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On the periodic spectrum of the 1-dimensional Schrödinger operator

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1. Introduction

Let us consider Hill's equation

$$-y''(x) + q(x)y(x) = \lambda y(x) \qquad (x \text{ in } \mathbf{R}) \tag{1}$$

$$y(x+2) = y(x) \qquad (x \text{ in } \mathbf{R}) \tag{2}$$

where q is a potential in $L^2[0, 1]$, periodically extended to all of $\mathbf{R}$. It is well known that those λ's for which (1)–(2) admits a non zero solution, form a non decreasing sequence of real numbers $\lambda_k = \lambda_k(q)$ $(k \geq 0)$, written with multiplicities. $(\lambda_k)_{k\geq 0}$ is called the periodic spectrum of q. Observe that, for convenience, the period in (2) has been chosen equal to 2 rather than 1 in order to include the so called antiperiodic eigenvalues as well. For q in $L^2[0, 1]$, the isospectral set $\operatorname{Iso}(q)$ is defined to be the set of all potentials p in $L^2[0, 1]$ such that $\lambda_k(p) = \lambda_k(q)$ $(k \geq 0)$ and $G(q)$ denotes the set of all potentials p in $L^2[0, 1]$ with the same gaps as q, i.e. $\lambda_0(p) = \lambda_0(q)$ and $\lambda_{2k}(p) - \lambda_{2k-1}(p) = \lambda_{2k}(q) - \lambda_{2k-1}(q)$ $(k \geq 1)$. Then $\operatorname{Iso}(q) \subseteq G(q)$.

This paper presents an elementary proof of a result due to J. Garnett and E. Trubowitz [GT1] which says that the converse inclusion $G(q) \subseteq \operatorname{Iso}(q)$ also holds:

THEOREM (Garnett, Trubowitz). *For all q in $L^2[0, 1]$,* $\operatorname{Iso}(q) = G(q)$.

In [GT1], this theorem is proved by applying harmonic measure arguments to the identification, due to Marcenko and Ostrovskii [MO], of band configurations with certain slit quarter planes. In this paper it is shown that the theorem is a direct consequence of the spectral theory for even potentials q in $L^2[0, 1]$ (i.e. $q(x) = q(1-x)$), as it is presented in the beautiful paper [GT2], using analysis in Hilbert space.

2. Proof of theorem

First observe that due to the fact that $\lambda_k(q+c)=\lambda_k(q)+c$ ($k\geq 0$; c real) it suffices to prove the theorem for potentials q in V where V is given by $V:=\{q\in L^2[0,1]:\lambda_0(q)=0\}$.

Let q be a fixed element in V. Clearly, for p in $G(q)$, $\operatorname{Iso}(p)\subset G(q)$ and thus $G(q)=\bigcup \operatorname{Iso}(p)$ where the union extends over all p in $G(q)$.

Denote by $\mu_n=\mu_n(p)$ $(n\geq 1)$ and $\nu_n=\nu_n(p)$ $(n\geq 0)$ the Dirichlet and Neuman spectrum of p in $L^2[0,1]$, that is the spectrum of (1) for the boundary conditions $y(0)=0$, $y(1)=0$ and $y'(0)=0$, $y'(1)=0$ respectively. It is well known (cf. e.g. [MW]) that $\nu_0\leq\lambda_0$ and $\lambda_{2n-1}\leq\mu_n$, $\nu_n\leq\lambda_{2n}$ $(n\geq 1)$. By the lemma below one can find for a given p in V an element $p_{\max}$ in $\operatorname{Iso}(p)\cap E$ with the properties that $\mu_n(p_{\max})=\lambda_{2n}(p)$ $(n\geq 1)$, $\nu_0(p_{\max})=\lambda_0(p)$ and $\nu_n(p_{\max})=\lambda_{2n-1}(p)$ $(n\geq 1)$ where E denotes the subspace of $L^2[0,1]$ of all even potentials p (i.e. $p(x)=p(1-x)$). Thus for all $n\geq 1$, $\mu_n(p_{\max})-\nu_n(p_{\max})=\lambda_{2n}(q)-\lambda_{2n-1}(q)$. For p in $L^2[0,1]$, define $\sigma(p)$ to be the sequence $(\mu_n(p)-\nu_n(p))_{n\geq 1}$. From the asymptotics (cf. e.g. [PT]) $\mu_n(p)$, $\nu_n(p)=n^2\pi^2+\int_0^1 p(x)\,dx+a_n$, b_n, where $\sum(a_n^2+b_n^2)<\infty$, one concludes that $\sigma(p)$ is an element in l^2. In [GT2] it is proved that the restriction of σ to $V\cap E$ is 1–1. Therefore $p_{\max}=q_{\max}$ for all p in $G(q)$ and thus $\operatorname{Iso}(p)=\operatorname{Iso}(q)$. This implies that $G(q)=\operatorname{Iso}(q)$.

LEMMA. *Let p be in $L^2[0,1]$. Then there exists a potential $p_{\max}$ in* Iso (p), *such that*

(1) $\mu_n(p_{\max})=\lambda_{2n}(p)$ $(n\geq 1)$

(2) $\nu_0(p_{\max})=\lambda_0(p)$ *and* $\nu_n(p_{\max})=\lambda_{2n-1}(p)$ $(n\geq 1)$

(3) $p_{\max}$ *is even, i.e. an element in E.*

Proof. For p in $L^2[0,1]$ with only a finite number of simple periodic eigenvalues, the existence of $p_{\max}$ with property (1) together with $\|p_{\max}\|_{L^2}=\|p\|_{L^2}$ is a direct consequence of results presented in [M, M]. By standard arguments one proves (2) and (3). To be more precise, denote by $y_1(x,\lambda)$ and $y_2(x,\lambda)$ the fundamental solutions of (1), i.e. the solutions $y(x,\lambda)$ of (1) with the initial conditions $y(0,\lambda)=1$, $y'(0,\lambda)=0$ and $y(0,\lambda)=0$, $y'(0,\lambda)=1$ respectively. For $\lambda_{2n}(p_{\max})=\mu_n(p_{\max})$, $y_2(x,\mu_n)$ is the corresponding eigenfunction and thus $y_2'(2,\mu_n)=1$. By investigating the Floquet matrix

$$F(\lambda)=\begin{pmatrix} y_1(1,\lambda) & y_2(1,\lambda)\\ y_1'(1,\lambda) & y_2'(1,\lambda)\end{pmatrix}$$

one concludes that $|y_2'(1,\mu_n)|=1$. Combining Corollary 2.2 and Lemma 3.4 in [PT], it follows that $p_{\max}$ is even. Using this fact together with reflection one now verifies that $\nu_n(p_{\max})$ is a periodic eigenvalue of $p_{\max}$ $(n \geq 0)$. From $\nu_0 \leq \lambda_0$ and $\lambda_{2n-1} \leq \mu_n$, $\nu_n \leq \lambda_{2n}$ $(n \geq 1)$ it then follows that $\nu_0(p_{\max}) = \lambda_0(p_{\max})$ and $\nu_n(p_{\max}) = \lambda_{2n-1}(p_{\max})$ $(n \geq 1)$.

Towards the general case, choose a sequence $(p_n)_{n\geq 1}$ of potentials in $L^2[0, 1]$ such that $p = \lim_{n\to\infty} p_n$ in the norm topology of $L^2[0, 1]$ and such that, for $n \geq 1$, p_n has only a finite number of periodic eigenvalues. (Cf. [CK] for an elementary proof concerning the existence of such a sequence). This implies that $\lim_{n\to\infty} \lambda_k(p_n) = \lambda_k(p)$ $(k \geq 0)$ as the eigenvalues depend continuously on the potential. Denote by q_n the potential $(p_n)_{\max}$ in Iso(p_n). Then $\lambda_{2k}(p_n) = \mu_k(q_n)$ $(k \geq 1)$, $\lambda_0(p_n) = \nu_0(q_n)$ and $\lambda_{2k-1}(p_n) = \nu_{2k-1}(q_n)$ $(k \geq 1)$. Moreover $\|p_n\|_{L^2[0,1]} = \|q_n\|_{L^2[0,1]}$. Thus $(q_n)_{n\geq 1}$ is a sequence, bounded in $L^2[0, 1]$; without loss of any generality we may assume that $(q_n)_{n\geq 1}$ converges weakly to a potential q in $L^2[0, 1]$. As E is a closed subspace of L^2, q must be an element of E. Now use that all eigenvalues λ_k $(k \geq 0)$, ν_k $(k \geq 0)$ and μ_k $(k \geq 1)$ depend continuously on the potential with respect to the weak topology of $L^2[0, 1]$ to conclude that $\lambda_k(q) = \lim_{n\to\infty} \lambda_k(q_n) = \lim_{n\to\infty} \lambda_k(p_n) = \lambda_k(p)$ $(k \geq 0)$ as well as $\mu_k(q) = \lambda_{2k}(p)$ $(k \geq 1)$, $\nu_0(q) = \lambda_0(p)$ and $\nu_k(q) = \lambda_{2k-1}(p)$ $(k \geq 1)$. Thus $p_{\max} := q$ has the desired properties.

REMARK. The proof of the lemma shows that for a given q in $L^2[0, 1]$ one has $\|p\|_{L^2} = \|q\|_{L^2}$ for all potentials p in Iso(q). For q in $C^\infty(\mathbf{R}/\mathbf{Z})$ this is a consequence of results in [MT].

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