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DOI: <https://doi.org/10.1016/j.jalgebra.2008.10.013>

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ZORA URL: <https://doi.org/10.5167/uzh-23594>

Journal Article

Originally published at:

Brodmann, M; Sharp, R Y (2009). Supporting degrees of multi-graded local cohomology modules. *Journal of Algebra*, 321(2):450-482.

DOI: <https://doi.org/10.1016/j.jalgebra.2008.10.013>

SUPPORTING DEGREES OF MULTI-GRADED LOCAL COHOMOLOGY MODULES

MARKUS P. BRODMANN AND RODNEY Y. SHARP

ABSTRACT. For a finitely generated graded module M over a positively-graded commutative Noetherian ring R , the second author established in 1999 some restrictions, which can be formulated in terms of the Castelnuovo regularity of M or the so-called a^* -invariant of M , on the supporting degrees of a graded-indecomposable graded-injective direct summand, with associated prime ideal containing the irrelevant ideal of R , of any term in the minimal graded-injective resolution of M . Earlier, in 1995, T. Marley had established connections between finitely graded local cohomology modules of M and local behaviour of M across $\text{Proj}(R)$.

The purpose of this paper is to present some multi-graded analogues of the above-mentioned work.

0. INTRODUCTION

Very briefly, the purpose of this paper is to explore multi-graded analogues of some results in the algebra of modules, and particularly local cohomology modules, over a commutative Noetherian ring that is graded by the additive semigroup \mathbb{N}_0 of non-negative integers.

To describe the results that we plan to generalize, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be such a ‘positively-graded’ commutative Noetherian ring. Any unexplained notation in this Introduction will be as in Chapters 12 and 13 of our book [4]. In particular, the $*$ injective envelope of a graded R -module M will be denoted by $*E(M)$ (see [4, §13.2]), and, for $t \in \mathbb{Z}$, the t th shift functor (on the category $*\mathcal{C}(R)$ of all graded R -modules and homogeneous R -homomorphisms) will be denoted by $(\cdot)(t)$ (see [4, §12.1]).

Let \mathbb{N} denote the set of positive integers; set $R_+ := \bigoplus_{n \in \mathbb{N}} R_n$, the irrelevant ideal of R . For a graded R -module M and $\mathfrak{p} \in * \text{Spec}(R)$ (the set of homogeneous prime ideals of R), we use $M_{(\mathfrak{p})}$ to denote the homogeneous localization of M at \mathfrak{p} . For $i \in \mathbb{N}_0$, the ordinary Bass number $\mu^i(\mathfrak{p}, M)$ is equal to the rank of the homogeneous localization $(* \text{Ext}_R^i(R/\mathfrak{p}, M))_{(\mathfrak{p})}$ as a (free) module over $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ (see R. Fossum and H.-B. Foxby [6, Corollary 4.9]).

Let $i \in \mathbb{N}_0$, and consider a direct decomposition given by a homogeneous isomorphism

$$*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha \in \Lambda_i} *E(R/\mathfrak{p}_\alpha)(-n_\alpha),$$

for an appropriate family $(\mathfrak{p}_\alpha)_{\alpha \in \Lambda_i}$ of graded prime ideals of R and an appropriate family $(n_\alpha)_{\alpha \in \Lambda_i}$ of integers. (See [4, §13.2].)

Suppose that the graded prime ideal \mathfrak{p} contains the irrelevant ideal R_+ . In this case, the graded ring $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ is concentrated in degree 0, and its 0th component is a field isomorphic to $k_{R_0}(\mathfrak{p}_0)$, the residue field of the local ring $(R_0)_{\mathfrak{p}_0}$. Thus,

$$\mu^i(\mathfrak{p}, M) = \dim_{k_{R_0}(\mathfrak{p}_0)} (* \text{Ext}_R^i(R/\mathfrak{p}, M))_{(\mathfrak{p})} = \sum_{t \in \mathbb{Z}} \dim_{k_{R_0}(\mathfrak{p}_0)} ((* \text{Ext}_R^i(R/\mathfrak{p}, M))_{(\mathfrak{p})})_t.$$

Date: October 24, 2008.

2000 Mathematics Subject Classification. Primary 13D45, 13E05, 13A02; Secondary 13C15.

Key words and phrases. Multi-graded commutative Noetherian ring, multi-graded local cohomology module, associated prime ideal, multi-graded injective module, Bass numbers, shifts, anchor points, finitely graded module, the Annihilator Theorem for local cohomology, grade.

The second author was partially supported by the Swiss National Foundation (Grant Number 20-103491/1), and the Engineering and Physical Sciences Research Council of the United Kingdom (Grant Number EP/C538803/1).

In [15], it was shown that the graded $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ -module $({}^*\mathrm{Ext}_R^i(R/\mathfrak{p}, M))_{(\mathfrak{p})}$ carries information about the shifts ‘ $-n_\alpha$ ’ for those $\alpha \in \Lambda_i$ for which $\mathfrak{p}_\alpha = \mathfrak{p}$. One has

$${}^*E(R/\mathfrak{p})(n) \not\cong {}^*E(R/\mathfrak{p})(m) \text{ in } {}^*\mathcal{C}(R) \quad \text{for } m, n \in \mathbb{Z} \text{ with } m \neq n,$$

and, for a given $t \in \mathbb{Z}$, the cardinality of the set $\{\alpha \in \Lambda_i : \mathfrak{p}_\alpha = \mathfrak{p} \text{ and } n_\alpha = t\}$ is equal to

$$\dim_{k_{R_0}(\mathfrak{p}_0)} \left(({}^*\mathrm{Ext}_R^i(R/\mathfrak{p}, M))_{(\mathfrak{p})} \right)_t,$$

the dimension of the t th component of $({}^*\mathrm{Ext}_R^i(R/\mathfrak{p}, M))_{(\mathfrak{p})}$.

Let ${}^*\mathrm{Var}(R_+) := \{\mathfrak{q} \in {}^*\mathrm{Spec}(R) : \mathfrak{q} \supseteq R_+\}$. Let $\mathfrak{p} \in {}^*\mathrm{Var}(R_+)$, let $i \in \mathbb{N}_0$ and let $t \in \mathbb{Z}$. We say that t is an *i th level anchor point of \mathfrak{p} for M* if

$$\left(({}^*\mathrm{Ext}_R^i(R/\mathfrak{p}, M))_{(\mathfrak{p})} \right)_t \neq 0;$$

the set of all i th level anchor points of \mathfrak{p} for M is denoted by $\mathrm{anch}^i(\mathfrak{p}, M)$; also, we write

$$\mathrm{anch}(\mathfrak{p}, M) = \bigcup_{j \in \mathbb{N}_0} \mathrm{anch}^j(\mathfrak{p}, M),$$

and refer to this as the set of *anchor points of \mathfrak{p} for M* . Thus $\mathrm{anch}^i(\mathfrak{p}, M)$ is the set of integers h for which, when we decompose

$${}^*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha \in \Lambda_i} {}^*E(R/\mathfrak{p}_\alpha)(-n_\alpha)$$

by means of a homogeneous isomorphism, there exists $\alpha \in \Lambda_i$ with $\mathfrak{p}_\alpha = \mathfrak{p}$ and $n_\alpha = h$. Note that $\mathrm{anch}^i(\mathfrak{p}, M) = \emptyset$ if $\mu^i(\mathfrak{p}, M) = 0$, and that $\mathrm{anch}^i(\mathfrak{p}, M)$ is a finite set when M is finitely generated.

It was also shown in [15] that, when the graded R -module M is non-zero and finitely generated, the Castelnuovo regularity $\mathrm{reg}(M)$ of M is an upper bound for the set

$$\bigcup_{\mathfrak{p} \in {}^*\mathrm{Var}(R_+)} \mathrm{anch}(\mathfrak{p}, M)$$

of all anchor points of M . Consequently, for each $i \geq 0$, every $*$ indecomposable $*$ injective direct summand F of ${}^*E^i(M)$ with associated prime containing R_+ must have $F_j = 0$ for all $j > \mathrm{reg}(M)$.

In §§2,3 we shall present an analogue of this theory for a standard multi-graded commutative Noetherian ring $S = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} S_{\mathbf{n}}$ (where $r \in \mathbb{N}$ with $r \geq 2$). There is a satisfactory generalization of anchor point theory to the multi-graded case, but we must stress now that we have not uncovered any links between our multi-graded anchor point theory and the fast-developing theory of multi-graded Castelnuovo regularity (see, for example, Huy Tài Hà [9] and D. Maclagan and G. G. Smith [12]). This may be because our multi-graded anchor point theory only yields information about multi-graded local cohomology modules with respect to \mathbb{N}_0^r -graded ideals of S that contain one of the components $S_{(0, \dots, 0, 1, 0, \dots, 0)}$, whereas the ideal $S_+ := \bigoplus_{\mathbf{n} \in \mathbb{N}^r} S_{\mathbf{n}}$, which is relevant to multi-graded Castelnuovo regularity, normally does not have that property.

The short §4 provides some motivation for our work in §5, where we provide multi-graded analogues of work of T. Marley [14] about finitely graded local cohomology modules. We say that a graded R -module $L = \bigoplus_{n \in \mathbb{Z}} L_n$ is *finitely graded* precisely when $L_n \neq 0$ for only finitely many $n \in \mathbb{Z}$. In [14], Marley defined, for a finitely generated graded R -module M ,

$$g_a(M) := \sup \{k \in \mathbb{N}_0 : H_a^i(M) \text{ is finitely graded for all } i < k\},$$

and he modified ideas of N. V. Trung and S. Ikeda in [16, Lemma 2.2] to prove that

$$g_a(M) := \sup \left\{ k \in \mathbb{N}_0 : R_+ \subseteq \sqrt{(0 :_R H_a^i(M))} \text{ for all } i < k \right\};$$

he then used Faltings’ Annihilator Theorem for local cohomology (see [5] and [4, Theorem 9.5.1]). In §5 below, we shall obtain some multi-graded analogues of some of Marley’s results in this area.

1. BACKGROUND RESULTS IN MULTI-GRADED COMMUTATIVE ALGEBRA

Let $R = \bigoplus_{g \in G} R_g$ be a commutative Noetherian ring graded by a finitely generated, additively-written, torsion-free Abelian group G . Some aspects of the G -graded analogue of the theory of Bass numbers have been developed by S. Goto and K.-i. Watanabe [8, §§1.2, 1.3], and it is appropriate for us to review some of those here.

We shall denote by ${}^*\mathcal{C}^G(R)$ (or sometimes by ${}^*\mathcal{C}(R)$ when the grading group G is clear) the category of all G -graded R -modules and G -homogeneous R -homomorphisms of degree 0_G between them. Projective (respectively injective) objects in the category ${}^*\mathcal{C}^G(R)$ will be referred to as * projective (respectively * injective) G -graded R -modules. Similarly, the attachment of * to other concepts indicates that they refer to the obvious interpretations of those concepts in the category ${}^*\mathcal{C}^G(R)$, although we shall sometimes use ‘ G ’ instead of * in order to emphasize the grading group. However, the following comments about ${}^*\mathrm{Hom}_R$ and the ${}^*\mathrm{Ext}_R^i$ ($i \geq 0$) may be helpful.

1.1. Reminders. Let $M = \bigoplus_{g \in G} M_g$ and $N = \bigoplus_{g \in G} N_g$ be G -graded R -modules.

- (i) Let $a \in G$. We say that an R -homomorphism $f : M \rightarrow N$ is G -homogeneous of degree a precisely when $f(M_g) \subseteq N_{g+a}$ for all $g \in G$. Such a G -homogeneous homomorphism of degree 0_G is simply called G -homogeneous. We denote by ${}^*\mathrm{Hom}_R(M, N)_a$ the R_{0_G} -submodule of $\mathrm{Hom}_R(M, N)$ consisting of all G -homogeneous R -homomorphisms from M to N of degree a . Then the sum $\sum_{a \in G} {}^*\mathrm{Hom}_R(M, N)_a$ is direct, and we set

$${}^*\mathrm{Hom}_R(M, N) := \sum_{a \in G} {}^*\mathrm{Hom}_R(M, N)_a = \bigoplus_{a \in G} {}^*\mathrm{Hom}_R(M, N)_a.$$

This is an R -submodule of $\mathrm{Hom}_R(M, N)$, and the above direct decomposition provides it with a structure as G -graded R -module. It is straightforward to check that

$${}^*\mathrm{Hom}_R(\bullet, \bullet) : {}^*\mathcal{C}^G(R) \times {}^*\mathcal{C}^G(R) \longrightarrow {}^*\mathcal{C}^G(R)$$

is a left exact, additive functor.

- (ii) If M is finitely generated, then $\mathrm{Hom}_R(M, N)$ is actually equal to ${}^*\mathrm{Hom}_R(M, N)$ with its G -grading forgotten.
- (iii) For $i \in \mathbb{N}_0$, the functor ${}^*\mathrm{Ext}_R^i$ is the i th right derived functor in ${}^*\mathcal{C}^G(R)$ of ${}^*\mathrm{Hom}_R$. We make two comments here about the case where M is finitely generated. In that case $\mathrm{Ext}_R^i(M, N)$ is actually equal to ${}^*\mathrm{Ext}_R^i(M, N)$ with its G -grading forgotten, and, second, one can calculate the ${}^*\mathrm{Ext}_R^i(M, N)$ by applying the functor ${}^*\mathrm{Hom}_R(M, \bullet)$ to a (deleted) * injective resolution of N in the category ${}^*\mathcal{C}^G(R)$ and then taking cohomology of the resulting complex.

For $a \in G$, we shall denote the a th shift functor by $(\bullet)(a) : {}^*\mathcal{C}^G(R) \rightarrow {}^*\mathcal{C}^G(R)$: thus, for a G -graded R -module $M = \bigoplus_{g \in G} M_g$, we have $(M(a))_g = M_{g+a}$ for all $g \in G$; also, $f(a) \upharpoonright_{(M(a))_g} = f \upharpoonright_{M_{g+a}}$ for each morphism f in ${}^*\mathcal{C}^G(R)$ and all $g \in G$.

1.2. Theorem (S. Goto and K.-i. Watanabe [8, §1.3]). *Let M be G -graded R -module, and denote by ${}^*\mathrm{Spec}(R)$ the set of G -graded prime ideals of R . We denote by ${}^*E(M)$ or ${}^*E_R(M)$ ‘the’ * injective envelope of M , and by ${}^*E^i(M)$ or ${}^*E_R^i(M)$ ‘the’ i th term in ‘the’ minimal * injective resolution of M (for each $i \geq 0$).*

- (i) $\mathrm{Ass}_R {}^*E_R(M) = \mathrm{Ass}_R M$.
- (ii) *We have that M is a * indecomposable * injective G -graded R -module if and only if M is isomorphic (in the category ${}^*\mathcal{C}^G(R)$) to ${}^*E(R/\mathfrak{q})(a)$ for some $\mathfrak{q} \in {}^*\mathrm{Spec}(R)$ and $a \in G$. In this case, $\mathrm{Ass}_R M = \{\mathfrak{q}\}$ and \mathfrak{q} is uniquely determined by M .*
- (iii) *Let $(M_\lambda)_{\lambda \in \Lambda}$ be a non-empty family of G -graded R -modules. Then $\bigoplus_{\lambda \in \Lambda} M_\lambda$ is * injective if and only if M_λ is * injective for all $\lambda \in \Lambda$.*
- (iv) *Each * injective G -graded R -module M is a direct sum of * indecomposable * injective G -graded submodules, and this decomposition is uniquely determined by M up to isomorphisms.*
- (v) *Let i be a non-negative integer. In view of part (iv) above, there is a family $(\mathfrak{p}_\alpha)_{\alpha \in \Lambda_i}$ of G -graded prime ideals of R and a family $(g_\alpha)_{\alpha \in \Lambda_i}$ of elements of G for which there is a G -homogeneous*

isomorphism

$$*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha \in \Lambda_i} *E(R/\mathfrak{p}_\alpha)(-g_\alpha).$$

Let $\mathfrak{p} \in *\mathrm{Spec}(R)$. Then the cardinality of the set $\{\alpha \in \Lambda_i : \mathfrak{p}_\alpha = \mathfrak{p}\}$ is equal to the ordinary Bass number $\mu^i(\mathfrak{p}, M)$ (that is, to $\dim_{k(\mathfrak{p})} \mathrm{Ext}_R^i(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p})$, where $k(\mathfrak{p})$ denotes the residue field of the local ring $R_\mathfrak{p}$).

A significant part of §2 of this paper is concerned with the shifts ‘ $-g_\alpha$ ’ in the statement of part (v) of Theorem 1.2. (The minus signs are inserted for notational convenience.) In [15], the second author obtained some results about such shifts in the special case in which R is graded by the semigroup \mathbb{N}_0 of non-negative integers, and in §2 below, we shall establish some multi-graded analogues.

We shall employ the following device used by Huy Tài Hà [9, §2].

1.3. Definition. Let $\phi : G \rightarrow H$ be a homomorphism of finitely generated torsion-free Abelian groups, and let $R = \bigoplus_{g \in G} R_g$ be a G -graded commutative Noetherian ring.

For each $h \in H$, set $R_h^\phi := \bigoplus_{g \in \phi^{-1}(\{h\})} R_g$; then

$$R^\phi := \bigoplus_{h \in H} R_h^\phi = \bigoplus_{h \in H} \left(\bigoplus_{g \in \phi^{-1}(\{h\})} R_g \right)$$

provides an H -grading on R , and we denote R by R^ϕ when considering it as an H -graded ring in this way.

Furthermore, for each G -graded R -module $M = \bigoplus_{g \in G} M_g$, set $M_h^\phi := \bigoplus_{g \in \phi^{-1}(\{h\})} M_g$ and $M^\phi := \bigoplus_{h \in H} M_h^\phi$; then M^ϕ is an H -graded R^ϕ -module. Also, if $f : M \rightarrow N$ is a G -homogeneous homomorphism of G -graded R -modules, then the same map f becomes an H -homogeneous homomorphism of H -graded R^ϕ -modules $f^\phi : M^\phi \rightarrow N^\phi$.

In this way, $(\cdot)^\phi$ becomes an exact additive covariant functor from $*C^G(R)$ to $*C^H(R)$.

1.4. Notation. We shall use \mathbb{N} and \mathbb{N}_0 to denote the sets of positive and non-negative integers, respectively, and r will denote a fixed positive integer. Throughout the remainder of the paper, $R := \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} R_{\mathbf{n}}$ will denote a commutative Noetherian ring, graded by the additively-written finitely generated free Abelian group \mathbb{Z}^r (with its usual addition). For $\mathbf{n} = (n_1, \dots, n_r)$, $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$, we shall write

$$\mathbf{n} \leq \mathbf{m} \quad \text{if and only if} \quad n_i \leq m_i \text{ for all } i = 1, \dots, r;$$

furthermore, $\mathbf{n} < \mathbf{m}$ will mean that $\mathbf{n} \leq \mathbf{m}$ and $\mathbf{n} \neq \mathbf{m}$. The zero element of \mathbb{Z}^r will be denoted by $\mathbf{0}$, and, for each $i = 1, \dots, r$, we shall use \mathbf{e}_i to denote the element of \mathbb{Z}^r which has 1 in the i th spot and all other components zero. Also, $\mathbf{1}$ will denote $(1, \dots, 1) \in \mathbb{Z}^r$. Thus $\mathbf{1} = \sum_{i=1}^r \mathbf{e}_i$, and $R_{\mathbf{e}_1} R_{\mathbf{e}_2} \dots R_{\mathbf{e}_r} \subseteq R_{\mathbf{1}}$.

We shall sometimes denote the i th component of a general member \mathbf{w} of \mathbb{Z}^r by w_i without additional explanation.

Comments made above that apply to the category $*C^{\mathbb{Z}^r}(R)$ will be used without further comment. For example, we shall say that a graded ideal of R is **maximal* if it is maximal among the set of proper \mathbb{Z}^r -graded ideals of R , and that R is **local* if it has a unique **maximal* ideal. We shall use $*\mathrm{Max}(R)$ to denote the set of **maximal* ideals of R .

We shall use $*\mathrm{Spec}(R)$ to denote the set of \mathbb{Z}^r -graded prime ideals of R ; for a \mathbb{Z}^r -graded ideal \mathfrak{a} of R , we shall set $*\mathrm{Var}(\mathfrak{a}) := \{\mathfrak{p} \in *\mathrm{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$.

The next three lemmas are multi-graded analogues of preparatory results in [15, §1].

1.5. Lemma. Let $\mathfrak{p} \in *\mathrm{Spec}(R)$ and let a be an \mathbb{Z}^r -homogeneous element of degree \mathbf{n} in $R \setminus \mathfrak{p}$. Then multiplication by a provides a \mathbb{Z}^r -homogeneous automorphism of degree \mathbf{n} of $*E(R/\mathfrak{p})$. Also, each element of $*E(R/\mathfrak{p})$ is annihilated by some power of \mathfrak{p} .

Consequently, if S is a multiplicatively closed subset of \mathbb{N}_0^r -homogeneous elements of R such that $S \cap \mathfrak{p} \neq \emptyset$, then $S^{-1}(*E(R/\mathfrak{p})) = 0$.

Proof. Multiplication by a provides a \mathbb{Z}^r -homogeneous R -homomorphism

$$\mu_a : *E(R/\mathfrak{p}) \longrightarrow *E(R/\mathfrak{p})(\mathbf{n}).$$

Since $\text{Ker } \mu_a$ has zero intersection with R/\mathfrak{p} , it follows that μ_a is injective. In view of Theorem 1.2(ii), $\text{Im } \mu_a$ is a non-zero $*$ injective \mathbb{Z}^r -graded submodule of the $*$ indecomposable $*$ injective \mathbb{Z}^r -graded R -module $*E(R/\mathfrak{p})(\mathbf{n})$. Hence μ_a is surjective.

The fact that each element of $*E(R/\mathfrak{p})$ is annihilated by some power of \mathfrak{p} follows from Theorem 1.2(i), which shows that \mathfrak{p} is the only associated prime ideal of each non-zero cyclic submodule of $*E(R/\mathfrak{p})$. The final claim is then immediate. \square

The next two lemmas below can be proved by making obvious modifications to the proofs of the (well-known) ‘ungraded’ analogues.

1.6. Lemma. *Let $f : L \longrightarrow M$ be a \mathbb{Z}^r -homogeneous homomorphism of \mathbb{Z}^r -graded R -modules such that M is a $*$ essential extension of $\text{Im } f$. Let S be a multiplicatively closed subset of \mathbb{Z}^r -homogeneous elements of R . Then $S^{-1}M$ is a $*$ essential extension of its \mathbb{Z}^r -graded submodule $\text{Im}(S^{-1}f)$.*

Proof. Modify the proof of [4, 11.1.5] in the obvious way. \square

1.7. Lemma. *Let S be a multiplicatively closed subset of \mathbb{Z}^r -homogeneous elements of R , and let $\mathfrak{p} \in * \text{Spec}(R)$ be such that $\mathfrak{p} \cap S = \emptyset$. Then*

- (i) *the natural map $*E_R(R/\mathfrak{p}) \longrightarrow S^{-1}(*E_R(R/\mathfrak{p}))$ is a \mathbb{Z}^r -homogeneous R -isomorphism, so that $*E_R(R/\mathfrak{p})$ has a natural structure as a \mathbb{Z}^r -graded $S^{-1}R$ -module;*
- (ii) *there is a \mathbb{Z}^r -homogeneous isomorphism (in $*\mathcal{C}(S^{-1}R)$)*

$$*E_R(R/\mathfrak{p}) \cong *E_{S^{-1}R}(S^{-1}R/S^{-1}\mathfrak{p});$$

- (iii) *$*E_{S^{-1}R}(S^{-1}R/S^{-1}\mathfrak{p})$, when considered as a \mathbb{Z}^r -graded R -module by means of the natural homomorphism $R \longrightarrow S^{-1}R$, is \mathbb{Z}^r -homogeneously isomorphic to $*E_R(R/\mathfrak{p})$;*
- (iv) *for each $\mathbf{n} \in \mathbb{Z}^r$, there is a \mathbb{Z}^r -homogeneous isomorphism (in $*\mathcal{C}(S^{-1}R)$)*

$$S^{-1}(*E_R(R/\mathfrak{p})(\mathbf{n})) \cong *E_{S^{-1}R}(S^{-1}R/S^{-1}\mathfrak{p})(\mathbf{n});$$

- (v) *if I is a $*$ injective \mathbb{Z}^r -graded R -module, then the \mathbb{Z}^r -graded $S^{-1}R$ -module $S^{-1}I$ is $*$ injective.*

Proof. (i) This is immediate from 1.5.

(ii) One can make the obvious modifications to the proof of [4, 10.1.11] to see that, as a \mathbb{Z}^r -graded $S^{-1}R$ -module, $*E_R(R/\mathfrak{p})$ is $*$ injective; it is also \mathbb{Z}^r -homogeneously isomorphic, as a \mathbb{Z}^r -graded $S^{-1}R$ -module, to $S^{-1}(*E_R(R/\mathfrak{p}))$. One can use 1.6 to see that $S^{-1}(*E_R(R/\mathfrak{p}))$ is a $*$ essential extension of $S^{-1}R/S^{-1}\mathfrak{p}$. The claim follows.

(iii), (iv) These are now easy.

(v) This can now be proved by making the obvious modifications to the proof of [4, 10.1.13(ii)]. \square

2. A MULTI-GRADED ANALOGUE OF ANCHOR POINT THEORY

2.1. Definition. We shall say that R is *positively graded* precisely when $R_{\mathbf{n}} = 0$ for all $\mathbf{n} \not\geq \mathbf{0}$. When that is the case, we say that R (as in 1.4) is *standard* precisely when $R = R_{\mathbf{0}}[R_{\mathbf{e}_1}, \dots, R_{\mathbf{e}_r}]$.

The main results of this paper will concern the case where R is positively graded and standard.

2.2. Lemma. *Suppose that $R := \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. If \mathfrak{a} is an \mathbb{N}_0^r -graded ideal of R such that $\mathfrak{a} \supseteq R_{\mathbf{t}}$ for some $\mathbf{t} \in \mathbb{N}_0^r$, then $\mathfrak{a} \supseteq R_{\mathbf{n}}$ for each $\mathbf{n} \in \mathbb{N}_0^r$ with $\mathbf{n} \geq \mathbf{t}$.*

Proof. Since R is standard, $R_{\mathbf{n}} = R_{\mathbf{t}}R_{\mathbf{n}-\mathbf{t}}$, and so is contained in \mathfrak{a} . \square

2.3. Definition. Suppose that $R := \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. Let $\mathfrak{p} \in * \text{Spec}(R)$. The set $\{j \in \{1, \dots, r\} : R_{\mathbf{e}_j} \subseteq \mathfrak{p}\}$ will be called *the set of \mathfrak{p} -directions* and will be denoted by $\text{dir}(\mathfrak{p})$.

Observe that, if $i \in \text{dir}(\mathfrak{p})$, then $\mathfrak{p} \supseteq R_{\mathbf{1}}$ by 2.2. Conversely, if $\mathfrak{p} \supseteq R_{\mathbf{1}}$, then, since $R_{\mathbf{1}} = R_{\mathbf{e}_1} \dots R_{\mathbf{e}_r}$, there exists $i \in \{1, \dots, r\}$ such that $R_{\mathbf{e}_i} \subseteq \mathfrak{p}$, and $i \in \text{dir}(\mathfrak{p})$. Thus $\text{dir}(\mathfrak{p}) \neq \emptyset$ if and only if $\mathfrak{p} \supseteq R_{\mathbf{1}}$.

More generally, let \mathfrak{b} be an \mathbb{N}_0^r -graded ideal of R . We define *the set of \mathfrak{b} -directions* to be

$$\text{dir}(\mathfrak{b}) := \left\{ j \in \{1, \dots, r\} : R_{\mathbf{e}_j} \subseteq \sqrt{\mathfrak{b}} \right\}.$$

The members of the set $\{1, \dots, r\} \setminus \text{dir}(\mathfrak{b})$ are called the *non- \mathfrak{b} -directions*. It is easy to see that $\text{dir}(\mathfrak{b}) = \bigcap_{\mathfrak{p} \in \text{Min}(\mathfrak{b})} \text{dir}(\mathfrak{p})$, where $\text{Min}(\mathfrak{b})$ denotes the set of minimal prime ideals of \mathfrak{b} .

2.4. *Remark.* It follows from Lemma 2.2 that, in the situation of Definition 2.3, each \mathbb{N}_0^r -homogeneous element of $R \setminus \mathfrak{p}$ has degree with i th component 0 for all $i \in \text{dir}(\mathfrak{p})$.

2.5. **Proposition.** *Suppose that $R := \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. Let $\mathfrak{p} \in {}^* \text{Var}(R_1 R)$. For notational convenience, suppose that $\text{dir}(\mathfrak{p}) = \{1, \dots, m\}$, where $0 < m \leq r$. For each $i \in \{1, \dots, r\} \setminus \text{dir}(\mathfrak{p}) = \{m+1, \dots, r\}$, select $u_i \in R_{\mathbf{e}_i} \setminus \mathfrak{p}$.*

Let $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{Z}^m$. For $\mathbf{c} = (c_{m+1}, \dots, c_r) \in \mathbb{Z}^{r-m}$, we shall denote by $\mathbf{a}|\mathbf{c}$ the element $(a_1, \dots, a_m, c_{m+1}, \dots, c_r)$ of \mathbb{Z}^r obtained by juxtaposition.

(i) *For all choices of $\mathbf{c}, \mathbf{d} \in \mathbb{Z}^{r-m}$, there is an isomorphism of R_0 -modules*

$$({}^*E_R(R/\mathfrak{p}))_{\mathbf{a}|\mathbf{c}} \cong ({}^*E_R(R/\mathfrak{p}))_{\mathbf{a}|\mathbf{d}}.$$

(Note that this does not say anything of interest if $m = r$.)

- (ii) *If $({}^*E_R(R/\mathfrak{p}))_{\mathbf{a}|\mathbf{c}} \neq 0$ for any $\mathbf{c} \in \mathbb{Z}^{r-m}$, then $\mathbf{a} \leq \mathbf{0}$.*
 (iii) *Let $T := R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$, where $R_{(\mathfrak{p})}$ is the \mathbb{Z}^r -homogeneous localization of R at \mathfrak{p} . Then*

- (a) *T is a simple \mathbb{Z}^r -graded ring in the sense of [8, Definition 1.1.1];*
 (b) *T_0 is a field;*
 (c) *for each $\mathbf{c} = (c_{m+1}, \dots, c_r) \in \mathbb{Z}^{r-m}$,*

$$T_{\mathbf{a}|\mathbf{c}} = \begin{cases} 0 & \text{if } \mathbf{a} \neq \mathbf{0}, \\ T_0(\overline{u_{m+1}/1})^{c_{m+1}} \dots (\overline{u_r/1})^{c_r} & \text{if } \mathbf{a} = \mathbf{0} \end{cases}$$

(where $\overline{}$ is used to denote natural images of elements of $R_{(\mathfrak{p})}$ in T); and

- (d) *every \mathbb{Z}^r -graded T -module is free.*
 (iv) *We have $(0 : {}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}))_{\mathfrak{p}R_{(\mathfrak{p})}} = R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$.*
 (v) *If $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^m$ and $\mathbf{c}, \mathbf{d} \in \mathbb{Z}^{r-m}$, and there is a \mathbb{Z}^r -homogeneous isomorphism*

$$({}^*E_R(R/\mathfrak{p}))_{\mathbf{a}|\mathbf{c}} \cong ({}^*E_R(R/\mathfrak{p}))_{\mathbf{b}|\mathbf{d}},$$

then $\mathbf{a} = \mathbf{b}$.

Note. The obvious interpretation of the above statement is to be made in the case where $m = r$.

Proof. It will be convenient to write \mathbf{v} for a general member of \mathbb{Z}^m and \mathbf{w} for a general member of \mathbb{Z}^{r-m} , and to use $\mathbf{v}|\mathbf{w}$ to indicate the element of \mathbb{Z}^r obtained by juxtaposition.

(i) By Lemma 1.5, for each $i = m+1, \dots, r$, multiplication by u_i provides a \mathbb{Z}^r -homogeneous automorphism of ${}^*E_R(R/\mathfrak{p})$ of degree \mathbf{e}_i ; the claim follows from this.

(ii) Set $\Delta := \{\mathbf{v} \in \mathbb{Z}^m : v_i > 0 \text{ for some } i \in \{1, \dots, m\}\}$. Since $R_{\mathbf{e}_i} \subseteq \mathfrak{p}$ for all $i = 1, \dots, m$, it follows from Lemma 2.2 that the \mathbb{Z}^r -graded R -module R/\mathfrak{p} has $(R/\mathfrak{p})_{\mathbf{v}|\mathbf{w}} = 0$ for all choices of $\mathbf{v}|\mathbf{w} \in \mathbb{Z}^r$ with $\mathbf{v} \in \Delta$. Therefore the \mathbb{Z}^r -graded submodule

$$\bigoplus_{\substack{\mathbf{v} \in \Delta \\ \mathbf{w} \in \mathbb{Z}^{r-m}}} (R/\mathfrak{p})_{\mathbf{v}|\mathbf{w}}$$

of R/\mathfrak{p} is zero. Since ${}^*E_R(R/\mathfrak{p})$ is a * essential extension of R/\mathfrak{p} , it follows that

$$\bigoplus_{\substack{\mathbf{v} \in \Delta \\ \mathbf{w} \in \mathbb{Z}^{r-m}}} ({}^*E_R(R/\mathfrak{p}))_{\mathbf{v}|\mathbf{w}} = 0.$$

(iii) By Remark 2.4, each \mathbb{N}_0^r -homogeneous element of $R \setminus \mathfrak{p}$ has degree $\mathbf{v}|\mathbf{w}$ with $\mathbf{v} = \mathbf{0}$. Also, $(R/\mathfrak{p})_{\mathbf{v}|\mathbf{w}} = 0$ for all $\mathbf{v} \in \mathbb{Z}^m$ with $\mathbf{v} > \mathbf{0}$. Now every non-zero \mathbb{Z}^r -homogeneous element of T is a unit of T , so that T is a simple \mathbb{Z}^r -graded ring. Furthermore, the subgroup

$$G := \{\mathbf{n} \in \mathbb{Z}^r : T_{\mathbf{n}} \text{ contains a unit of } T\}$$

is equal to $\{(n_1, \dots, n_m, n_{m+1}, \dots, n_r) \in \mathbb{Z}^r : n_1 = \dots = n_m = 0\}$. The claims in parts (b), (c) and (d) now follow from [8, Lemma 1.1.2, Corollary 1.1.3 and Theorem 1.1.4].

(iv) Recall that $T = R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$. Now the \mathbb{Z}^r -graded T -module $(0 : {}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}) \mathfrak{p}R_{(\mathfrak{p})})$ contains its \mathbb{Z}^r -graded T -submodule $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$, and cannot be strictly larger, by * essentiality and the fact (see part (iii)) that every \mathbb{Z}^r -graded T -module is free.

(v) By Lemma 1.7(iv), there is a \mathbb{Z}^r -homogeneous isomorphism of \mathbb{Z}^r -graded $R_{(\mathfrak{p})}$ -modules

$$({}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}))(\mathbf{a}|\mathbf{c}) \cong ({}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}))(\mathbf{b}|\mathbf{d}).$$

Abbreviate ${}^*E_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})$ by F . It follows from part (iv) that

$$\begin{aligned} T(\mathbf{a}|\mathbf{c}) &= (0 :_F \mathfrak{p}R_{(\mathfrak{p})})(\mathbf{a}|\mathbf{c}) = (0 :_{F(\mathbf{a}|\mathbf{c})} \mathfrak{p}R_{(\mathfrak{p})}) \\ &\cong (0 :_{F(\mathbf{b}|\mathbf{d})} \mathfrak{p}R_{(\mathfrak{p})}) = (0 :_F \mathfrak{p}R_{(\mathfrak{p})})(\mathbf{b}|\mathbf{d}) \\ &= T(\mathbf{b}|\mathbf{d}), \end{aligned}$$

where the isomorphism is \mathbb{Z}^r -homogeneous. But, for $\mathbf{n} = (n_1, \dots, n_m, n_{m+1}, \dots, n_r) \in \mathbb{Z}^r$, we have

$$T(\mathbf{a}|\mathbf{c})_{\mathbf{n}} \neq 0 \quad \text{if and only if} \quad (n_1, \dots, n_m) = -\mathbf{a}$$

(by part (iii)). Therefore $\mathbf{a} = \mathbf{b}$. \square

2.6. *Remark.* Suppose that $R := \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard, and let \mathfrak{b} be an \mathbb{N}_0^r -graded ideal of R for which $\text{dir}(\mathfrak{b}) \neq \emptyset$.

Write $\text{dir}(\mathfrak{b}) = \{i_1, \dots, i_m\}$, where $0 < m \leq r$ and $i_1 < \dots < i_m$. Let $\phi(\mathfrak{b}) : \mathbb{Z}^r \rightarrow \mathbb{Z}^m$ be the epimorphism of Abelian groups defined by

$$\phi(\mathfrak{b})((n_1, \dots, n_r)) = (n_{i_1}, \dots, n_{i_m}) \quad \text{for all } (n_1, \dots, n_r) \in \mathbb{Z}^r.$$

We can think of $\phi(\mathfrak{b}) : \mathbb{Z}^r \rightarrow \mathbb{Z}^m$ as the homomorphism which ‘forgets the co-ordinates in the non- \mathfrak{b} -directions’.

Now let $\mathfrak{p} \in {}^*\text{Var}(R_{\mathbf{1}}R)$. The above defines an Abelian group homomorphism $\phi(\mathfrak{p}) : \mathbb{Z}^r \rightarrow \mathbb{Z}^{\#\text{dir}(\mathfrak{p})}$. (For a finite set Y , the notation $\#Y$ denotes the cardinality of the set Y .) In the case where $\mathfrak{b} \subseteq \mathfrak{p}$, we have $\text{dir}(\mathfrak{b}) \subseteq \text{dir}(\mathfrak{p})$, and we define the Abelian group homomorphism $\phi(\mathfrak{p}; \mathfrak{b}) : \mathbb{Z}^{\#\text{dir}(\mathfrak{p})} \rightarrow \mathbb{Z}^{\#\text{dir}(\mathfrak{b})}$ to be the unique \mathbb{Z} -homomorphism such that $\phi(\mathfrak{p}; \mathfrak{b}) \circ \phi(\mathfrak{p}) = \phi(\mathfrak{b})$.

Now let $\mathfrak{p} \in {}^*\text{Var}(R_{\mathbf{1}}R)$ and $\#\text{dir}(\mathfrak{p}) = m$; we use the notation of 1.3. Let $T := R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$, and let L be a \mathbb{Z}^r -graded T -module.

(i) By Proposition 2.5(iii), for each $\mathbf{a} \in \mathbb{Z}^m$ and each $\mathbf{n} \in \mathbb{Z}^r$,

$$(T(-\mathbf{n})^{\phi(\mathfrak{p})})_{\mathbf{a}} = \begin{cases} 0 & \text{if } \phi(\mathfrak{p})(\mathbf{n}) \neq \mathbf{a}, \\ (T^{\phi(\mathfrak{p})})_{\mathbf{0}} & \text{if } \phi(\mathfrak{p})(\mathbf{n}) = \mathbf{a}. \end{cases}$$

In particular, the \mathbb{Z}^m -graded ring $T^{\phi(\mathfrak{p})}$ is concentrated in degree $\mathbf{0} \in \mathbb{Z}^m$.

(ii) Each component of the \mathbb{Z}^m -graded $T^{\phi(\mathfrak{p})}$ -module $L^{\phi(\mathfrak{p})}$ is a free $(T^{\phi(\mathfrak{p})})_{\mathbf{0}}$ -submodule of $L^{\phi(\mathfrak{p})}$.

(iii) If L is finitely generated, then

$$\text{rank}_{T^{\phi(\mathfrak{p})}} L^{\phi(\mathfrak{p})} = \sum_{\mathbf{a} \in \mathbb{Z}^m} \text{rank}_{(T^{\phi(\mathfrak{p})})_{\mathbf{0}}} (L^{\phi(\mathfrak{p})})_{\mathbf{a}};$$

since the left-hand side of the above equation is finite, all except finitely many of the terms on the right-hand side are zero.

2.7. **Theorem.** *Suppose that $R := \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. Let M be a \mathbb{Z}^r -graded R -module, and let*

$$I^\bullet : 0 \rightarrow {}^*E^0(M) \xrightarrow{d^0} {}^*E^1(M) \rightarrow \dots \rightarrow {}^*E^i(M) \xrightarrow{d^i} {}^*E^{i+1}(M) \rightarrow \dots$$

be the minimal * injective resolution of M . For each $i \in \mathbb{N}_0$, let

$$\theta_i : {}^*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha \in \Lambda_i} {}^*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha)$$

be a \mathbb{Z}^r -homogeneous isomorphism, where $\mathfrak{p}_\alpha \in {}^*\text{Spec}(R)$ and $\mathbf{n}_\alpha \in \mathbb{Z}^r$ for all $\alpha \in \Lambda_i$.

Let $\mathfrak{p} \in {}^*\text{Var}(R_{\mathbf{1}}R)$ and use the notation $\phi(\mathfrak{p}) : \mathbb{Z}^r \rightarrow \mathbb{Z}^m$ and $T := R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ of Remark 2.6, where m is the number of \mathfrak{p} -directions.

Let $i \in \mathbb{N}_0$ and let $\mathbf{a} \in \mathbb{Z}^m$. Then the cardinality of the set $\{\alpha \in \Lambda_i : \mathfrak{p}_\alpha = \mathfrak{p} \text{ and } \phi(\mathfrak{p})(\mathbf{n}_\alpha) = \mathbf{a}\}$ is equal to

$$\text{rank}_{(T^{\phi(\mathfrak{p})})_{\mathbf{0}}} \left(\left((* \text{Ext}_{R(\mathfrak{p})}^i (R(\mathfrak{p})/\mathfrak{p}R(\mathfrak{p}), M(\mathfrak{p})))^{\phi(\mathfrak{p})} \right)_{\mathbf{a}} \right).$$

Proof. By Lemmas 1.5, 1.6 and 1.7, there are \mathbb{Z}^r -homogeneous isomorphisms of graded $R(\mathfrak{p})$ -modules

$$*E_{R(\mathfrak{p})}^i (M(\mathfrak{p})) \cong (*E_R^i (M))_{(\mathfrak{p})} \cong \bigoplus_{\substack{\alpha \in \Lambda_i \\ \mathfrak{p}_\alpha \subseteq \mathfrak{p}}} *E(R(\mathfrak{p})/\mathfrak{p}_\alpha R(\mathfrak{p}))(-\mathbf{n}_\alpha).$$

One can calculate $*\text{Ext}_{R(\mathfrak{p})}^i (R(\mathfrak{p})/\mathfrak{p}R(\mathfrak{p}), M(\mathfrak{p}))$ (up to isomorphism in the category $*\mathcal{C}^{\mathbb{Z}^r}(R(\mathfrak{p}))$) by taking the i th cohomology module of the complex $(0 : (I^\bullet)_{(\mathfrak{p})} \mathfrak{p}R(\mathfrak{p}))$. Note that, by Lemma 1.6, for each $j \in \mathbb{N}_0$, the inclusion $\text{Ker}(d_{(\mathfrak{p})}^j) \subseteq *E^j(M)_{(\mathfrak{p})}$ is $*$ essential, so that the inclusion

$$\text{Ker}(d_{(\mathfrak{p})}^j) \cap (0 : *E^j(M)_{(\mathfrak{p})} \mathfrak{p}R(\mathfrak{p})) \subseteq (0 : *E^j(M)_{(\mathfrak{p})} \mathfrak{p}R(\mathfrak{p}))$$

is also $*$ essential. Because, by Proposition 2.5(iii)(d), each \mathbb{Z}^r -graded T -module is free, it follows that all the ‘differentiation’ maps in the complex $(0 : (I^\bullet)_{(\mathfrak{p})} \mathfrak{p}R(\mathfrak{p}))$ are zero. Hence

$$*\text{Ext}_{R(\mathfrak{p})}^i (R(\mathfrak{p})/\mathfrak{p}R(\mathfrak{p}), M(\mathfrak{p})) \cong \bigoplus_{\substack{\alpha \in \Lambda_i \\ \mathfrak{p}_\alpha \subseteq \mathfrak{p}}} (0 : *E(R(\mathfrak{p})/\mathfrak{p}_\alpha R(\mathfrak{p}))(-\mathbf{n}_\alpha) \mathfrak{p}R(\mathfrak{p})) \quad \text{in } *\mathcal{C}^{\mathbb{Z}^r}(R(\mathfrak{p})).$$

For $\alpha \in \Lambda_i$ such that $\mathfrak{p}_\alpha \subset \mathfrak{p}$ (the symbol ‘ \subset ’ is reserved to denote strict inclusion), there exists an \mathbb{N}_0^r -homogeneous element $u \in \mathfrak{p} \setminus \mathfrak{p}_\alpha$, and the fact (see Lemma 1.5) that multiplication by $u/1 \in R(\mathfrak{p})$ provides an automorphism of $*E(R(\mathfrak{p})/\mathfrak{p}_\alpha R(\mathfrak{p}))$ ensures that

$$(0 : *E(R(\mathfrak{p})/\mathfrak{p}_\alpha R(\mathfrak{p}))(-\mathbf{n}_\alpha) \mathfrak{p}R(\mathfrak{p})) = 0.$$

If $\alpha \in \Lambda_i$ is such that $\mathfrak{p}_\alpha = \mathfrak{p}$, then, by Proposition 2.5(iv),

$$(0 : *E_{R(\mathfrak{p})} (R(\mathfrak{p})/\mathfrak{p}R(\mathfrak{p}))(-\mathbf{n}_\alpha) \mathfrak{p}R(\mathfrak{p})) = (R(\mathfrak{p})/\mathfrak{p}R(\mathfrak{p}))(-\mathbf{n}_\alpha)$$

and, by Proposition 2.5(iii)(d), this is a free \mathbb{Z}^r -graded T -module.

Therefore there is a \mathbb{Z}^r -homogeneous isomorphism of \mathbb{Z}^r -graded T -modules

$$*\text{Ext}_{R(\mathfrak{p})}^i (R(\mathfrak{p})/\mathfrak{p}R(\mathfrak{p}), M(\mathfrak{p})) \cong \bigoplus_{\substack{\alpha \in \Lambda_i \\ \mathfrak{p}_\alpha = \mathfrak{p}}} (R(\mathfrak{p})/\mathfrak{p}R(\mathfrak{p}))(-\mathbf{n}_\alpha).$$

Now apply the functor $(\bullet)^{\phi(\mathfrak{p})}$ to obtain a \mathbb{Z}^m -homogeneous isomorphism of \mathbb{Z}^m -graded $T^{\phi(\mathfrak{p})}$ -modules

$$(* \text{Ext}_{R(\mathfrak{p})}^i (R(\mathfrak{p})/\mathfrak{p}R(\mathfrak{p}), M(\mathfrak{p})))^{\phi(\mathfrak{p})} \cong \bigoplus_{\substack{\alpha \in \Lambda_i \\ \mathfrak{p}_\alpha = \mathfrak{p}}} ((R(\mathfrak{p})/\mathfrak{p}R(\mathfrak{p}))(-\mathbf{n}_\alpha))^{\phi(\mathfrak{p})}.$$

But, by Remark 2.6(i), for an $\alpha \in \Lambda_i$,

$$\left((T(-\mathbf{n}_\alpha))^{\phi(\mathfrak{p})} \right)_{\mathbf{a}} = \begin{cases} 0 & \text{if } \phi(\mathfrak{p})(\mathbf{n}_\alpha) \neq \mathbf{a}, \\ (T^{\phi(\mathfrak{p})})_{\mathbf{0}} & \text{if } \phi(\mathfrak{p})(\mathbf{n}_\alpha) = \mathbf{a}. \end{cases}$$

The desired result now follows from Remark 2.6(iii) □

2.8. Definitions. Let the situation and notation be as in Theorem 2.7, so that, in particular, $\mathfrak{p} \in * \text{Var}(R_1 R)$ and m denotes the number of \mathfrak{p} -directions.

Let $i \in \mathbb{N}_0$. We say that $\mathbf{a} \in \mathbb{Z}^m$ is an *i th level anchor point of \mathfrak{p} for M* if

$$\left((* \text{Ext}_{R(\mathfrak{p})}^i (R(\mathfrak{p})/\mathfrak{p}R(\mathfrak{p}), M(\mathfrak{p})))^{\phi(\mathfrak{p})} \right)_{\mathbf{a}} \neq 0;$$

the set of all i th level anchor points of \mathfrak{p} for M is denoted by $\text{anch}^i(\mathfrak{p}, M)$; also, we write

$$\text{anch}(\mathfrak{p}, M) = \bigcup_{j \in \mathbb{N}_0} \text{anch}^j(\mathfrak{p}, M),$$

and refer to this as the set of *anchor points of \mathfrak{p} for M* .

Thus $\text{anch}^i(\mathfrak{p}, M)$ is the set of m -tuples $\mathbf{a} \in \mathbb{Z}^m$ for which, when we decompose

$$*E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha \in \Lambda_i} *E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha)$$

by means of a \mathbb{Z}^r -homogeneous isomorphism, there exists $\alpha \in \Lambda_i$ with $\mathfrak{p}_\alpha = \mathfrak{p}$ and $\phi(\mathfrak{p})(\mathbf{n}_\alpha) = \mathbf{a}$. Note that $\text{anch}^i(\mathfrak{p}, M) = \emptyset$ if $\mu^i(\mathfrak{p}, M) = 0$, and that, if M is finitely generated, then $\text{anch}^i(\mathfrak{p}, M)$ is a finite set, by Remark 2.6(iii).

The details in our present multi-graded situation are more complicated (and therefore more interesting!) than in the singly-graded situation studied in [15] because there might exist a $\mathfrak{p} \in *\text{Var}(R_1 R)$ for which the set of \mathfrak{p} -directions is a proper subset of $\{1, \dots, r\}$. This cannot happen when $r = 1$. It is worthwhile for us to draw attention to the simplifications that occur in the above theory when $\text{dir}(\mathfrak{p}) = \{1, \dots, r\}$, for that case provides a more-or-less exact analogue of the anchor point theory for the singly-graded case developed in [15].

2.9. Example. Suppose that $R := \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. Let M be a \mathbb{Z}^r -graded R -module, and let

$$I^\bullet : 0 \longrightarrow *E^0(M) \xrightarrow{d^0} *E^1(M) \longrightarrow \dots \longrightarrow *E^i(M) \xrightarrow{d^i} *E^{i+1}(M) \longrightarrow \dots$$

be the minimal $*$ injective resolution of M . For each $i \in \mathbb{N}_0$, let

$$\theta_i : *E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha \in \Lambda_i} *E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha)$$

be a \mathbb{Z}^r -homogeneous isomorphism, where $\mathfrak{p}_\alpha \in *\text{Spec}(R)$ and $\mathbf{n}_\alpha \in \mathbb{Z}^r$ for all $\alpha \in \Lambda_i$.

Let $\mathfrak{p} \in *\text{Spec}(R)$ be such that $\mathfrak{p} \supseteq R_{\mathbf{n}}$ for all $\mathbf{n} > \mathbf{0}$, so that $\text{dir}(\mathfrak{p}) = \{1, \dots, r\}$. In this case, $T := R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ is concentrated in degree $\mathbf{0}$, and $T_{\mathbf{0}}$ is a field isomorphic to $k_{R_{\mathbf{0}}}(\mathfrak{p}_{\mathbf{0}})$.

Let $i \in \mathbb{N}_0$. Then $\text{anch}^i(\mathfrak{p}, M)$ is the set of r -tuples $\mathbf{a} \in \mathbb{Z}^r$ for which there exists $\alpha \in \Lambda_i$ with $\mathfrak{p}_\alpha = \mathfrak{p}$ and $\mathbf{n}_\alpha = \mathbf{a}$. The cardinality of the set of such α s is

$$\dim_{k_{R_{\mathbf{0}}}(\mathfrak{p}_{\mathbf{0}})} \left(\left(* \text{Ext}_{R_{(\mathfrak{p})}}^i(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}) \right)_{\mathbf{a}} \right),$$

and we have

$$\sum_{\mathbf{a} \in \mathbb{Z}^r} \dim_{k_{R_{\mathbf{0}}}(\mathfrak{p}_{\mathbf{0}})} \left(\left(* \text{Ext}_{R_{(\mathfrak{p})}}^i(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}) \right)_{\mathbf{a}} \right) = \mu^i(\mathfrak{p}, M).$$

In particular, if M is finitely generated, then there are only finitely many i th level anchor points of \mathfrak{p} for M .

This reflects rather well the singly-graded anchor point theory studied in [15].

Our next aim is to extend (in a sense) the final result in Example 2.9 (namely that, when M (as in the example) is a finitely generated \mathbb{Z}^r -graded R -module and $\mathfrak{p} \in *\text{Spec}(R)$ is such that $\mathfrak{p} \supseteq R_{\mathbf{n}}$ for all $\mathbf{n} > \mathbf{0}$, then, for each $i \in \mathbb{N}_0$, there are only finitely many i th level anchor points of \mathfrak{p} for M) to all \mathbb{N}_0^r -graded primes of R that contain R_1 .

2.10. Remark. Let S be a multiplicatively closed set of \mathbb{Z}^r -homogeneous elements of R , and let M, N be \mathbb{Z}^r -graded R -modules with M finitely generated. Then, for each $i \in \mathbb{N}_0$, there is a \mathbb{Z}^r -homogeneous $S^{-1}R$ -isomorphism

$$S^{-1} \left(* \text{Ext}_R^i(M, N) \right) \cong * \text{Ext}_{S^{-1}R}^i(S^{-1}M, S^{-1}N).$$

2.11. Theorem. Assume that $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard, and let M be a \mathbb{Z}^r -graded R -module. Let $i \in \mathbb{N}_0$, and let $\mathfrak{p} \in *\text{Var}(R_1 R)$. Then

$$\text{anch}^i(\mathfrak{p}, M) = \text{anch}^i(\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})}),$$

and so is finite if M is finitely generated.

Proof. Suppose, for ease of notation, that $\text{dir}(\mathfrak{p}) = \{1, \dots, m\}$, where $0 < m \leq r$. Note that $\mathfrak{p}^{\phi(\mathfrak{p})}$ is a \mathbb{Z}^m -graded prime ideal of the \mathbb{Z}^m -graded ring $R^{\phi(\mathfrak{p})}$, and that $\text{dir}(\mathfrak{p}^{\phi(\mathfrak{p})}) = \{1, \dots, m\}$ (by Lemma 2.2).

Set $E := {}^* \text{Ext}_R^i(R/\mathfrak{p}, M)$. Let $\mathbf{a} \in \mathbb{Z}^m$. In view of 2.10, the m -tuple \mathbf{a} is an i th level anchor point of \mathfrak{p} for M if and only if $((E_{(\mathfrak{p})})^{\phi(\mathfrak{p})})_{\mathbf{a}} \neq 0$. Our initial task in this proof is to show that this is the case if and only if

$$({}^* \text{Ext}_{R^{\phi(\mathfrak{p})}}^i(R^{\phi(\mathfrak{p})}/\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})}))_{(\mathfrak{p}^{\phi(\mathfrak{p})})_{\mathbf{a}}} \neq 0.$$

Now the \mathbb{Z}^r -graded R -module E can be constructed by application of the functor ${}^* \text{Hom}_R(\cdot, M)$ to a (deleted) * free resolution of R/\mathfrak{p} by finitely generated * free \mathbb{Z}^r -graded modules in the category ${}^* \mathcal{C}^{\mathbb{Z}^r}(R)$ and then taking cohomology of the resulting complex. It follows that there is a \mathbb{Z}^m -homogeneous isomorphism of \mathbb{Z}^m -graded $R^{\phi(\mathfrak{p})}$ -modules

$$E^{\phi(\mathfrak{p})} \cong {}^* \text{Ext}_{R^{\phi(\mathfrak{p})}}^i(R^{\phi(\mathfrak{p})}/\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})}).$$

Suppose that $((E_{(\mathfrak{p})})^{\phi(\mathfrak{p})})_{\mathbf{a}} \neq 0$. Thus there exists $\mathbf{n} \in \mathbb{Z}^r$ such that $\phi(\mathfrak{p})(\mathbf{n}) = \mathbf{a}$ and $\xi \in (E_{(\mathfrak{p})})_{\mathbf{n}}$ such that $\xi \neq 0$. By Remark 2.4, there exists $\mathbf{n}' \in \mathbb{Z}^r$ such that $\phi(\mathfrak{p})(\mathbf{n}') = \mathbf{a}$ and $e \in E_{\mathbf{n}'}$ which is not annihilated by any \mathbb{Z}^r -homogeneous element of $R \setminus \mathfrak{p}$. Now any \mathbb{Z}^m -homogeneous element of $R^{\phi(\mathfrak{p})} \setminus \mathfrak{p}^{\phi(\mathfrak{p})}$ will, when written as a sum of \mathbb{Z}^r -homogeneous elements of R , have at least one component outside \mathfrak{p} , and so $0 \neq e/1 \in (E^{\phi(\mathfrak{p})})_{(\mathfrak{p}^{\phi(\mathfrak{p})})_{\mathbf{a}}}$. Hence $((E^{\phi(\mathfrak{p})})_{(\mathfrak{p}^{\phi(\mathfrak{p})})_{\mathbf{a}}}) \neq 0$, so that

$$({}^* \text{Ext}_{R^{\phi(\mathfrak{p})}}^i(R^{\phi(\mathfrak{p})}/\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})}))_{(\mathfrak{p}^{\phi(\mathfrak{p})})_{\mathbf{a}}} \neq 0.$$

Now suppose that $(({}^* \text{Ext}_{R^{\phi(\mathfrak{p})}}^i(R^{\phi(\mathfrak{p})}/\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})}))_{(\mathfrak{p}^{\phi(\mathfrak{p})})_{\mathbf{a}}} \neq 0$. Then $((E^{\phi(\mathfrak{p})})_{(\mathfrak{p}^{\phi(\mathfrak{p})})_{\mathbf{a}}}) \neq 0$. Since every \mathbb{Z}^m -homogeneous element of $R^{\phi(\mathfrak{p})} \setminus \mathfrak{p}^{\phi(\mathfrak{p})}$ has degree $\mathbf{0} \in \mathbb{Z}^m$, it follows that there exists $e \in (E^{\phi(\mathfrak{p})})_{\mathbf{a}}$ that is not annihilated by any \mathbb{Z}^m -homogeneous element of $R^{\phi(\mathfrak{p})} \setminus \mathfrak{p}^{\phi(\mathfrak{p})}$. In particular, e is not annihilated by any \mathbb{Z}^r -homogeneous element of $R \setminus \mathfrak{p}$. Therefore $0 \neq e/1 \in ((E_{(\mathfrak{p})})^{\phi(\mathfrak{p})})_{\mathbf{a}}$.

This proves that $\text{anch}^i(\mathfrak{p}, M) = \text{anch}^i(\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})})$. Finally, since $\text{dir}(\mathfrak{p}^{\phi(\mathfrak{p})}) = \{1, \dots, m\}$, it follows from Example 2.9 that $\text{anch}^i(\mathfrak{p}^{\phi(\mathfrak{p})}, M^{\phi(\mathfrak{p})})$ is finite when M is finitely generated. \square

The aim of the remainder of this section is to establish a multi-graded analogue of a result of Bass [1, Lemma 3.1]. However, there are some subtleties which mean that our generalization of [15, Lemma 1.8] is not completely straightforward.

2.12. Theorem. *Assume that $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard, and let M be a finitely generated \mathbb{Z}^r -graded R -module. Let $\mathfrak{p}, \mathfrak{q} \in {}^* \text{Spec}(R)$ be such that $R_{\mathbf{1}}R \subseteq \mathfrak{p} \subset \mathfrak{q}$ (we reserve the symbol ' \subset ' to denote strict inclusion) and that there is no \mathbb{Z}^r -graded prime ideal strictly between \mathfrak{p} and \mathfrak{q} . Note that $\text{dir}(\mathfrak{p}) \subseteq \text{dir}(\mathfrak{q})$: suppose, for ease of notation, that $\text{dir}(\mathfrak{p}) = \{1, \dots, m\}$ and $\text{dir}(\mathfrak{q}) = \{1, \dots, m, m+1, \dots, h\}$, where $0 < m \leq h \leq r$.*

Let $i \in \mathbb{N}_0$. Then, for each $\mathbf{a} = (a_1, \dots, a_m) \in \text{anch}^i(\mathfrak{p}, M)$, there exists

$$\mathbf{b} = (b_1, \dots, b_m, b_{m+1}, \dots, b_h) \in \text{anch}^{i+1}(\mathfrak{q}, M)$$

such that $(b_1, \dots, b_m) = (a_1, \dots, a_m) = \mathbf{a}$.

Proof. There exists an \mathbb{N}_0^r -homogeneous element $b \in \mathfrak{q} \setminus \mathfrak{p}$. By Remark 2.4, each \mathbb{N}_0^r -homogeneous element of $R \setminus \mathfrak{p}$ has degree with first m components 0. In particular, $\deg(b) = \mathbf{0}|\mathbf{v} \in \mathbb{Z}^m \times \mathbb{Z}^{r-m}$ for some $\mathbf{v} \in \mathbb{Z}^{r-m}$.

Since $\mathbf{a} \in \text{anch}^i(\mathfrak{p}, M)$, there exists $\mathbf{w} \in \mathbb{Z}^{r-m}$ such that $({}^* \text{Ext}_{R_{(\mathfrak{p})}}^i(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}))_{\mathbf{a}|\mathbf{w}} \neq 0$. Set $E := {}^* \text{Ext}_R^i(R/\mathfrak{p}, M)$. In view of Remark 2.10, we must have $(E_{(\mathfrak{p})})_{\mathbf{a}|\mathbf{w}} \neq 0$. Since each \mathbb{N}_0^r -homogeneous element of $R \setminus \mathfrak{p}$ has degree with first m components 0, this means that there exists a homogeneous element $e \in E$, with $\deg(e) = \mathbf{a}|\mathbf{w}'$ for some $\mathbf{w}' \in \mathbb{Z}^{r-m}$, that is not annihilated by any \mathbb{N}_0^r -homogeneous element of $R \setminus \mathfrak{p}$. But $R \setminus \mathfrak{q} \subseteq R \setminus \mathfrak{p}$, and so it follows that $(E_{(\mathfrak{q})})_{\mathbf{a}|\mathbf{w}'} \neq 0$. By Remark 2.10 again, $({}^* \text{Ext}_{R_{(\mathfrak{q})}}^i(R_{(\mathfrak{q})}/\mathfrak{p}R_{(\mathfrak{q})}, M_{(\mathfrak{q})}))_{\mathbf{a}|\mathbf{w}'} \neq 0$. Write $F := {}^* \text{Ext}_{R_{(\mathfrak{q})}}^i(R_{(\mathfrak{q})}/\mathfrak{p}R_{(\mathfrak{q})}, M_{(\mathfrak{q})})$.

There is an exact sequence

$$0 \longrightarrow (R_{(\mathfrak{q})}/\mathfrak{p}R_{(\mathfrak{q})})(-\mathbf{0}|\mathbf{v}) \xrightarrow{b/1} R_{(\mathfrak{q})}/\mathfrak{p}R_{(\mathfrak{q})} \longrightarrow R_{(\mathfrak{q})}/(\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}) \longrightarrow 0$$

in ${}^*\mathcal{C}^{\mathbb{Z}^r}(R_{(\mathfrak{q})})$, and this induces an exact sequence

$$F \xrightarrow{b/1} F(\mathbf{0}|\mathbf{v}) \longrightarrow {}^*\mathrm{Ext}_{R_{(\mathfrak{q})}}^{i+1}(R_{(\mathfrak{q})}/(\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}), M_{(\mathfrak{q})}).$$

Recall that $\deg(b) = \mathbf{0}|\mathbf{v}$. We claim that there exists $\mathbf{y} \in \mathbb{Z}^{r-m}$ such that $F_{\mathbf{a}|\mathbf{y}} \neq (b/1)F_{\mathbf{a}|\mathbf{y}-\mathbf{v}}$. To see this, note that $b/1 \in \mathfrak{q}R_{(\mathfrak{q})}$, the unique * maximal ideal of the homogeneous localization $R_{(\mathfrak{q})}$, and if we had $F_{\mathbf{a}|\mathbf{y}} = (b/1)F_{\mathbf{a}|\mathbf{y}-\mathbf{v}}$ for every $\mathbf{y} \in \mathbb{Z}^{r-m}$, then we should have $F_{\mathbf{a}|\mathbf{w}'} \subseteq \bigcap_{n \in \mathbb{N}} (b/1)^n F$, which is zero by the multi-graded version of Krull's Intersection Theorem. (One can show that $G := \bigcap_{n \in \mathbb{N}} (b/1)^n F$ satisfies $G = (b/1)G$, and then use the multi-graded version of Nakayama's Lemma.) Thus there exists $\mathbf{y} \in \mathbb{Z}^{r-m}$ such that $F_{\mathbf{a}|\mathbf{y}} \neq (b/1)F_{\mathbf{a}|\mathbf{y}-\mathbf{v}}$, and therefore, in view of the last exact sequence,

$$\left({}^*\mathrm{Ext}_{R_{(\mathfrak{q})}}^{i+1}(R_{(\mathfrak{q})}/(\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}), M_{(\mathfrak{q})}) \right)_{\mathbf{a}|\mathbf{y}} \neq 0.$$

Now $R_{(\mathfrak{q})}/(\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})})$ is concentrated in \mathbb{Z}^r -degrees whose first m components are all zero. Therefore all its \mathbb{Z}^r -graded R -homomorphic images and all its \mathbb{Z}^r -graded submodules are also concentrated in \mathbb{Z}^r -degrees whose first m components are all zero.

The only \mathbb{Z}^r -graded prime ideal of $R_{(\mathfrak{q})}$ that contains the ideal $\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}$ is $\mathfrak{q}R_{(\mathfrak{q})}$, and so $\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}$ is $\mathfrak{q}R_{(\mathfrak{q})}$ -primary. It follows that there is a chain of \mathbb{Z}^r -graded ideals of $R_{(\mathfrak{q})}$ from $\mathfrak{q}R_{(\mathfrak{q})}$ to $\mathfrak{p}R_{(\mathfrak{q})} + (b/1)R_{(\mathfrak{q})}$ with the property that each subquotient is $R_{(\mathfrak{q})}$ -isomorphic to $(R_{(\mathfrak{q})}/\mathfrak{q}R_{(\mathfrak{q})})(\mathbf{0}|\mathbf{z})$ for some $\mathbf{z} \in \mathbb{Z}^{r-m}$. It therefore follows from the half-exactness of ${}^*\mathrm{Ext}_{R_{(\mathfrak{q})}}^{i+1}$ that there exists $\mathbf{y}' \in \mathbb{Z}^{r-m}$ such that

$$\left({}^*\mathrm{Ext}_{R_{(\mathfrak{q})}}^{i+1}(R_{(\mathfrak{q})}/\mathfrak{q}R_{(\mathfrak{q})}, M_{(\mathfrak{q})}) \right)_{\mathbf{a}|\mathbf{y}'} \neq 0.$$

The claim then follows from Theorem 2.7. \square

2.13. Corollary. *Assume that $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard, and let M be a finitely generated \mathbb{Z}^r -graded R -module. Let $\mathfrak{p} \in {}^*\mathrm{Var}(R_1R)$, and suppose, for ease of notation, that $\mathrm{dir}(\mathfrak{p}) = \{1, \dots, m\}$.*

Let $\mathbf{a} \in \mathrm{anch}(\mathfrak{p}, M)$. Then there exists $\mathfrak{q} \in {}^\mathrm{Spec}(R)$ such that $\mathfrak{q} \supseteq R_{\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{N}_0^r$ with $\mathbf{n} > \mathbf{0}$ and $\mathbf{b} = (b_1, \dots, b_m, b_{m+1}, \dots, b_r) \in \mathrm{anch}(\mathfrak{q}, M)$ such that $\mathbf{a} = (b_1, \dots, b_m)$.*

Proof. There exists a saturated chain $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_t = \mathfrak{q}$ of \mathbb{Z}^r -graded prime ideals of R such that \mathfrak{q} is * maximal. Since \mathfrak{q} is contained in the \mathbb{Z}^r -graded prime ideal

$$(\mathfrak{q} \cap R_{\mathbf{0}}) \bigoplus_{\mathbf{n} > \mathbf{0}} \bigoplus R_{\mathbf{n}},$$

these two \mathbb{Z}^r -graded prime ideals must be the same; we therefore see that $\mathfrak{q} \supseteq R_{\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{N}_0^r$ with $\mathbf{n} > \mathbf{0}$. The claim is now immediate from Theorem 2.12. \square

3. THE ENDS OF CERTAIN MULTI-GRADED LOCAL COHOMOLOGY MODULES

We begin with a combinatorial lemma.

3.1. Lemma. *Let $\mathbf{a} := (a_1, \dots, a_r) \in \mathbb{Z}^r$ and let Σ be a non-empty subset of \mathbb{Z}^r such that $\mathbf{n} \leq \mathbf{a}$ for all $\mathbf{n} \in \Sigma$. Then Σ has only finitely many maximal elements.*

Note. We are grateful to the referee for drawing our attention to the following proof, which is shorter than our original.

Proof. The set $\Delta := \mathbf{a} - \Sigma := \{\mathbf{a} - \mathbf{n} : \mathbf{n} \in \Sigma\}$ is a non-empty subset of \mathbb{N}_0^r . Now \mathbb{N}_0^r is a Noetherian monoid with respect to addition, by [11, Proposition 1.3.5], for example. (All terminology concerning monoids in this proof is as in [11, Chapter 1].) Therefore the monoideal (Δ) of \mathbb{N}_0^r generated by Δ can be generated by finitely many elements of Δ , say by $\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(s)} \in \Delta$. Therefore

$$\Delta \subseteq (\Delta) = \left(\mathbf{m}^{(1)} + \mathbb{N}_0^r \right) \cup \dots \cup \left(\mathbf{m}^{(s)} + \mathbb{N}_0^r \right),$$

from which it follows that any minimal member of Δ must belong to the set $\{\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(s)}\}$. Therefore any maximal member of Σ must belong to the set $\{\mathbf{a} - \mathbf{m}^{(1)}, \dots, \mathbf{a} - \mathbf{m}^{(s)}\}$. \square

3.2. Notation. Let $\Sigma, \Delta \subseteq \mathbb{Z}^r$. We shall denote by $\max(\Sigma)$ the set of maximal members of Σ . (If Σ has no maximal member, then we interpret $\max(\Sigma)$ as the empty set.)

We shall write $\Sigma \preceq \Delta$ to indicate that, for each $\mathbf{n} \in \Sigma$, there exists $\mathbf{m} \in \Delta$ such that $\mathbf{n} \leq \mathbf{m}$; moreover, we shall describe this situation by the terminology ‘ Δ dominates Σ ’. We shall use obvious variants of this terminology. Observe that, if $\Sigma \preceq \Delta$ and $\Delta \preceq \Sigma$, then $\max(\Sigma) = \max(\Delta)$, and $\Sigma \preceq \max(\Sigma)$ if and only if $\Delta \preceq \max(\Delta)$.

3.3. Remark (Huy Tài Hà [9, §2]). Let $\phi : \mathbb{Z}^r \rightarrow \mathbb{Z}^m$, where m is a positive integer, be a homomorphism of Abelian groups. We use the notation R^ϕ , *etcetera*, of Definition 1.3. Let \mathfrak{a} be a \mathbb{Z}^r -graded ideal of R . Then $((H_{\mathfrak{a}}^i(\cdot))^\phi)_{i \in \mathbb{N}_0}$ and $((H_{\mathfrak{a}^\phi}^i(\cdot^\phi))_{i \in \mathbb{N}_0}$ are both negative strongly connected sequences of covariant functors from ${}^*\mathcal{C}^{\mathbb{Z}^r}(R)$ to ${}^*\mathcal{C}^{\mathbb{Z}^m}(R^\phi)$; moreover, the 0th members of these two connected sequences are the same functor, and, whenever, I is a $*$ -injective \mathbb{Z}^r -graded R -module and $i > 0$, we have $H_{\mathfrak{a}}^i(I) = 0$ when all gradings are forgotten, so that $(H_{\mathfrak{a}}^i(I))^\phi = 0$ and $H_{\mathfrak{a}^\phi}^i(I^\phi) = 0$. Consequently, the two above-mentioned connected sequences are isomorphic. Hence, for each \mathbb{Z}^r -graded R -module M , there is a \mathbb{Z}^m -homogeneous isomorphism of \mathbb{Z}^m -graded R^ϕ -modules

$$(H_{\mathfrak{a}}^i(M))^\phi \cong H_{\mathfrak{a}^\phi}^i(M^\phi) \quad \text{for each } i \in \mathbb{N}_0.$$

3.4. Notation. Throughout this section, we shall be concerned with the situation where

$$R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$$

is positively graded; we shall only assume that R is standard when this is explicitly stated.

We shall be greatly concerned with the \mathbb{N}_0^r -graded ideal

$$\mathfrak{c} := \mathfrak{c}(R) := \bigoplus_{\substack{\mathbf{n} \in \mathbb{N}_0^r \\ \mathbf{n} > \mathbf{0}}} R_{\mathbf{n}}.$$

We shall accord R_+ its usual meaning (see E. Hyry [10, p. 2215]), so that

$$R_+ := \bigoplus_{\substack{\mathbf{n} \in \mathbb{N}_0^r \\ \mathbf{n} \geq \mathbf{1}}} R_{\mathbf{n}} = \bigoplus_{\mathbf{n} \in \mathbb{N}^r} R_{\mathbf{n}}.$$

Observe that, when $r = 1$, we have $\mathfrak{c} = R_+$. However, in general this is not the case when $r > 1$.

3.5. Definition. Suppose that $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard; let $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$ be a finitely generated \mathbb{Z}^r -graded R -module, and let $j \in \mathbb{N}_0$.

Let \mathfrak{b} be an \mathbb{N}_0^r -graded ideal such that $\text{dir}(\mathfrak{b}) \neq \emptyset$, and let $i \in \text{dir}(\mathfrak{b})$; consider the Abelian group homomorphism $\phi_i : \mathbb{Z}^r \rightarrow \mathbb{Z}$ for which $\phi_i((n_1, \dots, n_r)) = n_i$ for all $(n_1, \dots, n_r) \in \mathbb{Z}^r$, which is just the i th coordinate function.

By Lemma 2.2, since $R_{\mathbf{e}_i} \subseteq \sqrt{\mathfrak{b}}$, we have

$$(R^{\phi_i})_+ = \bigoplus_{\substack{\mathbf{n} \in \mathbb{N}_0^r \\ n_i > 0}} R_{\mathbf{n}} \subseteq \sqrt{\mathfrak{b}^{\phi_i}}.$$

It therefore follows from [15, Corollary 2.5], with the notation of that paper, that the \mathbb{N}_0 -graded R^{ϕ_i} -module $(H_{\mathfrak{b}}^j(M))^{\phi_i} \cong H_{\mathfrak{b}^{\phi_i}}^j(M^{\phi_i})$, if non-zero, has finite end satisfying

$$\text{end}((H_{\mathfrak{b}}^j(M))^{\phi_i}) \leq a^*(M^{\phi_i}) = \sup\{\text{end}(H_{(R^{\phi_i})_+}^k(M^{\phi_i})) : k \in \mathbb{N}_0\} = \sup\{a_{(R^{\phi_i})_+}^k(M^{\phi_i}) : k \in \mathbb{N}_0\}.$$

(Note that, in these circumstances, the invariant $a^*(M^{\phi_i})$ is an integer.) Thus, if $\mathbf{n} := (n_1, \dots, n_r) \in \mathbb{Z}^r$ is such that $H_{\mathfrak{b}}^j(M)_{\mathbf{n}} \neq 0$, then $n_i \leq a^*(M^{\phi_i})$. Thus there exists $\mathbf{a} \in \mathbb{Z}^{\#\text{dir}(\mathfrak{b})}$ such that, for all $\mathbf{n} := (n_1, \dots, n_r) \in \mathbb{Z}^r$ with $H_{\mathfrak{b}}^j(M)_{\mathbf{n}} \neq 0$, we have $\phi(\mathfrak{b})(\mathbf{n}) \leq \mathbf{a}$. We define the *end* of $H_{\mathfrak{b}}^j(M)$ by

$$\text{end}(H_{\mathfrak{b}}^j(M)) := \max\left\{\phi(\mathfrak{b})(\mathbf{n}) : \mathbf{n} \in \mathbb{Z}^r \text{ and } H_{\mathfrak{b}}^j(M)_{\mathbf{n}} \neq 0\right\}.$$

By Lemma 3.1, if $H_{\mathfrak{b}}^j(M) \neq 0$ and $\text{dir}(\mathfrak{b}) \neq \emptyset$, then this end is a non-empty finite set of points of $\mathbb{Z}^{\#\text{dir}(\mathfrak{b})}$. Note that the end of $H_{\mathfrak{b}}^j(M)$ dominates $\phi(\mathfrak{b})(\mathbf{n})$ for every $\mathbf{n} \in \mathbb{Z}^r$ for which $H_{\mathfrak{b}}^j(M)_{\mathbf{n}} \neq 0$.

We draw the reader's attention to the fact that, when $r > 1$ and $R_{\mathbf{e}_i} \neq 0$ for all $i \in \{1, \dots, r\}$, the ideal $R_+ = \bigoplus_{\substack{\mathbf{n} \in \mathbb{N}_0^r \\ \mathbf{n} \geq \mathbf{1}}} R_{\mathbf{n}}$ has empty set of directions; consequently, we have not defined the end of the i th local cohomology module $H_{R_+}^i(M)$ of M with respect to R_+ . Thus we are not, in this paper, making any contribution to the theory of multi-graded Castelnuovo regularity, and, in particular, we are not proposing an alternative definition of a -invariant or a^* -invariant (see [9, Definitions 3.1.1 and 3.1.2]).

With this definition of the ends of (certain) multi-graded local cohomology modules, we can now establish multi-graded analogues of some results in [15, §2].

3.6. Theorem. *Suppose that $R := \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. Let M be a finitely generated \mathbb{Z}^r -graded R -module, and let*

$$I^\bullet : 0 \longrightarrow *E^0(M) \xrightarrow{d^0} *E^1(M) \longrightarrow \dots \longrightarrow *E^i(M) \xrightarrow{d^i} *E^{i+1}(M) \longrightarrow \dots$$

be the minimal $*$ injective resolution of M .

Let \mathfrak{b} be an \mathbb{N}_0^r -graded ideal such that $\text{dir}(\mathfrak{b}) \neq \emptyset$, and let $j \in \mathbb{N}_0$. Then

$$\begin{aligned} \max \left(\bigcup_{i=0}^j \text{end}(H_{\mathfrak{b}}^i(M)) \right) &= \max \{ \phi(\mathfrak{b})(\mathbf{n}) : \mathbf{n} \in \mathbb{Z}^r \text{ and } (\Gamma_{\mathfrak{b}}(*E^i(M)))_{\mathbf{n}} \neq 0 \text{ for some } i \in \{0, \dots, j\} \} \\ &= \max \left(\bigcup_{i=0}^j \bigcup_{\mathfrak{p} \in * \text{Var}(\mathfrak{b})} \phi(\mathfrak{p}; \mathfrak{b})(\text{anch}^i(\mathfrak{p}, M)) \right). \end{aligned}$$

Proof. Let $i \in \mathbb{N}_0$ and set

$$\Delta_i := \{ \phi(\mathfrak{b})(\mathbf{n}) : \mathbf{n} \in \mathbb{Z}^r \text{ and } H_{\mathfrak{b}}^i(M)_{\mathbf{n}} \neq 0 \}, \quad \Sigma_i := \{ \phi(\mathfrak{b})(\mathbf{n}) : \mathbf{n} \in \mathbb{Z}^r \text{ and } (\Gamma_{\mathfrak{b}}(*E^i(M)))_{\mathbf{n}} \neq 0 \}$$

and

$$\Phi_i := \bigcup_{\mathfrak{p} \in * \text{Var}(\mathfrak{b})} \phi(\mathfrak{p}; \mathfrak{b})(\text{anch}^i(\mathfrak{p}, M)).$$

Also, let

$$\theta_i : *E^i(M) \xrightarrow{\cong} \bigoplus_{\alpha \in \Lambda_i} *E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha)$$

be a \mathbb{Z}^r -homogeneous isomorphism, where $\mathfrak{p}_\alpha \in * \text{Spec}(R)$ and $\mathbf{n}_\alpha \in \mathbb{Z}^r$ for all $\alpha \in \Lambda_i$.

We shall first show that $\Delta_i \preceq \Sigma_i \preceq \Phi_i$. Now $H_{\mathfrak{b}}^i(M)$ is a homomorphic image, by a \mathbb{Z}^r -homogeneous epimorphism, of

$$\text{Ker} \left(\Gamma_{\mathfrak{b}}(d^i) : \Gamma_{\mathfrak{b}}(*E^i(M)) \longrightarrow \Gamma_{\mathfrak{b}}(*E^{i+1}(M)) \right).$$

Therefore, if $\mathbf{n} \in \mathbb{Z}^r$ is such that $H_{\mathfrak{b}}^i(M)_{\mathbf{n}} \neq 0$, then $(\Gamma_{\mathfrak{b}}(*E^i(M)))_{\mathbf{n}} \neq 0$. This proves that $\Delta_i \subseteq \Sigma_i$, so that $\Delta_i \preceq \Sigma_i$.

Furthermore, given $\mathbf{n} \in \mathbb{Z}^r$ such that $(\Gamma_{\mathfrak{b}}(*E^i(M)))_{\mathbf{n}} \neq 0$, we can see from the isomorphism θ_i that there must exist $\alpha \in \Lambda_i$ such that $\mathfrak{b} \subseteq \mathfrak{p}_\alpha$ and $(*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha))_{\mathbf{n}} \neq 0$. It now follows from Proposition 2.5(ii) that $\phi(\mathfrak{p}_\alpha)(\mathbf{n}) \leq \phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)$, so that

$$\phi(\mathfrak{p}_\alpha; \mathfrak{b})(\phi(\mathfrak{p}_\alpha)(\mathbf{n})) \leq \phi(\mathfrak{p}_\alpha; \mathfrak{b})(\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)).$$

Now $\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)$ is an i th level anchor point of \mathfrak{p}_α for M , and $\phi(\mathfrak{p}_\alpha; \mathfrak{b}) \circ \phi(\mathfrak{p}_\alpha) = \phi(\mathfrak{b})$. This is enough to prove that $\Sigma_i \preceq \Phi_i$.

In particular, we have proved that $\Delta_0 \preceq \Sigma_0 \preceq \Phi_0$. We shall prove the desired result by induction on j . We show next that $\Phi_0 \preceq \Delta_0$, and this, together with the above, will prove the claim in the case where $j = 0$. Let $\mathfrak{m} \in \Phi_0$. Thus $\mathfrak{m} \in \mathbb{Z}^{\#\text{dir}(\mathfrak{b})}$ and there exists $\alpha \in \Lambda_0$ such that $\mathfrak{p}_\alpha \in * \text{Var}(\mathfrak{b})$ and $\mathfrak{m} = \phi(\mathfrak{p}_\alpha; \mathfrak{b})(\phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha))$. Now the image of

$$\bigoplus_{\substack{\mathbf{n} \in \mathbb{Z}^r \\ \phi(\mathfrak{p}_\alpha)(\mathbf{n}) \geq \phi(\mathfrak{p}_\alpha)(\mathbf{n}_\alpha)}} (*E(R/\mathfrak{p}_\alpha)(-\mathbf{n}_\alpha))_{\mathbf{n}}$$

under θ_0^{-1} is a non-zero \mathbb{Z}^r -graded submodule of $\Gamma_{\mathbf{b}}(*E^0(M))$; as the latter is a $*$ essential extension of $\Gamma_{\mathbf{b}}(M)$, it follows that there exists $\mathbf{n} \in \mathbb{Z}^r$ with $\phi(\mathbf{p}_\alpha)(\mathbf{n}) \geq \phi(\mathbf{p}_\alpha)(\mathbf{n}_\alpha)$ such that $(\Gamma_{\mathbf{b}}(M))_{\mathbf{n}} \neq 0$. Moreover,

$$\phi(\mathbf{b})(\mathbf{n}) = \phi(\mathbf{p}_\alpha; \mathbf{b})(\phi(\mathbf{p}_\alpha)(\mathbf{n})) \geq \phi(\mathbf{p}_\alpha; \mathbf{b})(\phi(\mathbf{p}_\alpha)(\mathbf{n}_\alpha)) = \mathbf{m}.$$

It follows that $\Phi_0 \preceq \Delta_0$, so that $\max(\Delta_0) = \max(\Sigma_0) = \max(\Phi_0)$, and the desired result has been proved when $j = 0$.

Now suppose that $j > 0$ and make the obvious inductive assumption. As we have already proved that $\Delta_i \preceq \Sigma_i$ and $\Sigma_i \preceq \Phi_i$ for all $i = 0, \dots, j$, it will be enough, in order to complete the inductive step, for us to prove that $\Phi_j \preceq \bigcup_{k=0}^j \Delta_k$. So consider $\alpha \in \Lambda_j$ such that $\mathbf{p}_\alpha \in * \text{Var}(\mathbf{b})$; we shall show that $\phi(\mathbf{p}_\alpha; \mathbf{b})(\phi(\mathbf{p}_\alpha)(\mathbf{n}_\alpha))$ is dominated by a member of $\Delta_0 \cup \Delta_1 \cup \dots \cup \Delta_{j-1} \cup \Delta_j$.

Now the image of

$$\bigoplus_{\substack{\mathbf{n} \in \mathbb{Z}^r \\ \phi(\mathbf{p}_\alpha)(\mathbf{n}) \geq \phi(\mathbf{p}_\alpha)(\mathbf{n}_\alpha)}} (*E(R/\mathbf{p}_\alpha)(-\mathbf{n}_\alpha))_{\mathbf{n}}$$

under θ_j^{-1} is a non-zero \mathbb{Z}^r -graded submodule of $\Gamma_{\mathbf{b}}(*E^j(M))$; as the latter is a $*$ essential extension of $\text{Ker } \Gamma_{\mathbf{b}}(d^j)$, it follows that there exists $\mathbf{n} \in \mathbb{Z}^r$ with $\phi(\mathbf{p}_\alpha)(\mathbf{n}) \geq \phi(\mathbf{p}_\alpha)(\mathbf{n}_\alpha)$ such that $(\text{Ker } \Gamma_{\mathbf{b}}(d^j))_{\mathbf{n}} \neq 0$. There is an exact sequence

$$0 \longrightarrow \text{Im } \Gamma_{\mathbf{b}}(d^{j-1}) \longrightarrow \text{Ker } \Gamma_{\mathbf{b}}(d^j) \longrightarrow H_{\mathbf{b}}^j(M) \longrightarrow 0$$

of graded \mathbb{Z}^r -modules and homogeneous homomorphisms. Therefore either $H_{\mathbf{b}}^j(M)_{\mathbf{n}} \neq 0$ or

$$(\text{Im } \Gamma_{\mathbf{b}}(d^{j-1}))_{\mathbf{n}} \neq 0.$$

In the first case, $\phi(\mathbf{p}_\alpha; \mathbf{b})(\phi(\mathbf{p}_\alpha)(\mathbf{n})) = \phi(\mathbf{b})(\mathbf{n}) \in \Delta_j$. In the second case, $(\Gamma_{\mathbf{b}}(*E^{j-1}(M)))_{\mathbf{n}} \neq 0$, whence $\phi(\mathbf{b})(\mathbf{n}) \in \Sigma_{j-1}$, so that, by the inductive hypothesis, $\phi(\mathbf{b})(\mathbf{n})$ is dominated by an element of $\Delta_0 \cup \Delta_1 \cup \dots \cup \Delta_{j-1}$; thus, in this case also, $\phi(\mathbf{p}_\alpha; \mathbf{b})(\phi(\mathbf{p}_\alpha)(\mathbf{n}_\alpha))$ is dominated by an element of $\bigcup_{k=0}^j \Delta_k$. This is enough to complete the inductive step. \square

3.7. Notation. Suppose that $R := \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard, and let M be a finitely generated \mathbb{Z}^r -graded R -module. Let \mathcal{Q} be a non-empty subset of $\{1, \dots, r\}$. Define $\mathbf{c}^{\mathcal{Q}} := \sum_{i \in \mathcal{Q}} R_{\mathbf{e}_i} R$. Then $\text{dir}(\mathbf{c}^{\mathcal{Q}}) \supseteq \mathcal{Q}$, and $\mathbf{c}^{\mathcal{Q}}$ is the smallest ideal (up to radical) with set of directions containing \mathcal{Q} . We also define the \mathcal{Q} -bound $\text{bnd}^{\mathcal{Q}}(M)$ of M by

$$\text{bnd}^{\mathcal{Q}}(M) := \max \left(\bigcup_{i \in \mathbb{N}_0} \text{end}(H_{\mathbf{c}^{\mathcal{Q}}}^i(M)) \right).$$

Observe that $\text{bnd}^{\mathcal{Q}}(M)$ is a finite set of points in $\mathbb{Z}^{\#\text{dir}(\mathbf{c}^{\mathcal{Q}})}$, because $H_{\mathbf{c}^{\mathcal{Q}}}^i(M) = 0$ whenever i exceeds the arithmetic rank of $\mathbf{c}^{\mathcal{Q}}$.

For consistency with our earlier notation in 3.4, we abbreviate $\mathbf{c}^{\{1, \dots, r\}} = \sum_{\mathbf{n} > \mathbf{0}} R_{\mathbf{n}}$ by \mathbf{c} . Note that $\text{bnd}^{\{1, \dots, r\}}(M) = \max \left(\bigcup_{i \in \mathbb{N}_0} \text{end}(H_{\mathbf{c}}^i(M)) \right)$ is a finite set of points in \mathbb{Z}^r .

The following corollaries, which are multi-graded analogues of [15, Corollaries 2.5, 2.6], can now be deduced immediately from Theorem 3.6.

3.8. Corollary. *Suppose that $R := \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard. Let M be a finitely generated \mathbb{Z}^r -graded R -module, and let*

$$I^\bullet : 0 \longrightarrow *E^0(M) \xrightarrow{d^0} *E^1(M) \longrightarrow \dots \longrightarrow *E^i(M) \xrightarrow{d^i} *E^{i+1}(M) \longrightarrow \dots$$

be the minimal $$ injective resolution of M .*

Let \mathfrak{b} be an \mathbb{N}_0^r -graded ideal of R such that $\text{dir}(\mathfrak{b}) \neq \emptyset$, and let $j \in \mathbb{N}_0$. Then

$$\begin{aligned} \max \left(\bigcup_{i=0}^j \text{end}(H_{\mathfrak{b}}^i(M)) \right) &\preceq \max \left(\bigcup_{i=0}^j \bigcup_{\mathfrak{p} \in {}^* \text{Var}(\mathfrak{c}^{\text{dir}(\mathfrak{b})})} \phi(\mathfrak{p}; \mathfrak{c}^{\text{dir}(\mathfrak{b})})(\text{anch}^i(\mathfrak{p}, M)) \right) \\ &= \max \left\{ \phi(\mathfrak{b})(\mathfrak{n}) : \mathfrak{n} \in \mathbb{Z}^r \text{ and } (\Gamma_{\mathfrak{c}^{\text{dir}(\mathfrak{b})}}(*E^i(M)))_{\mathfrak{n}} \neq 0 \text{ for an } i \in \{0, \dots, j\} \right\} \\ &= \max \left(\bigcup_{i=0}^j \text{end}(H_{\mathfrak{c}^{\text{dir}(\mathfrak{b})}}^i(M)) \right) \preceq \text{bnd}^{\text{dir}(\mathfrak{b})}(M). \end{aligned}$$

3.9. Corollary. Suppose that $R := \bigoplus_{\mathfrak{n} \in \mathbb{N}_0^r} R_{\mathfrak{n}}$ is positively graded and standard. Let M be a finitely generated \mathbb{Z}^r -graded R -module.

Let \mathfrak{b} be an \mathbb{N}_0^r -graded ideal of R of arithmetic rank t such that $\text{dir}(\mathfrak{b}) \neq \emptyset$, and let $k \in \mathbb{N}$ with $k > t$. Then

$$\begin{aligned} \max \left(\bigcup_{i=0}^t \bigcup_{\mathfrak{p} \in {}^* \text{Var}(\mathfrak{b})} \phi(\mathfrak{p}; \mathfrak{b})(\text{anch}^i(\mathfrak{p}, M)) \right) &= \max \left(\bigcup_{i=0}^t \text{end}(H_{\mathfrak{b}}^i(M)) \right) = \max \left(\bigcup_{i=0}^k \text{end}(H_{\mathfrak{b}}^i(M)) \right) \\ &= \max \left(\bigcup_{i=0}^k \bigcup_{\mathfrak{p} \in {}^* \text{Var}(\mathfrak{b})} \phi(\mathfrak{p}; \mathfrak{b})(\text{anch}^i(\mathfrak{p}, M)) \right). \end{aligned}$$

Consequently, for a $\mathfrak{p} \in {}^* \text{Var}(\mathfrak{b})$ and $\mathfrak{a} \in \text{anch}(\mathfrak{p}, M)$, we can conclude that $\phi(\mathfrak{p}; \mathfrak{b})(\mathfrak{a})$ is dominated by

$$\max \left(\bigcup_{i=0}^t \bigcup_{\mathfrak{p} \in {}^* \text{Var}(\mathfrak{b})} \phi(\mathfrak{p}; \mathfrak{b})(\text{anch}^i(\mathfrak{p}, M)) \right),$$

a set of points of $\mathbb{Z}^{\# \text{dir}(\mathfrak{b})}$ which arises from consideration of just the 0th, 1st, ..., $(t-1)$ th and t th terms of the minimal $*$ -injective resolution of M .

Our next aim is the establishment of multi-graded analogues of [15, Corollaries 3.1 and 3.2].

3.10. Lemma. Suppose that $R := \bigoplus_{\mathfrak{n} \in \mathbb{N}_0^r} R_{\mathfrak{n}}$ is positively graded, and let \mathfrak{m} be a $*$ -maximal ideal of R . Then $\mathfrak{m}_0 := \mathfrak{m} \cap R_0$ is a maximal ideal of R_0 and $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{c}$, where \mathfrak{c} is as defined in Notation 3.4.

Proof. Recall that

$$\mathfrak{c} := \bigoplus_{\substack{\mathfrak{n} \in \mathbb{N}_0^r \\ \mathfrak{n} > \mathbf{0}}} R_{\mathfrak{n}}.$$

Since $\mathfrak{m}_0 \in \text{Spec}(R_0)$, it follows that $R \supset \mathfrak{m}_0 \oplus \mathfrak{c} \supseteq \mathfrak{m}$, so that $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{c}$. Furthermore, \mathfrak{m}_0 must be a maximal ideal of R_0 . \square

3.11. Corollary. Suppose that $R := \bigoplus_{\mathfrak{n} \in \mathbb{N}_0^r} R_{\mathfrak{n}}$ is positively graded and standard. Let M be a finitely generated \mathbb{Z}^r -graded R -module; let \mathfrak{b} be an \mathbb{N}_0^r -graded ideal of R such that $\text{dir}(\mathfrak{b}) \neq \emptyset$. Then

$$\max \left(\bigcup_{i \in \mathbb{N}_0} \text{end}(H_{\mathfrak{b}}^i(M)) \right) = \max \left(\bigcup_{\mathfrak{m} \in {}^* \text{Var}(\mathfrak{b}) \cap {}^* \text{Max}(R)} \bigcup_{i \in \mathbb{N}_0} \phi(\mathfrak{m}; \mathfrak{b}) \text{end}(H_{\mathfrak{m}}^i(M)) \right).$$

Proof. Let $\mathfrak{m} \in {}^* \text{Var}(\mathfrak{b}) \cap {}^* \text{Max}(R)$. By Lemma 3.10, $\text{dir}(\mathfrak{m}) = \{1, \dots, r\}$; therefore, by Theorem 3.6, $\max(\bigcup_{i \in \mathbb{N}_0} \text{end}(H_{\mathfrak{m}}^i(M))) = \max(\bigcup_{i \in \mathbb{N}_0} \text{anch}^i(\mathfrak{m}, M))$. Another use of Theorem 3.6 therefore shows

that

$$\begin{aligned} \max \left(\bigcup_{i \in \mathbb{N}_0} \phi(\mathfrak{m}; \mathfrak{b})(\text{end}(H_{\mathfrak{m}}^i(M))) \right) &= \max \left(\bigcup_{i \in \mathbb{N}_0} \phi(\mathfrak{m}; \mathfrak{b})(\text{anch}^i(\mathfrak{m}, M)) \right) \\ &\preceq \max \left(\bigcup_{i \in \mathbb{N}_0} \bigcup_{\mathfrak{p} \in {}^* \text{Var}(\mathfrak{b})} \phi(\mathfrak{p}; \mathfrak{b})(\text{anch}^i(\mathfrak{p}, M)) \right) \\ &= \max \left(\bigcup_{i \in \mathbb{N}_0} \text{end}(H_{\mathfrak{b}}^i(M)) \right). \end{aligned}$$

We have thus proved that

$$\max \left(\bigcup_{i \in \mathbb{N}_0} \text{end}(H_{\mathfrak{b}}^i(M)) \right) \succeq \max \left(\bigcup_{\mathfrak{m} \in {}^* \text{Var}(\mathfrak{b}) \cap {}^* \text{Max}(R)} \bigcup_{i \in \mathbb{N}_0} \phi(\mathfrak{m}; \mathfrak{b})(\text{end}(H_{\mathfrak{m}}^i(M))) \right).$$

Now let $\mathfrak{n} \in \mathbb{Z}^{\# \text{dir}(\mathfrak{b})}$ be a maximal member of $\bigcup_{i \in \mathbb{N}_0} \text{end}(H_{\mathfrak{b}}^i(M))$. By Theorem 3.6, there exist $s \in \mathbb{N}_0$ and $\mathfrak{p} \in {}^* \text{Var}(\mathfrak{b})$ such that $\mathfrak{n} = \phi(\mathfrak{p}; \mathfrak{b})(\mathfrak{w})$ for some s th level anchor point \mathfrak{w} of \mathfrak{p} for M . Now use Theorem 2.12 repeatedly, in conjunction with a saturated chain (of length t say) of \mathbb{N}_0^r -graded prime ideals of R with \mathfrak{p} as its smallest term and a * maximal ideal \mathfrak{m} as its largest term: the conclusion is that there exists $\mathfrak{v} \in \text{anch}^{s+t}(\mathfrak{m}, M)$ such that $\phi(\mathfrak{m}; \mathfrak{p})(\mathfrak{v}) = \mathfrak{w}$. Now

$$\mathfrak{n} = \phi(\mathfrak{p}; \mathfrak{b})(\mathfrak{w}) = \phi(\mathfrak{p}; \mathfrak{b})(\phi(\mathfrak{m}; \mathfrak{p})(\mathfrak{v})) = \phi(\mathfrak{m}; \mathfrak{b})(\mathfrak{v}).$$

But, by Theorem 3.6 again, \mathfrak{v} is dominated by $\max(\bigcup_{i \in \mathbb{N}_0} \text{end}(H_{\mathfrak{m}}^i(M)))$; it follows that

$$\max \left(\bigcup_{i \in \mathbb{N}_0} \text{end}(H_{\mathfrak{b}}^i(M)) \right) \preceq \max \left(\bigcup_{\mathfrak{m} \in {}^* \text{Var}(\mathfrak{b}) \cap {}^* \text{Max}(R)} \bigcup_{i \in \mathbb{N}_0} \phi(\mathfrak{m}; \mathfrak{b})(\text{end}(H_{\mathfrak{m}}^i(M))) \right).$$

The desired conclusion follows. \square

3.12. Corollary. *Let the situation be as in Corollary 3.11, but assume in addition that (R_0, \mathfrak{m}_0) is local and that \mathfrak{b} is proper; set $\mathfrak{m} := \mathfrak{m}_0 \oplus \mathfrak{c}$, where \mathfrak{c} is as defined in Notation 3.4. Then*

$$\max \left(\bigcup_{i \in \mathbb{N}_0} \text{end}(H_{\mathfrak{b}}^i(M)) \right) = \max \left(\bigcup_{i \in \mathbb{N}_0} \phi(\mathfrak{m}; \mathfrak{b})(\text{end}(H_{\mathfrak{m}}^i(M))) \right).$$

In particular,

$$\max \left(\bigcup_{i \in \mathbb{N}_0} \text{end}(H_{\mathfrak{c}}^i(M)) \right) = \max \left(\bigcup_{i \in \mathbb{N}_0} \text{end}(H_{\mathfrak{m}}^i(M)) \right).$$

4. SOME VANISHING RESULTS FOR MULTI-GRADED COMPONENTS OF LOCAL COHOMOLOGY MODULES

It is well known that, when $r = 1$, if M is a finitely generated \mathbb{Z} -graded R -module, then there exists $t \in \mathbb{Z}$ such that $H_{R_+}^i(M)_n = 0$ for all $i \in \mathbb{N}_0$ and all $n \geq t$; it then follows from [15, Corollary 2.5] that, if \mathfrak{b} is any graded ideal of R with $\mathfrak{b} \supseteq R_+$, then $H_{\mathfrak{b}}^i(M)_n = 0$ for all $i \in \mathbb{N}_0$ and all $n \geq t$. One of the aims of this section is to establish a multi-graded analogue of this result.

4.1. Notation. Throughout this section, we shall be concerned with the situation where

$$R = \bigoplus_{\mathfrak{n} \in \mathbb{N}_0^r} R_{\mathfrak{n}}$$

is positively graded; we shall only assume that R is standard when this is explicitly stated.

We shall be concerned with the \mathbb{N}_0^r -graded ideal R_+ of R given (see Notation 3.4) by

$$R_+ := \bigoplus_{\substack{\mathfrak{n} \in \mathbb{N}_0^r \\ \mathfrak{n} \geq \mathbf{1}}} R_{\mathfrak{n}}.$$

Although it is well known (see Hyry [10, Theorem 1.6]) that, if M is a finitely generated \mathbb{Z}^r -graded R -module, then $H_{R_+}^i(M)_{(n_1, \dots, n_r)} = 0$ for all $n_1, \dots, n_r \gg 0$, we have not been able to find in the literature a proof of the corresponding statement with R_+ replaced by an \mathbb{N}_0^r -graded ideal \mathfrak{b} that contains R_+ . We present such a proof below, because we think it is of interest in its own right.

4.2. Theorem. *Suppose that $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded; let M be a finitely generated \mathbb{Z}^r -graded R -module. Let \mathfrak{b} be an \mathbb{N}_0^r -graded ideal of R such that $\mathfrak{b} \supseteq R_+$. Then there exists $t \in \mathbb{Z}$ such that*

$$H_{\mathfrak{b}}^i(M)_{\mathbf{n}} = 0 \quad \text{for all } i \in \mathbb{N}_0 \text{ and all } \mathbf{n} \geq (t, t, \dots, t).$$

Proof. We shall prove this by induction on r . In the case where $r = 1$ the result follows from [15, Corollary 2.5], as was explained in the introduction to this section.

Now suppose that $r > 1$ and that the claim has been proved for smaller values of r . We define three more \mathbb{N}_0^r -graded ideals \mathfrak{a} , \mathfrak{c} and \mathfrak{d} of R , as follows. Set

$$\mathfrak{a} := \bigoplus_{\mathbf{n}=(n_1, \dots, n_r) \in \mathbb{N}_0^r} \mathfrak{a}_{\mathbf{n}} \quad \text{where } \mathfrak{a}_{\mathbf{n}} = \begin{cases} \mathfrak{b}_{\mathbf{n}} & \text{if } n_r = 0, \\ R_{\mathbf{n}} & \text{if } n_r > 0; \end{cases}$$

$$\mathfrak{c} := \bigoplus_{\mathbf{n}=(n_1, \dots, n_r) \in \mathbb{N}_0^r} \mathfrak{c}_{\mathbf{n}} \quad \text{where } \mathfrak{c}_{\mathbf{n}} = \begin{cases} \mathfrak{b}_{\mathbf{n}} & \text{if } (n_1, \dots, n_{r-1}) \not\geq (1, \dots, 1), \\ R_{\mathbf{n}} & \text{if } (n_1, \dots, n_{r-1}) \geq (1, \dots, 1); \end{cases}$$

and $\mathfrak{d} := \mathfrak{a} + \mathfrak{c}$.

Consider $\mathfrak{a} \cap \mathfrak{c}$: for each $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}_0^r$, the \mathbf{n} th component $(\mathfrak{a} \cap \mathfrak{c})_{\mathbf{n}}$ satisfies

$$(\mathfrak{a} \cap \mathfrak{c})_{\mathbf{n}} = \mathfrak{a}_{\mathbf{n}} \cap \mathfrak{c}_{\mathbf{n}} = \begin{cases} \mathfrak{b}_{\mathbf{n}} & \text{if } n_r = 0 \text{ or } (n_1, \dots, n_{r-1}) \not\geq (1, \dots, 1), \\ R_{\mathbf{n}} & \text{if } n_r > 0 \text{ and } (n_1, \dots, n_{r-1}) \geq (1, \dots, 1). \end{cases}$$

Since $\mathfrak{b} \supseteq R_+$, we see that $\mathfrak{a} \cap \mathfrak{c} = \mathfrak{b}$.

Let $\sigma : \mathbb{Z}^r \rightarrow \mathbb{Z}^{r-1}$ be the group homomorphism defined by

$$\sigma((n_1, \dots, n_r)) = (n_1 + n_r, \dots, n_{r-1} + n_r) \quad \text{for all } (n_1, \dots, n_r) \in \mathbb{Z}^r.$$

Note that, for $(n_1, \dots, n_r) \in \mathbb{N}_0^r$, we have $(n_1 + n_r, \dots, n_{r-1} + n_r) \geq \mathbf{1}$ in \mathbb{Z}^{r-1} if and only if $n_r \geq 1$ or $(n_1, \dots, n_{r-1}) \geq \mathbf{1}$; furthermore, if $n_r \geq 1$, then $\mathfrak{a}_{\mathbf{n}} = R_{\mathbf{n}}$, and if $(n_1, \dots, n_{r-1}) \geq \mathbf{1}$, then $\mathfrak{c}_{\mathbf{n}} = R_{\mathbf{n}}$. Let $\mathbf{m} \in \mathbb{Z}^{r-1}$ with $\mathbf{m} \geq \mathbf{1}$. Therefore, in the \mathbb{N}_0^{r-1} -graded ring R^σ , we have

$$(\mathfrak{d}^\sigma)_{\mathbf{m}} = \bigoplus_{\substack{\mathbf{n} \in \mathbb{Z}^r \\ \sigma(\mathbf{n}) = \mathbf{m}}} (\mathfrak{a}_{\mathbf{n}} + \mathfrak{c}_{\mathbf{n}}) = \bigoplus_{\substack{\mathbf{n} \in \mathbb{Z}^r \\ \sigma(\mathbf{n}) = \mathbf{m}}} R_{\mathbf{n}} = (R^\sigma)_{\mathbf{m}}.$$

Thus $\mathfrak{d}^\sigma \supseteq \bigoplus_{\mathbf{m} \geq \mathbf{1}} (R^\sigma)_{\mathbf{m}} = (R^\sigma)_+$.

It therefore follows from the inductive hypothesis that there exists $\tilde{t} \in \mathbb{Z}$ such that $(H_{\mathfrak{d}^\sigma}^j(M^\sigma))_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geq (\tilde{t}, \dots, \tilde{t})$ in \mathbb{Z}^{r-1} . In view of Remark 3.3, this means that $((H_{\mathfrak{d}^\sigma}^j(M^\sigma))^\sigma)_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geq (\tilde{t}, \dots, \tilde{t})$ in \mathbb{Z}^{r-1} , so that, for all $j \in \mathbb{N}_0$,

$$H_{\mathfrak{d}}^j(M)_{(n_1, \dots, n_r)} = 0 \quad \text{whenever } (n_1, \dots, n_{r-1}, n_r) \geq (\frac{1}{2}\tilde{t}, \dots, \frac{1}{2}\tilde{t}, \frac{1}{2}\tilde{t}) \text{ in } \mathbb{Z}^r.$$

We now give two similar, but simpler, arguments. Let $\pi : \mathbb{Z}^r \rightarrow \mathbb{Z}$ be the group homomorphism given by projection onto the r th co-ordinate. Note that, for $\mathbf{n} \in \mathbb{N}_0^r$, if $\pi(\mathbf{n}) \geq 1$, then $\mathfrak{a}_{\mathbf{n}} = R_{\mathbf{n}}$. Therefore $\mathfrak{a}^\pi \supseteq (R^\pi)_+$. It therefore follows from the case where $r = 1$ that there exists $\bar{t} \in \mathbb{Z}$ such that $(H_{\mathfrak{a}^\pi}^j(M^\pi))_{\mathbf{n}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{n} \geq \bar{t}$. In view of Remark 3.3, this means that $((H_{\mathfrak{a}^\pi}^j(M^\pi))^\pi)_{\mathbf{n}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{n} \geq \bar{t}$, that is,

$$H_{\mathfrak{a}}^j(M)_{(n_1, \dots, n_r)} = 0 \quad \text{whenever } j \in \mathbb{N}_0 \text{ and } n_r \geq \bar{t}.$$

Next, let $\theta : \mathbb{Z}^r \rightarrow \mathbb{Z}^{r-1}$ be the group homomorphism defined by

$$\theta((n_1, \dots, n_r)) = (n_1, \dots, n_{r-1}) \quad \text{for all } (n_1, \dots, n_r) \in \mathbb{Z}^r.$$

Note that, if $\mathbf{n} \in \mathbb{Z}^r$ has $\theta(\mathbf{n}) \geq \mathbf{1}$ in \mathbb{Z}^{r-1} , then $\mathfrak{c}_{\mathbf{n}} = R_{\mathbf{n}}$. Therefore, for $\mathbf{m} \in \mathbb{Z}^{r-1}$ with $\mathbf{m} \geq \mathbf{1}$, we have $(\mathfrak{c}^\theta)_{\mathbf{m}} = (R^\theta)_{\mathbf{m}}$. This means that, in the \mathbb{N}_0^{r-1} -graded ring R^θ , we have $\mathfrak{c}^\theta \supseteq \bigoplus_{\mathbf{m} \geq \mathbf{1}} (R^\theta)_{\mathbf{m}} = (R^\theta)_+$.

It therefore follows from the inductive hypothesis that there exists $\widehat{t} \in \mathbb{Z}$ such that $(H_{\widehat{c}^\theta}^j(M^\theta))_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geq (\widehat{t}, \dots, \widehat{t})$ in \mathbb{Z}^{r-1} . In view of Remark 3.3, this means that $((H_{\widehat{c}}^j(M))^\theta)_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geq (\widehat{t}, \dots, \widehat{t})$ in \mathbb{Z}^{r-1} , so that

$$H_{\widehat{c}}^j(M)_{(n_1, \dots, n_r)} = 0 \quad \text{whenever } j \in \mathbb{N}_0 \text{ and } (n_1, \dots, n_{r-1}) \geq (\widehat{t}, \dots, \widehat{t}).$$

We recall that $\mathbf{a} \cap \mathbf{c} = \mathbf{b}$. There is an exact Mayer–Vietoris sequence (in the category ${}^*\mathcal{C}^{\mathbb{Z}^r}(R)$)

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\mathfrak{b}}^0(M) & \longrightarrow & H_{\mathfrak{c}}^0(M) \oplus H_{\mathfrak{a}}^0(M) & \longrightarrow & H_{\mathfrak{b}}^0(M) \\ & & \longrightarrow & & \longrightarrow & & \longrightarrow \\ & & \longrightarrow & & \longrightarrow & & \longrightarrow \\ & & \longrightarrow & & \longrightarrow & & \longrightarrow \\ & & \longrightarrow & & \longrightarrow & & \longrightarrow \\ & & \longrightarrow & & \longrightarrow & & \longrightarrow \end{array}$$

It now follows from this Mayer–Vietoris sequence that, if we set $t := \max\{\frac{1}{2}\widehat{t}, \widehat{t}, \widehat{t}\}$, then

$$H_{\mathfrak{b}}^j(M)_{(n_1, \dots, n_r)} = 0 \quad \text{whenever } j \in \mathbb{N}_0 \text{ and } (n_1, \dots, n_r) \geq (t, \dots, t).$$

This completes the inductive step, and the proof. \square

We can deduce from the above Theorem 4.2 a vanishing result for multi-graded components of local cohomology modules with respect to a multi-graded ideal that has both directions and non-directions.

4.3. Corollary. *Suppose that $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard; let M be a finitely generated \mathbb{Z}^r -graded R -module. Let \mathfrak{b} be an \mathbb{N}_0^r -graded ideal of R that has some directions and some non-directions: to be precise, and for ease of notation, suppose that $\text{dir}(\mathfrak{b}) = \{m+1, \dots, r\}$, where $1 \leq m < r$. Then there exists $t \in \mathbb{Z}$ such that, for all $j \in \mathbb{N}_0$, and for all $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{Z}^r$ for which $(n_1, \dots, n_m, n_{m+1} + \dots + n_r) \geq (t, \dots, t)$ in \mathbb{Z}^{m+1} , we have $H_{\mathfrak{b}}^j(M)_{\mathbf{n}} = 0$.*

Note. As \mathfrak{b} has some directions and R is standard, it follows from Lemma 2.2 that $R_+ \subseteq \mathfrak{b}$, so that Theorem 4.2 yields a $t' \in \mathbb{Z}$ such that $H_{\mathfrak{b}}^i(M)_{\mathbf{n}} = 0$ for all $\mathbf{n} \geq (t', \dots, t')$. Thus, when $m = r - 1$, the conclusion of Corollary 4.3 already follows from Theorem 4.2.

Proof. Without loss of generality, we can, and do, assume that $\mathfrak{b} = \sqrt{\mathfrak{b}}$.

Let $\phi : \mathbb{Z}^r \rightarrow \mathbb{Z}^{m+1}$ be the group homomorphism defined by

$$\phi((n_1, \dots, n_r)) = (n_1, \dots, n_m, n_{m+1} + \dots + n_r) \quad \text{for all } (n_1, \dots, n_r) \in \mathbb{Z}^r.$$

Let $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ be such that $\phi(\mathbf{n}) \geq \mathbf{1}$ in \mathbb{Z}^{m+1} . Then $n_{m+1} + \dots + n_r \geq 1$, so that one of n_{m+1}, \dots, n_r is positive. Now $R_{\mathbf{e}_i} \subseteq \sqrt{\mathfrak{b}} = \mathfrak{b}$ for all $i = m+1, \dots, r$, and since $\mathbf{n} \geq \mathbf{e}_i$ for one of these i s, it follows from Lemma 2.2 that $\mathfrak{b} \supseteq R_{\mathbf{n}}$. It therefore follows that, in the \mathbb{N}_0^{m+1} -graded ring R^ϕ , we have $\mathfrak{b}^\phi \supseteq \bigoplus_{\mathbf{m} \geq \mathbf{1}} (R^\phi)_{\mathbf{m}} = (R^\phi)_+$.

We can now appeal to Theorem 4.2 to deduce that there exists $t \in \mathbb{Z}$ such that $(H_{\mathfrak{b}^\phi}^j(M^\phi))_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geq (t, \dots, t)$ in \mathbb{Z}^{m+1} . In view of Remark 3.3, this means that $((H_{\mathfrak{b}}^j(M))^\phi)_{\mathbf{h}} = 0$ for all $j \in \mathbb{N}_0$ and all $\mathbf{h} \geq (t, \dots, t)$ in \mathbb{Z}^{m+1} , so that

$$H_{\mathfrak{b}}^j(M)_{(n_1, \dots, n_r)} = 0 \quad \text{whenever } j \in \mathbb{N}_0 \text{ and } (n_1, \dots, n_m, n_{m+1} + \dots + n_r) \geq (t, \dots, t).$$

\square

One of the reasons why we consider that Theorem 4.2 is of interest in its own right concerns the structure of the (multi-)graded components $H_{\mathfrak{b}}^i(M)_{\mathbf{n}}$ ($\mathbf{n} \in \mathbb{Z}^r$) as modules over $R_{\mathbf{0}}$ (the hypotheses and notation here are as in Theorem 4.2). The example in [4, Exercise 15.1.7] shows that these graded components need not be finitely generated $R_{\mathbf{0}}$ -modules; however, it is always the case that (for a finitely generated \mathbb{Z}^r -graded R -module M) the (multi-)graded components $H_{R_+}^i(M)_{\mathbf{n}}$ ($\mathbf{n} \in \mathbb{Z}^r$) of the i th local cohomology module of M with respect to R_+ are finitely generated $R_{\mathbf{0}}$ -modules (for all $i \in \mathbb{N}_0$), as we now show.

4.4. Theorem. *Suppose that $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded; let M be a finitely generated \mathbb{Z}^r -graded R -module. Then $H_{R_+}^i(M)_{\mathbf{n}}$ is a finitely generated $R_{\mathbf{0}}$ -module, for all $i \in \mathbb{N}_0$ and all $\mathbf{n} \in \mathbb{Z}^r$.*

Note. In the case where $r = 1$, this result is well known: see [4, Proposition 15.1.5].

Proof. We use induction on i . When $i = 0$, the claim is immediate from the fact that $H_{R_+}^0(M)$ is isomorphic to a submodule of M , and so is finitely generated. So suppose that $i > 0$ and that the claim has been proved for smaller values of i , for all finitely generated \mathbb{Z}^r -graded R -modules.

Recall that all the associated prime ideals of M are \mathbb{N}_0^r -graded. Set $B(M) := \text{Ass}_R(M) \setminus \text{Var}(R_+)$, and denote $\#B(M)$ by $b(M)$; we shall argue by induction on $b(M)$. If $b(M) = 0$, then M is R_+ -torsion, so that $H_{R_+}^i(M) = 0$ and the desired result is clear in this case.

Now suppose that $b(M) = 1$: let \mathfrak{p} be the unique member of $B(M)$. Set $\overline{M} := M/\Gamma_{R_+}(M)$. We can use the long exact sequence of local cohomology modules induced by the exact sequence

$$0 \longrightarrow \Gamma_{R_+}(M) \longrightarrow M \longrightarrow \overline{M} \longrightarrow 0,$$

together with the fact that $H_{R_+}^j(\Gamma_{R_+}(M)) = 0$ for all $j \in \mathbb{N}$, to see that, in order to complete the proof in this case, it is sufficient for us to prove the result for \overline{M} . Now \overline{M} is R_+ -torsion-free, and $\text{Ass}(\overline{M}) = \{\mathfrak{p}\}$. (See [4, Exercise 2.1.12].) There exists a \mathbb{Z}^r -homogeneous element $a \in R_+ \setminus \mathfrak{p}$; note that a is a non-zero-divisor on \overline{M} . Let the degree of a be $\mathbf{v} = (v_1, \dots, v_r)$, and note that $v_j > 0$ for all $j = 1, \dots, r$. By Theorem 4.2, there exists $t \in \mathbb{Z}$ such that $H_{R_+}^j(\overline{M})_{\mathbf{n}} = 0$ for all $\mathbf{n} \geq (t, t, \dots, t)$.

Let $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{Z}^r$. Since $v_j > 0$ for all $j = 1, \dots, r$, there exists $w \in \mathbb{N}$ such that $n_j + v_j w \geq t$ for all $j = 1, \dots, r$. The exact sequence

$$0 \longrightarrow \overline{M} \xrightarrow{a^w} \overline{M}(w\mathbf{v}) \longrightarrow (\overline{M}/a^w\overline{M})(w\mathbf{v}) \longrightarrow 0$$

induces an exact sequence of $R_{\mathbf{0}}$ -modules

$$H_{R_+}^{i-1}(\overline{M}/a^w\overline{M})_{\mathbf{n}+w\mathbf{v}} \longrightarrow H_{R_+}^i(\overline{M})_{\mathbf{n}} \longrightarrow H_{R_+}^i(\overline{M})_{\mathbf{n}+w\mathbf{v}},$$

and since w was chosen to ensure that the rightmost term in this sequence is zero, it follows from the inductive hypothesis that $H_{R_+}^i(\overline{M})_{\mathbf{n}}$ is a finitely generated $R_{\mathbf{0}}$ -module. This completes the proof in the case where $b(M) = 1$.

Now suppose that $b(M) = b > 1$ and that it has been proved that all the graded components of $H_{R_+}^i(L)$ are finitely generated $R_{\mathbf{0}}$ -modules for all choices of finitely generated \mathbb{Z}^r -graded R -module L with $b(L) < b$. Let $\mathfrak{p}, \mathfrak{q} \in B(M)$ with $\mathfrak{p} \neq \mathfrak{q}$: suppose, for the sake of argument, that $\mathfrak{p} \not\subseteq \mathfrak{q}$. Consider the \mathfrak{p} -torsion submodule $\Gamma_{\mathfrak{p}}(M)$ of M . By [4, Exercise 2.1.12], $\text{Ass}(\Gamma_{\mathfrak{p}}(M))$ and $\text{Ass}(M/\Gamma_{\mathfrak{p}}(M))$ are disjoint and $\text{Ass} M = \text{Ass}(\Gamma_{\mathfrak{p}}(M)) \cup \text{Ass}(M/\Gamma_{\mathfrak{p}}(M))$. Now $\mathfrak{p} \in \text{Ass}(\Gamma_{\mathfrak{p}}(M))$ and $\mathfrak{q} \notin \text{Ass}(\Gamma_{\mathfrak{p}}(M))$; hence $b(\Gamma_{\mathfrak{p}}(M)) < b$ and $b(M/\Gamma_{\mathfrak{p}}(M)) < b$. Therefore, by the inductive hypothesis, both $H_{R_+}^i(\Gamma_{\mathfrak{p}}(M))_{\mathbf{n}}$ and $H_{R_+}^i(M/\Gamma_{\mathfrak{p}}(M))_{\mathbf{n}}$ are finitely generated $R_{\mathbf{0}}$ -modules, for all $\mathbf{n} \in \mathbb{Z}^r$. We can now use the long exact sequence of local cohomology modules (with respect to R_+) induced from the exact sequence $0 \longrightarrow \Gamma_{\mathfrak{p}}(M) \longrightarrow M \longrightarrow M/\Gamma_{\mathfrak{p}}(M) \longrightarrow 0$ to deduce that $H_{R_+}^i(M)_{\mathbf{n}}$ is a finitely generated $R_{\mathbf{0}}$ -module for all $\mathbf{n} \in \mathbb{Z}^r$. The result follows. \square

5. A MULTI-GRADED ANALOGUE OF MARLEY'S WORK ON FINITELY GRADED LOCAL COHOMOLOGY MODULES

As was mentioned in the Introduction, the purpose of this section is to obtain some multi-graded analogues of results about finitely graded local cohomology modules that were proved, in the case where $r = 1$, by Marley in [14]. We shall present a multi-graded analogue of one of Marley's results and some extensions of that analogue.

5.1. Notation. Throughout this section, we shall be concerned with the situation where $R = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^r} R_{\mathbf{n}}$ is positively graded and standard, and we shall let $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$ be a \mathbb{Z}^r -graded R -module. Also, \mathfrak{b} will always denote an \mathbb{N}_0^r -graded ideal of R .

For $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}_0^r$, we shall denote $\{i \in \{1, \dots, r\} : n_i \neq 0\}$ by $\mathcal{P}(\mathbf{n})$.

5.2. Definition. An r -tuple $\mathbf{n} \in \mathbb{Z}^r$ is called a *supporting degree* of M precisely when $M_{\mathbf{n}} \neq 0$; we denote the set of all supporting degrees of M by $\mathcal{S}(M)$.

Note that Theorem 4.2 imposes substantial restrictions on $\mathcal{S}(H_{\mathfrak{b}}^i(M))$ when $(i \in \mathbb{N}_0 \text{ and } \mathfrak{b} \supseteq R_+)$. The example below is included as motivation for the introduction of some notation.

5.3. Example. Let k be an algebraically closed field and let $A = k \oplus A_1 \oplus \cdots \oplus A_m \oplus \cdots$ and $B = k \oplus B_1 \oplus \cdots \oplus B_n \oplus \cdots$ be two normal Noetherian standard \mathbb{N}_0 -graded k -algebra domains with $w := \dim A > 1$ and $v := \dim B > 1$. We consider the \mathbb{N}_0^2 -graded k -algebra

$$R := A \otimes_k B = \bigoplus_{(m,n) \in \mathbb{N}_0^2} A_m \otimes_k B_n.$$

Clearly $R = k[R_{(1,0)}, R_{(0,1)}]$ is positively graded and standard, and, as a finitely generated k -algebra, is Noetherian. By [17, Chapter III, §15, Theorem 40, Corollary 1], R is again an integral domain. Observe that $R_+ = R_{(1,1)}R = A_+ \otimes_k B_+$. As A and B are normal and their dimensions exceed 1, we have $H_{A_+}^i(A) = H_{B_+}^i(B) = 0$ for $i = 0, 1$. The Künneth relations for tensor products (see [7] or [13, Theorem 10.1]) yield, for each $i \in \mathbb{N}_0$, an isomorphism of \mathbb{Z}^2 -graded R modules

$$H_{R_+}^i(R) \cong \left(A \otimes_k H_{B_+}^i(B) \right) \oplus \left(H_{A_+}^i(A) \otimes_k B \right) \oplus \left(\bigoplus_{\substack{j,l \in \mathbb{N} \setminus \{1\} \\ j+l=i+1}} \left(H_{A_+}^j(A) \otimes_k H_{B_+}^l(B) \right) \right).$$

As $\mathcal{S}(A) = \mathcal{S}(B) = \mathbb{N}_0$, it follows that, for each $i \in \mathbb{N}_0$,

$$\mathcal{S}(H_{R_+}^i(R)) = \left(\mathbb{N}_0 \times \mathcal{S}(H_{B_+}^i(B)) \right) \cup \left(\mathcal{S}(H_{A_+}^i(A)) \times \mathbb{N}_0 \right) \cup \left(\bigcup_{\substack{j,l \in \mathbb{N} \setminus \{1\} \\ j+l=i+1}} \left(\mathcal{S}(H_{A_+}^j(A)) \times \mathcal{S}(H_{B_+}^l(B)) \right) \right).$$

Observe, in particular, that $H_{R_+}^i(R) = 0$ for $i = 0, 1$ and for all $i \geq w + v$.

Appropriate choices for A and B yield many examples for R . We shall just concentrate on a class of examples obtained by this procedure when A and B are chosen in a particular way, which we now describe. We can use [2, Proposition (2.13)], in conjunction with the Serre–Grothendieck correspondence (see [4, 20.4.4]), to choose the algebra A (as above) so that, for a prescribed set $W \subseteq \{2, \dots, w-1\}$, we have

$$\mathcal{S}(H_{A_+}^i(A)) = \begin{cases} \emptyset & \text{for all } i \in \mathbb{N}_0 \setminus (W \cup \{w\}), \\ \{0\} & \text{for all } i \in W, \\ \{k \in \mathbb{Z} : k < 0\} & \text{for } i = w. \end{cases}$$

Similarly, for a prescribed set $V \subseteq \{2, \dots, v-1\}$, we choose B (as above) so that

$$\mathcal{S}(H_{B_+}^i(B)) = \begin{cases} \emptyset & \text{for all } i \in \mathbb{N}_0 \setminus (V \cup \{v\}), \\ \{0\} & \text{for all } i \in V, \\ \{k \in \mathbb{Z} : k < 0\} & \text{for } i = v. \end{cases}$$

With such a choice of A for $w = 5$ and $W = \{2\}$, and such a choice of B for $v = 5$ and $V = \{3\}$, the sets of supporting degrees $\mathcal{S}(H_{R_+}^i(R))$ for $i = 2, 3, 4, 5$ are as in Figure 1 below.

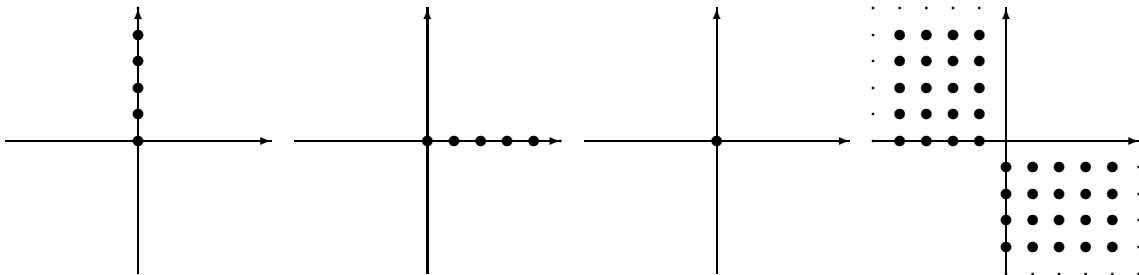


FIGURE 1. $\mathcal{S}(H_{R_+}^i(R))$ for $i = 2, 3, 4, 5$ respectively

In view of Theorem 4.2, the supporting set $\mathcal{S}(H_{R_+}^5(R))$ seems unremarkable. The local cohomology module $H_{R_+}^4(R)$ is finitely graded. Although neither $H_{R_+}^3(R)$ nor $H_{R_+}^2(R)$ is finitely graded, both have sets of supporting degrees that are quite restricted.

We now return to the general situation described in Notation 5.1. In the case where $r = 1$, one way of recording that a local cohomology module $H_{\mathfrak{b}}^i(M)$ is finitely graded is to state that there exist $s, t \in \mathbb{Z}$ with $s < t$ such that

$$\mathcal{S}(H_{\mathfrak{b}}^i(M)) = \{n \in \mathbb{Z} : H_{\mathfrak{b}}^i(M)_n \neq 0\} \subseteq \{n \in \mathbb{Z} : s \leq n < t\}.$$

One might expect the natural generalization to our multi-graded situation to involve conditions such as

$$\mathcal{S}(H_{\mathfrak{b}}^i(M)) = \{\mathbf{n} \in \mathbb{Z}^r : H_{\mathfrak{b}}^i(M)_{\mathbf{n}} \neq 0\} \subseteq \{\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{Z}^r : s_i \leq n_i < t_i \text{ for all } i = 1, \dots, r\},$$

where $\mathbf{s} = (s_1, \dots, s_r)$, $\mathbf{t} = (t_1, \dots, t_r) \in \mathbb{Z}^r$ satisfy $\mathbf{s} \leq \mathbf{t}$. However, in the light of evidence like that provided by Example 5.3 above, and other examples, we introduce the following.

5.4. Notation. Let $\mathbf{s} = (s_1, \dots, s_r)$, $\mathbf{t} = (t_1, \dots, t_r) \in \mathbb{Z}^r$ with $\mathbf{s} \leq \mathbf{t}$. We set

$$\mathbb{X}(\mathbf{s}, \mathbf{t}) := \{\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{Z}^r : \text{there exists } i \in \{1, \dots, r\} \text{ such that } s_i \leq n_i < t_i\}.$$

5.5. Example. Figure 2 below illustrates, in the case where $r = 2$, the set $\mathbb{X}((-2, 1), (0, 2))$.

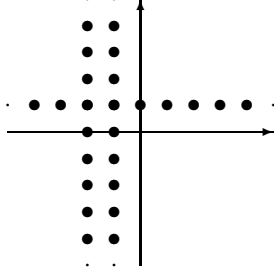


FIGURE 2. The set $\mathbb{X}((-2, 1), (0, 2))$ in \mathbb{Z}^2

5.6. Remark. Let $\mathbf{s}, \mathbf{s}', \mathbf{s}'', \mathbf{t}, \mathbf{t}', \mathbf{t}'' \in \mathbb{Z}^r$ with $\mathbf{s} \leq \mathbf{t}$, $\mathbf{s}' \leq \mathbf{t}'$ and $\mathbf{s}'' \leq \mathbf{t}''$. Let $\mathbf{m} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$.

(i) Clearly $(\mathbf{s} + \mathbb{N}_0^r) \setminus (\mathbf{t} + \mathbb{N}_0^r) \subseteq \mathbb{X}(\mathbf{s}, \mathbf{t})$.

(ii) Suppose that $\mathcal{P}(\mathbf{t} - \mathbf{s}) \subseteq \mathcal{P}(\mathbf{m})$. Let $\mathbf{w} \in \mathbb{Z}^r$ be such that $\mathcal{P}(\mathbf{w}) \subseteq \{1, \dots, r\} \setminus \mathcal{P}(\mathbf{m})$. Then

$$\mathbb{X}(\mathbf{s} + \mathbf{w}, \mathbf{t} + \mathbf{w}) = \mathbb{X}(\mathbf{s}, \mathbf{t}) = \{\mathbf{n} \in \mathbb{Z}^r : \text{there exists } i \in \mathcal{P}(\mathbf{m}) \text{ such that } s_i \leq n_i < t_i\}.$$

(iii) Clearly $\mathbb{X}(\mathbf{s}', \mathbf{t}') \cup \mathbb{X}(\mathbf{s}'', \mathbf{t}'') \subseteq \mathbb{X}(\min\{\mathbf{s}', \mathbf{s}''\}, \max\{\mathbf{t}', \mathbf{t}''\})$.

(iv) Assume that $\mathcal{P}(\mathbf{t}' - \mathbf{s}') \subseteq \mathcal{P}(\mathbf{m})$ and $\mathcal{P}(\mathbf{t}'' - \mathbf{s}'') \subseteq \mathcal{P}(\mathbf{m})$. For each $i \in \{1, \dots, r\}$, set

$$\tilde{s}_i := \min\{s'_i, s''_i\} \quad \text{and} \quad \tilde{t}_i := \begin{cases} \max\{t'_i, t''_i\} & \text{if } i \in \mathcal{P}(\mathbf{m}), \\ \tilde{s}_i & \text{if } i \in \{1, \dots, r\} \setminus \mathcal{P}(\mathbf{m}). \end{cases}$$

Set $\tilde{\mathbf{s}} := (\tilde{s}_1, \dots, \tilde{s}_r)$ and $\tilde{\mathbf{t}} := (\tilde{t}_1, \dots, \tilde{t}_r)$. Then

$$\tilde{\mathbf{s}} \leq \tilde{\mathbf{t}}, \quad \mathcal{P}(\tilde{\mathbf{t}} - \tilde{\mathbf{s}}) \subseteq \mathcal{P}(\mathbf{m}) \quad \text{and} \quad \mathbb{X}(\mathbf{s}', \mathbf{t}') \cup \mathbb{X}(\mathbf{s}'', \mathbf{t}'') \subseteq \mathbb{X}(\tilde{\mathbf{s}}, \tilde{\mathbf{t}}).$$

The next lemma provides a small hint about the importance of the sets $\mathbb{X}(\mathbf{s}, \mathbf{t})$ of Notation 5.4 for our work.

5.7. Lemma. Let $\mathbf{m} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$. Assume that M is finitely generated and that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R M)}$. Then there exist $\mathbf{s}, \mathbf{t} \in \mathbb{Z}^r$ such that $\mathbf{s} \leq \mathbf{t}$, $\mathcal{P}(\mathbf{t} - \mathbf{s}) \subseteq \mathcal{P}(\mathbf{m})$ and $\mathcal{S}(M) \subseteq (\mathbf{s} + \mathbb{N}_0^r) \setminus (\mathbf{t} + \mathbb{N}_0^r)$, so that $\mathcal{S}(M) \subseteq \mathbb{X}(\mathbf{s}, \mathbf{t})$ in view of Remark 5.6(i).

Proof. As M is finitely generated, there exist $\mathbf{s}, \mathbf{w} \in \mathbb{Z}^r$ such that $\mathbf{s} \leq \mathbf{w}$ and $M = \sum_{\mathbf{s} \leq \mathbf{n} \leq \mathbf{w}} RM_{\mathbf{n}}$. In particular, $\mathcal{S}(M) \subseteq \mathbf{s} + \mathbb{N}_0^r$.

Moreover, there exists $u \in \mathbb{N}$ such that $(R_{\mathbf{m}})^u \subseteq (0 :_R M)$; since R is standard, $(R_{\mathbf{m}})^u = R_{u\mathbf{m}}$; hence $R_{u\mathbf{m}}M_{\mathbf{n}} = 0$ for all $\mathbf{n} \in \mathbb{Z}^r$.

Let $\mathbf{t} = \mathbf{s} + \sum_{i \in \mathcal{P}(\mathbf{m})} (w_i - s_i + um_i)\mathbf{e}_i$. Now, let $\mathbf{h} = (h_1, \dots, h_r) \in \mathbf{t} + \mathbb{N}_0^r$. Our proof will be complete once we have shown that $M_{\mathbf{h}} = 0$. For each $i \in \mathcal{P}(\mathbf{m})$, we have $h_i \geq t_i = w_i + um_i$. Moreover,

$$M_{\mathbf{h}} = \sum_{\mathbf{n} \in \mathcal{T}} R_{\mathbf{h}-\mathbf{n}}M_{\mathbf{n}}, \quad \text{where } \mathcal{T} = \{\mathbf{n} \in \mathbb{Z}^r : \mathbf{s} \leq \mathbf{n} \leq \mathbf{w}, \mathbf{n} \leq \mathbf{h}\}.$$

Let $\mathbf{n} = (n_1, \dots, n_r) \in \mathcal{T}$. If $i \in \mathcal{P}(\mathbf{m})$, then $n_i + um_i \leq w_i + um_i \leq h_i$; if $i \in \{1, \dots, r\} \setminus \mathcal{P}(\mathbf{m})$, then $n_i + um_i = n_i \leq h_i$. Consequently $\mathbf{n} + u\mathbf{m} \leq \mathbf{h}$. Therefore $u\mathbf{m} \leq \mathbf{h} - \mathbf{n}$ for all $\mathbf{n} \in \mathcal{T}$, and hence

$$M_{\mathbf{h}} = \sum_{\mathbf{n} \in \mathcal{T}} R_{\mathbf{h}-\mathbf{n}}M_{\mathbf{n}} = \sum_{\mathbf{n} \in \mathcal{T}} R_{\mathbf{h}-\mathbf{n}-u\mathbf{m}}R_{u\mathbf{m}}M_{\mathbf{n}} = 0.$$

□

5.8. Definition. Let $\mathcal{Q} \subseteq \{1, \dots, r\}$. By a \mathcal{Q} -domain in \mathbb{Z}^r we mean a set of the form

$$\mathbb{X}(\mathbf{s}, \mathbf{t}) \quad \text{with } \mathbf{s}, \mathbf{t} \in \mathbb{Z}^r, \mathbf{s} \leq \mathbf{t} \text{ and } \mathcal{P}(\mathbf{t} - \mathbf{s}) \subseteq \mathcal{Q}.$$

5.9. Remarks. The following statements are immediate from the definition.

- (i) A \emptyset -domain in \mathbb{Z}^r is empty.
- (ii) If $\mathcal{Q} \subseteq \mathcal{Q}' \subseteq \{1, \dots, r\}$ and if \mathbb{X} is a \mathcal{Q} -domain in \mathbb{Z}^r , then \mathbb{X} is a \mathcal{Q}' -domain in \mathbb{Z}^r .
- (iii) If \mathbb{X} is a \mathcal{Q} -domain in \mathbb{Z}^r and $\mathbf{w} \in \mathbb{Z}^r$, then $\mathbf{w} + \mathbb{X} := \{\mathbf{w} + \mathbf{n} : \mathbf{n} \in \mathbb{X}\}$ is a \mathcal{Q} -domain in \mathbb{Z}^r .
- (iv) If $\mathbf{s}, \mathbf{t} \in \mathbb{Z}^r$ with $\mathbf{s} \leq \mathbf{t}$ and $\mathcal{P}(\mathbf{t} - \mathbf{s}) \subseteq \mathcal{Q}$, then $(\mathbf{s} + \mathbb{N}_0^r) \setminus (\mathbf{t} + \mathbb{N}_0^r)$ is contained in a \mathcal{Q} -domain in \mathbb{Z}^r , by Remark 5.6(i).
- (v) If \mathbb{X} is a \mathcal{Q} -domain in \mathbb{Z}^r and $\mathbf{w} \in \mathbb{Z}^r$ is such that $\mathcal{P}(\mathbf{w}) \cap \mathcal{Q} = \emptyset$, then $\mathbb{X} = \mathbf{w} + \mathbb{X}$, by Remark 5.6(ii).
- (vi) By Remark 5.6(iv), the union of finitely many \mathcal{Q} -domains in \mathbb{Z}^r is contained in a \mathcal{Q} -domain in \mathbb{Z}^r .

5.10. Lemma. Let $\mathbf{m}, \mathbf{k} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$, and let T be a \mathbb{Z}^r -graded R -module such that $R_{\mathbf{m}}T = 0$. Let $y \in R_{\mathbf{k}}$, and let K denote the kernel of the homogeneous R -homomorphism $T \rightarrow T(\mathbf{k})$ given by multiplication by y .

- (i) If $\mathcal{P}(\mathbf{m}) \subseteq \mathcal{P}(\mathbf{k})$, then there exists $v \in \mathbb{N}_0$ such that $\mathcal{S}(T) \subseteq \bigcup_{j=0}^v (\mathcal{S}(K) - j\mathbf{k})$.
- (ii) If $\mathcal{P}(\mathbf{m}) \not\subseteq \mathcal{P}(\mathbf{k})$, if multiplication by y provides an isomorphism $T \xrightarrow{\cong} T(\mathbf{k})$, and if T considered as an R_y -module is finitely generated, then $\mathcal{S}(T)$ is contained in a $(\mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k}))$ -domain in \mathbb{Z}^r .

Proof. Write $\mathbf{m} = (m_1, \dots, m_r)$ and $\mathbf{k} = (k_1, \dots, k_r)$. Let $u \in \mathbb{N}$ be such that $m_i \leq uk_i$ for all $i \in \mathcal{P}(\mathbf{k})$. Set $\mathbf{h} := \sum_{i \in \{1, \dots, r\} \setminus \mathcal{P}(\mathbf{k})} m_i \mathbf{e}_i$. Then, if $i \in \mathcal{P}(\mathbf{k})$, we have $(u\mathbf{k} + \mathbf{h})_i = uk_i \geq m_i$, whereas, if $i \in \{1, \dots, r\} \setminus \mathcal{P}(\mathbf{k})$, we have $(u\mathbf{k} + \mathbf{h})_i = m_i \geq m_i$. Therefore $\mathbf{m} \leq u\mathbf{k} + \mathbf{h}$.

Now, let $z \in R_{\mathbf{h}}$. Then, because R is standard, $y^u z \in R_{u\mathbf{k} + \mathbf{h}} = R_{u\mathbf{k} + \mathbf{h} - \mathbf{m}}R_{\mathbf{m}}$. As $R_{\mathbf{m}}T = 0$, it follows that $y^u zT = 0$. Therefore $y^u R_{\mathbf{h}}T = 0$.

(i) Assume that $\mathcal{P}(\mathbf{m}) \subseteq \mathcal{P}(\mathbf{k})$. Then $\mathcal{P}(\mathbf{h}) = \mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k}) = \emptyset$, so that $\mathbf{h} = \mathbf{0}$. Hence $y^u T = y^u R_{\mathbf{0}}T = 0$.

Now let $\mathbb{K} := \bigcup_{j=0}^{u-1} (\mathcal{S}(K) - j\mathbf{k})$, and let $\mathbf{n} \in \mathbb{Z}^r \setminus \mathbb{K}$. If we show that $T_{\mathbf{n}} = 0$, then we shall have proved part (i). Now $\mathbf{n} + j\mathbf{k} \notin \mathcal{S}(K)$ for all $j \in \{0, \dots, u-1\}$, and so the $R_{\mathbf{0}}$ -homomorphism $y^u : T_{\mathbf{n}} \rightarrow T_{\mathbf{n} + u\mathbf{k}}$, which is the composition of the $R_{\mathbf{0}}$ -homomorphisms $y : T_{\mathbf{n} + j\mathbf{k}} \rightarrow T_{\mathbf{n} + (j+1)\mathbf{k}}$ for $j = 0, \dots, u-1$, is injective. But $y^u T_{\mathbf{n}} = 0$, and so $T_{\mathbf{n}} = 0$.

(ii) Now assume that $\mathcal{P}(\mathbf{m}) \not\subseteq \mathcal{P}(\mathbf{k})$, that multiplication by y provides an isomorphism $T \xrightarrow{\cong} T(\mathbf{k})$, and that T considered as an R_y -module is finitely generated. As $y^u R_{\mathbf{h}}T = 0$, it follows that $R_{\mathbf{h}}T = 0$.

As T is finitely generated over R_y , there are finitely many r -tuples $\mathbf{g}^{(1)}, \dots, \mathbf{g}^{(q)} \in \mathbb{Z}^r$ such that $T = \sum_{j=1}^q R_y T_{\mathbf{g}^{(j)}}$. Now, for $i \in \{1, \dots, r\}$, set

$$s_i := \begin{cases} 0 & \text{if } i \notin \mathcal{P}(\mathbf{h}), \\ \min\{g_i^{(j)} : j = 1, \dots, q\} & \text{if } i \in \mathcal{P}(\mathbf{h}), \end{cases} \quad t_i := \begin{cases} 0 & \text{if } i \notin \mathcal{P}(\mathbf{h}), \\ \max\{g_i^{(j)} : j = 1, \dots, q\} + h_i & \text{if } i \in \mathcal{P}(\mathbf{h}), \end{cases}$$

and put $\mathbf{s} = (s_1, \dots, s_r)$, $\mathbf{t} = (t_1, \dots, t_r)$. Then $\mathbf{s} \leq \mathbf{t}$ and $\mathcal{P}(\mathbf{t} - \mathbf{s}) = \mathcal{P}(\mathbf{h}) = \mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k})$. Let $\mathbf{n} \in \mathbb{Z}^r \setminus \mathbb{X}(\mathbf{s}, \mathbf{t})$. If we show that $T_{\mathbf{n}} = 0$, then we shall have proved part (ii). Let $\alpha \in T_{\mathbf{n}}$. There exist integers v_1, \dots, v_q such that $\alpha \in \sum_{j=1}^q y^{v_j} R_{\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}} T_{\mathbf{g}^{(j)}}$.

Note that, for each $i \in \mathcal{P}(\mathbf{h}) = \mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k})$, we have either $n_i < s_i$ or $t_i \leq n_i$ (because $\mathbf{n} \notin \mathbb{X}(\mathbf{s}, \mathbf{t})$). Assume first that there is some $i \in \mathcal{P}(\mathbf{h})$ with $n_i < s_i$. As $i \notin \mathcal{P}(\mathbf{k})$, it follows that

$$(\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)})_i = n_i - v_j k_i - g_i^{(j)} = n_i - g_i^{(j)} < s_i - g_i^{(j)} \leq 0,$$

for all $j \in \{1, \dots, q\}$, so that $R_{\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}} = 0$ and $\alpha = 0$.

Therefore, we can, and do, assume that $t_i \leq n_i$ for all $i \in \mathcal{P}(\mathbf{h})$. In this case, for each $i \in \mathcal{P}(\mathbf{h})$ and each $j \in \{1, \dots, q\}$, we have

$$(\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)})_i = n_i - v_j k_i - g_i^{(j)} = n_i - g_i^{(j)} \geq t_i - g_i^{(j)} \geq h_i.$$

Therefore, for each $j \in \{1, \dots, q\}$, either $\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)} \geq \mathbf{h}$, or $\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}$ has a negative component and $R_{\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}} = 0$. This means that

$$\alpha \in \sum_{j=1}^q y^{v_j} R_{\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)}} T_{\mathbf{g}^{(j)}} = \sum_{\substack{j=1 \\ \mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)} \geq \mathbf{0}}}^q y^{v_j} R_{\mathbf{n} - v_j \mathbf{k} - \mathbf{g}^{(j)} - \mathbf{h}} R_{\mathbf{h}} T_{\mathbf{g}^{(j)}} = 0.$$

It follows that $T_{\mathbf{n}} = 0$, as required. \square

5.11. Lemma. *Let $\mathbf{m} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$ and $\mathbf{k} \in \mathbb{N}_0^r$. Assume that M is finitely generated and that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R M)}$. Let $y \in R_{\mathbf{k}}$. Then there exists a $(\mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k}))$ -domain \mathbb{X} in \mathbb{Z}^r such that $\mathcal{S}(H_{yR}^1(M)) \subseteq \mathbb{X}$.*

Proof. Assume first that $\mathbf{k} = \mathbf{0}$. Then $\mathcal{P}(\mathbf{k}) = \emptyset$ and, by the multi-graded analogue of [4, Lemma 13.1.10], there are $R_{\mathbf{0}}$ -isomorphisms $H_{yR}^1(M)_{\mathbf{n}} \cong H_{yR_0}^1(M_{\mathbf{n}})$ for all $\mathbf{n} \in \mathbb{Z}^r$. Therefore $\mathcal{S}(H_{yR}^1(M)) \subseteq \mathcal{S}(M)$, and the claim follows in this case from Lemma 5.7.

We now deal with the remaining case, where $\mathbf{k} \neq \mathbf{0}$. Since (by the multi-graded analogue of [4, 12.4.2]) there is a \mathbb{Z}^r -homogeneous epimorphism of \mathbb{Z}^r -graded R -modules $D_{yR}(M) \rightarrow H_{yR}^1(M)$, it suffices for us to show that $\mathcal{S}(D_{yR}(M))$ is contained in a $(\mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k}))$ -domain in \mathbb{Z}^r .

Recall that there is a homogeneous isomorphism $D_{yR}(M) \cong M_y$, and so the multiplication map $y : D_{yR}(M) \rightarrow D_{yR}(M)$ is an isomorphism, and $D_{yR}(M)$ is finitely generated as an R_y -module. Since $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R M)}$, there exists $u \in \mathbb{N}$ such that $R_{u\mathbf{m}}M = 0$, so that $R_{u\mathbf{m}}M_y = 0$ and $R_{u\mathbf{m}}D_{yR}(M) = 0$. Observe that $\mathcal{P}(u\mathbf{m}) = \mathcal{P}(\mathbf{m})$. We now apply Lemma 5.10, with $D_{yR}(M)$ as the module T and $u\mathbf{m}$ in the rôle of \mathbf{m} : if $\mathcal{P}(u\mathbf{m}) = \mathcal{P}(\mathbf{m}) \subseteq \mathcal{P}(\mathbf{k})$, then part (i) of Lemma 5.10 yields that $\mathcal{S}(D_{yR}(M)) = \emptyset$, while if $\mathcal{P}(u\mathbf{m}) = \mathcal{P}(\mathbf{m}) \not\subseteq \mathcal{P}(\mathbf{k})$, then it follows from part (ii) of Lemma 5.10 that $\mathcal{S}(D_{yR}(M))$ is contained in a $(\mathcal{P}(\mathbf{m}) \setminus \mathcal{P}(\mathbf{k}))$ -domain in \mathbb{Z}^r . \square

5.12. Lemma. *Let $\mathbf{m} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$. Assume that M is finitely generated and that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R M)}$. Then there exists a $\mathcal{P}(\mathbf{m})$ -domain \mathbb{X} in \mathbb{Z}^r such that $\mathcal{S}(H_{\mathbf{b}}^i(M)) \subseteq \mathbb{X}$ for all $i \in \mathbb{N}_0$.*

Proof. Since $H_{\mathbf{b}}^i(M) = 0$ for all $i > \text{ara}(\mathbf{b})$, it follows from Remark 5.9(vi) that it is sufficient for us to show that, for each $i \in \mathbb{N}_0$, there exists a $\mathcal{P}(\mathbf{m})$ -domain \mathbb{X}_i in \mathbb{Z}^r such that $\mathcal{S}(H_{\mathbf{b}}^i(M)) \subseteq \mathbb{X}_i$. For $i = 0$, this is immediate from Lemma 5.7.

Let y_1, \dots, y_s be \mathbb{N}_0^r -homogeneous elements of R that generate \mathbf{b} . We argue by induction on s . When $s = 1$ and $i = 1$, the desired result follows from Lemma 5.11; as we have already dealt, in the preceding paragraph, with the case where $i = 0$, and as $H_{y_1 R}^i(M) = 0$ for all $i > 1$, we have established the desired result in all cases when $s = 1$.

So suppose now that $s > 1$ and that the desired result has been proved in all cases where \mathfrak{b} can be generated by fewer than s \mathbb{N}_0^r -homogeneous elements. Again, we have already dealt with the case where $i = 0$. For $i \in \mathbb{N}$, there is an exact Mayer–Vietoris sequence (in the category ${}^*\mathcal{C}^{\mathbb{Z}^r}(R)$)

$$\cdots \longrightarrow H_{(y_1 y_s, \dots, y_{s-1} y_s)R}^{i-1}(M) \longrightarrow H_{\mathfrak{b}}^i(M) \longrightarrow H_{(y_1, \dots, y_{s-1})R}^i(M) \oplus H_{y_s R}^i(M) \longrightarrow \cdots$$

By the inductive hypothesis, there exist $\mathcal{P}(\mathfrak{m})$ -domains $\mathbb{X}'_i, \mathbb{X}''_i, \mathbb{X}'''_i$ in \mathbb{Z}^r such that

$$\mathcal{S}(H_{(y_1 y_s, \dots, y_{s-1} y_s)R}^{i-1}(M)) \subseteq \mathbb{X}'_i, \quad \mathcal{S}(H_{(y_1, \dots, y_{s-1})R}^i(M)) \subseteq \mathbb{X}''_i \quad \text{and} \quad \mathcal{S}(H_{y_s R}^i(M)) \subseteq \mathbb{X}'''_i.$$

Therefore $\mathcal{S}(H_{\mathfrak{b}}^i(M)) \subseteq \mathbb{X}'_i \cup \mathbb{X}''_i \cup \mathbb{X}'''_i$, and so the desired result follows from Remark 5.9(vi). \square

5.13. Lemma. *Let $\mathfrak{m} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be prime ideals of R such that $R_{\mathfrak{m}} \not\subseteq \mathfrak{p}_i$ for each $i = 1, \dots, n$. Then there exists $u \in \mathbb{N}$ such that $R_{u\mathfrak{m}} \not\subseteq \bigcup_{i=1}^n \mathfrak{p}_i$*

Proof. Consider the (Noetherian) \mathbb{N}_0 -graded ring $R_0[R_{\mathfrak{m}}] = \bigoplus_{j \in \mathbb{N}_0} R_{j\mathfrak{m}}$ (in which $R_{j\mathfrak{m}}$ is the component of degree j , for all $j \in \mathbb{N}_0$). Apply the ordinary Homogeneous Prime Avoidance Lemma (see [4, Lemma 15.1.2]) to the graded ideal $R_{\mathfrak{m}}R_0[R_{\mathfrak{m}}] = \bigoplus_{j \in \mathbb{N}} R_{j\mathfrak{m}}$ and the prime ideals $\mathfrak{p}_i \cap R_0[R_{\mathfrak{m}}]$ ($i = 1, \dots, n$). \square

5.14. Lemma. *Let $\mathfrak{m} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$ and let \mathbb{X} be a $\mathcal{P}(\mathfrak{m})$ -domain in \mathbb{Z}^r . Then there exists $u \in \mathbb{N}$ such that, for each $\mathfrak{w} \in \mathbb{Z}^r$, there is some $j \in \{0, \dots, \#\mathcal{P}(\mathfrak{m})\}$ with $\mathfrak{w} + j u \mathfrak{m} \notin \mathbb{X}$.*

Proof. There exist $\mathfrak{s}, \mathfrak{t} \in \mathbb{Z}^r$ with $\mathfrak{s} \leq \mathfrak{t}$ and $\mathcal{P}(\mathfrak{t} - \mathfrak{s}) \subseteq \mathcal{P}(\mathfrak{m})$ for which $\mathbb{X} = \mathbb{X}(\mathfrak{s}, \mathfrak{t})$. Choose $u \in \mathbb{N}$ such that $u\mathfrak{m} \geq \mathfrak{t} - \mathfrak{s}$.

For an arbitrary $\mathfrak{w} \in \mathbb{Z}^r$, set $\mathcal{I}(\mathfrak{w}) = \{i \in \{1, \dots, r\} : s_i \leq w_i < t_i\}$, and observe that $\mathcal{I}(\mathfrak{w}) \subseteq \mathcal{P}(\mathfrak{m})$, and that $\mathfrak{w} \in \mathbb{X}$ if and only if $\mathcal{I}(\mathfrak{w}) \neq \emptyset$. Note also that, for $i \in \mathcal{I}(\mathfrak{w})$ and $j \in \mathbb{N}$, we have

$$(\mathfrak{w} + j u \mathfrak{m})_i = w_i + j u m_i \geq s_i + u m_i \geq s_i + t_i - s_i = t_i,$$

so that $i \notin \mathcal{I}(\mathfrak{w} + j u \mathfrak{m})$. So, for each $i \in \mathcal{P}(\mathfrak{m})$, if there is a $j' \in \mathbb{N}_0$ with $i \in \mathcal{I}(\mathfrak{w} + j' u \mathfrak{m})$, then $i \notin \mathcal{I}(\mathfrak{w} + j u \mathfrak{m})$ for all $j > j'$. This means that, for each $i \in \mathcal{P}(\mathfrak{m})$, there is at most one $j' \in \mathbb{N}_0$ with $i \in \mathcal{I}(\mathfrak{w} + j' u \mathfrak{m})$. By the pigeon-hole principle, it is therefore possible to choose a $j \in \{0, \dots, \#\mathcal{P}(\mathfrak{m})\}$ for which $\mathcal{I}(\mathfrak{w} + j u \mathfrak{m}) \cap \mathcal{P}(\mathfrak{m}) = \emptyset$, and then $\mathfrak{w} + j u \mathfrak{m} \notin \mathbb{X}$. \square

The concept introduced in the next definition can be regarded as a multi-graded analogue of one defined by Marley in [14, §2].

5.15. Definition. Let $\mathcal{Q} \subseteq \{1, \dots, r\}$, and let \mathfrak{b} be an \mathbb{N}_0^r -graded ideal of R . We define the \mathcal{Q} -finiteness dimension $g_{\mathfrak{b}}^{\mathcal{Q}}(M)$ of M with respect to \mathfrak{b} by

$$g_{\mathfrak{b}}^{\mathcal{Q}}(M) := \sup\{k \in \mathbb{N}_0 : \text{for all } i < k, \text{ there exists a } \mathcal{Q}\text{-domain } \mathbb{X}_i \text{ in } \mathbb{Z}^r \text{ with } \mathcal{S}(H_{\mathfrak{b}}^i(M)) \subseteq \mathbb{X}_i\},$$

if this supremum exists, and ∞ otherwise.

5.16. Example. For R as in Example 5.3, we have

$$g_{R_+}^{\emptyset}(R) = 2, \quad g_{R_+}^{\{1\}}(R) = 3, \quad g_{R_+}^{\{2\}}(R) = 2, \quad g_{R_+}^{\{1,2\}}(R) = 5.$$

5.17. Remarks. The first three of the statements below are immediate from Remarks 5.9(i),(ii),(iii) respectively.

- (i) In the case where $\mathcal{Q} = \emptyset$, we have $g_{\mathfrak{b}}^{\emptyset}(M) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{b}}^i(M) \neq 0\}$ (with the usual convention that the infimum of the empty set of integers is interpreted as ∞).
- (ii) If $\mathcal{Q} \subseteq \mathcal{Q}' \subseteq \{1, \dots, r\}$, then $g_{\mathfrak{b}}^{\mathcal{Q}}(M) \leq g_{\mathfrak{b}}^{\mathcal{Q}'}(M)$.
- (iii) For $\mathfrak{n} \in \mathbb{Z}^r$, we have $g_{\mathfrak{b}}^{\mathcal{Q}}(M(\mathfrak{n})) = g_{\mathfrak{b}}^{\mathcal{Q}}(M)$.
- (iv) Let $(\mathcal{Q}_{\lambda})_{\lambda \in \Lambda}$ be a family of subsets of $\{1, \dots, r\}$. Set

$$\Omega := \left\{ \bigcap_{\lambda \in \Lambda} \mathbb{X}_{\lambda} : \mathbb{X}_{\lambda} \text{ is a } \mathcal{Q}_{\lambda}\text{-domain in } \mathbb{Z}^r \text{ for all } \lambda \in \Lambda \right\}.$$

It is straightforward to check that

$$\inf\{g_{\mathfrak{b}}^{\mathcal{Q}_{\lambda}}(M) : \lambda \in \Lambda\} = \sup\{k \in \mathbb{N}_0 : \text{for all } i < k, \text{ there exists } \mathbb{Y}_i \in \Omega \text{ with } \mathcal{S}(H_{\mathfrak{b}}^i(M)) \subseteq \mathbb{Y}_i\}.$$

(v) Since a subset of \mathbb{Z}^r is finite if and only if it is contained in a set of the form $\bigcap_{j=1}^r \mathbb{X}_j$, where \mathbb{X}_j is a $\{j\}$ -domain in \mathbb{Z}^r for all $j \in \{1, \dots, r\}$, it therefore follows from part (iv) that

$$\min\{g_b^{\{1\}}(M), \dots, g_b^{\{r\}}(M)\} = \sup\{k \in \mathbb{N}_0 : \mathcal{S}(H_b^i(M)) \text{ is finite for all } i < k\}.$$

Thus we can say that $\min\{g_b^{\{1\}}(M), \dots, g_b^{\{r\}}(M)\}$ identifies the smallest integer i (if there be any) for which $H_b^i(M)$ is not finitely graded.

5.18. Proposition. *Let $\mathbf{m} \in \mathbb{N}_0^r \setminus \{\mathbf{0}\}$, and let $f \in \mathbb{N}$. Assume that M is finitely generated. The following statements are equivalent:*

- (i) $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R H_b^i(M))}$ for all integers $i < f$;
- (ii) for each integer $i < f$, there is a $\mathcal{P}(\mathbf{m})$ -domain \mathbb{X}_i in \mathbb{Z}^r such that $\mathcal{S}(H_b^i(M)) \subseteq \mathbb{X}_i$, that is $f \leq g_b^{\mathcal{P}(\mathbf{m})}(M)$;
- (iii) there is a $\mathcal{P}(\mathbf{m})$ -domain \mathbb{X} in \mathbb{Z}^r such that $\mathcal{S}(H_b^i(M)) \subseteq \mathbb{X}$ for all integers $i < f$.

Proof. (ii) \Leftrightarrow (iii) This is immediate from Remark 5.9(vi).

(iii) \Rightarrow (i) Assume that statement (iii) holds. By Lemma 5.14, there exist $u, v := \#\mathcal{P}(\mathbf{m}) \in \mathbb{N}$ such that, for each $\mathbf{n} \in \mathbb{Z}^r$, there exists $j(\mathbf{n}) \in \{0, \dots, v\}$ with $\mathbf{n} + j(\mathbf{n})u\mathbf{m} \notin \mathbb{X}$. So, for each $\mathbf{n} \in \mathbb{Z}^r$ and each integer $i < f$, we have $H_b^i(M)_{\mathbf{n} + j(\mathbf{n})u\mathbf{m}} = 0$ and

$$R_{v\mathbf{u}\mathbf{m}}H_b^i(M)_{\mathbf{n}} = R_{v\mathbf{u}\mathbf{m} - j(\mathbf{n})u\mathbf{m}}R_{j(\mathbf{n})u\mathbf{m}}H_b^i(M)_{\mathbf{n}} \subseteq R_{v\mathbf{u}\mathbf{m} - j(\mathbf{n})u\mathbf{m}}H_b^i(M)_{\mathbf{n} + j(\mathbf{n})u\mathbf{m}} = 0.$$

Therefore $R_{v\mathbf{u}\mathbf{m}}H_b^i(M) = 0$ for all integers $i < f$, and hence

$$(R_{\mathbf{m}})^{vu} \subseteq R_{v\mathbf{u}\mathbf{m}} \subseteq (0 :_R H_b^i(M)) \quad \text{for all } i < f.$$

(i) \Rightarrow (ii) Assume that statement (i) holds. We argue by induction on f . When $f = 1$, the desired conclusion is immediate from Lemma 5.7 (applied to $H_b^0(M)$).

So assume now that $f > 1$ and that statement (ii) has been proved for smaller values of f . This inductive hypothesis implies that there exist $\mathcal{P}(\mathbf{m})$ -domains $\mathbb{X}_0, \dots, \mathbb{X}_{f-2}$ in \mathbb{Z}^r such that $\mathcal{S}(H_b^i(M)) \subseteq \mathbb{X}_i$ for all $i \in \{0, \dots, f-2\}$. It thus remains to find a $\mathcal{P}(\mathbf{m})$ -domain \mathbb{X}_{f-1} in \mathbb{Z}^r such that $\mathcal{S}(H_b^{f-1}(M)) \subseteq \mathbb{X}_{f-1}$.

Set $\overline{M} := M/\Gamma_{R_{\mathbf{m}}R}(M)$, and observe that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R \Gamma_{R_{\mathbf{m}}R}(M))}$. It therefore follows from Lemma 5.12 that there is a $\mathcal{P}(\mathbf{m})$ -domain \mathbb{X}' in \mathbb{Z}^r such that $\mathcal{S}(H_b^{f-1}(\Gamma_{R_{\mathbf{m}}R}(M))) \subseteq \mathbb{X}'$. In view of the exact sequence of \mathbb{Z}^r -graded R -modules

$$H_b^{f-1}(\Gamma_{R_{\mathbf{m}}R}(M)) \longrightarrow H_b^{f-1}(M) \longrightarrow H_b^{f-1}(\overline{M})$$

and Remark 5.9(vi), it is now enough for us to show that $\mathcal{S}(H_b^{f-1}(\overline{M}))$ is contained in a $\mathcal{P}(\mathbf{m})$ -domain in \mathbb{Z}^r .

As $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R H_b^j(\Gamma_{R_{\mathbf{m}}R}(M)))}$ for all $j \in \mathbb{N}_0$, the exact sequence

$$H_b^i(M) \longrightarrow H_b^i(\overline{M}) \longrightarrow H_b^{i+1}(\Gamma_{R_{\mathbf{m}}R}(M))$$

shows that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R H_b^i(\overline{M}))}$ for all integers $i < f$. Set $\text{Ass}_R(\overline{M}) =: \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$. As $R_{\mathbf{m}}R$ does not consist entirely of zero-divisors on \overline{M} , we have $R_{\mathbf{m}} \not\subseteq \mathfrak{p}_i$ for each $i = 1, \dots, k$. Therefore, by Lemma 5.13, there exists $u' \in \mathbb{N}$ such that $R_{u'\mathbf{m}} \not\subseteq \bigcup_{i=1}^k \mathfrak{p}_i$, and hence there exists $y' \in R_{u'\mathbf{m}}$ which is not a zero-divisor on \overline{M} . We can now take a sufficiently high power y of y' to find $u \in \mathbb{N}$ and $y \in R_{u\mathbf{m}}$ such that $R_{u\mathbf{m}}H_b^{f-1}(\overline{M}) = 0$ and y is a non-zero-divisor on \overline{M} , so that there is a short exact sequence of \mathbb{Z}^r -graded R modules

$$0 \longrightarrow \overline{M}(-u\mathbf{m}) \xrightarrow{y} \overline{M} \longrightarrow \overline{M}/y\overline{M} \longrightarrow 0.$$

It now follows from the long exact sequence of local cohomology modules induced from the above short exact sequence that $R_{\mathbf{m}} \subseteq \sqrt{(0 :_R H_b^i(\overline{M}/y\overline{M}))}$ for all integers $i < f - 1$. Therefore, by the inductive hypothesis, there is a $\mathcal{P}(\mathbf{m})$ -domain \mathbb{X}'' in \mathbb{Z}^r such that $\mathcal{S}(H_b^{f-2}(\overline{M}/y\overline{M})) \subseteq \mathbb{X}''$. Let K be the kernel of the map $H_b^{f-1}(\overline{M}) \longrightarrow H_b^{f-1}(\overline{M})(u\mathbf{m})$ provided by multiplication by y . The long exact sequence

of local cohomology modules induced from the last-displayed short exact sequence now shows that $\mathcal{S}(K) \subseteq \mathbb{X}'' - u\mathbf{m}$.

We now apply Lemma 5.10(i) to $H_{\mathfrak{b}}^{f-1}(\overline{M})$, with $u\mathbf{m}$ playing the rôles of both \mathbf{m} and \mathbf{k} : the conclusion is that there exists $v \in \mathbb{N}_0$ such that

$$\mathcal{S}(H_{\mathfrak{b}}^{f-1}(\overline{M})) \subseteq \bigcup_{j=0}^v (\mathcal{S}(K) - j\mathbf{u}\mathbf{m}) \subseteq \bigcup_{j=0}^v (\mathbb{X}'' - u\mathbf{m} - j\mathbf{u}\mathbf{m}).$$

We can now use Remarks 5.9(iii),(vi) to deduce the existence of a $\mathcal{P}(\mathbf{m})$ -domain \mathbb{X}_{f-1} in \mathbb{Z}^r such that $\mathcal{S}(H_{\mathfrak{b}}^{f-1}(\overline{M})) \subseteq \mathbb{X}_{f-1}$. With this, the proof is complete. \square

We now connect the concept of \mathcal{Q} -finiteness dimension of M with respect to \mathfrak{b} , introduced in Definition 5.17, with the concept of \mathfrak{a} -finiteness dimension of M relative to \mathfrak{b} (where \mathfrak{a} is a second ideal of R), studied by Faltings in [5]. (See also [4, Chapter 9].)

5.19. Reminder. Assume that M is finitely generated, and let $\mathfrak{a}, \mathfrak{d}$ be ideals of R (not necessarily graded).

The \mathfrak{a} -finiteness dimension $f_{\mathfrak{d}}^{\mathfrak{a}}(M)$ of M relative to \mathfrak{d} is defined by

$$f_{\mathfrak{d}}^{\mathfrak{a}}(M) = \inf \left\{ i \in \mathbb{N}_0 : \mathfrak{a} \not\subseteq \sqrt{(0 : H_{\mathfrak{d}}^i(M))} \right\}$$

and the \mathfrak{a} -minimum \mathfrak{d} -adjusted depth $\lambda_{\mathfrak{d}}^{\mathfrak{a}}(M)$ of M is defined by

$$\lambda_{\mathfrak{d}}^{\mathfrak{a}}(M) := \inf \{ \text{depth } M_{\mathfrak{p}} + \text{ht}(\mathfrak{d} + \mathfrak{p})/\mathfrak{p} : \mathfrak{p} \in \text{Spec}(R) \setminus \text{Var}(\mathfrak{a}) \}.$$

(Here, $\text{Var}(\mathfrak{a})$ denotes the *variety* $\{ \mathfrak{p} \in \text{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a} \}$ of \mathfrak{a} .) It is always the case that $f_{\mathfrak{d}}^{\mathfrak{a}}(M) \leq \lambda_{\mathfrak{d}}^{\mathfrak{a}}(M)$; Faltings' (Extended) Annihilator Theorem [5] states that if R admits a dualizing complex or is a homomorphic image of a regular ring, then $f_{\mathfrak{d}}^{\mathfrak{a}}(M) = \lambda_{\mathfrak{d}}^{\mathfrak{a}}(M)$. (See [3, Corollary 3.8] for an account of the extended version of Faltings' Annihilator Theorem.)

5.20. Remark. Let the situation be as in Reminder 5.19, let $K \subseteq R$, and let $(K_j)_{j \in J}$ be a family of subsets of R .

- (i) It is easy to deduce from the definition that $f_{\mathfrak{d}}^{KR}(M) = \inf \{ f_{\mathfrak{d}}^{aR}(M) : a \in K \}$.
- (ii) We can then deduce from part (i) that $f_{\mathfrak{d}}^{(\bigcup_{j \in J} K_j)R}(M) = \inf \{ f_{\mathfrak{d}}^{K_j R}(M) : j \in J \}$.
- (iii) Similarly, it is easy to deduce from the definition that $\lambda_{\mathfrak{d}}^{KR}(M) = \inf \{ \lambda_{\mathfrak{d}}^{aR}(M) : a \in K \}$.
- (iv) We can then deduce from part (iii) that $\lambda_{\mathfrak{d}}^{(\bigcup_{j \in J} K_j)R}(M) = \inf \{ \lambda_{\mathfrak{d}}^{K_j R}(M) : j \in J \}$.

5.21. Theorem. Assume that M is finitely generated, and let $\emptyset \neq \mathcal{T} \subseteq \mathbb{N}_0^r$.

(i) We have

$$\begin{aligned} & \sup \{ k \in \mathbb{N}_0 : \text{for all } i < k \text{ and all } \mathbf{m} \in \mathcal{T}, \text{ there exists a } \mathcal{P}(\mathbf{m})\text{-domain } \mathbb{X}_i^{(\mathbf{m})} \text{ in } \mathbb{Z}^r \\ & \quad \text{such that } \mathcal{S}(H_{\mathfrak{b}}^i(M)) \subseteq \mathbb{X}_i^{(\mathbf{m})} \} \\ &= \inf \left\{ g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M) : \mathbf{m} \in \mathcal{T} \right\} \\ &= f_{\mathfrak{b}}^{\sum_{\mathbf{m} \in \mathcal{T}} R_{\mathbf{m}}R}(M) \leq \lambda_{\mathfrak{b}}^{\sum_{\mathbf{m} \in \mathcal{T}} R_{\mathbf{m}}R}(M). \end{aligned}$$

(ii) If R admits a dualizing complex or is a homomorphic image of a regular ring, then we can replace the inequality in part (i) by equality.

Proof. Apply Remark 5.17(iv) to the family $(\mathcal{P}(\mathbf{m}))_{\mathbf{m} \in \mathcal{T}}$ of subsets of $\{1, \dots, r\}$ to conclude that

$$\begin{aligned} & \sup \{ k \in \mathbb{N}_0 : \text{for all } i < k \text{ and all } \mathbf{m} \in \mathcal{T}, \text{ there exists a } \mathcal{P}(\mathbf{m})\text{-domain } \mathbb{X}_i^{(\mathbf{m})} \text{ in } \mathbb{Z}^r \\ & \quad \text{such that } \mathcal{S}(H_{\mathfrak{b}}^i(M)) \subseteq \mathbb{X}_i^{(\mathbf{m})} \} \\ &= \inf \left\{ g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M) : \mathbf{m} \in \mathcal{T} \right\}. \end{aligned}$$

By Proposition 5.18, we have $g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M) = f_{\mathfrak{b}}^{R_{\mathbf{m}}R}(M)$ for all $\mathbf{m} \in \mathcal{T}$. Therefore, on use of Remark 5.20(ii), we deduce that

$$\inf\left\{g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M) : \mathbf{m} \in \mathcal{T}\right\} = \inf\left\{f_{\mathfrak{b}}^{R_{\mathbf{m}}R}(M) : \mathbf{m} \in \mathcal{T}\right\} = f_{\mathfrak{b}}^{\sum_{\mathbf{m} \in \mathcal{T}} R_{\mathbf{m}}R}(M).$$

We can now use Faltings' (Extended) Annihilator Theorem [5] (see Reminder 5.19) to complete the proof of part (i) and to obtain the statement in part (ii). \square

5.22. Corollary. *Assume that M is finitely generated.*

(i) *For each non-empty set $\mathcal{T} \subseteq \mathbf{1} + \mathbb{N}_0^r$, we have*

$$f_{\mathfrak{b}}^{\sum_{\mathbf{m} \in \mathcal{T}} R_{\mathbf{m}}R}(M) = g_{\mathfrak{b}}^{\{1, \dots, r\}}(M).$$

(ii) *For each set $\mathcal{T} \subseteq \mathbb{N}_0^r \setminus \{\mathbf{0}\}$ such that $\mathbb{N}e_i \cap \mathcal{T} \neq \emptyset$ for all $i \in \{1, \dots, r\}$, we have*

$$\begin{aligned} f_{\mathfrak{b}}^{\sum_{\mathbf{m} \in \mathcal{T}} R_{\mathbf{m}}R}(M) &= \sup\{k \in \mathbb{N}_0 : \mathcal{S}(H_{\mathfrak{b}}^i(M)) \text{ is finite for all } i < k\} \\ &= \sup\{k \in \mathbb{N}_0 : H_{\mathfrak{b}}^i(M) \text{ is finitely graded for all } i < k\}. \end{aligned}$$

(iii) *If $M \neq \mathfrak{b}M$, then $f_{\mathfrak{b}}^R(M) = g_{\mathfrak{b}}^{\emptyset}(M) = \text{grade}_M \mathfrak{b}$.*

Note. If, in the case where $r = 1$, we take $\mathcal{T} = \mathbb{N}$, so that $\sum_{m \in \mathcal{T}} R_m R = R_+$, then the statement in part (ii) becomes

$$f_{\mathfrak{b}}^{R_+}(M) = \sup\{k \in \mathbb{N}_0 : H_{\mathfrak{b}}^i(M) \text{ is finitely graded for all } i < k\},$$

a result proved by Marley in [14, Proposition 2.3].

Proof. (i) By Theorem 5.21(i), we have $f_{\mathfrak{b}}^{\sum_{\mathbf{m} \in \mathcal{T}} R_{\mathbf{m}}R}(M) = \inf\left\{g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M) : \mathbf{m} \in \mathcal{T}\right\}$. But $\mathcal{P}(\mathbf{m}) = \{1, \dots, r\}$ for all $\mathbf{m} \in \mathbf{1} + \mathbb{N}_0^r$.

(ii) By Theorem 5.21(i), we have

$$f_{\mathfrak{b}}^{\sum_{\mathbf{m} \in \mathcal{T}} R_{\mathbf{m}}R}(M) = \inf\left\{g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M) : \mathbf{m} \in \mathcal{T}\right\}.$$

By the hypothesis, for each $i \in \{1, \dots, r\}$, there exists $\mathbf{m}_i \in \mathcal{T}$ with $\mathcal{P}(\mathbf{m}_i) = \{i\}$. It therefore follows from Remark 5.17(ii) that $\inf\left\{g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{m})}(M) : \mathbf{m} \in \mathcal{T}\right\} = \min\left\{g_{\mathfrak{b}}^{\{1\}}(M), \dots, g_{\mathfrak{b}}^{\{r\}}(M)\right\}$. However, we noted in Remark 5.17(v) that

$$\min\left\{g_{\mathfrak{b}}^{\{1\}}(M), \dots, g_{\mathfrak{b}}^{\{r\}}(M)\right\} = \sup\{k \in \mathbb{N}_0 : \mathcal{S}(H_{\mathfrak{b}}^i(M)) \text{ is finite for all } i < k\}.$$

(iii) Since $R = R_{\mathbf{0}}R$, we can deduce from Theorem 5.21(i) and Remark 5.17(i) that

$$f_{\mathfrak{b}}^R(M) = f_{\mathfrak{b}}^{R_{\mathbf{0}}R}(M) = g_{\mathfrak{b}}^{\mathcal{P}(\mathbf{0})}(M) = g_{\mathfrak{b}}^{\emptyset}(M) = \sup\{k \in \mathbb{N}_0 : H_{\mathfrak{b}}^i(M) = 0 \text{ for all } i < k\} = \text{grade}_M \mathfrak{b}.$$

\square

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