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A CENTRAL LIMIT THEOREM FOR TWO-DIMENSIONAL RANDOM WALKS IN RANDOM SCENERIES

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Let S_n, $n \in \mathbb{N}$, be a recurrent random walk on $\mathbb{Z}^2$ ($S_0 = 0$) and $\xi(\alpha)$, $\alpha \in \mathbb{Z}^2$, be i.i.d. $\mathbb{R}$-valued centered random variables. It is shown that $\sum_{i=1}^n \xi(S_i)/\sqrt{n \log n}$ satisfies a central limit theorem. A functional version is presented.

1. Introduction. Let X_i, $i \in \mathbb{N}$, be a sequence of i.i.d. random vectors with values in $\mathbb{Z}^2$ which have mean 0 and a finite nonsingular covariance matrix Σ. We write

$$S_0 = 0, \qquad S_n = \sum_{i=1}^n X_i \quad \text{for } n \geq 1.$$

Furthermore, let $\xi(\alpha)$, $\alpha \in \mathbb{Z}^2$, be i.i.d. $\mathbb{R}$-valued random variables which are independent of the X_i, have mean 0 and a finite positive variance σ^2. We will prove that

$$Z_n = \sum_{i=0}^n \xi(S_i)$$

satisfies a central limit theorem.

When $(S_n)_{n \in \mathbb{N}}$ is a $\mathbb{Z}$-valued random walk, scaling limits of Z_n have been discussed by Kesten and Spitzer [6]. In this case, non-Gaussian limit laws appear. Kesten and Spitzer did not assume that the X and ξ have second moments, but that they belong to the domain of attraction of stable laws. There are some cases which are not covered by their results, e.g., if the X_i belong to the domain of attraction of the Cauchy law and the ξ have second moments. It is not difficult to modify the arguments presented here to prove that in this case Z_n is asymptotically Gaussian.

Obviously

$$E(Z_n) = 0$$

and it follows easily (see the Lemma 2.3) that

$$E(Z_n^2) \sim \text{const.}\, n \log n. \tag{1.1}$$

We use the notation $a_n \sim b_n$ for $a_n, b_n > 0$ if $\lim_{n \to \infty} a_n/b_n = 1$. We make the simplifying additional assumption that there is no proper subgroup L of $\mathbb{Z}^2$ such that for some $x \in \mathbb{Z}^2$ with $P(X_i = x) > 0$, one has $P(X_i - x \in L) = 1$. This assumption simplifies somewhat the statement and the proof of the theorem, but it is not really of importance.

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We define the random variables Y_n, $n \in \mathbb{N}$, with values in $D[0, \infty)$, the set of right continuous real-valued functions with left limits, by

$$Y_n(t) = \sqrt{\pi}\,(\det \Sigma)^{1/4} Z_{[nt]}/\sigma\sqrt{n \log n}\,.$$

For probability laws on $D[0, \infty)$ we have the standard notion of weak convergence (see [4]).

THEOREM. *The laws of Y_n converge weakly to the Wiener measure.*

The result has been conjectured by Kesten and Spitzer [6]. It is an instance where in the critical dimension between Gaussian and non-Gaussian behavior the scaling limits are still Gaussian with a slightly nonstandard normalizing factor (here $\sqrt{n \log n}$). It is easy to see that in dimensions greater than or equal to 3, $Z_n/\sqrt{n}$ is asymptotically normal. The argument essentially is contained in [6], page 10.

It has been brought to my attention by the referee that there exists an unpublished preprint by Borodin [3] where the result had earlier been proved. The proof given here is different from that of Borodin. Some features of the arguments here may be of independent interest (e.g., Lemma 2.4 and the proof of tightness).

2. Preliminary calculations.

We write $\chi(k)$, $k \in J = [-\pi, \pi)^2$, for the characteristic function of the X_i,

$$\chi(k) = E(\exp(i\langle k, X_1\rangle)).$$

$\langle\ ,\ \rangle$ is the inner product in $\mathbb{R}^2$. Our assumption on the X_i implies

$$\chi(k) = 1 \Leftrightarrow k = 0, \qquad k \in J \tag{2.1}$$

(see [1], Lemma 21.6). Obviously, $\chi(k)$ has the following expansion around 0:

$$\chi(k) = 1 - \tfrac{1}{2}\langle \Sigma k, k\rangle + R(k) \quad \text{where} \quad |R(k)| = o(|k|^2) \quad \text{for } k \to 0. \tag{2.2}$$

Let

$$V_n = \sum_{i,j=0}^{n} 1_{S_i = S_j}.$$

LEMMA 2.3.

$$E(V_n) \sim n \log n / 2\pi\sqrt{\det \Sigma}\,.$$

PROOF. This is well-known. For the convenience of the reader we give a proof which can be modified to derive also the variance (and higher moments). Let

$$\rho(\lambda) = \sum_{m=0}^{\infty} \lambda^m P(S_m = 0), \qquad |\lambda| < 1.$$

$\rho(\lambda)$ can be expressed in terms of $\chi(k)$:

$$\rho(\lambda) = (2\pi)^{-2} \int_J (1 - \lambda\chi(k))^{-1}\, dk.$$

As $\lambda \to 1$, we have $\rho(\lambda) \to \infty$.

We introduce the following notation: If $\alpha_\varepsilon(\lambda)$, $\beta_\varepsilon(\lambda)$ are positive functions for $\varepsilon > 0$ and $0 \leq \lambda < 1$ which diverge for $\lambda \to 1$, then we write

$$\alpha_\varepsilon(\lambda) \underset{\varepsilon\to 0}{\sim} \beta_\varepsilon(\lambda)$$

if

$$\lim_{\varepsilon\to 0} \liminf_{\lambda\to 1} \alpha_\varepsilon(\lambda)/\beta_\varepsilon(\lambda) = \lim_{\varepsilon\to 0} \limsup_{\lambda\to 1} \alpha_\varepsilon(\lambda)/\beta_\varepsilon(\lambda) = 1.$$

For $\varepsilon > 0$ let $U_\varepsilon = \{k \in J\colon |k| < \varepsilon\}$. Then by (2.1)

$$\rho(\lambda) \underset{\varepsilon\to 0}{\sim} (2\pi)^{-2} \int_{U_\varepsilon} (1 - \lambda\chi(k))^{-1}\, dk,$$

which by (2.2) is

$$\underset{\varepsilon\to 0}{\sim} \frac{1}{2}(2\pi)^{-1}(\det \Sigma)^{-1/2} \log\frac{1}{1-\lambda}.$$

By the Tauberian theorem for sequences (see [5], Theorem XIII 5.5)

$$\sum_{j=0}^{n} P(S_j = 0) \sim \tfrac{1}{2}(2\pi)^{-1}(\det \Sigma)^{-1/2} \log n.$$

Using

$$E(V_n) = n + 1 + 2\sum_{i=0}^{n-1} \sum_{j=1}^{n-i} P(S_j = 0)$$

the lemma follows. □

LEMMA 2.4.

$$\operatorname{var}(V_n) = O(n^2).$$

PROOF.

$$\operatorname{var}(V_n) = 4 \sum_{0\leq i_1<j_1\leq n} \sum_{0\leq i_2<j_2\leq n} \big[P(S_{i_1} = S_{j_1}, S_{i_2} = S_{j_2}) - P(S_{i_1} = S_{j_1})P(S_{i_2} = S_{j_2})\big].$$

The summands in this expression vanish if $j_1 \le i_2$ or $j_2 \le i_1$. Let I_1 be the set of 4-tuples (i_1, j_1, i_2, j_2) of indices which satisfy $0 \le i_1 \le i_2 < j_1 < j_2 \le n$ and I_2 the set where $0 \le i_1 < i_2 < j_2 \le j_1 \le n$. Then

$$\begin{aligned}\operatorname{var}(V_n) \le\ & 8\sum_{I_1} P(S_{i_1} = S_{j_1}, S_{i_2} = S_{j_2}) \\ & +8\sum_{I_2} P(S_{i_1} = S_{j_1}, S_{i_2} = S_{j_2}) \\ & +8\sum_{I_1} P(S_{i_1} = S_{j_1})P(S_{i_2} = S_{j_2}) \\ & +8\sum_{I_2} P(S_{i_1} = S_{j_1})P(S_{i_2} = S_{j_2}) \\ & +4\sum_{0\le i<j\le n} \left(P(S_i = S_j) - P(S_i = S_j)^2\right) \\ =\ & 8(a_1(n) + a_2(n) + a_3(n) + a_4(n)) + 4a_5(n),\end{aligned}$$

say.

$$a_5(n) = O(n^2) \quad \text{is obvious.}$$

First, we estimate

$$a_1(n) = \sum_{x\in\mathbf{Z}^2}\sum_{\mathbf{m}\in M_n} P(S_{m_2} = x)P(S_{m_3} = -x)P(S_{m_4} = x).$$

The second sum is over the set M_n of 5-tuples $\mathbf{m} = (m_1, m_2, \dots, m_5)$ satisfying $m_1, m_2, m_5 \ge 0$, $m_3, m_4 \ge 1$ and $m_1 + \cdots + m_5 = n$. Therefore,

$$\begin{aligned}\rho_1(\lambda) &= \sum_{n=0}^{\infty} a_1(n)\lambda^n \\ &= \sum_{x\in\mathbf{Z}^2} (1-\lambda)^{-2}\left(\sum_{m=0}^{\infty}\lambda^m P(S_m = x)\right) \\ &\quad\times\left(\sum_{m=1}^{\infty}\lambda^m P(S_m = -x)\right)\left(\sum_{m=1}^{\infty}\lambda^m P(S_m = x)\right) \\ &= (1-\lambda)^{-2}(2\pi)^{-4}\lambda^2 \\ &\quad\times\int_J\int_J dk_1\, dk_2 \frac{\chi(k_1)\chi(k_1+k_2)}{(1-\lambda\chi(k_1))(1-\lambda\chi(k_2))(1-\lambda\chi(k_1+k_2))}.\end{aligned}$$

If $\varepsilon > 0$ is fixed, then the integral over the set

$$\{(k_1, k_2) \in J^2 : |k_1| \ge \varepsilon \text{ or } |k_2| \ge \varepsilon\}$$

is of order $\log(1/(1-\lambda))$ by the same calculation as in Lemma 2.3. Let

$$U_\varepsilon = \{(k_1, k_2) \in J^2 : |k_1| < \varepsilon \text{ and } |k_2| < \varepsilon\}.$$

Then (using the same notation as in Lemma 2.3)

$$\int_{U_\varepsilon}\int dk_1\,dk_2 \frac{\lambda^2\chi(k_1)\chi(k_1+k_2)}{(1-\lambda\chi(k_1))(1-\lambda\chi(k_2))(1-\lambda\chi(k_1+k_2))}$$

$$\underset{\varepsilon\to 0}{\sim} \int_{U_\varepsilon}\int dk_1\,dk_2 \Bigg[\left((1-\lambda)+\frac{1}{2}\langle\Sigma k_1, k_1\rangle\right)\left((1-\lambda)+\frac{1}{2}\langle\Sigma k_2, k_2\rangle\right)$$

$$\times\left((1-\lambda)+\frac{1}{2}\langle\Sigma(k_1+k_2), k_1+k_2\rangle\right)\Bigg]^{-1}$$

$$\underset{\varepsilon\to 0}{\sim} \text{const.}(1-\lambda)^{-1}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{dk_1\,dk_2}{(1+|k_1|^2)(1+|k_2|^2)(1+|k_1+k_2|^2)}$$

$$\sim \text{const.}(1-\lambda)^{-1}.$$

Using the Tauberian theorem and the obvious fact that $a_1(n)$ is monotone, we conclude that

$$a_1(n) \sim \text{const.}\, n^2.$$

$a_2(n)$ can be treated similarly.

We come to $a_3(n)$. The argument is essentially the same as for $a_1(n)$: Let

$$\rho_3(\lambda) = \sum_{n=0}^{\infty}\lambda^n a_3(n)$$

$$= \sum_{n=0}^{\infty}\lambda^n \sum_{\mathbf{m}\in M_n} P(S_{m_2+m_3}=0)P(S_{m_3+m_4}=0)$$

$$= (1-\lambda)^{-2}(2\pi)^{-4}\lambda^2$$

$$\times\int_J\int_J \frac{\chi(k_1)\chi(k_2)^2\,dk_1\,dk_2}{(1-\lambda\chi(k_1))(1-\lambda\chi(k_2))(1-\lambda\chi(k_1)\chi(k_2))}$$

$$\underset{\varepsilon\to 0}{\sim} \text{const.}(1-\lambda)^{-3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{dk_1\,dk_2}{(1+|k_1|^2)(1+|k_2|^2)(1+|k_1|^2+|k_1|^2)}$$

$$\sim \text{const.}(1-\lambda)^{-3}$$

by the same type of arguments as for $a_1(n)$. So again we conclude that

$$a_3(n) \sim \text{const.}\, n^2.$$

$a_4(n)$ can be treated similarly. □

If $\alpha \in \mathbb{Z}^2$, let

$$N_\alpha(n) = \sum_{j=0}^{n} 1_{S_j=\alpha}.$$

LEMMA 2.5.

$$\sup_{\alpha\in\mathbb{Z}^2} N_\alpha(n) = o(n^\varepsilon) \quad a.s.\ \textit{for each}\ \varepsilon > 0.$$

PROOF. If $m \in \mathbb{N}$, then a straightforward computation gives

$$\sum_{n=0}^{\infty} \lambda^n E\big(N_0(n)^m\big) \sim \text{const.}(1-\lambda)^{-1}\left(\log\left(\frac{1}{1-\lambda}\right)\right)^m.$$

Therefore,

$$\sum_{k=0}^{n} E\big(N_0(k)^m\big) \sim \text{const. } n(\log n)^m.$$

As $E(N_0(k)^m)$ is increasing in k, we conclude that

$$E\big(N_0(n)^m\big) = o(n^{\varepsilon}) \quad \text{for any } \varepsilon > 0, \text{ and } m \in \mathbb{N}.$$

$N_0(n)$ obviously is the stochastically largest among the $N_\alpha(n)$. Therefore,

$$\begin{aligned}
&P\Big(\sup_{\alpha} N_\alpha(n) \geq t\Big) - O(n^{-3}) \\
&\quad = P\big(\sup\{N_\alpha(n)\colon |\alpha_1| \leq n^2, |\alpha_2| \leq n^2\} \geq t\big) \qquad \big[\text{where } \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2\big] \\
&\quad \leq (2n^2+1)^2 \sup_{\alpha} P(N_\alpha(n) \geq t) \leq (2n^2+1)^2 P(N_0(n) \geq t) \\
&\quad \leq (2n^2+1)^2 t^{-m} E\big(N_0(n)^m\big) = (2n^2+1)^2 t^{-m} o(n^{\varepsilon}) \quad \text{for any } m \in \mathbb{N},\ \varepsilon > 0.
\end{aligned}$$

From this inequality the lemma follows in a standard way. □

A straightforward calculation gives

LEMMA 2.6. *If* $0 < a < b$, *then*

$$\sum_{j=1}^{[an]} \sum_{i=[an]+1}^{[bn]} P(S_i = S_j) = o(n \log n).$$

3. Proof of the theorem. First, we prove the convergence of the finite dimensional distributions. Let $a_1, \ldots, a_m \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m$,

$$\begin{aligned}
&\sum_{j=1}^{m} a_j\big(Y_n(t_j) - Y_n(t_{j-1})\big) \\
&\qquad = \sum_{j=1}^{m} \sum_{\alpha \in \mathbb{Z}^2} a_j\big(N_\alpha([nt_j]) - N_\alpha([nt_{j-1}])\big)\xi(\alpha)/d_n,
\end{aligned}$$

where

$$d_n = \sigma\sqrt{n \log n}\,/\sqrt{2\pi}\,(\det \Sigma)^{1/4}.$$

Let $\mathscr{A}$ be the σ-field generated by $X_1, X_2, \ldots$. Conditioned on $\mathscr{A}$, the preceding expression is a sum of independent and nonidentically distributed random variables. Lemma 2.5 implies that almost surely the Lindeberg form of the central limit theorem applies (see [2], Theorem 7.2). Using the fact that

$$E(Z_n|\mathscr{A}) = 0, \qquad E\big(Z_n^2|\mathscr{A}\big) = \sigma^2 V_n,$$

we obtain

$$\lim_{n\to\infty} P\left(\frac{d_n \Sigma_{j=1}^m a_j\big(Y_n(t_j) - Y_n(t_{j-1})\big)}{\left\{\Sigma_{\alpha\in\mathbf{Z}^2}\left(\Sigma_{j=1}^m a_j\big(N_\alpha([nt_j]) - N_\alpha([nt_{j-1}])\big)\right)^2\right\}^{1/2}} \le x \,\middle|\, \mathscr{A} \right)$$

$$= (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-s^2/2}\,ds \quad \text{a.s.}$$

On the other hand, from Lemmas 2.3, 2.4 and 2.6 one concludes that

$$d_n^{-2}\sum_\alpha \left\{ \sum_{j=1}^m a_j\big(N_\alpha([nt_j]) - N_\alpha([nt_{j-1}])\big)\right\}^2$$

converge in probability to

$$\sum_{j=1}^m a_j(t_j - t_{j-1}).$$

Using this, one sees that

$$\sum_{j=1}^m a_j\big(Y_n(t_j) - Y_n(t_{j-1})\big)$$

is asymptotically normally distributed with mean 0 and variance $\Sigma_{j=1}^m a_j^2(t_j - t_{j-1})$. By the Cramér–Wold theorem (see [2], Theorem 7.7) $(Y_n(t_j))_{j=1,\dots,m}$ is asymptotically normally distributed with mean 0 and covariance matrix $(\min(t_i, t_j))_{i,j=1,\dots,m}$.

It remains to prove that the sequence Y_n is tight in $D[0,\infty)$. We closely follow an argument in [7].

The $(\xi(S_i))_{i\ge 0}$ form a stationary sequence. By the standard tightness conditions, it suffices to show that for any $\varepsilon > 0$ there exist arbitrary large $\lambda > 0$ such that for all large enough $n \in \mathbb{N}$

$$P\Big(\sup_{i\le n}|Z_i| \ge \lambda\sqrt{n\log n}\Big) \le \frac{\varepsilon}{\lambda^2}.$$

Let

$$Z_m^* = \max_{0\le i\le m} Z_i.$$

If $\rho > \sqrt{2}$, then

$$P\Big(Z_m^* \ge \rho\sigma\sqrt{V_m}\,|\mathscr{A}\Big) \le P\Big(Z_m \ge (\rho - \sqrt{2})\sigma\sqrt{V_m}\,|\mathscr{A}\Big) + P\Big(Z_{m-1}^* \ge \rho\sigma\sqrt{V_m},\, Z_{m-1}^* - Z_m \ge \sqrt{2}\,\sigma\sqrt{V_m}\,|\mathscr{A}\Big).$$

Conditioned on $\mathscr{A}$, Z_{m-1}^* and $Z_m - Z_{m-1}^*$ depend both nondecreasingly on each component of the independent random field $(\xi(\alpha))_{\alpha\in\mathbf{Z}^2}$. Therefore,

$$\begin{aligned} &P\Big(Z_{m-1}^* \ge \rho\sigma\sqrt{V_m},\, Z_{m-1}^* - Z_m \ge \sqrt{2}\,\sigma\sqrt{V_m}\,|\mathscr{A}\Big)\\ &\quad\le P\Big(Z_{m-1}^* \ge \rho\sigma\sqrt{V_m}\,|\mathscr{A}\Big)P\Big(Z_{m-1}^* - Z_m \ge \sqrt{2}\,\sigma\sqrt{V_m}\,|\mathscr{A}\Big)\\ &\quad\le P\Big(Z_m^* \ge \rho\sigma\sqrt{V_m}\,|\mathscr{A}\Big)\big(2\sigma^2 V_m\big)^{-1}E\Big((Z_{m-1}^* - Z_m)^2\,|\mathscr{A}\Big),\end{aligned}$$

$$E\Big((Z_{m-1}^* - Z_m)^2\,|\mathscr{A}\Big) \le E\big(Z_m^2|\mathscr{A}\big) = \sigma^2 V_m$$

(see Theorem 2 of [7]). Therefore,

$$P\left(Z_m^* \geq \rho\sigma\sqrt{V_m}\right) \leq 2P\left(Z_m \geq \left(\rho - \sqrt{2}\right)\sigma\sqrt{V_m}\right).$$

The same inequality for the field $(-\xi(\alpha))_{\alpha \in \mathbb{Z}^2}$ then gives

$$P\left(\max_{j \leq m}|Z_j| \geq \rho\sigma\sqrt{V_m}\right) \leq 2P\left(|Z_m| \geq \left(\rho - \sqrt{2}\right)\sigma\sqrt{V_m}\right).$$

As $V_m/m \log m$ converges in probability to $(2\pi)^{-1}(\det \Sigma)^{-1/2}$ there exists for each $\delta > 0$ a number $m_1(\delta)$ such that for $m \geq m_1(\delta)$ and $\rho > \sqrt{2}$,

$$P\left(\max_{j \leq m}|Z_j| \geq 2\rho\sigma\sqrt{m \log m}\,\pi^{-1/2}(\det \Sigma)^{-1/4}\right)$$
$$\leq 2P\left(|Z_m| \geq 2\left(\rho - \sqrt{2}\right)\sigma\sqrt{m \log m}\,\pi^{-1/2}(\det \Sigma)^{-1/4}\right) + \delta.$$

We put $\lambda = 2\rho\sigma\pi^{-1/2}(\det \Sigma)^{-1/4}$. Let $\varepsilon > 0$. As $\operatorname{var}(Z_m) \sim \text{const.}\, m \log m$, we can choose λ and m_2 large enough such that for $m \geq m_2$,

$$2P\left(|Z_m| \geq 2\left(\rho - \sqrt{2}\right)\sigma\sqrt{m \log m}\,\pi^{-1/2}(\det \Sigma)^{-1/4}\right) \leq \frac{\varepsilon}{2\lambda^2}.$$

Then if $m \geq \max(m_1(\varepsilon/2\lambda^2), m_2)$ we have

$$P\left(\max_{j \leq m}|Z_j| \geq \lambda\sqrt{m \log m}\right) \leq \varepsilon/\lambda^2$$

as required. □

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