpha, M) \longrightarrow \varinjlim_{\alpha\in\Lambda} T(W_\alpha, N).$$

It is now straightforward to check that, in this way, $\varinjlim_{\alpha\in\Lambda} T(W_\alpha, \bullet)$ becomes a covariant, R-linear functor from $\mathscr{C}(R)$ to itself. Observe that, since passage to direct limits preserves exactness, if T is left exact, then so too is this new functor.

1.2.9 Examples. (i) Probably the most important examples for us of the ideas of 1.2.8 concern the case where we take for Λ the set $\mathbb{N}$ of positive integers with its usual ordering and the inverse system $(R/\mathfrak{a}^n)_{n\in\mathbb{N}}$ of R-modules under the natural homomorphisms $h^n_m : R/\mathfrak{a}^n \to R/\mathfrak{a}^m$ (for $n, m \in \mathbb{N}$ with $n \geq m$) (in such circumstances, $\mathfrak{a}^n \subseteq \mathfrak{a}^m$, of course). In this way, we obtain covariant, R-linear functors

$$\varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(R/\mathfrak{a}^n, \bullet) \quad \text{and} \quad \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, \bullet) \quad (i \in \mathbb{N}_0)$$

from $\mathscr{C}(R)$ to itself. Of course, the natural equivalence between the left exact functors Hom_R and Ext^0_R leads to a natural equivalence between the left exact functors

$$\varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(R/\mathfrak{a}^n, \bullet) \quad \text{and} \quad \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^0_R(R/\mathfrak{a}^n, \bullet)$$

which we shall use without further comment.

(ii) Very similar considerations, this time based on the inclusion maps $\mathfrak{a}^n \to \mathfrak{a}^m$ (for $n, m \in \mathbb{N}$ with $n \geq m$), lead to functors (which are again covariant and R-linear)

$$\varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(\mathfrak{a}^n, \bullet) \quad \text{and} \quad \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(\mathfrak{a}^n, \bullet) \quad (i \in \mathbb{N}_0)$$

from $\mathscr{C}(R)$ to itself, and a natural equivalence between the left exact functors

$$\varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(\mathfrak{a}^n, \bullet) \quad \text{and} \quad \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^0_R(\mathfrak{a}^n, \bullet).$$

These functors will be considered in detail in Chapter 2.

It will be convenient for us to consider situations slightly more general than that studied in 1.2.9(i) above.

1.2.10 Definition and Example. Let $(\Lambda, \leq)$ be a (non-empty) directed partially ordered set. By an *inverse family of ideals (of R) over* Λ, we mean a family $(\mathfrak{b}_\alpha)_{\alpha\in\Lambda}$ of ideals of R such that, whenever $(\alpha,\beta) \in \Lambda \times \Lambda$ with $\alpha \geq \beta$, we have $\mathfrak{b}_\alpha \subseteq \mathfrak{b}_\beta$.

For example, if

$$\mathfrak{b}_1 \supseteq \mathfrak{b}_2 \supseteq \ldots \supseteq \mathfrak{b}_n \supseteq \mathfrak{b}_{n+1} \supseteq \ldots$$

is a descending chain of ideals of R, then $(\mathfrak{b}_n)_{n\in\mathbb{N}}$ is an inverse family of ideals over $\mathbb{N}$ (with its usual ordering). In particular, the family $(\mathfrak{a}^n)_{n\in\mathbb{N}}$ is an inverse family of ideals over $\mathbb{N}$.

Let $(\mathfrak{b}_\alpha)_{\alpha\in\Lambda}$ be an inverse family of ideals of R over Λ. Then the natural R-homomorphisms $h^\alpha_\beta : R/\mathfrak{b}_\alpha \to R/\mathfrak{b}_\beta$ (for $\alpha,\beta \in \Lambda$ with $\alpha \geq \beta$) turn $(R/\mathfrak{b}_\alpha)_{\alpha\in\Lambda}$ into an inverse system over Λ, and so we can apply the ideas of 1.2.8 to produce covariant, R-linear functors

$$\varinjlim_{\alpha\in\Lambda} \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, \,\bullet\,) \quad \text{and} \quad \varinjlim_{\alpha\in\Lambda} \operatorname{Ext}^i_R(R/\mathfrak{b}_\alpha, \,\bullet\,) \ \ (i \in \mathbb{N}_0)$$

(from $\mathscr{C}(R)$ to itself), the first two of which are left exact and naturally equivalent.

1.2.11 Theorem. *Let* $\mathfrak{B} = (\mathfrak{b}_\alpha)_{\alpha\in\Lambda}$ *be an inverse family of ideals of R over* Λ, *as in* 1.2.10.

(i) *There is a covariant, R-linear functor* $\Gamma_{\mathfrak{B}} : \mathscr{C}(R) \to \mathscr{C}(R)$ *which is such that, for an R-module M,*

$$\Gamma_{\mathfrak{B}}(M) = \bigcup_{\alpha\in\Lambda} (0 :_M \mathfrak{b}_\alpha),$$

and, for a homomorphism $f : M \longrightarrow N$ *of R-modules,* $\Gamma_{\mathfrak{B}}(f) : \Gamma_{\mathfrak{B}}(M) \longrightarrow \Gamma_{\mathfrak{B}}(N)$ *is just the restriction of f to the submodule* $\Gamma_{\mathfrak{B}}(M)$ *of M.*

(ii) *There is a natural equivalence*

$$\phi' \ (= \phi'_{\mathfrak{B}}) : \varinjlim_{\alpha\in\Lambda} \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, \,\bullet\,) \xrightarrow{\cong} \Gamma_{\mathfrak{B}}$$

(of functors from $\mathscr{C}(R)$ *to itself) which is such that, for an R-module M and* $\alpha \in \Lambda$, *the image under* ϕ'_M *of the natural image of an* $h \in \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, M)$ *is* $h(1 + \mathfrak{b}_\alpha)$. *Consequently,* $\Gamma_{\mathfrak{B}}$ *is left exact.*

(iii) *In particular, there is a natural equivalence*

$$\phi^0 \ (= \phi^0_{\mathfrak{a}}) : \varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(R/\mathfrak{a}^n, \,\bullet\,) \xrightarrow{\cong} \Gamma_{\mathfrak{a}}$$

which is such that, for an R-module M and $n \in \mathbb{N}$, *the image under* ϕ^0_M *of the natural image of an* $h \in \operatorname{Hom}_R(R/\mathfrak{a}^n, M)$ *is* $h(1 + \mathfrak{a}^n)$.

Proof. (i) This can be proved by straightforward modification of the ideas of 1.1.1, and so will be left to the reader.

(ii) Let $f : M \longrightarrow N$ be a homomorphism of R-modules. For each $\alpha \in \Lambda$, let $\phi_{\mathfrak{b}_\alpha, M} : \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, M) \longrightarrow (0 :_M \mathfrak{b}_\alpha)$ be the R-isomorphism for which $\phi_{\mathfrak{b}_\alpha, M}(h) = h(1 + \mathfrak{b}_\alpha)$ for all $h \in \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, M)$. Let $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$, and let $h^\alpha_\beta : R/\mathfrak{b}_\alpha \to R/\mathfrak{b}_\beta$ be as in 1.2.10. Since the diagram

$$\begin{tikzcd}
\operatorname{Hom}_R(R/\mathfrak{b}_\beta, M) \arrow[r, "\phi_{\mathfrak{b}_\beta, M}", "\cong"'] \arrow[d, "\operatorname{Hom}_R(h^\alpha_\beta, M)"'] & (0 :_M \mathfrak{b}_\beta) \arrow[d] \\
\operatorname{Hom}_R(R/\mathfrak{b}_\alpha, M) \arrow[r, "\phi_{\mathfrak{b}_\alpha, M}", "\cong"'] & (0 :_M \mathfrak{b}_\alpha)
\end{tikzcd}$$

(in which the right hand vertical map is inclusion) commutes, it follows that there is indeed an R-isomorphism

$$\phi'_M : \varinjlim_{\alpha \in \Lambda} \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, M) \xrightarrow{\cong} \Gamma_{\mathfrak{B}}(M) = \bigcup_{\alpha \in \Lambda} (0 :_M \mathfrak{b}_\alpha)$$

as described in the statement of the theorem. It is easy to check that the diagram

$$\begin{tikzcd}
\varinjlim_{\alpha \in \Lambda} \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, M) \arrow[r, "\phi'_M", "\cong"'] \arrow[d, "\varinjlim_{\alpha \in \Lambda} \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, f)"'] & \Gamma_{\mathfrak{B}}(M) \arrow[d, "\Gamma_{\mathfrak{B}}(f)"] \\
\varinjlim_{\alpha \in \Lambda} \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, N) \arrow[r, "\phi'_N", "\cong"'] & \Gamma_{\mathfrak{B}}(N)
\end{tikzcd}$$

commutes, and the final claim is then immediate from 1.2.10.

(iii) This is now immediate from (ii), since when we apply (ii) to the family of ideals $\mathfrak{B} := (\mathfrak{a}^n)_{n \in \mathbb{N}}$, the functor $\Gamma_{\mathfrak{B}}$ of (i) is precisely the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$. □

We commented earlier that it would in time become transparent that $\Gamma_{\mathfrak{a}}$ is left exact: we had 1.2.11 in mind when we made that comment.

1.2.12 ♯Exercise. Provide a proof for part (i) of 1.2.11.

1.3 Connected sequences of functors

In this section, we are going to use the concepts of 'connected sequence of functors' and 'strongly connected sequence of functors'. These are explained on p. 212 of Rotman's book [52]. For the reader's convenience, we recall here relevant definitions in the case of negative connected sequences, as we shall be particularly concerned with this case.

1.3.1 Definition. Let R' be a commutative ring.

A sequence $(T^i)_{i \in \mathbb{N}_0}$ of covariant functors from $\mathcal{C}(R)$ to $\mathcal{C}(R')$ is said to be a *negative connected sequence* (respectively, a *negative strongly connected sequence*) if the following conditions are satisfied.

(i) Whenever $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is an exact sequence in $\mathcal{C}(R)$, there are defined connecting R'-homomorphisms

$$T^i(N) \longrightarrow T^{i+1}(L) \qquad \text{for all } i \in \mathbb{N}_0$$

such that the long sequence

$$\begin{tikzcd}
0 \arrow[r] & T^0(L) \arrow[r, "T^0(f)"] & T^0(M) \arrow[r, "T^0(g)"] & T^0(N) \\
\arrow[r] & T^1(L) \arrow[r, "T^1(f)"] & T^1(M) \arrow[r, "T^1(g)"] & T^1(N) \\
\arrow[r] & \cdots & & \cdots \\
\arrow[r] & T^i(L) \arrow[r, "T^i(f)"] & T^i(M) \arrow[r, "T^i(g)"] & T^i(N) \\
\arrow[r] & T^{i+1}(L) \arrow[r] & \cdots &
\end{tikzcd}$$

is a complex (respectively, is exact).

(ii) Whenever

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "\lambda"] & M \arrow[r] \arrow[d, "\mu"] & N \arrow[r] \arrow[d, "\nu"] & 0 \\
0 \arrow[r] & L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of R-modules and R-homomorphisms with exact rows, then there is induced, by λ, μ and ν, a chain map of the long complex of (i) for the top row into the corresponding long complex for the bottom row.

It might help if we remind the reader of the convention regarding the raising and lowering of indices in a situation such as that of 1.3.1, under which T^i would be written as T_{-i}: with this convention, $(T^i)_{i \geq 0}$ can be written as $(T_j)_{j \leq 0}$.

We also point out that, if $T : \mathscr{C}(R) \to \mathscr{C}(R')$ is an additive covariant functor, such as $\Gamma_{\mathfrak{a}}$, then its sequence of right derived functors $(\mathscr{R}^i T)_{i \in \mathbb{N}_0}$ is a negative strongly connected sequence of covariant functors from $\mathscr{C}(R)$ to $\mathscr{C}(R')$; furthermore, if T is left exact, then $\mathscr{R}^0 T$ is naturally equivalent to T. We shall be concerned so often with left exact, additive, covariant functors that it will considerably simplify the exposition if we make now the following convention which will be in force for the rest of the book.

1.3.2 Convention. Whenever R' is a commutative ring and $T : \mathscr{C}(R) \longrightarrow \mathscr{C}(R')$ is a covariant, additive functor which is left exact, then we shall identify T with its 0-th right derived functor $\mathscr{R}^0 T$ in the natural way. Likewise, we shall identify Ext^0_R with Hom_R in the natural way.

1.3.3 Definition. Let R' be a commutative ring, and let $(T^i)_{i \in \mathbb{N}_0}$ and $(U^i)_{i \in \mathbb{N}_0}$ be two negative connected sequences of covariant functors from $\mathscr{C}(R)$ to $\mathscr{C}(R')$. A *homomorphism* $\Psi : (T^i)_{i \in \mathbb{N}_0} \longrightarrow (U^i)_{i \in \mathbb{N}_0}$ *of connected sequences* is a family $(\psi^i)_{i \in \mathbb{N}_0}$ where, for each $i \in \mathbb{N}_0$, $\psi^i : T^i \to U^i$ is a natural transformation of functors, and which is such that the following condition is satisfied: whenever $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence of R-modules and R-homomorphisms, then, for each $i \in \mathbb{N}_0$, the diagram

$$\begin{tikzcd}
T^i(N) \arrow[r] \arrow[d, "\psi^i_N"] & T^{i+1}(L) \arrow[d, "\psi^{i+1}_L"] \\
U^i(N) \arrow[r] & U^{i+1}(L)
\end{tikzcd}$$

(in which the horizontal maps are the appropriate connecting homomorphisms arising from the connected sequences) commutes.

A homomorphism $\Psi = (\psi^i)_{i \in \mathbb{N}_0} : (T^i)_{i \in \mathbb{N}_0} \longrightarrow (U^i)_{i \in \mathbb{N}_0}$ of connected sequences is said to be an *isomorphism (of connected sequences)* precisely when $\psi^i : T^i \to U^i$ is a natural equivalence of functors for each $i \in \mathbb{N}_0$.

We hope the reader is sufficiently adept at techniques similar to those on pp. 212–214 of [52] to find the following exercise straightforward; if not, he or she might like to study Theorem 10 (and its Corollary) of Section 6.5 of Northcott [43], which together provide a solution.

1.3.4 ♯Exercise. Let R' be a commutative ring, and let $(T^i)_{i \in \mathbb{N}_0}$ and $(U^i)_{i \in \mathbb{N}_0}$ be two negative connected sequences of covariant functors from $\mathscr{C}(R)$ to $\mathscr{C}(R')$.

(i) Let $\psi^0 : T^0 \to U^0$ be a natural transformation of functors. Assume that

(a) the sequence $(T^i)_{i\in\mathbb{N}_0}$ is strongly connected, and

(b) $T^i(I) = 0$ for all $i \in \mathbb{N}$ and all injective R-modules I.

Show that there exist uniquely determined natural transformations $\psi^i : T^i \to U^i$ $(i \in \mathbb{N})$ such that $(\psi^i)_{i\in\mathbb{N}_0} : (T^i)_{i\in\mathbb{N}_0} \longrightarrow (U^i)_{i\in\mathbb{N}_0}$ is a homomorphism of connected sequences.

(ii) Let $\psi : T^0 \to U^0$ be a natural equivalence of functors. Assume that

(a) the sequence $(T^i)_{i\in\mathbb{N}_0}$ is strongly connected,

(b) the sequence $(U^i)_{i\in\mathbb{N}_0}$ is strongly connected, and

(c) $T^i(I) = U^i(I) = 0$ for all $i \in \mathbb{N}$ and all injective R-modules I.

By part (i), there is a unique homomorphism of connected sequences $\Psi := (\psi^i)_{i\in\mathbb{N}_0} : (T^i)_{i\in\mathbb{N}_0} \longrightarrow (U^i)_{i\in\mathbb{N}_0}$ for which $\psi^0 = \psi$. Show that Ψ is actually an isomorphism of connected sequences.

We shall not state explicitly the analogues of 1.3.1, 1.3.3 and 1.3.4 for positive connected sequences, but we warn the reader now that we shall use such analogues in Chapter 11.

The following consequence of 1.3.4(ii) essentially provides a characterization of the right derived functors of a left exact, additive, covariant functor from $\mathscr{C}(R)$ to $\mathscr{C}(R')$, where R' is a commutative ring.

1.3.5 Theorem. *Let R' be a commutative ring, and let T be a left exact, additive, covariant functor from $\mathscr{C}(R)$ to $\mathscr{C}(R')$. Let $(T^i)_{i\in\mathbb{N}_0}$ be a negative strongly connected sequence of covariant functors from $\mathscr{C}(R)$ to $\mathscr{C}(R')$ such that there exists a natural equivalence $\psi : T^0 \xrightarrow{\cong} T$ and such that $T^i(I) = 0$ for all $i \in \mathbb{N}$ and all injective R-modules I.*

Then there is a unique isomorphism of connected sequences

$$\Psi = (\psi^i)_{i\in\mathbb{N}_0} : (T^i)_{i\in\mathbb{N}_0} \xrightarrow{\cong} (\mathscr{R}^i T)_{i\in\mathbb{N}_0}$$

(of functors from $\mathscr{C}(R)$ to $\mathscr{C}(R')$) such that $\psi^0 = \psi$. (Of course, we are employing Convention 1.3.2.) □

The next exercise strengthens Exercise 1.2.7.

1.3.6 Exercise. Let S be a multiplicatively closed subset of R. Show that

$$\left(S^{-1}(H^i_{\mathfrak{a}}(\,\cdot\,))\right)_{i\in\mathbb{N}_0} \quad \text{and} \quad \left(H^i_{\mathfrak{a}S^{-1}R}(S^{-1}(\,\cdot\,))\right)_{i\in\mathbb{N}_0}$$

are isomorphic connected sequences of functors (from $\mathscr{C}(R)$ to $\mathscr{C}(S^{-1}R)$).

1.3.7 Remarks. Let $\mathfrak{B} = (\mathfrak{b}_\alpha)_{\alpha\in\Lambda}$ be an inverse family of ideals of R over Λ, as in 1.2.10.

Let us temporarily write $U^i := \varinjlim_{\alpha\in\Lambda} \mathrm{Ext}^i_R(R/\mathfrak{b}_\alpha, \bullet)$ for $i \in \mathbb{N}_0$. These functors were introduced in 1.2.10. We are going to show now how they fit together into a negative strongly connected sequence of functors (from $\mathscr{C}(R)$ to itself).

First of all, whenever $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence of R-modules and R-homomorphisms, there are induced, for each $\alpha \in \Lambda$, connecting homomorphisms

$$\mathrm{Ext}^i_R(R/\mathfrak{b}_\alpha, N) \longrightarrow \mathrm{Ext}^{i+1}_R(R/\mathfrak{b}_\alpha, L) \qquad (i \in \mathbb{N}_0)$$

which make the induced long sequence

$$\begin{array}{lllll}
0 \longrightarrow & \mathrm{Hom}_R(R/\mathfrak{b}_\alpha, L) & \longrightarrow \mathrm{Hom}_R(R/\mathfrak{b}_\alpha, M) & \longrightarrow & \mathrm{Hom}_R(R/\mathfrak{b}_\alpha, N) \\
\longrightarrow & \mathrm{Ext}^1_R(R/\mathfrak{b}_\alpha, L) & \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{b}_\alpha, M) & \longrightarrow & \mathrm{Ext}^1_R(R/\mathfrak{b}_\alpha, N) \\
\longrightarrow & \cdots & & & \cdots \\
\longrightarrow & \mathrm{Ext}^i_R(R/\mathfrak{b}_\alpha, L) & \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{b}_\alpha, M) & \longrightarrow & \mathrm{Ext}^i_R(R/\mathfrak{b}_\alpha, N) \\
\longrightarrow & \mathrm{Ext}^{i+1}_R(R/\mathfrak{b}_\alpha, L) & \longrightarrow \quad \cdots & &
\end{array}$$

exact. Moreover, these connecting homomorphisms are such that, for $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$, the diagram

$$\begin{tikzcd}
\mathrm{Ext}^i_R(R/\mathfrak{b}_\beta, N) \arrow[r] \arrow[d, "\mathrm{Ext}^i_R(h^\alpha_\beta, N)"'] & \mathrm{Ext}^{i+1}_R(R/\mathfrak{b}_\beta, L) \arrow[d, "\mathrm{Ext}^{i+1}_R(h^\alpha_\beta, L)"] \\
\mathrm{Ext}^i_R(R/\mathfrak{b}_\alpha, N) \arrow[r] & \mathrm{Ext}^{i+1}_R(R/\mathfrak{b}_\alpha, L)
\end{tikzcd}$$

(in which the horizontal maps are the appropriate connecting homomorphisms and $h^\alpha_\beta : R/\mathfrak{b}_\alpha \rightarrow R/\mathfrak{b}_\beta$ is the natural homomorphism) commutes for each $i \in \mathbb{N}_0$. It follows that these diagrams induce 'connecting' R-homomorphisms

$$U^i(N) = \varinjlim_{\alpha\in\Lambda} \mathrm{Ext}^i_R(R/\mathfrak{b}_\alpha, N) \longrightarrow U^{i+1}(L) = \varinjlim_{\alpha\in\Lambda} \mathrm{Ext}^{i+1}_R(R/\mathfrak{b}_\alpha, L)$$

(for $i \in \mathbb{N}_0$); moreover, the fact that passage to direct limits preserves exactness ensures that the resulting long sequence

$$\begin{tikzcd}
0 \arrow[r] & U^0(L) \arrow[r] & U^0(M) \arrow[r] & U^0(N) \\
\arrow[r] & U^1(L) \arrow[r] & U^1(M) \arrow[r] & U^1(N) \\
\arrow[r] & \cdots & & \cdots \\
\arrow[r] & U^i(L) \arrow[r] & U^i(M) \arrow[r] & U^i(N) \\
\arrow[r] & U^{i+1}(L) \arrow[r] & \cdots &
\end{tikzcd}$$

is exact. Next, standard properties of the extension functors ensure that, whenever

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "\lambda"] & M \arrow[r] \arrow[d, "\mu"] & N \arrow[r] \arrow[d, "\nu"] & 0 \\
0 \arrow[r] & L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of R-modules and R-homomorphisms with exact rows, then, for all $\alpha \in \Lambda$, the diagram

$$\begin{tikzcd}
\operatorname{Ext}^i_R(R/\mathfrak{b}_\alpha, N) \arrow[r] \arrow[d, "\operatorname{Ext}^i_R(R/\mathfrak{b}_\alpha{,}\nu)"'] & \operatorname{Ext}^{i+1}_R(R/\mathfrak{b}_\alpha, L) \arrow[d, "\operatorname{Ext}^{i+1}_R(R/\mathfrak{b}_\alpha{,}\lambda)"] \\
\operatorname{Ext}^i_R(R/\mathfrak{b}_\alpha, N') \arrow[r] & \operatorname{Ext}^{i+1}_R(R/\mathfrak{b}_\alpha, L')
\end{tikzcd}$$

(in which the horizontal maps are the appropriate connecting homomorphisms) commutes for each $i \in \mathbb{N}_0$. It therefore follows that the diagram

$$\begin{tikzcd}
\varinjlim_{\alpha\in\Lambda} \operatorname{Ext}^i_R(R/\mathfrak{b}_\alpha, N) \arrow[r] \arrow[d, "\varinjlim_{\alpha\in\Lambda} \operatorname{Ext}^i_R(R/\mathfrak{b}_\alpha{,}\nu)"'] & \varinjlim_{\alpha\in\Lambda} \operatorname{Ext}^{i+1}_R(R/\mathfrak{b}_\alpha, L) \arrow[d, "\varinjlim_{\alpha\in\Lambda} \operatorname{Ext}^{i+1}_R(R/\mathfrak{b}_\alpha{,}\lambda)"] \\
\varinjlim_{\alpha\in\Lambda} \operatorname{Ext}^i_R(R/\mathfrak{b}_\alpha, N') \arrow[r] & \varinjlim_{\alpha\in\Lambda} \operatorname{Ext}^{i+1}_R(R/\mathfrak{b}_\alpha, L')
\end{tikzcd}$$

(in which the horizontal maps are again the appropriate connecting homomorphisms) commutes for all $i \in \mathbb{N}_0$.

We have thus made

$$\left(\varinjlim_{\alpha\in\Lambda} \operatorname{Ext}^i_R(R/\mathfrak{b}_\alpha, \bullet)\right)_{i\in\mathbb{N}_0}$$

into a negative strongly connected sequence of covariant functors from $\mathscr{C}(R)$ to $\mathscr{C}(R)$. Since $\varinjlim_{\alpha\in\Lambda} \operatorname{Ext}^i_R(R/\mathfrak{b}_\alpha, I) = 0$ for all $i \in \mathbb{N}$ whenever I is an injective R-module, it now follows from 1.3.5 that there is a unique isomorphism of connected sequences

$$\widetilde{\Psi} = \left(\widetilde{\psi}^i\right)_{i\in\mathbb{N}_0} : \left(\varinjlim_{\alpha\in\Lambda} \operatorname{Ext}^i_R(R/\mathfrak{b}_\alpha, \bullet)\right)_{i\in\mathbb{N}_0} \xrightarrow{\cong} \left(\mathscr{R}^i\Gamma_{\mathfrak{B}}\right)_{i\in\mathbb{N}_0}$$

for which $\widetilde{\psi}^0$ is the natural equivalence $\phi'_{\mathfrak{B}}$ of 1.2.11(ii); furthermore, both these connected sequences are isomorphic to the negative (strongly) connected sequence of functors formed by the right derived functors of

$$\varinjlim_{\alpha\in\Lambda} \operatorname{Hom}_R(R/\mathfrak{b}_\alpha, \bullet).$$

A special case of 1.3.7 describes local cohomology modules as direct limits of Ext modules. As this description is of crucial importance for our subject, we state it separately.

1.3.8 Theorem. *There is a unique isomorphism of connected sequences (of functors from $\mathscr{C}(R)$ to $\mathscr{C}(R)$)*

$$\Phi_{\mathfrak{a}} = \left(\phi^i_{\mathfrak{a}}\right)_{i\in\mathbb{N}_0} : \left(\varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, \bullet)\right)_{i\in\mathbb{N}_0} \xrightarrow{\cong} \left(H^i_{\mathfrak{a}}\right)_{i\in\mathbb{N}_0}$$

which extends the natural equivalence $\phi^0_{\mathfrak{a}} : \varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(R/\mathfrak{a}^n, \bullet) \xrightarrow{\cong} \Gamma_{\mathfrak{a}}$ *of* 1.2.11(iii). *Consequently, for each R-module M and each $i \in \mathbb{N}_0$,*

$$H^i_{\mathfrak{a}}(M) \cong \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M). \ \square$$

1.3.9 ♯Exercise. Let M be an R-module, not necessarily finitely generated. Recall the definition of M-sequence from, for example, [35, p. 123].

(i) Show that, if $a_1, \ldots, a_n$ and $a'_1, a_2, \ldots, a_n$ are M-sequences (of elements of R), then so too is $a_1a'_1, a_2, \ldots, a_n$.

(ii) Show that, if $a_1, \ldots, a_n$ is an M-sequence (of elements of R) and $h_1, \ldots, h_n$ are positive integers, then $a_1^{h_1}, \ldots, a_n^{h_n}$ is also an M-sequence.

(iii) Show that, if $a_1, \ldots, a_n$ is an M-sequence of elements of $\mathfrak{a}$, then we have $\operatorname{Ext}^i_R(R/\mathfrak{a}, M) = 0$ for all $i = 0, \ldots, n-1$.

(iv) Show that, if $\mathfrak{a}$ contains an M-sequence of length n, then $H^i_{\mathfrak{a}}(M) = 0$ for each $i = 0, \ldots, n-1$.

We shall pursue the ideas of Exercise 1.3.9 later in the book, particularly in Chapter 6.