WINDINGS OF PRIME GEODESICS

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Abstract. The winding of a closed oriented geodesic around the cusp of the modular orbifold is computed by the Rademacher symbol, a classical function from the theory of modular forms. For a general cusped hyperbolic orbifold, we have a procedure to associate to each cusp a Rademacher symbol. In this paper we construct winding numbers related to these Rademacher symbols.

In cases where the two functions coincide, access to the spectral theory of automorphic forms yields statistical results on the distribution of closed (primitive) oriented geodesics with respect to their winding.

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1. Introduction

The winding of a closed oriented geodesic around the cusp of the modular orbifold $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ can be expressed by an explicit function Ψ from the theory of modular forms called the Rademacher symbol.

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In Figure 1 below, the coloured path represents a portion of the dashed hyperbolic geodesic folded onto the standard fundamental domain for the modular group $\mathrm{PSL}_2(\mathbb{Z})$.

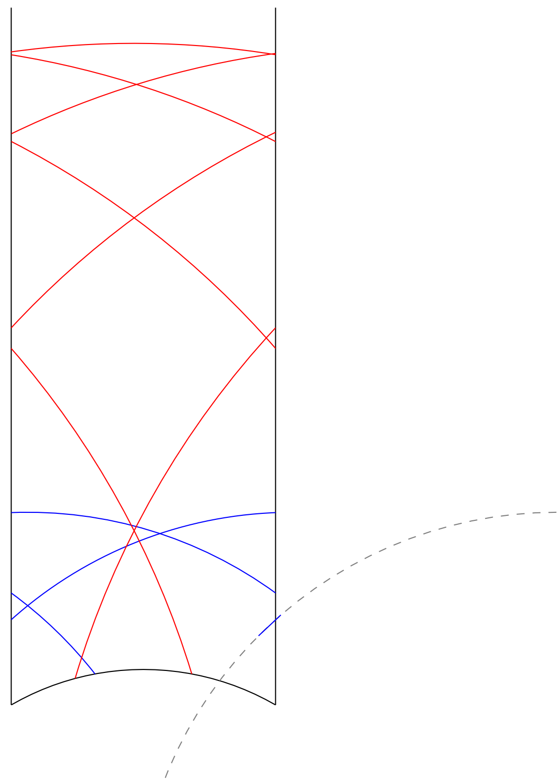


FIGURE 1

Travelling left to right along the blue segment, we record $a_1 = 3$ windings around the cusp until eventually hitting the unit circle $|z| = 1$. The geodesic trajectory then continues along the red segment, where we now go from right to left, and we record the $a_2 = 7$ next windings until again reaching the unit circle, and so on.

If the dashed geodesic is the axis of a closed geodesic C, this process yields a finite sequence $(a_1, \dots, a_n)$ that encodes the *oriented* winding around the cusp. For the (minimal even length) such sequence, the Rademacher symbol $\Psi(C)$ is the alternating sum

$$\Psi(C) = a_1 - a_2 + a_3 - a_4 + \cdots + a_n.$$

This observation — that the winding is realized by the Rademacher symbol — allows to deduce counting and distribution results for the windings of modular geodesics. This was implemented by Sarnak [Sar10] who obtained the following result, among others that we will discuss later in this introduction. Let $\Pi(T)$ be the countable set of prime (i.e., primitive, closed, and oriented) geodesics on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ of length up to T. For $n \in \mathbb{Z}$ fixed, the asymptotic density of prime geodesics with winding number $\Psi(C) = n$ is

$$\frac{\pi_n(T)}{\pi(T)} := \frac{\#\{C \in \Pi(T) : \Psi(C) = n\}}{\#\Pi(T)} \sim \frac{1}{3}\frac{T}{T^2 + \left(\frac{\pi n}{3}\right)^2}$$

as $T \to \infty$; see [Sar10, Corollary 2]. Together with the prime geodesic theorem, this shows that there are infinitely many prime geodesics winding n times for each choice of $n \in \mathbb{Z}$. Moreover it appears that the distribution of the winding numbers is symmetric about $n = 0$, with its peak at $n = 0$. We will soon see that this symmetry persists for a natural extension of this winding number to more general hyperbolic surfaces.

We recall the following notion of winding for closed curves on a cusped hyperbolic surface M, due to Reinhart [Rei60] and Chillingworth [Chi72]. Fix a non-vanishing continuous vector field X on M.[1] Then for each closed parametrized curve $C(t) = (c(t), \theta(t))$ in the unit tangent bundle T^1M, given by position $c(t)$ and direction $\theta(t)$ at time t, the winding number of C relative to the vector field X is defined to be the total variation of the angle function

$$\theta(t) := \measuredangle(\theta(t), X(c(t)))$$

over one revolution. This winding number is independent of the choice of parametrization and classifies nontrivial homotopy classes of closed curves in T^1M [Chi72, Sma58].

The main object of this paper is to construct a similar winding number function with the vector field X replaced by an automorphic form f, chosen such that the resulting winding number generalizes the Rademacher symbol for the modular orbifold.

Our construction goes as follows. Let $M = \Gamma\backslash\mathbb{H}$ be a cusped hyperbolic surface. Suppose that there exists an automorphic form f for Γ of even integer weight that is nowhere-vanishing and for which $f(x+iy) \to 0$ as $y \to \infty$.[2] By a standard procedure we may lift f to a (unique) continuous function $F : \mathrm{T}^1M \to \mathbb{C}^*$. Then each closed curve C in T^1M is mapped to a continuous oriented closed curve $F(C) \subset \mathbb{C}$. By construction, the curve $F(C)$ never passes through the origin of the plane, which corresponds to the cusp at ∞. We define the **winding of C relative to f** to be the topological index of the planar curve $F(C)$ about the origin, i.e.,

$$\mathrm{ind}(F(C), 0) = \frac{1}{2\pi i}\int_C \frac{dF}{F}.$$

This integer is independent of parametrization and homotopy-invariant. Since each homotopy class contains a unique geodesic representative, it suffices to restrict C to the family Π of prime geodesics on M.

[1] The restriction to noncompact surfaces is essential for this construction; compact surfaces may not admit such vector fields. For an algebraic construction of winding numbers for compact surfaces, see [Hub12].

[2] We will assume for simplicity that Γ has a cusp at ∞; this is true up to a conjugation.

There has recently been a renewed interest in variations around the Rademacher symbol, e.g., the linking number introduced by Duke, Imamoglu, and Toth in [DIT17], the Dedekind–Rademacher cocycle studied by Darmon, Pozzi, and Vonk in [DV21], or the extension of Rademacher symbols to Fuchsian groups introduced by the first named author [Bur22] in relation to the Manin–Drinfeld theorem, and studied by Matsusaka and Ueki [MU23] in the special case of triangle groups from the viewpoint of linking numbers.

From [Bur22] we know that each cusp $\mathfrak{a}$ of M gives rise to a generalization $\Psi_{\mathfrak{a}}$ of the Rademacher symbol. To simplify notation and presentation, we will assume that any cofinite Fuchsian group under consideration has a cusp at ∞ and only consider the Rademacher symbol $\Psi := \Psi_\infty$ associated to that cusp. For the case of a general cusp, the reader is referred to [Bur22]. This construction of Rademacher symbols builds on extracting an automorphic form from Kronecker's first limit formula, and this orients the choice of the automorphic form f with respect to which we will compute the windings of prime geodesics. For the purpose of this introduction, we will state the following simplified version of our main result, whose statement we postpone to §2.

Theorem 1. *Let $M = \Gamma\backslash\mathbb{H}$ be a cusped hyperbolic orbifold with finite area V. There exists an automorphic form f, possibly nonholomorphic, of even weight $k \in 2\mathbb{N}$ that is nowhere-vanishing, for which $f(z) \to 0$ as $y \to \infty$, and such that the winding number relative to f is either*

$$\operatorname{ind}(F(C), 0) = \frac{kV}{4\pi}\Psi(C) \tag{1}$$

or a linear combination of values of $\frac{kV}{4\pi}\Psi$.

For the modular orbifold $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ of area $\frac{\pi}{3}$, the modular form f coincides with the modular discriminant $\triangle$ of weight $k = 12$, and we recover the classical Rademacher symbol Ψ. Further, we will see that we are in the situation of (1) whenever $\Gamma \in \mathcal{G}$, where $\mathcal{G}$ is the collection of the following families of Fuchsian groups: congruence subgroups of the modular group, maximal arithmetic noncocompact Fuchsian groups (under inclusion), genus 0 noncocompact Fuchsian groups (e.g., Hecke triangle groups). In these situations we are able to extend Sarnak's strategy for counting.

We obtain the following results.

Theorem 2. *Let $\Gamma \in \mathcal{G}$ and consider the winding number given by (1). Recall that $\pi_n(T)$ is the number of prime geodesics in $\Pi(T)$ with winding n. There exists $\delta \in (0, \frac{1}{2}]$ such that*

$$\pi_n(T) = \frac{4}{kT}\int_2^{e^T} \frac{\log t}{(\log t)^2 + \left(\frac{4\pi n}{k}\right)^2}\,dt + O\left(\frac{e^{T(1-\delta/2)}}{T}\right)$$

as $T \to \infty$. The implied constant does not depend on n.

Corollary 1. *Let $\Gamma \in \mathcal{G}$ and consider the winding number given by (1). For $n \in \mathbb{Z}$ fixed, the asymptotic density of prime geodesics with n windings is*

$$\frac{\pi_n(T)}{\pi(T)} \sim \frac{4}{kT}\frac{T}{T^2 + \left(\frac{4\pi n}{k}\right)^2}$$

as $T \to \infty$.

Theorem 3. *Let $\Gamma \in \mathcal{G}$ and consider the winding number given by (1). The limiting distribution of the ratio of winding-to-length for prime geodesics is Cauchy. More precisely, for any interval $[a, b]$ of the real line we have*

$$\lim_{T\to\infty} \frac{\#\{C \in \Pi(T) : \frac{4\pi}{k}a \leq \frac{\operatorname{ind}(F(C),0)}{\ell_C} \leq \frac{4\pi}{k}b\}}{\pi(T)} = \int_a^b \frac{du}{\pi(1+u^2)}.$$

Theorem 2 and Theorem 3 generalize [Sar10, Theorem 1] and [Sar10, Theorem 3], which are stated for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, $\delta = \frac{1}{2}$, and $k = 12$. We discuss further the appearance of the Cauchy distribution at the end of §2.

We extend Sarnak's results with the following equidistribution theorem.

Theorem 4. *Let $\Gamma \in \mathcal{G}$ and consider the winding number given by (1). Let $A \subseteq \mathbb{Z}$ be a set with natural density $d(A)$. The density of prime geodesics with winding number in A is equal to $d(A)$. Explicitly, if $\pi_A(T)$ denotes the number of prime geodesics in $\Pi(T)$ with winding number in A, then*

$$\lim_{T\to\infty} \frac{\pi_A(T)}{\pi(T)} = d(A).$$

The analogon of this equidistribution theorem for prime geodesics in fixed homology classes was established by Petridis and Risager in [PR08]. For winding numbers, the result is new — even in the case of the modular orbifold — and includes a geometric analogue of the prime number theorem for arithmetic progressions. It would be interesting to identify other natural winding numbers for which this behavior persists. It is also not clear whether the restriction to the family $\mathcal{G}$ is necessary for these winding numbers. We discuss this point further in §2, as well as the occurence of the Cauchy distribution in Theorem 3.

2. Discussion of results and structure of paper

Let $G = \mathrm{SL}_2(\mathbb{R})$ and let $\Gamma < G$ be a discrete subgroup. We say that Γ is a cofinite Fuchsian group if the orbifold $\Gamma\backslash\mathbb{H}$ is noncompact and of finite area V. Section §3 collects preliminary results on Fuchsian groups and the spectral theory of automorphic forms for the convenience of the reader and to fix notation. After this, the paper is structured in three parts.

2.1. **Rademacher symbols.** Section §4 reviews key properties of the classical Rademacher symbol as well as various topological interpretations, following in particular Atiyah [Ati87] and Ghys [Ghy07]. The description of the Rademacher symbol $\Psi(C)$ as an alternating sum encoding the number of (oriented) turns around the cusp of the modular orbifold is given in Proposition 4.1 and the discussion that follows.

In Section §5, we recall the Rademacher symbols for Fuchsian groups Γ introduced in [Bur22]. We prove in Theorem 5.5 and Corollary 5.6 that Ψ is well defined on the set Π of prime geodesics on $M = \Gamma\backslash\mathbb{H}$.

For an oriented closed geodesic C on M, the value of the Rademacher symbol can be expressed by the real period

$$\Psi(C) = \int_C E_2(z)dz, \tag{2}$$

where the closed 1-form $E_2(z)dz$ is determined by a weight 2 nonholomorphic Eisenstein series; see [Bur22, Lemma 3.1]. When Γ is distinct from $\mathrm{SL}_2(\mathbb{Z})$ the period (2) is not necessarily an integer, and can in fact be irrational. The study of these periods is related to classical results in arithmetic geometry — such as the Manin–Drinfeld and Manin–Mumford theorems — and this is the object of [Bur22]. For our purposes, we only record that the period (2) is rational whenever $\Gamma \in \mathcal{G}$; see [Bur22, Theorem 1.2].

Section §5 also contains the main contribution of this work to the theory of automorphic forms. In Theorem 5.1 we construct, for each cofinite Fuchsian group Γ, each cusp $\mathfrak{a}$ of Γ, and each real weight $r \in \mathbb{R}$, an explicit multiplier system χ_r of weight r for Γ. This construction also yields an automorphic form $\triangle_r$. For $\mathrm{PSL}_2(\mathbb{Z})$, $\triangle_{12}$ is precisely the modular discriminant. It is a modification of this form that we use for the winding number.

2.2. **Winding numbers.** For ease of notation we assume in this discussion that Γ is torsionfree. Geometrically this means that the quotient space $M = \Gamma\backslash\mathbb{H}$ has the structure of a hyperbolic surface. However our results also hold for orbifolds.

In §6, we construct a nowhere-vanishing nonholomorphic automorphic form $\triangle_2^*$ of weight 2 for Γ such that $\triangle_2^*(z) \to 0$ as $y \to \infty$. We lift this form to the continuous function

$$\widetilde{\triangle}_2^* : G \to \mathbb{C}, \quad \widetilde{\triangle}_2^*(g) = \triangle_2^*(g(i))g'(i),$$

where $g(i)$ denotes the action of g by fractional linear transformation. The automorphic transformation of $\triangle_2^*$ guarantees that the lift $\widetilde{\triangle}_2^*$ quotients through $\Gamma\backslash G \cong \mathrm{T}^1 M$. By construction, the lift $\widetilde{\triangle}_2^*$ yields the winding number

$$\mathrm{ind}(\widetilde{\triangle}_2^*(C), 0) = \frac{1}{2\pi i}\int_C \frac{d\widetilde{\triangle}_2^*}{\widetilde{\triangle}_2^*}.$$

For the next statement we recall that there are bijections between the set of closed geodesics on $\Gamma\backslash G$, the set of oriented closed geodesics on M, and the set conjugacy classes of hyperbolic elements in Γ of positive trace.

Theorem 5. *Let M be a cusped hyperbolic surface with finite area V. Fix a generating set $\mathcal{S} = \{\gamma_i\}_{i\in I}$ for Γ. There exists a nonholomorphic automorphic form $\triangle_2^*$ of weight 2 for Γ, that is nowhere-vanishing, for which $\triangle_2^*(z) \to 0$ as $y \to \infty$, and there exists a conjugacy-class invariant function $\zeta : \Gamma \to \mathbb{Z}$ such that*

$$\zeta(C) = \mathrm{ind}(\widetilde{\triangle}_2^*(C), 0)$$

for each $C \in \Pi$, and

$$\zeta(\gamma) = \zeta(\gamma_{i_1}\cdots\gamma_{i_k}) = \frac{V}{2\pi}\left(\Psi(\gamma) - \Psi(\gamma_{i_1}) - \cdots - \Psi(\gamma_{i_k})\right)$$

for each $\gamma \in \Gamma$.

See Theorem 6.2 for the full statement for hyperbolic orbifolds. When Ψ is rational-valued, it is possible to replace the nonholomorphic form $\triangle_2^*$ by the holomorphic form $\triangle_k$ of weight k and degree $\frac{kV}{4\pi}$ from §5, at the expense of a nonexplicit and possibly higher weight k.

Theorem 6. *Let M be a cusped hyperbolic orbifold with finite area V. If the Rademacher symbol Ψ is rational-valued, then there exists $k \in 2\mathbb{N}$ such that*

$$\frac{kV}{4\pi}\Psi(C) = \mathrm{ind}(\widetilde{\triangle}_k(C), 0) \tag{3}$$

for each $C \in \Pi$.

We recall that by our previous discussion of Rademacher symbols, this situation arises for example when Γ is genus 0, maximal arithmetic, or a congruence subgroup of the modular group. In particular the assumption in Theorem 6 is satisfied for all $\Gamma \in \mathcal{G}$. The fact that the winding number (3) is defined relative to a holomorphic form is of particular importance. It provides us access to the spectral theory of automorphic forms; see Proposition 5.3.

2.3. **Counting results.** To prove Theorem 2 we first need the following twisted prime geodesic theorem for multiplier systems of arbitrary real weight, which we prove in §8.

Theorem 7. *Let Γ be a cofinite Fuchsian group and let χ be a multiplier system of weight r on Γ. Then as $T \to \infty$, we have*

$$\sum_{C_\gamma \in \Pi(T)} \chi(\gamma) = \mathrm{Li}\left(e^{s_0(\chi,r)T}\right) + \cdots + \mathrm{Li}\left(e^{s_k(\chi,r)T}\right) + O\left(e^{3T/4}(L(\chi,r)+1)\right), \tag{4}$$

where $s_0(\chi,r) \geq s_1(\chi,r) \geq \cdots \geq s_k(\chi,r) > \frac{1}{2}$ are the spectral eigenparameters determined by the small eigenvalues $\lambda_j(\chi,r) = s_j(\chi,r)(1-s_j(\chi,r))$ of the weight r Laplacian Δ_r, and $L(\chi,r)$ is defined by (33).

For $r = 0$ and a multiplier system χ that is trivial at $r = 0$, we recover the usual prime geodesic theorem; see, e.g., [Iwa02, Theorem 1.5]. The proof of Theorem 7 follows a routine application of Selberg's trace formula for arbitrary real weight as treated by Hejhal [Hej83]. To keep track of the dependence of the error term on

(χ, r) we derive in §7 upper bound estimates for the two spectral terms appearing in Weyl's law, which may be of independent interest; see Theorem 7.2.

With Theorem 7 in hand, we now review the strategy of Sarnak [Sar10] adapted to our setting. By construction, the winding number $\zeta(C)$ in Theorem 5 determines the multiplier systems $\chi(\gamma) = e^{i\pi r\,\zeta(\gamma)}$, $r \in \mathbb{R}$. Fix $n \in \mathbb{Z}$. To pick up all prime geodesics $C \in \Pi(T)$ with $\zeta(C) = n$ windings, we will want to integrate (4) as follows:

$$2\,\pi_n(T) = \sum_{C\in\Pi(T)} \int_{-1}^{1} e^{i\pi r(\zeta(C)-n)}\,dr = \int_{-1}^{1} \mathrm{Li}\left(e^{s_0(\chi,r)T}\right) e^{-i\pi rn}\,dr + \dots. \tag{5}$$

There are two obstacles to carry out this strategy in our setting. The first obstacle is that we don't *a priori* understand how the eigenparameters $s_j(\chi, r)$ vary as a function of r. In §9 we prove the existence of a small interval $|r| \le \delta$ in which the bottom eigenparameter $s_0(\chi, r)$ has multiplicity 1; see Corollary 9.3. Observe that the width of this interval is reflected in the growth exponent of the error term in Theorem 2. This result is obtained by studying the counting function for small eigenvalues of Δ_r under continuous deformation in r. Our strategy is inspired by the analogue for characters established by Risager [Ris11] with the addition of a crucial estimate of Jorgenson and Lundelius on the hyperbolic heat trace [JL97]. Whereas for $\mathrm{PSL}_2(\mathbb{Z})$, one can rely on a result of Bruggeman, which asserts that the spectrum of Δ_r has no nontrivial small eigenvalues [Bru86], the situation for more general cofinite Fuchsian groups requires the full force of the results derived in §9.

The second obstacle is the lack of information on the eigenparameter $s_0(\chi, r)$ to carry out the integration of the main term on the right hand side of (5). This issue can be circumvented if we restrict to surfaces for which $\Gamma \in \mathcal{G}$ and work with the winding number (3) relative to the holomorphic form $\triangle_k$. Then it follows from Proposition 5.3 that $s_0(\chi_r, r) = 1 - \frac{|r|}{2}$, and Theorem 2 follows.

The proofs of Theorem 2, Theorem 3 and Theorem 4 are contained in §10. The latter two results use are deduced by manipulating the counting result of Theorem 2. It would be interesting to approach these statements through different tools, in particular with the aim of extending them to the winding numbers in Theorem 5.

We conclude this discussion with a final comment on the appearance of the Cauchy distribution in Theorem 3. Consider a long closed geodesic winding many times high in the cusp. The winding number grows very quickly against the period, which is measured by hyperbolic length and grows slowly far in the cusp; we believe that the presence of cusps accounts for the fat tails of the Cauchy distribution. The following evidence supports this heuristic. By using word length instead of arc length (therefore insensitive to the cusps), Calegari [Cal09, Chapter 6.1] finds in this setting a Gaussian limiting distribution. In another variation of this problem, studying the asymptotic winding in homology of the geodesic flow, Guivarc'h and Le Jan [GLJ93] showed that the limiting distribution is Gaussian when the surface is compact and Cauchy otherwise. Finally, we show in Proposition 10.4 that if $M = \Gamma\backslash\mathbb{H}$ is a closed hyperbolic surface and ζ is a class invariant quasimorphism on Γ, then $|\zeta(C)| = O(\ell_C)$, which rules out a limiting Cauchy distribution.

3. Background

3.1. **Hyperbolic geometry.** We will work on the upper half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$. For each $z \in \mathbb{H}$, we have $\mathrm{T}_z\mathbb{H} \cong \mathbb{C}$ and the hyperbolic metric is given by the inner product $\langle v, w\rangle_z = y^{-2}\mathrm{Re}(v\overline{w})$. Geodesics in this model are either vertical half-lines or semi-circles orthogonal to the real axis.

The unit tangent bundle is given by $\mathrm{T}^1\mathbb{H} = \{(z, v) \in \mathbb{H} \times \mathbb{C} : |v| = y\}$. We parametrize elements of $\mathrm{T}^1\mathbb{H}$ by (z, θ) where $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is the angular variable of v as measured counterclockwise from the vertical. Each point $(z, \theta) \in \mathrm{T}^1\mathbb{H}$ determines a unique hyperbolic geodesic on $\mathrm{T}^1\mathbb{H}$ by parallel transport. The unit tangent bundle $\mathrm{T}^1\mathbb{H}$ may further be algebraically identified with the matrix group $\mathrm{PSL}_2(\mathbb{R})$ via the transitive action on $\mathrm{T}^1\mathbb{H}$ given by

$$(z, \theta) \mapsto (g(z), \theta - 2\arg j(g, z)),$$

where

$$\begin{aligned} g(z) &= \frac{az+b}{cz+d}, \\ j(g,z) &= cz + d, \\ \arg(z) &\in (-\pi, \pi]. \end{aligned} \tag{6}$$

The function $j(\cdot, z)$ defines a multiplicative cocycle on $\mathrm{SL}_2(\mathbb{R})$; i.e., it satisfies $j(gh, z) = j(g, hz)j(h, z)$ for all $g, h \in \mathrm{SL}_2(\mathbb{R})$. Since $\det(g) = 1$ and $z \in \mathbb{H}$, it is moreover nowhere vanishing. To see that the action is well defined, we remark that for all $g, h \in \mathrm{SL}_2(\mathbb{R})$, the difference

$$\omega(g, h) := \frac{1}{2\pi}\left(\arg j(g, hz) + \arg j(h, z) - \arg j(gh, z)\right) \tag{7}$$

is an integer in $\{0, \pm 1\}$; indeed since $\omega(g, h)$ is continuous as a function of z and integer-valued, it does not depend on the particular choice of z. The choice of the branch of logarithm ($\arg(z) \in (-\pi, \pi]$) is recorded for instance by the fact that $\omega(-I, -I) = 1$, where $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. For further reference, we also record that ω defines a 2-cocycle; indeed, direct computation establishes that

$$\omega(gh, k) + \omega(g, h) = \omega(g, hk) + \omega(h, k)$$

for all $g, h, k \in \mathrm{SL}_2(\mathbb{R})$.

3.2. **The geometry of Fuchsian groups.** By uniformization, every hyperbolic surface (or orbifold) may be realized as the space of orbits $\Gamma\backslash\mathbb{H}$ for some discrete subgroup $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ (under the action of Γ by fractional linear transformations as in (6)). This action has discrete orbits, which allows to represent the orbit space $\Gamma\backslash\mathbb{H}$ visually by a choice of fundamental domain $\mathcal{F}$ in $\mathbb{H}$. The hyperbolic area $\mathrm{vol}(\mathcal{F}) = \int_{\mathcal{F}} d\mu(z) = \int_{\mathcal{F}} \frac{dxdy}{y^2}$ does not depend on the particular choice of the fundamental domain and we set

$$V = \mathrm{vol}(\mathcal{F}).$$

We will say that Γ is a **cofinite Fuchsian group** if $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ is a discrete subgroup that admits a noncompact but finite-area fundamental domain in $\mathbb{H}$.

Elements of Γ are classified by their fixed point sets in $\mathbb{H}$; each γ is either the identity, elliptic (with one fixed point in $\mathbb{H}$), parabolic (with one fixed point in $\partial_\infty \mathbb{H} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$), or hyperbolic (with two distinct fixed points in $\partial \mathbb{H}_\infty$), and their conjugacy classes correspond to, respectively, the identity, torsion points of $\Gamma\backslash\mathbb{H}$, cusps of $\Gamma\backslash\mathbb{H}$, and closed geodesics on $\Gamma\backslash\mathbb{H}$. A Fuchsian group is cofinite if and only if it has finite area V and a finite (but nontrivial) number of (Γ-inequivalent) cusps. It can be presented as a finitely generated group generated by hyperbolic motions $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ (with g the genus of $\Gamma\backslash\mathbb{H}$), parabolic motions $c_1, \ldots, c_h$ (with h the number of cusps of $\Gamma\backslash\mathbb{H}$), and elliptic motions $e_1, \ldots, e_\ell$ of orders $m_i \geq 2$, $i = 1, \ldots, \ell$, satisfying the relations

$$[a_1, b_1][a_2, b_2] \cdots [a_g, b_g] c_1 \cdots c_h e_1 \cdots e_\ell = e_i^{m_i} = 1$$

for each $i = 1, \ldots, \ell$. Each cofinite Fuchsian group is in particular the free product of a free group and finite cyclic groups. Via the Gauss–Bonnet formula we have the identity

$$V = 2\pi \left(2g - 2 + h + \sum_{i=1}^{\ell} \left(1 - e_i^{-1}\right) \right).$$

For arithmetic applications, it is convenient to work with discrete subgroups $\Gamma < \mathrm{SL}_2(\mathbb{R})$ instead. The action of Γ on $\mathbb{H}$ factors through its image in $\mathrm{PSL}_2(\mathbb{R})$, which is given by either Γ or $\Gamma/\{\pm I\}$ depending on whether $-I \in \Gamma$. This distinction requires some caution. For instance, if $-I \in \Gamma$, we only have a bijection between the set of closed oriented geodesics and the conjugacy classes of hyperbolic elements $\gamma \in \Gamma$ with $\mathrm{tr}(\gamma) > 2$.

More explicitly, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a primitive hyperbolic matrix in $\mathrm{SL}_2(\mathbb{R})$; then $(a+d)^2 > 4$ and $c \neq 0$. The fixed points of γ in $\partial_\infty \mathbb{H}$ are given by

$$\alpha = \frac{a - d + \sqrt{(a+d)^2 - 4}}{2c}, \quad \overline{\alpha} = \frac{a - d - \sqrt{(a+d)^2 - 4}}{2c}.$$

As a hyperbolic motion, γ acts on the unique hyperbolic geodesic connecting α and $\overline{\alpha}$ via the geodesic flow, and the orientation of the resulting geodesic depends on $\mathrm{sign}(c(a+d))$. In fact, a computation of eigenvalues and eigenvectors for γ yields the diagonal form

$$a_\gamma = g^{-1} \gamma g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

where

$$g = \begin{pmatrix} \alpha & \overline{\alpha} \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \lambda = \frac{a + d + \sqrt{(a+d)^2 - 4}}{2}.$$

We now observe that $\lambda > 1$ iff $a + d > 2$ and that $\alpha > \overline{\alpha}$ iff $c > 0$. We summarize these observations in the following table.

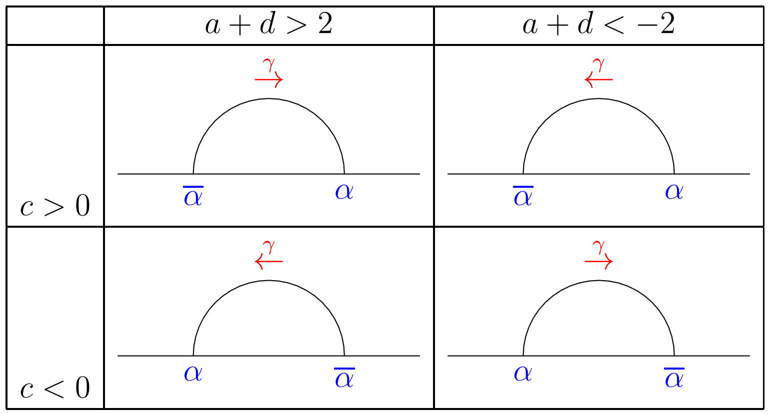


Table 1

The subgroup $\Gamma < \mathrm{SL}_2(\mathbb{R})$ is discrete if and only if its image in $\mathrm{PSL}_2(\mathbb{R})$ is discrete and we keep the terminology of cofinite Fuchsian group here as well.

3.3. **Multiplier systems.** The background material in this section follows [Roe66] and [Hej83].

Definition 3.1. *Let $r \in \mathbb{R}$. A function $\chi : \Gamma \to \mathbb{C}$ is a* multiplier system of weight r *if it verifies*

(1) $|\chi(\gamma)| = 1$;
(2) $\chi(-I) = e^{-\pi i r}$ *(if $-I \in \Gamma$);*
(3) *and* $\chi(\gamma_1\gamma_2)\chi(\gamma_1)^{-1}\chi(\gamma_2)^{-1} = e^{2\pi i r \omega(\gamma_1,\gamma_2)}$.

If $r = n \in \mathbb{Z}$ is even (respectively odd), then χ is an even (respectively odd) unitary character of Γ. By periodicity, any multiplier system χ is of weight r if and only if it is of weight $r + 2m$ for any $m \in \mathbb{Z}$. Thus up to multiplication by a character, we may assume that a multiplier system has weight $r \in (-1, 1]$. Further, given a multiplier system χ of weight r on Γ and an element $g \in \mathrm{GL}_2(\mathbb{R})$,

$$\chi^g(\tau) := \chi(g\tau g^{-1})e^{2\pi i r(\omega(g\tau g^{-1},g)-\omega(g,\tau))}$$

defines a multiplier system on $g^{-1}\Gamma g$.

Let χ be a multiplier system of weight r for Γ, and let $\mathcal{H}(\Gamma, \chi, r)$ be the space of all measurable functions $f : \mathbb{H} \to \mathbb{C}$ such that

$$f(\gamma z) = \chi(\gamma)\left(\frac{j(\gamma, z)}{|j(\gamma, z)|}\right)^r f(z) \tag{8}$$

and

$$\langle f, f\rangle = \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{f(z)}dz = \int_{\Gamma\backslash\mathbb{H}} |f(z)|^2 dz < +\infty.$$

Equipped with the inner product $\langle\cdot,\cdot\rangle$, $\mathcal{H}(\Gamma,\chi,r)$ has the structure of a Hilbert space and is isomorphic to the space $L^2(\Gamma\backslash\mathbb{H},\chi,r)$ of (equivalence classes of) square-integrable functions satisfying (8).

Acting on (a dense subspace of) $L^2(\Gamma\backslash\mathbb{H},\chi,r)$ we have the Maass raising and lowering operators

$$K_r = iy\partial_x + y\partial_y + \frac{r}{2},$$
$$\Lambda_r = iy\partial_x - y\partial_y + \frac{r}{2}.$$

For $f\in\mathcal{H}(\Gamma,\chi,r)$ and $g\in\mathcal{H}(\Gamma,\chi,r+2)$ both at least C^1, we have the identity

$$\langle K_r f, g\rangle = \langle f, \Lambda_{r+2} g\rangle .$$

The Maass operators relate to the weight r (geometric) Laplacian via

$$\Delta_r = \Lambda_{r+2}K_r - \frac{r}{2}\left(1+\frac{r}{2}\right) = K_{r-2}\Lambda_r + \frac{r}{2}\left(1-\frac{r}{2}\right) = -y^2\left(\partial_x^2+\partial_y^2\right) + iry\partial_x.$$

It follows that

$$\langle f,\Delta_r g\rangle = \langle K_r f, K_r g\rangle - \frac{r}{2}\left(1+\frac{r}{2}\right)\langle f,g\rangle = \langle \Lambda_r f,\Lambda_r g\rangle + \frac{r}{2}\left(1-\frac{r}{2}\right)\langle f,g\rangle \tag{9}$$

and that

$$\langle\Delta_r f, g\rangle = \langle f,\Delta_r g\rangle$$

for all C^2 functions $f,g\in\mathcal{H}(\Gamma,\chi,r)$. The bottom eigenvalue is given by

$$\lambda_0(\chi,r) = \inf_{f\in\mathcal{H}\cap C^2}\frac{\langle\Delta_r f,f\rangle}{\langle f,f\rangle}.$$

Specializing (9) to $f=g$ yields

$$\langle\Delta_r f,f\rangle = \|K_r f\|^2 - \frac{r}{2}\left(1+\frac{r}{2}\right)\|f\|^2 = \|\Lambda_r f\|^2 + \frac{r}{2}\left(1-\frac{r}{2}\right)\|f\|^2,$$

and implies that

$$\|K_r f\|^2 = \|\Lambda_r f\|^2 + r\|f\|^2.$$

That is, $\|K_r f\|^2 \geq \|\Lambda_r f\|^2$ if $r\geq 0$ and $\|\Lambda_r f\|^2\geq\|K_r f\|^2$ if $r\leq 0$. Hence

$$\lambda_0(\chi,r) = \frac{|r|}{2}\left(1-\frac{|r|}{2}\right) + \inf_{f\in\mathcal{H}\cap C^2}\begin{cases}\dfrac{\|\Lambda_r f\|^2}{\|f\|^2} & \text{if } r\geq 0,\\[2ex] \dfrac{\|K_r f\|^2}{\|f\|^2} & \text{if } r\leq 0.\end{cases}$$

Remark 3.2. *In §5, we will construct a continuous family $\{\chi_r\}_{r\in\mathbb{R}}$ of multiplier systems (for any cofinite Fuchsian group) and show that for each multiplier system χ_r, we have*

$$\lambda_0(\chi_r,r) = \frac{|r|}{2}\left(1-\frac{|r|}{2}\right),$$

or in other words that the minimal bottom eigenvalue for Δ_r is realized.

If $-I \notin \Gamma$, then each cusp $\mathfrak{a}$ has an infinite cyclic stabilizer group $\Gamma_\mathfrak{a} < \Gamma$. We will denote its generator by $\gamma_\mathfrak{a}$. If $-I \in \Gamma$, then $\Gamma_\mathfrak{a}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; in this case we will also denote the infinite cyclic generator by $\gamma_\mathfrak{a}$. If $\mathfrak{a}, \mathfrak{b}$ are equivalent cusps, then $\Gamma_\mathfrak{a}$ and $\Gamma_\mathfrak{b}$ are conjugate in Γ. Let $\mathfrak{a}$ be a cusp for Γ and let $\gamma_\mathfrak{a}$ be a cyclic generator of $\Gamma_\mathfrak{a}$. We say that $\mathfrak{a}$ is **singular** if $\chi(\gamma_\mathfrak{a}) = 1$ and **regular** otherwise.

With respect to (the unique self-adjoint extension of) Δ_r, the space $L^2(\Gamma\backslash\mathbb{H}, \chi, r)$ has a complete spectral resolution with pure point spectrum

$$\lambda_0(\chi, r) \leq \lambda_1(\chi, r) \leq \lambda_2(\chi, r) \leq \dots \quad \to \infty$$

and absolutely continuous spectrum $[\frac{1}{4}, \infty)$ with multiplicity $m = m(\Gamma, \chi, r)$ equal to the number of inequivalent singular cusps. The continuous spectrum is described by Eisenstein series.

3.4. **Eisenstein series and residual spectrum.** Computations as well as geometric considerations are easier when the cusp is at ∞, which we may assume up to conjugating Γ by a **scaling** $\sigma_\mathfrak{a} \in G$ such that $\sigma_\mathfrak{a}(\infty) = \mathfrak{a}$ and $(\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a})_\infty = \sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a}$ is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

if $-I \notin \Gamma$ or is otherwise isomorphic to $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ with the cyclic generator of $\mathbb{Z}$ given by T. The choice of scaling is not unique; indeed, each element of the continuous family $\sigma_\mathfrak{a} T^x = \sigma_\mathfrak{a} \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, $x \in \mathbb{R}$, showcases those same properties.

For each singular cusp $\mathfrak{a}$, we have a well defined Eisenstein series given by

$$E_\mathfrak{a}(z, s, \chi, r) = \sum_{\gamma \in \Gamma_\mathfrak{a}\backslash\Gamma} \overline{\chi(\gamma)} e(-r\omega(\sigma_\mathfrak{a}, \gamma)) e^{-ir \arg j(\sigma_\mathfrak{a}\gamma, z)} \operatorname{Im}(\sigma_\mathfrak{a}\gamma z)^s,$$

which converges absolutely and uniformly on compact sets for $\operatorname{Re}(s) > 1$, transforms according to (8), verifies $(\Delta_r - s(s-1))E_\mathfrak{a}(z, s, \chi, r) = 0$ and has a meromorphic continuation to all $s \in \mathbb{C}$.

The $m \times 1$ vector $\mathcal{E}(z, s, \chi, r)$ of all Eisenstein series at (inequivalent) singular cusps satisfies the functional equation

$$\mathcal{E}(z, 1-s, \chi, r) = \Phi(s, \chi, r)\mathcal{E}(z, s, \chi, r), \tag{10}$$

where $\Phi = (\varphi_{\mathfrak{a}\mathfrak{b}})_{\mathfrak{a},\mathfrak{b}}$ is the **scattering matrix**, and $\varphi = \det \Phi$ is the **scattering determinant**. From the functional equation (10), we see that

$$\Phi(s, \chi, r)\Phi(1-s, \chi, r) = I.$$

Furthermore, on the line $\operatorname{Re}(s) = \frac{1}{2}$, we have

$$\Phi\left(\frac{1}{2} + it, \chi, r\right) \Phi\left(\frac{1}{2} + it, \chi, r\right)^* = \Phi\left(\frac{1}{2} + it, \chi, r\right) \Phi\left(\frac{1}{2} - it, \chi, r\right) = I,$$

where $A^* = \overline{A}^T$ denotes the conjugate transpose of the matrix A. In particular, the entries of the scattering matrix are bounded.

Following Selberg [Sel89, pp. 655-656], the scattering determinant $\varphi = \det \Phi$ has the Dirichlet series expression

$$\varphi(s,\chi,r) = \left(\frac{\sqrt{\pi}4^{1-s}\Gamma(2s-1)}{\Gamma(s+\frac{r}{2})\Gamma(s-\frac{r}{2})}\right)^{m(\chi,r)} \sum_{n\geq 1} \frac{a_n}{b_n^{2s}} \tag{11}$$

where $0 < b_1 < b_2 < \dots$ are real coefficients, and where the series converges absolutely for $\mathrm{Re}(s) > 1$, and is moreover regular for $\mathrm{Re}(s) \geq 1/2$ except possibly for a finite number of poles $s_1(\chi,r), s_2(\chi,r), \dots, s_k(\chi,r)$ — each with multiplicity $\leq m(\chi,r)$ in the interval $(1/2,1]$. These poles are the eigenparameters of the residual eigenvalues

$$\lambda_j(\chi,r) = s_j(\chi,r)(1-s_j(\chi,r))$$

of the weight r Laplacian Δ_r.

If $E_{\mathfrak{a}}(z,s,\chi,r)$ has a pole at $s=s_0$, this pole is simple, $s_0 \in (1/2,1]$, and $\varphi_{\mathfrak{aa}}(s)$ also has a pole at s_0, see [Roe66, Sätze 10.3-4]. Conversely, if $\varphi_{\mathfrak{aa}}(s)$ has a pole at $s_0 \in (1/2,1]$, then the Maass–Selberg relations (see [Roe66, Lemma 11.2]) imply that $E_{\mathfrak{a}}(z,s,\chi,r)$ has a pole at s_0. The residue $\rho_{\mathfrak{a},s_0} \in L^2(\Gamma\backslash\mathbb{H},\chi,r)$ at this pole is an eigenfunction of Δ_r with eigenvalue $s_0(1-s_0)$. Moreover, if $\rho_{\mathfrak{b},s_0}$ is the residue of $E_{\mathfrak{b}}(z,s,\chi,r)$ at $s=s_0$, then

$$\langle \rho_{\mathfrak{a},s_0}, \rho_{\mathfrak{b},s_0}\rangle = \operatorname{Res} \varphi_{\mathfrak{ab}}(s)|_{s=s_0}, \tag{12}$$

see [Roe66, Satz 11, p.302]. Finally, we record the following lemma.

Lemma 3.3. *If the scattering determinant φ has a pole of order n at $s=s_0$, where $\mathrm{Re}(s_0) > \frac{1}{2}$, then there are n linearly independent eigenfunctions of Δ_r with eigenvalue $s_0(1-s_0)$.*

Proof. Suppose that for some subset $\mathfrak{a}_1,\dots,\mathfrak{a}_n$ of singular cusps, the residues $\rho_{\mathfrak{a}_i,s_0}(z)$ are not identically zero. By (12), we can transform $\Phi(s,\chi,r)$ via elementary row operations into a matrix that only has poles at $s=s_0$ in rows $\mathfrak{a}_1,\dots,\mathfrak{a}_n$. Then $\varphi(s,\chi,r)$ has a pole of order at most n at $s=s_0$. Since the order is n, the residues $\rho_{\mathfrak{a}_i,s_0}$ are linearly independent with eigenvalue $s_0(1-s_0)$. □

3.5. **Selberg's trace formula.** We consider again χ to be a multiplier system of weight r. We review Selberg's trace formula for $L^2(\Gamma\backslash\mathbb{H},\chi,r)$ for the convenience of the reader following [Hej83, pp. 412-413].

Let $h:\mathbb{C}\to\mathbb{C}$ be a holomorphic even function defined on

$$\left\{z\in\mathbb{C} : |y| < \max\{\tfrac{|r|-1}{2},\tfrac{1}{2}\}+\delta\right\}$$

for some $\delta>0$ such that $h(t) = O\left((1+t)^{-2-\delta}\right)$ and let

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)e^{-itu}du.$$

Then

$$\sum_{n\geq 0} h(t_n) = \frac{V}{4\pi}\int_{-\infty}^{\infty} t h(t) \frac{\sinh(2\pi t)}{\cosh(2\pi t)+\cos(\pi r)} dt \tag{13}$$

$$+ \frac{V}{4\pi} \sum_{\substack{l \text{ odd} \\ 1\leq l\leq |r|}} (|r|-l) h\left(\frac{i(|r|-l)}{2}\right) \tag{14}$$

$$+ \sum_{\substack{\{\gamma\} \\ \mathrm{tr}(\gamma)<2 \\ 0<\theta_\gamma<\pi}} \frac{\chi(\gamma) i e^{i(r-1)\theta}}{4|\gamma|\sin\theta_\gamma} \int_{-\infty}^{\infty} g(t) e^{(r-1)t/2} \frac{(e^t - e^{2i\theta})}{\cosh t - \cos 2\theta_\gamma} dt \tag{15}$$

$$+ \sum_{\substack{\{\gamma\}_{\mathrm{pr}} \\ \mathrm{tr}(\gamma)>2}} \sum_{k\geq 1} \frac{\chi(\gamma^k)\ell_\gamma g(k\ell_\gamma)}{\sinh(k\ell_\gamma/2)} \tag{16}$$

$$- g(0) \sum_{\alpha_{\mathfrak{a}}(\chi,r)\neq 0} \log|1 - e(\alpha_{\mathfrak{a}}(\chi,r))| \tag{17}$$

$$+ \frac{1}{2} \sum_{\alpha_{\mathfrak{a}}(\chi,r)\neq 0} (\tfrac{1}{2} - \alpha_{\mathfrak{a}}(\chi,r)) \mathrm{PV}\int_{-\infty}^{\infty} g(t) e^{(r-1)t/2} \frac{(e^t-1)}{\cosh(t)-1} dt \tag{18}$$

$$+ m(\chi,r) \int_0^{\infty} \frac{g(t)(1-\cosh(rt/2))}{e^{t/2}-e^{-t/2}} dt \tag{19}$$

$$- m(\chi,r)\left(g(0)\log 2 - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)\frac{\Gamma'(1+it)}{\Gamma(1+it)} dt\right) \tag{20}$$

$$+ \frac{1}{4} h(0) \mathrm{Tr}\left(I - \Phi(\tfrac{1}{2},\chi,r)\right) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\chi,r\right) dt, \tag{21}$$

where

- the sum on the LHS runs over the spectral eigenparameters $t_n = t_n(\chi, r) \in \mathbb{C}$ given by

$$\lambda_n(\chi,r) = s_n(\chi,r)(1-s_n(\chi,r)) \tag{22}$$

$$= \left(\frac{1}{2}+it_n(\chi,r)\right)\left(\frac{1}{2}-it_n(\chi,r)\right) = \frac{1}{4}+t_n(\chi,r)^2 \tag{23}$$

  and counted with multiplicity;
- $|\gamma|$ in (15) is the order of the elliptic element γ, and θ_γ is the angle in $(0, 2\pi)$ for which $\left(\begin{smallmatrix}\cos\theta_\gamma & -\sin\theta_\gamma \\ \sin\theta_\gamma & \cos\theta_\gamma\end{smallmatrix}\right)$ is a $\mathrm{SL}_2(\mathbb{R})$-conjugate of γ;
- ℓ_γ in (16) denotes the hyperbolic length of the unique closed geodesic corresponding to the conjugacy class $\{\gamma\}_{\mathrm{pr}}$ of the primitive hyperbolic element γ;
- the sums in (17), (18) are taken over the set of regular cusps with $\alpha_{\mathfrak{a}}(\chi, r) \in [0, 1)$ determined by $\chi(\gamma_{\mathfrak{a}}) = e(\alpha_{\mathfrak{a}})$;

- and the terms (19), (20), (21) only appear in the presence of singular cusps, whereby $m(\chi, r)$ is the number of inequivalent singular cusps.

4. Classical Rademacher symbol and windings of modular geodesics

In this section we fix $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. In the classical theory of modular forms, the modular discriminant function $\triangle(z)$ is the prototype of a holomorphic cusp form for Γ. Jacobi proved that $\triangle$ admits the infinite product expansion

$$\triangle(z) = q\prod_{n\geq 1}(1-q^n)^{24}$$

(with $q = e^{2\pi i z}$ and $\mathrm{Im}(z) > 0$). It follows that $\triangle$ is nowhere-vanishing with $\triangle(z) \to 0$ as $y \to \infty$. We fix the branch of logarithm

$$\log \triangle(z) = 2\pi i z - 24\sum_{m,n\geq 1}\frac{q^{mn}}{n}.$$

Then a simple direct computation shows that

$$\frac{d}{dz}\log\triangle(z) = 2\pi i\left(1 - 24\sum_{n\geq 1}\left(\sum_{d\mid n} d\right)q^n\right) = 2\pi i\left(E_2(z) + \frac{3}{\pi y}\right), \tag{24}$$

where E_2 is Hecke's modular, nonholomorphic, Eisenstein series of weight two, given by

$$E_2(z) = \lim_{\varepsilon\to 0^+}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\frac{1}{(cz+d)^2|cz+d|^{\varepsilon}}.$$

For each $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$, we have the transformation law

$$\log\triangle(\gamma z) - \log\triangle(z) \;=\; 6\log(-(cz+d)^2) + 2\pi i\Phi(\gamma), \tag{25}$$

where, on the right-hand side (RHS), log denotes the principal branch of logarithm and the **Dedekind symbol** $\Phi(\gamma)$ is explicitly determined by the matrix entries of γ and involves Dedekind sums [Ded92]. The transformation law can be deduced either from (24) or via Kronecker's first limit formula; see [Sie65, Chapter 1.2].

In his study of Dedekind sums, Rademacher [Rad56] put the emphasis on a second invariant, the **Rademacher symbol** Ψ, given by

$$\Psi(\gamma) \;=\; \Phi(\gamma) - 3\,\mathrm{sign}(c(a+d)),$$

which he shows to be integer-valued and conjugacy class invariant.

Several topological interpretations of Ψ were identified by Atiyah [Ati87, (5.60)]. For instance, for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ hyperbolic, the normalization $-\frac{1}{3}\Psi(\gamma)$ coincides with Meyer's function in the study of signatures of torus bundles over surfaces, with Hirzebruch's signature defect, with the logarithmic monodromy of Quillen's determinant line bundle, or with the Atiyah–Patodi–Singer eta-invariant and its adiabatic limit, to name just a few.

Closer to our interest in this paper, $\Psi(\gamma)$ also coincides (again for γ hyperbolic, and possibly up to a scalar multiple) with the signature of an oriented link on the braid group B_3 with three strands [GG05], with the linking number for modular knots with the trefoil knot τ in $S^3 \setminus \tau$, and with the planar winding number

$$\operatorname{ind}(\widetilde{\triangle}(C), 0) \;=\; \frac{1}{2\pi i}\int_C \frac{d\widetilde{\triangle}}{\widetilde{\triangle}}, \tag{26}$$

where $\widetilde{\triangle}(C) \subset \mathbb{C} \setminus \{0\}$ is the continuous closed curve obtained by taking the lift $\widetilde{\triangle} : \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}^*$ of the modular discriminant $\triangle(z)$, and evaluating it along a parametrized closed oriented orbit C of the geodesic flow on $\Gamma\backslash G$ [Ghy07]. The equivalence of these last three interpretations boils down to the remarkable fact that the homogeneous space $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})$ is diffeomorphic to $S^3\setminus\tau$ with common fundamental group B_3; see [Mil71, p. 84].

In the rest of this section, we give an explicit interpretation of the winding number Ψ, which corresponds to the description given in the introduction. The explicit formula for the Rademacher symbol in terms of Dedekind sums can be evaluated via the Euclidean algortihm, and thus in terms of continued fraction expansions; see [Mey57, Zag75, Hic77, Kel12].

Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ be a primitive hyperbolic element with positive trace. Its fixed points $x_+ > x_-$ are quadratic irrationals, hence admit an eventually periodic simple continued fraction expansion. Let

$$x_+ = [a_1, \dots, a_m, \overline{a_{m+1}, \dots, a_{m+n}}],$$

where $a_i \in \mathbb{Z}$ and $a_i \geq 1$ for $i \geq 2$. The stabilizer of the purely periodic part $x = [\overline{a_{m+1}, \dots, a_{m+n}}]$ in $\mathrm{GL}_2(\mathbb{Z})$ is an infinite cyclic group generated by

$$B = A_{a_{m+1}} A_{a_{m+2}} \cdots A_{a_{m+n}} = \begin{pmatrix} a_{m+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{m+2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{m+n} & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that if n is even, we have $B \in \mathrm{SL}_2(\mathbb{Z})$. The Rademacher function Ψ behaves nicely on such group elements..

Proposition 4.1. *Let $n \in \mathbb{N}$ be even, and let $a_1, \dots, a_n$ be a finite set of positive integers. Then*

$$\Psi(A_{a_1} \cdots A_{a_n}) = a_1 - a_2 + \cdots - a_n.$$

Proof. Let $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $U = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. The proof will proceed by induction via repeated applications of the following formulas

$$\begin{aligned} \Psi(T^a) = a, \qquad \Psi(U^a) &= \Psi(ST^{-a}S^{-1}) = -a; \\ \Phi(\gamma\gamma') &= \Phi(\gamma) + \Phi(\gamma') - 3\mathrm{sign}(c_\gamma c_{\gamma'} c_{\gamma\gamma'}); \\ \Psi(\gamma) &= \Phi(\gamma) - 3\mathrm{sign}(c(a+d)); \end{aligned}$$

see [RG72, (59), (62)–(63)].

We first consider $n = 2$ and write down explicitly

$$A_a A_b = \begin{pmatrix} ab+1 & a \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = T^a U^b.$$

Then

$$\Psi(A_a A_b) = \Phi(A_a A_b) - 3 = \Phi(T^a) + \Phi(U^b) - 3 = \Psi(T^a) + \Psi(U^b) = a - b.$$

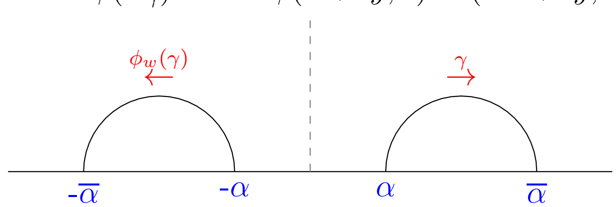
More generally, for any even product $A_{a_1} \cdots A_{a_n}$, all entries are positive and writing $A = A_{a_3} \cdots A_{a_n}$, we have

$$\begin{aligned} \Psi(A_{a_1} \cdots A_{a_n}) &= \Psi(A_{a_1} A_{a_2} A) = \Phi(A_{a_1} A_{a_2} A) - 3 \\ &= \Phi(A_{a_1} A_{a_2}) - 3 + \Phi(A) - 3 = \Psi(A_{a_1} A_{a_2}) + \Psi(A). \end{aligned}$$

We conclude by induction. □

Up to replacing γ by a conjugate, we may assume that the quadratic irrational x_+ is reduced, i.e., that $x_+ > 1$, $x_- \in (-1, 0)$, and as such admits a purely periodic continued fraction expansion. We review this in detail, and take the opportunity to fix some notation. Let $B = A_{a_{m+1}} \cdots A_{a_{m+n}}$ as above and set $A = A_{a_1} \cdots A_{a_m}$. We find that $\gamma = ABA^{-1}$. Then

- If both m and n are even, then both $A, B \in \mathrm{SL}_2(\mathbb{Z})$ and we have that
$$\Psi(\gamma) = \Psi(B) = a_{m+1} - a_{m+2} + \cdots - a_{m+n},$$
where $(a_{m+1}, \dots, a_{m+n})$ is the periodic part of the continued fraction expansion of the fixed point x_+ of γ.
- If n is even and m is odd, then γ and B are not conjugate in $\mathrm{SL}_2(\mathbb{Z})$. Let
$$w = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
and define $\phi_w(A) = w^{-1}Aw$. Explicitly, we have
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$
Then γ is conjugate to $\phi_w(B) = A_{-a_{m+1}} \cdots A_{-a_{m+n}}$ in $\mathrm{SL}_2(\mathbb{Z})$ and hence
$$\Psi(\gamma) = \Psi(\phi_w(B)) = -a_{m+1} + a_{m+2} - \cdots + a_{m+n} = -\Psi(B).$$
- Considering the closed oriented geodesic C_γ associated to γ, one sees that $\phi_w(\gamma)$ corresponds to $\phi(C_\gamma)$ where $\phi(x + iy, \theta) = (-x + iy, \pi - \theta)$.



- If n is odd, then $B \notin \mathrm{SL}_2(\mathbb{Z})$ but $B^2 \in \mathrm{SL}_2(\mathbb{Z})$, and we find that γ is conjugate to either B^2 or $\phi_w(B^2)$ (depending on the parity of m). As already observed by Kelmer [Kel12, Proposition 3.3], n is odd exactly when γ is inert, i.e., invariant under the orientation-reversal involution ϕ_w. Moreover, we have that

$$\Psi(\gamma) = \pm(a_{m+1} - a_{m+2} + \cdots + a_{m+n} - a_{m+1} + \cdots - a_{m+n}) = 0.$$

There are yet other geometric interpretations. Consider $\mathrm{SL}_2(\mathbb{Z})$ as the free product of the finite cyclic groups generated by S and TS. Every nontrivial primitive hyperbolic element of $\mathrm{SL}_2(\mathbb{Z})$ can be seen to be conjugate to a word of the form $A = SU^{\varepsilon_1}SU^{\varepsilon_2}\cdots SU^{\varepsilon_r}$, with $\varepsilon_j = \pm 1$. The Rademacher symbol for A can be expressed as

$$\Psi(\gamma) = \sum_{j=1}^{r} \varepsilon_j.$$

See [RG72, Chapter 4-C] for proofs and [BG92, Section B-4] for a geometric description of Ψ recording the (oriented) turns of the corresponding geodesic in the Basse–Serre tree of $\mathrm{SL}_2(\mathbb{Z})$.

5. Rademacher symbols for Fuchsian groups

Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ be a cofinite Fuchsian group. From [Bur22, Section 2] we have a holomorphic function

$$f(z) = iz + \sum_{n\geq 1} c(n)q^n,$$

where the $c(n)$ are real coefficients growing at most polynomially in n. These coefficients can be written down explicitly, but we will not need this information. Instead what we need are the two quasimorphisms

$$S : \Gamma \to \mathbb{R}, \quad S(\gamma) = \frac{1}{i}\left(f(\gamma z) - f(z) - 2V^{-1}\log j(\gamma, z)\right),$$

where the branch of logarithm is chosen such that $\arg(z) \in (-\pi, \pi]$, and

$$\Phi : \Gamma \to \mathbb{R}, \quad \Phi(\gamma) = \frac{1}{i}\left(f(\gamma z) - f(z) - V^{-1}\operatorname{sign}(c)^2\log\left(-j(\gamma, z)^2\right)\right),$$

where the negative sign is introduced to correct the fact that $\arg(cz + d)^2 \in (0, 2\pi)$ for $c \neq 0$. The choice of branch has the effect that $S(-\gamma) \neq S(\gamma)$ if $-I \in \Gamma$, while $\Psi(-\gamma) = \Psi(\gamma)$. We record for reference that

$$S(\gamma_1\gamma_2) - S(\gamma_1) - S(\gamma_2) = 4\pi V^{-1}\omega(\gamma_1, \gamma_2) \tag{27}$$

for all $\gamma_1, \gamma_2 \in \Gamma$, where ω is the integer-valued 2-cocycle discussed in §3.1, and that for each $\gamma_1 = \left(\begin{smallmatrix} * & * \\ c_1 & d_1 \end{smallmatrix}\right)$, $\gamma_2 = \left(\begin{smallmatrix} * & * \\ c_2 & d_2 \end{smallmatrix}\right)$ in Γ, we have

$$\Phi(\gamma_1\gamma_2) - \Phi(\gamma_1) - \Phi(\gamma_2) = -\pi V^{-1}\mathrm{sign}(c_1c_2c_3), \tag{28}$$

where $\gamma_1\gamma_2 = \left(\begin{smallmatrix} * & * \\ c_3 & d_3 \end{smallmatrix}\right)$. The reader can further verify that

$$\Phi(\pm I) = S(I) = 0, \quad S(-I) = -2\pi V^{-1},$$
$$\Phi(\gamma^{-1}) = -\Phi(\gamma), \quad S(\gamma^{-1}) = -S(\gamma)$$

for each $\gamma \in \Gamma$.

Both S and Φ are regularly (and perhaps confusingly) referred to in the literature as the Dedekind symbol. We note that it is the function Φ that generalizes the Dedekind symbol of Dedekind and Rademacher.

We use the Dedekind symbol to construct in the proof of the next theorem an explicit continuous family of multiplier systems for any cofinite Fuchsian group.

Theorem 5.1. *Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ be a cofinite Fuchsian group. For each $r \in \mathbb{R}$, there exists a multiplier system χ_r of weight r for Γ and a holomorphic automorphic form $\triangle_r$ that transforms with respect to χ_r.*

Proof. The existence statement is classical, see [Pet38, p. 534], [Hej83, pp. 334-335]. Our point is to record here an explicit construction of such a multiplier system.

Since we recall that a multiplier system χ of weight r on Γ induces a multiplier system χ^g on $g^{-1}\Gamma g$ (cf. §3.3) we will from here on assume that Γ has a cusp at ∞ with stabilizer group Γ_∞ generated by $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. We claim that $\chi_r(\gamma) = e^{irVS(\gamma)/2}$ defines a multiplier system of weight r for Γ. Indeed, equation (27) implies that

$$\chi_r(\gamma_1\gamma_2)\chi_r(\gamma_1)^{-1}\chi_r(\gamma_2)^{-1} = e^{2\pi i r\omega(\gamma_1,\gamma_2)}$$

and that $S(-I) = -2\pi V^{-1}$ if $-I \in \Gamma$. (Recall that the choice of principal branch implies that $\omega(-I,-I) = 1$.)

We now consider

$$\triangle_r(z) = e^{rVf(z)/2}$$

We can directly check that $\triangle_r(\gamma z) = \chi_r(\gamma) j(\gamma,z)^r \triangle_r(z)$ for each $\gamma \in \Gamma$. The definition of f implies that $\triangle_r$ is holomorphic. That f is holomorphic at each cusp can be deduced from a more careful study of the Kronecker limit formula. □

Remark 5.2. *Observe that if $r > 0$, we have $\triangle_r(z) \neq 0$ and $\triangle_r(z) \to 0$ as $y \to \infty$.*

Let

$$u_0(z) := \begin{cases} y^{r/2}\triangle_r(z) & \text{if } r \geq 0, \\ y^{-r/2}\overline{\triangle_r(z)} & \text{if } r \leq 0. \end{cases}$$

The following observation will play an important rôle later on.

Proposition 5.3. *We have $u_0 \in L^2(\Gamma\backslash\mathbb{H}, \chi_r, r)$ and*

$$\Delta_r u_0 = \frac{|r|}{2}\left(1 - \frac{|r|}{2}\right) u_0,$$

where Δ_r is the weight r Laplacian.

Proof. The transformation formula for $\triangle_r$ implies that u_0 satisfies (8), while the definition of f guarantees that $\langle u_0, u_0\rangle < \infty$. A direct computation shows that for $u_0 = y^{r/2}\triangle_r$, we have

$$\Lambda_r u_0 = y^{1+r/2}\left(i\frac{\partial}{\partial x}\triangle_r - \frac{\partial}{\partial y}\triangle_r\right)$$

while for $u_0 = y^{-r/2}\overline{\triangle}_r$, we have

$$K_r u_0 = y^{1-r/2}\left(i\frac{\partial}{\partial x}\overline{\triangle}_r - \frac{\partial}{\partial y}\overline{\triangle}_r\right).$$

Since $\triangle$ is holomorphic, we can conclude that $\Lambda_r u_0 = K_r u_0 = 0$. □

By analogy with the classical Rademacher symbol, see [RG72, Chapter 4C], the **Rademacher symbol** associated to Φ is then defined by

$$\Psi : \Gamma \to \mathbb{R}, \quad \Psi(\gamma) = \Phi(\gamma) - \pi V^{-1}\mathrm{sign}(c(a+d)), \tag{29}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Proposition 5.4. *By comparison with the properties of the Dedekind symbols S and Φ, we find that for all $\gamma \in \Gamma$,*

(i) $\Psi(-\gamma) = \Psi(\gamma)$;
(ii) $\Psi(\gamma^{-1}) = -\Psi(\gamma)$;
(iii) $\Psi(\gamma) = \Phi(\gamma) = S(\gamma)$ *whenever* $\gamma \in \Gamma_\infty = \{T^n : n \in \mathbb{Z}\}(\times\mathbb{Z}/2\mathbb{Z})$;
(iv) *and*

$$\Psi(\gamma) = \begin{cases} S(\gamma) & a+d > 0, \\ S(\gamma) + \pi V^{-1}\mathrm{sign}(c) & a+d = 0, \\ S(\gamma) + 2\pi V^{-1}\mathrm{sign}(c) & a+d < 0, \end{cases}$$

whenever $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \notin \Gamma_\infty$.

Proof. Statements (i) and (ii) are immediate. For (iii) and (iv), we recall that $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty$ if and only if $c = 0$. This can be seen by looking at the fixed points of γ in $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$. Indeed, the quadratic equation $\gamma z = z$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has solutions

$$z_\pm = \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c}$$

and so $c = 0$ corresponds uniquely to the situation in which z is the point at infinity and $\gamma \in \Gamma_\infty$. Finally, comparing definitions, we find that for $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $c \neq 0$ we have

$$\begin{aligned} \Psi(\gamma) - S(\gamma) &= \Phi(\gamma) - S(\gamma) - \pi V^{-1}\mathrm{sign}(c(a+d)) \\ &= -V^{-1}\arg(-(cz+d)^2) + 2V^{-1}\arg(cz+d) - \pi V^{-1}\mathrm{sign}(c(a+d)) \\ &= \pi V^{-1}\mathrm{sign}(c)\left(1 - \mathrm{sign}(a+d)\right). \end{aligned}$$

Statement (iv) follows. □

One may also write down an algebraic relation of the form of (27) or (28) for Ψ but we omit to do so — the stated formula for Φ is usually more convenient to manipulate. To conclude this section we prove that Ψ is well defined on the set Π of prime geodesics on $M = \Gamma\backslash\mathbb{H}$. Since there is bijection between closed geodesics on M and Γ-conjugacy classes of elements in Γ, it suffices to show that Ψ is conjugacy class invariant and homogeneous.

Theorem 5.5. *For each $\gamma \in \Gamma$ such that $|\mathrm{tr}(\gamma)| \geq 2$, we have*

$$\Psi(\gamma) = \lim_{n\to\infty} \frac{\Phi(\gamma^n)}{n}.$$

Proof. By recursion on (28), we have that for each $n \geq 1$,

$$\Phi(\gamma^n) = n \cdot \Phi(\gamma) - \pi V^{-1} \sum_{k=1}^{n-1} \mathrm{sign}(c_\gamma c_{\gamma^k} c_{\gamma^{k+1}}), \tag{30}$$

where $\gamma^k = \left(\begin{smallmatrix} * & * \\ c_{\gamma^k} & * \end{smallmatrix}\right)$. We will compute $\Phi(\gamma^n)$ using (30).

Suppose first that $|\mathrm{tr}(\gamma)| > 2$. Then γ has two distinct real eigenvalues $\lambda_1 \neq \lambda_2 = \frac{1}{\lambda_1}$. Up to replacing γ by γ^{-1}, we may assume that $\lambda_1 > 1$. Then $\gamma^n = g^{-1} \left(\begin{smallmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{smallmatrix}\right) g$ for some $g \in G$. Upon comparing matrix coefficients, we find that

$$c_{\gamma^n} = c_\gamma \cdot \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}.$$

This implies

$$\begin{aligned} \mathrm{sign}(c_\gamma c_{\gamma^n} c_{\gamma^{n+1}}) &= \mathrm{sign}(c_\gamma)\mathrm{sign}((\lambda_1^n - \lambda_2^n)(\lambda_1^{n+1} - \lambda_2^{n+1})) \\ &= \begin{cases} \mathrm{sign}(c_\gamma) & \text{if } \lambda_1 > 1, \\ -\mathrm{sign}(c_\gamma) & \text{if } \lambda_1 < -1. \end{cases} \end{aligned}$$

The latter conditions being equivalent to $\mathrm{tr}(\gamma) > 0$ and $\mathrm{tr}(\gamma) < 0$ respectively, we conclude that

$$\lim_{n\to\infty} \frac{\Phi(\gamma^n)}{n} = \Phi(\gamma) - \pi V^{-1} \mathrm{sign}(c(a+d)) = \Psi(\gamma). \tag{31}$$

Suppose now that $|\mathrm{tr}(\gamma)| = 2$. Then γ is conjugate (in G) to a matrix of the form $\pm \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$ for some nonzero $h \neq 0$. Hence γ^n is conjugate to $\left(\begin{smallmatrix} (\pm1)^n & (\pm1)^{n-1} nh \\ 0 & (\pm1)^n \end{smallmatrix}\right)$ and comparing matrix coefficients, we now find that $c_{\gamma^n} = c_\gamma (\pm 1)^{n-1} n$. Again, we find $\mathrm{sign}(c_\gamma c_{\gamma^n} c_{\gamma^{n+1}}) = \mathrm{sign}(c_\gamma \mathrm{tr}(\gamma))$, and consequently, (31) holds as well. □

Corollary 5.6. *For any $\gamma \in \Gamma$ with $|\mathrm{tr}(\gamma)| \geq 2$, we have*

(1) $\Psi(\gamma^n) = n\Psi(\gamma)$ *for each* $n \in \mathbb{Z}$*,*
(2) $\Psi(\gamma_1) = \Psi(\gamma_2)$ *whenever* γ_1, γ_2 *are in the same* Γ*-conjugacy class.*

Proof. Since $\Psi(\gamma^{-1}) = -\Psi(\gamma)$ and $\Psi(I) = 0$, it suffices to proves (1) for $n \in \mathbb{N}$. This follows from

$$\Psi(\gamma^n) = n \cdot \lim_{m\to\infty} \frac{\Phi(\gamma^{mn})}{mn} = n \cdot \Psi(\gamma).$$

To prove (2), we use that by (28), we have the uniform upper bound

$$|\Psi(\tau^{-1}\gamma\tau) - \Psi(\gamma)| \le 4\pi V^{-1},$$

for all $\gamma, \tau \in \Gamma$. Therefore, for any elements $\gamma_1, \gamma_2 \in \Gamma$ in the same Γ-conjugacy class, we have

$$|\Psi(\gamma_1) - \Psi(\gamma_2)| = \lim_{n\to\infty} \frac{|\Psi(\gamma_1^n) - \Psi(\gamma_2^n)|}{n} \le \lim_{n\to\infty} \frac{4\pi V^{-1}}{n} = 0.$$

□

6. Winding numbers for prime geodesics

Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ be a cofinite Fuchsian group.

Definition 6.1. *We call the* **torsion index** *of* Γ *the smallest positive integer* $\tau \in \mathbb{N}$ *such that* $\frac{\tau V}{2\pi} \in \mathbb{N}$.

For example the torsion index of $\mathrm{SL}_2(\mathbb{Z})$ is $\tau = 6$. If Γ is torsionfree then $\tau = 1$.

Theorem 6.2 (cf. Theorem 5)**.** *Let M be a cusped hyperbolic orbifold with finite area V. Fix a generating set $\mathcal{S} = \{\gamma_i\}_{i\in I}$ for Γ. There exists a nonholomorphic automorphic form $\triangle^*_{2\tau}$ of weight 2τ for Γ, that is nowhere-vanishing, for which $\triangle^*_{2\tau}(z) \to 0$ as $y \to \infty$, and there exists a function $\zeta : \Gamma \to \mathbb{Z}$ such that*

$$\zeta(\gamma) = \mathrm{ind}(\widetilde{\triangle}^*_{2\tau}(C_\gamma), 0)$$

for each hyperbolic element $\gamma \in \Gamma$ of positive trace, and

$$\zeta(\gamma) = \zeta(\gamma_{i_1} \cdots \gamma_{i_k}) = \frac{\tau V}{2\pi} \left(\Psi(\gamma) - \sum_{|\mathrm{tr}(\gamma_{i_j})| \ge 2} \Psi(\gamma_{i_j}) \right)$$

for each $\gamma \in \Gamma$.

Proof. The starting point of our construction is the observation that

$$dS(\gamma_1, \gamma_2) \coloneqq S(\gamma_1\gamma_2) - S(\gamma_1) - S(\gamma_2) = 4\pi V^{-1}\omega(\gamma_1, \gamma_2)$$

takes values in $\mathbb{Q}$. By definition, dS is a 2-cocycle on Γ. Since $\mathrm{H}^2(\Gamma, \mathbb{Q}) = \{0\}$ (see [Pat75]), there exists $\zeta : \Gamma \to \mathbb{Q}$ satisfying $d\zeta = dS$. The function ζ is not uniquely determined. We choose ζ as follows. Set $g(\gamma) = -S(\gamma)$ for each nonelliptic generator $\gamma \in \mathcal{S}$, and $g(\gamma) = 0$ for each elliptic generator $\gamma \in \mathcal{S}$, then extend g to a homomorphism on Γ. We show that the function $\zeta \coloneqq \frac{\tau V}{2\pi}(S + g)$ is integer-valued.

By recursion on (27) we have

$$S(\gamma^m) = m\, S(\gamma) + 4\pi V^{-1} \sum_{k=1}^{m-1} \omega(\gamma^k, \gamma).$$

Thus if $\gamma^m = \pm I$, then $S(\gamma) \in \frac{2\pi}{mV}\mathbb{Z}$. Then for any elliptic element $\gamma \in \Gamma$, we have $S(\gamma) \in \frac{2\pi}{\tau V}$. Hence if $\gamma \in \Gamma$ then $\zeta(\gamma)$ is a linear combination of values of $\frac{\tau V}{2\pi} S$ on elliptic generators and values of $\frac{\tau V}{2\pi} dS$, hence an integer.

Since $d\zeta = \frac{\tau V}{2\pi} dS$, the difference $h(\gamma) \coloneqq \zeta(\gamma) - \frac{\tau V}{2\pi} S(\gamma)$ is a homomorphism that further quotients through the projection of Γ in $\mathrm{PSL}_2(\mathbb{R})$, which we will again denote by Γ. The classical theorems of Hurewiecz and de Rham yield the isomorphism

$$\mathrm{Hom}(\Gamma, \mathbb{R}) \cong \mathrm{Hom}(\Gamma/[\Gamma,\Gamma], \mathbb{R}) \cong \mathrm{Hom}(\mathrm{H}_1(M), \mathbb{R}) \cong \mathrm{H}^1_{\mathrm{dR}}(M).$$

In particular there exists a unique smooth closed 1-form $\alpha \in \Omega^1(M)$ such that

$$h(\gamma) = \zeta(\gamma) - \tfrac{\tau V}{2\pi} S(\gamma) = \int_{[\gamma]} \alpha,$$

where on the right the integral is taken over a cycle representative of $[\gamma]$, the image of γ under the projection $\Gamma \to \Gamma/[\Gamma,\Gamma] \cong \mathrm{H}_1(M)$. Let $\pi : \mathbb{H} \to \Gamma\backslash\mathbb{H}$ be the canonical covering map. On the universal cover $\mathbb{H}$, the pull-back form $\pi^*\alpha$ is exact (since $\mathbb{H}$ is contractible) and so there exists a smooth real-valued function $a \in C^\infty(\mathbb{H})$ such that

$$a(\gamma z) - a(z) = \int_z^{\gamma z} \pi^*\alpha = h(\gamma).$$

We now define $\triangle^*_{2\tau}(z) \coloneqq \triangle_{2\tau}(z) e^{2\pi i\, a(z)}$. Then $\triangle^*_{2\tau}$ is real-analytic and nowhere-vanishing and $\triangle^*_{2\tau}(z) \to 0$ as $y \to \infty$. Next we set $\widetilde{\triangle}^*_{2\tau}(g) \coloneqq \triangle^*_{2\tau}(gi) j(g,i)^{-2\tau}$ for each $g \in G$. By Theorem 5.1 and Proposition 5.4, we have

$$\widetilde{\triangle}^*_{2\tau}(\gamma g) = \chi_{2\tau}(\gamma) e^{2\pi i h(\gamma)} \widetilde{\triangle}^*_{2\tau}(g) = e^{2\pi i \left(\frac{\tau V}{2\pi} S(\gamma) + h(\gamma)\right)} \widetilde{\triangle}^*_{2\tau}(g) = \widetilde{\triangle}^*_{2\tau}(g)$$

for all $\gamma \in \Gamma$ and $g \in G$. Hence $\widetilde{\triangle}^*_{2\tau}$ quotients through $\Gamma\backslash G$ and we conclude with the following direct computation. Let γ be a hyperbolic element of positive trace. Then

$$\begin{aligned}
\frac{1}{2\pi i} \int_{C_\gamma} \frac{d\widetilde{\triangle}^*_{2\tau}}{\widetilde{\triangle}^*_{2\tau}} &= \frac{1}{2\pi i} \int \frac{d\left(\triangle_{2\tau}(C(t)i) e^{2\pi i a(C(t)i)} j(C(t),i)^{-2\tau}\right)}{\triangle_{2\tau}(C(t)i) e^{2\pi i a(C(t)i)} j(C(t),i)^{-2\tau}} \\
&= \frac{1}{2\pi i} \left(\log \triangle_{2\tau}(\gamma z) - \log \triangle_{2\tau}(z) + 2\pi i h(\gamma) - 2\tau \log j(\gamma, z)\right) \\
&= \frac{1}{2\pi i} \left(\tau V f(\gamma z) - \tau V f(z) + 2\pi i h(\gamma) - 2\tau \log j(\gamma, z)\right) = \zeta(\gamma).
\end{aligned}$$

□

Theorem 6.3 (Theorem 6)**.** *Let M be a cusped hyperbolic orbifold with finite area V. If the Rademacher symbol Ψ is rational-valued, then there exists $k \in 2\mathbb{N}$ such that*

$$\mathrm{ind}(\widetilde{\triangle}_k(C), 0) = \frac{kV}{4\pi} \Psi(C)$$

for each $C \in \Pi$.

Proof of Theorem 6. By assumption and (27), the function S is rational-valued. On the other hand, by Gauss–Bonnet, the constant $\frac{V}{2\pi}$ is rational as well. In fact, considering (27), we conclude that $\frac{V}{2\pi} S$ takes values in $\frac{1}{m}\mathbb{Z}$ for some minimal $m \in \mathbb{N}$. Then χ_r is trivial for $r = 2m$.

For each $\gamma \in \Gamma$ of positive trace, we have the identity $\frac{mV}{2\pi} S(\gamma) = \frac{mV}{2\pi}\Psi(\gamma) \in \mathbb{Z}$ and the same computation as above shows that this integer is equal to $\operatorname{ind}(\widetilde{\triangle}_{2m}(C_\gamma), 0)$. □

7. Estimates on Weyl's law

In this section we gather estimates from the second author's PhD thesis that will be needed to prove the twisted prime geodesic theorem for multiplier systems. On a first read, the reader is invited to skip directly to §8 and come back to this section for reference on the relevant estimates.

Evaluating the trace formula reviewed in §3.5 for the test function $h(t) = e^{-t^2/T^2}$ yields Weyl's law on the asymptotic distribution of eigenvalues, i.e.,

$$\sum_{|t_n|\le T} 1 - \frac{1}{4\pi}\int_{-T}^{T} \frac{\varphi'}{\varphi}\left(\frac{1}{2} + it, \chi, r\right) dt \;\sim\; \frac{V}{4\pi}\, T^2 \tag{32}$$

as $T \to \infty$; see [Hej83, p. 414]. In general, we have no means of separately estimating the two terms on the left hand-side. (If this were the case, we would obtain from the first term an asymptotic formula for the distribution of the eigenvalues.) We obtain upper bounds on both terms of the LHS of (32) as well as their explicit dependence on χ and r. For this purpose, set

$$L(\chi, r) \;=\; 1 + \sum_{\alpha_{\mathfrak{a}}(\chi,r)\neq 0} \log\left(\alpha_{\mathfrak{a}}(\chi, r)^{-1}\right) \tag{33}$$

where the sum is over all regular cusps $\mathfrak{a}$ of Γ. The following first result follows by a careful evaluation of the trace formula.

Theorem 7.1. *Let $|r| \le 1$ and $T \ge 2$. Then*

$$\sum_{n\ge 0} e^{-t_n^2/T^2} - \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-t^2/T^2}\frac{\varphi'}{\varphi}\left(\frac{1}{2} + it, \chi, r\right) dt \;\ll\; T^2 + T(L(\chi, r) + 1).$$

Proof. We evaluate the terms in the trace formula against the test-function $h(t) = e^{-t^2/T^2}$ and its Fourier transform

$$g(t) \;=\; \frac{T}{2\sqrt{\pi}} e^{-(tT)^2/4} \;\ll\; Te^{-t^2}.$$

To evaluate (13), we note that $|\sinh(t)/\cosh(t)| \to 1$ as $|t| \to \infty$, and that $\cosh(t)-1$ is positive except in $t = 0$, where it has a double zero. Since $\sinh(t)$ has a zero at $t = 0$, the integrand

$$\frac{t\sinh(2\pi t)}{\cosh(2\pi t) + \cos(\pi r)}$$

can be continuously extended to $(t, r) = (0, \pm 1)$ and is uniformly bounded for $|t| \le 1$. Hence

$$(13) \ll \int_{-\infty}^{\infty} (|t| + 1)e^{-t^2/T^2} dt \;\ll\; T^2.$$

Since we only consider $|r| \leq 1$, (14) is 0. For (15), there are only finitely many terms, and

$$(15) \ll T\int_{-\infty}^{\infty} e^{-t^2+(r-1)t/2}(e^t+1)dt \ll T.$$

The same bound also applies to the terms (18) and (19). We conclude that

$$|(17)+(18)+(19)+(20)| \ll T(L(\chi,r)+1)+\int_{-\infty}^{\infty} e^{-t^2/T^2}\left|\frac{\Gamma'}{\Gamma}(1+it)\right| dt.$$

Further, we rely on the standard approximation [Iwa02, (B.11)]

$$\frac{\Gamma'}{\Gamma}(1+it) = \log(s)-\frac{1}{2s}+O\left(\frac{1}{|s|^2}\right)$$

to bound

$$\int_{-\infty}^{\infty} e^{-t^2/T^2}\left|\frac{\Gamma'}{\Gamma}(1+it)\right| dt \ll \int_{-\infty}^{\infty} e^{-t^2/T^2}(|t|+1)dt \ll T^2.$$

For the hyperbolic contribution, we have

$$(16) \ll T\sum_{\substack{\{\gamma\}_{\mathrm{pr}}\\ \mathrm{tr}(\gamma)>2}}\frac{\ell_\gamma}{\sinh(\ell_\gamma/2)}\sum_{k\geq 1}e^{-k\ell_\gamma^2} \ll T\sum_{\substack{\{\gamma\}_{\mathrm{pr}}\\ \mathrm{tr}(\gamma)>2}}\frac{\ell_\gamma e^{-\ell_\gamma}}{\sinh(\ell_\gamma/2)} = O(T),$$

where the last equality follows from the prime geodesic theorem. Finally, using that the entries in the scattering matrix $\Phi(\frac{1}{2},\chi,r)$ are bounded and combining these estimates, we conclude that Theorem 7.1 holds. □

We show next how to obtain separate upper bounds for the two terms on the LHS of (32) using the Dirichlet series representation (11) and a positivity argument.

Theorem 7.2. *If $|r| \leq 1$, we have*

$$|\{n\in\mathbb{N}: |t_n|\leq T\}| \ll T^2+T(L(\chi,r)+1), \tag{34}$$

and

$$\int_{-T}^{T}\left|\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\chi,r\right)\right| dt \ll T^2+T(L(\chi,r)+1). \tag{35}$$

Proof. We consider the function

$$\varphi^*(s,\chi,r) = b_1^{2s-1}\prod_{j=1}^{k}\frac{s_j(\chi,r)-s}{s_j(\chi,r)+s-1}\varphi(s,\chi,r), \tag{36}$$

where $b_1 > 0$ is the smallest positive denominator in the Dirichlet series given by (11) and is bounded below. The local factors are repeated with multiplicity for each pole $s_j(\chi, r) \in (1/2, 1]$ — see Lemma 3.3 —, and $\varphi^*(s, \chi, r)$ is regular and uniformly bounded for $\mathrm{Re}(s) \geq 1/2$. Selberg [Sel89, pp. 655-656] shows that

$$-\frac{\varphi^{*\,\prime}}{\varphi^*}\left(\frac{1}{2} + it, \chi, r\right) > 0$$

for all $t \in \mathbb{R}$. By taking the logarithmic derivative of (36), we have

$$-\frac{\varphi^{*\,\prime}}{\varphi^*}(s, \chi, r) = -\frac{\varphi'}{\varphi}(s, \chi, r) - 2\log b_1 + \sum_{j=1}^{k} \frac{2s_j(\chi, r) - 1}{(s_j(\chi, r) - s)(s_j(\chi, r) + s - 1)}. \tag{37}$$

Since b_1 is uniformly bounded below, the term $-\log b_1$ is uniformly bounded above. For $|t| \leq T$, $e^{-t^2/T^2} \geq e^{-1}$, and using that $-\frac{\varphi^{*\,\prime}}{\varphi^*} > 0$, we have

$$\sum_{|t_n| \leq T} 1 - \frac{1}{4\pi}\int_{-T}^{T} \frac{\varphi^{*\,\prime}}{\varphi^*}\left(\frac{1}{2} + it, \chi, r\right) dt \ll \sum_{|t_n(r)| \leq T} e^{-\frac{t_n^2}{T^2}} - \frac{1}{4\pi}\int_{-T}^{T} e^{-\frac{t^2}{T^2}} \frac{\varphi^{*\,\prime}}{\varphi^*}\left(\frac{1}{2} + it, \chi, r\right) dt.$$

Using the relation (37), the last line implies the estimate

$$\sum_{|t_n| \leq T} 1 - \frac{1}{4\pi}\int_{-T}^{T} \frac{\varphi'}{\varphi}\left(\frac{1}{2} + it, \chi, r\right) dt \ll \sum_{|t_n| \geq 0} e^{-t_n(r)^2/T^2} - \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-t^2/T^2} \frac{\varphi'}{\varphi}\left(\frac{1}{2} + it, \chi, r\right) dt + E(T)$$

for

$$E(T) = \sum_{j=1}^{k} \frac{2s_j(\chi, r) - 1}{4\pi} \int_{-T}^{T} \frac{e^{-t^2/T^2} - 1}{(\sigma_j(r) - \frac{1}{2})^2 + t^2} dt - \frac{\log b_1}{2\pi}\left(2T - \int_{-T}^{T} e^{-t^2/T^2} dt\right).$$

We have $E(T) = O(T)$. Theorem 7.2 then follows from evaluating the terms individually using the positivity of $-\frac{\varphi^{*\,\prime}}{\varphi^*}$. ☐

8. Prime geodesic theorem for multiplier systems

We prove Theorem 7.

Theorem 8.1 (Theorem 7)**.** *Let Γ be a cofinite Fuchsian group and let χ be a multiplier system of weight r on Γ. Then as $T \to \infty$, we have*

$$\sum_{C_\gamma \in \Pi(T)} \chi(\gamma) = \mathrm{Li}\left(e^{s_0(\chi, r)T}\right) + \cdots + \mathrm{Li}\left(e^{s_k(\chi, r)T}\right) + O\left(e^{3T/4}(L(\chi, r) + 1)\right), \tag{38}$$

where $s_0(\chi, r) \geq s_1(\chi, r) \geq \cdots \geq s_k(\chi, r) > \frac{1}{2}$ are the spectral eigenparameters determined by the small eigenvalues $\lambda_j(\chi, r) = s_j(\chi, r)(1 - s_j(\chi, r))$ of the weight r Laplacian Δ_r, and $L(\chi, r)$ is defined by (33).

8.1. **Choice of test-function.** We choose g to be the mollified characteristic function on the symmetric interval $[-T, T]$. More precisely, we choose a smooth even function $k \in C_c^\infty(\mathbb{R})$ of compact support such that $\int_{\mathbb{R}} k(t)dt = 1$ and $k(t) \geq 0$. Then fix $\varepsilon > 0$, let $k_\varepsilon(t) = \frac{1}{\varepsilon}k(\frac{t}{\varepsilon})$, and set up the convolution product

$$g(t) = (\mathbf{1}_{[-T,T]} * k_\varepsilon)(t) = \int_{-T}^{T} k_\varepsilon(t-u)du.$$

Observe that ε controls how closely g approximates $\mathbf{1}_{[-T,T]}$; indeed, we have

$$g(t) = \begin{cases} 1 & \text{if } |t| \leq T - \varepsilon, \\ 0 & \text{if } |t| \geq T + \varepsilon, \end{cases}$$

and g is smoothly decaying on $(T-\varepsilon, T+\varepsilon)$. The parameter ε will be chosen later on as a function of T. The inverse Fourier transform h is given by

$$h(t) \;=\; \frac{2\sin(tT)}{t}\widehat{k}(\varepsilon t).$$

We again underline that both g and h depend on the parameters T, ε. In particular, $g(0) = 1$ if $\varepsilon \leq T$ and $h(0) = 2T\widehat{k}(0) = 2T$. By integration by parts, we have

$$|\widehat{k}(\varepsilon t)| \leq \frac{1}{(\varepsilon t)^j}$$

for any $j \geq 1$. As $|t| \to 0$, we have

$$\widehat{k}(\varepsilon t) = 1 + O(\varepsilon |t|).$$

8.2. **Basic estimates.** We now collect estimates for the trace formula. We have

$$\begin{aligned} (13) &\ll \int_{-\infty}^{\infty} |h(t)|(|t|+1)dt \ll T\int_{-\infty}^{\infty}(|t|+1)\frac{|\sin(tT)|}{|tT|}|\widehat{k}(\varepsilon t)|dt \\ &\ll \; T\int_{-1}^{1}\left|\frac{\sin(tT)}{tT}\widehat{k}(\varepsilon t)\right|dt + 2\int_{1}^{\infty}\left|\sin(tT)\widehat{k}(\varepsilon t)\right|dt \\ &\ll T + \varepsilon^{-1}\int_{1}^{\infty}|\widehat{k}(t)|dt \;\ll\; T + \varepsilon^{-1}, \\ (15) \;&\ll\; \int_{-\infty}^{\infty}|g(t)|dt \;\ll\; T, \end{aligned}$$

where we used that the sum is finite and that for fixed θ, the term

$$e^{(r-1)t/2}\frac{e^t - e^{2i\theta}}{\cosh(t) - \cos(2\theta)}$$

is uniformly bounded. Similarly, (18) and (19) are $O(T)$. For (20), we have

$$\begin{aligned}\int_{-\infty}^{\infty} |h(t)| \left|\frac{\Gamma'}{\Gamma}(1+it)\right| dt &\ll \int_{-\infty}^{\infty} |h(t)\log(1+it)|dt + \int_{-1}^{1} |h(t)|dt \\ &\ll T + \int_{-\infty}^{\infty} |\widehat{k}(\varepsilon t)|dt \\ &\ll T + \varepsilon^{-1}\int_{-\infty}^{\infty} (1+t^2)^{-1}dt \ll T + \varepsilon^{-1}.\end{aligned}$$

For the hyperbolic contribution, we use the estimate

$$(16) = \sum_{C_\gamma \in \Pi(T)} \frac{\chi(\gamma)\ell_\gamma}{\sinh(\ell_\gamma/2)} + O(\varepsilon e^{T/2} + T^2).$$

Finally, using Theorem 7.2, we have

$$\begin{aligned}(21) \ll\ & m(\chi,r)T + \int_{-\infty}^{\infty} |h(t)| \left|\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\chi,r\right)\right| dt \\ \ll\ & m(\chi,r)T + \int_{-1}^{1}(1+O(\varepsilon|t|)) \left|\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\chi,r\right)\right| dt \\ & + \int_{1}^{\varepsilon^{-1}} \frac{1}{t} \left|\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\chi,r\right)\right| dt + \varepsilon^{-2}\int_{\varepsilon^{-1}}^{\infty} \frac{1}{t^3} \left|\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\chi,r\right)\right| dt \\ \ll\ & (m(\chi,r) + L(\chi,r) + 1)T + \varepsilon^{-1}\left(L(\chi,r)+1\right).\end{aligned}$$

Collecting estimates, the trace formula yields

$$\sum_{n\geq 0} h(t_n) = \sum_{C_\gamma\in\Pi(T)} \frac{\chi(\gamma)\ell_\gamma}{\sinh(\ell_\gamma/2)} + O(\varepsilon e^{T/2} + T^2 + \varepsilon^{-1}(L(\chi,r)+1)).$$

8.3. **Small eigenvalues.** For each eigenvalue $\lambda_j(\chi,r) < \frac{1}{4}$, the parametrization $\lambda_j(\chi,r) = (\frac{1}{2}+it_j)(\frac{1}{2}-it_j) = \frac{1}{4}+t_j^2$ implies that we may choose $t_j \in -i(0,\frac{1}{2}]$. Then

$$h(t_j) \;=\; \frac{e^{it_jT} - e^{-it_jT}}{it_j(r)}\widehat{k}(\varepsilon t_j) \;=\; \frac{e^{it_jT}}{it_j} + O(\varepsilon e^{|t_j|T})$$

and hence

$$\sum_{\lambda_j(\chi,r)<1/4} h(t_j) \;=\; \sum_j \frac{e^{it_jT}}{it_j} + O(\varepsilon e^{T/2}),$$

where the implied constant is independent of χ and r.

8.4. **Cuspidal eigenvalues.** Consider now $\lambda_j(\chi,r) \geq 1/4$, and hence $t_j \in \mathbb{R}$. Then

$$\sum_{0\leq|t_j|<1} h(t_j) = 2T \sum_{0\leq|t_j|<1} \frac{\sin(t_jT)}{t_jT}\widehat{k}(\varepsilon t_j) = O(T),$$

while by summation by parts and Theorem 7.2, we have

$$\sum_{1\leq|t_j|<\varepsilon^{-1}} h(t_j) \ll \sum_{1\leq|t_j|<\varepsilon^{-1}} \frac{1}{|t_j|} \ll \varepsilon^{-1}(L(\chi,r)+1),$$

and using additionally the fast decay of Fourier coefficients, we have

$$\sum_{1\leq\varepsilon|t_j|} h(t_j) \ll \varepsilon \sum_{1\leq\varepsilon|t_j|} \frac{1}{(\varepsilon|t_j|)^3} \ll \varepsilon^{-1}(L(\chi,r)+1).$$

8.5. **Bootstrapping.** Collecting estimates, we conclude that

$$\sum_{C_\gamma\in\Pi(T)} \frac{\chi(\gamma)\ell_\gamma}{\sinh(\ell_\gamma/2)} = \sum_j \frac{e^{it_jT}}{it_j)} + O(\varepsilon e^{T/2} + T^2 + \varepsilon^{-1}(L(\chi,r)+1)),$$

where the error term can be optimized to $O(e^{T/4}(L(\chi,r)+1))$. Applying summation by parts we find

$$\begin{aligned}\sum_{C_\gamma\in\Pi(T)} \chi(\gamma)\ell_\gamma &= \sum_j \frac{e^{(it_j+1/2)T}}{it_j+1/2} + O(e^{3T/4}(L(\chi,r)+1)) \\ &= \frac{e^{s_0(\chi,r)T}}{s_0(\chi,r)} + \frac{e^{s_1(\chi,r)T}}{s_1(\chi,r)} + \cdots + \frac{e^{s_k(\chi,r)T}}{s_k(\chi,r)} + O\left(e^{3T/4}(L(\chi,r)+1)\right)\end{aligned} \tag{39}$$

and

$$\sum_{C_\gamma\in\Pi(T)} \chi(\gamma) = \mathrm{Li}\left(e^{s_0(\chi,r)T}\right) + \cdots + \mathrm{Li}\left(e^{s_k(\chi,r)T}\right) + O\left(e^{3T/4}(L(\chi,r)+1)\right),$$

which concludes the proof of Theorem 7.

9. Small eigenvalues and continuous deformations

We consider a continuous perturbation of (39) in the weight r. For this we need to understand how the small eigenvalues behave under perturbation. The main result of this section is

Lemma 9.1. *Let Γ be a cofinite Fuchsian group and let χ be a multiplier system of weight r for Γ such that $\chi(\gamma)$ is continuous as a function of r, for any fixed $\gamma\in\Gamma$. Fix $T<1/4$ such that it is not an eigenvalue for $\Delta_0=\Delta$. Then the count $\#\{\lambda_n(\chi,r)\leq T\}$ is continuous at $r=0$.*

Our proof follows the strategy of [Ris11, Lemma 3.3] (which proves the same result for automorphic forms transforming with a unitary character) together with an estimate from [JL97] for the hyperbolic heat trace. We will need the following facts pertaining to the Laplace transform. For a sufficiently nice function $f:\mathbb{R}_{>0}\to\mathbb{C}$, its Laplace transform is given by

$$\mathscr{L}(f)(z) = \int_0^\infty e^{-zt} f(t)dt$$

with inverse transform

$$f(u) = \frac{1}{2\pi i}\int_{(a)} e^{zu}\mathscr{L}(f)(z)dz.$$

Let $\rho > 0$ and define

$$f_\rho(t) = \int_0^t \frac{(t-u)^{\rho-1}}{\Gamma(\rho)} f(u)du.$$

Then following [Wid41, Thm. 8.1, 8.2, p. 73], we have

$$\frac{1}{2\pi i}\int_{(a)} e^{zu}\frac{\mathscr{L}(f)(z)}{z^\rho}dz = \begin{cases} f_\rho(u) & \text{if } u \geq 0, \\ 0 & \text{if } u < 0, \end{cases} \tag{40}$$

whenever $a > 0$ is large enough for $\mathscr{L}(f)(z)$ to converge absolutely in the half-plane $\mathrm{Re}(z) \geq a > a_0$.

Proof of Lemma 9.1. Let $f(t) = t^{w-1}$ with $w \geq 1$. Its Laplace transform is

$$\mathscr{L}(f)(z) = \frac{\Gamma(w)}{z^w}.$$

We consider the trace formula for the test function $h_z(t) = e^{-zt^2}$ for a fixed $z \in \mathbb{C}$ with fixed positive real part $a = \mathrm{Re}(z) > 0$ and we integrate the trace formula against

$$\frac{\mathscr{L}(f)(z)e^{z(T-1/4)}}{2\pi i z} = \frac{\Gamma(w)e^{z(T-1/4)}}{2\pi i z^{w+1}} \tag{41}$$

along $(a) = a + i\mathbb{R}$. On the left hand-side of the trace formula, this yields with (40)

$$\begin{aligned} \frac{1}{2\pi i}\sum_{n\geq 0}\int_{(a)} e^{z(T-(\frac{1}{4}+t_n^2))}\frac{\mathscr{L}(f)(z)}{z}dz &= \sum_{\lambda_n(\chi,r)\leq T} f_1(T-\lambda_n(r)) \\ &= \sum_{\lambda_n(\chi,r)\leq T}\int_0^{T-\lambda_n(\chi,r)} f(u)du \\ &= \frac{1}{w}\sum_{\lambda_n(\chi,r)\leq T}(T-\lambda_n(\chi,r))^w. \end{aligned}$$

Recall that $a = \mathrm{Re}(z) > 0$. The Fourier transform of $h_z(t)$ is given by

$$g_z(t) = \sqrt{\frac{\pi}{z}}e^{-t^2/(4z)}.$$

The integration against (41) annihilates most terms of the trace formula. Indeed, for all $t \in \mathbb{R}$ and $T < \frac{1}{4}$, we have

$$\frac{1}{2\pi i}\int_{(a)} h_z(t)\frac{e^{z(T-1/4)}}{z^2}dz = \frac{1}{2\pi i}\int_{(a)}\frac{e^{z(T-\frac{1}{4}-t^2)}}{z^2}dz = 0,$$

and

$$\frac{1}{2\pi i}\int_{(a)} g_z(0)\frac{e^{z(T-1/4)}}{z^2}dz = \frac{1}{2\sqrt{\pi}i}\int_{(a)}\frac{e^{z(T-\frac{1}{4})}}{z^{5/2}}dz = 0.$$

By (6.43)–(6.44) in [Hej83, p.401], the integrals given by (15) and (18) can be expressed in terms of $h_z(t)$ rather than $g_z(t)$. So these terms vanish as well after integration. We are left with

$$N_w(T,r) := \frac{1}{w}\sum_{\lambda_n(\chi,r)\le T}(T-\lambda_n(r))^w \;=\; \frac{1}{2\pi i}\int_{(a)} F_r(z)\frac{\Gamma(w)e^{z(T-1/4)}}{z^{w+1}}dz,$$

where

$$F_r(z) = \sum_{\{\gamma\}_{\mathrm{pr}}}\sum_{k\ge 1}\frac{\chi(\gamma)\ell_\gamma g_z(k\ell)}{e^{k\ell/2}-e^{-k\ell/2}} + m(\chi,r)\int_0^\infty \frac{g_z(u)(1-\cosh(\frac{ru}{2}))}{e^{u/2}-e^{-u/2}}du.$$

We first claim that $N_w(T,r)$ is continuous at $r=0$ by dominated convergence for $w>\frac{3}{2}$. First, we see that, as $r\to 0$,

$$\frac{g_z(u)(1-\cosh(\frac{r}{2}u))}{e^{u/2}-e^{-u/2}}\frac{e^{z(T-1/4)}}{z^{w+1}} \;\to\; 0$$

uniformly for $z\in(a)$, and $t\in\mathbb{R}_{>0}$. By dominated convergence, the second term of $F_r(z)$ is then continuous as a function of r at $r=0$, and in fact 0 at $r=0$. (In a small interval $|r|\le\delta$, $r\ne 0$, the count $m(\chi,r)$ is constant.) To control the hyperbolic contribution, we use the bound

$$|H(a+it)| \ll (1+|t|)^{3/2}.$$

(see (4.2) in [JL97]) for the hyperbolic heat trace

$$H(z) = \frac{e^{-z/4}}{4\sqrt{\pi z}}\sum_{\{\gamma\}_{\mathrm{pr}}}\sum_{k\ge 1}\frac{\ell_\gamma e^{-(k\ell_\gamma)^2/(4z)}}{\sinh(k\ell_\gamma/2)}.$$

Then

$$\frac{1}{2\pi i}\int_{(a)} H(z)\frac{e^{z(T-1/4)}}{z^{w+1}}dz$$

is bounded by

$$\int_1^\infty \frac{(1+t)^{3/2}}{(a^2+t^2)^{\frac{1+w}{2}}}dt + O(1) \ll \int_1^\infty t^{1/2-w}dt.$$

This proves that $N_w(T,r)$ is continuous at $r=0$ for $w>3/2$. To extend the claim to all $w\ge 1$, we first observe that by definition, the function $N_w(T,r)$ is positive and monotone increasing in T and that

$$\frac{d}{dT}N_{w+1}(T,r) = wN_w(T,r), \tag{42}$$

where we have used that $\Gamma(w+1) = w\Gamma(w)$. Hence by the mean value theorem, we have

$$N_w(T,r) \le \frac{N_{w+1}(T+\delta,r) - N_{w+1}(T,r)}{w\delta} \le N_w(T+\delta,r)$$

for $T < T+\delta < 1/4$. So for any $w \ge 1$, using the continuity of N_{w+1} at $r = 0$, we have

$$\limsup_{r\to 0} N_w(T,r) \;\le\; \frac{N_{w+1}(T+\delta,0) - N_{w+1}(T,0)}{w\delta} \;=\; N_w(T,0),$$

where the last equality follows from (42). Similarly, we have

$$\liminf_{r\to 0} N_w(T,r) \ge \frac{1}{w+1}\frac{N_{w+1}(T,0) - N_{w+1}(T-\delta,0)}{\delta} = N_w(T,0),$$

and hence $N_w(T,r)$ is continuous at $r = 0$ for all $w \ge 1$ and in particular for $w = 1$. By positivity, we have

$$\frac{1}{\delta}(N_1(T,r) - N_1(T-\delta,r)) \;\le\; \#\{\lambda_n(r) \le T\} \;\le\; \frac{1}{\delta}(N_1(T+\delta,r) - N_1(T,r))$$

for $0 < T+\delta < \frac{1}{4}$. It follows that the counting function in the middle is continuous at $r = 0$. □

Corollary 9.2. *Fix $T < 1/4$ that is not an eigenvalue for $\Delta_0 = \Delta$. There exists $\delta > 0$ such that for $|r| \le \delta$, the number $\#\{\lambda_n(\chi,r) \le T\}$ of small eigenvalues counted with multiplicity is constant.*

Recall that $\lambda_0(\chi,0) = 0$ for $L^2(\Gamma\backslash\mathbb{H},\chi,0)$ if and only if χ is trivial at $r = 0$.

Corollary 9.3. *If χ is trivial at $r = 0$, then the bottom eigenvalue $\lambda_0(\chi,r)$ has multiplicity 1 in some small interval $|r| \le \delta$.*

10. Statistics for winding numbers

Theorem 10.1 (Theorem 2)**.** *Let $\Gamma \in \mathcal{G}$ and consider the winding number given by Theorem 6. Then $\pi_n(T)$ is the number of prime geodesics in $\Pi(T)$ with winding*

$$\mathrm{ind}(\widetilde{\triangle}_k(C),0) = n$$

and there exists $\delta \in (0,\frac{1}{2}]$ such that

$$\pi_n(T) = \frac{4}{kT}\int_2^{e^T} \frac{\log t}{(\log t)^2 + \left(\frac{4\pi n}{k}\right)^2}\,dt \,+ O\left(\frac{e^{T(1-\delta/2)}}{T}\right)$$

as $T \to \infty$. The implied constant does not depend on n.

Proof of Theorem 2. We specialize the twisted prime geodesic Theorem 7 to the multiplier system χ_r constructed in Theorem 5.1. Since $\Gamma \in \mathcal{G}$, Theorem 6 guarantees the existence of a weight $k \in 2\mathbb{N}$ such that χ_k precisely encodes the winding number

$$\frac{kV}{4\pi}\Psi(\gamma) = \mathrm{ind}(\widetilde{\triangle}_k(C_\gamma),0)$$

in the following way. Choosing for each C_γ a representative $\gamma \in \Gamma$ with positive trace, we have

$$\chi_r(\gamma) = e^{2\pi i \frac{rV}{4\pi}\Psi} = e^{2\pi i \frac{r}{k}\mathrm{ind}(\widetilde{\triangle}_k(C_\gamma),0)}.$$

By periodicity, we may restrict r to the interval $I = (-k/2, k/2]$. We have

$$\sum_{C_\gamma \in \Pi(T)} \ell_\gamma \int_I e^{2\pi i \frac{r}{k}\left(\mathrm{ind}(\widetilde{\triangle}_k(C_\gamma),0)-n\right)} dr \;=\; k \sum_{\substack{C_\gamma \in \Pi(T) \\ \mathrm{ind}(\widetilde{\triangle}_k(C_\gamma),0)=n}} \ell_\gamma.$$

Integrating on the RHS of (39) accordingly will require more care.

First consider the case of $|r| < 1$. By Proposition 5.3, we have $\lambda_0(\chi_r, r) = \frac{|r|}{2}(1 - \frac{|r|}{2})$ or, equivalently, $s_0(\chi_r, r) = 1 - \frac{|r|}{2}$. In particular, $\lambda_0(\chi_0, 0) = 0$ with multiplicity 1 and we may thus choose T small enough so that Lemma 9.1 asserts the existence of $\delta > 0$ such that $N_r(T) = N_0(T) = 1$ whenever $|r| < \delta$. That is, $\lambda_1(\chi_r, r) \geq T$ whenever $|r| < \delta$, or equivalently, there is a constant c (depending only on T) such that $s_1(\chi_r, r) \leq 1 - \frac{c}{2}$ whenever $|r| < \delta$. Up to choosing a smaller δ, we may assume that $\delta \leq \min\{c, \frac{1}{2}\}$. Then whenever $|r| < \delta$, the twisted prime geodesic theorem in the form given by (39) yields

$$\sum_{C_\gamma \in \Pi(T)} \chi_r(\gamma)\ell_\gamma = \frac{e^{T(1-\frac{|r|}{2})}}{1 - \frac{|r|}{2}} + O\left(e^{T(1-\frac{\delta}{2})}(L(\chi_r, r) + 1)\right) \tag{43}$$

whereas the LHS is trivially bounded by the error term alone when $\delta \leq |r| \leq 1$, since $s_0(\chi_r, r) \leq 1 - \frac{|r|}{2} \leq 1 - \frac{\delta}{2}$.

If $|r| > 1$, we fix $\rho \in (-1, 1]$ to be adjusted weight uniquely determined by $\rho \equiv r \pmod 2$; that is, $\chi_r = \chi_\rho \chi$ for some nontrivial unitary character χ. Since then $\chi_r = \chi \neq 1$ at $\rho = 0$, we have $\lambda_1(\chi, 0) \geq \lambda_0(\chi, 0) > 0$ and again there exists $\delta > 0$ such that $\lambda_0(\chi_r, r) \geq T$ whenever $|\rho| < \delta$, and we conclude as above that the LHS in (43) is $O(e^{T(1-\delta/2)})$ for $|r| > 1$. This proves that there exists $\delta \in (0, 1/2]$ such that

$$\sum_{C_\gamma \in \Pi(T)} \chi_r(\gamma)\ell_\gamma \;=\; \begin{cases} \dfrac{e^{T(1-\frac{|r|}{2})}}{1 - \frac{|r|}{2}} + O\left(e^{T(1-\frac{\delta}{2})}(L(\chi_r, r) + 1)\right) & \text{if } |r| \leq \delta, \\ O\left(e^{T(1-\frac{\delta}{2})}(L(\chi_r, r) + 1)\right) & \text{if } |r| > \delta. \end{cases} \tag{44}$$

Now integrating the RHS of (44) against $e^{-2\pi i \frac{rn}{k}} dr$ over the interval I yields

$$\begin{aligned}\int_{-\delta}^{\delta} \frac{e^{T(1-|r|/2)}}{1-|r|/2} e^{-2\pi i \frac{rn}{k}}\, dr &= 2\mathrm{Re} \int_0^{\delta} \frac{e^{T(1-r/2)}}{1-r/2} e^{-2\pi i \frac{rn}{k}}\, dr \\ &= 2\mathrm{Re} \int_0^{\delta} \left(\int_2^{e^T} \frac{dy}{y^{r/2}} \right) e^{-2\pi i \frac{rn}{k}}\, dr + O(1) \\ &= 2\mathrm{Re} \int_2^{e^T} \int_0^{\delta} e^{-\frac{r}{2}\left(\log y - 4\pi i \frac{n}{k}\right)} dr\ dy + O(1) \\ &= 4 \int_2^{e^T} \frac{\log y}{(\log y)^2 + (\frac{4\pi}{k} n)^2}\, dy + O\left(\int_2^{e^T} \frac{e^{-\frac{\delta}{2}\log y}}{\log y}\, dy \right) \\ &= 4 \int_2^{e^T} \frac{\log y}{(\log y)^2 + (\frac{4\pi}{k} n)^2}\, dy + O\left(\frac{e^{T(1-\delta/2)}}{T} \right).\end{aligned}$$

It remains to integrate the error term in Theorem 7. Using [JOS20, Proposition 5.6] we can check that each cusp of Γ is singular except the cusp at infinity, for which we have $\alpha_\infty(\chi_r, r) = \frac{rV}{4\pi}$. Then for every $\varepsilon > 0$ we have

$$\int_{\varepsilon}^{1} L(\chi_r, r)\, dr = -\int_{\varepsilon}^{1} \log(r)\, dr + O(1) = O(1)$$

as $\varepsilon \to 0^+$. We conclude with an application of summation by parts. □

Theorem 10.2 (Theorem 3)**.** *Let $\Gamma \in \mathcal{G}$ and consider the winding number given by Theorem 6. The limiting distribution of the ratio of winding-to-length for prime geodesics is Cauchy. More precisely, for each $t \in \mathbb{R}$ we have*

$$\lim_{T\to\infty} \frac{\#\{C \in \Pi(T) : \mathrm{ind}(\widetilde{\triangle}_k(C), 0) \le \frac{4\pi t}{k} \ell_C\}}{\pi(T)} = \int_{-\infty}^{t} \frac{du}{\pi(1+u^2)}.$$

Proof of Theorem 3. An application of integration by parts yields

$$\pi_n(T) = \frac{4}{kT} \int_2^{e^T} \frac{\log y}{(\log y)^2 + (\frac{4\pi}{k} n)^2}\, dy = \frac{4}{k} \frac{Te^T}{T^2 + (\frac{4\pi}{k} n)^2} + O\left(\frac{e^T}{T^3} \right). \tag{45}$$

We see the Cauchy distribution appear in

$$\begin{aligned}\sum_{aT \le n \le bT} \pi_n(T) &= \frac{4}{k} \frac{e^T}{T^2} \left(\sum_{aT \le n \le bT} \frac{1}{1 + \left(\frac{4\pi}{kT} n\right)^2} + O(1) \right) \\ &= \frac{4}{k} \frac{e^T}{T^2} \left(\int_{aT}^{bT} \frac{dx}{1 + \left(\frac{4\pi}{kT} x\right)^2} + O(1) \right) \\ &= \frac{e^T}{T} \int_{4\pi a/k}^{4\pi b/k} \frac{du}{\pi(1+u^2)} + O\left(\frac{e^T}{T^2} \right),\end{aligned}$$

where we have used Euler–Maclaurin and a change of variable.

For simplicity, we may assume that $b > 0$. Let $\varepsilon > 0$. Then

$$\frac{1}{\pi(T)} \sum_{\substack{C \in \Pi(T) \\ b\ell_C \leq \mathrm{ind}(C) \leq bT}} 1 \leq \frac{\pi(T(1-\varepsilon))}{\pi(T)} + \frac{1}{\pi(T)} \sum_{\substack{C \in \Pi(T) \setminus \Pi(T(1-\varepsilon)) \\ T(1-\varepsilon) < \frac{\mathrm{ind}(C)}{b} < T}} 1$$

$$\sim \int_{\frac{4\pi b}{k}(1-\varepsilon)}^{\frac{4\pi b}{k}} \frac{du}{\pi(1+u^2)}$$

as $T \to \infty$. Since this holds for any choice of $\varepsilon > 0$, we conclude that

$$\lim_{T \to \infty} \frac{1}{\pi(T)} \sum_{\substack{C \in \Pi(T) \\ \mathrm{ind}(C) \leq b\ell_C}} 1 = \int_{-\infty}^{4\pi \frac{b}{k}} \frac{du}{\pi(1+u^2)}.$$

□

Definition 10.3. *A function $f : \Gamma \to \mathbb{R}$ is a* **quasimorphism** *if there exists $D > 0$ such that*

$$\sup_{\gamma_1, \gamma_2 \in \Gamma} |f(\gamma_1 \gamma_2) - f(\gamma_1) - f(\gamma_2)| \ \leq \ D \ < +\infty.$$

The constant D is called the defect of the quasimorphism f.

From the properties proved in §5, we can see that the Dedekind and Rademacher symbols are examples of quasimorphisms. To support our heuristic that the appearance of the Cauchy law reflects the behavior of long geodesics winding high in the cusp, we observe that conjugacy class invariant quasimorphisms on torsionfree cocompact Fuchsian groups grow at most linearly in ℓ_γ.

Proposition 10.4. *Let $M = \Gamma \backslash \mathbb{H}$ be a closed hyperbolic surface, and let $f : \Gamma \to \mathbb{Z}$ be a conjugacy class invariant quasimorphism with defect D. Then*

$$|f(\gamma)| \ll (1 + D)\ell_\gamma.$$

Proof. Fix a finite generating set $\mathcal{S}$ for Γ, and let $d_{\mathcal{S}}$ denote the word metric on Γ with respect to $\mathcal{S}$. Each $\gamma \in \Gamma$ can be written as a finite reduced word in $\mathcal{S}$ so that we have the immediate bound

$$|f(\gamma)| \leq \sum_{\gamma_i \in \mathcal{S}} |f(\gamma_i)| + D\ell_{\mathcal{S}}(\gamma) \ \leq \ (C + D)\ell_{\mathcal{S}}(\gamma),$$

where D is the quasimorphism defect of f, and C is an absolute constant that depends on f and $\mathcal{S}$. Since Γ acts cocompactly on $\mathbb{H}$, the Milnor–Švarc lemma (or, as it is sometimes called, the fundamental observation of geometric group theory, see, e.g. [FM12, Theorem 8.2]) says that the metric spaces $(\Gamma, d_{\mathcal{S}})$ and $(\mathbb{H}, d_{\mathbb{H}})$ are quasi-isometric, with the quasi-isometry given by $\Gamma \to \mathbb{H}$, $\gamma \mapsto \gamma z$ for any fixed $z \in \mathbb{H}$. In particular, there exist further constants $A \geq 1$, $B \geq 0$ such that for any z in a fixed fundamental domain for Γ, we have $\ell_{\mathcal{S}}(\gamma) \leq A \cdot d_{\mathbb{H}}(\gamma z, z) + B$. □

We use both Theorem 2 and Theorem 3 to prove the equidistribution result of Theorem 4.

Theorem 10.5 (Theorem 4)**.** *Let $\Gamma \in \mathcal{G}$ and consider the winding number given by Theorem 6. Let $A \subseteq \mathbb{Z}$ be a set with natural density $d(A)$. The density of prime geodesics with winding number in A is equal to $d(A)$. Explicitly, if $\pi_A(T)$ denotes the number of prime geodesics in $\Pi(T)$ with winding number in A, then*

$$\lim_{T\to\infty} \frac{\pi_A(T)}{\pi(T)} \;=\; d(A).$$

Proof of Theorem 4. Fix $\varepsilon > 0$. There exist positive constant K_0, K_1 such that...

(1) ...for all $K \geq K_0$,

$$\left|\frac{|\{n \in A : |n| \leq K\}|}{|\{n \in \mathbb{Z} : |n| \leq K\}|} - d(A)\right| < \varepsilon$$

(2) ...and for all $K \geq K_1$,

$$\int_{2\pi \frac{K_1}{\tau\nu}}^{\infty} \frac{du}{\pi(1+u^2)} \;<\; \varepsilon.$$

Choose T sufficiently large that $K_1 T > K_0$. We decompose the set A as

$$A \;=\; A_1 \cup A_2 \cup A_3 \;:=\; A \cap \left(\{|n| \leq K_0\} \cup \{K_0 < |n| \leq K_1 T\} \cup \{K_1 T < |n|\}\right).$$

Using (1) and applying summation by parts twice, we have

$$\begin{aligned}
\pi_{A_2}(T) \;&=\; \sum_{\substack{|n|\leq K_1 T\\ n\in A}} 1 \cdot \pi_{K_1 T}(T) - \sum_{\substack{|n|\leq K_0\\ n\in A}} 1 \cdot \pi_{K_0}(T) - \int_{K_0}^{K_1 T} \left(\sum_{\substack{|n|\leq t\\ n\in A}} 1\right) d\pi_t(T)\\
&=\; d(A)\left(\sum_{|n|\leq K_1 T} 1 \cdot \pi_{K_1 T}(T) - \sum_{|n|\leq K_0} 1 \cdot \pi_{K_0}(T) - \int_{K_0}^{K_1 T} \left(\sum_{|n|\leq t} 1\right) d\pi_t(T)\right)\\
&\quad + O\left(\varepsilon \pi(T)\right) \;=\; d(A) \sum_{K_0 < |n| \leq K_1 T} \pi_n(T) + O\left(\varepsilon \pi(T)\right).
\end{aligned}$$

Then

$$\begin{aligned}
\left|\frac{\pi_A(T)}{\pi(T)} - d(A)\right| \;&\leq\; \frac{\pi_{A_1\cup A_3}(T)}{\pi(T)} + d(A)\left|\sum_{K_0<|n|\leq K_1 T} \frac{\pi_n(T)}{\pi(T)} - 1\right| + O(\varepsilon)\\
&=\; \frac{\pi_{A_1\cup A_3}(T)}{\pi(T)} + d(A)\left(\sum_{|n|\leq K_0} \frac{\pi_n(T)}{\pi(T)} + \sum_{|n|>K_1 T} \frac{\pi_n(T)}{\pi(T)}\right) + O(\varepsilon)\\
&\leq\; (1+d(A))\left(\sum_{|n|\leq K_0} \frac{\pi_n(T)}{\pi(x)} + \sum_{|n|>K_1 T} \frac{\pi_n(T)}{\pi(T)}\right) + O(\varepsilon).
\end{aligned}$$

From (45) and the prime geodesic theorem, the first sum is of growth order $O(\frac{1}{T})$ as $T \to \infty$. Combining Theorem 3 and (2), we have that

$$\sum_{|n|>K_1 T} \frac{\pi_n(T)}{\pi(T)} = 2\int_{4\pi\frac{K_1}{k}}^{\infty} \frac{du}{\pi(1+u^2)} + O\left(\frac{1}{T}\right) < \varepsilon + O\left(\frac{1}{T}\right).$$

We conclude by letting $T \to \infty$ and choosing ε to be arbitrarily small. □

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