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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE COMPRESSIBLE REYNOLDS LUBRICATION EQUATION*

MICHEL CHIPOT[†] AND MITCHELL LUSKIN[‡]

Abstract. We prove the existence of a solution to the compressible Reynolds lubrication equation and we show that our solution is unique in the class of nonnegative solutions (under some additional hypotheses, we prove that our solution is unique among all weak solutions). We also prove the strong result that the mapping from the boundary data to the solution is monotonic.

Key words. compressible Reynolds lubrication equation, nonlinear elliptic boundary value problem

AMS(MOS) subject classifications. Primary 76N99, 35J65

1. Introduction. The compressible Reynolds lubrication equation [7, p. 63] is the nonlinear elliptic partial differential equation

$$(1.1) \quad \begin{aligned} 6\mu\mathbf{V} \cdot \nabla(Ph) &= \nabla \cdot (h^3 P \nabla P), & x &= (x_1, x_2) \in \Omega, \\ P &= P_a, & x &\in \partial\Omega, \end{aligned}$$

which gives the pressure, $P = P(x)$, that develops in a layer of air of thickness, $h = h(x)$, which is confined between two solid bodies when the average of the velocities of the upper and lower bodies is $\mathbf{V} = (V_1, V_2)$. The air is assumed to be isothermal and to be an ideal gas (the density is taken to be proportional to the pressure) [7]. Here, $\mu > 0$ is the dynamic viscosity of the fluid, $\Omega \subseteq \mathbb{R}^2$ is the region (with smooth boundary, $\partial\Omega$) where the upper and lower bodies are in proximity, and $P_a > 0$ is the ambient pressure.

When the thickness of a gaseous fluid layer is of the order of the molecular mean free path of the gas, then the compressible Reynolds equation becomes a poor model for the pressure in the fluid layer. In many applications in the modeling of the mechanical systems of magnetic recording the following modified Reynolds equation [3] has been found to be a good model equation for predicting the pressure in the fluid layer

$$(1.2) \quad \begin{aligned} 6\mu\mathbf{V} \cdot \nabla(Ph) &= \nabla \cdot (h^3 P \nabla P) + \nabla \cdot (6\lambda_a h^2 P_a \nabla P), & x &\in \Omega, \\ P &= P_a, & x &\in \partial\Omega. \end{aligned}$$

Here $\lambda_a \geq 0$ is the mean free path of the gas at ambient pressure. Note that $\lambda_a = 0$ gives the compressible Reynolds equation (1.1) and all the analysis and results which we give for the modified Reynolds equation, (1.2), are valid for the modified Reynolds equation, (1.1), unless we state otherwise.

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We assume only that $h=h(x)$ is a Lipschitz continuous function such that for positive constants h_{\min}, h_{\max} , and H we have the bounds

$$(1.3) \quad \begin{aligned} 0 < h_{\min} \leq h(x) \leq h_{\max}, \quad x \in \Omega, \\ |\nabla h(x)| \leq H, \quad \text{a.e. } x \in \Omega. \end{aligned}$$

We prove the existence of a nonnegative weak solution to the Reynolds equation (1.2), and we prove that this solution is unique in the class of nonnegative functions. We actually prove the stronger result that the mapping $S: P_a \rightarrow P(x)$ from the boundary data to the solution satisfies the monotonicity result

$$P_a \geq Q_a \text{ implies } P(x) \geq Q(x) \text{ for all } x \in \Omega$$

where $Q = S(Q_a)$. Moreover, if

$$\mathbf{V} \cdot \nabla h \leq 0,$$

we prove that our solution to the Reynolds equation (1.1) is unique among all weak solutions.

We note that the implicit function theorem has been used in [5] to obtain the existence of solutions to (1.1) for small values of the velocity, $V = (V_1^2 + V_2^2)^{1/2}$. However, our techniques give the existence and uniqueness of solutions to (1.1) for all values of \mathbf{V} .

In the mechanical systems of magnetic recording, the pressure developed in the fluid layer is coupled to the deformation of the confining solid bodies (such as a disk or a tape) [8]. In a forthcoming paper, we will give an analysis of the system of coupled partial differential equations for the pressure and the deformation. In this case, h will depend on the deformation and the present results will be useful there.

2. Existence of solutions. In what follows, the $L^2(\Omega)$ inner product for real-valued functions $\psi, \zeta \in L^2(\Omega)$ is denoted by

$$(\psi, \zeta) = \int_{\Omega} \psi(x) \zeta(x) dx$$

with corresponding norm

$$\|\psi\|^2 = (\psi, \psi).$$

If $\psi, \zeta: \Omega \rightarrow \mathbb{R}^2$, then the $L^2(\Omega)$ inner product for $\psi, \zeta \in L^2(\Omega)$ is similarly denoted by

$$(\psi, \zeta) = \int_{\Omega} \psi(x) \cdot \zeta(x) dx$$

with corresponding norm

$$\|\psi\|^2 = (\psi, \psi),$$

where $\psi(x) \cdot \zeta(x)$ denotes the usual Euclidean inner product in \mathbb{R}^2 . Also, we define the Sobolev spaces

$$\begin{aligned} H^1(\Omega) &= \{ \psi \in L^2(\Omega) \mid \nabla \psi \in L^2(\Omega) \}, \\ H_0^1(\Omega) &= \{ \psi \in H^1(\Omega) \mid \psi \equiv 0 \text{ on } \partial\Omega \}, \end{aligned}$$

with norm

$$\|\psi\|_{H^1(\Omega)}^2 = \|\psi\|^2 + \|\nabla\psi\|^2,$$

where equality on the boundary is understood in the trace sense [1].

We first prove the existence of a solution to (1.2). Set $\Lambda = 6\mu V$ and $\lambda = 6\lambda_a P_a$ and introduce the dependent variable

$$u = \frac{1}{2}P^2 + \frac{\lambda}{h}P.$$

If P is a nonnegative solution to (1.2), then we see that u is a nonnegative solution to

$$(2.1) \quad \begin{aligned} \nabla \cdot (h^3 \nabla u) &= \nabla \cdot (\beta(x, u) \Lambda - \lambda \beta(x, u) \nabla h(x)), & x \in \Omega, \\ u(x) &= \frac{1}{2}P_a^2 + \frac{\lambda}{h(x)}P_a \equiv u_a(x), & x \in \partial\Omega, \end{aligned}$$

where

$$\beta(x, u) = \begin{cases} -\lambda + \sqrt{\lambda^2 + 2h^2u}, & u \geq 0, \\ 0, & u \leq 0. \end{cases}$$

Conversely, if u is a nonnegative of (2.1), then

$$P(x) = -\frac{\lambda}{h(x)} + \sqrt{\lambda^2/h(x)^2 + 2u(x)}$$

is a nonnegative solution to (1.2). Thus, we shall show that (2.1) has a nonnegative solution.

We set

$$\alpha(x, u) = \beta(x, u) \Lambda - \lambda \beta(x, u) \nabla h(x).$$

We define a weak solution, u , to the problem:

$$(2.2) \quad \begin{aligned} \nabla \cdot (h^3(x) \nabla u) &= \nabla \cdot \alpha(x, u), & x \in \Omega, \\ u &= \varphi, & x \in \partial\Omega, \end{aligned}$$

for $\varphi \in H^1(\Omega)$ to be u such that

$$(2.3) \quad \begin{aligned} u - \varphi &\in H_0^1(\Omega), \\ \int_{\Omega} h^3(x) \nabla u \cdot \nabla \xi \, dx &= \int_{\Omega} \alpha(x, u) \cdot \nabla \xi \, dx, & \xi \in H_0^1(\Omega). \end{aligned}$$

Using the inequality $\sqrt{A} + \sqrt{B} \geq \sqrt{A+B}$, it is easy to check that

$$\begin{aligned} |\beta(x, v)|^2 &\leq 2h^2|v|, & x \in \Omega, \quad v \in \mathbb{R}, \\ |\beta(x, v) - \beta(x, w)|^2 &\leq 2h^2|v - w|, & x \in \Omega, \quad v, w \in \mathbb{R}. \end{aligned}$$

Hence for some constants c_1, c_2 we have

$$(2.4) \quad \begin{aligned} |\alpha(x, v)|^2 &\leq c_1|v|, & x \in \Omega, \quad v \in \mathbb{R}, \\ |\alpha(x, v) - \alpha(x, w)|^2 &\leq c_2|v - w|, & x \in \Omega, \quad v, w \in \mathbb{R}. \end{aligned}$$

We shall prove the following theorem.

THEOREM 1. *There exists a weak solution to (2.2).*

In §3, we shall prove that a (weak) solution u , to (2.1) is nonnegative almost everywhere. Thus, we can define

$$(2.5) \quad P(x) = \frac{-\lambda}{h(x)} + \sqrt{\lambda^2/h(x)^2 + 2u(x)}$$

to be a weak solution to (1.2). Hence, we have the following theorem.

THEOREM 2. *There exists a nonnegative, weak solution, P , to (1.2).*

We now turn to the proof of Theorem 1. We shall use the Schauder fixed point theorem. We denote by $T: L^2(\Omega) \rightarrow L^2(\Omega)$ the map $u = T(v)$ where $u \in H^1(\Omega)$ is the solution to $u = \varphi$ on $\partial\Omega$,

$$(2.6) \quad \int_{\Omega} h^3(x) \nabla u \cdot \nabla \xi \, dx = \int_{\Omega} \alpha(x, v) \cdot \nabla \xi \, dx, \quad \xi \in H_0^1(\Omega).$$

The Schauder fixed point theorem states that if T is continuous and if there exists a closed, convex set B such that $T(B) \subset B$ and $\overline{T(B)}$ is compact, then there exists a fixed point, $u \in B$, of T , i.e., $T(u) = u$. We note that a fixed point of T is a weak solution of (2.2).

Set

$$B_R = \{v \in L^2(\Omega) \mid \|v\| \leq R\}.$$

We shall show that there exists a positive constant R_1 such that

$$T(B_R) \subset B_R \quad \text{if } R \geq R_1.$$

Further, we shall show that there exists a positive constant, $c_3 = c_3(R)$, such that

$$(2.7) \quad \|T(v)\|_{H^1(\Omega)} \leq c_3, \quad v \in B_R.$$

Thus, it follows from Rellich's theorem [1] that $\overline{T(B_R)}$ is compact.

The conditions (2.6) are equivalent to the conditions

$$(2.8) \quad \begin{aligned} u - \varphi \in H_0^1(\Omega), \\ \int_{\Omega} h^3(x) \nabla(u - \varphi) \cdot \nabla \xi \, dx = - \int_{\Omega} h^3(x) \nabla \varphi \cdot \nabla \xi \, dx \\ + \int_{\Omega} \alpha(x, v) \cdot \nabla \xi \, dx, \quad \xi \in H_0^1(\Omega). \end{aligned}$$

It follows from standard elliptic theory [2] that (2.8) has a unique solution, $u \in H^1(\Omega)$. (Note that $\alpha(x, v) \in L^2(\Omega)$ thanks to (2.4).) We can set $\xi = u - \varphi$ in (2.8) and use the Cauchy-Schwarz inequality to obtain the bound (see (1.3))

$$(2.9) \quad \|\nabla(u - \varphi)\| \leq c(\|\nabla \varphi\| + \|\alpha(x, v)\|).$$

Here and in what follows, c will denote a positive constant which can change from equation to equation. Now it follows from (2.4) that

$$(2.10) \quad \|\alpha(x, v)\| \leq c_1^{1/2} \|v\|_{L^1(\Omega)} \leq c_1^{1/2} |\Omega|^{1/2} \|v\|^{1/2},$$

where $|\Omega|$ is the measure of Ω . Hence, we obtain from (2.9) and the triangle inequality

$$(2.11) \quad \|\nabla u\| \leq \|\nabla \varphi\| + \|\nabla(u - \varphi)\| \leq c(\|\nabla \varphi\| + \|v\|^{1/2}).$$

Also,

$$(2.12) \quad \|\psi\| \leq \nu^{-1/2} \|\nabla \psi\|, \quad \psi \in H_0^1(\Omega),$$

where $\nu > 0$ is the smallest eigenvalue of the problem

$$\begin{aligned} -\Delta \psi &= \nu \psi, & x \in \Omega, \\ \psi &= 0, & x \in \partial\Omega. \end{aligned}$$

Now

$$(2.13) \quad \|u\| \leq \|\varphi\| + \|u - \varphi\| \leq \|\varphi\| + \nu^{1/2} \|\nabla(u - \varphi)\|.$$

Thus, it follows from (2.9), (2.10), and (2.13) that there exists positive constants c_4 and c_5 such that

$$(2.14) \quad \|u\| \leq c_4 \|\varphi\|_{H^1(\Omega)} + c_5 \|v\|^{1/2}.$$

We see from (2.14) that $\|v\| \leq R$ implies that $\|u\| \leq R$ if $R \geq R_1$ where

$$R_1 = \left(\frac{c_5 + \sqrt{c_5^2 + 4c_4 \|\varphi\|_{H^1(\Omega)}^2}}{2} \right).$$

Thus, $T(B_R) \subseteq B_R$ if $R \geq R_1$. Further, it follows from (2.11) that there exists $c_3 = c_3(R)$ such that

$$\|T(v)\|_{H^1(\Omega)} \leq c_3(R), \quad v \in B_R.$$

All of the hypotheses of Schauder’s theorem have now been satisfied except for the continuity of T . It follows from (2.6) that for $v, w \in L^2(\Omega)$,

$$(2.15) \quad T(v) - T(w) \in H_0^1(\Omega),$$

$$\int_{\Omega} h^3(x) \nabla(T(v) - T(w)) \cdot \nabla \xi \, dx = \int_{\Omega} (\alpha(x, v) - \alpha(x, w)) \cdot \nabla \xi \, dx, \quad \xi \in H_0^1(\Omega).$$

Thus, we can set $\xi = T(v) - T(w)$ above to obtain

$$(2.16) \quad \|\nabla(T(v) - T(w))\| \leq c \|\alpha(x, v) - \alpha(x, w)\|.$$

Now by (2.5)

$$\|\alpha(x, v) - \alpha(x, w)\| \leq c_2^{1/2} \|v - w\|_{L^1(\Omega)} \leq c_2^{1/2} |\Omega|^{1/2} \|v - w\|^{1/2}.$$

Hence, it follows from (2.12), (2.16), and (2.17) that

$$\|T(v) - T(w)\| \leq \|v - w\|^{1/2},$$

i.e., T is Hölder continuous with exponent 1/2. This completes the proof that T has a fixed point.

Remark. One could weaken the assumption (2.4) in various directions and still get a solution of (2.3).

3. Uniqueness of solutions. In this section, we prove a uniqueness and monotonicity result for weak solutions to the problem (2.2). More precisely, we have:

THEOREM 3. *The weak solution to (2.2) is unique. Further, suppose that u_1 is a weak solution to (2.2) corresponding to boundary data φ_1 and u_2 is a weak solution to (2.2) corresponding to boundary data φ_2 . If $\varphi_1 \geq \varphi_2$ a.e. on $\partial\Omega$, then $u_1 \geq u_2$ a.e. in Ω .*

Proof. The uniqueness of weak solutions clearly follows from the monotonicity result. We will use here an argument due to Carillo and Chipot. See [4] for a variant.

We assume that $\varphi_1 \geq \varphi_2$ a.e. on $\partial\Omega$. First, we prove that for all $\zeta \in C^\infty(\bar{\Omega})$ and $\zeta > 0$ we have

$$(3.1) \quad \int_{[u_1 - u_2 > 0]} h^3(x) \nabla(u_2 - u_1) \cdot \nabla \zeta - (\alpha(x, u_2) - \alpha(x, u_1)) \cdot \nabla \zeta \, dx \leq 0$$

where

$$[u_2 - u_1 > 0] = \{x \in \Omega \mid u_2(x) - u_1(x) > 0\}.$$

To do that, we consider for $\varepsilon > 0$

$$(3.2) \quad \xi = \min\left(\frac{(u_2 - u_1)^+}{\varepsilon}, \zeta\right)$$

where

$$\psi^+(x) = \max(\psi(x), 0).$$

Note that $\xi \in H_0^1(\Omega)$ since for $x \in \partial\Omega$,

$$\xi(x) = \min\left(\frac{(\varphi_2 - \varphi_1)^+}{\varepsilon}, \zeta\right) = 0.$$

It follows from subtracting (2.3) with $u = u_2$ from (2.3) with $u = u_1$ that

$$(3.3) \quad \int_{\Omega} h^3(x) \nabla(u_2 - u_1) \nabla \xi - (\alpha(x, u_2) - \alpha(x, u_1)) \cdot \nabla \xi \, dx = 0,$$

which for ξ given by (3.2) is equivalent to

$$(3.4) \quad \begin{aligned} & \int_{[u_2 - u_1 > \varepsilon \zeta]} h^3(x) \nabla(u_2 - u_1) \nabla \zeta - (\alpha(x, u_2) - \alpha(x, u_1)) \cdot \nabla \zeta \, dx \\ & + \frac{1}{\varepsilon} \int_{[0 < u_2 - u_1 \leq \varepsilon \zeta]} h^3(x) |\nabla(u_2 - u_1)|^2 \, dx \\ & - \frac{1}{\varepsilon} \int_{[0 < u_2 - u_1 \leq \varepsilon \zeta]} (\alpha(x, u_2) - \alpha(x, u_1)) \cdot \nabla(u_2 - u_1) \, dx = 0 \end{aligned}$$

where

$$[u_2 - u_1 > \varepsilon \zeta] = \{x \in \Omega \mid u_2(x) - u_1(x) > \varepsilon \zeta\}$$

and the other sets $[\cdot]$ are defined likewise. We estimate the last integral above by

$$\begin{aligned}
 I &\equiv \int_{[0 < u_2 - u_1 \leq \varepsilon \zeta]} (\alpha(x, u_2) - \alpha(x, u_1)) \cdot \nabla(u_2 - u_1) \, dx \\
 &\leq \left(\int_{[0 < u_2 - u_1 \leq \varepsilon \zeta]} h(x)^{-3} |\alpha(x, u_2) - \alpha(x, u_1)|^2 \, dx \right)^{1/2} \\
 (3.5) \quad &\times \left(\int_{[0 < u_2 - u_1 \leq \varepsilon \zeta]} h^3(x) |\nabla(u_2 - u_1)|^2 \, dx \right)^{1/2} \\
 &\leq \frac{1}{4} \int_{[0 < u_2 - u_1 \leq \varepsilon \zeta]} h(x)^{-3} |\alpha(x, u_2) - \alpha(x, u_1)|^2 \, dx \\
 &\quad + \int_{[0 < u_2 - u_1 \leq \varepsilon \zeta]} h^3(x) |\nabla(u_2 - u_1)|^2 \, dx.
 \end{aligned}$$

Using the estimate (3.5) in (3.4) we obtain from (1.3) and (2.4)

$$\begin{aligned}
 (3.6) \quad &\int_{[u_2 - u_1 > \varepsilon \zeta]} h^3(x) \nabla(u_2 - u_1) \nabla \zeta - (\alpha(x, u_2) - \alpha(x, u_1)) \cdot \nabla \zeta \, dx \\
 &\leq \frac{1}{4\varepsilon} \int_{[0 < u_2 - u_1 \leq \varepsilon \zeta]} h(x)^{-3} |\alpha(x, u_2) - \alpha(x, u_1)|^2 \, dx \\
 &\leq \frac{c_2 M}{4h_{\min}^3} \int_{[0 < u_2 - u_1 \leq \varepsilon \zeta]} dx
 \end{aligned}$$

where $M = \max \zeta$. Now the measure of the set $[0 < u_2 - u_1 \leq \varepsilon \zeta]$ goes to zero as $\varepsilon \rightarrow 0$. Thus, the estimate (3.1) follows from (3.6).

Now set $\mathbf{n} = (n_1, n_2) \equiv (-\Lambda_2, \Lambda_1)$ and $s > 0$. We then set

$$(3.7) \quad \zeta(x_1, x_2) = W - \exp(s(n_1 x_1 + n_2 x_2))$$

where W is a constant chosen large enough so that $\zeta > 0$. If we set ζ from (3.7) in (3.1) we obtain (see the definition of α)

$$(3.8) \quad \int_{[u_2 - u_1 > 0]} h^3(x) \nabla(u_2 - u_1) \nabla \zeta + \lambda(\beta(x, u_2) - \beta(x, u_1)) \nabla h \cdot \nabla \zeta \, dx \leq 0$$

since $\Lambda \cdot \nabla \zeta = 0$ for all $x \in \Omega$. Now it follows from integration by parts that

$$\begin{aligned}
 (3.9) \quad &\int_{[u_2 - u_1 > 0]} h^3(x) \nabla(u_2 - u_1) \nabla \zeta \, dx = \int_{\Omega} h^3(x) \nabla(u_2 - u_1)^+ \nabla \zeta \, dx \\
 &= - \int_{\Omega} (u_2 - u_1)^+ \nabla \cdot (h^3(x) \nabla \zeta) \, dx \\
 &= - \int_{[u_2 - u_1 > 0]} (u_2 - u_1) \nabla \cdot (h^3(x) \nabla \zeta) \, dx.
 \end{aligned}$$

So, from (3.8) we obtain

$$\int_{[u_2 - u_1 > 0]} (u_2 - u_1) [-\nabla \cdot (h^3(x) \nabla \zeta) + g(x) \nabla h \cdot \nabla \zeta] \, dx \leq 0$$

where

$$g(x) = \begin{cases} \frac{\lambda(\beta(x, u_2(x))) - \beta(x, u_1(x))}{(u_2(x) - u_1(x))} & \text{if } u_2(x) \neq u_1(x), \\ 0 & \text{if } u_2(x) = u_1(x). \end{cases}$$

If $\lambda = 0$, then $g = 0$. Further, from the definition of β , for $\lambda \neq 0$

$$|\beta(x, v) - \beta(x, w)| \leq \frac{h_{\max}^2}{\lambda} |v - w|, \quad x \in \Omega, \quad v, w \in \mathbb{R}.$$

Thus, $g \in L^\infty(\Omega)$. Also, we have that

$$\begin{aligned} -\nabla \cdot (h^3(x) \nabla \zeta) + g(x) \nabla h \cdot \nabla \zeta &= -h^3(x) \Delta \zeta - \nabla h^3 \cdot \nabla \zeta + g(x) \nabla h \cdot \nabla \zeta \\ &= \exp(s(n_1 x_1 + n_2 x_2)) [h^3(x) s^2 n^2 + (\nabla h^3 \cdot \mathbf{n}) s - g(x) (\nabla h \cdot \mathbf{n}) s] \end{aligned}$$

where $n^2 = n_1^2 + n_2^2$. Hence, it follows that for s sufficiently large

$$(3.10) \quad -\nabla \cdot (h^3(x) \nabla \zeta) + g(x) \nabla h \cdot \nabla \zeta > 0$$

for all $x \in \Omega$. The inequalities (3.9) and (3.10) thus allow us to conclude that $(u_2 - u_1)^+ = 0$ a.e. This concludes the proof of Theorem 3.

4. Nonnegativity and regularity of solutions. We have

$$(4.1) \quad \alpha(x, 0) = 0, \quad x \in \Omega.$$

Thus, Theorem 3 allows us to conclude that the weak solution, u , to (2.2) with boundary data $\varphi > 0$ on $\partial\Omega$ is nonnegative, i.e., $u \geq 0$ a.e. in Ω . (Take $\varphi_1 = \varphi$, $u_1 = u$, $\varphi_2 = 0$, $u_2 = 0$ in Theorem 3.) Thus, we have the following result.

COROLLARY 1. *If $\varphi > 0$ a.e. in $\partial\Omega$, the weak solution, u , to (2.2) satisfies $u \geq 0$ a.e. in Ω .*

If $\lambda \neq 0$ and $h(x) \in C^\infty(\bar{\Omega})$, then $\alpha(x, v)$ has bounded derivatives of all orders for $v \geq 0$. Hence, the standard techniques for the analysis of the regularity of solutions to elliptic boundary value problems [2], [6] can be used to prove that u is a smooth, classical solution to (2.1). However, if $\lambda = 0$, then $\alpha(x, v)$ is not differentiable at $v = 0$, and it is necessary to show that u is bounded away from zero to be able to prove that u is smooth. Thus, we demonstrate the following theorem which gives a condition on h which guarantees that u is bounded away from zero.

COROLLARY 2. *If $\varphi \geq \Phi > 0$ on $\partial\Omega$ where Φ is a constant and if*

$$\nabla \cdot \alpha(x, \Phi) \leq 0, \quad x \in \Omega,$$

then the weak solution, u , to (2.2) satisfies the bound $u \geq \Phi > 0$ on Ω .

Proof. We will use the method of Theorem 3 with $\varphi_1 = \varphi$, $u_1 = u$, $\varphi_2 = \Phi$, $u_2 = \Phi$. Now $u_2 = \Phi$ is not the weak solution of (2.2) with boundary data, $\varphi_2 = \Phi$, but we have instead by integration by parts that for all $\zeta \in H_0^1(\Omega)$, $\zeta \geq 0$,

$$(4.2) \quad \int_{\Omega} h^3(x) \nabla u_2 \cdot \nabla \zeta - \alpha(x, u_2) \cdot \nabla \zeta \, dx = \int_{\Omega} \nabla \cdot \alpha(x, \Phi) \zeta \, dx \leq 0.$$

From (4.2) we obtain in place of (3.3) that

$$(4.3) \quad \int_{\Omega} h^3(x) \nabla(u_2 - u_1) \cdot \nabla \zeta \, dx - (\alpha(x, u_2) - \alpha(x, u_1)) \cdot \nabla \zeta \, dx \leq 0$$

for all $\zeta \in H_0^1(\Omega)$, $\zeta \geq 0$.

The proof of Corollary 2 now follows from the observation that the inequality (4.3) can replace the equality (3.3) in the argument of Theorem 3.

An argument similar to that given in Corollary 2 can be used to prove the following result.

COROLLARY 3. *If $0 \leq \varphi \leq \Phi$ on $\partial\Omega$ and if*

$$\nabla \cdot \alpha(x, \Phi) \geq 0, \quad x \in \Omega,$$

then the weak solution, u , to (2.2) satisfies the bound $0 \leq u \leq \Phi$ on Ω .

Note that the technique of Corollary 2 and Corollary 3 could provide some other comparison results by choosing φ_2 and u_2 to satisfy a suitable differential inequality. The bound given in Corollary 2 allows the standard regularity theory for elliptic partial differential equations [2], [6] to be used to prove the following theorem.

COROLLARY 4. *Assume that the boundary of Ω , $\partial\Omega$, is infinitely differentiable, that $h \in C^\infty(\bar{\Omega})$, and that*

$$(4.4) \quad \Lambda \cdot \nabla h \leq 0, \quad x \in \Omega.$$

Then the Reynolds lubrication equation (1.1) has a nonnegative classical solution, P , such that $P \in C^\infty(\bar{\Omega})$ and such that P is unique in the class of nonnegative solutions.

We note that it is proven in [5] under the hypothesis (4.4) that solutions to (1.1) are positive, smooth, and unique in the class of smooth solutions. Our results here prove that such a solution exists and that the stronger bound of Corollary 2 holds. Further, if (4.4) holds, the following result proves that our solution to (1.1) is unique in the class of weak solutions.

COROLLARY 5. *Assume that P is a weak solution to (1.1) in the sense that $P \equiv P_a > 0$ on $\partial\Omega$ and*

$$(4.5) \quad \int_{\Omega} h^3 P \nabla P \cdot \nabla \xi - Ph \Lambda \cdot \nabla \xi \, dx = 0, \quad \xi \in H_0^1(\Omega),$$

where $P, P^2 \in H^1(\Omega)$. If (4.4) holds, then $P > 0$ a.e. in Ω .

Proof. Take $\xi = P^- = \min(P, 0) \in H_0^1(\Omega)$. Then by (4.5)

$$(4.6) \quad \begin{aligned} \int_{\Omega} h^3 P \nabla P \cdot \nabla P^- - Ph \Lambda \cdot \nabla P^- \, dx \\ = \int_{\Omega} h^3 P^- |\nabla P^-|^2 + \frac{1}{2} h \Lambda \cdot \nabla (P^-)^2 \, dx \\ = \int_{\Omega} h^3 P^- |\nabla P^-|^2 - \frac{1}{2} (\Lambda \cdot \nabla h) (P^-)^2 \, dx = 0. \end{aligned}$$

Hence, by (4.4),

$$\int_{\Omega} h^3 P^- |\nabla P^-|^2 \, dx = \frac{1}{2} \int_{\Omega} (\Lambda \cdot \nabla h) (P^-)^2 \, dx \leq 0.$$

Thus, $P^- = 0$ a.e. in Ω .

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