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Imperfections in Random Tournaments

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ABSTRACT

A *tournament* T on a set V of n players is an orientation of the edges of the complete graph K_n on V; T will be called a *random tournament* if the directions of these edges are determined by a sequence $\{Y_j : j = 1, \ldots, \binom{n}{2}\}$ of independent coin flips. If (y, x) is an edge in a (random) tournament, we say that y *beats* x. A set $A \subset V, |A| = k$, is said to be *beaten* if there exists a player $y \notin A$ such that y beats x for each $x \in A$. If such a y does not exist, we say that A is *unbeaten*. A (random) tournament on V is said to have property S_k if each k-element subset of V is beaten. In this paper, we use the Stein-Chen method to show that the probability distribution of the number W_0 of unbeaten k-subsets of V can be well-approximated by that of a Poisson random variable with the same mean; an improved condition for the existence of tournaments with property S_k is derived as a corollary. A multivariate version of this result is proved next: with W_j representing the number of k-subsets that are beaten by precisely j external vertices, $j = 0, 1, \ldots, b$, it is shown that the

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joint distribution of $(W_0, W_1, \ldots, W_b)$ can be approximated by a multidimensional Poisson vector with independent components, provided that b is not too large.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $V = \{1, 2, \ldots, n\}$ be a finite set. A *tournament* T on V is an orientation of the $\binom{n}{2}$ edges of the complete graph K_n on V. If (x, y) is an oriented edge in a tournament, we say that x *beats* y. A *random tournament* is a tournament on V obtained by choosing independently, for each $1 \le x < y \le n$, one of the edges (x, y) or (y, x) with equal probability. Moon (1968) describes applications of the combinatorics of tournaments to areas as diverse as the method of paired comparisons, voting schemes, choice functions, and dominance relations in sociometric groups.

T is said to have property S_k (written $T \in S_k$) if for each set $A, |A| = k \ge 1$, there exists $y \notin A$ such that (y, x) is an oriented edge for each $x \in A$. In other words, $T \in S_k$ if each member of each group $A = \{a_1, \ldots, a_k\}$ of k players is beaten by a player $y = y_A \notin A$. If such a y exists, we shall say that the set A is *beaten*, or *dominated*; if not, it will be said to be *unbeaten*. It is clear that when k is fixed and n is sufficiently large, a *random* tournament is very likely to have property S_k: Erdős (1963) showed that if $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$, then there is a tournament on n vertices with property S_k; a proof of this fact may be found in Alon, Spencer and Erdős (1992), p. 5. If we define $W_0 = \sum_{j=1}^{\binom{n}{k}} I_j$ as the number of unbeaten k-sets in a random tournament, where I_j is the indicator of the event that the jth set of size k is unbeaten, then it is clear that $\mathbf{E}(W_0) = \binom{n}{k}(1 - 2^{-k})^{n-k}$, and Erdős' result may be reformulated as the elementary (but far-reaching) observation that

$$\lambda = \mathbf{E}(W_0) < 1 \Rightarrow \mathbf{P}(W_0 = 0) > 0 \tag{1.1}$$

Throughout this paper, j will represent a k-subset as well as its index; we trust that this dual use of notation will not be confusing.

Let $f(k)$ denote the minimum possible number of vertices of a tournament $T \in S_k$. It can readily be verified that $f(1) = 3$ and $f(2) = 7$; a general lower bound due to Szekeres is $f(k) \ge ck2^k$ (see Moon (1968)). On the other hand, it is an easy exercise to deduce from Erdős's result that $f(k) \le k^2 2^k (\log 2)(1 + o(1)), k \to \infty$: in fact, for $n = k^2 2^k \log 2$, one has $\binom{n}{k}(1 - 2^{-k})^{n-k} \sim k^k (\log 2)^k e^k \to \infty$, while for $n = k^2 2^k$, $\binom{n}{k}(1 - 2^{-k})^{n-k} \sim k^k e^k (2/e)^{k^2} \to 0$, showing that $f(k) \le k^2 2^k$. A more careful analysis yields

$$\begin{aligned} f(k) \le& k^2 2^k \log 2 + (1 + \varepsilon) k 2^k \log k \\ =& k^2 2^k \log 2 (1 + o(1)), \end{aligned}$$

whatever the value of $\varepsilon > 0$, if k is large enough. The above observations reveal how rapidly decreasing the function $\phi(n) = \binom{n}{k}(1-2^{-k})^{n-k}$ is, and provide a natural reason why the elementary result of Erdős has not been bettered (thus far).

Property S_k is undesirable, in a very natural sense; a tournament $T \in S_k$ can, consequently, be thought of as being rather imperfect. We have seen, moreover, that a random tournament with an excessively large number of players has the potential to have the blemish S_k, i.e., satisfies the condition $\mathbf{P}(W_0 = 0) > 0$. The random variable W_0 may, in fact, be used as a measure of the *degree* of imperfection of a random tournament T on n vertices. What, then, can one assert about the probability distribution of W_0? Exact expressions are clearly quite intractable, but, since $W_0 = \sum_{j=1}^{\binom{n}{k}} I_j$, where $\mathbf{P}(I_j = 1) = (1-2^{-k})^{n-k}$ is small if n is substantially larger than 2^k, one might expect that the distribution of W_0 is close to Poisson for such values of n. On the other hand, the dependence between the I_j's is global, which might lead to the failure of the Poisson paradigm unless n is *even larger* than indicated above. Our first result, proved in Section 2, makes the above statements precise:

Theorem 1. *Let W_0 denote the number of unbeaten k-sets in a random tournament T on n vertices, and let $\mathrm{Po}(\lambda)$ denote the Poisson distribution with mean $\lambda = \binom{n}{k}(1-2^{-k})^{n-k}$. Assume that $k \geq 9$ and $2^k \leq n \leq k^2 2^k \log 2$, or that $k \geq 32$ and $2^k \leq n \leq k^2 2^k \log 4$. Then*

$$d_{\mathrm{TV}}\big(\mathcal{L}(W_0), \mathrm{Po}(\lambda)\big) = \sup_{A \subseteq \mathbf{Z}^+} |\mathbf{P}(W_0 \in A) - \mathrm{Po}(\lambda)\{A\}|$$
$$\leq (1 - \frac{1}{2^k})^{n-k} + \frac{\lambda k^2}{4^k} A_k \{2 + A_k + 2B_{nk}\},$$

where $A_k = (1-2^{-k})^{-k} = 1 + O(k2^{-k})$ $(k \to \infty)$, and $B_{nk} = O(k^3 2^{-k})$ is given explicitly in (2.9) below.

The proof of Theorem 1 relies on the Stein-Chen method of Poisson approximation; see Barbour, Holst and Janson (1992) for theoretical aspects (and applications) of the *coupling* approach to Stein-Chen approximation that we employ here. We note that, asymptotically as $k \to \infty$, W_0 can be well-approximated by a Poisson r.v. provided that $\lambda = o(4^k/k^2)$. Furthermore, Theorem 1 reveals that

$$\mathbf{P}(W_0 = 0) \geq e^{-\lambda} - \frac{3\lambda k^2}{4^k}(1 + o(1))$$

uniformly in $n \geq 2^k$, so that a tournament with property S_k exists if

$$\frac{3\lambda e^{\lambda} k^2}{4^k}(1 + o(1)) < 1,$$

that is, if $\lambda \leq (\log 4 - o(1))k$. We thus see that Theorem 1 yields, as a corollary, a significant improvement over Erdős's result (1.1). A specific example follows the proof of Theorem 1.

We next consider a more complicated question that is motivated by the notion of tournaments with "property $S(k,m)$", i.e., one for which each k-set is beaten by at least m players [see Moon (1968) for more details]: Given a random tournament on n vertices, what can be asserted about the overall structure of the numbers of groups of k players that are each beaten by exactly j others, $(j = 0, 1, \ldots, b)$? Specifically, how does one approximate the *joint* distribution of the random vector $(W_0, W_1, \ldots, W_b)$, where W_j denotes the number of k-subsets of $\{1, 2, \ldots, n\}$ that are beaten by precisely j external vertices? Our second result is in the spirit of other multivariate Poisson approximations obtained, e.g., by Arratia and Tavaré (1992), Arratia, Barbour and Tavaré (1992), and Godbole, Skipper and Sunley (1995), in the context of random permutations, the Ewens sampling formula, and random edge colourings of K_n respectively; given the nature of this result, moreover, we must evidently restrict ourselves to values of b that are smaller, by an order of magnitude, than the expected number $(n-k)/2^k$ of vertices that beat any particular k-set. Letting $\lambda_j = \mathbf{E}(W_j) = \binom{n}{k}\binom{n-k}{j}2^{-kj}(1-2^{-k})^{n-k-j}$ denote the expected number of k-subsets that are dominated by j players, and setting $\Lambda_b = \sum_{j=0}^{b} \lambda_j$, we have

Theorem 2. *Let W_j and λ_j be as in the preceding paragraph. Assume that $k \geq 9$ and $2^k \leq n \leq k^2 2^k \log 2$, or that $k \geq 32$ and $2^k \leq n \leq k^2 2^k \log 4$. Then the joint distribution of $(W_0, W_1, \ldots, W_b)$ can be approximated by a multivariate Poisson distribution $(\mathrm{Po}(\lambda_0), \mathrm{Po}(\lambda_1), \ldots, \mathrm{Po}(\lambda_b))$ with independent components, with total variation error given by*

$$d_{\mathrm{TV}}\Big(\mathcal{L}(\{W_j\}_{j=0}^{b}), \prod_{j=0}^{b} \mathrm{Po}(\lambda_j)\Big)$$
$$\leq \Lambda_b\Big\{2R_b + \frac{\Lambda_b k^2}{4^k} A_k^2 \Big(1 + \frac{2^k b}{n-k-b}\Big)\Big(5 + 4B_{nk} + \frac{2k^2}{n-k} + \frac{R_b 2^k b}{n-k-b}\Big)\Big\},$$

where $A_k = (1-2^{-k})^{-k}$ and $B_{nk} = 0(k^3 2^{-k})$ are as in Theorem 1, and $R_b = \Lambda_b / \binom{n}{k}$.

For fixed k and n, the bound in Theorem 2 increases with b, as is to be expected. Clearly, if the bound is to be small, the quantity $\Lambda_b k 2^{-k}$ must be small. Since Λ_b can be expressed in terms of binomial probabilities as

$$\Lambda_b = \binom{n}{k} \mathrm{Bi}\,(n-k, 2^{-k})\{[0,b]\},$$

and since $n \geq 2^k$, this implies that b must certainly be smaller than $n2^{-k}$. Under these circumstances, the bound is of order $\Lambda_b^2 k^2 4^{-k}$, and takes the value $5\Lambda_b^2 k^2 4^{-k}\{1+o(1)\}$ if $b = o(n2^{-k})$; good Poisson approximation is obtained whenever $\Lambda_b \ll k^{-1}2^k$. The order of the error bound is the same as that of Theorem 1, but for the presence of a Λ_b^2 multiplier in place of the λ. The square is due to the fact that the univariate Theorem 2.C of Barbour, Holst and Janson (1992) has the "magic" factor $(1-e^{-\lambda})/\lambda$, which is not available in their multivariate Theorem 10.J. Partial reinstatement of the magic factor is possible, however, as in Theorem 10.K in Barbour, Holst and Janson (1992), although it may be verified that our results do not improve significantly as a result.

To summarize, Theorem 1 reveals that the distribution of the number of unbeaten k-sets is approximately Poisson, and yields an improved condition for the existence of tournaments with property S_k. Theorem 2, on the other hand, provides an overall view of the numbers of k-sets that are beaten by various numbers of external vertices, showing that the "big picture" can be modeled by a Poisson process with independent components. Given these results, it is not surprising that the variables W_j can, for $j \geq 1$, be *individually* approximated by Poisson random variables with an even higher degree of precision than that given by Theorem 2. Results along these lines have been proved in Qian (1993), but are not included in this paper.

2. PROOF OF THEOREM 1

By Theorem 2.C in Barbour, Holst and Janson (1992),

$$\begin{aligned} d_{\mathrm{TV}}(\mathcal{L}(W_0), \mathrm{Po}(\lambda)) \leq & \frac{1-e^{-\lambda}}{\lambda} \sum_{j=1}^{\binom{n}{k}} \left(\pi_j^2 + \pi_j \sum_{i \neq j} \mathbf{P}(I_i \neq J_{ij}) \right) \\ \leq & \pi_1 + \sum_{i \neq 1} (P_{1i} + P_{2i}) \end{aligned} \tag{2.1}$$

where $\pi_j = \mathbf{P}(I_j = 1) = (1-2^{-k})^{n-k}$, the variables $\{J_i\}_{i=1}^{\binom{n}{k}} = \{J_{ij}\}_{i=1}^{\binom{n}{k}}$ satisfy

$$\mathcal{L}\Big(J_{ij} : 1 \leq i \leq \binom{n}{k}\Big) = \mathcal{L}\Big(I_i : 1 \leq i \leq \binom{n}{k} \Big| I_j = 1\Big) \tag{2.2}$$

and P_{1i} and P_{2i} are defined by

$$P_{1i} = \mathbf{P}(I_i = 1, J_{i1} = 0)$$

and

$$P_{2i} = \mathbf{P}(I_i = 0, J_{i1} = 1)$$

respectively. The coupled random variables $\{J_{i1}\}$ are defined as follows: If $I_1 = 1$, i.e., if the 1st set is unbeaten, we "do nothing", letting $J_{i1} = I_i$ for each i. If $I_1 = 0$, on the other hand, we consider the set $C(\neq \emptyset)$ of players who beat the 1st set. For each $c \in C$, we independently choose, with equal probability, one of the $2^k - 1$ orientations of the edges between c and members of the first set that lead to c *not* dominating the set 1, and redirect the edges accordingly, setting $J_{i1} = 1$ if the ith k-set is unbeaten after the adjustment ($J_{i1} = 0$ otherwise). It is evident that the J_{i1}'s satisfy (2.2), providing an accurate model for the global behaviour of the indicators, conditional on the first k-set being unbeaten. We begin by estimating P_{1i}. It is clear that the event $\{I_i = 1, J_{i1} = 0\} = \{I_i = 1, I_1 = 0, J_{i1} = 0\}$ occurs only if there is at least one player in i that beats 1 and, in the process of "redirecting edges" (i.e., achieving the coupling), the set i is beaten by a member of 1. It follows, since there are $\binom{k}{r}\binom{n-k}{k-r}$ sets i such that $|i \cap 1| = r$, that

$$\begin{aligned}
\sum_{i\neq 1} P_{1i} &\leq \sum_{r=0}^{k-1} \binom{k}{r}\binom{n-k}{k-r}(1-\frac{1}{2^k})^{n-2k+r} \sum_{s=1}^{k-r} \binom{k-r}{s} \frac{1}{2^{ks}}(1-\frac{1}{2^k})^{k-r-s} \frac{k-r}{2^{k-s}}(\frac{2^{k-1}}{2^k-1})^s \\
&= \sum_{r=0}^{k-1} \binom{k}{r}\binom{n-k}{k-r}(1-\frac{1}{2^k})^{n-k} \frac{k-r}{2^k}\Big\{(1+\frac{2^k}{(2^k-1)^2})^{k-r} - 1\Big\} \\
&\leq \lambda \sum_{r=0}^{k-1} \frac{\binom{k}{r}\binom{n-k}{k-r}}{\binom{n}{k}} (\frac{k}{2^k-1})^2 (1+\frac{2^k}{(2^k-1)^2})^{k-1} \\
&\leq \frac{\lambda k^2}{4^k} A_k^2,
\end{aligned} \tag{2.3}$$

with $A_k = (1-2^{-k})^{-k}$. The first inequality above follows since the s $(1 \leq s \leq k-r)$ elements of i that beat 1 originally must now be beaten by some element of $1 \setminus i$ (there are $k-r$ of these). The probability of this event is $(2^{k-1}/(2^k-1))^s$. Also, the other $k-s$ elements of i must have been beaten by the above element of $1 \setminus i$ *before* the arrow reversal.

Notice next that the event $\{I_i = 0, J_{i1} = 1\} = \{I_i = 0, I_1 = 0, J_{i1} = 1\}$ can only occur provided that (a) the sets 1 and i have a non-empty intersection; (b) there is at least one player who beats both 1 and i, and, (c) when the coupling is implemented, these players do not beat 1 or i any more. Finally, observe that $\mathbf{P}(I_i = 0, J_{i1} = 1) = 0$ if i is beaten by any point which does not beat 1. It follows that

$$\sum_{i\neq 1} P_{2i} \leq \sum_{r=1}^{k-1} \binom{k}{r}\binom{n-k}{k-r} \frac{n}{2^{2k-r}} \left\{ \frac{1}{2^{2k-r}} + (1-\frac{1}{2^k}) \right\}^{n-2k+r-1}$$

$$\leq \frac{n\lambda}{4^k} \sum_{r=1}^{k-1} \frac{\binom{k}{r}\binom{n-k}{k-r}}{\binom{n}{k}} 2^r \{1+\frac{2^r}{2^k(2^k-1)}\}^n A_k, \tag{2.4}$$

where A_k is as in (2.3). Now $\binom{k}{r}\binom{n-k}{k-r}/\binom{n}{k} \leq k^{2r}/\{n^r r!\}$, so that we thus have

$$\sum_{i\neq 1} P_{2i} \leq \frac{2k^2\lambda}{4^k} A_k \sum_{r=1}^{k-1} \phi(r,n), \tag{2.5}$$

where

$$\phi(r,n) = \frac{1}{r!}(\frac{2k^2}{n})^{r-1}\{1+\frac{2^r}{2^k(2^k-1)}\}^n.$$

Now, taking logarithms, we can write

$$\log\phi(r,n) \leq b_n(r) := (r-1)\log(\frac{2k^2}{n}) + \frac{n2^r}{2^k(2^k-1)} - \log(r!).$$

The function b_n has derivative

$$b_n'(r) = \log(\frac{2k^2}{n}) + \frac{n2^r\log 2}{2^k(2^k-1)} - \psi(r+1),$$

where $\psi(x)$ denotes $\frac{d}{dx}\log\Gamma(x)$, and this is a convex function in $r \geq 0$, satisfying

$$b_n'(1) < \log(\frac{2k^2}{n}) + \frac{2n\log 2}{2^k(2^k-1)} \leq \frac{2n\log 2}{2^k(2^k-1)} - 1 \leq 0$$

if

$$2k^2 e \leq n \leq \frac{2^k(2^k-1)}{2\log 2}. \tag{2.6}$$

Thus for such n and k,

$$\max_{2\leq r\leq k-1} \log\phi(r,n) \leq \max\{b_n(2), b_n(k-1)\};$$

note that (2.6) is satisfied if, for example, $2^k \leq n \leq k^2 2^k \log 4$ and $k \geq 9$. Now

$$c_k(n) = b_n(2) - b_n(k-1) = (k-3)\log(\frac{n}{2k^2}) + \log((k-1)!) - \log 2 - \frac{1}{2}(\frac{n}{2^k})(\frac{2^k-8}{2^k-1})$$

is a concave function of n $(n > 0)$, so that

$$\min_{n_1 \le n \le n_2} c_k(n) = \min\{c_k(n_1), c_k(n_2)\},$$

and hence

$$\sum_{r=2}^{k-1} \phi(r,n) \le (k-2)e^{b_n(2)} \tag{2.7}$$

for all $n_1 \le n \le n_2$, if $\min\{c_k(n_1), c_k(n_2)\} > 0$ and (2.6) is satisfied.

For $n_1 = 2^k$, we have $c_k(2^k) \sim k^2 \log 2$ $(k \to \infty)$, and calculation shows that in fact $c_k(2^k) > 0$ for all $k \ge 4$. For $n_2 = k^2 2^k \log 2$, $c_k(k^2 2^k \log 2) \sim \frac{1}{2} k^2 \log 2$ $(k \to \infty)$, and calculation shows that $c_k(k^2 2^k \log 2) > 0$ for all $k \ge 5$. For the critical $n_2 = k^2 2^k \log 4$, $c_k(k^2 2^k \log 4) \sim k \log k$ $(k \to \infty)$ and $c_k(k^2 2^k \log 4) > 0$ for all $k \ge 32$. Thus, collecting (2.5), (2.7) and the above, we have the estimate

$$\sum_{i \ne 1} P_{2i} \le \frac{2k^2 \lambda}{4^k} A_k \{\phi(1,n) + (k-2)e^{b_n(2)}\} \tag{2.8}$$

for $k \ge 9$ and $2^k \le n \le k^2 2^k \log 2$ or $k \ge 32$ and $2^k \le n \le k^2 2^k \log 4$. This in turn implies that, in these ranges of n,

$$\begin{aligned}
\sum_{i \ne 1} P_{2i} \le & \frac{2k^2 \lambda}{4^k} A_k \Big\{(1 + \frac{2}{2^k(2^k-1)})^n + \frac{(k-2)k^2}{n} e^{\frac{4n}{2^k(2^k-1)}}\Big\} \\
\le & \frac{2k^2 \lambda}{4^k} A_k \Big\{1 + \Big(\frac{2n}{2^k(2^k-1)} + \frac{(k-2)k^2}{n}\Big) e^{\frac{4n}{2^k(2^k-1)}}\Big\} \\
= & \frac{2k^2 \lambda}{4^k} A_k \{1 + B_{nk}\},
\end{aligned} \tag{2.9}$$

with $B_{nk} = O(k^3 2^{-k})$ $(k \to \infty)$ for $2^k \le n \le k^2 2^k \log 4$. Theorem 1 follows on combining (2.1), (2.3) and (2.9).

Example. Consider a random round-robin tournament between 6693 equally capable players. Let $k = 7$; this choice is made for computational convenience as well as to exhibit one of several *alternative* bounds that may be extracted from the proof of Theorem 1. The expected number of unbeaten 7-subsets is $\binom{6693}{7}(1 - 1/2^7)^{6686} \approx 2$. Erdős's result is not applicable, since $\lambda > 1$. Note next that (2.6) holds, the fact that k is smaller than 9 notwithstanding. It may be verified, moreover, that $c_7(6693) < 0$, so that we replace (2.9) by

$$\sum_{i \ne 1} P_{2i} \le \frac{2k^2 \lambda}{4^k} A_k \{\phi(1,n) + (k-2)e^{b_n(k-1)}\}.$$

Computation of the quantities in question reveals that $A_k \leq 1.0565$, $\phi(1,n) \leq 2.279$, and $e^{b_n(k-1)} = e^{b_n(6)} \leq 0.260$ and hence, using (2.1) and (2.3), $|\mathbf{P}(W_0 = 0) - e^{-\lambda}| \leq 0.05192$. Since also $e^{-\lambda} = e^{-2} \approx 0.135$, it follows that $P(W_0 = 0) > 0$. Of particular significance is the fact that Theorem 1 provides a total variation error bound that enables one to estimate more than just point probabilities: for example, the probability that there are between 2 and 5 unbeaten 7-subsets satisfies

$$0.52548 < P(W_0 \in \{2,3,4,5\}) \leq 0.62932.$$

3. PROOF OF THEOREM 2

By Theorem 10.J in Barbour, Holst and Janson (1992),

$$d_{\mathrm{TV}}\Big(\mathcal{L}(\{W_j\}_{j=0}^b), \prod_{j=0}^{b} \mathrm{Po}(\lambda_j)\Big) \leq \sum_{\alpha \in \Gamma} \left\{ {\pi_\alpha}^2 + \sum_{\beta \neq \alpha} \pi_\alpha \mathbf{P}(J_{\beta\alpha} \neq I_\beta) \right\} \tag{3.1}$$

where

$$\Gamma = \left\{(i,j), 1 \leq i \leq \binom{n}{k}, 0 \leq j \leq b\right\};$$

$$I_{i,j} = \begin{cases} 1 & \text{if the } i^{\text{th}}\ k\text{-subset is beaten by exactly } j \text{ external vertices,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\pi_{i,j} = P(I_{i,j} = 1) = \binom{n-k}{j} 2^{-kj}(1-2^{-k})^{n-k-j} := \rho_j;$$

$$W_j = \sum_{i=1}^{\binom{n}{k}} I_{i,j}; \qquad \lambda_j = E(W_j) = \binom{n}{k}\rho_j,$$

and where, for each (i,j), the coupled indicator variables $J_{i',j'} = J_{(i',j')(i,j)}$ satisfy the condition

$$\mathcal{L}\big\{(J_{(i',j')(i,j)} : (i',j') \in \Gamma)\big\} = \mathcal{L}\big\{(I_{i',j'}, (i',j') \in \Gamma)\big| I_{i,j} = 1\big\}. \tag{3.2}$$

The coupling is constructed as follows. If $I_{i,j} = 1$, we do nothing, letting $J_{(i',j')(i,j)} = I_{i',j'}$ for each (i',j'). If, on the other hand, $I_{i,j} = 0$, then i is either beaten by the s vertices in a set $A := \{a_1, a_2, \ldots, a_s\}$ where $s < j$, or by the t vertices in a set $B := \{b_1, b_2, \ldots, b_t\}$ where $t > j$. In the former case, we choose $j-s$ vertices randomly from the $n-k-s$ vertices in $\{1,2,\ldots,n\}\backslash(i \cup A)$ (so that the probability that any one is picked is $(j-s)/(n-k-s)$)

and "redirect arrows" so that each of these $j-s$ vertices beats the set i. In the latter case, we randomly select j of the t vertices in B, and continue to have the selected vertices beat the set i. For each of the other $t-j$ vertices in B, we realign the arrows to one of the 2^k-1 possibilities as in the proof of Theorem 1, chosen at random. Finally, we let $J_{(i',j')(i,j)}=1$ if the set i' is beaten by exactly j' vertices after the above-described coupling is implemented, with $J_{(i',j')(i,j)}=0$ otherwise. A little reflection reveals that (3.2) is indeed satisfied. Furthermore, since the intersection patterns of each of the $\binom{n}{k}$ k-subsets of V are identical, and since, for a fixed j, $\pi_{i,j}=\rho_j$ is the same for each i, (3.1) reduces to

$$\begin{aligned}
d_{\mathrm{TV}}\Big(\mathcal{L}(\{W_j\}_{j=0}^b),\prod_{j=0}^{b}\mathrm{Po}(\lambda_j)\Big)=&\binom{n}{k}\sum_{j=0}^{b}\rho_j^2+\binom{n}{k}\sum_{j=0}^{b}\sum_{(i',j')\neq(1,j)}\rho_j\mathbf{P}(I_{i',j'}\neq J_{i',j'})\\
\leq&\Lambda_bR_b+\sum_{j=0}^{b}\lambda_j\sum_{(i',j')\neq(1,j)}\mathbf{P}(I_{i',j'}\neq J_{i',j'}), \qquad (3.3)
\end{aligned}$$

where $J_{i',j'}=J_{(i',j')(1,j)}$.

Consider the second term on the right side of (3.3). Clearly,

$$\begin{aligned}
\sum_{(i',j')\neq(1,j)}\mathbf{P}(I_{i',j'}\neq J_{i',j'})=&\sum_{i'=1,j'\neq j}\mathbf{P}(I_{i',j'}\neq J_{i',j'})+\sum_{i'\neq1,0\leq j'\leq b}\mathbf{P}(I_{i',j'}\neq J_{i',j'})\\
=&\sum_{\substack{j'=0\\ j'\neq j}}^{b}\mathbf{P}(I_{1,j'}=1)+\sum_{i'\neq1,0\leq j'\leq b}\mathbf{P}(I_{i',j'}\neq J_{i',j'})\\
\leq&R_b+\Sigma_1+\Sigma_2, \qquad (3.4)
\end{aligned}$$

where

$$\Sigma_1=\sum_{|i'\cap1|=0,0\leq j'\leq b}\mathbf{P}(I_{i',j'}\neq J_{i',j'})$$

and

$$\Sigma_2=\sum_{i'\neq1,|i'\cap1|\neq0,0\leq j'\leq b}\mathbf{P}(I_{i',j'}\neq J_{i',j'})$$

represent, respectively, the contributions to (3.4) of k-sets that are disjoint from the set 1 or that intersect it. We begin by analysing Σ_1. Notice that our coupling has been constructed in a way that guarantees [when $i'\cap1=\emptyset$] that its effect on i' is one-directional. Specifically, we find, that after the coupling is implemented, i' is either beaten by fewer (or the same number of) vertices, or by more (or the same number of vertices) according as 1 was dominated by fewer than, or more than, j players. We thus have

$$\Sigma_1 = \Sigma_{11} + \Sigma_{12}, \tag{3.5}$$

where Σ_{11} and Σ_{12} correspond to 1 being dominated by fewer than or more than j vertices, respectively. So, if $N(1)$ denotes the number of vertices which initially dominate 1, we can write

$$\begin{aligned}\Sigma_{11} =& \sum_{|i'\cap 1|=0, 0\le j'\le b} \mathbf{P}(I_{i',j'} = 0, J_{i',j'} = 1, N(1) < j)\\ &+ \sum_{|i'\cap 1|=0, 0\le j'\le b} \mathbf{P}(I_{i',j'} = 1, J_{i',j'} = 0, N(1) < j)\\ =&\binom{n-k}{k}\sum_{j'=0}^{b}[\mathbf{P}(A_{j,j'}) + \mathbf{P}(B_{j,j'})].\end{aligned} \tag{3.6}$$

In (3.6), $A_{j,j'}$ represents the event that 1 is beaten by fewer than j, and i' by more than j' vertices initially, and that exactly j' players best i' after the coupling is implemented. Likewise, $B_{j,j'}$ is defined as the event that 1 is beaten by fewer than j players, and that i' is beaten by j' (less than j') individuals before (after) the coupling.

Notice that $A_{j,j'}$ occurs only if (i) 1 is initially dominated by fewer than j vertices, none of which are in the set i'; (ii) i' is initially beaten by j' vertices outside 1 and at least one from within 1; and, finally, (iii) at least one vertex in i' is chosen to bring the number beating 1 up to j. It follows that

$$\begin{aligned}\mathbf{P}(A_{j,j'}) \le& \sum_{s=0}^{j-1}\binom{n-2k}{s}\frac{1}{2^{ks}}\Big(1-\frac{1}{2^k}\Big)^{n-2k-s}\binom{n-2k}{j'}\frac{1}{2^{kj'}}\\ &(1-\frac{1}{2^k})^{n-2k-j'}(1-(1-\frac{1}{2^k})^k)(\frac{k(j-s)}{n-k-s})\\ \le& A_k^2 R_b \rho_{j'}\frac{k}{2^k}\frac{kb}{n-k-b};\end{aligned} \tag{3.7}$$

in (3.7), $A_k = (1-\frac{1}{2^k})^{-k}$ as before, while $(1-(1-2^{-k})^k)$ represents the probability that i' is beaten by at least one member of 1.

In a similar fashion, one may see that a necessary condition for $B_{j,j'}$ to occur is that 1 is beaten by fewer than j players, none of whom are from i'; i' is initially beaten by exactly j' vertices, of which at most $j'-1$, are from outside 1; and that at least one vertex in i' is chosen to beat 1. It follows, under the above set of conditions, that no member of 1 dominates i' any more, and that i' is beaten by at most $j'-1$ vertices. We thus have

$$\begin{aligned}
\mathbf{P}(B_{j,j'}) \leq & \sum_{s=0}^{j-1} \binom{n-2k}{s} \frac{1}{2^{ks}} (1-\frac{1}{2^k})^{n-2k-s} \sum_{r=0}^{j'-1} \binom{n-2k}{r} \frac{1}{2^{kr}} (1-\frac{1}{2^k})^{n-2k-r} \\
& \binom{k}{j'-r} \frac{1}{2^{k(j'-r)}} (1-\frac{1}{2^k})^{k-(j'-r)} (\frac{k(j-s)}{n-k-s}) \\
\leq & A_k (\sum_{s=0}^{j-1} \rho_s) \binom{n-k}{j'} \sum_{r=0}^{j'-1} \frac{\binom{n-2k}{r}\binom{k}{j'-r}}{\binom{n-k}{j'}} \frac{1}{2^{kj'}} (1-\frac{1}{2^k})^{n-k-j'} (\frac{kb}{n-k-b}) \\
\leq & A_k (\frac{kb}{n-k-b}) \rho_{j'} R_b \Big(\frac{kb}{n-k}\Big). \qquad (3.8)
\end{aligned}$$

Inequalities (3.6) through (3.8) now yield

$$\begin{aligned}
\Sigma_{11} \leq & \binom{n-k}{k} A_k^2 \sum_{j'=0}^{b} \frac{R_b k^2 b}{2^k (n-k-b)} \rho_{j'} \Big\{1 + \frac{2^k b}{n-k}\Big\} \\
\leq & A_k^2 \frac{\Lambda_b R_b k^2 b}{2^k (n-k-b)} \Big\{1 + \frac{2^k b}{n-k}\Big\}. \qquad (3.9)
\end{aligned}$$

For Σ_{12}, recall that if 1 is dominated by more than j vertices, i' cannot be beaten by fewer players than before. The contribution of Σ_{12} to (3.5) is thus given by

$$\begin{aligned}
\Sigma_{12} = & \sum_{\substack{|i' \cap 1| = 0 \\ 0 \leq j' \leq b}} \mathbf{P}(I_{i',j'} = 1, J_{i',j'} = 0, N(1) > j) + \sum_{\substack{|i' \cap 1| = 0 \\ 0 \leq j' \leq b}} \mathbf{P}(I_{i',j'} = 0, J_{i',j'} = 1, N(1) > j) \\
= & \binom{n-k}{k} \sum_{j'=0}^{b} [\mathbf{P}(C_{j,j'}) + \mathbf{P}(D_{j,j'})], \qquad (3.10)
\end{aligned}$$

where $C_{j,j'}$ (resp. $D_{j,j'}$) is the event that 1 is beaten by more than j vertices, and that i' is dominated by exactly j' (fewer than j') players *initially*, but by more than j' (exactly j') vertices *after* the coupling is carried out. Notice that for $C_{j,j'}$ to occur requires that 1 is beaten by at least one vertex from i'; that i' is initially dominated by exactly j' players, none of whom are members of 1; and that redirection leads to at least one vertex in 1 beating i'. It follows that

$$\begin{aligned}
\mathbf{P}(C_{j,j'}) \leq & \sum_{\ell=1}^{k} \binom{k}{\ell} \frac{1}{2^{k\ell}} (1-\frac{1}{2^k})^{k-\ell} \binom{n-2k}{j'} \frac{1}{2^{kj'}} (1-\frac{1}{2^k})^{n-2k-j'} \frac{k}{2^{k-\ell}} (\frac{2^{k-1}}{2^k-1})^{\ell} \\
\leq & A_k^2 \sum_{\ell=1}^{k} \binom{k}{\ell} \frac{1}{2^{k\ell}} (1-\frac{1}{2^k})^{k-\ell} \rho_{j'} \frac{k}{2^k}
\end{aligned}$$

$$
\begin{aligned}
&= A_k^2 (1 - (1 - \frac{1}{2^k})^k) \frac{k}{2^k} \rho_{j'} \\
&\leq \frac{k^2 A_k^2}{4^k} \rho_{j'}. \qquad (3.11)
\end{aligned}
$$

Finally, a necessary condition for $D_{j,j'}$ to occur is as follows. Initially, set 1 must be beaten by more than j vertices, at least one of which must be from i'. The set i', in turn, is to be dominated by $r < j'$ vertices, which must *all* be outside the set 1. Finally, in the process of redirecting arrows, exactly $j' - r$ vertices in 1 end up beating i', so that $J_{i'j'} = 1$. We thus have

$$
\begin{aligned}
\mathbf{P}(D_{j,j'}) \leq & \sum_{\ell=1}^{k} \binom{k}{\ell} \frac{1}{2^{k\ell}} (1 - \frac{1}{2^k})^{k-\ell} \sum_{r=0}^{j'-1} \binom{n-2k}{r} \frac{1}{2^{kr}} (1 - \frac{1}{2^k})^{n-2k-r} \\
& \binom{k}{j'-r} (\frac{1}{2^{k-\ell}})^{j'-r} \left\{ \frac{2^{k-(j'-r)}}{2^k - 1} \right\}^{\ell} \\
\leq & A_k \sum_{\ell=1}^{k} \binom{k}{\ell} \frac{1}{2^{k\ell}} (1 - \frac{1}{2^k})^{k-\ell} \sum_{r=0}^{j'-1} \binom{n-2k}{r} \frac{1}{2^{kr}} (1 - \frac{1}{2^k})^{n-2k-r} \binom{k}{j'-r} (\frac{1}{2^k})^{j'-r} \\
\leq & A_k \frac{k}{2^k} \binom{n-k}{j'} \frac{1}{2^{kj'}} \sum_{r=0}^{j'-1} (1 - \frac{1}{2^k})^{n-2k-r} \frac{\binom{n-2k}{r}\binom{k}{j'-r}}{\binom{n-k}{j'}} \\
\leq & A_k \frac{k}{2^k} \binom{n-k}{j'} \frac{1}{2^{kj'}} (1 - \frac{1}{2^k})^{n-2k-j'} (\frac{kj'}{n-k}) \\
\leq & \frac{A_k^2 k^2 b}{2^k (n-k)} \rho_{j'}. \qquad (3.12)
\end{aligned}
$$

Inequalities (3.10) through (3.12) thus reveal that

$$
\begin{aligned}
\Sigma_{12} = & \binom{n-k}{k} \sum_{j'=0}^{b} (\mathbf{P}(C_{j,j'}) + \mathbf{P}(D_{j,j'})) \\
\leq & \binom{n-k}{k} A_k^2 \sum_{j'=0}^{b} \rho_{j'} \frac{k^2}{4^k} \left\{ 1 + \frac{2^k b}{n-k} \right\} \\
\leq & A_k^2 \frac{\Lambda_b k^2}{4^k} \left\{ 1 + \frac{2^k b}{n-k} \right\}, \qquad (3.13)
\end{aligned}
$$

so that the total contribution of the non-overlapping sets to (3.4) is given, on using (3.9) and (3.13), by

$$
\Sigma_1 = \Sigma_{11} + \Sigma_{12} \leq A_k^2 \frac{\Lambda_b k^2}{4^k} \left\{ 1 + \frac{R_b 2^k b}{n-k-b} \right\} \left\{ 1 + \frac{2^k b}{n-k} \right\}. \qquad (3.14)
$$

We now turn to the case when i' and 1 have a non-empty overlap, and bound the contribution $\Sigma_2 = \sum_{i' \neq 1, i' \cap 1 \neq \emptyset, 0 \leq j' \leq b} \mathbf{P}(I_{i',j'} \neq J_{i',j'})$ to the discrepancy (3.4). In this case, the situation is far more complicated than in the non-overlapping case; in particular, the "one-directional" effect exploited systematically in the argument leading to (3.14) no longer holds. We bypass this difficulty, however, by bounding $\sum_{j'} \mathbf{P}(I_{i',j'} \neq J_{i',j'})$ by twice the probability that i' gets beaten by *a different number of vertices than before*. This latter quantity is far easier to control, as we shall see. Accordingly, we define $E_{j,i'}$ (resp. $F_{j,i'}$) as the event that i' is beaten by a different number of players after the coupling, on the event that 1 is beaten by more than (resp. less than) j vertices initially, and set

$$\mathbf{P}(E_{j,i'}) = \mathbf{P}(G_{j,i'}) + \mathbf{P}(H_{j,i'}) \quad \text{and} \quad \mathbf{P}(F_{j,i'}) = \mathbf{P}(K_{j,i'}) + \mathbf{P}(L_{j,i'}) \tag{3.15}$$

where $G_{j,i'}$ represents the event that 1 is dominated by more than j vertices, and that i' is subsequently beaten by *more players than before*, and where $H_{j,i'}$, $K_{j,i'}$ and $L_{j,i'}$ have similar interpretations. For $G_{j,i'}$, we dissect according to the size r of the overlap between 1 and i' and the number s of vertices in i' initially beating 1, and observe that there must be at least one vertex in 1 which afterwards dominates i'. This yields

$$\begin{aligned}
\sum_{i' \neq 1, i' \cap 1 \neq \emptyset} \mathbf{P}(G_{j,i'}) \leq & \sum_{r=1}^{k-1} \binom{k}{r} \binom{n-k}{k-r} \sum_{s=1}^{k-r} \binom{k-r}{s} \frac{1}{2^{ks}} (1 - \frac{1}{2^k})^{k-r-s} (k-r) \Big(\frac{2^{k-1}}{2^k - 1} \Big)^k \\
& \sum_{\ell=0}^{b} \binom{n-2k+r}{\ell} \frac{1}{2^{k\ell}} (1 - \frac{1}{2^k})^{n-2k+r-\ell} \\
\leq & A_k^2 \binom{n}{k} \sum_{\ell=0}^{b} \rho_\ell \sum_{r=1}^{k-1} \frac{\binom{k}{r}\binom{n-k}{k-r}}{\binom{n}{k}} \frac{(k-r)^2}{4^k} \\
\leq & A_k^2 \frac{k^4}{(n-k)4^k} \Lambda_b.
\end{aligned} \tag{3.16}$$

The quantity $\sum_{\ell=0}^{b} \binom{n-2k+r}{\ell} (1/2^{k\ell})(1-2^{-k})^{n-2k+r-\ell}$ in the above chain of inequalities represents the probability that i' is initially beaten by fewer than b vertices from outside $i' \cup 1$, which must be the case, since otherwise both $I_{i',j'}$ and $J_{i',j'}$ are zero.

For $H_{j,i'}$, we similarly obtain

$$\begin{aligned}
\sum_{i' \neq 1, i' \cap 1 \neq \emptyset} \mathbf{P}(H_{j,i'}) \leq & \sum_{r=1}^{k-1} \binom{k}{r} \binom{n-k}{k-r} \frac{n}{2^{2k-r}} \\
& \sum_{\ell=0}^{b} \binom{n-2k+r-1}{\ell} \Big(\frac{1}{2^k} - \frac{1}{2^{2k-r}} \Big)^\ell \Big(1 - \frac{1}{2^k} + \frac{1}{2^{2k-r}} \Big)^{n-2k+r-1-\ell}
\end{aligned}$$

$$\leq \frac{n\Lambda_b}{4^k} A_k \sum_{r=1}^{k-1} \frac{\binom{k}{r}\binom{n-k}{k-r}}{\binom{n}{k}} 2^r \left[1 + \frac{2^r}{2^k(2^k-1)}\right]^n, \tag{3.17}$$

because at least one vertex must initially beat both 1 and i', and at most b points outside $i' \cup 1$ may initially beat i' but not 1, since otherwise i' would afterwards still be dominated by more than b vertices, forcing both $I_{i',j'}$ and $J_{i',j'}$ to be zero. Comparing (3.17) with (2.4), we see that

$$\sum_{i' \neq 1, i' \cap 1 \neq \emptyset} \mathbf{P}(H_{j,i'}) \leq \frac{2\Lambda_b k^2}{4^k} A_k \{1 + B_{nk}\}, \tag{3.18}$$

where B_{nk} is as in Theorem 1.

We next turn to the computation of $\mathbf{P}(K_{j,i'})$ and $\mathbf{P}(L_{j,i'})$, which represent, respectively, the probabilities that i' is afterwards beaten by more or fewer vertices, both on the event that fewer than j players beat 1 initially. For $K_{j,i'}$, note that i' must initially be dominated by at most b vertices outside $1 \cup i'$ which do not dominate 1, and that at least one of the additional points chosen in the coupling to dominate 1 must also dominate i'. Hence

$$\begin{aligned}
&\sum_{i' \neq 1, i' \cap 1 \neq \emptyset} \mathbf{P}(K_{j,i'}) \\
&\leq \sum_{r=1}^{k-1} \binom{k}{r}\binom{n-k}{k-r} \sum_{l=0}^{b} \binom{n-2k+r}{l} \left[\frac{1}{2^k}\left(1 - \frac{1}{2^{k-r}}\right)\right]^l \left(1 - \frac{1}{2^k} + \frac{1}{2^{2k-r}}\right)^{n-2k+r-l} \frac{j}{2^{k-r}} \\
&\leq 2^{-k} b A_k \binom{n}{k} \sum_{r=1}^{k-1} \frac{\binom{k}{r}\binom{n-k}{k-r}}{\binom{n}{k}} 2^r \sum_{l=0}^{b} \rho_l \left[1 + \frac{2^r}{2^k(2^k-1)}\right]^n \\
&\leq \frac{2\Lambda_b k^2 b}{2^k n} A_k \{1 + B_{nk}\},
\end{aligned} \tag{3.19}$$

again using the argument following (2.4).

For $L_{j,i'}$, dissect according to the number s of vertices outside i' which initially dominate 1, and note that some vertex of 1 must initially dominate i', whereas some vertex of i' must afterwards dominate 1. This gives

$$\begin{aligned}
\sum_{i' \neq 1, i' \cap 1 \neq \emptyset} \mathbf{P}(L_{j,i'}) \leq &\sum_{r=1}^{k-1} \binom{k}{r}\binom{n-k}{k-r} \sum_{s=0}^{j-1} \binom{n-2k+r}{s} \frac{1}{2^{ks}} (1 - \frac{1}{2^k})^{n-2k+r-s} \\
&\left\{1 - (1 - \frac{1}{2^k})^{k-r}\right\} \frac{(k-r)b}{n-k-b} \\
\leq &A_k \binom{n}{k} \sum_{r=1}^{k-1} \frac{\binom{k}{r}\binom{n-k}{k-r}}{\binom{n}{k}} \sum_{s=0}^{b} \rho_s \frac{k}{2^k} \frac{kb}{n-k-b}
\end{aligned}$$

$$\leq A_k \Lambda_b \frac{k^2 b}{2^k(n-k-b)} \frac{k^2}{n-k}. \tag{3.20}$$

Inequalities (3.15), (3.16), (3.18), (3.19) and (3.20) reveal that

$$\Sigma_2 \leq \frac{2\Lambda_b k^2}{4^k} A_k^2 \Big\{1 + \frac{2^k b}{n-k-b}\Big\}\Big\{2(1+B_{nk}) + \frac{k^2}{n-k}\Big\}. \tag{3.21}$$

Equation (3.21), together with (3.3), (3.4) and (3.14), now proves Theorem 2.

Example. Consider a random tournament on $n = 9.4545719 \bullet 10^{33}$ vertices; suppose that $k = 100$ and that we wish to approximate the joint distribution of the vector (W_0, W_1, W_2). Then

$$\lambda_0 = \binom{n}{k}(1-\frac{1}{2^k})^{n-k} \approx 3,$$

$$\lambda_1 = \binom{n}{k}\binom{n-k}{1}\frac{1}{2^k}(1-\frac{1}{2^k})^{n-k-1} \approx 22387,$$

and

$$\lambda_2 = \binom{n}{k}\binom{n-k}{2}\frac{1}{2^{2k}}(1-\frac{1}{2^k})^{n-k-2} \approx 8.34832 \bullet 10^7,$$

while the error bound given by Theorem 2 is less than $2.2 \bullet 10^{-40}$; the bound from Theorem 2 is valid even though $n > k^2 2^k \log 4$, since (2.6) holds and $c_{100}(9.45... \bullet 10^{33}) > 0$.

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