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# A few remarks on blowing-up and connectedness

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## 1. Introduction

In this note we present some results which are related to the connectedness of fibers obtained by blowing-up an algebraic variety. The paper has two parts, according to the two types of questions which shall be treated, namely: bounds for the connectedness-dimension of fibers and calculation of Stein-factors of certain blowing-up morphisms.

More precisely, in the first part we consider the blowing-up $X \xrightarrow{\pi_I} \mathrm{Spec}(R)$ of a noetherian local ring $R$ with respect to an ideal $I$ of $R$. We then choose a proper ideal $J$ of $R$, which contains $I$ and give bounds for the connectedness-dimension of the fiber $Y := \pi_I^{-1}(V(J))$. Here, for a closed subset $Z$ of a noetherian scheme $X$, the *connectedness-dimension* $c(Z)$ of $Z$ is defined by:

$$(1.1) \qquad c(Z) := \min\{\dim(W) \mid W \subseteq Z \textit{ closed}, \ Z - W \textit{ disconnected}\}.$$

Thereby, the dimension of the empty set $\Phi$ is defined as $-1$ and $\Phi$ is considered as disconnected. So it always holds $c(Z) \geqq -1$ with equality iff $Z$ is disconnected. Often, we also shall use the following description of $c(Z)$:

**(1.2)** $c(Z) = \min\{\dim(Z_1 \cap Z_2)\}$, *where $Z_1$ and $Z_2$ are both unions of irreducible components of $Z$ and satisfy $Z = Z_1 \cup Z_2$.*

The starting point to obtain our bounds is [5], (3.4), which gives a criterion for the connectedness and a bound on the connectedness-dimension for the fibers of an arbitrary quasihomogeneous morphism $\pi: X \to \mathrm{Spec}(R)$. We first deduce a bound which is formulated in terms of the formal extension of our blowing-up, (2.5). Then we prove the lemma (2.7) which — for any arbitrary noetherian scheme — gives a bound for the connectedness-dimension in terms of the depths of the local rings of the scheme. (Bounds similar to (2.7) have been given by Hartshorne [11], (2.3), (2.4).) Finally we get a bound for the connectedness-dimension of our fiber $Y$, which depends only on the depths of the localisations of $R$, provided that $R$ is excellent, (2.9). We apply this to get very explicite bounds in case $R$ is either normal or satisfies the Serre-property $S_2$, (2.10). Thereby the normal case may be considered as a sharpened version of Zariski's connectedness theorem [14]. Finally we get bounds for the connectedness-dimensions of tangent cones, (2.11).

In the second part we consider *Macaulayfications* $\tilde{V} \xrightarrow{\tilde{\pi}} V$ of the type introduced by Faltings [6]. In particular we show that — even for surfaces — *there is in general no minimal proper birational Cohen-Macaulay* (*CM*) *model* (3.10), (3.11). More precisely we will see that if there is a non-trivial *finite* birational CM-model which is minimal among all such models, it is not minimal among all *proper* birational CM-models. This answers to the negative a corresponding question of [2]. It also shows that the Serre-property $S_2$ alone (contrary to normality) in general may not be realized by a minimal proper birational model. This answers to the negative a corresponding question of [1]. For simplicity we prove the needed triviality of Grothendieck-Stein-factorizations only for a special class of Macaulayfications. In fact this result holds for all Macaulayfications introduced in [2], [4] and [6].

We also want to point out that the hypothesis of excellence in (2.9), (2.10) (ii), (2.11) (iii) may be replaced by the weaker condition that $R$ is universally catenary and that its formal fibers have the Serre-property $S_1$. Consequently, the estimates (2.10) (ii) and (2.11) (iii) hold for an arbitrary CM-ring resp. an arbitrary locally noetherian CM-scheme.

As for the unexplained terminology we refer to [10] and [12].

## 2. Blowing-up and tangent cones

Let $(R, \mathfrak{m})$ be a local noetherian ring. Let $I \subseteqq R$ be an ideal. The *arithmetic rank* $r(I)$ of $I$ is defined as the minimal number of elements of $I$ which span an ideal whose radical is $\sqrt{I}$. Thus:

$$(2.1) \qquad r(I) = \min\{r \mid \exists a_1, \ldots, a_r \in I \text{ with } \sqrt{(a_1, \ldots, a_r)} = \sqrt{I}\}.$$

Let $X$ be a noetherian scheme, $Z \subseteqq X$ a closed subset and $x \in Z$ a point. Let $\mathscr{I}_x \subseteqq \mathscr{O}_{X,x}$ be the vanishing ideal of $Z$ at $x$. Then $r(\mathscr{I}_x)$ is called the *arithmetic rank of $Z$ at $x$*, and denoted by $r_{X,x}(Z)$. Furthermore we put

$$(2.2) \qquad r_X(Z) := \max\{r_{X,x}(Z) \mid x \in Z\}.$$

Using this notation, we have the following result, which has been shown in [5].

**(2.3) Proposition.** *Let $S = R \oplus R_1 \oplus \cdots$ be a graded noetherian $R$-algebra and assume that $\hat{X} := \operatorname{Proj}(\hat{R} \otimes_R S)$ is connected. Let $Z \subseteqq \operatorname{Spec}(R)$ be a closed set and let $\pi: X := \operatorname{Proj}(S) \to \operatorname{Spec}(R)$ be the canonical morphism. Then the fiber $Y := \pi^{-1}(Z)$ is connected and satisfies the inequality $c(Y) \geqq c(\hat{X}) - r_X(Y) - 1$.*

If $I \subseteq R$ is an ideal, $\mathfrak{R}(I)$ shall denote the *Rees-algebra* of $I$, e.g. the graded $R$-algebra $\bigoplus_{n \geq 0} I^n$. The *blowing-up* of $\operatorname{Spec}(R)$ with respect to $I$ is defined as the canonical morphism

$$(2.4) \qquad Bl_R(I) := \operatorname{Proj}(\mathfrak{R}(I)) \xrightarrow{\pi_I} \operatorname{Spec}(R).$$

As an application of (2. 3) we obtain

**(2. 5) Proposition.** *Let $I \subseteq J \subseteq \mathfrak{m}$ be ideals. Assume that $\hat{X} := Bl_{\hat{R}}(I\hat{R})$ is connected. Then $Y := \pi_I^{-1}(V(J))$ is connected and satisfies the inequality $c(Y) \geqq c(X) - r(J/I) - 2$.*

*Proof.* Applying (2. 3) with $S = R(I)$ and $Z = V(I)$ it remains to show that $r_{X,x}(Y) \leqq r(J/I) + 1$ for all $x \in Y$ (where $X = Bl_R(I)$). So, put $r = r(J/I)$ and let $a_1, \ldots, a_r \in J$ be such that $\sqrt{(a_1, \ldots, a_r)/I} = \sqrt{J/I}$. Let $x \in Y$. Then $I\mathcal{O}_{X,x}$ is a principal ideal of $\mathcal{O}_{X,x}$, [7]. Choosing a generator $b \in \mathcal{O}_{X,x}$ of $I\mathcal{O}_{X,x}$ we obtain $\sqrt{J\mathcal{O}_{X,x}} = \sqrt{(a_1, \ldots, a_r, b)\mathcal{O}_{X,x}}$. This proves our claim as $\sqrt{J\mathcal{O}_{X,x}}$ is the vanishing ideal of $Y$ at $x$.

We now want to give a connectedness-result for the fiber $Y$ in which the formal extension $Bl_{\hat{R}}(I\hat{R})$ of $Bl_R(I)$ does not occur. First we will prove some auxiliary results.

For a noetherian scheme $X$ we put

$$(2.6) \qquad \delta(X) := \min\{\dim \overline{\{x\}} \mid \operatorname{depth}(\mathcal{O}_{X,x}) \leqq 1\}.$$

**(2. 7) Lemma.** *$X$ connected $\Rightarrow c(X) \geqq \delta(X)$.*

*Proof.* Let $X_1, X_2 \subseteq X$ both be unions of irreducible components of $X$ such that $X_1 \cup X_2 = X$. We have to show $d := \dim(X_1 \cap X_2) \geqq \delta(X)$. Let $x$ be a generic point of $X_1 \cap X_2$ such that $\dim \overline{\{x\}} = d$. We must show $\operatorname{depth}(R) \leqq 1$, where $R = \mathcal{O}_{X,x}$. We may assume that $\operatorname{depth}(R) > 0$. Let $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be an irredundant primary decomposition of 0 in $R$. After an appropriate reordering of the primary components $\mathfrak{q}_i$ we find a natural number $s < n$ such that $\sqrt{\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s}$ and $\sqrt{\mathfrak{q}_{s+1} \cap \cdots \cap \mathfrak{q}_n}$ are the vanishing ideals of $X_1$ resp. of $X_2$ at $x$. Put $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$, $J = \mathfrak{q}_{s+1} \cap \cdots \cap \mathfrak{q}_n$. It follows $I \cap J = (0)$, $\sqrt{I + J} = \mathfrak{m}$, $\operatorname{depth}(R/I) > 0$, $\operatorname{depth}(R/J) > 0$. Applying local cohomology to the exact sequence $0 \to R \to R/I \oplus R/J \to R/(I + J) \to 0$ we obtain an exact sequence

$$H^0_{\mathfrak{m}}(R/I) \oplus H^0_{\mathfrak{m}}(R/J) \to H^0_{\mathfrak{m}}(R/(I + J)) \to H^1_{\mathfrak{m}}(R).$$

As $\operatorname{depth}(R/I)$, $\operatorname{depth}(R/J) > 0$ here the left hand term vanishes. As $\sqrt{I + J} = \mathfrak{m}$ the middle term equals $R/(I + J) \neq 0$. It follows $H^1_{\mathfrak{m}}(R) \neq 0$, thus $\operatorname{depth}(R) \leqq 1$.

In the sequel we use the notations

$$c(R) := c(\operatorname{Spec}(R)), \quad \delta(R) := \delta(\operatorname{Spec}(R)).$$

**(2. 8) Lemma.** *Let $I \subseteq \mathfrak{m}$ be an ideal such that $ht(I) > 0$ and such that $\dim(R/I) < c(R)$. Then $c(Bl_R(I)) \geqq c(R)$.*

*Proof.* Put $X = Bl_R(I)$ and let $X = X_1 \cup X_2$, where $X_1$ and $X_2$ are unions of irreducible components of $X$. We must show $\dim(X_1 \cap X_2) \geqq c(R)$. As $ht(I) > 0$, $\pi: X \to \operatorname{Spec}(R)$ is birational (and proper). So we have $\operatorname{Spec}(R) = \pi(X_1) \cup \pi(X_2)$ and both $\pi(X_1)$ and $\pi(X_2)$ are unions of irreducible components of $\operatorname{Spec}(R)$. So there is a point $x \in \pi(X_1) \cap \pi(X_2)$ such that $\dim \overline{\{x\}} \geqq c(R)$. As $\dim(R/I) < c(R)$ we have $x \notin V(I)$. So $\pi^{-1}(\{x\})$ consists of a single point $y$, and it holds $\dim \overline{\{y\}} \geqq \dim \overline{\{x\}}$. In particular we have $y \in X_1 \cap X_2$, which proves our claim.

**(2. 9) Proposition.** *Let $R$ be excellent and let $I \subseteq J \subseteq \mathfrak{m}$ be ideals such that $\dim(R/I) < \delta(R)$. Then $Y := \pi_I^{-1}(V(J))$ is connected and satisfies $c(Y) \geqq \delta(R) - r(J/I) - 2$.*

*Proof.* In view of (2. 5) we have to show that $c(Bl_R(I\hat{R})) \geqq \delta(R)$. In view of (2. 7) and (2. 8) it remains to show that $ht(I\hat{R}) > 0$ and that $\delta(\hat{R}) \geqq \delta(R)$. As $\dim(R/I) < \delta(R)$ we have $ht(I) > 0$, thus $ht(I\hat{R}) > 0$. It remains to prove the second inequality. So, let $\hat{\mathfrak{p}} \in \operatorname{Spec}(\hat{R})$ such that $\dim(\hat{R}/\hat{\mathfrak{p}}) = \delta(\hat{R})$ and $\operatorname{depth}(\hat{R}_{\hat{\mathfrak{p}}}) \leqq 1$. Put $\mathfrak{p} = \hat{\mathfrak{p}} \cap R$. As $R$ is excellent, one of the two following statements holds [8], [12]:

a) $\operatorname{depth}(R_{\mathfrak{p}}) = \operatorname{depth}(\hat{R}_{\hat{\mathfrak{p}}})$, $\dim(R/\mathfrak{p}) = \dim(\hat{R}/\hat{\mathfrak{p}})$.

b) $\operatorname{depth}(R_{\mathfrak{p}}) = 0$, $\dim(R/\mathfrak{p}) = \dim(\hat{R}/\hat{\mathfrak{p}}) + 1$.

a) immediately induces $\delta(\hat{R}) \geqq \delta(R)$. If b) holds, we have $\dim(\hat{R}/\hat{\mathfrak{p}}) > 0$, thus $\dim(R/\mathfrak{p}) > 1$. We thus find a $\mathfrak{q} \in V(\mathfrak{p}) - \operatorname{Ass}(R)$ such that $ht(\mathfrak{q}/\mathfrak{p}) = 1$. It follows $\operatorname{depth}(R_{\mathfrak{q}}) = 1$, $\dim(R/\mathfrak{q}) = \dim(R/\mathfrak{p}) - 1 = \delta(\hat{R})$, thus $\delta(\hat{R}) \geqq \delta(R)$.

**(2. 10) Corollary.** *Let $R$ be excellent, let $I \subset J \subset \mathfrak{m}$ be ideals and put $Y = \pi_I^{-1}(V(J))$.*

(i) *If $R$ is normal and $I \neq 0$, then $Y$ is connected and satisfies $c(Y) \geqq \dim(R) - r(J/I) - 2$.*

(ii) *If $R$ satisfies the Serre-property $S_2$ and if $ht(I) > 1$, then $Y$ is connected and satisfies $c(Y) \geqq \dim(R) - r(J/I) - 3$.*

*Proof.* (i) If $R$ is normal, $\hat{R}$ is a domain [12] and so (as $I \neq 0$) $Bl_{\hat{R}}(I\hat{R})$ is integral and of the same dimension as $R$. Now apply (2. 5).

(ii) As $R$ satisfies $S_2$ it follows $\delta(R) \geqq \dim(R) - 1$. Now we may conclude by (2. 9).

We denote by $CT_x(X)$ the *tangent cone* at a point $x$ of a noetherian scheme $X$.

**(2. 11) Corollary.** *Let $X$ be excellent, $x \in X$. Then*

(i) $c(CT_x(X)) \geqq \delta(R) - 2$.

(ii) *$X$ normal at $x \Rightarrow c(CT_x(X)) \geqq \dim(\mathcal{O}_{X,x}) - 1$.*

(iii) *$X$ satisfies $S_2$ at $x \Rightarrow c(CT_x(X)) \geqq \dim(\mathcal{O}_{X,x}) - 2$.*

*Proof.* Put $\mathcal{O}_{X,x} = R$. Then

$$CT_x(X) = \operatorname{Spec}(\mathfrak{R}(\mathfrak{m})/(\mathfrak{m})) \text{ and } Y := \pi_{\mathfrak{m}}^{-1}(\operatorname{Spec}(R/\mathfrak{m})) = \operatorname{Proj}(\mathfrak{R}(\mathfrak{m})/(\mathfrak{m}))$$

show that $c(Y) = c(CT_x(X)) - 1$ and allow to prove our statements applying (2. 9) and (2. 10) in the special case where $I = J = \mathfrak{m}$.

## 3. Stein-factors and Macaulayfication

As in the previous section, let $(R, \mathfrak{m})$ be a local noetherian ring and let $S = R \oplus S_1 \oplus \cdots$ be a graded noetherian $R$-algebra. We put $X = \operatorname{Proj}(S)$. Then $\Gamma(\mathcal{O}_X)$ is a finite $R$-algebra and defines the Grothendieck-Stein-factor of the canonical morphism $\pi: X \longrightarrow \operatorname{Spec}(R)$, [13], [7].

So, there is a commutative diagram

(3. 1)
$$\begin{array}{ccc} X := \operatorname{Proj}(S) & \xrightarrow{\quad \pi \quad} & \operatorname{Spec}(R) \\ & \searrow^{\pi_0} \qquad \nearrow_{\gamma} & \\ & X_0 = \operatorname{Spec}(\Gamma(\mathcal{O}_X)) & \end{array}$$

such that $\pi_0$ is proper, $\gamma$ is finite and the fiber $\pi_0^{-1}(\{y\})$ is connected for each $y \in \pi(X)$. We want to determine the Stein-factorization (3. 1) for certain classes of blowing-up.

First we recall an algebraic description of $\Gamma(\mathcal{O}_X)$. We use the functor $D_J := \varinjlim_n \operatorname{Hom}(J^n, \cdot)$ of $J$-transform, $J$ being an ideal of a noetherian ring $A$ [8], [1]. Obviously $D_J(A)$ is an $A$-algebra. Now, putting $S_+ = S_1 \oplus S_2 \oplus \cdots$ the $S$-algebra $D_{S_+}(S)$ is canonically graded (as an $S$-module) and it holds [7], [13]:

(3. 2)
$$\Gamma(\mathcal{O}_X) = D_{S_+}(S)_0.$$

Thereby — for a graded $S$-module $M$ — $M_n$ denotes the $R$-module of elements of degree $n$.

From now on, let $I \subseteqq R$ be an ideal. For an $R$-module $M$, $\Gamma_I(M)$ shall denote the $I$-torsion of $M$. Recall that $\Gamma_I(R) \subseteqq R = \Gamma(\mathcal{O}_{\operatorname{Spec}(R)})$ is the ideal of sections having support in $V(I)$ and that $D_I(R)$ is canonically isomorphic to $\Gamma(\mathcal{O}_{\operatorname{Spec}(R) - V(I)})$ [8], [1]. Moreover, there is a natural exact sequence [9], [1]:

(3. 3)
$$0 \longrightarrow \Gamma_I(M) \longrightarrow M \longrightarrow D_I(M) \longrightarrow H_I^1(M) \longrightarrow 0.$$

We recall another property of $I$-transforms, namely (cf. [8], [1]):

**(3. 4)** *Let $x \in I$ be a non-zero-divisor with respect to $\bar{R} = R/\Gamma_I(R)$. Then*

$$D_I(R) = \bigcup_{n \geqq 0} (\bar{R} : I^n)_{\bar{R}_x}.$$

**(3. 5) Lemma.** *Let $ht(I) > 0$ and put $\bar{R} = R/\Gamma_I(R)$, $D = D_I(R)$, $X = Bl_R(I)$. Then*

$$R \subseteqq \bigcup_{n \geqq 0} (I^n \bar{R} : I^n)_D = \Gamma(\mathcal{O}_X) \subseteqq D = \Gamma(\mathcal{O}_{\mathrm{Spec}(R) - V(I)}).$$

*Proof.* By (3. 3) it holds $\bar{R} \subseteqq D$. As $ht(I) > 0$ there is a $c \in I$ which is non-zero divisor with respect to $\bar{R}$. By (3. 4) it follows $D = \bigcup_{n \geqq 0} (\bar{R} : I^n)_{\bar{R}_c}$. So it remains to verify the equality $\bigcup_{n \geqq 0} (I^n \bar{R} : I^n)_{\bar{R}_c} = \Gamma(\mathcal{O}_X)$. Let $c^* \in \mathfrak{R}(I) = \bigoplus_{n \geqq 0} I^n =: S$ be the element $c \in I$ considered as homogeneous of degree one. Thus $c^* = (0, c, 0, \ldots) \in S$. Then by our choice of $c$, $c^*$ belongs to $S_+$ and is a non-zero divisor with respect to $\bar{S} := S/\Gamma_{S_+}(S)$. So, by (3. 2) and (3. 4) we obtain $\Gamma(\mathcal{O}_X) = [\bigcup_{n \geqq 0} (\bar{S} : (S_+)^n)_{(S_+)_{c^*}}]_0$. Observing that $\bar{S} = \mathfrak{R}(I\bar{R})$ it follows

$$\Gamma(\mathcal{O}_X) = \bigcup_{n \geqq 0} ((I\bar{R})^n : (I\bar{R})^n)_{R_c},$$

thus our claim.

Now, let $V$ be an irreducible variety of dimension $d > 1$ and let $p \in V$ be a closed point. *Assume that $V - \{p\}$ consists only of Cohen-Macaulay points.* We put $R = \mathcal{O}_{V,p}$, $\mathfrak{m} = \mathfrak{m}_{V,p}$. Then, according to Grothendieck's finiteness-theorem [9] the local cohomology-modules $H^i_{\mathfrak{m}}(R)$ are finitely generated for $i = 0, 1, \ldots, d-1$. As a consequence there is an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ of $R$ such that for each system of parameters $x_1, \ldots, x_d \in \mathfrak{q}$ it holds [6], [2]:

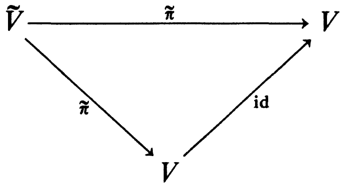
(3. 6) $$(x_1, \ldots, x_d)\, H^i_{\mathfrak{m}}(R/(x_1, \ldots, x_j)) = 0, \quad \text{whenever} \quad i + j < d.$$

We fix such a system $x_1, \ldots, x_d \in \mathfrak{q}$ and put $I = (x_1, \ldots, x_d)$. Then the blowing-up $\pi_I : Bl_R(I) \to \mathrm{Spec}(R)$ is a *Macaulayfication*, e.g. $Bl_R(I)$ is a CM-scheme [6], [2]. Now let $\mathcal{I} \subseteqq \mathcal{O}_V$ be the ideal defined by

$$\mathcal{I}_x = \begin{cases} \mathcal{O}_{V,x}, & \text{for} \quad x \neq p, \\ I, & \text{for} \quad x = p \end{cases}$$

and put $\tilde{V} := \mathrm{Proj}(\mathfrak{R}(\mathcal{I}) = \bigoplus_{n \geqq 0} \mathcal{I}^n)$. Then clearly the blowing-up $\tilde{\pi} : \tilde{V} \to V$ is a Macaulayfication.

**(3. 7) Proposition.** *The Grothendieck-Stein-factorization of the Macaulayfication* $\tilde{\pi}\colon \tilde{V} \to V$ *is given by*




*Proof.* As $\tilde{\pi}\colon \tilde{V} - \tilde{\pi}^{-1}(\{p\}) \to V - \{p\}$ is an isomorphism, it suffices to show that the Grothendieck-Stein-factor of $\pi_I\colon Bl_R(I) \to \operatorname{Spec}(R)$ is given by $\gamma = \mathrm{id}\colon \operatorname{Spec}(R) \to \operatorname{Spec}(R)$. As $V$ is integral we have $\Gamma_I(R) = 0$. So, as $D := D_{\mathfrak{m}}(R) = D_I(R)$ it remains to show that $\bigcup_{n \geqq 0} (I^n : I^n)_D = R$, (3. 5) (observe that $R \subseteqq D$ by (3. 4)). So we have to prove $(I^n : I^n)_D \subseteqq R$ for all $n \geqq 0$. To do so, put $L = (x_1, \ldots, x_{d-1})R$. Then we know, that $(L^m : x_d)_D \subseteqq L^{m-1}$ for all $m \in \mathbb{N}$, [2]. It follows

$$(I^m : x_d)_D = ((L^m + x_d I^{m-1}) : x_d)_D = (L^m : x_d)_D + I^{m-1} = I^{m-1},$$

thus $(I^m : I)_D = I^{m-1}$ for all $m \in \mathbb{N}$. By induction on $n$ we get immediately $(I^n : I^n)_D = R$ for all $n \geqq 0$. This proves our claim.

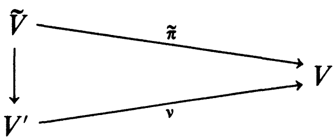
In view of (3. 3) it is clear that $D = D_{\mathfrak{m}}(R)$ is a finite birational integral extension of $R$ and that

(3. 8) $$H^i_{\mathfrak{m}}(D) = \begin{cases} 0, & i \leqq 1 \\ H^i_{\mathfrak{m}}(R), & i > 1 \end{cases} \quad \text{(cf. [3])}.$$

So, putting $V' = \operatorname{Spec}(\mathcal{O}_{V-\{p\}})$, the canonical map $\nu\colon V' \to V$ is finite, birational and $V'$ is of depth $> 1$ in all closed points.

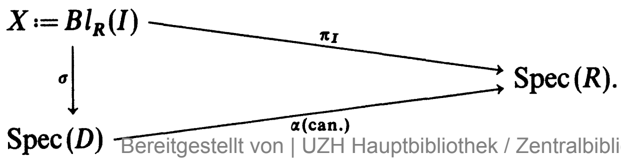
**(3. 9) Corollary.** *Let* $\operatorname{depth}(\mathcal{O}_{V,p}) = 1$. *Then the Macaulayfication* $\tilde{\pi}\colon \tilde{V} \to V$ *does not factor through the finite morphism* $\nu\colon V' \to V$.

*Proof.* A factorization

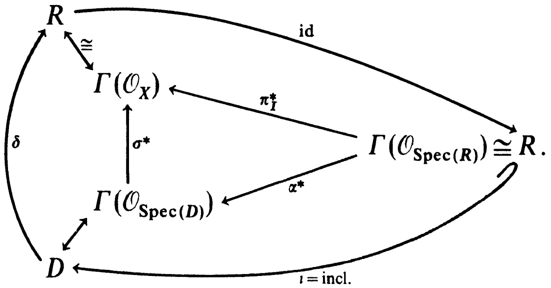



would induce a commutative diagram

(∗)



From the previous proof we have the canonical isomorphism $\Gamma(\mathcal{O}_X) \cong R$. So, passing to global sections in (*) gives the commutative diagram




By (3. 3) we have $\operatorname{coker} \iota = H^1_{\mathfrak{m}}(R)$ and thus get a splitting exact sequence of $R$-modules

$$0 \to R \xrightarrow{\iota} D \to H^1_{\mathfrak{m}}(R) \to 0.$$

As $D$ is a birational extension of $R$, $\iota$ must be an isomorphism. It follows $H^1_{\mathfrak{m}}(R) = 0$, thus $\operatorname{depth}(\mathcal{O}_{V,p}) > 1$.

Keep the above hypotheses and notations. Then $D = D_{\mathfrak{m}}(R)$ is the least $R$-module of depth $>1$, containing $R$ and contained in the quotient field of $R$ [1], [2]. This shows that $V' \xrightarrow{\nu} V$ is the (essentially unique) minimal finite birational model of $V$ whose closed points are of depth $>1$. In the special case, where $H^i_{\mathfrak{m}}(R) = 0$ for $i \neq 1, d$, (3. 8) shows that $D$ is the least CM-$R$-module containing $R$ and being contained in the quotient field of $R$. So, in this situation, $V' \xrightarrow{\nu} V$ is the minimal *finite* birational model of $V$, which is CM (thus the minimal *finite* Macaulayfication). Observing this we get

**(3. 10) Corollary.** *Let* $\operatorname{depth}(\mathcal{O}_{V,p}) = 1$. *Then*

(i) *There is no minimal proper birational model* $V^* \xrightarrow{\mu} V$, *whose closed points are of* depth $>1$.

(ii) *If* $H^i_{\mathfrak{m}_{V,p}}(\mathcal{O}_{V,p}) = 0$ *for all* $i \neq 1, d$, *there is no minimal proper birational model* $V^* \xrightarrow{\mu} V$ *such that* $V^*$ *is* CM.

*Proof.* Indeed, such a minimal model $V^* \xrightarrow{\mu} V$ would be a factor of the corresponding minimal finite birational model $V' \xrightarrow{\nu} V$, thus coincide with this latter. But then the Macaulayfication $\tilde{V} \xrightarrow{\tilde{\pi}} V$ would factor through $V' \xrightarrow{\nu} V$. This contradicts (3. 9).

**(3. 11) Examples.** (i) Let $z$ and $w$ be indeterminates and let

$$V = \operatorname{Spec}(\mathbb{C}[z, w^2, zw, w^3]).$$

Let $p \in V$ be the closed point $z = w = 0$. It is immediate that

$$D_{\mathfrak{m}_{V,p}}(\mathcal{O}_{V,p}) = \mathbb{C}[z, w]_{(z,w)} = \mathcal{O}_{\mathbb{A}^2, o}$$

(where $o$ denotes the origin of $\mathbb{A}^2$). So the normalization morphism $\nu: \mathbb{A}^2 \to V$ (which is induced by the inclusion $\mathbb{C}[z, w^2, zw, w^3] \hookrightarrow \mathbb{C}[z, w]$) is the minimal finite Macaulayfication of $V$. It is immediate to check that $H^1_{\mathfrak{m}}(\mathcal{O}_{V,p}) \cong \mathbb{C}$. This allows to choose the previously introduced ideal $\mathfrak{q}$ as $\mathfrak{m} = \mathfrak{m}_{V,p}$, [3]. Putting

$$J = (z, w^2)\, \mathbb{C}[z, w^2, zw, w^3],$$

$\tilde{V} := \operatorname{Proj}(\bigoplus_{n \geq 0} J^n) \xrightarrow{\tilde{\pi}} V$ is a Macaulayfication of $V$ which does not factor through $\nu: \mathbb{A}^2 \to V$. It is easy to see that $\nu$ is a homeomorphism. So, *topologically* $\tilde{\pi}$ *factors through* $\nu$.

(ii) Let $V = \operatorname{Spec}(\mathbb{C}[z, zw, w^2 - w, w^3 - w])$ and let $p$ be the point $z = w = 0$. It is easy to check that the normalization morphism $\nu: \mathbb{A}^2 \to V$ (induced by the inclusion $A := \mathbb{C}[z, zw, w^2 - w, w^3 - w] \hookrightarrow \mathbb{C}[z, w]$) is the minimal finite Macaulayfication of $V$, and that $H^1_{\mathfrak{m}_{V,p}}(\mathcal{O}_{V,p}) \cong \mathbb{C}$. So, putting $J = (z, w^2 - w)A$, $\tilde{\pi}: \tilde{V} = \operatorname{Proj}(\bigoplus_{n \geq 0} J^n) \to V$ is a Macaulayfication which does not factor through $\nu: \mathbb{A}^2 \to V$. Here, *there is no surjective topological factorization*, as the fiber $\nu^{-1}(\{p\})$ consists of the two points $(0, 0), (1, 0) \in \mathbb{A}^2$, whereas the fiber $\tilde{\pi}^{-1}(\{p\})$ is homeomorphic to $\mathbb{P}^1$, thus connected. Note, that $\operatorname{Proj}(\bigoplus_{n \geq 0} \mathfrak{m}^n_{V,p}) \to V$ defines a Macaulayfication which factors through $\nu$ by a proper map and whose exceptional fiber consists of two copies of $\mathbb{P}^1$.

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