Reductions of non-separable approximate linear programs for network revenue management

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Decision Support

Reductions of non-separable approximate linear programs for network revenue management

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We suggest to group resources into non-separable subnetworks. For each chosen subnetwork, basis functions either span the whole function space or consist of linear functions. Given this more general choice of basis functions, we extend existing reductions of approximate linear programs. If there is only one subnetwork, for which the basis functions span the whole function space, we prove the equivalence to a compact linear program of polynomial size. For the general case, we suggest an approximate reduction. Numerical examples illustrate our novel upper bounds for the maximum expected revenue and the corresponding competitive policies. In particular, we find that the added benefit of non-separability heavily depends on the network structure and the capacity.

Our work helps to better understand the impact of assuming separability in network revenue management. The polynomial sized reductions make it possible to estimate the added average revenue resulting from incorporating interactions between resources. The theory we develop demonstrates how the interpretation of dual variables as state-action probabilities can be applied to reduce exponentially large approximate linear programs via variable aggregation.

1. Introduction and literature review

Network revenue management (NRM) problems are prevalent in various sectors of the transportation industry, the hospitality industry and even in medical services, see, e.g., McGill & van Ryzin (1999) or Diamant, Milner, & Quereshy (2018). To overcome the infamous curse of dimensionality faced in problems of realistic size, approximate dynamic programming approaches have been suggested. One particular stream of this research focuses on reducing the linear programming formulation of the approximated problem, which has few variables and many constraints, to an equivalent optimization problem, which can be solved without column or row generation. Such reductions are usually based on separable approximations, i.e., interactions between different resources are ignored. In some realistic networks, however, there are considerable interactions between some of the resources. We propose a very general non-separable approximation, where the number and size of non-separable subnetworks can be chosen arbitrarily. We show that our reduction can lead to a significant improvement both in the upper bound and in policy performance compared to previously known reductions.

In particular, we make the following contributions:

1. We introduce a novel general type of value function approximation, which is partially non-separable and can thus capture dependencies between several resources. The whole network is partitioned into several subnetworks. For each subnetwork, the basis functions either span the whole function space or consist of linear functions.
2. If there is only one subnetwork for which the basis functions span the whole function space, we can reduce the corresponding approximate linear program (ALP) to a linear program (LP) which can be solved directly using a commercial solver instead of applying row generation or similar tech-

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niques. Viewing decision variables as state-action probabilities, the corresponding proof provides insights into the nature of reductions.
3. A heuristic approximative reduction for the general case extends the compact ALP for the separable piecewise linear approximation by Vossen & Zhang (2015) (Proposition 3) to a partially non-separable form leading to novel upper bounds, which are tighter than any reduction we are aware of. Our reductions enable us to efficiently analyze the benefit of non-separability in various network types.
4. We demonstrate that given the same approximation, two equivalent formulations of a problem (the pre- and post-arrival formulation) can lead to different bounds of different qualities.
5. We provide structural results of optimal solutions to our reductions, which highlight the intuitive interpretation.
6. In order to efficiently partition the network into non-separable subnetworks of a given size such that the upper bound is as tight as possible, we suggest a mapping which assigns to each partition a value which can be computed efficiently, and which strongly correlates with the tightness of the upper bound.
7. In numerical experiments, we demonstrate that non-separability can significantly improve both the upper bound and the corresponding policy. Our examples include a realistic dynamic example using data from a large European bus company.

1.1. Literature review

A systematic introduction into NRM problems can be found in Chapter 3 of Talluri & van Ryzin (2004b). Historically, NRM models assume that customers request a particular origin-destination pair at a particular fare and will not buy at all if that fare is not offered. This assumption is often referred to as the independent demand assumption. Since the publication of the seminal paper by Talluri & van Ryzin (2004a), including customer choice has become increasingly popular. This stream of literature allows customer purchase decisions to depend on the set of available products. A special case of these customer choice models only considers the set of available products for one particular origin-destination pair the customer is interested in buying. Walczak, Mardan, & Kallesen (2010) show that as long as the same resources are used, this case can be transformed into an equivalent problem with independent demand by introducing artificial demands and prices. If a company decides which price to offer within a discrete set of prices, pricing problems fall into this category of customer choice models as well. Models assuming independent demand are hence much more applicable than previously believed.

If NRM problems are formulated as Markov decision processes, the value function $v_t(\mathbf{r})$ typically denotes the optimal expected revenue starting with remaining capacity $\mathbf{r}$ at time t. The resulting curse of dimensionality is often addressed by approximate dynamic programming (ADP). In order to approximately solve the resulting dynamic program, different decomposition methods have been suggested, see Talluri & van Ryzin (2004b) for an introduction. Powell (2011) outlines simulation-based solution methods.

Approximate linear programming (ALP) uses the fact that the value function $v_t(\mathbf{r})$ is the optimal solution to a linear program. Solving this linear program exactly is infeasible due to the large number of variables and constraints. In order to reduce the number of variables, $v_t(\mathbf{r})$ is approximated by a linear combination of a small number of basis functions, see, e.g., Schweitzer & Seidmann (1985), de Farias & Roy (2003), or Adelman (2007). This approximation, however, does not usually decrease the number of constraints. The resulting problem can be solved by constraint or column generation, see, e.g., Trick & Zin (1997), Kunnumkal & Talluri (2016), Adelman (2004), Adelman (2007) or Adelman & Klabjan (2012). Other approaches discussed in the literature include constraint sampling (de Farias & Roy, 2004) or constraint-violation learning (Lin, Nadarajah, & Soheili, 2019). Imposing a certain structure on the value function is equivalent to reducing the function space out of which the value function is chosen. As a result, the ALP yields an upper bound on the optimal expected revenue. Conversely, enlarging the function space tightens this upper bound. The following paragraphs describe such a gradual loosening of the structure imposed on the value function.

Commonly, the value function is approximated as a separable function

$$v_t(\mathbf{r}) \approx \sum_i v_{t,i}(r_i).$$

If $v_{t,i}(r_i)$ is approximated by an affine function with time-independent slope, Adelman (2007) shows that the well-known deterministic linear program (DLP), see, e.g., Talluri & van Ryzin (1998), can be obtained from the ALP. Adelman (2007) also considers more general basis functions and shows that for affine basis functions with time-dependent slopes, the column generation subproblem can be solved efficiently. In the following, we refer to this case as *AL*. The variables obtained by the *AL* approximation can actually be interpreted as bid prices of the remaining resources. Many authors suggest bid price methods, see, e.g., Hosseinalifam, Marcotte, & Savard (2016). Farias & Roy (2007) reduce *AL* to a linear program with fewer constraints. The number of constraints, however, is still intractable for problems of realistic size. Most relevant for our work is the reduction of Tong & Topaloglu (2014), who find an equivalent linear program with much fewer constraints. Vossen & Zhang (2015) and Kunnumkal & Talluri (2019a) suggest *AL* reductions for NRM with customer choice.

Instead of using an affine approximation in this separable ALP framework, implementing the approximation $\sum_i v_{t,i}(r_i)$ without any further simplification means that all values $v_{t,i}(r_i), t = 1,\dots,T+1, i = 1,\dots,I, r_i = 0,\dots,c_i,$ are viewed as variables of the ALP. This more general approximation is often referred to as separable piecewise linear approximation *SPL*. Since the key property of this approximation is separability, the term "piecewise linear" is slightly misleading. But we use this term to be consistent with this stream of literature. To tackle this case, Farias & Roy (2007) use constraint sampling and a relaxed reduced linear program. Kunnumkal & Talluri (2016) prove the equivalence of the *SPL* approximation and the Lagrangian relaxation introduced by Topaloglu (2009). They also show that the column generation subproblem is solvable in polynomial time. Meissner & Strauss (2012) implement an *SPL* approach for customer choice based on column generation.

In a seminal paper, Vossen & Zhang (2015) present a framework based on the Dantzig-Wolfe decomposition principle (Dantzig & Wolfe, 1960) which enables them to even prove a reduction for *SPL*. The work of Vossen & Zhang (2015) has been applied and refered to multiple times, see, e.g., Yuan, Nie, Wu, & Fu (2018), Ke, Zhang, & Zheng (2019), Sayah (2015) or Diamant, Milner, & Quereshy (2018). An even more elegant reduction framework with broader application scope is outlined in Ke, Zhang, & Zheng (2021). Both frameworks do not allow a direct application in our case. Rather, we base our reduction proof on the interpretation of dual variables as state-action probabilities and apply the technique of variable aggregation.

Vossen & Zhang (2015) also prove a reduction of *SPL* for the general customer choice model, in which the number of constraints increases exponentially in the product space. Another reduction of the general discrete choice network revenue management problem was suggested by Kunnumkal & Talluri (2019b) using Lagrangian multipliers. Feldman & Topaloglu (2017), Gallego, Ratliff, & Shebalov (2015), Strauss & Talluri (2017) as well as Cao, Rusmevichientong, & Topaloglu (2020) reduce deterministic linear programs associated with various customer choice models. Other techniques have been used to reduce ALPs in various applications, see, e.g., Topaloglu (2013), Adelman & Barz (2013), Morrison & Kumar (1999) or Guestrin, Koller, Parr, & Venkataraman (2003).

Concerning non-separable approximations in ADP, Cooper & Homem-de Mello (2007) base a dynamic programming decomposition approach on partitioning the network into two-leg subnetworks. Their work is relevant for us because they suggest a mapping of each partition to a value that should correlate with the quality of the resulting approximation. Since their suggested mapping is tailored to hub-and-spoke networks, it only considers two-leg subnetworks. In contrast, we introduce a novel mapping and show that it works well for various network types and general subnetwork sizes. Ma, Rusmevichientong, & Topaloglu (2020) suggest value function approximations based on various types of non-separable basis functions. Their approach assigns a basis function to each product, and the coefficients are computed by a backward recursion.

Very few authors have considered non-separable approximations in ALP: Lin, Nadarajah, & Soheili (2019), Pakiman, Nadarajah, Soheili, & Lin (2019), Sun, Wang, & Zipkin (2016), Adelman & Klabjan (2007) use various types of non-separable basis functions and apply constraint-violation learning, constraint sampling or constraint generation to address the problem of the infeasibly large number of constraints. To our knowledge, no literature has addressed non-separable approximations in ALP for NRM, which can be reduced to a linear program of manageable size. In this paper, we fill this gap and suggest a novel upper bound for NRM that is tighter than *SPL*. For large capacities, the computational complexity of our upper bound problem can be large. As time and capacity tend to infinity, however, Talluri & van Ryzin (1998) show that there always exists a bid price policy which is asymptotically optimal. Since bid prices are by definition separable, we also expect the advantage of modeling non-separability to fade as the capacity grows large.

1.2. Outline

The rest of the paper is organized as follows: In Section 2, we revisit the necessary preliminaries and state the two reductions for *AL* and *SPL* which were already proven by Tong & Topaloglu (2014) and Vossen & Zhang (2015). In Section 3, we propose a general non-separable approximation which we call *NSEP*. This approximation can be viewed as an extension of *AL* and *SPL* to a setting with non-separable subnetworks. In Section 4, we then provide a reduction of the resulting linear program for one special case of *NSEP*. In Section 5, we use the post-arrival formulation to propose non-equivalent problems which have fewer constraints than their pre-arrival counterparts. In Section 6, we develop a linear program that heuristically implements insights obtained from the previously discussed upper bound problems. Section 7 discusses our proposal for a mapping that associates a value to each partition of subnetworks capturing its quality with respect to non-separability. We conclude the paper with numerical experiments in Section 8. We demonstrate that the novel upper bounds are tighter than the ones discussed in the literature, and that there are settings in which assuming separability can have a significant impact on both the upper bound and the resulting policy.

2. Preliminaries

In this section, we formulate our model and summarize well-known approximations in the context of ALP in NRM. In this development, we always refer to the so-called pre-arrival formulation that determines decisions before customer arrivals are observed. We will discuss insights that can be gained from an alternative post-arrival formulation in Section 5.

2.1. Model

We consider a network revenue management problem with independent demand as outlined in Adelman (2007).

Products $j \in \{1, \ldots, J\} =: \mathcal{J}$ requiring resources $i \in \{1, \ldots, I\} =: \mathcal{I}$ are sold during a finite discrete-time selling horizon with time units $t \in \{1, \ldots, T\} =: \mathcal{T}$. Remaining units of resources are considered to be worthless at time $t = T + 1$. Whenever we refer to these sets, we briefly write for all i, j or t and use $\max_i, \prod_j, \sum_t$ etc. instead of $\max_{i\in\mathcal{I}}, \prod_{j\in\mathcal{J}}, \sum_{t\in\mathcal{T}}$.

At time $t = 1$, the seller has c_i units of resource i. We refer to c_i as the capacity of resource i. To simplify notation, we define the capacity vector $\mathbf{c} = (c_1, \ldots, c_I)^T$ and $C := \max_i c_i$. At each time t, at most one customer requesting a particular product j arrives with probability $p_{t,j}$. Selling product j at time t generates a revenue of $f_{t,j}$ but reduces the number of available units of resource i by $a_{ij} \in \{0, 1\}$. The values of a_{ij} are summarized in the consumption matrix $A \in \{0, 1\}^{I\times J}$. Column j of this matrix contains information on which resources are used by product j. We refer to this vector as $\mathbf{a}_j := (a_{1j}, \ldots a_{Ij})^T$. Throughout the paper, we assume that the event of no customer arrival is included in the enumeration of $j = 1, \ldots, J$ by a dummy product with $f_{t,j} = 0$ and $\mathbf{a}_j = (0, \ldots, 0)^T$. As a consequence, $\sum_j p_{t,j} = 1$ for all t. Let $d_j := \sum_t p_{t,j}$ denote the total demand for each product j. We say that demand is stationary if $p_{t,j} = p_j$ for all t, j and hence $d_j = T p_j$.

The remaining capacity of resource i at any given point in time is denoted by r_i. As before, we define $\mathbf{r} = (r_1, \ldots, r_I)^T$. Obviously, at time $t = 1$ the remaining capacity of every resource is equal to its capacity, hence $r_i = c_i$ for all i. In every period t, the seller must determine whether a customer requesting product j is served ($u_j = 1$) or not ($u_j = 0$). Given a customer request for product j the revenue obtained is $u_j f_{t,j}$ and the remaining capacity is reduced by $u_j \mathbf{a}_j$. So, for $t \geq 2$, the remaining capacity r_i of a resource can be lower than c_i for all i. Assuming that the seller cannot sell more resources than we have and summarizing the decisions as a vector $\mathbf{u} \in \{0, 1\}^J$, the set of feasible actions and the set of feasible remaining resources are then given by

$$\mathfrak{U}_{\mathbf{r}} := \left\{ \mathbf{u} \in \{0, 1\}^J \mid u_j \mathbf{a}_j \leq \mathbf{r} \quad \forall j \right\} \quad \text{and}$$

$$\mathfrak{R}_t := \begin{cases} \prod_i \{c_i\}, & t = 1 \\ \prod_i \{0, \ldots, c_i\}, & t \geq 2. \end{cases}$$

In line with the standard model, we do not consider cancellations. The seller's goal is to maximize the total expected revenue that can be gained given initial capacity $\mathbf{c}$ over the selling horizon $t = 1, \ldots, T$.

From our explanation above, we know that the maximum expected revenue that can be obtained over the selling horizon $t = 1, \ldots, T$ given remaining resources $\mathbf{r}$ can be computed using the Bellman equation

$$v_t(\mathbf{r}) = \max_{\mathbf{u}\in\mathfrak{U}_{\mathbf{r}}} \sum_j p_{t,j} \big(u_j f_{t,j} + v_{t+1}(\mathbf{r} - u_j \mathbf{a}_j) \big) \text{ for all } \mathbf{r} \in \mathfrak{R}_t,$$
$$t = 1, \ldots, T$$
$$v_{T+1}(\mathbf{r}) = 0 \text{ for all } \mathbf{r} \in \mathfrak{R}_{T+1}.$$

To illustrate the main ideas, we use the following running example throughout the first part of the paper.

Running Example. Consider a small bus line as depicted in Fig. 1: $I = 3$ legs (AB, BC, CD) connect four cities A, B, C and D. The bus line offers tickets for the origin-destination pairs AB, BC, CD, BD

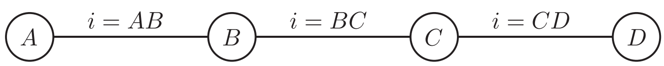

Fig. 1. Small bus line connecting four cities A, B, C and D.

and AD with two fare classes each. Including the dummy product, this translates into $J = 11$ products. To improve readability, we use descriptive instead of numerical indices. For example, $d_{\text{BD,high}}$ denotes the total demand for the high fare class of origin-destination pair BD. High and low fares are 5 and 10 for AB, BC and CD, 15 and 30 for BD and 25 and 50 for AD. (Our main findings do not depend on this specific choice of fares.) The capacity of the bus is $c = c_i = 4, i = \text{AB, BC, CD}$ and the selling horizon is $T = 20$. Demand is stationary with $d_{i,\text{high}} = d_{i,\text{low}}$ for all i. To achieve a load of 1.3 on all legs for given demand on AD and BD, we let

$$d_{\text{AB,high}} + d_{\text{AB,low}} = 1.3 \cdot c - \sum_{j \in \{high,low\}} d_{\text{AD},j}$$

and

$$d_{\text{BC,high}} + d_{\text{BC,low}} = d_{\text{CD,high}} + d_{\text{CD,low}} = 1.3 \cdot c - \sum_{j \in \{high,low\}} (d_{AD,j} + d_{BD,j}).$$

We refer to the setting with $d_{\text{AD,high}} = d_{\text{AD,low}} = 0.5$ and $d_{\text{BD,high}} = d_{\text{BD,low}} = 1$ as the base case. For this base case, the maximum expected revenue can be determined as $v_1(\mathbf{c}) = 105.84$ (rounded to two decimals) by recursively using the Bellman equation.

It is well-known that a solution of the Bellman equation can be found by the following linear program

$$\begin{aligned} D: \quad & \min_{v(\cdot)} v_1(\mathbf{c}) \\ \text{s.t.} \quad & v_t(\mathbf{r}) \geq \sum_j p_{t,j}\big(u_j f_{t,j} + v_{t+1}(\mathbf{r} - u_j \mathbf{a}_j)\big), \\ & \qquad \forall t = 1, \dots, T, \mathbf{r} \in \mathfrak{R}_t, \mathbf{u} \in \mathfrak{U}_\mathbf{r} \\ & v_{T+1}(\mathbf{r}) \geq 0, \qquad \forall \mathbf{r} \in \mathfrak{R}_{T+1}. \end{aligned}$$

Its dual can be written as

$$\begin{aligned} P: \quad & \max_{X \geq 0} \sum_{t,\mathbf{r} \in \mathfrak{R}_t, \mathbf{u} \in \mathfrak{U}_\mathbf{r}} p_{t,j} f_{t,j} u_j X_{t,\mathbf{r},\mathbf{u}} \\ \text{s.t.} \quad & \sum_{\mathbf{u} \in \mathfrak{U}_\mathbf{s}} X_{t,\mathbf{s},\mathbf{u}} = \begin{cases} 1, & t=1 \\ \sum_j p_{t-1,j} \sum_{\mathbf{r} \in \mathfrak{R}_{t-1}, \mathbf{u} \in \mathfrak{U}_\mathbf{r}} 1_{\{\mathbf{r} - u_j \mathbf{a}_j = \mathbf{s}\}} X_{t-1,\mathbf{r},\mathbf{u}}, & t \geq 2 \end{cases} \quad \forall t, \mathbf{s} \in \mathfrak{R}_t. \end{aligned}$$

Problem D has at most $T(C+1)^I$ variables and $T(C+1)^I 2^J$ constraints, problem P has at most $T(C+1)^I 2^J$ variables and $T(C+1)^I$ constraints. Denoting the maximum of the number of variables and the number of constraints as the size of the linear program, the size of these linear programs is $\mathcal{O}(T(C+1)^I 2^J)$, i.e., exponential in the number of products and resources. Solving the Bellman equation directly involves solving $T(C+1)^I$ equations iterating over all 2^J feasible actions for every equation. This phenomenon is known as the curse of dimensionality. It is the reason why for many problems of realistic size, a solution of the Bellman equation via backwards recursion or the solution of these linear programs is intractable. This is why we resort to approximate dynamic programming, in particular approximate linear programming (ALP).

2.2. A general ALP framework

Replacing the value function by a linear combination of $\ell = 1, \dots, L$ basis functions $\phi_\ell(\mathbf{r})$,

$$v_t(\mathbf{r}) \approx \sum_{\ell=1}^{L} V_{t,\ell} \phi_\ell(\mathbf{r}), \qquad \forall t, \mathbf{r} \in \mathfrak{R}_t \tag{1}$$

in D, we obtain an ALP which we denote by (D_ϕ). The number of variables in D is reduced to (at most) $T \cdot L$, see, e.g., de Farias & Roy (2003) or Adelman (2007). Adelman (2007) interprets the constraints of the dual problem

$$\begin{aligned} P_\phi: \quad & \max_{X \geq 0} \sum_{j,t,\mathbf{r} \in \mathfrak{R}_t, \mathbf{u} \in \mathfrak{U}_\mathbf{r}} p_{t,j} f_{t,j} u_j X_{t,\mathbf{r},\mathbf{u}} \\ \text{s.t.} \quad & \sum_{\mathbf{r} \in \mathfrak{R}_t, \mathbf{u} \in \mathfrak{U}_\mathbf{r}} \phi_\ell(\mathbf{r}) X_{t,\mathbf{r},\mathbf{u}} = \begin{cases} \phi_\ell(\mathbf{c}), & t = 1 \\ \sum_j \sum_{\mathbf{r} \in \mathfrak{R}_{t-1}, \mathbf{u} \in \mathfrak{U}_\mathbf{r}} p_{t-1,j} \phi_\ell(\mathbf{r} - u_j \mathbf{a}_j) X_{t-1,\mathbf{r},\mathbf{u}}, & t \geq 2, \end{cases} \\ & \forall t, \ell \end{aligned}$$

as basis-weighted flow-balance constraints. This is because if the basis function $\phi_\ell(\mathbf{r}) = 1$ is included, the nonnegative variables X must sum to 1 for all t and can hence be viewed as probabilities of being in state $\mathbf{r}$ and taking action $\mathbf{u}$ at that time t. The linear program P_ϕ has $\mathcal{O}(TL)$ constraints, but still $\mathcal{O}(T(C+1)^I 2^J)$ variables. Column generation can be used to solve P_ϕ if the column generation subproblem is easy to solve. A different and more efficient approach to solving P_ϕ is to find an equivalent optimization problem with fewer variables. We refer to such an equivalent problem as a reduction if neither the number of variables nor the number of constraints grows exponentially in T, I, J or C.

2.3. A few well-known reductions

For two important classes of basis functions, Tong & Topaloglu (2014) and Vossen & Zhang (2015) suggest reductions.

2.3.1. The AL reduction:

The first class approximates the value function by an affine function, i.e.,

$$AL: \qquad v_t(\mathbf{r}) \approx \theta_t + \sum_i V_{t,i} r_i, \qquad \forall t, \mathbf{r} \in \mathfrak{R}_t.$$

The corresponding linear program P_ϕ with $L = I + 1$ basis functions $\phi_i(\mathbf{r}) = r_i$ and $V_{t,I+1} = \theta_t$, $\phi_{I+1}(\mathbf{r}) = 1$, is equivalent to the reduction

$$\begin{aligned} \widehat{P}_{AL}: \quad & \max_{\rho, \mu \geq 0} \sum_{t,j} p_{t,j} f_{t,j} \mu_{t,j} \\ \text{s.t.} \quad & \rho_{t,i} = \begin{cases} c_i, & t = 1 \\ \rho_{t-1,i} - \sum_j p_{t-1,j} a_{ij} \mu_{t-1,j}, & t \geq 2 \end{cases} \qquad \forall t, i \\ & \mu_{t,j} \leq 1, \qquad \forall t, j \\ & a_{ij} \mu_{t,j} \leq \rho_{t,i}, \qquad \forall t, i, j, \end{aligned} \tag{2}$$

see Tong & Topaloglu (2014) Proposition 2 and Vossen & Zhang (2015) Proposition 1. The size of this linear program is $\mathcal{O}(TIJ)$, i.e., it is linear in the number of products and resources. Intuitively, this reduction captures the idea that for $\phi_i(\mathbf{r}) = r_i$, the left-hand side of the constraint in P_ϕ is a convex combination of the values $r_i = 0, \dots, c_i$ with weights $X_{t,\mathbf{r},\mathbf{u}}$, which we can interpret as an expected value of r_i at time t. In $\widehat{P}_{AL}$ the variable $\rho_{t,i}$ denotes this expected value of r_i at time t and $\mu_{t,j}$ represents the expected value of u_j at time t. Obviously, any probability distribution X in P_ϕ yields corresponding values of $\rho_{t,i}$ and $\mu_{t,j}$. To prove equivalence of the two problems, one can then show that for any feasible solution $\rho_{t,i}, \mu_{t,j}$ in $\widehat{P}_{AL}$ a corresponding probability distribution $X_{t,\mathbf{r},\mathbf{u}}$ can be constructed.

2.3.2. The SPL reduction

The second class approximates the value function by a separable (piecewise linear) function, i.e.,

$$SPL: \qquad v_t(\mathbf{r}) \approx \sum_i v_{t,i}(r_i) = V_{t,1} + \sum_i \sum_{r=1}^{C} V_{t,(C+1)(i-1)+r+1} 1_{\{r_i = r\}}.$$

Since there is one value of $V_{t,\ell}$ for every combination of i and r, this approximation only assumes separability across resources and does not restrict the shape of the value function for any particular resource. We define

$$\mathcal{R}_{t,i} := \begin{cases} \{c_i\}, & t=1 \\ \{0,\dots,c_i\}, & t>1 \end{cases}, \qquad \mathcal{U}_{i,j,r} := \{u \in \{0,1\} \mid a_{ij}u \le r\}$$

and consider the linear program

$$\widehat{P}_{SPL}: \quad \max_{\sigma,\zeta,\mu\ge 0} \sum_{t,j} \sum_{u\in\{0,1\}} p_{t,j} f_{t,j} u \mu_{t,j,u}$$

$$\text{s.t.} \sum_{r\in\mathcal{R}_{t,i}} 1_{\{r=r'\}}\sigma_{t,i,r} = \begin{cases} 1, & t=1 \\ \sum_j \sum_{r\in\mathcal{R}_{t-1,i}, u\in\mathcal{U}_{i,j,r}} p_{t-1,j} 1_{\{r-ua_{ij}=r'\}} \zeta_{t-1,i,j,r,u} & t\ge 2, \end{cases}$$
$$\forall t, i, r' \in \mathcal{R}_{t,i}$$

$$\sum_{r\in\mathcal{R}_{t,i}} \sigma_{t,i,r} = 1, \qquad \forall t, i$$

$$\sum_{u\in\mathcal{U}_{i,j,r}} \zeta_{t,i,j,r,u} = \sigma_{t,i,r}, \qquad \forall t, i, j, r$$

$$\sum_{\substack{r\in\mathcal{R}_{t,i} \\ \text{s.t. } u\in\mathcal{U}_{i,j,r}}} \zeta_{t,i,j,r,u} = \mu_{t,j,u}, \qquad \forall t, i, j, u.$$

The optimal value obtained by $\widehat{P}_{SPL}$ is equal to the optimal value of P_ϕ using the approximation *SPL*. A proof, which is based on the result by Vossen & Zhang (2015), can be found in Appendix A. The size of $\widehat{P}_{SPL}$ is $\mathcal{O}(TIJC)$. The values of $\sigma_{t,i,r}$ can be interpreted as the probability that at time t, resource i is in state r. The values of $\mu_{t,j,u}$ represent the probability of taking action u for product j in time t. The probability that resource i is in state r and action u is taken for product j in time t is $\zeta_{t,i,j,r,u}$. Taking up the above-mentioned idea of constructing a probability distribution X, every solution to P_ϕ leads to corresponding values of $\sigma_{t,i,r}, \mu_{t,j,u}$ and $\zeta_{t,i,j,r,u}$. But since there exist values of $\sigma_{t,i,r}, \mu_{t,j,u}$ and $\zeta_{t,i,j,r,u}$ without a corresponding probability distribution X, problem $\widehat{P}_{SPL}$ is only weakly equivalent to P_ϕ, i.e., although the optimal objective values are equal, not every feasible solution to $\widehat{P}_{SPL}$ has a corresponding feasible solution to P_ϕ.

2.4. Corresponding policies

For every value function approximation, we simulate its corresponding policy as follows:

In the original Bellman equation, we replace the value function by the approximation. We then use the optimality principle that in a given state the action is chosen that maximizes the right-hand side. That means that for any approximation $v_t(\mathbf{r}) \approx \sum_{\ell=1}^{L} V_{t,\ell}\phi_\ell(\mathbf{r})$ a customer requesting product j at time t is accepted if and only if

$$f_j \ge \sum_{\ell=1}^{L} V_{t+1,\ell}\Big(\phi_\ell(\mathbf{r}) - \phi_\ell(\mathbf{r}-\mathbf{a}_j)\Big).$$

Running Example. For the base case with $d_{\text{AD,high}} = d_{\text{AD,low}} = 0.5$ and $d_{\text{BD,high}} = d_{\text{BD,low}} = 1$, the optimal value of $\widehat{P}_{AL}$ is 118.74. Simulating 10'000 repetitions of the selling horizon, the average revenue gained by the corresponding policy is 99.66 (standard error 0.26). As expected, we have $99.66 \le v_1(\mathbf{c}) = 105.84 \le 118.74$.

The optimal value of $\widehat{P}_{SPL}$ is 110.25. The average revenue gained by the corresponding policy is 104.24 (standard error 0.25). Both values are closer to the optimal expected revenue $v_1(\mathbf{c}) = 105.84$ when compared to *AL*.

If there is only demand for single-leg products, i.e., $d_{\text{AD,high}} = d_{\text{AD,low}} = d_{\text{BD,high}} = d_{\text{BD,low}} = 0$, we expect the network to be separable. Hence, *SPL* should be optimal, *AL* still represents an approximation. Indeed, the optimal value of $\widehat{P}_{SPL}$ is equal to $86.73 = v_1(\mathbf{c})$, and the average revenue is 86.67 (standard error 0.12). On the other hand, the optimal value of $\widehat{P}_{AL}$ is equal to 91.95, and the average revenue is 83.36 (standard error 0.11).

3. Non-separable approximations

As mentioned in the previous section, the *SPL* approximation $v_t(\mathbf{r}) \approx \sum_i v_{t,i}(r_i)$ only assumes separability and does not make any further assumptions concerning the shape of the value function. The corresponding upper bound can hence only be tightened by including non-separable basis-functions.

We illustrate the potential of including non-separable basis functions in the context of our running example:

Running Example. Consider the base case and assume that $t = 10$. If the remaining capacities for legs AB, BC and CD are 1, 0 and 1, respectively, multi-leg products cannot be offered. Hence, an expected revenue maximizing policy clearly accepts a high fare request for origin-destination pair CD. If the remaining capacities for legs AB, BC and CD are all equal to 1, however, an optimal policy declines even a high fare request for origin-destination CD. Since $v_{t+1}(1,1,1) - v_{t+1}(1,1,0) = 14.88$, the optimal policy saves the seat on leg CD for a potential future sale of a higher-revenue multi-leg product. Using a separable approximation of the value function $v_t(\mathbf{r}) \approx \sum_{i\in\{\text{AB,BC,CD}\}} v_{t,i}(r_i)$ in this context, the decision to accept a customer requesting origin destination pair CD only depends on $v_{t+1,\text{CD}}(r_{\text{CD}}) - v_{t+1,\text{CD}}(r_{\text{CD}} - 1)$, i.e., it only considers the remaining capacity of this particular leg. The optimal policy can hence not be mirrored. A non-separable approximation such as $v_t(\mathbf{r}) \approx v_{t,\text{AB}}(r_{\text{AB}}) + v_{t,\text{BC,CD}}(r_{\text{BC}}, r_{\text{CD}})$ on the other hand, could include the dependency on the remaining capacity on leg BC.

In this section, we propose such a general class of non-separable approximations. The idea is to approximate the value function on subsets of resources by non-separable basis functions and on the remaining resources by *AL* or *SPL*. Therefore, our approximations can be viewed as an extension to the two approximations discussed in the previous section.

In order to lay out our proposed approximations more precisely, we introduce the following notation: Let $\mathcal{I}_n \subseteq \mathcal{I}, n = 1,\dots,N$ be mutually exclusive subsets of $\mathcal{I}$, i.e., $\mathcal{I}_n \cap \mathcal{I}_{n'} = \emptyset, \forall n \ne n'$. Define $\mathcal{I}_{AL} := \mathcal{I} \setminus \bigcup_n \mathcal{I}_n$, where we abbreviate $\bigcup_n$ for $\bigcup_{n=1}^N$ as usual. Then, denote

$$\mathbf{r}_n := (r_i)^T_{i\in\mathcal{I}_n} \qquad \mathbf{c}_n := (c_i)^T_{i\in\mathcal{I}_n}$$
$$\mathbf{a}_{n,j} := (a_{ij})^T_{i\in\mathcal{I}_n} \qquad \mathcal{R}_i := \{0,\dots,c_i\}$$
$$\mathfrak{R}_{t,n} := \prod_{i\in\mathcal{I}_n} \mathcal{R}_{t,i} \qquad \mathfrak{R}_n := \prod_{i\in\mathcal{I}_n} \mathcal{R}_i$$
$$\mathfrak{U}_{\mathbf{r}_n} := \Big\{\mathbf{u} \in \{0,1\}^J \mid u_j\mathbf{a}_{n,j} \le \mathbf{r}_n \quad \forall j\Big\} \qquad \mathfrak{U}_{i,r} := \Big\{\mathbf{u} \in \{0,1\}^J \mid u_j a_{ij} \le r \quad \forall j\Big\}$$
$$\mathcal{U}_{j,\mathbf{r}_n} := \{u \in \{0,1\} \mid u\mathbf{a}_{n,j} \le \mathbf{r}_n\}.$$

The index n of a vector or set hence indicates that the vector or set only considers the resources $i \in \mathcal{I}_n$. The typeface indicates if the elements of a set are vectors or scalars.

For each $\mathcal{I}_n$, we use basis functions spanning the whole function space on $\mathfrak{R}_{t,n}$. On $\mathcal{I}_{AL}$ however, the basis functions are restricted to linear functions. This means, we consider

$$NSEP: \qquad v_t(\mathbf{r}) \approx \sum_{n=1}^{N} W_{t,n,\mathbf{r}_n} + \sum_{i\in\mathcal{I}_{AL}} V_{t,i} r_i$$

with values $W_{t,n,\mathbf{r}_n}$ defining an $|\mathcal{I}_n|$-dimensional non-separable value function for resources $i \in \mathcal{I}_n$. The approximation provided by *NSEP* is a generalization of both *AL* and *SPL* since chosing $N = 0$

leads to *AL* and chosing $N = I$ with $\mathcal{I}_i = \{i\}$ for all $i = 1, \dots, N$ yields *SPL*.

Using *NSEP* in P_ϕ, we obtain the following linear program:

$$P_{NSEP}: \quad \max_{X\geq 0} \sum_{t,j} \sum_{\mathbf{r}\in\mathfrak{R}_t, \mathbf{u}\in\mathfrak{U}_\mathbf{r}} p_{t,j} f_{t,j} u_j X_{t,\mathbf{r},\mathbf{u}}$$

$$\text{s.t.} \sum_{\substack{\mathbf{r}\in\mathfrak{R}_t, \mathbf{u}\in\mathfrak{U}_\mathbf{r} \\ \text{s.t. } \mathbf{r}_n=\mathbf{s}_n}} X_{t,\mathbf{r},\mathbf{u}} = \begin{cases} 1, & t=1 \\ \sum_j \sum_{\substack{\mathbf{r}\in\mathfrak{R}_{t-1}, \mathbf{u}\in\mathfrak{U}_\mathbf{r} \\ \text{s.t. } \mathbf{r}_n - u_j \mathbf{a}_{n,j} = \mathbf{s}_n}} p_{t-1,j} X_{t-1,\mathbf{r},\mathbf{u}}, & t\geq 2, \end{cases} \quad \forall t, n, \mathbf{s}_n \in \mathfrak{R}_{t,n}$$

$$\sum_{\mathbf{r}\in\mathfrak{R}_t, \mathbf{u}\in\mathfrak{U}_\mathbf{r}} r_i X_{t,\mathbf{r},\mathbf{u}} = \begin{cases} c_i, & t=1 \\ \sum_j \sum_{\mathbf{r}\in\mathfrak{R}_{t-1}, \mathbf{u}\in\mathfrak{U}_\mathbf{r}} p_{t-1,j}(r_i - u_j a_{ij}) X_{t-1,\mathbf{r},\mathbf{u}}, & t\geq 2, \end{cases} \quad \forall t, i\in\mathcal{I}_{AL}$$

$$\sum_{\mathbf{r}\in\mathfrak{R}_t, \mathbf{u}\in\mathfrak{U}_\mathbf{r}} X_{t,\mathbf{r},\mathbf{u}} = 1, \qquad \forall t. \tag{3}$$

The number of variables of P_{NSEP} is $\mathcal{O}(TI2^J(C+1)^I)$. The number of constraints is $\mathcal{O}(T(I+\sum_n (C+1)^{|\mathcal{I}_n|})$. For $N = 1$, we prove a reduction in Section 4. In Section 6, we then propose a heuristic reduction for P_{NSEP} that works for any choice of subsets and hence any value of N.

Running Example. Consider $N = 1$ with $\mathcal{I}_1 = \{\text{BC}, \text{CD}\}$ in the base case. The corresponding approximation $v_t(\mathbf{r}) \approx W_{t,1,(r_{\text{BC}}, r_{\text{CD}})} + V_{t,\text{AB}} r_{\text{AB}}$ has one parameter for every pair of remaining resources on legs BC and CD and one for the marginal seat value for leg AB for every time index. The optimal value of P_{NSEP} is 109.54 in this setting. The average revenue from simulating the corresponding policy (10'000 repetitions) equals 102.22 (standard error 0.26). If we choose $N = 2$ and $\mathcal{I}_1 = \{\text{AB}\}, \mathcal{I}_2 = \{\text{BC}, \text{CD}\}$, we have $v_t(\mathbf{r}) \approx W_{t,1,(r_{\text{AB}})} + W_{t,2,(r_{\text{BC}}, r_{\text{CD}})}$. Using this approximation in the base case, the value of P_{NSEP} is 107.75 resulting in an average revenue of 105.19 (standard error 0.26). Comparing this with the most general separable approximation *SPL*, this is an improvement in revenue of 0.91%.

If we consider the case $d_{\text{AD,high}} + d_{\text{AD,low}} = 0$, the bus line can be decomposed into two separate bus lines, one from A to B and the other from B to D. In this case, we expect the exact value function $v_t(\mathbf{r})$ to have the form $W_{t,1,(r_{\text{AB}})} + W_{t,2,(r_{\text{BC}}, r_{\text{CD}})}$. As a consequence, our approximation should yield the optimal solution. Indeed, setting for example $d_{\text{BD,high}} = d_{\text{BD,low}} = 1.5$, the optimal value of P_{NSEP} is $101.76 = v_1(\mathbf{c})$ and the corresponding policy is the optimal policy.

4. A reduction for $N = 1$: a single non-affine subnetwork

In this section, we introduce a reduction of P_{NSEP} given $N = 1$. Even though the index $n = 1$ is superflous in this setting, we keep the index for the sake of consistency and later references. The intuition of the following reduction is to introduce variables with similar interpretations as in $\widehat{P}_{AL}$ and $\widehat{P}_{SPL}$. Applying variable aggregation, we introduce

$$\varrho_{t,n,\mathbf{r}_n} = \sum_{\substack{\mathbf{r}'\in\mathfrak{R}_t, \mathbf{u}\in\mathfrak{U}_{\mathbf{r}'} \\ \text{s.t. } \mathbf{r}'_n=\mathbf{r}_n}} X_{t,\mathbf{r}',\mathbf{u}}, \tag{4}$$

the probability that the resources $i \in \mathcal{I}_n$ are in state $\mathbf{r}_n$ at time t,

$$\mu_{t,j,u} = \sum_{\substack{\mathbf{r}\in\mathfrak{R}_t, \mathbf{u}\in\mathfrak{U}_\mathbf{r} \\ \text{s.t. } u_j=u}} X_{t,\mathbf{r},\mathbf{u}}, \tag{5}$$

the probability that action u for product j is taken at time t,

$$\xi_{t,n,j,\mathbf{r}_n,u} = \sum_{\substack{\mathbf{r}'\in\mathfrak{R}_t, \mathbf{u}\in\mathfrak{U}_{\mathbf{r}'} \\ \text{s.t. } \mathbf{r}'_n=\mathbf{r}_n \\ \text{s.t. } u_j=u}} X_{t,\mathbf{r}',\mathbf{u}}, \tag{6}$$

the probability that the resources $i \in \mathcal{I}_n$ are in state $\mathbf{r}_n$ and action u for product j is taken at time t, and

$$\rho_{t,i} = \sum_{\mathbf{r}\in\mathfrak{R}_t, \mathbf{u}\in\mathfrak{U}_\mathbf{r}} r_i X_{t,\mathbf{r},\mathbf{u}} \qquad (\text{for } i \in \mathcal{I}_{AL}), \tag{7}$$

which represents the expected number of remaining units of resource $i \in \mathcal{I}_{AL}$ at time t. In addition, we introduce $\Upsilon_{t,i,n,\mathbf{r}_n} \geq \max_j a_{ij} \xi_{t,n,j,\mathbf{r}_n,1}$, which represents an upper bound on the probability of subnetwork $\mathcal{I}_n$ being in state $\mathbf{r}_n$ and taking an action that requires resource $i \in \mathcal{I}_{AL}$ at time t. Now consider the following linear program:

$$\widehat{P}^{(\star)}_{NSEP}: \quad \max_{\rho,\varrho,\xi,\mu,\Upsilon\geq 0} \sum_{j,t} \sum_{u\in\{0,1\}} p_{t,j} f_{t,j} u \; \mu_{t,j,u} \tag{8}$$

$$\text{s.t.} \quad \varrho_{t,n,\mathbf{s}_n} = \begin{cases} 1, & t=1 \\ \sum_j \sum_{\substack{\mathbf{r}_n\in\mathfrak{R}_{t-1,n}, u\in\mathcal{U}_{j,\mathbf{r}_n} \\ \text{s.t. } \mathbf{r}_n - u\mathbf{a}_{n,j}=\mathbf{s}_n}} p_{t-1,j} \xi_{t-1,n,j,\mathbf{r}_n,u}, & t\geq 2, \end{cases} \quad \forall t, n, \mathbf{s}_n \in \mathfrak{R}_{t,n} \tag{9}$$

$$\rho_{t,i} = \begin{cases} c_i, & t=1 \\ \rho_{t-1,i} - \sum_j \sum_{u\in\{0,1\}} a_{ij} p_{t-1,j} u \mu_{t-1,j,u}, & t\geq 2, \end{cases} \quad \forall t, i\in\mathcal{I}_{AL} \tag{10}$$

$$\sum_{\mathbf{r}_n\in\mathfrak{R}_{t,n}} \varrho_{t,n,\mathbf{r}_n} = 1, \qquad \forall t, n \tag{11}$$

$$\sum_{u\in\mathcal{U}_{j,\mathbf{r}_n}} \xi_{t,n,j,\mathbf{r}_n,u} = \varrho_{t,n,\mathbf{r}_n}, \qquad \forall t, n, j, \mathbf{r}_n \in \mathfrak{R}_{t,n} \tag{12}$$

$$\sum_{\substack{\mathbf{r}_n\in\mathfrak{R}_{t,n} \\ \text{s.t. } \mathbf{r}_n\geq \mathbf{a}_{n,j}u}} \xi_{t,n,j,\mathbf{r}_n,u} = \mu_{t,j,u}, \qquad \forall t, n, j, u \tag{13}$$

$$\sum_{u\in\mathcal{U}_{j,\mathbf{r}_n}} a_{ij} u \xi_{t,n,j,\mathbf{r}_n,u} \leq \Upsilon_{t,i,n,\mathbf{r}_n}, \qquad \forall t, i\in\mathcal{I}_{AL}, n, j, \mathbf{r}_n \in \mathfrak{R}_{t,n} \tag{14}$$

$$\sum_{\mathbf{r}_n\in\mathfrak{R}_{t,n}} \Upsilon_{t,i,n,\mathbf{r}_n} \leq \rho_{t,i}, \qquad \forall t, i\in\mathcal{I}_{AL}, n. \tag{15}$$

Since $\mu_{t,j,u}$ represents the probability that action u for product j is taken at time t, the objective of $\widehat{P}^{(\star)}_{NSEP}$ can be interpreted as the maximization of the expected revenue over the entire time horizon. Constraints (9) and (10) are flow-balance equations similar to the constraint of P_ϕ. Constraint (11) allows us to interpret the values of ϱ as probabilities on the state-spaces $\mathfrak{R}_{t,n}$. (11) is indeed redundant, which can be verified by induction using constraint (9). Constraints (12) and (13) enforce consistency with the probabilistic interpretation of the decision variables. For example, regarding (12), the probability that the resources $i \in \mathcal{I}_n$ are in state $\mathbf{r}_n$ at time t must be the sum over all actions u of the probabilities that the resources $i \in \mathcal{I}_n$ are in state $\mathbf{r}_n$ and action u for product j is taken at time t. The definition of Υ is summarized in (14). Constraint (15) formulates an availability constraint similar to the last constraint of $\widehat{P}_{AL}$: The sum of all $\Upsilon_{t,n,i,\mathbf{r}_n}$ over all states $\mathbf{r}_n$ can be viewed as an upper bound on the expected usage of resource i at time t. This expected usage may not exceed the expected number of available units of resource i. Since the size of $\widehat{P}^{(\star)}_{NSEP}$ is $\mathcal{O}(TIJ(C+1)^{|\mathcal{I}_1|})$, this is a reduction.

Theorem 1. *For $N = 1$, $\widehat{P}^{(\star)}_{NSEP}$ and P_{NSEP} are equivalent.*

The following lemma is central for proving Theorem 1. Its proof can be found in Appendix B. The idea is that given a feasible solution to the model with aggregated variables, $\widehat{P}^{(\star)}_{NSEP}$, we can always construct a corresponding feasible solution to the unreduced ALP P_{NSEP}, and vice versa. Our constructive proof provides insights about the probability distributions X_t corresponding to feasible solutions to the reduced problem $\widehat{P}^{(\star)}_{NSEP}$. The proof suggests, e.g., that the distributions X_t have to provide the appropriate expected values $\rho_{t,i}$ for all $i \in \mathcal{I}_{AL}$. Apart from this constraint, the probability distributions on the state spaces $\mathcal{R}_{t,i}$ for $i \in \mathcal{I}_{AL}$ are almost arbitrary.

Lemma 1. *Consider the case* $N = 1$.

(i) *For every* $\rho, \varrho, \xi, \mu, \Upsilon$ *satisfying (11)–(15), there exists X satisfying (3)–(7).*
(ii) *Every X satisfying (3) determines values* $\rho, \varrho, \xi, \mu, \Upsilon$ *by (4)–(7) and* $\Upsilon_{t,n,i,\mathbf{r}_n} := \max_j a_{ij}\xi_{t,n,j,\mathbf{r}_n,1}$, *which satisfy (11)–(15).*

Proof of Theorem 1. Let $\rho, \varrho, \xi, \mu, \Upsilon$ be a feasible solution to $\widehat{P}^{(\star)}_{NSEP}$, and let X be the distribution given by Lemma 1. By (3)–(7) and the first two constraints of $\widehat{P}^{(\star)}_{NSEP}$, X is a feasible solution to P_{NSEP} with the same objective value.

To show the other direction, let X be a feasible solution to P_{NSEP}, and let $\rho, \varrho, \xi, \mu, \Upsilon$ be according to (4)–(7) and $\Upsilon_{t,i,n,\mathbf{r}_n} := \max_j a_{ij}\xi_{t,n,j,\mathbf{r}_n,1}$. Lemma 1 combined with the first two constraints of P_{NSEP} imply that $\rho, \varrho, \xi, \mu, \Upsilon$ is a feasible solution to $\widehat{P}^{(\star)}_{NSEP}$ with the same objective value. □

To understand why the proof of Lemma 1 and hence Theorem 1 cannot be transferred to the general case with $N > 1$, note that the most important part of proving Lemma 1 is constructing a probability distribution $X_{t,\mathbf{r},\mathbf{u}}$ on the state-action space from a given feasible solution of aggregated variables. We do this construction in two main steps (for details, see Appendix B): First, we construct probabilities for all pairs of state vectors on $\mathcal{I}_1$ and actions, i.e., for all tuples $(\mathbf{r}_1, \mathbf{u})$. Second, we refine the distribution such that it also includes the *AL*-part of the state space $\mathfrak{R}_t$. It is the first of these two steps for which the generalization fails. Constructing a distribution on $\{\mathbf{r}_1, \mathbf{u}\}$ from the values $\xi_{t,1,j,\mathbf{r}_n,u}$ only requires the extension of the reduced index pairs j, u to the full decision vector $\mathbf{u}$. The simultaneous extension of state- and product-space is not possible. (This difficulty also is the reason why the reduction for *SPL* in Vossen & Zhang (2015) is tedious and only yields weak equivalence.)

5. Reduction of the post-arrival formulation

In the pre-arrival formulation of the revenue management problem discussed so far, the value function represents the optimal expected future revenue that can be gained from selling remaining resources $\mathbf{r}$ at time t. Since the action $\mathbf{u}$ in state $\mathbf{r}$ is determined before a customer requesting product j has arrived, $\mathbf{u}$ is a vector summarizing which action to take in each of the J potential cases. In the post-arrival formulation, the value function represents the optimal expected future revenue at time t given remaining resources $\mathbf{r}$ after a customer requesting product j has arrived. As a consequence, the post-arrival state consists of the tuple $(\mathbf{r}, j)$ instead of $\mathbf{r}$ only, the action is either 0 or 1. See Fig. 2 for a visualization of the different formulations. Moreover, the post-arrival formulation is not to be confused with the post-decision formulation, a term often used in the dynamic programming community to define a function representing the maximum expected reward given both the state and the action taken. Our pre- and the post-arrival formulation both use pre-decision states since the optimal expected future revenue is determined before making the decision.

The Bellman equation of the post-arrival formulation reads

$$v'_t(\mathbf{r}, j) = \max_{u \in \mathcal{U}_{j,\mathbf{r}}} \left(u f_{t,j} + \sum_{j'} p_{t+1,j'} v'_{t+1}(\mathbf{r} - u\mathbf{a}_j, j') \right) \quad \forall t, j, \mathbf{r} \in \mathfrak{R}_t$$

$$v'_{T+1}(\mathbf{r}, j) = 0 \quad \forall j, \mathbf{r} \in \mathfrak{R}_{T+1}.$$

Again, the value function can be obtained by solving the linear program

$$D': \min_{v'(\cdot)} \sum_j p_{1,j} v'_1(\mathbf{c}, j)$$

$$\text{s.t.} \quad v'_t(\mathbf{r}, j) \geq u f_{t,j} + \sum_{j'} p_{t+1,j'} v'_{t+1}(\mathbf{r} - u\mathbf{a}_j, j'), \quad \forall t, j, \mathbf{r} \in \mathfrak{R}_t, u \in \mathcal{U}_{j,\mathbf{r}}.$$

In the exact problem formulation, we have that $v_t(\mathbf{r}) = \sum_j p_{t,j} v'_t(\mathbf{r}, j)$. Hence, both formulations lead to the same policy and the same optimal expected revenue. In this section, we discuss a reduction that can be done if we use the *NSEP* approximation in this post-arrival formulation. In particular, we will see that the approximation yields different results when applied to the pre- or post-arrival formulation: Post-arrival approximations always yield upper bounds on their pre-arrival counterparts since every post-arrival approximation $v'_t(\mathbf{r}, j)$ which is feasible to D' yields an approximation $v_t(\mathbf{r}) := \sum_j p_{t,j} v'_t(\mathbf{r}, j)$, which is feasible to D. An example demonstrating non-equivalence is outlined in Appendix C.

Plugging *NSEP* into D', we obtain a problem with dual P'_{NSEP}. This dual can be reformulated in the spirit of the proof of Theorem 1. Indeed, the reduction is facilitated by the fact that only the state space must be reduced. As a consequence, the reduction, which we denote by $\widehat{P}'_{NSEP}$, is valid for general $N \geq 1$. Problem $\widehat{P}'_{NSEP}$ is very similar but not identical to $\widehat{P}^{(\star)}_{NSEP}$. In particular, it has the same objective function (8) and constraints (9)–(13). Constraints (14) and (15) are replaced by the availability-constraint

$$\sum_u a_{ij} u \mu_{t,j,u} \leq \rho_{t,i}, \quad \forall t, i \in \mathcal{I}_{AL}, j, \tag{16}$$

which is familiar from $\widehat{P}_{AL}$. The variable Υ is not needed in this formulation. Compared to the pre-arrival formulation, the size of the reduction is hence reduced to $\mathcal{O}(TJ(I + \sum_n (C+1)^{|\mathcal{I}_n|}))$.

Theorem 2. *Problems* P'_{NSEP} *and* $\widehat{P}'_{NSEP}$ *are equivalent (for arbitrary* $N \geq 1$*).*

The objective value obtained from substituting the value function by an approximation of a particular form in D always yields an upper bound to the optimal value of the objective function $v_1(\mathbf{c})$ of D. Also, enlarging the function space spanned by the basis functions tightens the upper bound. In addition, the post-arrival formulation provides a looser bound than the pre-arrival formulation. All these considerations are summarized in the following lemma:

Lemma 2. *Let Z be the optimal value of P and D,* Z_{AL} *be the optimal value of* $\widehat{P}_{AL}$, Z_{SPL} *be the optimal value of* $\widehat{P}_{SPL}$, Z_{NSEP} *be the optimal value of* P_{NSEP}, *and* Z'_{NSEP} *be the optimal value of* P'_{NSEP} *and* $\widehat{P}'_{NSEP}$. *Then:*

$$Z \leq Z_{NSEP} \leq Z'_{NSEP} \leq Z_{AL} \qquad \text{for all partitions } (\mathcal{I}_n)_n$$
$$Z \leq Z_{NSEP} \leq Z'_{NSEP} \leq Z_{SPL} \qquad \text{if } \mathcal{I}_{AL} = \emptyset.$$

Running Example. Using the approximation $v_t(\mathbf{r}) \approx W_{t,1,(r_{BC},r_{CD})} + V_{t,AB} r_{AB}$ in the base case of our running example, we have $Z_{NSEP} = Z'_{NSEP} = 109.54$.

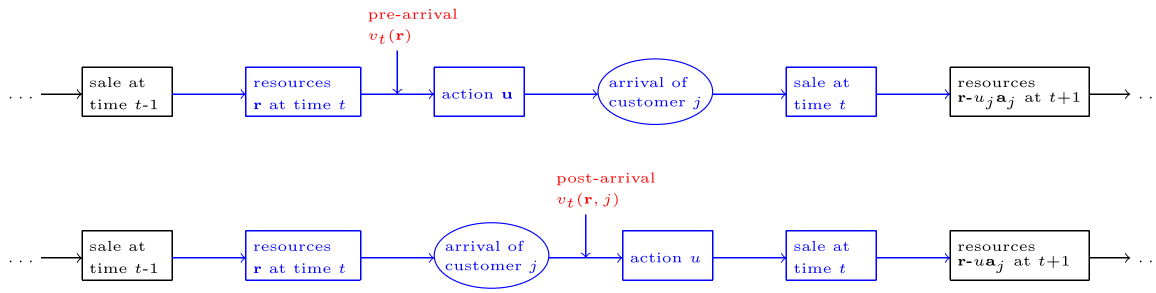

Fig. 2. Pre-arrival (top) and post-arrival (bottom) selling process.

6. A heuristic reduction

The pre-arrival reduction presented in Section 4 is only valid for $N = 1$. The post-arrival reduction has fewer constraints but also leads to a looser bound. Generalizing the differences between the pre- and post-arrival formulations $\widehat{P}^{(\star)}_{NSEP}$ and $\widehat{P}'_{NSEP}$ in the case $N = 1$ to the general case $N \geq 1$, we now formulate a linear program that might not lead to the same value as plugging $NSEP$ into P_ϕ. This is why we call it a heuristic reduction. This heuristic reduction provides a bound that is tighter than $\widehat{P}'_{NSEP}$. Its dual values can also be used to construct a revenue management policy.

Comparing the pre- and post-arrival reductions $\widehat{P}^{(\star)}_{NSEP}$ and $\widehat{P}'_{NSEP}$, the difference is that the two constraints (14) and (15) including variable Υ in $\widehat{P}^{(\star)}_{NSEP}$ are replaced by the less restrictive constraint (16) in $\widehat{P}'_{NSEP}$. To obtain the heuristic reduction H_{NSEP}, we hence consider $\widehat{P}^{(\star)}_{NSEP}$ but 1) enforce (14) for all $i \in \mathcal{I} \setminus \mathcal{I}_n$ instead of $i \in \mathcal{I}_{AL}$, and 2) complement constraint (15), which connects resources $i \in \mathcal{I}_n$ with resources $i \in \mathcal{I}_{AL}$, with the following similar constraint, which connects resources $i \in \mathcal{I}_n$ with resources $i \in \mathcal{I}_{n'}, n \neq n'$:

$$\sum_{\mathbf{r}_n \in \mathfrak{R}_{t,n}} \Upsilon_{t,i,n,\mathbf{r}_n} \leq \sum_{\substack{\mathbf{r}_{n'} \in \mathfrak{R}_{t,n'} \\ \text{s.t. } r'_i \geq 1}} \varrho_{t,n',\mathbf{r}_{n'}} \qquad \forall t, n \neq n', i \in \mathcal{I}_{n'}. \tag{17}$$

The size of H_{NSEP} is $\mathcal{O}(TIJ \sum_n (C+1)^{|\mathcal{I}_n|})$. For $N = 1$, H_{NSEP} is equal to $\widehat{P}^{(\star)}_{NSEP}$ and thus equivalent to P_{NSEP} by Theorem 1. In general, H_{NSEP} is neither equivalent to the pre- nor to the post-arrival formulation of our NRM problem. However, we can prove that the bound obtained by H_{NSEP} always falls in between those two bounds. For an example demonstrating non-equivalence of P_{NSEP}, H_{NSEP} and P'_{NSEP}, see Appendix C.

Theorem 3. *Let Z be the optimal value of P and D, Z_{AL} be the optimal value of $\widehat{P}_{AL}$, Z_{SPL} be the optimal value of $\widehat{P}_{SPL}$, Z_{NSEP} be the optimal value of P_{NSEP}, Z'_{NSEP} be the optimal value of P'_{NSEP} and $\widehat{P}'_{NSEP}$, and Z^H_{NSEP} be the optimal value of H_{NSEP}. Then,*

$$Z \leq Z_{NSEP} \leq Z^H_{NSEP} \leq Z'_{NSEP} \leq Z_{AL} \qquad \text{for all partitions } (\mathcal{I}_n)_n$$
$$Z \leq Z_{NSEP} \leq Z^H_{NSEP} \leq Z'_{NSEP} \leq Z_{SPL} \qquad \text{if } \mathcal{I}_{AL} = \emptyset.$$

Adelman (2007) shows that for AL the marginal seat revenues $V_{t,i}$ of resource i are decreasing in t. In addition to showing this for all $i \in \mathcal{I}_{AL}$, we can show that for all n the value function modeling the resources $i \in \mathcal{I}_n$ is jointly decreasing in t for fixed resources and increasing in every resource for fixed time:

Theorem 4. *Consider the general case $N \geq 1$. Let W be the dual variable of (9) and V be the dual variable of (10). Then, there exists an optimal solution to (H_{NSEP}) and dual values W and V, such that*

$$V_{t,i} \geq V_{t+1,i}, \qquad \forall t, i \in \mathcal{I}_{AL}$$
$$W_{t,n,\mathbf{r}_n} \geq W_{t+1,n,\mathbf{r}_n}, \qquad \forall t, n, \mathbf{r}_n \in \mathfrak{R}_{t,n}$$
$$W_{t,n,\mathbf{r}_n+\mathbf{e}} \geq W_{t,n,\mathbf{r}_n}, \qquad \forall t, n, \mathbf{e} \in \mathfrak{E}_n, \mathbf{r}_n \in \mathfrak{R}_{t,n} \text{ s.t. } \mathbf{r}_n + \mathbf{e} \in \mathfrak{R}_{t,n},$$

where $\mathfrak{E}_n := \{\mathbf{e} \in \{0,1\}^{\mathcal{I}_n} \mid \sum_{i \in \mathcal{I}_n} e_i = 1\}$ denotes the set of canonical unity vectors.

The monotonicity results summarized in Theorem 4 can also be verified for $\widehat{P}'_{NSEP}$.

Running Example. Using the approximation $v_t(\mathbf{r}) \approx W_{t,1,(r_{BC},r_{CD})} + V_{t,AB} r_{AB}$ in the base case of our running example, we have $Z_{NSEP} = Z^H_{NSEP} = Z'_{NSEP} = 109.54$. Illustrating the properties summarized in Theorem 4, an optimal solution exists with

$$V_{16,AB} = 5 \geq V_{17,AB} = 5 \geq V_{18,AB} = 4.34 \geq V_{19,AB} = 3.49$$
$$\geq V_{20,AB} = 2.43,$$

$$W_{9,1,(3,2)} = 81.99 \leq W_{9,1,(4,2)} = 84.19 \leq W_{9,1,(4,3)} = 92.33$$

and

$$W_{9,1,(2,2)} = 77.10 \geq W_{10,1,(2,2)} = 71.86.$$

7. A novel network measure for non-separable partitions

In the previous sections, we considered approximations for a given network partition $(\mathcal{I}_n)_n$. We did not, however, discuss how to choose such a partition. The choice of a partition includes the number N and sizes $|\mathcal{I}_n|$ of the subnetworks as well as the elements of each set $\mathcal{I}_n$. We numerically examine the impact of N and $|\mathcal{I}_n|$ in Section 8.3. In this section, we try to provide guidance on how to choose $\mathcal{I}_n$ for given N and $|\mathcal{I}_n|$. Cooper & Homem-de Mello (2007) were the first to discuss a mapping to measure the quality of partitions with respect to non-separability for $|\mathcal{I}_n| \leq 2$. Translated to our notation, it reads

$$(\mathcal{I}_n)_n \mapsto M^{\text{C\&M}}_{(\mathcal{I}_n)_n} := \sum_{n,j,t} 1_{\{|\mathcal{I}_n|=2\}} \left(\prod_{i \in \mathcal{I}_n} a_{ij} \right) p_{t,j} f_{t,j}.$$

For a given partition, this mapping assigns a value that is equal to the sum of the total expected revenue of products that use both resources of subnetworks $\mathcal{I}_n$ of size 2. Maximizing $M^{\text{C\&M}}_{(\mathcal{I}_n)_n}$ tends to avoid splitting products between subnetworks.

In our numerical experiments in Section 8, we find that this technique works well for hub-and-spoke networks but not for various other network types, including bus lines. This is why we derive a novel mapping, which can be extended to subnetworks of arbitrary sizes. Obviously, the subnetworks $\mathcal{I}_n$ should be chosen such that we can benefit most from non-separability within $\mathcal{I}_n$. In the following, we motivate a heuristic mapping to capture this. To simplify the exposition of the main idea, we focus on the case where $|\mathcal{I}_n|$ is equal for all n, $|\mathcal{I}_n| =: \bar{q}$, and $\mathcal{I}_{AL} = \emptyset$.

Our approach is based on the following two assumptions: The benefit of non-separability of resources within the subset $\mathcal{I}_n$ can be

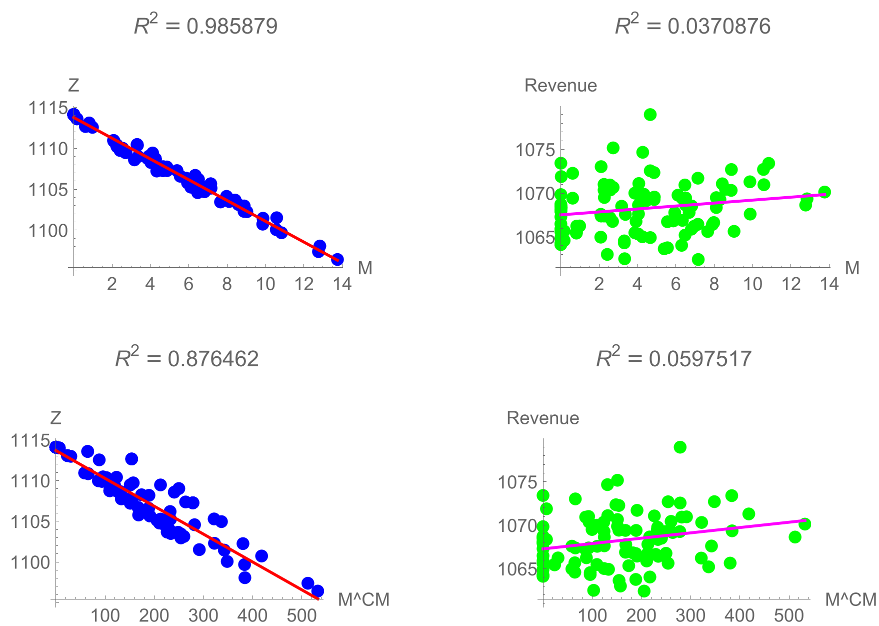

Fig. 3. Correlation of $M_{(\mathcal{I}_n)_n}$ (top) resp. $M^{C\&M}_{(\mathcal{I}_n)n}$ (bottom) with Z'_{NSEP} (left) and corresponding average revenue (right, maximum standard error 2.8, 10′000 simulations), network setting H&S with $(L,T,C)=(4,30,3)$ given partitions $(\mathcal{I}_n)_n$ with $|\mathcal{I}_n|=2,\forall n$.

captured by the difference between the optimal value of the upper bound problem $\widehat{P}_{SPL}$ (based on a partition that consists of I subnetworks of size 1) and $\widehat{P}'_{NSEP}$ given a partition that consists of the subnetwork $\mathcal{I}_n$ and subnetworks of size 1 otherwise. Denoting the optimal value of $\widehat{P}'_{NSEP}$ given such an auxiliary partition by $Z'^n{}_{NSEP}$ and the optimal value of $\widehat{P}_{SPL}$ by Z_{SPL}, we further assume that the best partition can then be found by finding the partition that yields the largest sum of benefits, $\sum_n(Z_{SPL}-Z'^n{}_{NSEP})$. The main challenge is hence to find a good approximation of $Z_{SPL}-Z'^n{}_{NSEP}$.

To motivate the details of our approximation, we fix n and consider the auxiliary partition $(\mathcal{I}^n_{\tilde{n}})_{\tilde{n}}$ with $\mathcal{I}^n_1=\mathcal{I}_n=\{i^n_1,\dots,i^n_{\bar{q}}\}$ and $N-\bar{q}$ subnetworks of size 1, i.e., $|\mathcal{I}^n_{\tilde{n}}|=1$ for all $\tilde{n}>1$. Now consider the dual of $\widehat{P}'_{NSEP}$ (given $\mathcal{I}_{AL}=\emptyset$):

$$\widehat{D}'_{NSEP}:\quad \min_{W,\alpha,\beta}\sum_{\tilde{n}}W_{1,\tilde{n},\mathbf{c}_{\tilde{n}}}$$

$$\text{s.t.}\quad W_{t,\tilde{n},\mathbf{r}_{\tilde{n}}}\geq\sum_j\alpha_{t,\tilde{n},j,\mathbf{r}_{\tilde{n}}},\qquad \forall t,\tilde{n},\mathbf{r}_{\tilde{n}}$$

$$\alpha_{t,\tilde{n},j,\mathbf{r}_{\tilde{n}}}+\beta_{t,\tilde{n},j,u}\geq p_{t,j}W_{t+1,\tilde{n},\mathbf{r}_{\tilde{n}}-u\mathbf{a}_{\tilde{n},j}},\qquad \forall t,\tilde{n},j,\mathbf{r}_{\tilde{n}}\in\mathfrak{R}_{t,\mathbf{r}_{\tilde{n}}},u\in\mathcal{U}_{j,\mathbf{r}_{\tilde{n}}}$$

$$-\sum_{\tilde{n}}\beta_{t,\tilde{n},j,u}\geq p_{t,j}f_{t,j}u,\qquad \forall t,j,u.$$

For the separable partition corresponding to SPL, $\widehat{D}'_{NSEP}$ is equal to $\widehat{D}_{SPL}$. Let $\overline{W},\overline{\alpha},\overline{\beta}$ be an optimal solution of $\widehat{D}'_{NSEP}$ in this separable case, and let $\overline{W}^n,\overline{\alpha}^n,\overline{\beta}^n$ be an optimal solution of $\widehat{D}'_{NSEP}$ given the partition $(\mathcal{I}^n_{\tilde{n}})_{\tilde{n}}$. To reduce the computational complexity of determining $\overline{W}^n,\overline{\alpha}^n,\overline{\beta}^n$, we resort to an approximation by setting all values of decision variables with $i\notin\mathcal{I}_n$ to their separable counterparts, i.e., $W=\overline{W}$, $\alpha=\overline{\alpha}$ and $\beta=\overline{\beta}$. The remaining problem with variables W,α,β for $\tilde{n}=n$ then reads:

$$\min_{W,\alpha,\beta} W_{1,n,\mathbf{c}_n}+\sum_{i\in\mathcal{I}\setminus\mathcal{I}_n}\overline{W}_{t,i,c_i} \tag{18}$$

$$\text{s.t.}\quad W_{t,n,\mathbf{r}_n}\geq\sum_j\alpha_{t,n,j,\mathbf{r}_n},\quad\forall t,\mathbf{r}_n \tag{19}$$

$$\alpha_{t,n,j,\mathbf{r}_n}+\beta_{t,n,j,u}\geq p_{t,j}W_{t+1,n,\mathbf{r}_n-u\mathbf{a}_{n,j}},$$

$$\forall t,j,\mathbf{r}_n\in\mathfrak{R}_{t,\mathbf{r}_n},u\in\mathcal{U}_{j,\mathbf{r}_n} \tag{20}$$

$$-\beta_{t,n,j,u}-\sum_{i\in\mathcal{I}\setminus\mathcal{I}_n}\overline{\beta}_{t,i,j,u}\geq p_{t,j}f_{t,j}u,\quad\forall t,j,u. \tag{21}$$

Plugging (21) into (20) and (20) into (19) yields the following set of constraints:

$$W_{t,n,\mathbf{r}_n}\geq\sum_j\left(p_{t,j}f_{t,j}u_j+p_{t,j}W_{t+1,n,\mathbf{r}_n-u_j\mathbf{a}_{n,j}}+\sum_{i\in\mathcal{I}\setminus\mathcal{I}_n}\overline{\beta}_{t,i,j,u_j}\right),$$
$$\forall t,\mathbf{r}_n\in\mathfrak{R}_{t,\mathbf{r}_n},\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_n}.$$

Minimizing (18) subject to this set of constraints yields the optimal solution $W^\star_{t,n,\mathbf{r}_n}$ with

$$W^\star_{t,n,\mathbf{r}_n}=\max_{\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_n}}\sum_j\left(p_{t,j}f_{t,j}u_j+p_{t,j}W^\star_{t+1,n,\mathbf{r}_n-u_j\mathbf{a}_{n,j}}+\sum_{i\in\mathcal{I}\setminus\mathcal{I}_n}\overline{\beta}_{t,i,j,u_j}\right)$$
$$W^\star_{T+1,n,\mathbf{r}_n}=0.$$

The computation of $W^\star_{t,n,\mathbf{r}_n}$ can be done efficiently since the state space $\mathfrak{R}_n$ is only $\bar{q}$-dimensional, and the maximization over all actions $\mathbf{u}$ can be done separately for each product j. The difference $Z_{SPL}-Z'^n{}_{NSEP}$ can then be estimated as

$$M_{\mathcal{I}_n}:=\sum_i\overline{W}_{1,i,c_i}-\left(W^\star_{1,n,\mathbf{c}_n}+\sum_{i\in\mathcal{I}\setminus\mathcal{I}_n}\overline{W}_{1,i,c_i}\right)=\sum_{q=1}^{\bar{q}}\overline{W}_{1,i^n_q,c_{i^n_q}}-W^\star_{1,n,\mathbf{c}_n}.$$

Hence, given a partition $(\mathcal{I}_n)_n$ with $\mathcal{I}_n=\{i^n_1,\dots,i^n_{\bar{q}}\}$, the corresponding network measure is the sum of all single-subnetwork measures $M_{\mathcal{I}_n}$, i.e.,

$$(\mathcal{I}_n)_n\mapsto M_{(\mathcal{I}_n)_n}:=\sum_n M_{\mathcal{I}_n}=\sum_n\left(\sum_{q=1}^{\bar{q}}\overline{W}_{1,i^n_q,c_{i^n_q}}-W^\star_{1,n,\mathbf{c}_n}\right).$$

Running Example. For the base case, we obtain $M_{(\{AB\},\{BC,CD\})}=2.31$, $M_{(\{AB,BC\},\{CD\})}=1.49$ and $M_{(\{AB,CD\},\{BC\})}=1.51$. These values suggest to partition the network into the subnetworks {AB} and {BC,CD} yielding the optimal value (upper bound) of 107.75 for $\widehat{P}'_{NSEP}$ and an average revenue of 105.19 (standard error 0.26). Note that the upper bounds provided by $\widehat{P}'_{NSEP}$ are larger (108.28) for both of the other two partitions and the corresponding average revenues are lower (104.44 and 104.29 with standard error 0.26).

8. Numerical experiments

In the previous sections, we have introduced numerous linear programs that can be used to provide both upper bounds and approximations of the value function. While we were able to prove certain hierarchies between bounds, the size of the difference in these bounds is unclear from our theoretical development. In addition, we only obtained a partial order of the bounds, and we cannot make any conclusions about the performance of the policy one would derive from our approximations. We furthermore discussed a mapping $(\mathcal{I}_n)_n \mapsto M_{(\mathcal{I}_n)_n}$ to capture the quality of non-separable network partitions. The motivation of this mapping is heuristic and should be tested in experiments.

In this section we use numerical experiments to close these gaps in our analysis. In particular, we (1) demonstrate that our network value $M_{(\mathcal{I}_n)_n}$ indeed strongly correlates with the tightness of the corresponding upper bound; (2) illustrate the hierarchy of the bounds we established before; (3) show that the quality of the policies is similar to the quality of the bounds; (4) highlight that some of the suggested approximations can be used in problems of realistic size and provide tighter bounds and better policies than their separable counterparts.

Talluri & van Ryzin (1998) suggest that the bound and policy corresponding to the DLP are asymptotically tight. Since it is well-known that an affine approximation with time-independent slope in P_ϕ is equivalent to the DLP, we expect the same for the more general approximation *SPL*. Hence, we suspect potential improvements in situations where capacity is scarce and the time horizon is small. In our simulations, we focus on such settings.

In Section 8.1, we describe the various network types which we use for our experiments. In Section 8.2, we examine the correlation between $M_{(\mathcal{I}_n)_n}$ and the upper bound obtained from $\widehat{P}'_{NSEP}$. The encouraging results will allow us to choose the partition $(\mathcal{I}_n)_n$ for which $M_{(\mathcal{I}_n)_n}$ is largest. In Section 8.3, we take a closer look at the influence of the size and the number of non-affine subnetworks on both upper bound and average revenues. Finally, in Section 8.4, we numerically benchmark our non-separable approximations against *AL* and *SPL*.

8.1. Description of networks

In the following, we present four network settings, "Hub-and-Spoke" (H&S), "Simple Bus Line" (SBL), "Consecutive Bus Line" (CBL) and "Realistic Bus Line" (RBL). For all instances, the capacity on each leg is equal, $c_i = C\ \forall i$.

8.1.1. Hub-and-spoke network (H&S)

In this setup, we closely mimic the generation of hub-and-spoke networks described in Adelman (2007). There is a single hub and L non-hub locations, which implies $I = 2L$. There are $2L$ single-leg itineraries and $L(L-1)$ two-leg itineraries. For each itinerary, there is a high- and a low-fare class. The revenue f_j for each low-fare class is generated randomly from a discrete uniform distribution on the interval [15,49]. The corresponding high fares are five times the low fares. Demand is stationary and random, such that the non-arrival probability is equal to 0.2, and such that for each itinerary, the demand for the low fare is three times the demand for the high fare. Except for the random parts, the parameters L, T, C completely specify such a network.

8.1.2. Simple bus line (SBL)

This setup consists of I consequtive legs. There is only one fare class per origin-destination pair, and products include all possible origin-destination-pairs with a certain minimal length $\underline{l}$ and maximum length $\bar{l}$. Demand is stationary and equal for all products, with non-arrival probability 0.2. We define $f_j := \sqrt{\sum_i a_{ij}}$. The parameters $I, T, C, \underline{l}, \bar{l}$ completely specify such a network.

8.1.3. Consecutive bus lines (CBL)

This setup is generated by stringing together m SBLs with parameters $I, T, C, \underline{l}, \bar{l}$ each. The CBL has $m \cdot I$ legs and, to keep the load unchanged, $m \cdot T$ time steps. The parameters $m, m \cdot I, m \cdot T, C, \underline{l}, \bar{l}$ completely specify such a network. We expect that *NSEP* solves this setting to optimality when using the partition $\mathcal{I}_1 = \{1, \dots, I\}, \mathcal{I}_2 = \{I+1, \dots, 2I\}, \dots, \mathcal{I}_m = \{(m-1)I+1, \dots, mI\}$.

8.1.4. Realistic bus line (RBL)

We demonstrate the performance of our proposed reductions in a real-world example with data from a large European bus company. The bus line we consider has $C = 46$ seats and stops 4 times between its origin and final destination. It hence has 5 legs with a capacity $c_i = 46$ on each leg $i = 1, \dots, 5$. A total of 11 OD-pairs is offered at one of 8 predetermined prices per OD-pair. In each time step t and for each of the 11 OD-pairs, the bus company must decide which of the 8 prices is to be offered. Following Walczak, Mardan, & Kallesen (2010), we transform this discrete pricing problem into an equivalent independent demand problem with $11 \cdot 8$ products. More details about the data can be found in Appendix H.

8.2. Quality of network measures for non-separable partitions

In Figs. 3–6, we show scatterplots for one H&S, one CBL and two SBL settings. The horizontal axis represents the network quantity $M_{(\mathcal{I}_n)_n}$ resp. $M^{\text{C\&M}}_{(\mathcal{I}n)n'}$, and the vertical axis represents the optimal value Z'_{NSEP} of $\widehat{P}'_{NSEP}$ or the average revenue of the induced policy. For $\mathcal{I}_{AL} = \emptyset$ and given $|\mathcal{I}_n| = \bar{q}$ for all n, each point in such a scatterplot represents one partition $(\mathcal{I}_n)n$. Compared to $M^{\text{C\&M}}_{(\mathcal{I}n)n'}$, the quantity $M_{(\mathcal{I}n)n}$ explains a larger percentage of the variation in Z'_{NSEP} for both network types. On the right hand side of the figures, we can observe that the correlation between both measures and the average revenue is smaller. Figs. 3–5 also highlight that tighter upper bounds are not guaranteed to provide larger expected revenues even though a tendency is evident. In Fig. 5, we also see that upper bound and average revenue coincide for the partition with the largest $M_{(\mathcal{I}_n)_n}$. As we expected for the CBL setting, *NSEP* achieves optimality.

Given the large correlation between $M_{(\mathcal{I}_n)_n}$ and Z'_{NSEP}, the value of $M_{(\mathcal{I}_n)_n}$ could guide the choice of the partition $(\mathcal{I}_n)_n$. Calculating $M_{(\mathcal{I}_n)_n}$ for every possible partition is infeasible in practice, however, since the number of possible partitions grows exponentially in the number of legs I. As a remedy, we suggest using a greedy algorithm, where $\mathcal{I}_1, \mathcal{I}_2, \dots$ are successively chosen such that their single-subnetwork measure $M_{\mathcal{I}_n}$ is largest given the remaining resources $\mathcal{I} \setminus \bigcup_{n'=1}^{n-1} \mathcal{I}_{n'}$. In numerical experiments, the performance of this greedy algorithm is comparable to maximizing $M_{(\mathcal{I}_n)_n}$ over all partitions. For example, in Figs. 3–6, the partitions with the largest $M_{(\mathcal{I}_n)_n}$ are identical to the partitions found by the greedy algorithm.

8.3. Influence of the size and the number of non-affine subnetworks

In this section, we investigate the influence of $|\mathcal{I}_n| = \bar{q}$ and N on the optimal value of $\widehat{P}'_{NSEP}$ and the corresponding average revenue. We illustrate our findings using the setting SBL with $(I, T, C, \underline{l}, \bar{l}) = (10, 12, 3, 1, 10)$.

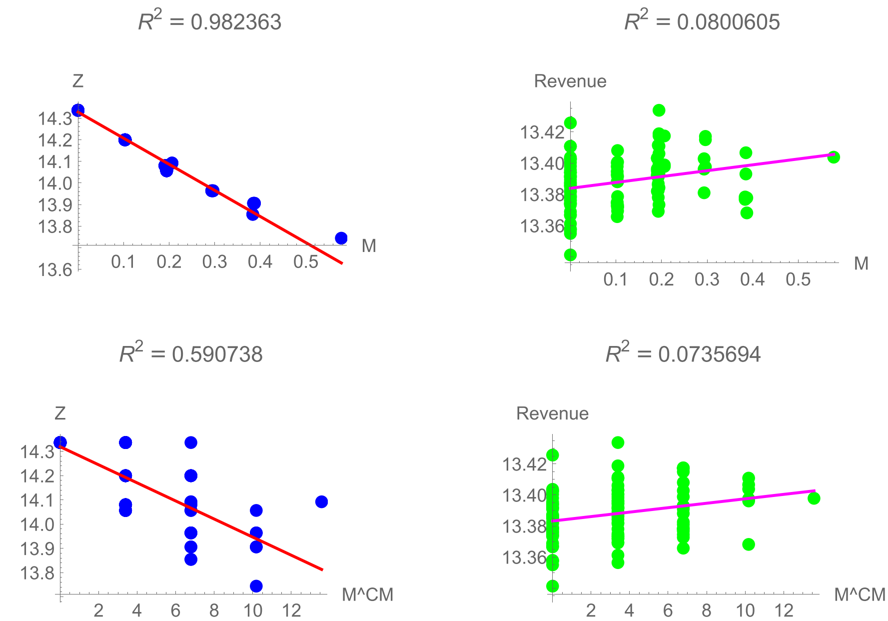

Fig. 4. Correlation of $M_{(\mathcal{I}_n)_n}$ (top) resp. $M^{C\&M}_{(\mathcal{I}_n)n}$ (bottom) with Z'_{NSEP} (left) and corresponding average revenue (right, maximum standard error 0.014, 10′000 simulations), network setting SBL with $(I, T, C, \underline{l}, \bar{l}) = (8, 21, 3, 2, 2)$ given partitions $(\mathcal{I}_n)_n$ with $|\mathcal{I}_n| = 2, \forall n$.

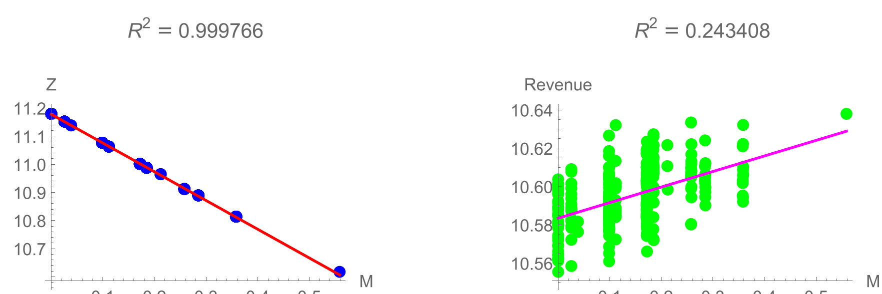

Fig. 5. Correlation of $M_{(\mathcal{I}_n)_n}$ with Z'_{NSEP} (left) and corresponding average revenue (right, maximum standard error 0.014,10′000 simulations), network setting CBL with $(m, m \cdot I, m \cdot T, C, \underline{l}, \bar{l}) = (3, 3 \cdot 3, 3 \cdot 6, 2, 1, 3)$ given partitions $(\mathcal{I}_n)_n$ with $|\mathcal{I}_n| = 3, \forall n$.

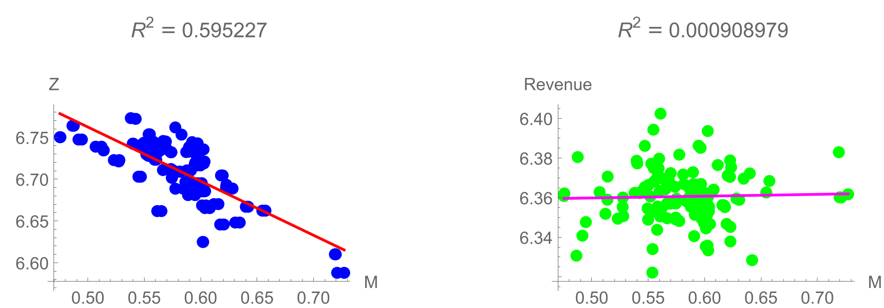

Fig. 6. Correlation of $M_{(\mathcal{I}_n)_n}$ with Z'_{NSEP} (left) and corresponding average revenue (right, maximum standard error 0.013, 10′000 simulations), network setting SBL with $(I, T, C, \underline{l}, \bar{l}) = (10, 8, 2, 1, 10)$ given partitions $(\mathcal{I}_n)_n$ with $|\mathcal{I}_n| = 5, \forall n$.

To specify the details of the partition underlying our *NSEP* approximation, we add the number of subnetworks N, their size $\bar{q}$, as well as the method to choose elements of a partition in the following. In this spirit, $NSEP(N, \bar{q}, \text{gr.})$ represents an approximation based on a partition with N subnetworks of size $\bar{q}$ with elements chosen using the greedy method described in Section 8.2. Furthermore, $NSEP(\lceil I/\bar{q} \rceil, \bar{q}, \text{gl.})$ denotes the approximation where all resources are grouped into $\lfloor I/\bar{q} \rfloor$ subnetworks of size $\bar{q}$, additional resources (if any) are placed in an extra non-separable subnetwork and the partition is chosen maximizing $M_{(\mathcal{I}_n)_n}$.

First, consider $NSEP(\lceil I/\bar{q} \rceil, \bar{q}, \text{gl.})$ and let $\bar{q}$ run from 1 to 5. The case $\bar{q} = 1$ coincides with *SPL*, and $\bar{q} = 5$ results in only two non-separable subnetworks. The results are shown in Fig. 7. We observe diminishing returns for increasing $\bar{q}$, i.e., significant improvements can be achieved even for small subnetwork sizes $\bar{q}$.

In Fig. 8, we report the results for $\bar{q} = 1, \ldots, 5$ and varying number of subnetworks $N = 0, \ldots, \lfloor \frac{I}{\bar{q}} \rfloor$ using $NSEP(N, \bar{q}, \text{gr.})$. The case $N = 0$ coincides with *AL*, and the case $\bar{q} = 1, N = I$ coincides with *SPL*. We again observe diminishing returns for increasing N.

Table 1
Sizes of the linear programs with respect to T, I, J and C.

$\widehat{P}_{AL}$	$\widehat{P}_{SPL}$	$\widehat{P}'_{NSEP}$, $NSEP(1,\bar{q},\text{gr.})$	$\widehat{P}'_{NSEP}$, $NSEP(\lceil I/\bar{q}\rceil,\bar{q},\text{gl.})$	H_{NSEP}, $NSEP(1,\bar{q},\text{gr.})$	H_{NSEP}, $NSEP(\lceil I/\bar{q}\rceil,\bar{q},\text{gl.})$
$\mathcal{O}(TIJ)$	$\mathcal{O}(TIJC)$	$\mathcal{O}(TJ(I+C^{\bar{q}}))$	$\mathcal{O}(TIJC^{\bar{q}})$	$\mathcal{O}(TIJC^{\bar{q}})$	$\mathcal{O}(TI^2JC^{\bar{q}})$

Table 2
Computing times (in seconds) for SBL $(I, T, C, \underline{l}, \bar{l}) = (8, T, C, 1, 8)$ with varying capacity C and time T. Technical details: Virtual machine with 256 gigabyte RAM and 32 cores of 2.59 GHz processors. The linear programs are solved with the CPLEX-solver “barrier”.

(C, T)	$\widehat{P}_{AL}$	$\widehat{P}_{SPL}$	$\widehat{P}'_{NSEP}$, $NSEP(1,2,\text{gr.})$	$\widehat{P}'_{NSEP}$, $NSEP(\lceil I/2\rceil,2,\text{gl.})$	H_{NSEP}, $NSEP(1,2,\text{gr.})$	H_{NSEP}, $NSEP(\lceil I/2\rceil,2,\text{gl.})$
(2,8)	0.08	0.61	0.44	0.83	1.30	1.25
(4,16)	0.08	1.75	2.63	4.16	5.28	9.14
(8,32)	0.22	7.63	15.47	30.88	41.34	86.97
(16,64)	0.31	36.94	180.11	468.39	340.39	889.58
(32,128)	0.72	115.98	647.47	10’551.34	4’455.06	79’748.27

Table 3
Upper bounds and relative improvement (in %) compared to *AL*. Since the RBL fares are not stationary, the static DLP is not solved and the result is not reported.

Network Setting	Upper Bound Z Using				
	AL	*SPL*	$NSEP(1,2,\text{gr.})$ $NSEP(1,3,\text{gr.})$	$NSEP(\lceil I/2\rceil,2,\text{gl.})$ $NSEP(\lceil I/3\rceil,3,\text{gl.})$	DLP
H&S $(L,T,C) =$ (3,50,6), $\widehat{P}'_{NSEP}$	1989.8	1893.7 (-4.83%)	1923.2 (-3.34%) 1913.6 (-3.83%)	1879.1 (-5.56%) 1873.9 (-5.82%)	2072.0
H&S $(L,T,C) =$ (3,50,6), H_{NSEP}	1989.8	1893.7 (-4.83%)	1923.2 (-3.34%) 1913.4 (-3.84%)	1879.1 (-5.56%) 1873.9 (-5.82%)	
SBL $(I,T,C,\underline{l},\bar{l}) =$ (8,40,10,1,8), $\widehat{P}'_{NSEP}$	38.791	37.915 (-2.26%)	37.862 (-2.39%) 36.951 (-4.74%)	37.015 (-4.58%) 36.451 (-6.03%)	39.660
SBL $(I,T,C,\underline{l},\bar{l}) =$ (8,40,10,1,8), H_{NSEP}	38.791	37.915 (-2.26%)	37.848 (-2.43%) 36.951 (-4.74%)	37.010 (-4.59%) 36.444 (-6.05%)	
SBL $(I,T,C,\underline{l},\bar{l}) =$ (8,20,5,1,8), $\widehat{P}'_{NSEP}$	18.944	18.290 (-3.45%)	18.278 (-3.51%) 17.669 (-6.73%)	17.658 (-6.79%) 17.287 (-8.75%)	19.830
SBL $(I,T,C,\underline{l},\bar{l}) =$ (8,20,5,1,8), H_{NSEP}	18.944	18.290 (-3.45%)	18.271 (-3.56%) 17.669 (-6.73%)	17.651 (-6.83%) 17.279 (-8.79%)	
RBL, $\widehat{P}'_{NSEP}$ & H_{NSEP}	699.83	685.21 (-2.09%)	682.53 (-2.47%) -	- -	-

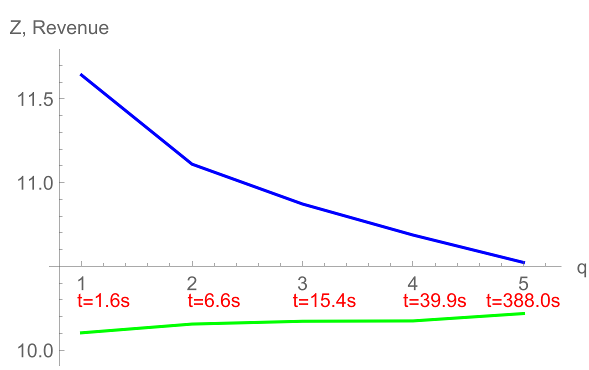

Fig. 7. Application of $NSEP(\lceil I/\bar{q}\rceil,\bar{q},\text{gl.})$ to the network setting SBL with $(I,T,C,\underline{l},\bar{l}) = (10,12,3,1,10)$. The optimal value of $\widehat{P}'_{NSEP}$ (blue), the corresponding average revenues (green, maximum standard error 0.005, 100’000 simulations) and computing times (red) for varying $\bar{q}$.

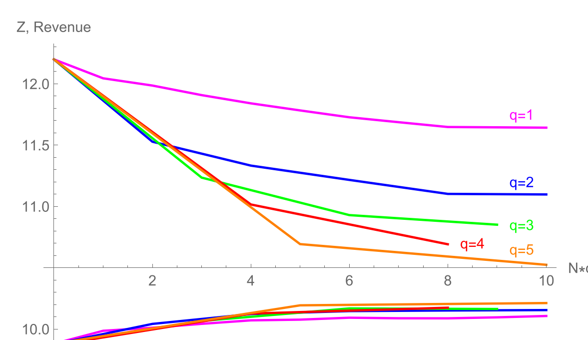

Fig. 8. Application of $NSEP(N,\bar{q},\text{gr.})$ to the network setting SBL with $(I,T,C,\underline{l},\bar{l}) = (10,12,3,1,10)$. The optimal value of $\widehat{P}'_{NSEP}$ (top) and the corresponding average revenues (bottom, maximum standard error 0.005, 100’000 simulations) for varying $N \cdot \bar{q}$ with $\bar{q} \in \{1,2,3,4,5\}$.

8.4. Upper bounds and policy performances

For given $\bar{q}$, we compare the benchmark policies *AL* and *SPL* with the approximations $NSEP(1,\bar{q},\text{gr.})$ and $NSEP(\lceil I/\bar{q}\rceil,\bar{q},\text{gl.})$ defined in the previous section.

Table 1 compares the problem sizes of $\widehat{P}'_{NSEP}$ and H_{NSEP} including also $\widehat{P}_{AL}$ and $\widehat{P}_{SPL}$. It is evident that the computational burden of both $NSEP(1,\bar{q},\text{gr.})$ and $NSEP(\lceil I/\bar{q}\rceil,\bar{q},\text{gl.})$ rapidly increases in C. In Table 2, we report the resulting run-times for an SBL network with scaled capacity C and time T given $\bar{q} = 2$.

For each of the approximations *AL*, *SPL*, $NSEP(1,\bar{q},\text{gr.})$ and $NSEP(\lceil I/\bar{q}\rceil,\bar{q},\text{gl.})$, we solve both $\widehat{P}'_{NSEP}$ and H_{NSEP} for $\bar{q} = 2, 3$ and simulate the corresponding policies. In Table 3, we present the upper bounds for various network settings. In Table 4, we report the average revenues generated by the simulation of the corresponding policies. We include the upper bounds obtained from the standard static deterministic linear program (DLP) in Table 3. To com-

Table 4
Average revenue (100'000 simulations) and relative improvement (in %) compared to *AL*. The maximum standard error is 1.1 for H&S, 0.01 for all SBL and 0.23 for RBL. Since the RBL fares are not stationary, the static DLP is not solved and the performance of corresponding policies is not reported.

Network Setting	Average Revenue Using				
	AL	*SPL*	$NSEP(1,2,\text{gr.})$ $NSEP(1,3,\text{gr.})$	$NSEP(\lceil I/2\rceil,2,\text{gl.})$ $NSEP(\lceil I/3\rceil,3,\text{gl.})$	DLP DPD
H&S $(L,T,C)=$ (3,50,6), $\widehat{P}'_{NSEP}$	1701.8	1855.5 (+9.03%)	1757.7 (+3.28%) 1786.2 (+4.96%)	1864.1 (+9.54%) 1865.1 (+9.60%)	1659.9 1807.8
H&S $(L,T,C)=$ (3,50,6), H_{NSEP}	1701.8	1855.5 (+9.03%)	1761.4 (+3.50%) 1796.5 (+5.57%)	1864.1 (+9.54%) 1865.1 (+9.60%)	
SBL $(I,T,C,\underline{l},\bar{l})=$ (8,40,10,1,8), $\widehat{P}'_{NSEP}$	34.795	35.074 (+0.80%)	34.868 (+0.21%) 35.062 (+0.77%)	35.266 (+1.35%) 35.370 (+1.65%)	34.430 34.289
SBL $(I,T,C,\underline{l},\bar{l})=$ (8,40,10,1,8), H_{NSEP}	34.795	35.074 (+0.80%)	34.863 (+0.19%) 35.062 (+0.77%)	35.270 (+1.36%) 35.378 (+1.67%)	
SBL $(I,T,C,\underline{l},\bar{l})=$ (8,20,5,1,8), $\widehat{P}'_{NSEP}$	16.136	16.446 (+1.93%)	16.352 (+1.34%) 16.482 (+2.15%)	16.540 (+2.51%) 16.592 (+2.83%)	16.115 15.636
SBL $(I,T,C,\underline{l},\bar{l})=$ (8,20,5,1,8), H_{NSEP}	16.136	16.446 (+1.93%)	16.354 (+1.35%) 16.482 (+2.15%)	16.544 (+2.53%) 16.595 (+2.85%)	
RBL, $\widehat{P}'_{NSEP}$ & H_{NSEP}	635.71	681.88 (+7.26%)	682.79 (+7.41%) -	- -	-

pare our approach to other bid price policies, we report the average revenues resulting from a direct implementation of the DLP bid prices as well as the results from the corresponding dynamic programming decomposition (DPD) in Table 4. For details, we refer to Talluri & van Ryzin (2004b), Sections 3.3.1 and 3.4.4.

We observe that $NSEP(1,\bar{q},\text{gr.})$ provides a tighter upper bound than *SPL* for all bus line settings but is outperformed by the *SPL* policy in all examples except for RBL. This observation emphasizes the importance of marginal seat revenues: non-linearity and non-separability of one single two-leg subnetwork $\mathcal{I}_1$ cannot always compensate the drawback of linearity on $\mathcal{I}_{AL}$.

The benefit of a non-linear approximation (assuming separability) can be quantified by the difference between *AL* and *SPL*. To quantify the benefit of non-separability, $NSEP(\lceil I/\bar{q}\rceil,\bar{q},\text{gl.})$ can be compared to *SPL*. For the hub-and-spoke network H&S the non-separable approximation $NSEP(\lceil I/\bar{q}\rceil,\bar{q},\text{gl.})$ leads to a smaller bound and larger average revenue than the separable counterparts. Comparing the size of the different effects, non-linearity, however, seems to be a more important factor than non-separability. This might be due to the small average number of legs used per product in H&S. Comparing SBL for $C=5$ and $C=10$, we see that both the benefits of non-separability and non-linearity decrease in capacity. The effects of both generalizations seem to be comparable as the upper bounds are concerned. The extra benefit of non-separability on average revenue is smaller.

For the realistic bus line example, the 256 gigabyte RAM used by our virtual machine was not enough to solve $NSEP(\lceil I/2\rceil,2,\text{gl.})$. However, $NSEP(1,2,\text{gr.})$ already obtains optimality.

9. Conclusion

Given a partition of the resources, we suggest a novel, non-separable value function approximation.

Instead of solving the proposed non-separable ALPs via row generation, row sampling or similar approaches, we suggest reductions as well as a compact heuristic linear program which extend the work of Vossen & Zhang (2015). Our reductions yield the tightest upper bound of all linear programs of polynomial size we are aware of. Our reduction method gives insights into the nature of reductions, in particular by interpreting the decision variables as state-action probabilities and applying variable aggregation.

We show that starting with the same approximation, the ALPs derived from the pre- and post-arrival formulation result in different upper bounds and policies. More precisely, the post-arrival ALP always provides an upper bound on the solution of the pre-arrival ALP.

The novel approximations yield optimality gaps which are significantly reduced compared to the standard *AL* or *SPL* approximations. The size of the benefit, however, clearly depends on both the network structure and the partition used. Comparing the bounds and expected revenues generated by an affine, a separable piecewise linear, and non-separable approximations, we investigate the relative importance of modeling capacity-dependent marginal seat revenues of individual resources as compared to modeling non-separability between resources. We find that in linear network settings with small capacities, the added benefit of non-separability can be compared to the effect of capacity-dependent marginal seat revenues.

In addition, we investigate the quality of different partitions of different network types and suggest a network measure that has a larger correlation to the quality of the corresponding upper bound than the mapping suggested by Cooper & Homem-de Mello (2007), in particular for linear networks. Since our measure is heuristic, more examination, both theoretical and experimental, is needed. We observe diminishing returns with respect to the number and sizes of the subnetworks. For the sake of managerial insights, further research could exploit this effect to improve our partitioning algorithm.

To our knowledge, we present the first non-separable value function approximation in the context of compact representations of approximate linear programs. For non-separable subnetworks $\mathcal{I}_n$, we use the basis functions $\{1_{\{\mathbf{r}_n=\mathbf{s}_n\}}(\mathbf{r}_n) \mid \mathbf{s}_n \in \mathfrak{R}_n\}$ which span the whole function space on $\mathfrak{R}_n$. For our reduction, the consideration of a smaller set of basis functions would only decrease the number of constraints but not the number of variables. Future research could analyze if a smaller set together with a column generation procedure could increase the computational efficiency while maintaining the structural advantages of non-separability.

Appendix A. Equality of optimal values of $\widehat{P}_{SPL}$ and P_ϕ using (*SPL*)

With slight adaptation of notation, the reduction of P_ϕ using *SPL* as stated in Vossen & Zhang (2015) reads:

$$\widehat{P}_{VZ}: \quad \max_{y,z,q} \sum_{t,j} p_{t,j} f_j q_{t,j}$$

$$\text{s.t.} \quad y_{t,i,r} = \begin{cases} 1, & t=1 \\ y_{t-1,i,r} - \sum_{j:a_{ij}=1} p_{t-1,j}(z_{t-1,i,j,r} - z_{t-1,i,j,r+1}), & t \geq 2, \end{cases} \quad \forall t,i,r$$

$$q_{t,j} = z_{t,i,j,1}, \quad \forall t,i,j: a_{ij}=1$$

$$z_{t,i,j,r+1} \leq z_{t,i,j,r}, \quad \forall t,i,j,r: a_{ij}=1$$

$$z_{t,i,j,r} \leq y_{t,i,r}, \quad \forall t,i,j,r: a_{ij}=1.$$

Let Z_{SPL}, Z_{VZ} and Z_ϕ be the optimal values of $\widehat{P}_{SPL}, \widehat{P}_{VZ}$ and P_ϕ using *SPL*, respectively. Vossen & Zhang (2015) have shown that $Z_{VZ} = Z_\phi$. To show $Z_\phi = Z_{SPL}$, it hence suffices to prove (i) $Z_\phi \leq Z_{SPL}$ and (ii) $Z_{SPL} \leq Z_{VZ}$.

Given a feasible solution X to P_ϕ using *SPL*, we get a feasible solution σ, ζ, μ to $\widehat{P}_{SPL}$ by defining

$$\zeta_{t,i,j,r,u} := \sum_{\substack{\mathbf{r}\in\mathfrak{R}_t, \mathbf{u}\in\mathfrak{U}_\mathbf{r} \\ \text{s.t. } r_i=r, u_j=u}} X_{t,\mathbf{r},\mathbf{u}}$$

and σ, μ as in the constraints of $\widehat{P}_{SPL}$. This proves (i). Given a feasible solution σ, ζ, μ to $\widehat{P}_{SPL}$, we get a feasible solution y, z, q to $\widehat{P}_{VZ}$ by defining

$$y_{t,i,r} := \sum_{r'=r}^{c_i} \sigma_{t,i,r'}, \qquad z_{t,i,j,r} := \sum_{r'=r}^{c_i} \zeta_{t,i,j,r',1}, \qquad q_{t,j} := \mu_{t,j,1}.$$

This proves (ii).

Appendix B. Proof of Lemma 1

We use the following notation:

$$\mathbf{r}_{(i)} := (r_1, \ldots, r_i)^T, \quad i=1,\ldots,I$$

$$\mathbf{a}_{(i)j} := (a_{1j}, \ldots, a_{ij})^T, \quad i=1,\ldots,I,\ j=1,\ldots,J$$

$$\mathfrak{R}_{t,(i)} := \prod_{i'=1}^{i} \mathcal{R}_{t,i'}, \quad i=1,\ldots,I$$

$$\mathfrak{U}_{\mathbf{r}_{(i)}} := \{\mathbf{u}\in\{0,1\}^J \mid u_j a_{i',j} \leq r_{i'}, \forall i'=1,\ldots,i,\ j=1,\ldots,J\}, \quad i=1,\ldots,I$$

$$\mathcal{U}_{j,\mathbf{r}_{(i)}} := \{u\in\{0,1\} \mid u a_{i',j} \leq r_{i'}, \forall i'=1,\ldots,i\}, \quad i=1,\ldots,I,\ j=1,\ldots,J.$$

Without loss of generality, we may assume that $\mathcal{I}_n = \mathcal{I}_1 = \{1,\ldots,m\}$ and $\mathcal{I}_{AL} = \{m+1,\ldots,I\}$ (where $\mathcal{I}_{AL} = \emptyset$ in case $m=I$). Given this assumption and the notation introduced above, $\mathbf{r}_1 = \mathbf{r}_{(m)}$.

Note that whenever product j cannot be offered, i.e., $1 \notin \mathcal{U}_{j,\mathbf{r}_{(m)}}$, the variable $\xi_{t,1,j,\mathbf{r}_1,1}$ is neither defined by (6) nor by $\widehat{P}^{(\star)}_{NSEP}$. We define $\xi_{t,1,j,\mathbf{r}_1,1} := 0$ in this case and use similar conventions for all variables used in this proof.

B1. Proof of part (i)

Let $\rho, \varrho, \xi, \mu, \Upsilon$ satisfy (11)–(15). Part (i) of Lemma 1 claims that $X = (X_{t,\mathbf{r},\mathbf{u}})_{t,\mathbf{r},\mathbf{u}}$ satisfying (3)–(7) exist.

We prove this claim by providing a recipe how to construct such values $X = (X_{t,\mathbf{r},\mathbf{u}})_{t,\mathbf{r},\mathbf{u}}$ given $\rho, \varrho, \xi, \mu, \Upsilon$. This construction of X is done separately for each t. Hence, t can be considered fixed in the following.

The construction of X is done in two steps: In the first step, we construct a new auxiliary variable $X^{(m)} = \left(X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}}\right)_{t,\mathbf{r}_{(m)}\in\mathfrak{R}_{t,(m)},\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(m)}}}$ with indices $t, \mathbf{r}_{(m)}, \mathbf{u}$ instead of $t, \mathbf{r}, \mathbf{u}$, i.e., the variable $X^{(m)}$ is restricted to the non-separable subnetwork. In the second step, we consecutively add the remaining resources of the network to obtain X. The details of those two steps are provided below.

Step 1: Since the final variable X should satisfy (3)–(7), we require the auxiliary variable $X^{(m)}$ to satisfy the following properties which are analogue to (3)–(6) (property (7) is only relevant on the subnetwork $\mathcal{I}_{AL}$):

$$1 = \sum_{\mathbf{r}_{(m)}\in\mathfrak{R}_{t,(m)},\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(m)}}} X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}} \tag{B.1}$$

$$\varrho_{t,1,\mathbf{r}_1} = \sum_{\substack{\mathbf{r}'_{(m)}\in\mathfrak{R}_{t,(m)},\mathbf{u}\in\mathfrak{U}_{\mathbf{r}'_{(m)}} \\ \text{s.t. } \mathbf{r}'_1=\mathbf{r}_1}} X^{(m)}_{t,\mathbf{r}'_{(m)},\mathbf{u}} \left(= \sum_{\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(m)}}} X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}} \right) \forall \mathbf{r}_1 \in \mathfrak{R}_{t,1} \tag{B.2}$$

$$\mu_{t,j,u} = \sum_{\substack{\mathbf{r}_{(m)}\in\mathfrak{R}_{t,(m)},\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(m)}} \\ \text{s.t. } u_j=u}} X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}} \quad \forall j\in\{1,\ldots,J\}, u\in\{0,1\} \tag{B.3}$$

$$\xi_{t,1,j,\mathbf{r}_1,u} = \sum_{\substack{\mathbf{r}'_{(m)}\in\mathfrak{R}_{t,(m)},\mathbf{u}\in\mathfrak{U}_{\mathbf{r}'_{(m)}} \\ \text{s.t. } \mathbf{r}'_1=\mathbf{r}_1 \\ \text{s.t. } u_j=u}} X^{(m)}_{t,\mathbf{r}'(m),\mathbf{u}} \left(= \sum_{\substack{\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(m)}} \\ \text{s.t. } u_j=u}} X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}} \right) \forall j, \quad \forall \mathbf{r}_1 \in \mathfrak{R}_{t,1}, u\in\mathcal{U}_{j,\mathbf{r}_1}. \tag{B.4}$$

Indeed, it is enough to construct a variable $X^{(m)}$ satisfying (B.4) because (B.1)–(B.3) follow from (B.4) given (11)–(13).

We now outline a construction of $X^{(m)}$ such that it satisfies (B.4):

For all $\mathbf{r}_{(m)} \in \mathfrak{R}_{t,(m)}$, let $\jmath(\mathbf{r}_{(m)},1), \ldots, \jmath(\mathbf{r}_{(m)},J)$ be a permutation of $1,\ldots,J$ such that

$$\xi_{t,1,\jmath(\mathbf{r}_1,1),\mathbf{r}_{(1)},1} \leq \cdots \leq \xi_{t,1,\jmath(\mathbf{r}_1,J),\mathbf{r}_1,1}.$$

Furthermore, let $\mathbf{u}^{\mathbf{r}_{(m)},j} \in \{0,1\}^J$ be the decision vector with components defined by

$$u^{\mathbf{r}_{(m)},j}_{\jmath(\mathbf{r}_{(m)},j')} := \begin{cases} 0, & \text{if } j' < j \\ 1, & \text{if } j' \geq j \end{cases} \tag{B.5}$$

for every $j, j' \in \{1,\ldots,J\}$ and $\mathbf{r}_{(m)} \in \mathfrak{R}_{t,(m)}$. Letting $\xi_{t,1,\jmath(\mathbf{r}_1,0),\mathbf{r}_{(1)},1} = 0$, we define the values of $X^{(m)}$ by setting

$$X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}} := \begin{cases} \xi_{t,1,\jmath(\mathbf{r}_1,j),\mathbf{r}_1,1} - \xi_{t,1,\jmath(\mathbf{r}_1,j-1),\mathbf{r}_{(m)},1}, & \text{if } \mathbf{u} = \mathbf{u}^{\mathbf{r}_{(m)},j}, \\ \varrho_{t,1,\mathbf{r}_1} - \xi_{t,1,\jmath(\mathbf{r}_1,J),\mathbf{r}_1,1}, & \text{if } \mathbf{u} = \mathbf{0}, \\ 0, & \text{otherwise.} \end{cases} \tag{B.6}$$

We now demonstrate that this choice of $X^{(m)}$ satisfies property (B.4): First, consider (B.4) for $u=1$: For all $j=1,\ldots,J$ and $\mathbf{r}_{(m)} \in \mathfrak{R}_{t,(m)}$ such that $1 \in \mathcal{U}_{\jmath(\mathbf{r}_{(m)},j),\mathbf{r}_{(m)}}$, we have

$$\xi_{t,1,\jmath(\mathbf{r}_1,j),\mathbf{r}_1,1} = \sum_{j'=1}^{j} \left(\xi_{t,1,\jmath(\mathbf{r}_1,j'),\mathbf{r}_1,1} - \xi_{t,1,\jmath(\mathbf{r}_1,j'-1),\mathbf{r}_1,1}\right) = \sum_{j'=1}^{j} X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}^{\mathbf{r}_{(m)},j'}} = \sum_{\substack{\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(m)}} \\ \text{s.t. } u_{\jmath(\mathbf{r}_{(m)},j)}=1}} X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}}, \tag{B.7}$$

where the first equality is a telescoping sum, the second equality follows from definition (B.6), and the third equality follows by the construction of $\mathbf{u}^{\mathbf{r}_{(m)},j'}$, see (B.5). Since $\jmath(\mathbf{r}_{(m)},\cdot)$ is a permutation, we have

$$\xi_{t,1,j,\mathbf{r}_1,1} = \sum_{\substack{\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(m)}} \\ \text{s.t. } u_j=1}} X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}}. \tag{B.8}$$

If $1 \notin \mathcal{U}_{j,\mathbf{r}_{(m)}}$, we have $\xi_{t,1,j,\mathbf{r}_1,1} = 0$ in (B.6) and hence by construction $X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}} = 0$ for all $\mathbf{u}$ with $u_j = 1$. As a consequence, $\xi_{t,1,j,\mathbf{r}_1,1} = 0 = \sum_{\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(m)}} \text{ s.t. } u_j=1} X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}}$ is true.

To show (B.4) for $u = 0$, we can conclude from (B.6) that $\sum_{\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(m)}}} X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}} = \varrho_{t,1,\mathbf{r}_1}$. Therefore, we can compute

$$\xi_{t,1,j,\mathbf{r}_1,0} \overset{(12)}{=} \varrho_{t,1,\mathbf{r}_1} - \xi_{t,1,j,\mathbf{r}_1,1} \overset{(B.8)}{=} \sum_{\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(m)}}} X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}} - \sum_{\substack{\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(m)}} \\ \text{s.t. } u_j=1}} X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}} = \sum_{\substack{\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(m)}} \\ \text{s.t. } u_j=0}} X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}}.$$

Putting the cases $u = 0$ and $u = 1$ together, we have shown that $X^{(m)}$ satisfies property (B.4).

Step 2: We now successively add resources to obtain auxiliary variables $X^{(i)} = \left(X^{(i)}_{t,\mathbf{r}_{(i)},\mathbf{u}}\right)_{t,\mathbf{r}_{(i)}\in\mathfrak{R}_{t,(i)},\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(i)}}}$ for $i = m+1,\dots,I$ until we finally have $X^{(I)} = X$.

Starting from $X^{(i-1)}$, we obtain $X^{(i)}$ by splitting each $X^{(i-1)}_{t,\mathbf{r}_{(i-1)},\mathbf{u}}$ into values $(X^{(i)}_{t,(\mathbf{r}^T_{(i-1)},r_i)^T,\mathbf{u}})_{r_i\in\mathcal{R}_{t,i}}$ such that

$$\sum_{r_i\in\mathcal{R}_{t,i} \text{ s.t. } \mathbf{u}\in\mathfrak{U}_{(\mathbf{r}^T_{(i-1)},r_i)^T}} X^{(i)}_{t,(\mathbf{r}^T_{(i-1)},r_i)^T,\mathbf{u}} = X^{(i-1)}_{t,\mathbf{r}_{(i-1)},\mathbf{u}}. \tag{B.9}$$

By requiring (B.9), one can show by induction starting with (B.1)–(B.4) that the same equations hold with all (m) replaced by (i) (except for the simplifications stated in brackets). In order to achieve (B.1)–(B.4), it is therefore enough to require (B.9).

At the same time, we have to ensure (7), i.e., the expected value of r_i (using probabilities given by $X^{(i)}$) should be equal to $\rho_{t,i}$:

$$\sum_{\mathbf{r}_{(i)}\in\mathfrak{R}_{t,(i)},\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(i)}}} r_i X^{(i)}_{t,\mathbf{r}_{(i)},\mathbf{u}} = \rho_{t,i}. \tag{B.10}$$

By induction using (B.9), property (B.10) ensures that (7) holds. The following construction of $X^{(i)}, i = m+1,\dots,I$, which ensures (B.9) and (B.10), hence yields the desired $X^{(I)} = X$.

We now demonstrate how such values $X^{(i)}, i = m+1,\dots,I$ can be constructed based on $X^{(m)}$, which has been constructed in Step 1.

Let $i > m$ and suppose that $X^{(i-1)}_{t,\mathbf{r}_{(i-1)},\mathbf{u}}$ has been constructed. To construct $X^{(i)}$ such that (B.9) and (B.10) hold, we split the value of $X^{(i-1)}_{t,\mathbf{r}_{(i-1)},\mathbf{u}}$ into the variables $(X^{(i)}_{t,\mathbf{r}_{(i)},\mathbf{u}})_{r_i\in\mathcal{R}_{t,i} \text{ s.t. } \mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(i)}}}$ such that (B.10) is fulfilled. We distinguish three different cases with respect to the value of $\rho_{t,i}$:

1. $\rho_{t,i} \in \mathbb{N}$: We set

$$X^{(i)}_{t,(r_1,\dots,r_i)^T,\mathbf{u}} = \begin{cases} X^{(i-1)}_{t,(r_1,\dots,r_{i-1})^T,\mathbf{u}}, & \text{if } r_i = \rho_{t,i}, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, (B.9) and (B.10) are satisfied by this definition. Furthermore, the definition is valid because positive values are only assigned to variables $X^{(i)}_{t,(r_1,\dots,r_i)^T,\mathbf{u}}$ with $r_i = \rho_{t,i}$, and $\mathbf{u} \in \mathfrak{U}_{(r_1,\dots,r_{i-1},\rho_i)^T}$ follows from $\mathbf{u} \in \mathfrak{U}_{(r_1,\dots,r_{i-1})^T}$ and $\rho_i \geq 1$.

2. $\rho_{t,i} > 1, \rho_{t,i} \notin \mathbb{N}$: We set

$$X^{(i)}_{t,(r_1,\dots,r_i)^T,\mathbf{u}} = \begin{cases} (\lceil\rho_{t,i}\rceil - \rho_{t,i})X^{(i-1)}_{t,(r_1,\dots,r_{i-1})^T,\mathbf{u}}, & \text{if } r_i = \lfloor\rho_{t,i}\rfloor, \\ (\rho_{t,i} - \lfloor\rho_{t,i}\rfloor)X^{(i-1)}_{t,(r_1,\dots,r_{i-1})^T,\mathbf{u}}, & \text{if } r_i = \lceil\rho_{t,i}\rceil, \\ 0, & \text{otherwise.} \end{cases}$$

Again, it can quickly be verified that (B.9) and (B.10) hold, and that the definition is valid.

3. $0 \leq \rho_{t,i} < 1$: In this case, the expected value of r_i given probabilities $X^{(i)}_{t,\mathbf{r}_{(i)},\mathbf{u}}$ should be smaller than 1. Therefore, it is necessary that some variables $X^{(i)}_{t,(r_1,\dots,r_{i-1},0)^T,\mathbf{u}}$ are assigned positive values. However, this may only happen if $\mathbf{u} \in \mathfrak{U}_{(r_1,\dots,r_{i-1},0)^T}$, i.e., if $\mathbf{u}$ does not use resource i. For all $r_i > 1$, we set $X^{(i)}_{t,(r_1,\dots,r_i)^T,\mathbf{u}} := 0$. Whenever $\mathbf{u}$ uses resource i, we must therefore set

$$X^{(i)}_{t,(r_1,\dots,r_{i-1},0)^T,\mathbf{u}} := 0$$
$$X^{(i)}_{t,(r_1,\dots,r_{i-1},1)^T,\mathbf{u}} := X^{(i-1)}_{t,(r_1,\dots,r_{i-1})^T,\mathbf{u}},$$

i.e., the variable $X^{(i)}_{t,(r_1,\dots,r_{i-1},1)^T,\mathbf{u}}$ is bound to take the value $X^{(i-1)}_{t,(r_1,\dots,r_{i-1})^T,\mathbf{u}}$. On the other hand, whenever $\mathbf{u}$ does not use resource i, the split of $X^{(i-1)}_{t,(r_1,\dots,r_{i-1})^T,\mathbf{u}}$ among $X^{(i)}_{t,(r_1,\dots,r_{i-1},0)^T,\mathbf{u}}$ and $X^{(i)}_{t,(r_1,\dots,r_{i-1},1)^T,\mathbf{u}}$ is not subject to any constraints, i.e., the variable $X^{(i)}_{t,(r_1,\dots,r_{i-1},1)^T,\mathbf{u}}$ may take any value between 0 and $X^{(i-1)}_{t,(r_1,\dots,r_{i-1})^T,\mathbf{u}}$. We rewrite (B.10) by distinguishing between those $\mathbf{u}$ which use resource i and those which do not:

$$\begin{aligned} &\sum_{\mathbf{r}_{(i)}\in\mathfrak{R}_{t,(i)},\mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(i)}}} r_i X^{(i)}_{t,\mathbf{r}_{(i)},\mathbf{u}} \\ &= \sum_{\substack{(r_1,\dots,r_{i-1})^T\in\mathfrak{R}_{t,(i-1)}, \\ \mathbf{u}\in\mathfrak{U}_{(r_1,\dots,r_{i-1})^T}}} 1\cdot X^{(i)}_{t,(r_1,\dots,r_{i-1},1)^T,\mathbf{u}} \\ &= \underbrace{\sum_{\substack{(r_1,\dots,r_{i-1})^T\in\mathfrak{R}_{t,(i-1)}, \\ \mathbf{u}\in\mathfrak{U}_{(r_1,\dots,r_{i-1})^T} \\ \mathbf{u} \text{ uses resource } i}} X^{(i-1)}_{t,(r_1,\dots,r_{i-1})^T,\mathbf{u}}}_{S_1} \\ &\quad + \underbrace{\sum_{\substack{(r_1,\dots,r_{i-1})^T\in\mathfrak{R}_{t,(i-1)}, \\ \mathbf{u}\in\mathfrak{U}_{(r_1,\dots,r_{i-1})^T} \\ \mathbf{u} \text{ uses resource } i \text{ not}}} X^{(i)}_{t,(r_1,\dots,r_{i-1},1)^T,\mathbf{u}}}_{S_2} = \rho_{t,i}. \end{aligned} \tag{B.11}$$

If all values $X^{(i)}_{t,(r_1,\dots,r_{i-1},1)^T,\mathbf{u}}$ in S_2 are chosen to be $X^{(i-1)}_{t,(r_1,\dots,r_{i-1})^T,\mathbf{u}}$, then we get $S_1 + S_2 = 1 > \rho_{t,i}$, see (B.1). If, on the other hand, all values $X^{(i)}_{t,(r_1,\dots,r_{i-1},1)^T,\mathbf{u}}$ in S_2 are chosen to equal 0, then $S_1 + S_2 = S_1$. Therefore, if S_1 does not exceed $\rho_{t,i}$, it is possible to assign appropriate values to the variables $X^{(i)}_{t,(r_1,\dots,r_{i-1},1)^T,\mathbf{u}}$ in S_2 such that (B.11) holds. The proof is thus complete if we can show $S_1 \leq \rho_{t,i}$.

By induction using (B.9), we have

$$S_1 = \sum_{\substack{(r_1,\dots,r_{i-1})^T\in\mathfrak{R}_{t,(i-1)}, \\ \mathbf{u}\in\mathfrak{U}_{(r_1,\dots,r_{i-1})^T} \\ \mathbf{u} \text{ uses resource } i}} X^{(i-1)}_{t,(r_1,\dots,r_{i-1})^T,\mathbf{u}} = \sum_{\substack{\mathbf{r}_{(m)}\in\mathfrak{R}_{t,(m)}, \\ \mathbf{u}\in\mathfrak{U}_{\mathbf{r}_{(m)}} \\ \mathbf{u} \text{ uses resource } i}} X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}}. \tag{B.12}$$

Fix $\mathbf{r}_{(m)}$ for a moment. By construction of $X^{(m)}$, see (B.5) and (B.6), the set of all $\mathbf{u}$ that use resource i and for which $X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}}$ is positive, consists of the decision vectors $\mathbf{u}^{\mathbf{r}_{(m)},j}, j = 1,\dots,\hat{\jmath}(\mathbf{r}_{(m)})$, where $\hat{\jmath}(\mathbf{r}_{(m)})$ is the largest index

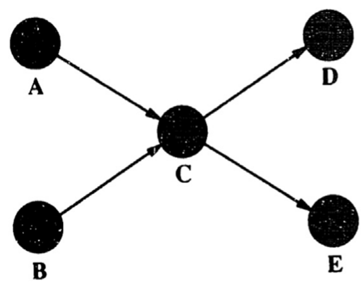

Fig. C.9. Structure: Small hub-and-spoke network, adapted from Williamson (1992), page 94, Fig. 4.4.

j such that $a_{i,j(\mathbf{r}_{(m)},j)} = 1$. Therefore, we can estimate

$$\sum_{\substack{\mathbf{r}_{(m)} \in \mathfrak{R}_{t,(m)}, \\ \mathbf{u} \in \mathfrak{U}_{\mathbf{r}_{(m)}} \\ \mathbf{u} \text{ uses resource } i}} X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}} = \sum_{\mathbf{r}_{(m)} \in \mathfrak{R}_{t,(m)}} \sum_{j=1}^{\hat{j}(\mathbf{r}_{(m)})} X^{(m)}_{t,\mathbf{r}_{(m)},\mathbf{u}^{\mathbf{r}_{(m)},j}}$$

$$\overset{(B.7)}{=} \sum_{\mathbf{r}_1 \in \mathfrak{R}_{t,1}} \xi_{t,1,j(\mathbf{r}_1,\hat{j}(\mathbf{r}_1)),\mathbf{r}_1,1}$$

$$\overset{(14)}{\leq} \sum_{\mathbf{r}_1 \in \mathfrak{R}_{t,1}} \Upsilon_{t,i,1,\mathbf{r}_1} \overset{(15)}{\leq} \rho_{t,i},$$

which proves $S_1 \leq \rho_{t,i}$. Therefore, (B.11) can be satisfied, which completes the construction of $X^{(i)}$.

B2. Proof of part (ii)

Let X be satisfying (3), let ρ, ϱ, ξ, μ be defined by (4)–(7), and let $\Upsilon_{t,n,i,\mathbf{r}_n} := \max_j a_{ij}\xi_{t,n,j,\mathbf{r}_n,1}$. We have to show that (11)–(15) hold. (11) follows from plugging (4) into (11) and using (3). (12) follows from plugging (6) into (12) and using (4). (13) follows from plugging (6) into (13) and using (5). (14) follows from the definition of Υ. To verify (15), we estimate

$$\sum_{\mathbf{r}_1 \in \mathfrak{R}_{t,1}} \Upsilon_{t,i,1,\mathbf{r}_1} = \sum_{\mathbf{r}_1 \in \mathfrak{R}_{t,1}} \max_j a_{ij}\xi_{t,n,j,\mathbf{r}_n,1}$$

$$\overset{(6)}{=} \sum_{\mathbf{r}_1 \in \mathfrak{R}_{t,1}} \max_j a_{ij} \sum_{\substack{\mathbf{r}' \in \mathfrak{R}_t, \mathbf{u} \in \mathfrak{U}_{\mathbf{r}'} \\ \text{s.t. } \mathbf{r}'_1 = \mathbf{r}_1 \\ \text{s.t. } u_j = 1}} X_{t,\mathbf{r}',\mathbf{u}}$$

$$= \sum_{\mathbf{r}_1 \in \mathfrak{R}_{t,1}} \max_j \sum_{\substack{\mathbf{r}' \in \mathfrak{R}_t, \mathbf{u} \in \mathfrak{U}_{\mathbf{r}'} \\ \text{s.t. } \mathbf{r}'_1 = \mathbf{r}_1}} \underbrace{a_{ij}u_j}_{\leq r'_i} X_{t,\mathbf{r}',\mathbf{u}}$$

$$\leq \sum_{\mathbf{r}_1 \in \mathfrak{R}_{t,1}} \sum_{\substack{\mathbf{r}' \in \mathfrak{R}_t, \mathbf{u} \in \mathfrak{U}_{\mathbf{r}'} \\ \text{s.t. } \mathbf{r}'_1 = \mathbf{r}_1}} r'_i X_{t,\mathbf{r}',\mathbf{u}} \overset{(7)}{=} \rho_{t,i}.$$

Appendix C. Non-equivalence of pre- and post-arrival ALPs and the heuristic reduction

In this section, we provide an example for which the inequalities $Z_{NSEP} \leq Z^H_{NSEP} \leq Z'_{NSEP}$ from Theorem 3 are strict. Thus, we not only demonstrate that pre- and post-arrival formulations are not equivalent, but also that the heuristic reduction H_{NSEP} is not an exact reduction.

The basic setup is taken from Williamson (1992) (Section 4.3). As illustrated in Fig. C.9, there is one hub, two non-hub origins and two non-hub destinations. The resulting eight OD-pairs AE, AD, BD, BE, AC, BC, CE, CD hence represent eight products, for each of which there is only one fare class. Fares f_j for each product are given in Table C.5. We set $C = 3, T = 18$ and $p_{t,j} = 1.5/T, \forall t, j$. The resulting upper bounds are displayed in Table C.6.

Table C.5
Fares: Product prices in hub-and-spoke network.

product j:	1	2	3	4	5	6	7	8
OD-pair:	AE	AD	BD	BE	AC	BC	CE	CD
fare f_j:	140	150	160	150	100	100	100	100

Table C.6
Strict inequalities: Bounds for an example where all inequalities of Theorem 3 are strict.

	$NSEP(1, 2, \text{gr.})$	$NSEP(\lceil I/2 \rceil, 2, \text{gl.})$
$\widehat{P}'_{NSEP}$	947.611	919.296
H_{NSEP}	942.596	919.164
P_{NSEP}	942.596	919.130

Since N is equal to 1 for $NSEP(1, 2, \text{gr.})$, Theorem 1 states that H_{NSEP} is equivalent to P_{NSEP} in this case. Thus, the corresponding upper bounds are equal. Table C.6 demonstrates strict inequalities of the upper bounds provided by $\widehat{P}'_{NSEP}$, H_{NSEP} and P_{NSEP} using the approximation $NSEP(\lceil I/2 \rceil, 2, \text{gl.})$. Furthermore, we can observe that the objective value of the heuristic reduction H_{NSEP} is very close to the non-reduced ALP P_{NSEP}.

Appendix D. Proof of Theorem 2

We obtain P'_{NSEP} by inserting the approximation

$$NSEP': \qquad \widetilde{v}_t(\mathbf{r}, j) \approx \sum_n W_{t,n,j,\mathbf{r}_n} + \sum_{i \in \mathcal{I}_{AL}} V_{t,i,j} r_i$$

into D', and computing the dual of the resulting linear program:

$$P'_{NSEP}: \qquad \max_{\widetilde{X} \geq 0} \sum_{t,j} \sum_{\mathbf{r} \in \mathfrak{R}_t, u \in \mathcal{U}_{j,\mathbf{r}}} f_{t,j} u \widetilde{X}_{t,j,\mathbf{r},u}$$

$$\sum_{\substack{\mathbf{r} \in \mathfrak{R}_t, u \in \mathcal{U}_{j,\mathbf{r}} \\ \text{s.t. } \mathbf{r}_n = \mathbf{s}_n}} \widetilde{X}_{t,j,\mathbf{r},u} = \begin{cases} p_{1,j}, & t=1 \\ p_{t,j} \sum_{j'} \sum_{\substack{\mathbf{r} \in \mathfrak{R}_{t-1}, u \in \mathcal{U}_{j',\mathbf{r}} \\ \text{s.t. } \mathbf{r}_n - \mathbf{a}_{n,j'} u = \mathbf{s}_n}} \widetilde{X}_{t-1,j',\mathbf{r},u}, & t \geq 2, \end{cases} \quad \forall t, j, n, \mathbf{s}_n \in \mathfrak{R}_{t,n}$$

$$\sum_{\mathbf{r} \in \mathfrak{R}_t, u \in \mathcal{U}_{j,\mathbf{r}}} r_i \widetilde{X}_{t,j,\mathbf{r},u} = \begin{cases} p_{1,j} c_i, & t=1 \\ p_{t,j} \sum_{j'} \sum_{\mathbf{r} \in \mathfrak{R}_{t-1}, u \in \mathcal{U}_{j',\mathbf{r}}} (r_i - a_{ij'} u) \widetilde{X}_{t-1,j',\mathbf{r},u}, & t \geq 2, \end{cases}$$

$$\forall t, i \in \mathcal{I}_{AL}, j$$

$$\sum_{\mathbf{r} \in \mathfrak{R}_t, u \in \mathcal{U}_{j,\mathbf{r}}} \widetilde{X}_{t,j,\mathbf{r},u} = p_{t,j}, \qquad \forall t, j.$$

Variable aggregation similar to (4)–(7) results in variables $\tilde{\varrho}_{t,n,j,\mathbf{r}_n}, \tilde{\mu}_{t,j,u}, \tilde{\xi}_{t,n,j,\mathbf{r}_n,u}, \tilde{\rho}_{t,i,j}$ and the following reduced linear program:

$$\widehat{\widetilde{P}}_{NSEP}: \qquad \max_{\tilde{\rho}, \tilde{\varrho}, \tilde{\mu}, \tilde{\xi} \geq 0} \sum_{t,j} \sum_{u \in \{0,1\}} f_{t,j} u \tilde{\mu}_{t,j,u}$$

$$\text{s.t. } \tilde{\varrho}_{t,n,j,\mathbf{s}_n} = \begin{cases} p_{1,j}, & t = 1 \\ p_{t,j} \sum_{j'} \sum_{\substack{\mathbf{r}_n \in \mathfrak{R}_{t-1,n}, u \in \mathcal{U}_{j',\mathbf{r}_n} \\ \text{s.t. } \mathbf{r}_n - u\mathbf{a}_{n,j'} = \mathbf{s}_n}} \tilde{\xi}_{t-1,n,j',\mathbf{r}_n,u}, & t \geq 2, \end{cases}$$
$$\forall t, j, n, \mathbf{s}_n \in \mathfrak{R}_{t,n} \tag{D.1}$$

$$\tilde{\rho}_{t,i,j} = \begin{cases} p_{1,j} c_i, & t = 1 \\ p_{t,j} \sum_{j'} \left(\tilde{\rho}_{t-1,i,j'} - \sum_{u \in \{0,1\}} a_{ij'} u \tilde{\mu}_{t-1,j',u} \right), & t \geq 2, \end{cases}$$
$$\forall t, i \in \mathcal{I}_{AL}, j \tag{D.2}$$

$$\sum_{\mathbf{r}_n \in \mathfrak{R}_{t,n}} \tilde{\varrho}_{t,n,j,\mathbf{r}_n} = p_{t,j}, \qquad \forall t, n, j \tag{D.3}$$

$$\sum_{u \in \mathcal{U}_{j,\mathbf{r}_n}} \tilde{\xi}_{t,n,j,\mathbf{r}_n,u} = \tilde{\varrho}_{t,n,j,\mathbf{r}_n}, \qquad \forall t, j, \mathbf{r}_n \tag{D.4}$$

$$\sum_{\substack{\mathbf{r}_n \in \mathfrak{R}_{t,n} \\ \text{s.t. } u \in \mathcal{U}_{j,\mathbf{r}_n}}} \tilde{\xi}_{t,n,j,\mathbf{r}_n,u} = \tilde{\mu}_{t,j,u}, \qquad \forall t, j, u \tag{D.5}$$

$$\sum_{u \in \{0,1\}} u a_{ij} \tilde{\mu}_{t,j,u} \le \tilde{\rho}_{t,i,j}, \qquad \forall t, i \in \mathcal{I}_{AL}, j. \tag{D.6}$$

The equivalence of P'_{NSEP} and $\widehat{\widehat{P}}_{NSEP}$ can be shown along the lines of the proof of Theorem 1 and Lemma 1. However, there is a major difference concerning Step 1 of the proof of Part (i) of Lemma 1, see Appendix B.1. In Lemma 1, we only consider the case $N = 1$, and Step 1 consists of constructing an auxiliary variable $(X^{(m)}_{t,\mathbf{r}_1,\mathbf{u}})_{t,\mathbf{r}_1,\mathbf{u}}$ representing probabilities for state-action pairs $(\mathbf{r}_1, \mathbf{u})$. Here, instead, we need to construct an auxiliary variable $(\tilde{X}^{(N)}_{t,j,\mathbf{r}_1,\ldots,\mathbf{r}_N,u})_{t,j,\mathbf{r}_1,\ldots,\mathbf{r}_N,u}$ representing probabilities for all feasible tuples $(j, \mathbf{r}_1, \ldots, \mathbf{r}_N, u)$. This is done by simple multiplication:

$$\tilde{X}^{(N)}_{t,j,\mathbf{r}_1,\ldots,\mathbf{r}_N,u} := \prod_{n=1}^{N} \tilde{\xi}_{t,n,j,\mathbf{r}_n,u}.$$

Then, the remaining resources $i \in \mathcal{I}_{AL}$ are added analogue to Appendix B.1, Step 2.

It remains to show that $\widehat{\widehat{P}}_{NSEP}$ is equivalent to $\widehat{P}'_{NSEP}$. For ease of exposition, we assume $p_{t,j} > 0, \forall t, j$.

Given a feasible solution $\tilde{\rho}, \tilde{\varrho}, \tilde{\mu}, \tilde{\xi}$ to $\widehat{\widehat{P}}_{NSEP}$, we define

$$\rho_{t,i} := \sum_j \tilde{\rho}_{t,i,j} \qquad \varrho_{t,n,\mathbf{r}_n} := \sum_j \tilde{\varrho}_{t,n,j,\mathbf{r}_n}$$
$$\xi_{t,n,j,\mathbf{r}_n,u} := \frac{\tilde{\xi}_{t,n,j,\mathbf{r}_n,u}}{p_{t,j}} \qquad \mu_{t,j,u} := \frac{\tilde{\mu}_{t,j,u}}{p_{t,j}}.$$

We argue that ρ, ϱ, ξ, μ is feasible to $\widehat{P}'_{NSEP}$ with equal objective value: Summing (D.1)–(D.3) over $j = 1, \ldots, J$, we directly obtain (9)–(11). (13) follows from (D.5) by definition. If we can show that $\tilde{\varrho}_{t,n,j,\mathbf{r}_n} = p_{t,j}\varrho_{t,n,\mathbf{r}_n}$ and $\tilde{\rho}_{t,i,j} = p_{t,j}\rho_{t,i}$, then (12) and (16) follow from (D.4) and (D.6). To show $\tilde{\varrho}_{t,n,j,\mathbf{r}_n} = p_{t,j}\varrho_{t,n,\mathbf{r}_n}$, we use constraint (D.1):

$$\tilde{\varrho}_{t,n,j,\mathbf{s}_n} = p_{t,j} \sum_{j'} \sum_{\substack{\mathbf{r}_n \in \mathfrak{R}_{t-1,n}, u \in \mathcal{U}_{j',\mathbf{r}_n} \\ \text{s.t. } \mathbf{r}_n - u\mathbf{a}_{n,j'} = \mathbf{s}_n}} \tilde{\xi}_{t-1,n,j',\mathbf{r}_n,u} \tag{D.7}$$

$$\overset{\sum_j}{\Rightarrow} \quad \varrho_{t,n,\mathbf{s}_n} = \sum_{j'} \sum_{\substack{\mathbf{r}_n \in \mathfrak{R}_{t-1,n}, u \in \mathcal{U}_{j',\mathbf{r}_n} \\ \text{s.t. } \mathbf{r}_n - u\mathbf{a}_{n,j'} = \mathbf{s}_n}} \tilde{\xi}_{t-1,n,j',\mathbf{r}_n,u}. \tag{D.8}$$

Inserting (D.8) into (D.7) produces $\tilde{\varrho}_{t,n,j,\mathbf{s}_n} = p_{t,j}\varrho_{t,n,\mathbf{s}_n}$. The proof for $\tilde{\rho}_{t,i,j} = p_{t,j}\rho_{t,i}$ is done similarly using (D.2).

On the other hand, given a feasible solution ρ, ϱ, μ, ξ to $\widehat{P}'_{NSEP}$, a feasible solution to $\widehat{\widehat{P}}_{NSEP}$ with equal objective value is given by

$$\tilde{\rho}_{t,i,j} := p_{t,j}\rho_{t,i} \qquad \tilde{\varrho}_{t,n,j,\mathbf{r}_n} := p_{t,j}\varrho_{t,n,\mathbf{r}_n}$$
$$\tilde{\xi}_{t,n,j,\mathbf{r}_n,u} := p_{t,j}\xi_{t,n,j,\mathbf{r}_n,u} \qquad \tilde{\mu}_{t,j,u} := p_{t,j}\mu_{t,j,u},$$

which follows by multiplying the constraints of $\widehat{P}'_{NSEP}$ with $p_{t,j}$.

Appendix E. Proof of Lemma 2

The inequalities $Z \le Z_{NSEP} \le Z'_{NSEP}$ have already been discussed in the text. It remains to show that $Z'_{NSEP} \le Z_{AL}$ and $Z'_{NSEP} \le Z_{SPL}$ (if $\mathcal{I}_{AL} = \emptyset$).

Let (P'_{AL}) and (P'_{SPL}) be the duals of the linear programs we obtain by inserting the approximations AL and SPL into D'. Since AL and SPL are special cases of $NSEP$, we obtain reductions $\widehat{P}'_{AL}$ and $\widehat{P}'_{SPL}$ by Theorem 2. It can be verified that for the special cases AL and SPL, the linear programs $\widehat{P}'_{AL}$ and $\widehat{P}'_{SPL}$ coincide with the pre-decision reductions $\widehat{P}_{AL}$ and $\widehat{P}_{SPL}$.

Let Z'_{AL} and Z'_{SPL} be the optimal values of $\widehat{P}'_{AL}$ and $\widehat{P}'_{SPL}$. By the above discussion, we have that

$$Z'_{AL} = Z_{AL}, \quad Z'_{SPL} = Z_{SPL}. \tag{E.1}$$

Moreover, analogue to the inequalities $Z_{NSEP} \le Z_{AL}$ and $Z_{NSEP} \le Z_{SPL}$ (if $\mathcal{I}_{AL} = \emptyset$), we have the corresponding post-decision inequalities

$$Z'_{NSEP} \le Z'_{AL} \quad \text{and} \quad Z'_{NSEP} \le Z'_{SPL} \text{ (if } \mathcal{I}_{AL} = \emptyset), \tag{E.2}$$

which follow from the fact that enlarging the function space spanned by the basis functions tightens the upper bound. Combining (E.1) and (E.2) yields the desired inequalities $Z'_{NSEP} \le Z_{AL}$ and $Z'_{NSEP} \le Z_{SPL}$.

Appendix F. Proof of Theorem 3

All inequalities except $Z_{NSEP} \le Z^H_{NSEP} \le Z'_{NSEP}$ are covered by Lemma 2.

The inequality $Z^H_{NSEP} \le Z'_{NSEP}$ follows from the fact that H_{NSEP} has the same objective function as $\widehat{P}'_{NSEP}$ and contains the constraints of $\widehat{P}'_{NSEP}$.

The proof to show $Z_{NSEP} \le Z^H_{NSEP}$ is practically identical to the proof of Part (ii) of Lemma 1, see Appendix B.2, and the corresponding part of Theorem 1: Given a feasible solution X to P_{NSEP}, we construct a feasible solution $\rho, \varrho, \xi, \mu, \Upsilon$ to H_{NSEP} with equal objective value. The only part which is not covered by Appendix B.2 is the verification of the additional constraint (17), which is done similarly to the verification of (15):

$$\begin{aligned}
\sum_{\mathbf{r}_n \in \mathfrak{R}_{t,n}} \Upsilon_{t,n,i,\mathbf{r}_n} &= \sum_{\mathbf{r}_n \in \mathfrak{R}_{t,n}} \max_j \sum_{u \in \mathcal{U}_{j,\mathbf{r}_n}} a_{ij} u \xi_{t,n,j,\mathbf{r}_n,u} \\
&= \sum_{\mathbf{r}_n \in \mathfrak{R}_{t,n}} \max_j \sum_{u \in \mathcal{U}_{j,\mathbf{r}_n}} \sum_{\substack{\mathbf{r}' \in \mathfrak{R}_t, \mathbf{u} \in \mathfrak{U}_{\mathbf{r}'} \\ \text{s.t. } \mathbf{r}'_n = \mathbf{r}_n, u_j = u}} \underbrace{a_{ij} u}_{\le \min(r'_i, 1)} X_{t,\mathbf{r}',\mathbf{u}} \\
&\le \sum_{\mathbf{r}_n \in \mathfrak{R}_{t,n}} \max_j \sum_{\substack{\mathbf{r}' \in \mathfrak{R}_t, \mathbf{u} \in \mathfrak{U}_{\mathbf{r}'} \\ \text{s.t. } \mathbf{r}'_n = \mathbf{r}_n}} \min(r'_i, 1) X_{t,\mathbf{r}',\mathbf{u}} \\
&\le \sum_{\substack{\mathbf{r}' \in \mathfrak{R}_t, \mathbf{u} \in \mathfrak{U}_{\mathbf{r}'} \\ \text{s.t. } r'_i \ge 1}} X_{t,\mathbf{r}',\mathbf{u}} = \sum_{\substack{\mathbf{r}_{n'} \in \mathfrak{R}_{t,n'} \\ \text{s.t. } r_i \ge 1}} \varrho_{t,n',\mathbf{r}_{n'}}.
\end{aligned}$$

Appendix G. Proof of Theorem 4

As noted in Section 4, constraint (11) is redundant. For the sake of simplicity, we therefore omit it in this proof.

We denote by $W, V, \alpha, \beta, \gamma, \delta, \epsilon$ the dual variables of (9), (10), (12), (13), (14), (15), (17). The dual of H_{NSEP} is given by

$$D^H_{NSEP}: \quad \min_{W,V,\alpha,\beta,\gamma,\delta,\epsilon} \sum_n W_{1,n,\mathbf{c}_n} + \sum_{i \in \mathcal{I}_{AL}} V_{1,i} c_i$$

$$\text{s.t.} \quad W_{t,n,\mathbf{r}_n} - \sum_j \alpha_{t,n,j,\mathbf{r}_n} - \sum_{i \in \mathcal{I}_n} \sum_{n'} 1_{\{n' \ne n\}} 1_{\{r_i \ge 1\}} \epsilon_{t,n',n,i} \ge 0, \qquad \forall t, n, \mathbf{r}_n \tag{G.1}$$

$$\alpha_{t,n,j,\mathbf{r}_n} + \beta_{t,n,j,u} + \sum_{i\in\mathcal{I}\setminus\mathcal{I}_n} a_{ij}u\gamma_{t,n,i,j,\mathbf{r}_n} - p_{t,j}W_{t+1,n,\mathbf{r}_n-u\mathbf{a}_{n,j}} \geq 0, \quad \forall t,n,j,\mathbf{r}_n,u \tag{G.2}$$

$$-\sum_n \beta_{t,n,j,u} + \sum_{i\in\mathcal{I}_{AL}} p_{t,j}a_{ij}uV_{t+1,i} \geq p_{t,j}f_{t,j}u, \qquad \forall t,j,u \tag{G.3}$$

$$V_{t,i} - V_{t+1,i} - \sum_n \delta_{t,n,i} \geq 0, \qquad \forall t, i\in\mathcal{I}_{AL} \tag{G.4}$$

$$-\sum_j \gamma_{t,n,i,j,\mathbf{r}_n} + 1_{\{i\in\mathcal{I}_{AL}\}}\delta_{t,n,i} + \sum_{n'} 1_{\{i\in\mathcal{I}_{n'}\}}\epsilon_{t,n,n',i} \geq 0, \quad \forall t,n,\mathbf{r}_n, i\in\mathcal{I}\setminus\mathcal{I}_n \tag{G.5}$$

$$W_{T+1} = V_{T+1} = 0$$
$$\gamma,\delta,\epsilon \geq 0.$$

We prove that there exists an optimal solution satisfying the properties stated in Theorem 4 in three steps. In Step 1, we establish the auxiliary property $\beta_{t,n,j,0} \leq 0$ which is useful for the remainder of the proof. In Step 2, we construct an optimal solution satisfying $W_{t,n,\mathbf{r}_n} \leq W_{t,n,\mathbf{r}_n+\mathbf{e}}$. Finally, in Step 3, we prove the properties $W_{t,n,\mathbf{r}_n} \geq W_{t+1,n,\mathbf{r}_n}$ and $V_{t,i} \geq V_{t+1,i}$.

Step 1: Let $W^*, V^*, \alpha^*, \beta^*, \gamma^*, \delta^*, \epsilon^*$ be an optimal solution to D^H_{NSEP}. Based on this, we construct another optimal solution $\tilde{W}, V^*, \tilde{\alpha}, \tilde{\beta}, \gamma^*, \delta^*, \epsilon^*$ which satisfies $\tilde{\beta}_{t,n,j,0} \leq 0$.

To pave the way for defining $\tilde{\beta}$, we define $\overline{\beta}_{t,n,j} := \beta^*_{t,n,j,0} - \frac{1}{N}\sum_{n'}\beta^*_{t,n',j,0}$. Setting $u=0$ in (G.3), we see that $\sum_n \beta^*_{t,n,j,0} \leq 0, \forall j$. This implies $\sum_n \overline{\beta}_{t,n,j} = 0, \forall j$, and $\beta^*_{t,n,j,0} - \overline{\beta}_{t,n,j} \leq 0, \forall n,j$.

We now iteratively construct $\tilde{W},\tilde{\alpha},\tilde{\beta}$. We initialize $\tilde{W}^{(T+1)} = W^*, \tilde{\alpha}^{(T+1)} = \alpha^*, \tilde{\beta}^{(T+1)} = \beta^*$ and then iterate over $t' = T,\dots,1$. At each stage t', we define

$$\tilde{\beta}^{(t')}_{t,n,j,u} := \begin{cases} \tilde{\beta}^{(t'+1)}_{t,n,j,u}, & \text{if } t \neq t', \\ \tilde{\beta}^{(t'+1)}_{t,n,j,u} - \overline{\beta}_{t',n,j}, & \text{if } t = t', \end{cases}$$

$$\tilde{\alpha}^{(t')}_{t,n,j,\mathbf{r}_n} := \begin{cases} \tilde{\alpha}^{(t'+1)}_{t,n,j,\mathbf{r}_n}, & \text{if } t > t', \\ \tilde{\alpha}^{(t'+1)}_{t,n,j,\mathbf{r}_n} + \overline{\beta}_{t',n,j}, & \text{if } t = t', \\ \tilde{\alpha}^{(t'+1)}_{t,n,j,\mathbf{r}_n} + p_{t,j}\sum_{j'}\overline{\beta}_{t',n,j'}, & \text{if } t < t', \end{cases}$$

$$\tilde{W}^{(t')}_{t,n,\mathbf{r}_n} := \begin{cases} \tilde{W}^{(t'+1)}_{t,n,\mathbf{r}_n}, & \text{if } t > t', \\ \tilde{W}^{(t'+1)}_{t,n,\mathbf{r}_n} + \sum_j \overline{\beta}_{t',n,j}, & \text{if } t \leq t'. \end{cases}$$

We verify by induction that at each stage t', the solution $\tilde{W}^{(t')}, V^*, \tilde{\alpha}^{(t')}, \tilde{\beta}^{(t')}, \gamma^*, \delta^*, \epsilon^*$ is feasible: Assume that $\tilde{W}^{(t'+1)}, V^*, \tilde{\alpha}^{(t'+1)}, \tilde{\beta}^{(t'+1)}, \gamma^*, \delta^*, \epsilon^*$ is feasible. Then, the values of $\tilde{W}^{(t')}, \tilde{\alpha}^{(t')}, \tilde{\beta}^{(t')}$ only differ from $\tilde{W}^{(t'+1)}, \tilde{\alpha}^{(t'+1)}, \tilde{\beta}^{(t'+1)}$ for $t \leq t'$, so we have to verify the constraints (G.1)–(G.3) for $t \leq t'$.

- (G.1) follows from the observation that $\forall t \leq t' : \tilde{W}^{(t')}_{t,n,\mathbf{r}_n} - \sum_j \tilde{\alpha}^{(t')}_{t,n,j,\mathbf{r}_n} = \tilde{W}^{(t'+1)}_{t,n,\mathbf{r}_n} - \sum_j \tilde{\alpha}^{(t'+1)}_{t,n,j,\mathbf{r}_n}$.
- (G.2) follows from the observation that for $t = t'$, we have $\tilde{\alpha}^{(t')}_{t,n,j,\mathbf{r}_n} + \tilde{\beta}^{(t')}_{t,n,j,u} = \tilde{\alpha}^{(t'+1)}_{t,n,j,\mathbf{r}_n} + \tilde{\beta}^{(t'+1)}_{t,n,j,u}$, and for $t < t'$, we have $\tilde{\alpha}^{(t')}_{t,n,j,\mathbf{r}_n} - p_{t,j}\tilde{W}^{(t')}_{t+1,n,\mathbf{r}_n-u\mathbf{a}_{n,j}} = \tilde{\alpha}^{(t'+1)}_{t,n,j,\mathbf{r}_n} - p_{t,j}\tilde{W}^{(t'+1)}_{t+1,n,\mathbf{r}_n-u\mathbf{a}_{n,j}}$.
- (G.3) follows from $\sum_n \overline{\beta}_{t,n,j} = 0$.

Finally, we set $\tilde{W} := \tilde{W}^{(1)}, \tilde{\alpha} := \tilde{\alpha}^{(1)}, \tilde{\beta} := \tilde{\beta}^{(1)}$. From the above discussion and by construction, $\tilde{\beta}$ satisfies $\tilde{\beta}_{t,n,j,0} \leq 0$. Also, $\sum_n \overline{\beta}_{t,n,j} = 0$ implies that $\tilde{W}, V^*$ provides the same objective value as W^*, V^*.

Step 2: Let $W^*, V^*, \alpha^*, \beta^*, \gamma^*, \delta^*, \epsilon^*$ be an optimal solution which satisfies $\beta^*_{t,n,j,0} \leq 0$ (see Step 1). Based on this, we construct another optimal solution $\tilde{W}, V^*, \tilde{\alpha}, \beta^*, \tilde{\gamma}, \delta^*, \epsilon^*$ which now also satisfies $\tilde{W}_{t,n,\mathbf{r}_n} \leq \tilde{W}_{t,n,\mathbf{r}_n+\mathbf{e}}$.

In the following, it is useful to classify the capacity vectors $\mathbf{r}_n$ by the sum of their entries, i.e., we define $\mathcal{W}_{n,s} := \{\mathbf{r}_n \in \mathfrak{R}_n \mid \sum_{i\in\mathcal{I}_n} r_i = s\}$ where $s = 0,1,\dots,s_{max} = \sum_{i\in\mathcal{I}_n} c_i$. We observe that two vectors $\mathbf{r}_n$ and $\mathbf{r}_n + \mathbf{e}$ always belong to different sets $\mathcal{W}_{n,s}$.

We again iteratively construct $\tilde{W},\tilde{\alpha},\tilde{\gamma}$. We iterate over time $t' = T,\dots,1$ and, for each t', we make a subiteration over the sum of entries $s' = s_{max},\dots,1,0$. At each stage (t',s'), we make sure that $\tilde{W}$ satisfies $\tilde{W}_{t,n,\mathbf{r}_n} \leq \tilde{W}_{t,n,\mathbf{r}_n+\mathbf{e}}, \forall \mathbf{e}\in\mathfrak{E}_n$. This is achieved by setting $\forall \mathbf{r}_n \in \mathcal{W}_{n,s'}$:

$$\left(\tilde{W}_{t',n,\mathbf{r}_n}, \tilde{\alpha}_{t',n,j,\mathbf{r}_n}, \tilde{\gamma}_{t',n,i,j,\mathbf{r}_n}\right) := \begin{cases} \left(W^*_{t',n,\mathbf{r}_n}, \alpha^*_{t',n,j,\mathbf{r}_n}, \gamma^*_{t',n,i,j,\mathbf{r}_n}\right), & \text{if } W^*_{t',n,\mathbf{r}_n} \leq \tilde{W}_{t',n,\mathbf{r}_n+\mathbf{e}_\star} \text{ or } s' = s_{max}, \\ \left(\tilde{W}_{t',n,\mathbf{r}_n+\mathbf{e}_\star}, \tilde{\alpha}_{t',n,j,\mathbf{r}_n+\mathbf{e}_\star}, \tilde{\gamma}_{t',n,i,j,\mathbf{r}_n+\mathbf{e}_\star}\right), & \text{otherwise,} \end{cases}$$

where $\mathbf{e}_\star := \arg\min_{\mathbf{e}\in\mathfrak{E}_n} \tilde{W}_{t',n,\mathbf{r}_n+\mathbf{e}}$.

To show that these changes still result in a feasible solution, we introduce variables $\tilde{W}^{(t',s')}, \tilde{\alpha}^{(t',s')}, \tilde{\gamma}^{(t',s')}$ which coincide

- with $\tilde{W},\tilde{\alpha},\tilde{\gamma}$ for $t > t'$ and $t = t'$, $\mathbf{r}_n \in \mathcal{W}_{n,s}$, $s \geq s'$, and
- with W^*,α^*,γ^* for $t < t'$ and $t = t'$, $\mathbf{r}_n \in \mathcal{W}_{n,s}$, $s < s'$.

Note that the values of $\tilde{W}^{(t',s')}, \tilde{\alpha}^{(t',s')}, \tilde{\gamma}^{(t',s')}$ only differ from $\tilde{W}^{(t',s'+1)}, \tilde{\alpha}^{(t',s'+1)}, \tilde{\gamma}^{(t',s'+1)}$ for $t = t'$ and $\mathbf{r}_n \in \mathcal{W}_{n,s'}$, and the values of $\tilde{W}^{(t',s_{max})}, \tilde{\alpha}^{(t',s_{max})}, \tilde{\gamma}^{(t',s_{max})}$ are identical to the values of $\tilde{W}^{(t'+1,0)}, \tilde{\alpha}^{(t'+1,0)}, \tilde{\gamma}^{(t'+1,0)}$.

We show by induction that $\tilde{W}^{(t',s')}, V^*, \tilde{\alpha}^{(t',s')}, \beta^*, \tilde{\gamma}^{(t',s')}, \delta^*, \epsilon^*$ is feasible. Assume that $\tilde{W}^{(t',s'+1)}, V^*, \tilde{\alpha}^{(t',s'+1)}, \beta^*, \tilde{\gamma}^{(t',s'+1)}, \delta^*, \epsilon^*$ is feasible. We only have to check that $\tilde{W}^{(t',s')}, V^*, \tilde{\alpha}^{(t',s')}, \beta^*, \tilde{\gamma}^{(t',s')}, \delta^*, \epsilon^*$ satisfies the constraints (G.1) and (G.5) for $t = t'$ and the constraint (G.2) for $t = t'$ and $t = t'-1$.

- The verification of (G.5) is trivial.
- (G.1) follows by observing $-1_{\{r_i\geq 1\}} \geq -1_{\{r_i+e_{\star,i}\geq 1\}}$, and so (G.1) is valid for $\mathbf{r}_n \in \mathcal{W}_{n,s'}$ because (G.1) is valid for $\mathbf{r}_n \in \mathcal{W}_{n,s'+1}$ by the induction hypothesis.
- (G.2) for $t = t'-1$ follows from the induction hypothesis and the inequality $-\tilde{W}^{(t',s')}_{t+1,n,\mathbf{r}_n-u\mathbf{a}_{n,j}} \geq -\tilde{W}^{(t',s'+1)}_{t+1,n,\mathbf{r}_n-u\mathbf{a}_{n,j}}$ which is true by construction.
- (G.2) for $t = t'$ follows by observing that by the induction hypothesis, we have $-\tilde{W}^{(t',s')}_{t+1,n,\mathbf{r}_n-u\mathbf{a}_{n,j}} \geq -\tilde{W}^{(t',s')}_{t+1,n,\mathbf{r}_n+\mathbf{e}_\star-u\mathbf{a}_{n,j}}$, and so (G.2) is valid for $\mathbf{r}_n \in \mathcal{W}_{n,s'}$ because (G.2) is valid for $\mathbf{r}_n \in \mathcal{W}_{n,s'+1}$ by the induction hypothesis.

At the end of the iteration, it holds that $(\tilde{W}^{(1,0)}, \tilde{\alpha}^{(1,0)}, \tilde{\gamma}^{(1,0)}) = (\tilde{W}, \tilde{\beta}, \tilde{\gamma})$, so we conclude that $\tilde{W}, V^*, \tilde{\alpha}, \beta^*, \tilde{\gamma}, \delta^*, \epsilon^*$ is feasible. By construction, $\tilde{W}$ satisfies $\tilde{W}_{t,n,\mathbf{r}_n} \leq \tilde{W}_{t,n,\mathbf{r}_n+\mathbf{e}_\star} \leq \tilde{W}_{t,n,\mathbf{r}_n+\mathbf{e}}, \forall \mathbf{e}\in\mathfrak{E}_n$. Finally, since $\tilde{W}_{1,n,\mathbf{c}_n} = W^*_{1,n,\mathbf{c}_n}$, $\tilde{W}, V^*$ provides the same objective value as W^*, V^*.

Step 3: Let $W^*, V^*, \alpha^*, \beta^*, \gamma^*, \delta^*, \epsilon^*$ be an optimal solution which satisfies $\beta^*_{t,n,j,0} \leq 0$ and $W^*_{t,n,\mathbf{r}_n} \leq W^*_{t,n,\mathbf{r}_n+\mathbf{e}}$ (see Step 1 and 2). Setting $u=0$ in (G.2) implies $\alpha^*_{t,n,j,\mathbf{r}_n} \geq p_{t,j}W^*_{t+1,n,\mathbf{r}_n}$. Inserting this into (G.1) yields $W^*_{t,n,\mathbf{r}_n} \geq W^*_{t+1,n,\mathbf{r}_n}$ by the nonnegativity of ϵ^*. Finally, the property $V^*_{t,i} \geq V^*_{t+1,i}$ follows directly from (G.4) by the nonnegativity of δ^*.

Appendix H. Parameters used in the realistic example

The probabilities $p_{t,j}$ were estimated using data from 2394 actual bookings of a given month in 2018. The time intervals are chosen such that the non-arrival probability per time step is equal to 0.2. Actual fares have been scaled to comply with company policies and were transformed following Walczak, Mardan, & Kallesen (2010). The transformed prices $f_{t,j}$ as well as the probabilities $p_{t,j}$ can be found in the following GitHub repository: https://github.com/slaume/Reductions-of-Non-Separable-ALPs-for-NRM. AMPL code necessary to reproduce our results can also be found in the same repository.

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