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On the rate of Poisson convergence

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1. Introduction

Let X_1, \dots, X_n be independent Bernoulli random variables, and let $p_i = P[X_i = 1]$, $\lambda = \sum_{i=1}^n p_i$ and $W = \sum_{i=1}^n X_i$. Successively improved estimates of the total variation distance between the distribution $\mathcal{L}(W)$ of W and a Poisson distribution P_λ with mean λ have been obtained by Prohorov [5], Le Cam [4], Kerstan [3], Vervaat [8], Chen [2], Serfling [7] and Romanowska [6]. Prohorov, Vervaat and Romanowska discussed only the case of identically distributed X_i 's, whereas Chen and Serfling were primarily interested in more general, dependent sequences. Under the present hypotheses, the following inequalities, here expressed in terms of the total variation distance

$$d(\mu, \nu) \equiv \sup_{A \subset \mathbb{Z}} |\mu(A) - \nu(A)|,$$

were established respectively by Le Cam, Kerstan and Chen:

$$\left. \begin{aligned} d(\mathcal{L}(W), P_\lambda) &\leq \sum_{i=1}^n p_i^2; \\ d(\mathcal{L}(W), P_\lambda) &\leq 1.05\lambda^{-1} \sum_{i=1}^n p_i^2, \quad \text{if } \max_i p_i \leq \frac{1}{4}; \\ d(\mathcal{L}(W), P_\lambda) &\leq 5\lambda^{-1} \sum_{i=1}^n p_i^2. \end{aligned} \right\} \quad (1.1)$$

(Kerstan's published estimate of $2d \leq 1.2\lambda^{-1} \sum_{i=1}^n p_i^2$ ([3], p. 174, equation (1)) is a misprint for $2d \leq 2.1\lambda^{-1} \sum_{i=1}^n p_i^2$, the constant 2.1 appearing twice on p. 175 of his paper.) Here, we use Chen's [2] elegant adaptation of Stein's method to improve the estimates given in (1.1), and we complement these estimates with a reverse inequality expressed in similar terms. Second order estimates, and the case of more general non-negative integer valued X_i 's, are also discussed.

In the latter case, it is natural to expect the distribution of W to be almost Poisson only if the contribution to W from X_i 's taking values other than 1 is in some sense small. As observed by Serfling, the X_i 's can be reduced to 0-1 random variables by replacing all values greater than 1 by 0, at a cost in total variation distance of no more than $\sum_{j=1}^n P[X_j \geq 2]$. Thus close approximation to the Poisson is possible when the chance of $\max_i X_i$ exceeding 1 is small. It is shown here that good Poisson approxi-

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mation can also be achieved in another natural situation, in which the expected contribution to W from those X_i 's greater than 1 is negligible compared to

$$\left\{ \sum_{j=1}^n P[X_j = 1] \right\}^{\frac{1}{2}},$$

the standard deviation of the approximating Poisson distribution; upper and lower distance estimates are derived to quantify this approximation.

2. Upper and lower bounds

Let x be any real valued function on the non-negative integers. Then, as in Chen [2],

$$\begin{aligned} E\{\lambda x(W + 1) - Wx(W)\} &= \sum_{j=1}^n \{p_j E x(W + 1) - E(X_j x(W))\} \\ &= \sum_{j=1}^n p_j E\{x(W + 1) - x(W_j + 1)\} \\ &= \sum_{j=1}^n p_j^2 E\{x(W_j + 2) - x(W_j + 1)\}, \end{aligned} \tag{2.1}$$

where $W_j = W - X_j$. The following theorem is derived from (2.1) by choosing a suitable function x .

THEOREM 1. *Under the hypotheses set out in the Introduction,*

$$d(\mathcal{L}(W), P_\lambda) \leq \lambda^{-1}(1 - e^{-\lambda}) \sum_{j=1}^n p_j^2. \tag{2.2}$$

Proof. For any $A \subseteq \mathbb{Z}$, define $x = x_{\lambda, A}$ by

$$x(0) = 0; \quad x(m + 1) = \lambda^{-m-1} e^\lambda m! [P_\lambda(A \cap U_m) - P_\lambda(A) P_\lambda(U_m)], \quad m \geq 0, \tag{2.3}$$

where $U_m = \{0, 1, \dots, m\}$. For this x ,

$$\lambda x(m + 1) - mx(m) = I[m \in A] - P_\lambda(A),$$

and so, from (2.1),

$$|P[W \in A] - P_\lambda(A)| \leq \sum_{j=1}^n p_j^2 E|x(W_j + 2) - x(W_j + 1)|. \tag{2.4}$$

It is shown in the appendix to Barbour and Eagleson [1], that, uniformly in A ,

$$\|x\| \equiv \sup_{m \geq 0} |x(m)| \leq 1 \wedge (1.4\lambda^{-\frac{1}{2}}), \tag{2.5}$$

and

$$\Delta x \equiv \sup_{m \geq 0} |x(m + 1) - x(m)| \leq \lambda^{-1}(1 - e^{-\lambda}). \tag{2.6}$$

The theorem follows immediately from (2.4) and (2.6). \square

Remark. Theorem 1 improves upon each of the estimates given in (1.1). For $0 \leq \lambda \leq 1$, set $p_1 = \lambda; p_j = 0, 2 \leq j \leq n$. Then

$$d(W, P_\lambda) = \lambda(1 - e^{-\lambda}) = \lambda^{-1}(1 - e^{-\lambda}) \sum_{j=1}^n p_j^2,$$

so that the inequality (2.2) is sharp in this case. For integral $\lambda \geq 1$, set $p_j = 1, 1 \leq j \leq \lambda; p_j = 0, \lambda < j \leq n$. Then

$$d(\mathcal{L}(W), P_\lambda) = 1 - \lambda^\lambda e^{-\lambda} / \lambda! \approx 1 - 1/\sqrt{2\pi\lambda},$$

as compared to the right hand side of (2.2) which takes the value $1 - e^{-\lambda}$, both of which tend to one as $\lambda \rightarrow \infty$. Thus (2.2) also comes close to being sharp for large λ . There is, however, no reason to suppose that (2.2) could not be improved under added restrictions on the p_i 's; for instance, Romanowska's [6] inequality for all p_i 's equal to p is sharper than (2.2) when $\frac{1}{2}(1-p)^{-\frac{1}{2}} < 1 - e^{-\lambda}$.

As a complement to Theorem 1, we prove the following result: note that

$$(1 \wedge \lambda^{-1}) \geq \lambda^{-1}(1 - e^{-\lambda}).$$

THEOREM 2. *Under the hypotheses set out in the Introduction,*

$$d(\mathcal{L}(W), P_\lambda) \geq \frac{1}{32}(1 \wedge \lambda^{-1}) \sum_{j=1}^n p_j^2. \tag{2.7}$$

Proof. Take x defined by

$$x(m) = (m - \lambda) e^{-(m-\lambda)^2/\theta\lambda}, \quad m \geq 0, \tag{2.8}$$

in (2.1), where the constant θ will be chosen later. Since, for P a Poisson variate with mean λ , $E\{\lambda x(P+1) - Px(P)\} = 0$, Equation (2.1) yields the equation

$$E\{[\lambda x(W+1) - Wx(W)] - [\lambda x(P+1) - Px(P)]\} = \sum_j p_j^2 E\{x(W_j+2) - x(W_j+1)\},$$

from which it follows that

$$2d(\mathcal{L}(W), P_\lambda) \sup_j |\lambda x(j+1) - jx(j)| \geq \sum_j p_j^2 E\{x(W_j+2) - x(W_j+1)\}. \tag{2.9}$$

Our first task is to bound the supremum on the left in (2.9). Since

$$(d/dw)(we^{-w^2/\theta\lambda}) = (1 - 2w^2/\theta\lambda) e^{-w^2/\theta\lambda},$$

which takes only values in the interval $[-2e^{-\frac{1}{2}}, 1]$, then

$$-2e^{-\frac{1}{2}} \leq x(w+1) - x(w) \leq 1, \tag{2.10}$$

and so

$$\begin{aligned} |\lambda x(j+1) - jx(j)| &= |\lambda\{x(j+1) - x(j)\} - (j-\lambda)^2 \exp\{- (j-\lambda)^2/\theta\lambda\}| \\ &\leq \lambda \max(1, 2e^{-\frac{1}{2}} + \theta e^{-1}). \end{aligned} \tag{2.11}$$

Next we treat the series on the right hand side of (2.9). Now,

$$1 - e^{-w^2/\theta\lambda}(1 - 2w^2/\theta\lambda) \leq 3w^2/\theta\lambda,$$

whence, writing $U_j = W_j - \lambda$,

$$1 - \{x(W_j+2) - x(W_j+1)\} \leq \int_{U_j+1}^{U_j+2} (3w^2/\theta\lambda) dw = (\theta\lambda)^{-1} (3U_j^2 + 9U_j + 7). \tag{2.12}$$

Therefore

$$1 - E\{x(W_j+2) - x(W_j+1)\} \leq (\theta\lambda)^{-1} (3\lambda + 7),$$

since $E(U_j^2) = \sum_{j+i} p_j(1-p_j) + p_i^2$ and $E(U_i) = -p_i$. Consequently,

$$\sum_j p_j^2 E\{x(W_j+2) - x(W_j+1)\} \geq \sum_j p_j^2 \{1 - (\theta\lambda)^{-1} (3\lambda + 7)\}. \tag{2.13}$$

Combining (2.9), (2.11) and (2.13), we see that if $\theta \geq e$, $d(\mathcal{L}(W), P_\lambda) \geq k \sum_{j=1}^n p_j^2$, where

$$k = \{1 - (\theta\lambda)^{-1} (3\lambda + 7)\} / 2(2e^{-\frac{1}{2}} + \theta e^{-1}). \tag{2.14}$$

If $\lambda \geq 1$ we take $\theta = 21$, which gives

$$\lambda k \geq (1 - 10/\theta) / 2(2e^{-\frac{1}{2}} + \theta e^{-1}) \geq 1/32,$$

while if $\lambda < 1$ we take $\theta = 21/\lambda$, obtaining

$$k \geq (1 - 10/\theta\lambda)/2(2e^{-\frac{1}{2}} + \theta\lambda e^{-1}) \geq 1/32.$$

Theorem 2 follows. |

3. Second-order estimates

For many choices of $(p_i)_{i=1}^n$, the bounds given in (2.2) and (2.7) can usefully be replaced by an estimate of $d(\mathcal{L}(W), P_\lambda)$ together with a bound on the error of the estimate. This approach was considered by Prohorov [5], Kerstan [3] and Chen [2]: our argument is similar to Chen's, though the error estimate is improved. The method is to take, for any $A \subseteq \mathbb{Z}^+$, the function x defined in (2.3), and then to approximate the right hand side of (2.1) by $(\sum_{j=1}^n p_j^2) E\{x(P+2) - x(P+1)\}$, where P , here and subsequently, denotes a Poisson variate with mean λ . Let $\Delta(A)$ denote the error in this approximation:

$$\begin{aligned} \Delta(A) &\equiv P[W \in A] - P_\lambda(A) - \left(\sum_{j=1}^n p_j^2\right) E\{x(P+2) - x(P+1)\} \\ &= P[W \in A] - P_\lambda(A) + \frac{1}{2}\lambda^{-2} \left(\sum_{j=1}^n p_j^2\right) E\{I[P \in A](P^2 - (2\lambda + 1)P + \lambda^2)\}. \end{aligned} \tag{3.1}$$

THEOREM 3. For any $A \subseteq \mathbb{Z}^+$, under the hypotheses set out in the Introduction,

$$|\Delta(A)| \leq 2\lambda^{-1}(1 - e^{-\lambda})(1 \wedge 1.4\lambda^{-\frac{1}{2}}) \sum_{j=1}^n p_j^3 + 2\{\lambda^{-1}(1 - e^{-\lambda}) \sum_{j=1}^n p_j^2\}^2. \tag{3.2}$$

Proof. Taking in (2.1) the function x defined in (2.3), it follows immediately that

$$|\Delta(A)| \leq 2\lambda^{-1}(1 - e^{-\lambda}) \sum_{j=1}^n p_j^2 d(\mathcal{L}(W_j), P_\lambda).$$

However, analogously to (2.1), for any $B \subseteq \mathbb{Z}^+$,

$$\begin{aligned} P[W_j \in B] - P_\lambda(B) &= E\{\lambda x_{\lambda, B}(W_j + 1) - W_j x_{\lambda, B}(W_j)\} \\ &= p_j E x_{\lambda, B}(W_j + 1) + \sum_{k \neq j} p_k^2 E\{x_{\lambda, B}(W_{jk} + 2) - x_{\lambda, B}(W_{jk} + 1)\}, \end{aligned}$$

where $W_{jk} = W_j - X_k$. Thus it follows from (2.5) and (2.6) that

$$d(\mathcal{L}(W_j), P_\lambda) \leq p_j(1 \wedge 1.4\lambda^{-\frac{1}{2}}) + \lambda^{-1}(1 - e^{-\lambda}) \sum_{k \neq j} p_k^2, \tag{3.3}$$

establishing (3.2). |

Remark. Since, by Schwarz's inequality, $\lambda^{-1}(\sum_{j=1}^n p_j^2)^2 \leq \sum_{j=1}^n p_j^3$, equation (3.2) implies that $|\Delta(A)| \leq 4(1 - e^{-\lambda})(1 - \frac{1}{2}e^{-\lambda})\lambda^{-1}\sum_{j=1}^n p_j^3$, which improves on Chen's estimate of $(12 + 48\sqrt{2})\lambda^{-1}\sum_{j=1}^n p_j^3$.

COROLLARY. Let $\delta(\lambda) \equiv -\frac{1}{2}\lambda^{-1}E\{(P^2 - (2\lambda + 1)P + \lambda^2)\}$.

Then

$$\begin{aligned} |d(\mathcal{L}(W), P_\lambda) - \lambda^{-1} \left(\sum_{j=1}^n p_j^2\right) \delta(\lambda)| &\leq 2\lambda^{-1}(1 - e^{-\lambda})(1 \wedge 1.4\lambda^{-\frac{1}{2}}) \sum_{j=1}^n p_j^3 \\ &\quad + 2 \left\{ \lambda^{-1}(1 - e^{-\lambda}) \sum_{j=1}^n p_j^2 \right\}^2. \end{aligned}$$

Remark. Kerstan gives instead the upper bound

$$0.65\lambda^{-1} \sum_{j=1}^n p_j^3 + 1.95 \left(\lambda^{-1} \sum_{j=1}^n p_j^2 \right)^2, \quad \text{if } \max_{1 \leq j \leq n} p_j \leq \frac{1}{4},$$

which is sometimes better than that of the Corollary, and sometimes worse.

The quantity $\delta(\lambda)$ is not in general very neatly expressible, except for moderately small values of λ : for example,

$$\delta(\lambda) = \begin{cases} \lambda(1 - \frac{1}{2}\lambda)e^{-\lambda}, & 0 \leq \lambda \leq 2 - \sqrt{2} \\ \lambda\left(\frac{1}{2} + \frac{\lambda}{2} - \frac{\lambda^2}{4}\right)e^{-\lambda}, & 2 - \sqrt{2} \leq \lambda \leq 3 - \sqrt{3}. \end{cases}$$

However, $y^2 - (2\lambda + 1)y + \lambda^2 \geq -(\lambda + \frac{1}{4})$ for all y , and so

$$\delta(\lambda) \leq \frac{1}{2} + \frac{1}{8\lambda},$$

for all λ . Furthermore, as $\lambda \rightarrow \infty$, $\delta(\lambda) \sim 0.242$.

The Corollary enables some further evaluation of the relative precision of the bounds in Theorems 1 and 2 to be made, in the following sense. Suppose that $\{X_{in}, 1 \leq i \leq n < \infty\}$ is a double array of Bernoulli random variables, independent within rows, and set $p_{jn} = P[X_{jn} = 1]$, $\lambda_n = \sum_{j=1}^n p_{jn}$ and $W_n = \sum_{j=1}^n X_{jn}$. Suppose also that as $n \rightarrow \infty$, $d(\mathcal{L}(W_n), P_{\lambda_n}) \rightarrow 0$, or, equivalently, that $\lambda_n^{-1}(1 - e^{-\lambda_n}) \sum_{j=1}^n p_{jn}^2 \rightarrow 0$. Then the error estimate given in the Corollary is of asymptotically smaller order as $n \rightarrow \infty$ than $\lambda_n^{-1}\delta(\lambda_n) \sum_{j=1}^n p_{jn}^2$, so that

$$d(\mathcal{L}(W_n), P_{\lambda_n}) \sim \lambda_n^{-1}\delta(\lambda_n) \sum_{j=1}^n p_{jn}^2,$$

as $n \rightarrow \infty$. Thus, for example, if also $\lambda_n \rightarrow \infty$,

$$u_n \equiv \lambda_n d(\mathcal{L}(W_n), P_{\lambda_n}) / \{(1 - e^{-\lambda_n}) \sum_{j=1}^n p_{jn}^2\} \rightarrow 0.242,$$

whereas Theorems 1 and 2 guarantee that $\frac{1}{3^2} \leq u_n \leq 1$. Actually, the choice of $\theta = 6.55$ in (2.14) gives $u_n \geq \frac{1}{11}$, provided that λ is sufficiently large, which is not too great a deviation from 0.242. Similar comparisons when $\lambda_n \leq 2 - \sqrt{2}$ yield

$$u_n \sim \lambda_n(1 - e^{-\lambda_n})^{-1} (1 - \frac{1}{2}\lambda_n) e^{-\lambda_n},$$

and so $u_n \rightarrow 1$ if $\lambda_n \rightarrow 0$. Again, for small enough λ , the choice of $\theta (= 14.6)$ in (2.14) improves the bound given by Theorem 2, but this time to $u_n \geq \frac{1}{2^3}$.

4. Non-negative integer variates

Let $(Y_j)_{j=1}^n$ be independent non-negative integer valued random variables, and let $p_j = P[Y_j = 1]$, $q_j = P[Y_j \geq 2]$, $\lambda = \sum_{j=1}^n p_j$ and $V = \sum_{j=1}^n Y_j$. Define the zero-one random variables $(X_j)_{j=1}^n$ by

$$X_j = \begin{cases} Y_j & \text{if } Y_j = 0 \text{ or } 1; \\ 0 & \text{otherwise,} \end{cases}$$

and set $W = \sum_{j=1}^n X_j$. Note that the X_j 's satisfy the conditions outlined in the Introduction. Then, as observed by Serfling [7],

$$d(\mathcal{L}(V), \mathcal{L}(W)) \leq \sum_{j=1}^n q_j,$$

and so (2.2) and (2.7) yield the inequalities

$$d(\mathcal{L}(V), P_\lambda) \leq \lambda^{-1}(1 - e^{-\lambda}) \sum_{j=1}^n p_j^2 + \sum_{j=1}^n q_j, \tag{4.1}$$

and

$$d(\mathcal{L}(V), P_\lambda) + \sum_{j=1}^n q_j \geq \frac{1}{3^2}(1 \wedge \lambda^{-1}) \sum_{j=1}^n p_j^2. \tag{4.2}$$

If the Y_j 's have finite second moments, the Stein-Chen method can be applied to get alternatives to (4.1) and (4.2). Let $\nu_j = EY_j$, $\kappa_j = E(Y_j(Y_j - 1))$, $\mu_j = E\{Y_j I[Y_j \geq 2]\}$ and $\nu = \sum_{j=1}^n \nu_j$.

THEOREM 4. *If $(Y_j)_{j=1}^n$ satisfy the above hypotheses, and if $\nu < \infty$,*

$$\begin{aligned}
 d(\mathcal{L}(V), P_\nu) &\leq \nu^{-1}(1 - e^{-\nu}) \sum_{i=1}^n \nu_i p_i \\
 &\quad + \sum_{i=1}^n \nu_i [\{2(1 \wedge 1.4\nu^{-\frac{1}{2}}) q_i\} \wedge \{\nu^{-1}(1 - e^{-\nu}) \mu_i\}] \\
 &\quad + [\{2(1 \wedge 1.4\nu^{-\frac{1}{2}}) \sum_{i=1}^n \mu_i\} \wedge \{\nu^{-1}(1 - e^{-\nu}) \sum_{i=1}^n \kappa_i\}] \tag{4.3}
 \end{aligned}$$

$$\leq \nu^{-1}(1 - e^{-\nu}) \left\{ \sum_{i=1}^n \nu_i^2 + \sum_{i=1}^n \kappa_i \right\}, \tag{4.4}$$

and

$$\begin{aligned}
 d(\mathcal{L}(V), P_\lambda) &\leq \lambda^{-1}(1 - e^{-\lambda}) \sum_{i=1}^n p_i^2 \\
 &\quad + \sum_{i=1}^n p_i [\{2(1 \wedge 1.4\lambda^{-\frac{1}{2}}) q_i\} \wedge \{\lambda^{-1}(1 - e^{-\lambda}) \mu_i\}] + (1 \wedge 1.4\lambda^{-\frac{1}{2}}) \sum_{i=1}^n \mu_i \\
 &\quad + [\{2(1 \wedge 1.4\lambda^{-\frac{1}{2}}) \sum_{i=1}^n \mu_i\} \wedge \{\lambda^{-1}(1 - e^{-\lambda}) \sum_{i=1}^n \kappa_i\}] \tag{4.5}
 \end{aligned}$$

$$\leq \lambda^{-1}(1 - e^{-\lambda}) \left(\sum_{i=1}^n \nu_i p_i + \sum_{i=1}^n \kappa_i \right) + (1 \wedge 1.4\lambda^{-\frac{1}{2}}) \sum_{i=1}^n \mu_i. \tag{4.6}$$

Remark. Estimate (4.1) is clearly better than (4.5) if $\nu = \infty$. On the other hand, for large λ , (4.5) can improve upon (4.1), in circumstances where $\sum_{j=1}^n q_j$ is not small but $\lambda^{-\frac{1}{2}} \sum_{j=1}^n \mu_j$ is: the latter condition is natural, in that it is simply requiring that the expected contribution to V from Y 's not taking the values 0 or 1 should be small when compared to the spread $\lambda^{\frac{1}{2}}$ of P_λ . If the Y_j 's take only the values 0, 1 and 2, estimate (4.4) reduces to $\nu^{-1}(1 - e^{-\nu}) \{\sum_{i=1}^n \nu_i^2 + 2\sum_{i=1}^n q_i\}$, and estimate (4.6) to

$$\lambda^{-1}(1 - e^{-\lambda}) \left\{ \sum_{i=1}^n p_i^2 + 2 \sum_{i=1}^n q_i(1 + p_i) \right\} + 2(1 \wedge 1.4\lambda^{-\frac{1}{2}}) \sum_{i=1}^n q_i,$$

enabling comparison with (4.1) to be easily made.

Estimate (4.4) is typically smaller than (4.6), because $\nu^{-1}(1 - e^{-\nu}) \leq \lambda^{-1}(1 - e^{-\lambda})$ and, usually, $\nu^{-1} \sum_{i=1}^n \mu_i \nu_i \leq (1 \wedge 1.4\lambda^{-\frac{1}{2}}) \sum_{i=1}^n \mu_i$. However, in view of (4.1), this does not necessarily imply that it is better to use P_ν than P_λ to approximate the distribution of V .

Proof. Pick any $A \subseteq \mathbb{Z}^+$, set $V_j = V - Y_j$, define x as in (2.3) but with ν for λ , and observe that

$$\begin{aligned}
 P[V \in A] - P_\nu(A) &= \sum_{j=1}^n \nu_j E\{x(V + 1) - x(V_j + 1)\} \\
 &\quad + \sum_{j=1}^n E\{Y_j(x(V_j + 1) - x(V))\} \\
 &= \sum_{j=1}^n \nu_j p_j E\{x(V_j + 2) - x(V_j + 1)\} \\
 &\quad + \sum_{j=1}^n \nu_j E\{(x(V + 1) - x(V_j + 1)) I[Y_j \geq 2]\} \\
 &\quad + \sum_{j=1}^n E\{Y_j(x(V_j + 1) - x(V))\}.
 \end{aligned}$$

The three terms are now estimated using (2.5) and (2.6), again with ν for λ , giving (4.3). The proof of (4.5) is similar, starting from the equation

$$P[V \in A] - P_\lambda(A) = \sum_{j=1}^n p_j E\{x(V+1) - x(V_j+1)\} + \sum_{j=1}^n E\{Y_j(x(V_j+1) - x(V))\} - \sum_{j=1}^n E\{(Y_j - p_j)x(V_j+1)\}. \quad |$$

It is also possible to adapt the proof of Theorem 2 so as to get lower bounds for $d(\mathcal{L}(V), P_\nu)$ and $d(\mathcal{L}(V), P_\lambda)$, to contrast with (4.2). The following Theorem establishes such a result, without, however, retaining the elegance of (2.7) or (4.2).

THEOREM 5. *If the hypotheses of Theorem 4 are satisfied, and if, in addition, $\kappa_j < \infty$, $1 \leq j \leq n$, then*

$$d(\mathcal{L}(V), P_\nu) + \frac{1}{2}\nu^{-1}\{2e^{-\frac{1}{2}} + 21e^{-1}(1 \wedge \nu)^{-1}\}^{-1} \times \left\{ 2e^{-\frac{1}{2}} \sum_{j=1}^n \mu_j \nu_j + \sum_{j=1}^n \kappa_j + \frac{1}{7}(1 \wedge \nu^{-1}) \right. \\ \left. \times \left[\left(\sum_{j=1}^n \nu_j p_j \right) \sum_{j=1}^n (\kappa_j - \nu_j^2) + 2 \sum_{j=1}^n p_j \nu_j^2 (\nu_j - 2) \right] \right\} \geq \frac{1}{32}(1 \wedge \nu^{-1}) \sum_{j=1}^n \nu_j p_j, \quad (4.7)$$

and

$$d(\mathcal{L}(V), P_\lambda) + \frac{1}{2}\lambda^{-1}\{2e^{-\frac{1}{2}} + 21e^{-1}(1 \wedge \lambda)^{-1}\}^{-1} \times \left\{ 2e^{-\frac{1}{2}} \sum_{j=1}^n \mu_j p_j + \sum_{j=1}^n \kappa_j \right. \\ \left. + (21/2e)^{\frac{1}{2}}(\lambda \vee 1)^{\frac{1}{2}} \sum_{j=1}^n \mu_j + \frac{1}{7}(1 \wedge \lambda^{-1}) \left[\left(\sum_{j=1}^n p_j^2 \right) \sum_{j=1}^n (\kappa_j - \nu_j^2) \right. \right. \\ \left. \left. + 2 \sum_{j=1}^n p_j^2 \nu_j (\nu_j - 2) \right] \right\} \geq \frac{1}{32}(1 \wedge \lambda^{-1}) \sum_{j=1}^n p_j^2. \quad (4.8)$$

Remark. If the Y_j 's are in fact 0-1 random variables, the complicated additional term on the left hand side of (4.7) is negative: it only becomes important when the Y_j 's are too far from being 0-1 variates. Some such term has to be present, since, if each Y_j is a Poisson variate, $d(\mathcal{L}(V), P_\nu) = 0$, whereas the right hand side of (4.7) is positive.

Proof. Take x as defined in (2.8), but with ν for λ , and deduce, from the proofs of Theorems 2 and 4, that, for $\theta \geq e$,

$$2d(\mathcal{L}(V), P_\nu) \nu(2e^{-\frac{1}{2}} + \theta e^{-1}) \geq \sum_{j=1}^n \nu_j p_j E\{x(V_j+2) - x(V_j+1)\} \\ + \sum_{j=1}^n \nu_j E\{(x(V+1) - x(V_j+1)) I[Y_j \geq 2]\} + \sum_{j=1}^n E\{Y_j(x(V_j+1) - x(V))\}.$$

Thus, from the argument leading to (2.10), it follows that

$$2d(\mathcal{L}(V), P_\nu) \nu(2e^{-\frac{1}{2}} + \theta e^{-1}) + 2e^{-\frac{1}{2}} \sum_{j=1}^n \nu_j \mu_j + \sum_{j=1}^n \kappa_j \\ \geq \sum_{j=1}^n \nu_j p_j E\{x(V_j+2) - x(V_j+1)\} \\ \geq \sum_{j=1}^n \nu_j p_j \{1 - (\theta\nu)^{-1} (3E(V_j - \nu)^2 + 9E(V_j - \nu) + 7)\}, \quad (4.9)$$

where the last line is a consequence of (2.12). Evaluating the moments of $V_j - \nu$ enables the right hand side of (4.9) to be estimated as no smaller than

$$\sum_{j=1}^n \nu_j p_j \{1 - (\theta\nu)^{-1} (3\nu + 7)\} - 3(\theta\nu)^{-1} \sum_{j=1}^n \nu_j p_j \left\{ \sum_{j=1}^n (\kappa_j - \nu_j^2) + 2\nu_j(\nu_j - 2) \right\},$$

and the proof of (4.7) is concluded in the same way as the proof of Theorem 2, taking $\theta = 21(1 \wedge \nu)^{-1}$. The upper bound (4.8) is proved in a similar way.]

Some simplification of (4.7) and (4.8) can often be achieved. The next Corollary, which follows directly from (4.7), illustrates the possibilities. The inequality obtained reduces to (2.7) when the Y_j 's are 0-1 random variables.

COROLLARY. *If, in addition to the conditions of Theorem 5, $\nu_j \leq 2$ for all j ,*

$$d(\mathcal{L}(V), P_\nu) + \frac{3}{100} \left\{ \sum_{j=1}^n \mu_j \nu_j + \frac{9}{4} \left(1 + \frac{1}{7} \sum_{j=1}^n p_j \nu_j \right) \sum_{j=1}^n \kappa_j \right\} \geq \frac{1}{32} \sum_{j=1}^n \nu_j p_j \quad \text{if } \nu < 1 \quad (4.10)$$

and

$$d(\mathcal{L}(V), P_\nu) + \frac{11\nu^{-1}}{400} \left\{ \sum_{j=1}^n \mu_j \nu_j + \frac{9}{4} \left(1 + \frac{\nu^{-1}}{7} \sum_{j=1}^n p_j \nu_j \right) \sum_{j=1}^n \kappa_j \right\} \geq \frac{\nu^{-1}}{32} \sum_{j=1}^n \nu_j p_j \quad \text{if } \nu \geq 1. \quad (4.11)$$

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REFERENCES

- [1] A. D. BARBOUR and G. K. EAGLESON. Poisson approximation for some statistics based on exchangeable trials. *Adv. in Appl. Probab.* **15** (1983), 585-600.
- [2] L. H. Y. CHEN. Poisson approximation for dependent trials. *Ann. Probab.* **3** (1975), 534-545.
- [3] J. KERSTAN. Verallgemeinerung eines Satzes von Procharov und Le Cam. *Z. Wahrsch. Verw. Gebiete* **2** (1964), 173-179.
- [4] L. LE CAM. An approximation theorem for the Poisson binomial distribution. *Pacific J. Math.* **10** (1960), 1181-1197.
- [5] J. V. PROHOROV. Asymptotic behaviour of the binomial distribution. *Uspekhi Mat. Nauk* **8** (1953), 135-142.
- [6] M. ROMANOWSKA. A note on the upper bound for the distance in total variation between the binomial and the Poisson distribution. *Statist. Neerlandica* **31** (1977), 127-130.
- [7] R. J. SERFLING. A general Poisson approximation theorem. *Ann. Probab.* **3** (1975), 726-731.
- [8] W. VERVAAT. Upper bounds for the distance in total variation between the binomial or negative binomial and the Poisson distribution. *Statist. Neerlandica* **23** (1969), 79-86.