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Debt Sustainability with Involuntary Default

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June 2, 2022

Abstract

This paper studies public debt sustainability under the assumption that a country always tries to service its debt obligations. We assume that default decreases the level of resources available for debt service, which consist of the country's primary surplus and the proceeds from new debt issuance. We show that our model encompasses the well-known result that, as long as $r < g$, countries permanently can run small deficits. In our model, this result holds if there is no decrease in resource availability following default. We thus show that, in the presence of involuntary default, a lesser decrease in resource availability in default–a lower cost of default–increases maximum sustainable debt. This is the opposite of what it is normally found in models that assume limited commitment and strategic default. We calibrate our model using data for the Eurozone and find that many countries have actual debt levels that are higher than their maximum sustainable debt. In discussing possible reasons for these high observed debt levels, we emphasize the role of expected GDP growth, growth volatility, and resource availability. We also model the role of the European Stability Mechanism (ESM). We show that while the ESM can increase the level of maximum sustainable debt, it also crowds out private lending.

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Introduction

The sustainability of sovereign debt in industrialized countries has received much attention in recent years. This is as it should be, in view of the explosion in sovereign debt to levels previously reached only in time of war. Thus, at the end of 2019, sovereign debt levels were 238% of GDP in Japan, 185% in Greece, 135% in Italy, 109% in the United States, 98% in France, and 85% in the United Kingdom.[1] It is natural to ask whether such high debt levels are sustainable, in the sense of not relying for repayment on ever increasing borrowing reminiscent of a Ponzi scheme, with naturally attending ever increasing probability of default.[2]

We seek to determine a country's maximum sustainable debt. For that purpose, we assume that the country's government does its utmost to service its debt. This assumption is consistent with our focus on *maximum* debt: lenders naturally lend more to those they expect to strive to avoid default than to those they fear may strategically default on the debt.[3] We characterize maximum sustainable debt as the fixed point that equates *(i)* debt beyond which default occurs and *(ii)* the maximum level of resources available for debt service. These resources in turn are the sum of *(ii–a)* the maximum primary surplus that the government can achieve, that which maximizes revenues and minimizes expenses to those strictly necessary to the functioning of the State, and *(ii–b)* the maximum proceeds that can be had from new debt issuance.

New debt proceeds naturally depend on the probability of default associated with the level of newly issued debt, that is, they depend on the level of new debt beyond which default will occur. There is therefore a recursive equation, which relates present and future debt beyond which default occurs. That debt is *maximum* as it equals the maximum level of resources available for debt service. It is *sustainable* only if future debt remains finite: debt that relies for repayment on the issuance of infinite debt, even if only far into the future, cannot be considered sustainable. When both the risk-free interest rate and the rate of growth in GDP are stationary, maximum sustainable debt is the fixed point of the equation that equates the maximum debt that can be owed to the maximum resources that are available to service such debt, where these resources in turn depend on the maximum debt that can be issued, and consequently owed. We derive a necessary and sufficient condition for a fixed point to exist.[4] When that condition is true, no debt is sustainable that exceeds the fixed point; this motivates our characterization of that point as maximum sustainable debt (MSD). When the condition is false, a case that recalls Blanchard's (2019) famous $g > r$ condition, growth can be counted upon to resorb large temporary increases in debt, or to make permanent small deficits feasible.

Our condition differs from Blanchard's (2019) in that it explicitly allows for default: default disrupts the process of debt rollover, which is central to debt service and repayment. Lenders provide less new debt in default; they thereby constrain the extent to which future primary surpluses can be relied upon for debt service. We show this constraint effectively to amount to

[1] Our analysis excludes the recent covid crisis.

[2] We assume an independent central bank that prevents the government from resorting to inflation for servicing its debt

[3] We further justify this assumption below.

[4] When the risk-free interest rate is stochastic rather than constant, the corresponding condition is sufficient only.

a reduction in the growth rate to be compared to the interest rate in Blanchard's (2019) $g > r$ condition. Default tightens that condition by replacing the growth rate g by a rate γ made lower by constrained reliance on future primary surpluses: $\gamma < g$. We find that, whilst Blanchard's $g > r$ condition is satisfied by all but two of the twelve Eurozone countries we consider, namely Greece and Italy, the corresponding $\gamma > r$ condition is satisfied by none:[5] accounting for the possibility and indeed the reality of government default makes for a less sanguine assessment of sovereign debt sustainability.[6] This is so because failure of the $\gamma > r$ condition implies the existence of finite MSD: growth will neither resorb large temporary increases in debt nor make permanent small deficits feasible if these should increase a country's debt above that country's MSD. The MSD we estimate for the Eurozone countries we consider are, in the case of Greece and Italy at least, well below these two countries' actual debt levels. When using data for the twenty-year period 1999-2019, thereby including the financial crisis but excluding the most recent Covid period, we estimate Greece's MSD at 49% of GDP and Italy's at 88%. We further show that accounting for the stochastic nature of the risk-free rate lowers MSD estimates, which become 45% for Greece and 66% for Italy.

The finding of low MSD for Greece and Italy begs the question of what makes these two countries' present high debt levels feasible. One possible explanation relies on the central role of growth in determining MSD: MSD is higher for higher growth; it is lower for more volatile growth. Using data for the period 1999-2007 only, that is, excluding the financial crisis and its aftermath, we estimate Greece and Italy's MSD to be 197% and 134%, respectively, for constant risk-free rate, and 155% and 110% for stochastic risk-free rate. The financial crisis and its aftermath saw markedly lower and more volatile growth.

Another explanation is the extent of refinancing in default: the larger the fraction of resources available for refinancing, the more debt financing effectively resembles equity financing, and the higher is maximum sustainable debt. The recovery rate we use in our baseline estimations is the 30% received by the private sector holders of the restructured Greek Government bonds.[7,8] For a hypothetical average Eurozone country, doubling the recovery rate to 60% increases MSD from 86% to 96% of GDP.[9] Yet another explanation is the risk-free rate, which discounts future resources available for debt service. For the average Eurozone country, decreasing the risk-free rate from 1.09% to zero percent increases MSD from 86% to 108% of GDP.[10] The final explanation we consider is support by institutions such as the European Stability Mechanism (ESM). These provide financing at subsidized rates, thereby making possible an in-

[5]The other ten Eurozone countries we consider are Austria, Belgium, Finland, France, Germany, Ireland, Luxembourg, Netherlands, Portugal, and Spain. All twelve countries' debt-to-GDP ratios at end 2019 are shown in Table 2.

[6]We are of course not the first ones to do so; we discuss related work in Section 1.

[7]The corresponding haircut of 70% is the average of the face value haircut (c. 75%) and the market value haircut (c. 65%). See Gulati, Trebesch, and Zettelmeyer (2013).

[8]The term 'recovery rate' is somewhat of a misnomer, for it applies not to the face value of the debt due but to the fraction of resources–primary surplus and proceeds from new debt issuance–available for debt service. We use the term recovery rate for simplicity.

[9]The growth rate of the average Eurozone country is obtained by weighting the growth rates of the twelve countries we consider by these countries' GDPs.

[10]Zero percent is the yield on two-year German Government bonds at the time of writing.

crease in maximum sustainable debt.[11] For the average Eurozone country, having the ESM hold debt that amounts to 50% of the country's GDP increases MSD to 108% of GDP, when ESM financing is interest-free. Both results suggest a lesser role for the recovery rate and the ESM than for growth in increasing MSD. An interesting result of our analysis is that ESM financing displaces private sector financing: ESM financing increases maximum sustainable debt, but it also decreases maximum sustainable borrowing from the private sector.[12] The increase in MSD from 86% to 108% of GDP made possible by ESM financing is accompanied by a 28 percentage point decrease in the debt due to the private sector, from 86% to 58% of GDP. Another result is that ESM financing can be viewed as being in the nature of a negative NPV project, in that the present value of the subsidy the ESM provides a borrowing country is higher than the increase in borrowing proceeds that ESM financing provides. This is because ESM financing increases the probability of default at maximum sustainable debt; such increase is undesirable as default decreases the level of resources available for refinancing, thereby decreasing the debt that can be raised from lenders.

Default in our model is involuntary: governments do their utmost to avoid default. How realistic is that assumption? Very! Levy Yeyati and Panizza (2011) report numerous instances of government reluctance to default: governments appear to default only as a last resort, after they have tried every possible way of staving default off. Defaults delayed well past the point at which throwing in the towel would have been optimal have prompted the IMF (2013, p.1) to comment that "debt restructurings have often been too little and too late." While debt service is costly, default can be even costlier, especially from the point of view of a government that can expect to lose power in the aftermath of default (Borensztein and Panizza (2009); Malone (2011)), and whose members' prospects for alternative employment would be jeopardized were they deemed too prone to default. Even a less than fully self-interested government may do its utmost to avoid default: Grossman and van Huyck (1988) and Tomz (2007) have argued that creditors are much more lenient towards borrowers for whom default was clearly unavoidable than those who are perceived to have been too quick to default; Bolton and Jeanne (2011) and Gennaioli, Martin, and Rossi (2014, 2018) have noted the potential of sovereign default to jeopardize the proper functioning of an entire banking system, in view of government bonds' importance as collateral for bank loans.

The paper proceeds as follows. Section 1 reviews the literature. Section 2 considers the case of constant risk-free rate; it derives our condition for sustainability and the expression for MSD; it relates maximum sustainable borrowing proceeds to the present value of future maximum primary surpluses. Section 3 considers the case of stochastic risk-free rate; it extends the condition for sustainability and the expression for MSD to that case. Section 4 considers the role of the ESM. Section 5 presents the data and Section 6 the results. Section concludes. The Appendix contains the proofs.

[11]We need not assume that the ESM has infinite resources, only that these be such that the ESM can offer subsidized–lower than market–interest rates.

[12]We identify a condition for the reverse to be true, but such condition rarely can be expected to hold.

1 Literature review

There is an immense literature on sovereign debt sustainability, to which the present literature review cannot but fail to do justice. Recent surveys are those of D'Erasmo, Mendoza, and Zhang (2016), Debrun, Ostry, Willems, and Wyplosz (2019), and Mitchener and Trebesch (2021). Here, we limit ourselves to discussing those papers directly related to our own: our purpose is to place our work in the context of previous work.

Central to work on sustainable government debt has been the intertemporal government budget constraint (IGBC), which relates present and future debt, the growth and interest rates, and a country's primary balance. Sustainable debt has generally been viewed as the fixed point of the IGBC, and sustainability conditions correspondingly have been viewed as those that ensure the existence of such fixed point.[13] Although Bohn (2007) has shown that debt diverging to infinity may nonetheless satisfy the IGBC, he implicitly assumes that the primary balance may grow unbounded if necessary, a rather implausible assumption.

Our derivation of maximum sustainable debt follows that previous work. The equation we consider, however, relates not actual but maximum debt: we derive a recursive equation which, as noted in the Introduction, equates present maximum debt to maximum resources available, which depend on the primary balance as well as the proceeds from the issuance of new debt, themselves a function of future maximum debt. As does the IGBC, our recursive equation includes the interest and growth rates, the former used to discount future maximum debt and the latter reflecting the expression of that debt as a fraction of future GDP. As do Ghosh, Kim, Mendoza, Ostry, and Qureshi (2013, GKMOQ) and Collard, Habib, and Rochet (2015, CHR), and unlike Tanner (2013) or Blanchard (2019) for example, we explicitly account for the possibility of default that arises from the nature of government liabilities, specifically debt. We differ from GKMOQ and CHR in allowing for refinancing in default, and from GKMOQ in assuming a constant maximum primary balance; in contrast, GKMOQ's primary balance is estimated along the lines of Bohn's (1998) fiscal reaction function.[14]

We have noted the relation of our work to Blanchard's (2019). Recent work has argued that the proper discount rate to be compared to the growth rate is not the risk free rate considered by Blanchard (2019), but a rate that includes a risk premium as compensation for output risk. This is because government debt is the present value of future primary surpluses, which are correlated with output, indeed are made riskier than output by the stronger procyclicality of taxes than of government spending. De Vette, Olijslagers, and Van Wijnbergen (2020) find that the discounted value of Dutch primary surpluses is roughly half the market value of Dutch government debt. They examine, and dismiss, a number of potential explanations for the gap between present and market values. The one explanation they do not dismiss is that of future fiscal adjustments that would increase the primary surplus; these range from 1.20% of GDP under partial equilibrium, that is, neglecting the effect of the adjustment in decreasing the

[13] See for example the discussion in Section 2.1 of D'Erasmo, Mendoza, and Zhang (2016).

[14] A fiscal reaction function relates a country's primary balance to its past debt. Bohn (1998) shows that a sufficient condition for debt sustainability is that the coefficient of past debt in the reaction function be strictly positive.

risk premium, to 0.28% of GDP under general equilibrium. Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) find that the PV of future surpluses in the US is *negative* 155% of GDP, in contrast to government debt that has market value equal to 37% of GDP. The surprising result that the PV of future surpluses is negative can be explained as follows. Surpluses are the difference between taxes and government spending. Both are procyclical at medium to long horizons, but the former more so than the latter; indeed, government spending is countercyclical at short horizons. This implies that the risk premium for taxes is higher than the risk premium for government spending. As taxes and spending are more or less equal, the PV of taxes is lower than that of spending, making the PV of primary surpluses negative. As do De Vette, Olijslagers, and Van Wijnbergen (2020), Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) consider a number of potential explanations for the gap between present and market values; none appears to be able to close the gap between values if considered in isolation. For example, Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) estimate that a spending cut of 7.7% of GDP would have to occur with a probability of 42% in order to reconcile the present value of future surpluses with the market value of the debt; they deem such probability unrealistic.

In contrast to such work, ours uses the risk-free rate to discount future primary surpluses: although not all Eurozone countries can be considered small open economies, most can; investors can diversify the risk presented by fluctuations in these countries' primary surpluses to an extent they cannot that presented by the US primary surplus. We share the use of the risk-free rate with Mehrotra and Sergeyev (2021), with whom we further share the assumption of a maximum primary surplus and an explicit allowance for the possibility of default. We differ from Mehrotra and Sergeyev (2021) in that default in their paper is due to extraordinary growth disasters, whereas it is due to ordinary output fluctuations in ours.

Reis (2020) decomposes government debt into the sum of (i) the present value of future primary surpluses, discounted at the marginal product of capital (MPK) and (ii) a bubble term, made possible by the fact that investors are willing to hold government bonds despite such bonds paying an interest rate lower than the MPK; the interest rate on government bonds is lowered by the convenience yield received for holding these uniquely safe assets which constitute highly desirable collateral.[15] Reis (2020) shows that a government can run a perpetual deficit if the MPK exceeds the growth rate, but that this deficit can be no larger than a fraction $g - r$ of the assets in the economy. Cochrane (2019) notes that forecasted deficits in the United States are well above the $g - r$ fraction of GDP that is consistent with a stable debt-to-GDP ratio.

Brunnermeier, Merkel, and Sannikov (2020) consider the role of government bonds in partially insuring otherwise uninsurable idiosyncratic risk: investors trade safe government bonds to smooth consumption when productive capital is subject to uninsurable idiosyncratic shocks; investors therefore are willing to accept a lower return on government bonds. This 'premium' for partial insurance combines with the convenience yield to form what Brunnermeier, Merkel, and Sannikov (2020) refer to as the service flow of government bonds; they argue that such flows can serve to reconcile the pattern of quasi continual primary deficits that characterizes countries such as Japan with the positive market value of these countries' government bonds. We do not

[15] See for example Krishnamurthy and Vissing-Jorgensen (2012) .

explicitly consider service flows in our analysis; we note, however, that a rough estimate of the value of service flows can be obtained from the difference between the maximum sustainable debt we compute and actual debt, when the latter is larger than the former.

Our work assumes involuntary default; this assumption distinguishes our work from the work on strategic default in the mold of Eaton and Gersovitz (1981).[16] The latter type of default has governments repeatedly compare the payoffs from debt service and default to choose that course of action which maximizes the present and future welfare of their population. Both types of default define a level of maximum sustainable debt: it is the level beyond which the government *cannot* pay under involuntary default, the level beyond which the government *will not* pay under strategic default. We believe our assumption of involuntary default is more appropriate given our focus on maximum debt: lenders naturally lend more to those they expect will do their utmost to service the debt that those they suspect continually will weigh the costs and benefits of servicing the debt.[17] Thus, in line with Gelpern and Panizza (2019), we turn our attention from sovereign authority's discretion to renege on debt to authority's power to mobilize resources to service debt.

2 Constant risk-free rate

2.1 Basic model

We assume one-period, zero-coupon debt. We consider a government that has issued in period $t-1$ debt of face value D_t to be paid in period t. The government is unable to service the debt, and default occurs, if the resources available for debt service to the government in period t are smaller than D_t. The government strives to avoid default.[18] It therefore attempts to maximize both the primary surplus, αY_t, and the proceeds from new debt issuance, $b_{M,t}Y_t$; α denotes the maximum primary surplus and $b_{M,t}$ the maximum proceeds from debt issuance, both expressed as a fraction of GDP Y_t. Together, maximum primary surplus (MPS) and maximum debt issuance proceeds define maximum debt $\omega_t Y_t$ in period t: $\omega_t Y_t = \alpha Y_t + b_{M,t} Y_t$ or, dividing by GDP Y_t

$$\omega_t = \alpha + b_{M,t}. \tag{1}$$

Default occurs when $D_t > \omega_t Y_t$. We express the face value of debt D_t due in period t but issued in period $t-1$ as a fraction d_t of period $t-1$ GDP: $D_t = d_t Y_{t-1}$. Default therefore occurs when $d_t Y_{t-1} > \omega_t Y_t$ or, rearranging

$$G_t \equiv \frac{Y_t}{Y_{t-1}} < \frac{d_t}{\omega_t}. \tag{2}$$

The preceding formulation makes clear the central role of the GDP growth rate. We assume that rate is independently and identically distributed and denote F its c.d.f. and f its p.d.f.

[16]See the surveys by Aguiar and Amador (2014) and Aguiar, Chatterjee, Cole, and Stangebye (2016).

[17]We nonetheless acknowledge the limited effectiveness of reputation at deterring strategic default: quantitative models of sovereign debt show that reputation has almost no impact on willingness to pay (Schmitt-Grohé and Uribe (2017)).

[18]Recall the discussion in the Introduction.

It remains to define maximum borrowing proceeds $b_{M,t}$. For that purpose, we denote R_f the risk-free interest rate between periods t and $t+1$. The interest rate is assumed to be constant in the present section. We further assume that, if default should occur in period $t+1$, only a fraction $\kappa < 1$ of resources available for debt service absent default, $\alpha Y_{t+1} + b_{M,t+1} Y_{t+1}$, would in fact be available for debt service: default limits both the availability of the primary surplus and the ability to issue new debt. Using (1) and $Y_{t+1} = Y_t G_{t+1}$ to write

$$\alpha Y_{t+1} + b_{M,t+1} Y_{t+1} = \omega_{t+1} Y_{t+1} = \omega_{t+1} Y_t G_{t+1},$$

we have

$$b_{M,t} Y_t = \frac{1}{R_f} \max_{d_{t+1}} \left[d_{t+1} Y_t \left(1 - F\left(\frac{d_{t+1}}{\omega_{t+1}}\right)\right) + \kappa \int_0^{\frac{d_{t+1}}{\omega_{t+1}}} \omega_{t+1} Y_t G \mathrm{d}F(G) \right], \tag{3}$$

where we have used the i.i.d. property of growth to drop the time subscript for growth. With risk-neutral lenders, period t proceeds equal period $t+1$ expected payment discounted at the risk-free rate R_f.[19] Payment absent default is $D_{t+1} = d_{t+1} Y_t$; from (2), such payment is received with probability $1 - F(d_{t+1}/\omega_{t+1})$. Payment in case of default is $\kappa \omega_{t+1} Y_t G_{t+1}$; it has expected value $\kappa \int_0^{d_{t+1}/\omega_{t+1}} \omega_{t+1} Y_t G \mathrm{d}F(G)$ over the range of growth rates for which default occurs $[0, d_{t+1}/\omega_{t+1})$. There is maximization over d_{t+1} because $b_{M,t}$ represents *maximum* borrowing proceeds.

Dividing (3) by Y_t and using (1), we obtain the recursive equation

$$\omega_t = \alpha + \frac{1}{R_f} \max_{d_{t+1}} \left[d_{t+1} \left(1 - F\left(\frac{d_{t+1}}{\omega_{t+1}}\right)\right) + \kappa \int_0^{\frac{d_{t+1}}{\omega_{t+1}}} \omega_{t+1} G \mathrm{d}F(G) \right]. \tag{4}$$

We say that a given debt-to-GDP ratio ω is sustainable if the sequence ω_t defined by $\omega_0 = \omega$ and (4) for all $t \geq 1$ is bounded above. We wish to determine the conditions for sustainability.

2.2 A condition for sustainability

Rewrite (4) as

$$\omega_t = \alpha + \phi(\omega_{t+1}), \tag{5}$$

where

$$\phi(\omega) \equiv \frac{1}{R_f} \max_d d \left[1 - \ell\left(\frac{d}{\omega}\right)\right] \tag{6}$$

and

$$\ell(\rho) \equiv F(\rho) - \frac{\kappa}{\rho} \int_0^\rho G \mathrm{d}F(G). \tag{7}$$

The function ϕ represents maximum proceeds from new debt issuance; these depend on expected loss from default, represented by the function ℓ. Thus, in line with our previous discussion, (5) expresses maximum debt as the sum of maximum primary surplus and maximum new debt issuance proceeds.

We note that ϕ is linear. To see this, make the change of variable $G \equiv d/\omega$ to rewrite (6) as

$$\phi(\omega) = \frac{1}{R_f} \max_{G \geqslant 0} \omega G \left[1 - \ell(G)\right] = \frac{\gamma}{R_f} \omega, \tag{8}$$

[19] The assumption of risk-neutrality could be relaxed under the alternative assumption of complete markets; the analysis would proceed along very similar lines, with risk-adjusted probabilities replacing actual probabilities.

where

$$\gamma \equiv \max_G G\left[1-\ell\left(G\right)\right] = G_M\left[1-\ell\left(G_M\right)\right], \tag{9}$$

with G_M the solution to the maximization problem. We can thus rewrite (5) as the linear difference equation

$$\omega_t = \alpha + \frac{\gamma}{R_f}\omega_{t+1}; \tag{10}$$

it has solution

$$\omega_t = \left(\frac{R_f}{\gamma}\right)^t\left(\omega - \omega_M\right) + \omega_M, \tag{11}$$

where ω_M is the fixed point of (10).[20]

Recall from Section 2.1 that sustainability requires ω_t to be bounded above. There are two cases to consider. If $R_f < \gamma$, any debt-to-GDP ratio ω is sustainable because $R_f/\gamma < 1$ can be counted upon gradually to whittle down $\omega - \omega_M$; there is no maximum debt in such case. By contrast, if $R_f > \gamma$, then ω is sustainable if and only if it is less than ω_M, which thereby constitutes the maximum sustainable debt-to-GDP ratio.

In order to test the condition $\gamma < R_f$, we assume that the growth rate G is lognormally distributed, $\log\left(G\right) \sim N\left(\mu, \sigma^2\right)$. The condition $\gamma < R_f$ becomes[21]

$$\exp\left[\mu + \frac{\sigma^2}{2}\right]\left\{\max_x \exp\left[\sigma x - \frac{\sigma^2}{2}\right]\left[1 - N\left(x\right)\right] + \kappa N\left(x-\sigma\right)\right\} < R_f. \tag{12}$$

Defining $\overline{G} \equiv \mathbb{E}\left[G\right] = \exp\left[\mu + \frac{\sigma^2}{2}\right]$, this rewrites

$$\overline{G} < \frac{R_f}{\max_x \exp\left[\sigma x - \frac{\sigma^2}{2}\right]\left[1 - N\left(x\right)\right] + \kappa N\left(x-\sigma\right)} \equiv G_c\left(\sigma\right). \tag{13}$$

Inequality (13) is the condition we shall test empirically. It is similar to the Blanchard condition in that it compares an expected growth rate, $\overline{G}$, to an interest rate, the risk-free rate grossed up by what is in effect the recovery-adjusted probability of repayment; this interest rate defines the cutoff growth rate, $G_c\left(\sigma\right)$.

What if $\overline{G} > G_c\left(\sigma\right)$ or, equivalently, $\gamma > R_f$? Then there is no maximum debt-to-GDP ratio, as high growth can be counted upon to resorb large temporary increases in debt, or to make permanent small deficits feasible. This naturally recalls the environment analyzed by Blanchard (2019) under the condition $\overline{G} > R_f$. We show in Remark 1 that our inequality and Blanchard's are in fact identical when there is full recovery in default.

Remark 1 *Assume full recovery, $\kappa = 1$; the inequality $\gamma > R_f$ is equivalent to the Blanchard Condition $\overline{G} > R_f$.*

When there is no loss in default, borrowing proceeds are maximized by issuing debt of infinite face value, thereby ensuring that there is default in every period, in turn ensuring that debtholders receive the entirety of the primary surplus and of new issuance proceeds in every period. Debtholders effectively become shareholders. This explains the replacement of γ by $\overline{G}$.

We conclude this section by noting that $\gamma < \overline{G}$ when $\kappa < 1$.[22] The upper bound on debt in

[20] We provide the expression for ω_M in Section 2.3.
[21] The derivation is in the Appendix.
[22] The derivation is in the Appendix.

case $\kappa < 1$ combines with partial recovery in default to limit both the level of resources available for debt service and the claim debtholders can make on these resources. This diminishes the effectiveness of growth in resorbing debt, thereby making the attainment of the environment analyzed by Blanchard (2019) more difficult: $\gamma > R_f$ implies $\overline{G} > R_f$ but the converse is not true.

2.3 Perspectives on maximum sustainable debt ω_M and maximum sustainable borrowing b_M

From (10), maximum sustainable debt (MSD) ω_M is the fixed point

$$\omega_M = \alpha + \frac{\gamma}{R_f}\omega_M \tag{14}$$

$$\Rightarrow \omega_M = \frac{\alpha R_f}{R_f - \gamma}. \tag{15}$$

Similarly, from (1) evaluated at maximum sustainable debt ω_M, we define maximum sustainable borrowing (MSB) $b_M \equiv \omega_M - \alpha$, which from (14) and (15) equals

$$b_M = \frac{\alpha\gamma}{R_f - \gamma}. \tag{16}$$

It is interesting to note that maximum sustainable borrowing b_M can be decomposed into the difference of two terms, the present value of future primary surpluses α and the present value of expected default costs. To see this, use (9), (14), and the definition of b_M to write

$$b_M = \frac{\omega_M G_M \left[1 - \ell\left(G_M\right)\right]}{R_f} = \frac{\left(\alpha + b_M\right) G_M \left[1 - \ell\left(G_M\right)\right]}{R_f}.$$

Iterating forward, we have[23]

$$b_M = \alpha \sum_{n=1}^{\infty} \left(\frac{G_M}{R_f}\right)^n \left[1 - \ell\left(G_M\right)\right]^n.$$

Replacing $\left[1 - \ell\left(G_M\right)\right]^n$ by $1 - \left(1 - \left[1 - \ell\left(G_M\right)\right]^n\right)$ we obtain

$$b_M = \alpha \sum_{n=1}^{\infty} \left(\frac{G_M}{R_f}\right)^n - \alpha \sum_{n=1}^{\infty} \left(\frac{G_M}{R_f}\right)^n \left(1 - \left[1 - \ell\left(G_M\right)\right]^n\right). \tag{17}$$

The first term on the RHS is the present value of future primary surpluses, the second is the present value of expected default costs. The latter interpretation is suggested by noting that (i) the function ℓ in (7) is a cumulative distribution function in its own right, a form of recovery-adjusted probability of default, and (ii) $\left[1 - \ell\left(G_M\right)\right]^n$ therefore can be interpreted as the probability that no default occurs up to and including period $t+n$ and $1 - \left[1 - \ell\left(G_M\right)\right]^n$ the probability that default has occurred in any period up to and including that period.[24] Default in any period up to and including $t + n$ makes the primary surplus in that period no longer

[23] Note that the inequality $\gamma = G_M\left[1 - \ell\left(G_M\right)\right] < R_f$ ensures convergence.

[24] To see that ℓ is a cumulative distribution function, note that

$$l'\left(\rho\right) = \left(1 - \kappa\right) f\left(\rho\right) + \frac{\kappa}{\rho^2}\int_0^{\rho} G \mathrm{d}F\left(G\right) > 0,$$

available for debt service. The expected cost in period $t+n$ of default in any period up to and including that period is therefore $\left(1-\left[1-\ell\left(G_M\right)\right]^n\right)\alpha G_M^n$, expressed as a fraction of current period GDP Y_t: it is the MPS in period $t+n$, foregone because of default in any period up to and including that period. The present value of these costs is the second term on the RHS of (17).

3 Stochastic risk-free rate

We now consider the case where the risk-free interest rate is stochastic. We denote R_t the risk-free interest rate between periods t and $t+1$ and $\mathbb{E}_t\left[.\right]$ the expectation over R_{t+1}, formed in period t. We assume the growth and interest rates are independent. We thus rewrite (3) as

$$b_{M,t}Y_t=\frac{1}{R_t}\max_{d_{t+1}}\mathbb{E}_t\left[d_{t+1}Y_t\left(1-F\left(\frac{d_{t+1}}{\omega_{t+1}}\right)\right)+\kappa\int_0^{\frac{d_{t+1}}{\omega_{t+1}}}\omega_{t+1}Y_tG\mathrm{d}F\left(G\right)\right],\qquad(18)$$

where we have used the independence of the growth and interest rates. The expectation over R_{t+1} is made necessary by the dependence of ω_{t+1} on R_{t+1} through $b_{M,t+1}$, similarly to that of $b_{M,t}$ on R_t in (18).

We assume that R_t is a stationary Markov Chain with K states and transition probability $\pi_{k'|k}$ from state k to state k', $k,k'=1,\ldots,K$. Thus, when in state k with risk-free rate R^k, dividing (18) by Y_t and using (1) we obtain

$$\begin{aligned}\omega_t^k&=\alpha+\frac{1}{R^k}\max_{d_{t+1}}\sum_{k'=1}^{K}\pi_{k'|k}\left[d_{t+1}\left(1-F\left(\frac{d_{t+1}}{\omega_{t+1}^{k'}}\right)\right)+\kappa\omega_{t+1}^{k'}\int_0^{\frac{d_{t+1}}{\omega_{t+1}^{k'}}}G\mathrm{d}F\left(G\right)\right]\\&=\alpha+\frac{1}{R^k}\max_{d_{t+1}}d_{t+1}\left(1-\sum_{k'=1}^{K}\pi_{k'|k}\,\ell\left(\frac{d_{t+1}}{\omega_{t+1}^{k'}}\right)\right)&(19)\\&=\alpha+\phi^k\left(\omega_{t+1}\right),&(20)\end{aligned}$$

where $\ell\left(\rho\right)$ is as in (7) and

$$\begin{aligned}\phi^k\left(\omega\right)&\equiv\frac{1}{R^k}\max_d d\left(1-\sum_{k'=1}^{K}\pi_{k'|k}\left[\ell\left(\frac{d}{\omega^{k'}}\right)\right]\right)&(21)\\&\equiv\frac{1}{R^k}\max_d d\left(1-\mathbb{E}_{\omega|k}\left[\ell\left(\frac{d}{\omega}\right)\right]\right),&(22)\end{aligned}$$

$\omega_{t+1}^{k'}$ is defined analogously to ω_t^k, ω_{t+1} is the K-dimensional vector $\left[\omega_{t+1}^{k'}\right]$, and $\mathbb{E}_{\omega|k}\left[.\right]$ is the expectation over the next period interest rate and its corresponding MSD given that the current period interest rate is R^k. The function ϕ^k represents maximum proceeds from new debt issuance. We shall need the following result

---

$$\lim_{\rho\to 0}\ell\left(\rho\right)=F\left(0\right)=0,$$

and

$$\lim_{\rho\to\infty}\ell\left(\rho\right)\equiv F\left(\infty\right)=1,$$

where we have used l'Hospital rule to obtain the first limit:

$$\lim_{\rho\to 0}\frac{\kappa}{\rho}\int_0^{\rho}G\mathrm{d}F\left(G\right)=\lim_{\rho\to 0}\kappa\rho f\left(\rho\right)=0.$$

Lemma 2 *For all k, the function $\phi^k(\omega)$ is (i) increasing and (ii) homogeneous of degree* 1.

We denote $\phi(\omega)$ the K-dimensional vector $\left[\phi^k(\omega)\right]$. We show

Proposition 3 *If $\gamma < \min_k \left[R^k\right]$ and $\exists k \in \{1,\dots,K\}$ such that $\omega_t^k > \omega^* \equiv \frac{\alpha \min_k\left[R^k\right]}{\min_k\left[R^k\right]-\gamma}$, then* $\lim_{s\to\infty}\max_k\left[\omega_s\right]=\infty$.

Proposition 3 states that, if $\gamma < \min_k \left[R^k\right]$, then maximum debt larger than ω^* is not sustainable. Put differently, if $\gamma < \min_k \left[R^k\right]$, then maximum sustainable debt can be no larger than ω^*. The condition $\gamma < \min_k \left[R^k\right]$ extends the earlier condition $\gamma < R_f$ from the case of constant to that of stochastic risk-free rate; $\omega^* = \alpha \min_k \left[R^k\right] / \left(\min_k \left[R^k\right] - \gamma\right)$ naturally recalls $\omega_M = \alpha R_f / (R_f - \gamma)$ in (15). We stress that the inequality $\gamma < \min_k \left[R^k\right]$ is a sufficient condition; Section 6.2 will show that it is not necessary, in the sense that we shall compute finite MSD even for those two countries for which $\gamma > \min_k \left[R^k\right]$.

Proposition 3 naturally begs the question of whether $\gamma < \min_k \left[R^k\right]$. In order to answer that question, we proceed as in Section 2.2 to derive the testable condition

$$\overline{G} \equiv \exp\left[\mu + \frac{\sigma^2}{2}\right] < \frac{\min_k\left[R^k\right]}{\max\limits_x \exp\left[\sigma x - \frac{\sigma^2}{2}\right]\left[1 - N(x)\right] + \kappa N(x-\sigma)} \equiv G_{c,s}(\sigma). \qquad (23)$$

4 European Stability Mechanism

4.1 Modeling the effect of the ESM

We now extend our analysis to consider the role of the European Stability Mechanism (ESM). We assume the ESM lends a Eurozone government an amount that makes up a constant fraction m of that country's GDP. What distinguishes the ESM as lender from other lenders is that it charges the government the subsidized interest rate R_S, as opposed to the market interest rate R_M, that which compensates risk-neutral investors for expected default loss: $R_M = R_f / \left[1 - \ell(G_M)\right]$. Concessionary financing naturally involves the charging of a lower interest rate, $R_S < R_M$. We wish to determine to extent to which ESM lending on concessionary terms increases a country maximum sustainable debt ω_M, at what cost to the ESM.

Default in period t occurs when the sum of zero coupon debt $D_t = d_t Y_{t-1}$ due to market lenders and subsidized interest bearing debt $mR_S Y_{t-1}$ due to the ESM, both raised in the previous period $t-1$ and expressed as a fraction of that period's GDP Y_{t-1}, is larger than the combined resources available to the government in the current period t. These are the sum of the maximum primary surplus αY_t, new ESM lending mY_t and maximum proceeds from new market debt issuance $b_{M,t}Y_t$. Together, these define maximum debt $\omega_t Y_t = \alpha Y_t + mY_t + b_{M,t}Y_t$. The analogue to (1) is therefore

$$\omega_t = \alpha + m + b_{M,t} \qquad (24)$$

and default occurs if and only if

$$G_t < \frac{d_t + mR_S}{\omega_t}.$$

The analogue to (3), divided by Y_t, in turn is

$$\begin{aligned}
b_{M,t} =& \frac{1}{R_f} \max_{d_{t+1}} \left[d_{t+1} \left[1 - F\left(\frac{d_{t+1} + mR_S}{\omega_{t+1}} \right) \right] + \kappa \frac{d_{t+1}}{d_{t+1} + mR_S} \int_0^{\frac{d_{t+1}+mR_S}{\omega_{t+1}}} \omega_{t+1} G \mathrm{d}F(G) \right] \\
=& \frac{1}{R_f} \max_{d_{t+1}} d_{t+1} \left[1 - F\left(\frac{d_{t+1} + mR_S}{\omega_{t+1}} \right) + \kappa \frac{\omega_{t+1}}{d_{t+1} + mR_S} \int_0^{\frac{d_{t+1}+mR_S}{\omega_{t+1}}} G \mathrm{d}F(G) \right] \\
=& \frac{1}{R_f} \max_{d_{t+1}} d_{t+1} \left[1 - \ell\left(\frac{d_{t+1} + mR_S}{\omega_{t+1}} \right) \right],
\end{aligned}$$

where we have assumed (i) that resources available in default ($\kappa\omega_{t+1}$) are shared in proportion to the amounts due private creditors (d_{t+1}) and the ESM (mR_S) and (ii) that default decreases the availability of ESM financing (from m to κm) to the same extent as it does other financing sources (from $\alpha + b_{M,t}$ to $\kappa(\alpha + b_{M,t})$).[25] This makes the analogues to (5) and (6)

$$\omega_t = \alpha + m + \phi(\omega_{t+1}), \tag{25}$$

and

$$\phi(\omega) \equiv \frac{1}{R_f} \max_d d \left[1 - \ell\left(\frac{d + mR_S}{\omega} \right) \right]. \tag{26}$$

Similarly to Section 2, we make the change of variable $G \equiv (d + mR_S)/\omega$ to rewrite (26) as

$$\phi(\omega) = \frac{1}{R_f} \max_G (\omega G - mR_S)[1 - \ell(G)]. \tag{27}$$

Unlike in Section 2, the function ϕ is no longer linear; this is because $\arg\max_G (\omega G - mR_S)[1 - \ell(G)]$ is a function of ω. There is therefore no analogue to the explicit solution (11).

When $\phi'(\cdot) = G_M [1 - \ell(G_M)]/R_f < 1$, ϕ is a contraction, (25) has a unique fixed point ω_M with corresponding $G_M = \arg\max_G (\omega_M G - mR_S)[1 - \ell(G)]$, and (25) can be rewritten as

$$\omega_t - \omega_M = \phi(\omega_{t+1}) - \phi(\omega_M). \tag{28}$$

As ϕ is increasing, the sign of $\omega_t - \omega_M$ is the same for all t and is equal to the sign of $\omega - \omega_M$. Moreover, as ϕ is a contraction, its inverse ϕ^{-1} is an expansion and the forward looking dynamics in equation (28) imply that $|\omega_t - \omega_M| \to \infty$.[26] Thus, similarly to Section 2, ω is sustainable if and only if it is less than the fixed point ω_M.

4.2 Perspectives on ω_M and b_M under ESM support

We first confirm that ESM financing does indeed increase MSD ω_M. We also show that $\partial\omega_M/\partial\kappa > 0$: the greater availability of resources in default increases MSD.

Proposition 4 *$\partial\omega_M/\partial m > 0$ and $\partial\omega_M/\partial\kappa > 0$.*

[25] These two assumptions imply that the single difference between ESM and private sector financing is the ESM's subsidized interest rate, $R_S < R_M$. ESM loans are thus *pari passu* with their private sector counterparts.

[26] To see this, use (25) to write $\omega_{t+1} = \phi^{-1}(\omega_t - \alpha - m)$; likewise write $\omega_M = \phi^{-1}(\omega_M - \alpha - m)$ and subtract to obtain

$$\omega_{t+1} - \omega_M = \phi^{-1}(\omega_t - \alpha - m) - \phi^{-1}(\omega_M - \alpha - m).$$

What is true of MSD ω_M does not necessarily extend to b_M. From (24) evaluated at MSD ω_M, we obtain

$$b_M = \omega_M - \alpha - m.$$

Differentiating with respect to m, we obtain

$$\frac{\partial b_M}{\partial m} = \frac{\partial \omega_M}{\partial m} - 1 = \frac{R_f - R_S\left[1 - \ell\left(G_M\right)\right]}{R_f - G_M\left[1 - \ell\left(G_M\right)\right]} - 1.$$

We have

$$\frac{\partial b_M}{\partial m} > 0 \Leftrightarrow G_M > R_S.$$

The intuition for this result is as follows: G_M is the minimum rate of growth necessary to avoid default when debt owed equals MSD ω_M, R_S is the interest rate charged by the ESM; the former measures the rate of growth in the resources available for debt service, the latter the rate at which these resources are claimed by the ESM (recall that all variables are expressed as fractions of GDP). If the latter is larger than the former, ESM financing decreases resources available to private creditors, who consequently decrease the amount they are willing to lend. In contrast, it is easy to see that $\partial b_M / \partial \kappa = \partial \omega_M / \partial \kappa > 0$.

We now derive the analogue to (17). Maximum borrowing proceeds are now the sum of proceeds from the private sector b_M and proceeds from the ESM m. Using $b_M = \phi\left(\omega_M\right)$ and (27) we can write[27]

$$b_M + m = \frac{\left[\alpha + b_M + m + \frac{m}{G_M}\left[R_M - R_S\right]\right] G_M\left[1 - \ell\left(G_M\right)\right]}{R_f}; \tag{29}$$

the term $\left(m/G_M\right)\left[R_M - R_s\right]$ represents the value of the annual subsidy provided by the ESM. The division by G_M reflects the fact that the subsidy is a fraction of GDP in the period that precedes the period in which debt is serviced. Iterating, we obtain

$$\begin{aligned} b_M + m &= \left[\alpha + \frac{m\left(R_M - R_S\right)}{G_M}\right] \sum_{n=1}^{\infty}\left(\frac{G_M}{R_f}\right)^n \left[1 - \ell\left(G_M\right)\right]^n \\ &= \left[\alpha + \frac{m\left(R_M - R_S\right)}{G_M}\right] \sum_{n=1}^{\infty}\left(\frac{G_M}{R_f}\right)^n \\ &\quad - \left[\alpha + \frac{m\left(R_M - R_S\right)}{G_M}\right] \sum_{n=1}^{\infty}\left(\frac{G_M}{R_f}\right)^n \left(1 - \left[1 - \ell\left(G_M\right)\right]^n\right); \end{aligned}$$

the first term in the last expression is the present value of future primary surpluses and subsidies, the second is the present value of expected default costs, which include foregone subsidies: recall our assumption that default decreases the availability of both public and private financing. The PV of the subsidy provided by the ESM, expressed as a fraction of current period GDP, equals[28]

$$PVS = \frac{m\left(R_M - R_S\right)}{R_M - G_M}. \tag{30}$$

The PV of the subsidy equals the annual value of the subsidy expressed as a fraction of GDP, growing at G_M and discounted at R_M.

[27] The derivation is in the Appendix.

[28] The derivation is in the Appendix.

We now compare the increase in borrowing proceeds made possible by ESM financing to the value of the subsidy PVS. The former is $b_M + m - b_M^0$, where

$$b_M^0 = \alpha \sum_{n=1}^{\infty} \left(\frac{G_M^0}{R_f} \right)^n \left[1 - \ell \left(G_M^0 \right) \right]^n ,$$

with $G_M^0 = \arg\max_G G \left[1 - \ell \left(G \right) \right]$.[29] We show[30]

$$b_M + m - b_M^0 = \alpha \left[\frac{G_M \left[1 - \ell \left(G_M \right) \right]}{R_f - G_M \left[1 - \ell \left(G_M \right) \right]} - \frac{G_M^0 \left[1 - \ell \left(G_M^0 \right) \right]}{R_f - G_M^0 \left[1 - \ell \left(G_M^0 \right) \right]} \right] + PVS. \tag{31}$$

Intuitively, any net benefit of ESM financing, that is, any increase in borrowing proceeds over and above the subsidy, depends on the extent to which ESM financing makes MPS α more frequently available for debt service.

Now, recalling that $G_M^0 = \underset{G}{\text{argmax}}\ G \left[1 - \ell \left(G \right) \right]$ whereas $G_M = \underset{G}{\text{argmax}}\ \left(\omega_M G - m R_S \right) \left[1 - \ell \left(G \right) \right]$ with $m > 0$, it is clear that $G_M^0 \left[1 - \ell \left(G_M^0 \right) \right] > G_M \left[1 - \ell \left(G_M \right) \right]$: MPS α is made less frequently available as result of ESM financing.

The preceding appears to suggest that ESM financing is actually detrimental to total payoff, in the sense that its cost, the present value of the subsidy provided by the ESM, is larger than its benefit, the increase in maximum borrowing proceeds that ESM financing makes possible. Still, as shown in Proposition 4 and as will be confirmed by our simulations below, ESM increases MSD ω_M.

5 Data

This section presents summary statistics for the main data used in the paper. All data are sourced from the World Economic Outlook Database published by the IMF. As mentioned in the Introduction, we do not consider the Covid period and end our analysis in 2019.

We start with real GDP growth, computed as the percentage change in real GDP in Euros. Over the full period under analysis, the simple average of the twelve countries' average annual growth rates was nearly 1.9% (Panel (a) of Table 1). There is, however, substantial cross-country variation, with average growth ranging from about 0.5% in Greece and Italy to more than 5% in Ireland. There is also substantial within-country growth volatility, with the within-country standard deviation often surpassing average growth.

If we consider only the pre-GFC period (Panel (b) of Table 1) we find substantially higher average growth at 3.1% and lower cross-country and within-country growth volatility. In this case too, however, there are large cross-country differences with average growth ranging from 1.5% in Italy and Germany to more than 5% in Luxembourg and Ireland.

Our second key variable is the debt-to-GDP ratio. Panel (a) of Table 2 shows that average gross public debt ranges from 15% of GDP in Luxembourg to 140% GDP in Greece. As mentioned in the Introduction, in 2019 the debt-to-GDP ratios in Greece and Italy were 185% and

[29] The values b_M^0 and G_M^0 are those derived in Section 2; the superscript is needed to distinguish these values from those derived in the present section.

[30] The derivation is in the Appendix.

Table 1: Real GDP Growth

	(a) Full Sample (1999–2019)				(b) Pre-GFC Sample (1999–2007)			
	$E[\Delta y/y]$	$\sigma[\Delta y/y]$	Min.	Max.	$E[\Delta y/y]$	$\sigma[\Delta y/y]$	Min.	Max.
Austria	1.73	1.64	−3.76	3.72	2.55	1.06	0.94	3.73
Belgium	1.79	1.35	−2.02	3.72	2.58	1.11	1.04	3.72
Finland	1.78	2.96	−8.07	5.77	3.62	1.43	1.71	5.77
France	1.54	1.39	−2.78	4.09	2.30	1.02	0.84	3.81
Germany	1.39	2.10	−5.70	4.19	1.59	1.50	−0.70	3.81
Greece	0.53	4.32	−10.15	5.79	3.94	1.58	0.60	5.79
Ireland	5.25	6.22	−5.10	25.18	6.32	2.31	3.01	10.52
Italy	0.46	1.94	−5.28	3.79	1.48	1.08	0.14	3.79
Luxembourg	3.36	3.07	−4.36	8.44	5.02	2.71	1.63	8.44
Netherlands	1.72	1.90	−3.67	5.03	2.58	1.70	0.16	5.03
Portugal	1.05	2.15	−4.06	3.91	1.80	1.52	−0.93	3.91
Spain	1.98	2.46	−3.77	5.05	3.77	0.78	2.73	5.05
Simple Average	1.88	3.14			3.13	2.05		

135%, respectively. The GFC led to a sharp increase in debt ratios in all countries considered, with the average debt-to-GDP ratio increasing from 62% to 77%.[31,32]

Table 2: Gross Debt over GDP (%)

	(a) Full Sample (1999–2019)				(b) Pre-GFC Sample (1999–2007)				2019
	Mean	Std.Dev.	Min.	Max.	Mean	Std.Dev.	Min.	Max.	
Austria	73.29	8.01	61.10	84.4	65.55	2.06	61.10	68.32	70.51
Belgium	101.71	6.53	87.32	115.36	101.26	9.21	87.32	115.36	98.07
Finland	48.12	10.05	32.56	63.64	40.49	3.07	33.90	44.01	59.51
France	78.86	15.82	58.34	98.31	62.76	3.29	58.34	67.38	97.62
Germany	68.07	7.80	58.19	82.47	62.80	3.44	58.19	67.54	59.24
Greece	140.65	37.23	99.74	180.89	104.68	2.69	99.74	108.31	184.90
Ireland	60.15	32.58	23.65	120.04	30.99	7.21	23.64	46.57	57.28
Italy	118.79	12.65	103.89	135.37	107.26	2.80	103.89	113.29	134.56
Luxembourg	15.11	6.71	7.43	23.69	7.88	0.38	7.42	8.37	22.00
Netherlands	54.90	7.98	41.97	68.01	48.50	4.35	41.97	57.53	47.43
Portugal	92.36	32.31	50.34	132.94	60.12	8.20	50.33	72.73	116.61
Spain	68.2	24.32	35.76	100.70	48.43	8.79	35.76	62.46	95.53
Simple Average	76.68	37.60			61.73	29.66			86.94

Another key variable of interest for our analysis is the primary balance over GDP. Panel (b) of Table 3 shows that before the global financial crisis, most countries in our sample ran often substantial primary surpluses (the exceptions are Austria, France, and Germany which essentially had a primary balance close to zero and Portugal and Greece which ran primary deficits). Many countries reacted to the GFC by running large primary deficits. In some cases these deficits were driven by the desire to stabilize the economy and in other cases by the need to recapitalize the banking system (this was the case in Ireland and Spain especially). The only

[31] Specifically, the debt-to-GDP ratio for all twelve countries was higher in the aftermath of the crisis than it had been in 2007 (not shown).

[32] It is sometimes argued that the relevant debt ratio should be the net debt-to-GDP. Table 5 in the Appendix reports summary statistics for net debt. While focusing on net debt makes a large difference for a few low-debt countries such as Finland and Luxembourg, there are no large differences in high-debt countries such as Italy, Portugal, and Spain. Net debt data are not available for Greece.

country that almost never had a large primary deficit was Italy.[33]

Table 3: Primary Balance over GDP (%)

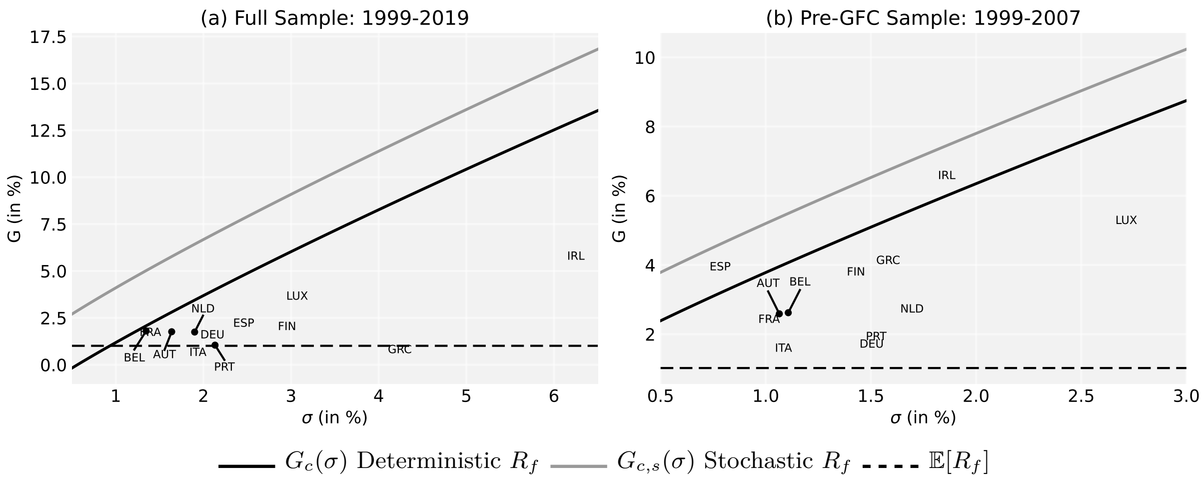
	(a) Full Sample (1999–2019)				*(b)* Pre-GFC Sample (1999–2007)			
	Mean	Std.Dev.	Min.	Max.	Mean	Std.Dev.	Min.	Max.
Austria	0.06	1.34	−3.16	2.04	0.27	1.28	−2.58	2.04
Belgium	1.86	2.66	−1.99	6.24	4.43	1.63	1.32	6.24
Finland	0.94	3.16	−2.85	7.62	3.92	1.78	2.08	7.62
France	−1.3	1.61	−4.94	1.25	−0.09	1.04	−1.51	1.25
Germany	0.85	1.49	−2.24	2.67	0.15	1.33	−1.23	2.66
Greece	−0.79	3.61	−10.19	4.19	−0.81	2.27	−4.08	2.81
Ireland	−1.74	8.06	−29.90	6.46	2.81	2.04	0.61	6.46
Italy	1.58	1.20	−0.93	4.33	2.13	1.39	0.24	4.33
Luxembourg	1.21	1.72	−2.15	4.42	1.45	2.37	−2.15	4.42
Netherlands	0.31	2.34	−3.91	3.88	1.38	1.66	−1.19	3.88
Portugal	−1.7	2.87	−8.68	2.89	−1.95	1.37	−3.82	−0.34
Spain	−1.47	4.13	−9.96	3.38	2.21	0.64	1.64	3.38
Simple Average	−0.02	3.54			1.33	2.40		

6 Quantitative results

6.1 Constant risk-free rate

We initially consider the case of constant risk-free rate over the period 1999-2019. Panel (a) of Figure 1 plots the twelve Eurozone countries in our sample in the $\sigma - \overline{G}$ space, where σ (x-axis) is the standard deviation of lognormally-distributed growth and $\overline{G} = \exp\left[\mu + \sigma^2/2\right]$ (y-axis) is the mean growth rate. The figure also plots the Eurozone's real risk-free interest rate, specifically Germany's. The mean growth rates of all countries but Greece and Italy are

Figure 1: Sustainability in the Euro Zone (1999-2019)

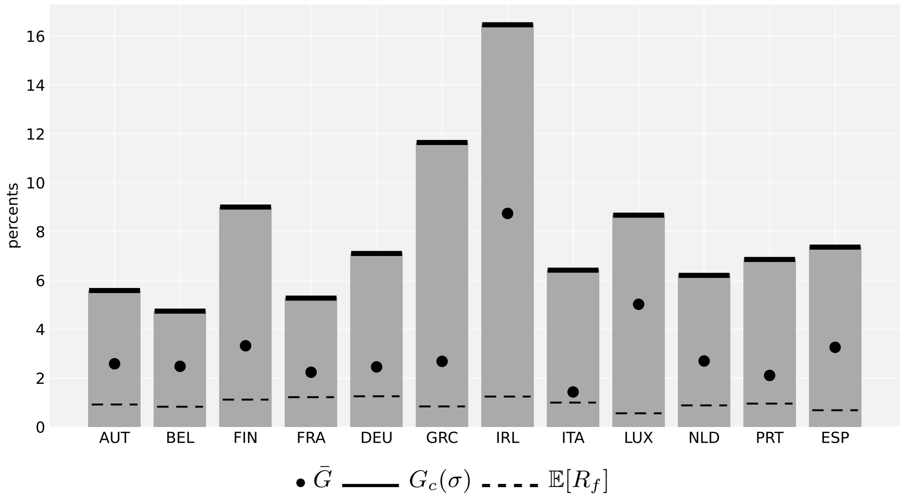


higher than the risk-free rate. This suggests that, for all but these two countries, growth can be counted upon to resorb large temporary increases in debt or to make permanent small deficits feasible. Figure 2 shows that using country-specific inflation rates rather than the Eurozone

[33] Italy had two years with small primary deficits: 2009 (0.9% of GDP) and 2010 (0.13% of GDP).

average makes even the debt of Greece and Italy satisfy the Blanchard Condition, by virtue of these two countries' higher inflation rates. Figure 2 plots each country's risk-free rate, its mean growth rate $\overline{G}$, and its cutoff growth rate $G_c(\sigma)$.

Figure 2: Sustainability in the Euro Zone (Country specific Interest Rate)



As shown in both figures, however, for no country is growth $\overline{G}$ higher than $G_c(\sigma)$, which is recalled to be the risk-free rate grossed up to account for the recovery-adjusted probability of repayment. This is the lower curve in Figure 1, and the top of the bars in Figure 2. Recalling from (13) that $\overline{G} < G_c(\sigma)$ is equivalent to $\gamma < R_f$, we conclude from Section 2.2 in the existence of an upper bound on debt, maximum sustainable debt ω_M in (15): accounting for default makes for a less sanguine assessment of sovereign debt sustainability. MSD are shown in Panel (a) of Table 4, in the penultimate column for the constant risk-free rate. The mean and standard

Table 4: Maximal Sustainable Debt

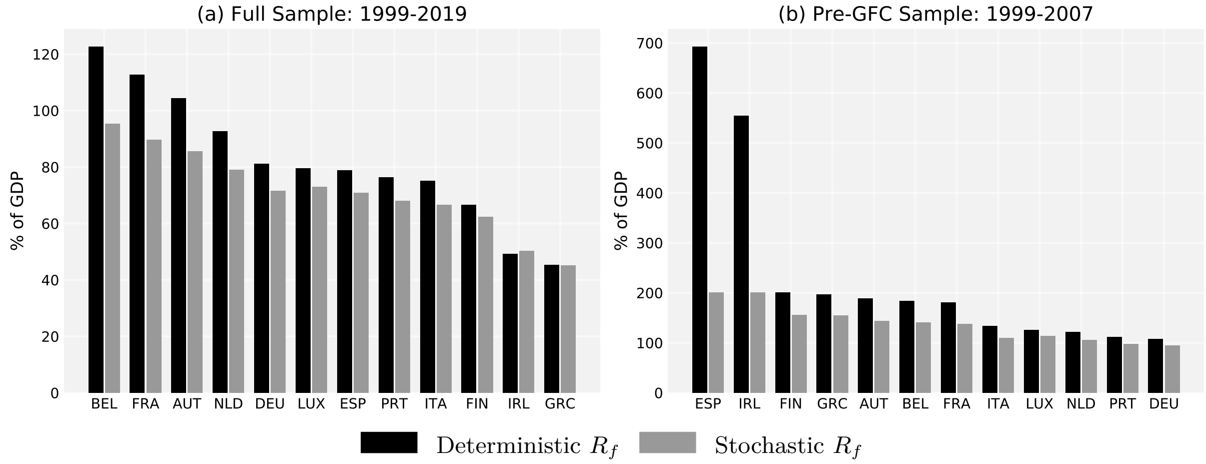
	(a) Full Sample (1999–2019)				*(b)* Pre-GFC Sample (1999–2007)			
Country	$E[\Delta y/y]$	$\sigma[\Delta y/y]$	Det. R_f	Stoch. R_f	$E[\Delta y/y]$	$\sigma[\Delta y/y]$	Det. R_f	Stoch. R_f
Austria	1.73	1.64	131.94	85.66	2.55	1.06	189.14	143.36
Belgium	1.78	1.35	163.21	95.33	2.58	1.11	183.87	141.00
Finland	1.79	2.95	76.46	62.26	3.62	1.43	200.87	155.47
France	1.52	1.39	145.77	89.74	2.29	1.02	180.49	137.80
Germany	1.37	2.10	96.78	71.61	1.59	1.50	108.14	94.84
Greece	0.53	4.24	49.49	45.12	3.94	1.58	196.73	155.17
Ireland	5.25	6.25	54.37	50.33	6.26	1.86	554.64	200.72
Italy	0.45	1.94	88.04	66.64	1.48	1.08	133.93	110.15
Luxembourg	3.36	3.07	94.51	73.02	5.02	2.71	125.40	113.59
Netherlands	1.71	1.90	113.64	79.00	2.58	1.70	121.48	105.81
Portugal	1.02	2.13	89.81	68.07	1.80	1.52	111.88	97.75
Spain	1.98	2.46	93.35	70.80	3.77	0.78	692.83	200.78

Note: The deterministic (Det.) real interest rates corresponds to the average over each sample. In the case of stochastic (Stoch.) real interest rates, we report the average maximal sustainable debt obtained using the ergodic distribution of the Markov Chain for the interest rate. Average growth rates and their volatility are computed over each sample and are expressed in percentages.

deviation of growth are reproduced from Table 1. Panel (a) of Figure 3 graphically reproduces

Table 4's estimates of MSD, in decreasing order. Belgium, France, and Austria have the highest MSD, due not so much to the mean of these countries' growth rates as to their low volatility. The simulations in Panels (a) and (b) of Figure 4 show that MSD is higher for higher growth

Figure 3: Maximal Sustainable Debt



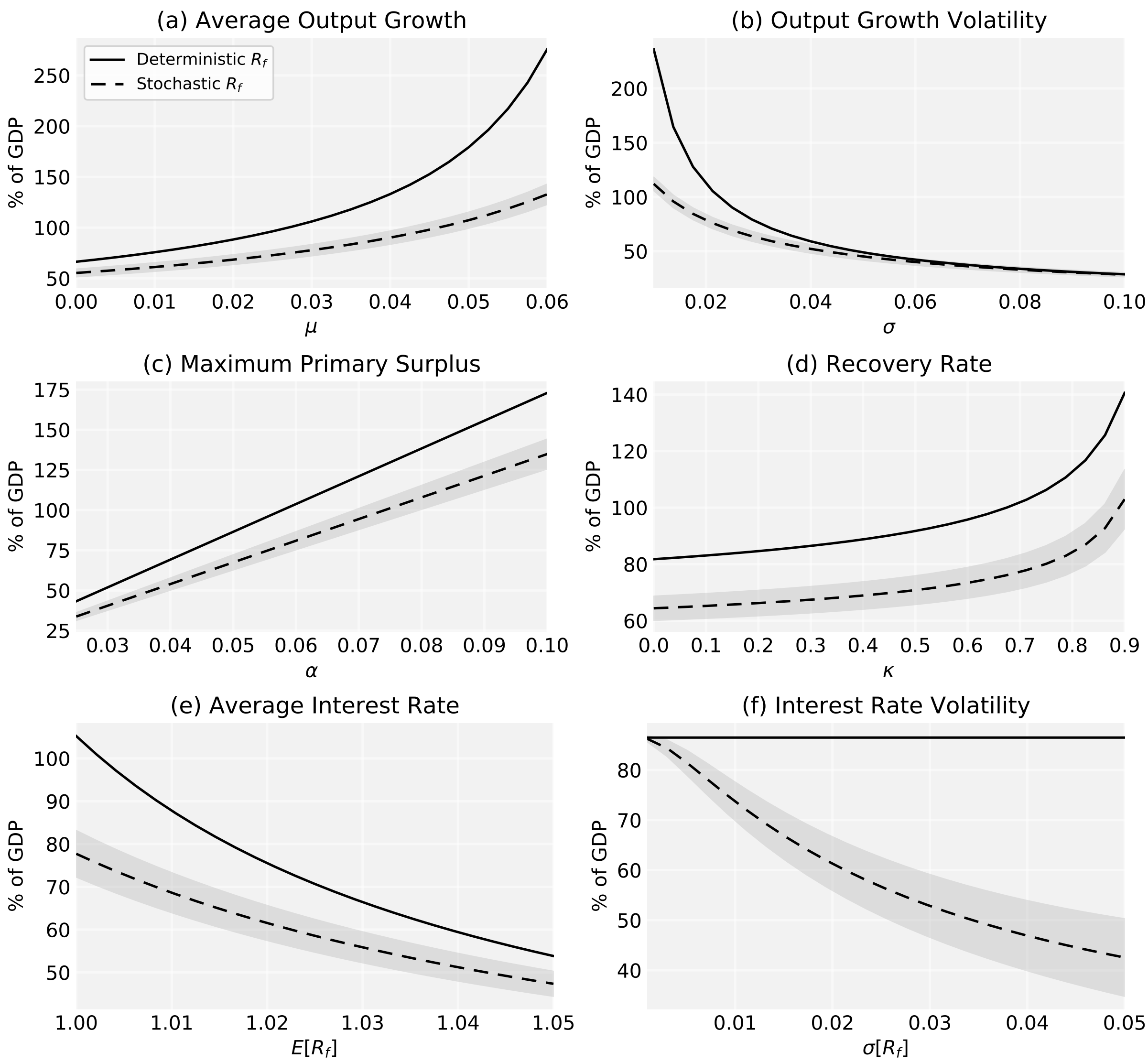
and lower for more volatile growth; the decrease in volatility is quite dramatic over the range of volatilities observed, specifically 1.35% (Belgium) to 6.25% (Ireland).

Greece's MSD (49%) and Italy's (88%) are well below these countries actual debt-to-GDP ratios (185% and 135% at end 2019, respectively), Portugal's only slightly so (MSD 90% and actual debt-to-GDP ratio 117%). One possible explanation for such discrepancy is the choice of period: our period includes the Great Financial Crisis (GFC) and its aftermath, which saw lower and more volatile growth. Notwithstanding the Covid crisis, which we do not consider, growth may recover and crisis-induced volatility may abate, to some extent at least. To obtain an indication of what MSD may be under more normal circumstances, we consider the period 1999-2007, which excludes the GFC and its aftermath. The results are shown in Panels (b) of Figures 1 and 3 and of Table 4; they are quite different from those in the corresponding Panels (a). All countries but Portugal have MSD above their end-2019 debt, sometimes very much so: compare the numbers in the penultimate column of Table 4 with those in Table 2. Portugal's MSD (112%) is slightly below its debt-to-GDP ratio (117%), Italy's is essentially the same (MSD 134% and debt-to-GDP ratio 135%), and Greece's MSD (197%) is somewhat above its debt-to-GDP ratio (185%). Spain and Ireland's MSD tower over those of the other ten Eurozone countries, thanks to their high growth rates and low growth volatilities. The importance of volatility is illustrated by the case of Luxembourg, which despite having the second highest average growth rate at 5.02%, has a very modest 125% MSD, at least in comparison to Spain's 693% and Ireland's 555%; this is because Luxembourg's growth volatility is 2.71%, well above Ireland's 1.86% and, especially, Spain's 0.78%. High volatility makes default more likely, thereby disrupting the crucial process of new debt issuance for the purpose of servicing existing debt.

Our purpose in comparing the 1999-2019 and 1999-2007 periods is not to argue that the latter somehow is more representative of future growth than the former. Rather, it is to obtain what may be considered a lower and an upper bound on MSD, the former obtained from the lower-growth, higher-volatility 1999-2019 period, the latter from the higher-growth, lower-volatility pre-GFC 1999-2007 period. Under such an interpretation, two countries that are close to their upper bound, and therefore may be cause for concern, are Italy and Portugal, in addition of course to Greece.

We examine the impact of other determinants of MSD. One is the maximum primary surplus α. MSD increases linearly in α, an immediate implication of (15). Our simulation of an average Eurozone country, one whose growth rate is the GDP-weighted average of the growth rates of the twelve countries we consider, shows in Panel (c) of Figure 4 that a doubling of the MPS from 5% to 10% increases MSD from 86% to 173%. That a country can sustain a primary surplus of 10% of GDP every time this is made necessary by debt service requirements is however open to doubt: Eichengreen and Panizza (2016) show that large and persistent primary surpluses are rare.

Figure 4: Maximal Sustainable Debt: Comparative Statics



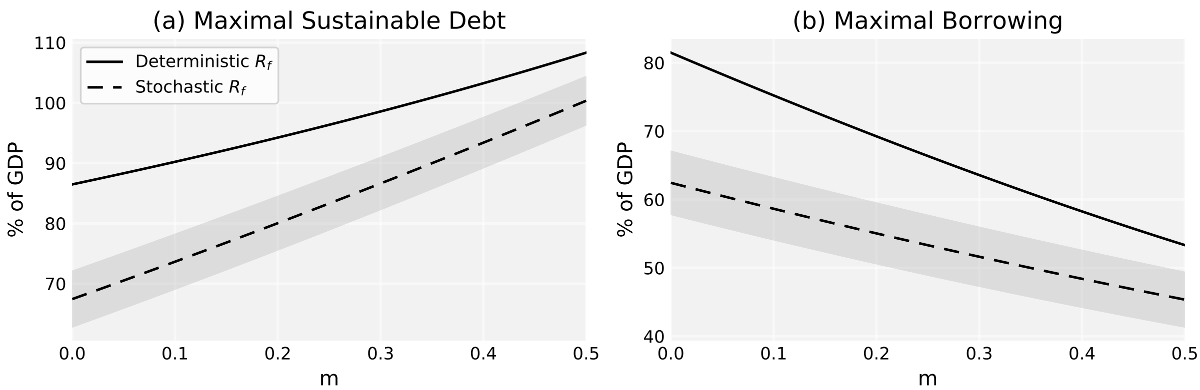
Note: In the case of stochastic real interest rates, we report the average maximal sustainable debt obtained using the ergodic distribution of the Markov Chain for the interest rate, along with the 2 standard deviation band around it (shaded area).

Another determinant of MSD is the recovery rate κ. Panel (d) of Figure 4 shows that an increase in the recovery rate increases MSD, at an accelerating rate. Consistently with Remark 1 and $\overline{G} > R_f$ for the average Eurozone country, MSD asymptotically tends to infinity as the recovery rate tends to one. What the recovery rates would be on the bonds of the eleven Eurozone countries other than Greece we consider if these were to default, and how different these rates would be from Greece's 30%, must however be considered open questions.

Yet another determinant is the risk-free rate. Panel (e) of Figure 4 shows how the MSD of the average Eurozone country varies with the risk-free interest rate. Clearly, the very low German borrowing rates of the past few years–recall that the yield on German Government bonds defines our risk-free rate–has helped sustain the present high levels of debt. A yield of zero percent increases MSD from 86% to 105% of GDP. Whether real interest rates will remain at their current, depressed levels also is an open question.

Despite or perhaps because of the controversies that have accompanied its birth (Tooze (2019)), the ESM appears to be moderately effective at increasing MSD. Panel (a) of Figure 5 shows that having the ESM hold government debt equal to 50% of GDP increases MSD from 86% to 108% of GDP. This is a far from negligible increase, yet it amounts to less than one-half

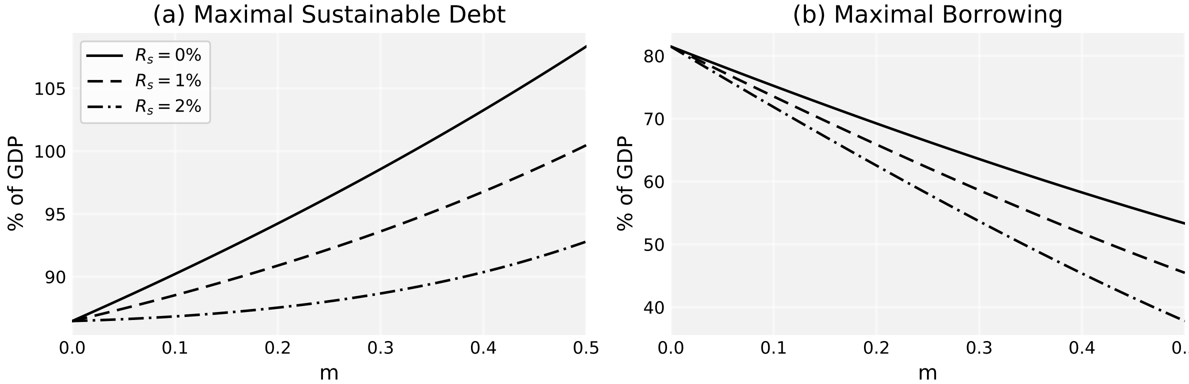
Figure 5: European Stability Mechanism



Note: In the case of stochastic real interest rates, we report the average maximal sustainable debt obtained using the ergodic distribution of the Markov Chain for the interest rate, along with the 2 standard deviation band around it (shaded area). This graph was generated assuming $R_S =$ 0%.

of the debt subscribed by the ESM. This is because ESM financing displaces private sector financing: Panel (b) of Figure 5 shows that maximum sustainable borrowing from the private sector, b_M in (16), decreases in ESM financing, from 81% to 53% of GDP. The reason, discussed in Section 4.2, is that higher total debt increases the probability of default, thereby decreasing the resources available for refinancing, in turn decreasing the resources available for debt service. When the subsidized interest R_S charged by the ESM increases from zero to 2%, the increase in MSD is only to 93%, a very modest 7 percentage point increase. This is shown in Panel (a) of Figure 6; Panel (b) shows the corresponding changes in MSB from the private sector.

Figure 6: European Stability Mechanism: Sensitivity to R_S



Note: Without loss of generality, we only report variations in MSD and borrowing in the deterministic model.

6.2 Stochastic risk-free rate

We now turn to the case of stochastic risk-free rate. The upper curves in both panels of Figure 1 show the results of testing the condition $\overline{G} < G_{c,s}(\sigma)$ for both the 1999-2019 and the 1999-2007 periods. The condition is true for all twelve countries for the former period, and for all countries but Ireland and Spain for the latter period. Recall that these are the two countries with markedly higher MSD under constant risk-free rate over the 1999-2007 period; they likely are the countries closest to attaining an environment such as analyzed by Blanchard (2019), in which growth's effectiveness at resorbing indebtedness removes any finite upper bound on debt.

The MSD under stochastic risk-free rate are shown in Table 4. That the 1999-2007 MSD exist for Ireland and Spain too, despite the failure of the condition $\overline{G} < G_{c,s}(\sigma)$ to hold over this period, confirms the sufficient but not necessary nature of that condition. Clearly, for both periods and for all countries, MSD is lower when the risk-free rate is stochastic. This is to some extent to be expected: debt has a concave payoff; its expected value decreases when the risk-free rate is stochastic; expected proceeds from new borrowing are therefore lower in such case, thereby lowering MSD. This interpretation is confirmed by Panel (f) of Figure 4, which shows MSD to be decreasing in the volatility of the stochastic risk-free rate.

The qualitative effects of the mean and the volatility of growth, the maximum primary surplus, the recovery rate, and the level of the risk-free rate are the same as in the case of constant risk-free rate; this is shown in panels (a) to (e) of Figure 4. The same is true of the effect of ESM financing (Figure 5).

The difference between the constant and stochastic cases is of a quantitative rather than a qualitative nature: the levels of MSD and of MSB are lower in the stochastic case, as are generally the (absolute values of the) slopes of the changes in the various parameters of interest.

7 Conclusion

We study public debt sustainability under the assumption that countries always try to repay but might be pushed into default because they lack the resources necessary to service existing debt. In our set up, default is always involuntary but it nevertheless limits both the availability of the primary surplus and the ability to issue new debt. These limits determine the cost of default. We derive a simple formula for maximum sustainable debt; its six main determinants are expected growth, growth volatility, maximum primary surplus, the risk-free interest rate and its volatility, and the recovery rate in the immediate aftermath of default. Our model encompasses and clarifies Blanchard's well-known result that, as long as $r < g$, countries can permanently run small deficits. We show that this result holds if there is no cost of default.

In models rooted in the Eaton and Gersovitz (1981) tradition, a higher cost of default increases willingness to pay and leads to a higher debt limit. In our model, willingness to pay is always there and a higher cost of default leads to lower maximum sustainable debt. The result that a higher cost of default increases maximum sustainable debt has important policy implications. Reforms of the international financial architecture aimed at reducing the costs of default have often been criticized on the grounds that they would be inefficient ex-ante because they reduce willingness to pay and thus limit countries' ability to borrow (Dooley, 2000). We show that in the presence of involuntary default (an assumption in line with the new consensus that countries tend to default too little and too late, IMF (2013)), such reforms might be efficient both ex-post and ex-ante.

However, we also show that such reforms need to be carefully calibrated. We study the case of an institution that provides loans at a subsidized rate (as we focus on the Eurozone, we call this institution ESM, in a global setting our analysis could be generalized to the IMF). While the presence of such an institution can increase a country's debt limit, we show that there is no free lunch. In fact, in our model, this is an expensive lunch, with the present value of the subsidy provided by the ESM being larger than the increase in maximum sustainable debt brought about by ESM intervention.[34]

Given our choice of parameters, we find that there are several Eurozone countries which have debt levels that are well above our estimated maximum sustainable debt (this was the case before the Covid pandemic and it is even more so now). We explore possible explanations for this disconnect and find that the high debt levels that we observe could be driven by the expectation of higher future GDP growth, lower growth volatility, and higher recovery rates in case of default (maximum sustainable debt increases rapidly when the recovery rate surpasses 75%).[35] The service flow value of government debt (Brunnermeier, Merkel, and Sannikov (2020)) could be another explanation.

We focus on advanced economies and assume a fully credible independent central bank that is unwilling to monetize the debt. Our model could thus be immediately applied to an emerging market country that only borrows abroad or that has a credible fixed exchange rate.[36] Extending the model to an emerging market country that borrows in both foreign and domestic currency and that does not have a fully credible central bank would require modeling both debt

[34]Note that we assume that the ESM is not senior to private creditors. It would be interesting to explore the consequence of making subsidized lending senior to market lending.

[35]We also explore the role of the maximum primary surplus and of the average interest rate and its volatility and we conclude that these factors are unlikely drivers of the high debt ratios that we observe in the Eurozone.

[36]In the first case, we would need to replace real GDP growth and its volatility with the growth and the volatility of GDP measured in foreign currency.

composition (i.e., the choice between domestic and foreign currency debt) and the temptation to inflate away domestic currency debt. Such a model could yield new avenues for studying debt sustainability in emerging market countries and for jointly analyzing external and domestic debt sustainability.

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—APPENDIX—

Derivation of (12): We have

$$\gamma = \max_G G\left[1 - \ell\left(G\right)\right]$$
$$= \max_x \exp\left[\mu + \sigma x\right]\left[1 - F\left(\exp\left[\mu + \sigma x\right]\right)\right] + \kappa \int_{-\infty}^{x} \exp\left[\mu + \sigma s\right] \mathrm{d}N\left(s\right)$$
$$= \exp\left[\mu + \frac{\sigma^2}{2}\right]\left\{\max_x \exp\left[\sigma x - \frac{\sigma^2}{2}\right]\left[1 - N\left(x\right)\right] + \kappa N\left(x - \sigma\right)\right\}.$$

■

Proof of Remark 1: (6) and (7) become when $\kappa = 1$

$$\phi\left(\omega\right) = \frac{1}{R_f} \max_d d\left[1 - F\left(\frac{d}{\omega}\right) + \frac{\omega}{d}\int_0^{d/\omega} G \mathrm{d}F\left(G\right)\right],$$

which has FOC

$$1 - F\left(\frac{d}{\omega}\right) + \frac{\omega}{d}\int_0^{d/\omega} G \mathrm{d}F\left(G\right) + d\left[-f\left(\frac{d}{\omega}\right)\frac{1}{\omega} - \frac{\omega}{d^2}\int_0^{d/\omega} G \mathrm{d}F\left(G\right) + \frac{\omega}{d}\frac{1}{\omega}\frac{d}{\omega} f\left(\frac{d}{\omega}\right)\right] = 0$$

$$\Leftrightarrow 1 - F\left(\frac{d}{\omega}\right) = 0,$$

which in turn has solution $d = \infty$.

Using (1) and (4), we can write

$$b_{M,t} = \frac{1}{R_f}\int_0^{\infty}\left(\alpha + b_{M,t+1}\right) G \mathrm{d}F\left(G\right) = \frac{E\left[G\right]}{R_f}\left(\alpha + b_{M,t+1}\right),$$

where $\mathbb{E}\left[G\right] = \int_0^{\infty} G \mathrm{d}F\left(G\right)$. By analogy to the analysis in Section 2.2, we conclude that there is no maximum debt when $\mathbb{E}\left[G\right] > R_f$; this is the Blanchard Condition. ■

Derivation of $\gamma < \mathbb{E}\left[G\right]$ when $\kappa < 1$: From (7) and (9), we can write

$$\begin{aligned}
\gamma &= \max_G G\left[1 - \ell\left(G\right)\right] \\
&= \max_G \left\{G\left[1 - F\left(G\right)\right] + \kappa \int_0^{G} g \mathrm{d}F\left(g\right)\right\} \\
&< \max_G \left\{G\left[1 - F\left(G\right)\right] + \int_0^{G} g \mathrm{d}F\left(g\right)\right\} \\
&= \int_0^{\infty} g \mathrm{d}F\left(g\right) \\
&= \mathbb{E}\left[G\right].
\end{aligned}$$

■

Proof of Lemma 2: (i) Immediate from the observation that $l'\left(\rho\right) > 0$ in Footnote 24.

(ii) From (22) we have

$$\phi^k\left(\lambda\omega\right) = \frac{1}{R^k}\max_d d\left(1 - \mathbb{E}_{\omega|k}\left[\ell\left(\frac{d}{\lambda\omega}\right)\right]\right)$$

Replace d by $\lambda d'$ to rewrite

$$\begin{aligned}\phi^k(\lambda\omega) &= \frac{1}{R^k}\max_{\lambda d'}\lambda d'\left(1-\mathbb{E}_{\omega|k}\left[\ell\left(\frac{\lambda d'}{\lambda\omega}\right)\right]\right)\\ &= \frac{\lambda}{R^k}\max_{\lambda d'} d'\left(1-\mathbb{E}_{\omega|k}\left[\ell\left(\frac{d'}{\omega}\right)\right]\right)\\ &= \frac{\lambda}{R^k}\max_{d'} d'\left(1-\mathbb{E}_{\omega|k}\left[\ell\left(\frac{d'}{\omega}\right)\right]\right)\\ &= \lambda\phi^k(\omega).\end{aligned}$$

■

Proof of Proposition 3: We denote 1_K the K-dimensional vector of ones and define the norm $\|u\|_\infty \equiv \max_k |u_k|$ for the vector $u \in \mathbb{R}^K$. From (20) we can write

$$\begin{aligned}\omega_t^k &= \alpha + \phi^k(\omega_{t+1})\\ &\leqslant \alpha + \phi^k\left(\|\omega_{t+1}\|_\infty 1_K\right)\\ &= \alpha + \|\omega_{t+1}\|_\infty \phi^k(1_K).\end{aligned}$$

where the inequality is from the result in Lemma 2 that ϕ^k is increasing and the second equality from the result that ϕ^k is homogeneous of degree 1.

Since the preceding inequality is true for every $k \in \{1,\ldots,K\}$, we can in turn write

$$\|\omega_t\|_\infty \leqslant \alpha + \|\omega_{t+1}\|_\infty \|\phi(1_K)\|_\infty$$

$$\Leftrightarrow \|\omega_{t+1}\|_\infty \geqslant \frac{1}{\|\phi(1_K)\|_\infty}\left(\|\omega_t\|_\infty - \alpha\right).$$

Define

$$\omega^* \equiv \frac{\alpha}{1-\|\phi(1_K)\|_\infty} > 0$$

and note that

$$\begin{aligned}\|\omega_{t+1}\|_\infty - \omega^* &\geqslant \frac{1}{\|\phi(1_K)\|_\infty}\left(\|\omega_t\|_\infty - \alpha\right) - \frac{\alpha}{1-\|\phi(1_K)\|_\infty}\\ &= \frac{1}{\|\phi(1_K)\|_\infty}\left(\|\omega_t\|_\infty - \alpha - \frac{\alpha\|\phi(1_K)\|_\infty}{1-\|\phi(1_K)\|_\infty}\right)\\ &= \frac{1}{\|\phi(1_K)\|_\infty}\left(\|\omega_t\|_\infty - \omega^*\right)\end{aligned}$$

This implies that, if $\|\phi(1_K)\|_\infty < 1$ and $\|\omega_t\|_\infty > \omega^*$, then $\lim_{s\to\infty}\|\omega_s\|_\infty = \infty$.

We further note from (9) and (21) that

$$\phi^k(1_K) = \frac{1}{R^k}\max_d d(1-\ell(d)) = \frac{\gamma}{R^k}$$

$$\Rightarrow \|\phi(1_K)\|_\infty = \frac{\gamma}{\min_k\left[R^k\right]}.$$

Therefore $\|\phi(1_K)\|_\infty < 1$ is equivalent to $\gamma < \min_k\left[R^k\right]$; furthermore, $\omega^* = \frac{\alpha\min_k\left[R^k\right]}{\left(\min_k\left[R^k\right]-\gamma\right)}$. ■

Proof of Proposition 4: Differentiate

$$\omega_M = \alpha + m + \phi(\omega_M),$$

with ϕ in (27) with respect to m to obtain

$$\frac{\partial\omega_M}{\partial m} = 1 + \left(\frac{G_M}{R_f}\frac{\partial\omega_M}{\partial m} - \frac{R_S}{R_f}\right)\left[1-\ell(G_M)\right]$$

$$
\begin{aligned}
\Leftrightarrow \quad \frac{\partial \omega_M}{\partial m} &= \frac{1-\frac{R_S}{R_f}\left[1-\ell\left(G_M\right)\right]}{1-\frac{G_M}{R_f}\left[1-\ell\left(G_M\right)\right]} \\
&= \frac{R_f - R_S\left[1-\ell\left(G_M\right)\right]}{R_f - G_M\left[1-\ell\left(G_M\right)\right]} \qquad (32)\\
&> 0.
\end{aligned}
$$

In order to determine the sign of $\partial\omega_M/\partial m$, we have used the two inequalities $G_M\left[1-\ell\left(G_M\right)\right] < R_f$ and $R_S < R_M$. The first inequality is the condition for ϕ to be a contraction. The second inequality reflects the provision of ESM financing on concessionary terms.[37]

Now differentiate ω_M with respect to κ to obtain

$$
\begin{aligned}
\frac{\partial \omega_M}{\partial \kappa} &= \frac{G_M}{R_f}\frac{\partial \omega_M}{\partial \kappa}\left[1-\ell\left(G_M\right)\right] - \left(\frac{\omega_M G_M - mR_S}{R_f}\right)\frac{\partial \ell\left(G_M\right)}{\partial \kappa} \\
\Leftrightarrow \left[1-\frac{G_M}{R_f}\left[1-\ell\left(G_M\right)\right]\right]\frac{\partial \omega_M}{\partial \kappa} &= -\left(\frac{\omega_M G_M - mR_S}{R_f}\right)\frac{\partial \ell\left(G_M\right)}{\partial \kappa} \\
\Leftrightarrow \frac{\partial \omega_M}{\partial \kappa} &= -\frac{\omega_M G_M - mR_S}{R_f - G_M\left[1-\ell\left(G_M\right)\right]}\frac{\partial \ell\left(G_M\right)}{\partial \kappa} > 0,
\end{aligned}
$$

where we have used $\partial\ell/\partial\kappa < 0$, $\omega_M G_M > mR_S$, and $G_M\left[1-\ell\left(G_M\right)\right] < R_f$: the first inequality is immediate from (7), the second from the first-order condition for G_M at ω_M and $\ell' > 0$ from Lemma 2, and the third from the condition for ϕ to be a contraction. ■

Derivation of (29): We have

$$
\begin{aligned}
b_M + m &= \frac{\left[\omega_M G_M - mR_S\right]\left[1-\ell\left(G_M\right)\right]}{R_f} + m \\
&= \frac{\left[\left(\alpha + b_M + m\right)G_M - mR_S + \frac{mR_f}{1-\ell(G_M)}\right]\left[1-\ell\left(G_M\right)\right]}{R_f} \\
&= \frac{\left[\alpha + b_M + m + \frac{m}{G_M}\left[R_M - R_S\right]\right]G_M\left[1-\ell\left(G_M\right)\right]}{R_f}.
\end{aligned}
$$

■

Derivation of (30): We have

$$
\begin{aligned}
PVS &= \frac{m\left(R_M - R_S\right)}{G_M}\sum_{n=1}^{\infty}\left(\frac{G_M}{R_f}\right)^n\left[1-\ell\left(G_M\right)\right]^n \\
&= \frac{m\left(R_M - R_S\right)}{G_M}\sum_{n=1}^{\infty}\left(\frac{G_M}{R_M}\right)^n \\
&= \frac{m\left(R_M - R_S\right)}{R_M}\frac{1}{1-\frac{G_M}{R_M}} \\
&= \frac{m\left(R_M - R_S\right)}{R_M - G_M}.
\end{aligned}
$$

■

Derivation of (31): We have

[37] Note that there would be no increase in MSD if there were no subsidy: $\partial\omega_M/\partial m = 0$ for $R_S = R_M$.

$$
\begin{aligned}
b_M + m - b_M^0 &= \left[\alpha + \frac{m\left(R_M - R_S\right)}{G_M}\right] \sum_{n=1}^{\infty} \left(\frac{G_M}{R_f}\right)^n \left[1 - \ell\left(G_M\right)\right]^n - \alpha \sum_{n=1}^{\infty} \left(\frac{G_M^0}{R_f}\right)^n \left[1 - \ell\left(G_M^0\right)\right]^n \\
&= \alpha \left[\sum_{n=1}^{\infty} \left(\frac{G_M}{R_M}\right)^n - \sum_{n=1}^{\infty} \left(\frac{G_M^0}{R_M^0}\right)^n\right] + \underbrace{\frac{m\left(R_M - R_S\right)}{G_M} \sum_{n=1}^{\infty} \left(\frac{G_M}{R_M}\right)^n}_{PVS} \\
&= \alpha \left[\frac{G_M}{R_M - G_M} - \frac{G_M^0}{R_M^0 - G_M^0}\right] + PVS \\
&= \alpha \left[\frac{G_M\left[1 - \ell\left(G_M\right)\right]}{R_f - G_M\left[1 - \ell\left(G_M\right)\right]} - \frac{G_M^0\left[1 - \ell\left(G_M^0\right)\right]}{R_f - G_M^0\left[1 - \ell\left(G_M^0\right)\right]}\right] + PVS.
\end{aligned}
$$

■

FOR APPENDIX

Table 5: Net Debt over GDP (%)

	(a) Full Sample (1999–2019)				*(b)* Pre-GFC Sample (1999–2007)				2019
	Mean	Std.Dev.	Min.	Max.	Mean	Std.Dev.	Min.	Max.	
Austria	51.63	6.92	40.38	60.48	45.33	3.49	40.38	50.03	47.90
Belgium	90.65	6.4	78.69	106.43	92.35	8.98	78.69	106.43	85.11
Finland	7.00	10.95	−8.10	26.93	−0.12	4.58	−5.17	7.82	26.93
France	69.46	15.45	49.52	89.38	54.18	3.92	49.52	58.88	88.88
Germany	52.53	6.24	40.83	62.39	51.43	4.55	44.92	57.44	40.83
Greece	na	[na]	na	na	na	na	na	na	n.a.
Ireland	45.46	26.54	14.58	90.05	21.35	6.05	14.59	34.46	49.25
Italy	108.19	10.63	95.66	122.24	98.55	2.28	95.65	103.63	122.08
Luxembourg	−18.17	8.56	−32.57	−8.4	−28.62	3.46	−32.57	−25.18	-8.41
Netherlands	43.54	6.46	33.23	54.84	38.42	3.09	33.23	44.61	41.49
Portugal	81.81	31.69	41.49	121.05	50.61	7.62	41.49	60.54	109.84
Spain	55.48	23.32	22.44	86.07	37.87	9.75	22.44	52.61	82.22
Simple Average	54.36	37.22			44.14	33.55			62.37