On calmness of the optimal value function

Klatte, Diethard

DOI: https://doi.org/10.23952/asvao.5.2023.2.09

Appl. Set-Valued Anal. Optim. 5 (2023), No. 2, pp. 253-264
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.5.2023.2.09

ON CALMNESS OF THE OPTIMAL VALUE FUNCTION

DIETHARD KLATTE

Institut für Betriebswirtschaftslehre, Universität Zürich, Plattenstrasse 14, CH-8032 Zürich, Switzerland

Dedicated to Professor Henry Wolkowicz on the occasion of his 75th birthday

Abstract. The paper is devoted to the calmness from below/from above for the optimal value function φ of parametric optimization problems, where we are specifically interested in perturbed semi-infinite programs. A main intention is to revisit classical results and to derive refinements of them. In particular, we show in the context of semi-infinite optimization that calmness from below for φ holds under quasi-convexity of the data functions and compactness of the solution set, which extends results on the lower semicontinuity of φ. Illustrative examples are given, which demonstrate the significance of the imposed assumptions even in the case of linear and quadratic programs.
Keywords. Calmness; Lipschitz stability of feasible set mappings; Optimal value function; Parametric models; Semi-infinite programs under data perturbation.
2020 Mathematics Subject Classification. 49J53; 90C34, 90C31.

1. Introduction

Given a metric space $(T, d(\cdot,\cdot))$, a function $f : \mathbb{R}^n \times T \to \mathbb{R}$, a multifunction $M : T \rightrightarrows \mathbb{R}^n$, and a reference point $\bar{t} \in T$, our *basic model* in this paper is the parametric optimization problem

$$\begin{array}{c} P(t): \quad f(x,t) \to \min_x \quad \text{s.t. } x \in M(t), \quad t \text{ varies near } \bar{t}, \\ \text{where } f \text{ is continuous and } M \text{ is closed.} \end{array} \tag{1.1}$$

Recall that the *graph* and the *domain* of a multifunction $\Gamma : T \rightrightarrows \mathbb{R}^n$ are defined by $\operatorname{gph}\Gamma := \{(t,x) \in T \times \mathbb{R}^n \mid x \in \Gamma(t)\}$ and $\operatorname{dom}\Gamma := \{t \in T \mid \Gamma(t) \neq \emptyset\}$, respectively, and Γ is *closed* if $\operatorname{gph}\Gamma$ is a closed set.

In particular, we study the case of parametric *semi-infinite constraints*, where the *feasible set mapping* M is given by

$$\begin{array}{c} M(t) := \{x \in \mathbb{R}^n \mid g_i(x,t) \le 0, \, i \in I\}, \, t \in T, \\ I \text{ is an arbitrary set, } g_i : \mathbb{R}^n \times T \to \mathbb{R} \, (i \in I) \text{ are continuous,} \end{array} \tag{1.2}$$

which implies that $\operatorname{gph} M$ is a closed set. If the feasible set mapping M is defined by (1.2), model (1.1) becomes a parametric semi-infinite program (SIP), and we will speak of the *standard parametric SIP* (1.1)∧(1.2). Later, assumptions like Lipschitz continuity, differentiability or (quasi-) convexity of the data will be added. Note that for finite I we have a usual parametric nonlinear program.

E-mail address: diethard.klatte@uzh.ch
Received June 2, 2023; Accepted June 19, 2023.

In this note, we intend to revisit classical results and to give various refinements of them. The main purpose of this paper is to present sufficient conditions for calmness from above and from below of the *optimal value function* (also called *marginal function*)

$$t \in T \mapsto \varphi(t) := \inf_{x \in M(t)} f(x,t) \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\},$$

with respect to the models (1.1) and (1.1)∧(1.2). The related *argmin mapping* is given by

$$t \in T \mapsto \Psi(t) = \operatorname*{argmin}_{x \in M(t)} f(x,t) := \{x \in M(t) \mid f(x,t) = \varphi(t)\}.$$

Following the terminology in [1], we say that a function $\tau : T \to \overline{\mathbb{R}}$ is *calm* at $\bar{t} \in T$ *from above* [*from below*] *with a constant* $\rho > 0$ if $\tau(\bar{t})$ is finite and for some $\delta > 0$,

$$\forall t \in B(\bar{t}, \delta): \quad \tau(t) \le \tau(\bar{t}) + \rho\, d(t, \bar{t}) \quad [\tau(t) \ge \tau(\bar{t}) - \rho\, d(t, \bar{t})], \tag{1.3}$$

and τ is said to be *calm at* $\bar{t}$ if τ is both calm from above and below: for some $\delta, \rho > 0$, one has

$$|\tau(t) - \tau(\bar{t})| \le \rho\, d(t, \bar{t}) \quad \text{when } t \in B(\bar{t}, \delta), \tag{1.4}$$

$B(t, \delta)$ denotes the closed δ-ball around $t \in T$ with respect to $d(\cdot,\cdot)$. The infimum of all $\rho > 0$ in (1.3) or (1.4), respectively, is called the *modulus* of calmness (from above/below).

The concept of calmness from below for the optimal value function was originally introduced in [2] under the name "calm", while calmness from above is also known as "quiet", see [3]. Calmness of the optimal value function φ is often a hidden subject in the stability analysis of optimization problems, because it is related to other properties of φ (e.g. convexity, differentiability, local Lipschitz continuity) as well as to constraint qualifications, duality theory, embedding theorems, and more. For many classes of optimization problems, there is a well-developed theory of that type, we refer e.g. to the standard monographs [1, 4, 5, 6, 7, 8, 9].

For references which are explicitly concerned with calmness of φ, cf. e.g. [2, 3, 6, 10] (for general models) and [11, 12, 13, 14] (for linear and quadratic optimization problems). One inspiration to write this paper comes from the interesting paper [11], where calmness of φ from above and below as well as the computation/estimation of their moduli are thoroughly studied in the framework of (finite) linear programs under so-called canonical perturbation. Though we will not consider moduli in what follows, the question arises, which conditions ensure calmness from below/above in nonlinear settings, and this particularly in the case of infinitely many constraints. Another impulse comes from the study of q-order calmness of the optimal set mapping in the framework (1.1)∧(1.2), where calmness of φ is used in the proofs, see both classical references as [5, 15, 16] and more recent papers as [10, 17, 18].

The paper is organized as follows. After introducing our terminology and discussing some preliminaries in Section 2, we present our main results in Section 3. It turns out that calmness from above and/or below of φ (or some restriction of φ to a subset) is ensured if f and M satisfy appropriate Lipschitz properties. For the calmness from below, compactness assumptions play a crucial role. We will show that these results have similarities to classical characterizations of upper or lower semicontinuity of φ for the models (1.1) and (1.1)∧(1.2), as given in the theory of point-to-set maps in optimization, cf. e.g. [4, 19, 20]. As a main result we show that under quasi-convexity of the data functions and compactness of $\Psi(\bar{t})$, the optimal value function φ is calm from below provided that M is calm. The needed assumptions on the feasible set mapping M for parametric SIPs will be discussed, too. Section 4 gives some concluding remarks.

2. Terminology and Preliminaries

We start with some notation. $\|\cdot\|$ denotes any norm in $\mathbb{R}^n$, B° is the open unit ball in $\mathbb{R}^n$ in this norm, B is the closed unit ball, $X+Y$ denotes the Minkowski sum of $X,Y \subset \mathbb{R}^n$, and we put $B(x,\varepsilon) := \{x\} + \varepsilon B$ for $\varepsilon > 0$. For $X \subset \mathbb{R}^n$, $\operatorname{int} X$ and $\operatorname{bd} X$ are the interior and the boundary of X, respectively, $\operatorname{conv} X$ denotes the convex hull of X, and $\operatorname{dist}(x,X) := \inf_{y\in X} \|y-x\|$ for $x \in \mathbb{R}^n$. The parameter space $(T,d(\cdot,\cdot))$ of our parametric models is a metric space, the product space $Z = \mathbb{R}^n \times T$ is equipped with a metric defined by $d_Z((x,t),(x',t')) = \|x-x'\| + d(t,t')$. Given a compact subset I of a metric space, $C(I)$ denotes the linear space of continuous functions $i \in I \mapsto b_i \in \mathbb{R}$ equipped with the norm $\|b\| = \max_{i\in I} b_i$. By $h \in C^1$ we abbreviate the property that h is a continuously differentiable function. Let $\mathbb{R}_+$ be the set of nonnegative real numbers, and $\mathbb{R}_- = -\mathbb{R}_+$.

Next we recall (Lipschitz) continuity concepts for functions and multifunctions, assuming $(T,d(\cdot,\cdot))$ is a metric space.

A function $\tau : T \to \overline{\mathbb{R}}$ is called *upper* [*lower*] *semicontinuous* (*u.s.c.* [*l.s.c.*]) at $\bar t \in T$ if

$$\limsup_{t\to\bar t} \tau(t) \le \tau(\bar t) \quad [\liminf_{t\to\bar t} \tau(t) \ge \tau(\bar t)].$$

By definition, one has that τ is u.s.c. at $\bar t$ if τ is calm at $\bar t$ from above, while τ is l.s.c. at $\bar t$ if τ is calm at $\bar t$ from below.

A function $F : T \to \mathbb{R}$ is called *Lipschitz* on $V \subset T$ *with a constant* $\rho > 0$ if

$$|F(t'') - F(t')| \le \rho\, d(t'',t') \qquad \text{for all } t'',t' \in V, \tag{2.1}$$

and is called *Lipschitz around* $\bar t$ if V is a neighborhood of $\bar t$. F is said to be *locally Lipschitz* if for each $t \in T$ there are a neighborhood V of t and some $\rho > 0$ satisfying (2.1).

Let $\Gamma : T \rightrightarrows \mathbb{R}^n$ be a given multifunction, and let $(\bar t,\bar x)$ in $\operatorname{gph}\Gamma$ be fixed. Γ is said to be *Lipschitz lower semicontinuous (Lipschitz l.s.c.)* at $(\bar t,\bar x)$ if, for some $\delta, L > 0$,

$$\operatorname{dist}(\bar x,\Gamma(t)) \le L d(t,\bar t) \qquad \forall t \in B(\bar t,\delta),$$

which includes $\Gamma(t) \ne \emptyset$ for all $t \in B(\bar t,\delta)$. Γ is called *calm* at $(\bar t,\bar x)$ (equivalently, Γ^{-1} is *metrically subregular* at $(\bar x,\bar t)$) if there are constants $\varepsilon,\delta,L > 0$ such that

$$\Gamma(t) \cap B(\bar x,\varepsilon) \subset \Gamma(\bar t) + L d(t,\bar t) B \qquad \forall t \in B(\bar t,\delta), \tag{2.2}$$

where $\Gamma(t) \cap B(\bar x,\varepsilon) = \emptyset$ for $t \ne \bar t$ is possible. Γ has the *Aubin property* at $(\bar t,\bar x)$ (equivalently, Γ is *pseudo-Lipschitz* at $(\bar t,\bar x)$, or Γ^{-1} is *metrically regular* at $(\bar x,\bar t)$) if there are constants $\varepsilon,\delta,L > 0$ such that

$$\Gamma(t) \cap B(\bar x,\varepsilon) \subset \Gamma(t') + L\|t-t'\| B \qquad \forall t,t' \in B(\bar t,\delta).$$

Given a nonempty set $\Omega \subset \operatorname{gph}\Gamma$, we say that Γ is *calm on* Ω if there are $\varepsilon,\delta,L > 0$ (uniformly on Ω) such that (2.2) holds for all $(\bar t,\bar x) \in \Omega$. Obviously, if Γ has the Aubin property at some point, then Γ calm and Lipschitz l.s.c. there.

Some of our results concern a restricted optimal value function for the model (1.1) or the setting (1.1)$\wedge$(1.2), respectively, defined for $Q \subset \mathbb{R}^n$ by

$$t \in T \mapsto \varphi_Q(t) := \inf_{x\in M_Q(t)} f(x,t), \quad \text{where } M_Q(t) := M(t) \cap Q.$$

The corresponding optimal solution set is $\Psi_Q(t) := \{x \in M_Q(t) \mid f(x,t) = \varphi_Q(t)\}$, $t \in T$.

We finish this section by recalling classical results on upper and lower semicontinuity of the optimal value function φ for the the basic model (1.1) and the standard parametric SIP (1.1)∧(1.2). The first lemma concerns the basic model (1.1), and applies, of course, to the parametric SIP (1.1)∧(1.2). It goes back to classical parametric optimization and set-valued analysis, as given, e.g., in [4, 19, 20]. Recall that, by the assumptions in (1.1), the objective function f is continuous, and the feasible set mapping M is closed.

Lemma 2.1. *Consider the basic model (1.1) and assume $\Psi(\bar{t}) \neq \emptyset$.*

(i) [4, Thm. 4.2.2 (1)'] *If M is l.s.c. at $(\bar{t}, \bar{x})$ for some $\bar{x} \in \Psi(\bar{t})$, then φ is u.s.c. at $\bar{t}$.*

(ii) [4, Cor. 4.2.2 (2)] *If $Q \cap \Psi(\bar{t}) \neq \emptyset$ for some compact set $Q \subset \mathbb{R}^n$, then the restricted optimal value function φ_Q is l.s.c. at $\bar{t}$.*

The assumptions of Lemma 2.1 (ii) do not guarantee that the function φ itself is l.s.c. at $\bar{t}$, see e.g. [4, Example 1.1], or the Examples 3.1 and 3.2 below, where in Example 3.2 the set $\Psi(\bar{t})$ is even a singleton. However, an obvious consequence of (ii) is that φ is l.s.c. at $\bar{t}$, provided $\Psi(t) \subset Q$ (with compact Q) holds for all t in a neighborhood of $\bar{t}$.

With respect to the l.s.c. property for φ, we remind of the next result which holds in the framework of perturbed (quasi-)convex semi-infinite optimization problems.

Lemma 2.2. [4, Thm. 4.3.4], [21] *Consider the standard parametric SIP (1.1)∧(1.2). Suppose that for each $t \in T$, the functions $f(\cdot, t)$, $g_i(\cdot, t)$, $i \in I$, are quasiconvex. If $\Psi(\bar{t})$ is nonempty and bounded, then φ is l.s.c. at $\bar{t}$.*

3. Calmness of the Optimal Value Function

In this section, we study conditions which guarantee calmness of φ from above or below, by additionally assuming that objective function f is locally Lipschitz. First we present a counterpart to Lemma 2.1 concerning calmness from above/below of the (possibly restricted) optimal value function of model (1.1). Similar results have appeared in different frameworks already in the past, see e.g. [2, 3, 10, 22]. Our method of proof goes back to the idea of proving Lipschitz continuity of a (restricted) optimal value function under the Aubin property of the feasible set mapping M at some point in $\operatorname{gph}\Psi$, see e.g. [15, 16].

Theorem 3.1. *Consider the basic model (1.1) and suppose that $\Psi(\bar{t}) \neq \emptyset$.*

(i) *Let $\bar{x} \in \Psi(\bar{t})$. If f is Lipschitz on some neighborhood V of $(\bar{x}, \bar{t})$ and M is Lipschitz l.s.c. at $(\bar{t}, \bar{x})$, then φ is calm at $\bar{t}$ from above.*

(ii) *Let $S \subset \Psi(\bar{t})$ be nonempty and compact. If f is Lipschitz on some open set $V \supset S \times \{\bar{t}\}$ and M is calm on $\{\bar{t}\} \times S$, then there is some $\varepsilon > 0$ such that, for $Q := S + \varepsilon B$, the restricted optimal value function φ_Q is calm at $\bar{t}$ from below.*

Proof. (i) Let $\rho_f > 0$ be a Lipschitz constant for f on V. Since M is Lipschitz l.s.c. at $(\bar{t}, \bar{x}) \in \operatorname{gph}\Psi$, there are $\delta_M, \rho_M > 0$ such that, for each $t \in B(\bar{t}, \delta_M)$,

$$\exists z_t \in M(t): \quad \|z_t - \bar{x}\| = \operatorname{dist}(\bar{x}, M(t)) \leq \rho_M d(t, \bar{t}). \tag{3.1}$$

Choose a positive $\delta \leq \delta_M$ small enough such that, with $\varepsilon := \rho_M \delta$, also $Q \times U \subset V$ is satisfied, where $Q := B(\bar{x}, \varepsilon)$ and $U := B(\bar{t}, \delta)$. Let $t \in U$. Then, with some z_t satisfying (3.1), one has

$z_t \in Q$ and

$$\begin{aligned} \varphi(t) \le \varphi_Q(t) \le f(z_t,t) &\le f(\bar{x},\bar{t}) + |f(z_t,t) - f(\bar{x},\bar{t})| \\ &\le f(\bar{x},\bar{t}) + \rho_f(\|z_t - \bar{x}\| + d(t,\bar{t})) \\ &\le \varphi(\bar{t}) + \rho_f(\rho_M + 1)d(t,\bar{t}), \end{aligned} \tag{3.2}$$

since $\varphi(\bar{t}) = f(\bar{x},\bar{t})$. So, φ is calm from above at $\bar{t}$ with a constant $\rho = \rho_f(\rho_M + 1)$.

(ii) Let $\rho_f > 0$ be a Lipschitz constant for f on V. As assumed, $S \subset \Psi(\bar{t})$ is nonempty and compact, and M is calm on $\{\bar{t}\} \times S$. Hence, there are $\delta_M, \varepsilon_M, \rho_M > 0$ such that

$$M(t) \cap (S + \varepsilon_M B) \subset M(\bar{t}) + \rho_M d(t,\bar{t}) B \quad \forall t \in B(\bar{t}, \delta_M). \tag{3.3}$$

Choose positive constants $\varepsilon \le \varepsilon_M$ and $\delta \le \min\{\delta_M, \varepsilon/\rho_M\}$ such that $(S + 2\varepsilon B) \times B(\bar{t}, \delta)$ is a subset of V. Put $Q := S + \varepsilon B$ and $U := B(\bar{t}, \delta)$. Note that $\varphi(\bar{t}) = \varphi_Q(\bar{t})$. We will show that

$$\exists \rho > 0 \; \forall t \in U: \quad \varphi_Q(t) \ge \varphi_Q(\bar{t}) - \rho d(t,\bar{t}), \tag{3.4}$$

which means that φ_Q is calm at $\bar{t}$ from below. Let $t \in U$. (3.4) automatically holds if $M_Q(t) = \emptyset$ because $\varphi_Q(t) = +\infty$ in this case. Let $M_Q(t) \ne \emptyset$ which is a compact set as the set $M(t)$ is closed and Q is compact. By Weierstrass' Theorem, it follows $\Psi_Q(t) \ne \emptyset$. Given any $y \in \Psi_Q(t)$, there is some $\bar{y} \in M(\bar{t})$ satisfying $\|y - \bar{y}\| \le \rho_M d(t,\bar{t})$ by (3.3). This entails $\|y - \bar{y}\| \le \rho_M \delta \le \varepsilon$ and $\operatorname{dist}(\bar{y}, S) \le \|\bar{y} - y\| + \operatorname{dist}(y, S) \le 2\varepsilon$. Hence $(\bar{y}, \bar{t}), (y, t) \in V$. Therefore,

$$\begin{aligned} \varphi_Q(\bar{t}) = \varphi(\bar{t}) \le f(\bar{y},\bar{t}) &\le f(y,t) + |f(y,t) - f(\bar{y},\bar{t})| \\ &\le f(y,t) + \rho_f(\|y - \bar{y}\| + d(t,\bar{t})) \\ &\le \varphi_Q(t) + \rho_f(\rho_M + 1)\, d(t,\bar{t}), \end{aligned}$$

so $\rho = \rho_f(\rho_M + 1)$ yields (3.4). □

Remark 3.1. Theorem 3.1 implies a known result (cf. e.g. [10, Lemma 1]), which follows immediately by putting $S = \{\bar{x}\}$ in (ii) and using the estimate (3.2) together with $\varphi(\bar{t}) = \varphi_Q(\bar{t})$: *Let $\bar{x} \in \Psi(\bar{t})$, and suppose that f is Lipschitz around $(\bar{x}, \bar{t})$. If M is calm and Lipschitz l.s.c. at $(\bar{t}, \bar{x})$, then, for some $\varepsilon > 0$, $\varphi_{B(\bar{x},\varepsilon)}$ is calm at $\bar{t}$.* □

The following counterpart to Lemma 2.2 is new. It holds in the framework of the semi-infinite model (1.1)∧(1.2) under quasiconvexity of f and g_i, $i \in I$, with respect to x.

Theorem 3.2. *Consider the standard parametric SIP (1.1)∧(1.2), let $S = \Psi(\bar{t})$. Suppose that f is Lipschitz on some open set $V \supset S \times \{\bar{t}\}$, and for each $t \in T$, the functions $f(\cdot,t)$, $g_i(\cdot,t)$, $i \in I$, are quasiconvex. If S is bounded and M is calm on $\{\bar{t}\} \times S$, then φ is calm at $\bar{t}$ from below.*

Proof. By assumption, $S = \Psi(\bar{t})$ is convex and compact. Since all assumptions of assertion (ii) in Theorem 3.1 are fulfilled, we know by this theorem that there are positive constants $\varepsilon, \delta, \rho$ such that, with $Q := S + \varepsilon B$,

$$\frac{\varphi_Q(\bar{t}) - \varphi_Q(t)}{d(t,\bar{t})} \le \rho \quad \forall t \in B(\bar{t}, \delta) \setminus \{\bar{t}\}, \tag{3.5}$$

since φ_Q is calm at $\bar{t}$ from below. Note that Q is convex and compact, and $\varphi(\bar{t}) = \varphi_Q(\bar{t})$. To prove by contradiction that φ is calm from below at $\bar{t}$, we assume

$$\text{there is some sequence } t^k \to \bar{t},\, t^k \neq \bar{t}, \text{ such that for each } k \geq 1,$$
$$\frac{\varphi(\bar{t}) - \varphi(t^k)}{d(t^k, \bar{t})} > k, \text{ and therefore, in particular, } \varphi(t^k) < \varphi(\bar{t}).$$

Hence, $\limsup_{k\to\infty} \varphi(t^k) \leq \varphi(\bar{t})$.

On the other hand, φ is l.s.c. at $\bar{t}$ according to Lemma 2.2, so $\liminf_{k\to\infty} \varphi(t^k) \geq \varphi(\bar{t})$. Thus,

$$\varphi(\bar{t}) = \lim_{k\to\infty} \varphi(t^k), \text{ and } \varphi(t^k) \text{ is finite for sufficiently large } k.$$

Therefore, we may choose some k' and points $x^k \in M(t^k)$ satisfying for $k \geq k'$,

$$\varphi(t^k) \leq f(x^k, t^k) \leq \varphi(t^k) + d(t^k, \bar{t}), \tag{3.6}$$

by definition of an infimum. Let $k' \geq \rho + 1$.

Next we prove that $x^k \notin Q$ for all $k \geq k'$. Indeed, if $x^k \in M(t^k) \cap Q$ for some $k \geq k'$, then (3.5) - (3.6) as well as $\varphi_Q(t^k) \leq f(x^k, t^k)$ and $\varphi(\bar{t}) = \varphi_Q(\bar{t})$ would imply that

$$\begin{aligned} k' \leq k < \frac{\varphi(\bar{t}) - \varphi(t^k)}{d(t^k, \bar{t})} \quad &\leq \quad \frac{\varphi(\bar{t}) - f(x^k, t^k) + d(t^k, \bar{t})}{d(t^k, \bar{t})} \\ &\leq \quad \frac{\varphi_Q(\bar{t}) - \varphi_Q(t^k)}{d(t^k, \bar{t})} + 1 \quad \leq \quad \rho + 1, \end{aligned}$$

a contradiction. So, given any $k \geq k'$, we have

$$x^k \in M(t^k) \setminus Q.$$

Now we consider the segment $\sigma^k := \text{conv}\{x^k, \bar{x}\}$, where $\bar{x} \in S = \Psi(\bar{t})$ is fixed. By definition, $\bar{x}$ belongs to the interior of the convex, compact set $Q = S + \varepsilon B$. Thus the intersection of σ^k with the boundary $\text{bd}\, Q$ of Q is a singleton $\{z^k\}$ such that $\text{dist}(z^k, S) = \varepsilon > 0$. Using the quasiconvexity of $f(\cdot, t^k)$ and (3.6), one has

$$\begin{aligned} f(z^k, t^k) \quad &\leq \quad \max\{f(x^k, t^k), f(\bar{x}, t^k)\} \\ &\leq \quad \max\{\varphi(t^k) + d(t^k, \bar{t}), f(\bar{x}, t^k)\}. \end{aligned} \tag{3.7}$$

Since $\text{bd}\, Q$ is compact, we may assume with no loss of generality that z^k converges to some point $\bar{z} \in \text{bd}\, Q$. It follows $\text{dist}(\bar{z}, S) = \varepsilon > 0$. Moreover, recall that $\lim \varphi(t^k) = \varphi(\bar{t})$ and $f(\bar{x}, \bar{t}) = \varphi(\bar{t})$. By passing to the limits in (3.7), we therefore have

$$f(\bar{z}, \bar{t}) \leq \max\{\varphi(\bar{t}), f(\bar{x}, \bar{t})\}, \text{ i.e., } f(\bar{z}, \bar{t}) \leq \varphi(\bar{t}).$$

We finish the proof by demonstrating $\bar{z} \in M(\bar{t})$, which implies $\bar{z} \in S$, contradicting $\text{dist}(\bar{z}, S) > 0$. Indeed, the functions $g_i(\cdot, t^k)$, $i \in I$, are quasiconvex, and so, by taking $x^k \in M(t^k)$ into account,

$$g_i(z^k, t^k) \leq \max\{g_i(x^k, t^k), g_i(\bar{x}, t^k)\} \leq \max\{0, g_i(\bar{x}, t^k)\} \text{ for all } i \in I.$$

So, by passing to the limits and using $\bar{x} \in M(\bar{t})$,

$$g_i(\bar{z}, \bar{t}) = \lim g_i(z^k, t^k) \leq \max\{0, g_i(\bar{x}, \bar{t})\} = 0 \text{ for all } i \in I,$$

hence $\bar{z} \in M(\bar{t})$, which completes the proof. □

Now we give two examples which illustrate the relevance of the assumptions in Theorem 3.1(ii) and Theorem 3.2 (similarly, for Lemma 2.1(ii), Lemma 2.2).

The first example demonstrates that the restriction of φ to a compact set Q in Lemma 2.1(ii) and Theorem 3.1(ii) as well as the boundedness assumption on $S = \Psi(\bar{t})$ in Lemma 2.2 and Theorem 3.2 are essential even in the case of a parametric linear program with an unperturbed feasible set.

Example 3.1. (φ is not l.s.c., unperturbed constraints, parametric LP) Consider the parametric linear program with fixed constraint set,

$$\min_x \; t \cdot x \quad \text{s.t.} \;\; x \le 0, \qquad \text{where } t \text{ varies near } 0,$$

which is a special realization of the standard SIP model (1.1)∧(1.2), and the constraint system fits into the class of continuous constraint models (3.8) below. One has

$$M(t) \equiv \mathbb{R}_-, \quad \Psi(t) = \begin{cases} \{0\} & \text{if } t < 0, \\ \mathbb{R}_- & \text{if } t = 0, \\ \emptyset & \text{if } t > 0. \end{cases} \qquad \varphi(t) = \begin{cases} 0 & \text{if } t \le 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

i.e., M is a constant multifunction (hence in particular calm and Lipschitz l.s.c. on $\{0\} \times \Psi(0)$), but $\Psi(0)$ is unbounded. φ is at $\bar{t} = 0$ u.s.c., but not l.s.c., and calm from above, but not from below.

However, if we consider a nonempty compact set $S \subset \Psi(0)$ (in accordance with (ii) in Theorem 3.1), say with $\min S = -\alpha$, $\alpha \ge 0$, then, after putting $Q := S + \varepsilon B$ ($\varepsilon > 0$), one has $\varphi_Q(t) = -(\alpha + \varepsilon)t$ if $t \ge 0$, and $\varphi(t) = 0$ else, so φ_Q is at $\bar{t} = 0$ calm from below and above. □

By the second example, which is taken from [4], we mainly intend to demonstrate that the (quasi-)convexity assumption in Lemma 2.2 and Theorem 3.2 cannot be avoided, even in the case of a parametric quadratic program with fixed constraint set.

Example 3.2. [4, Example 4.2.1] (φ is not l.s.c., parametric QP, non-convex objective function, fixed linear constraints) Consider the parametric program with (non-convex) quadratic objective function and fixed constraint polyhedron $M_\circ := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1 \le 1, \; x_2 \ge 0\}$,

$$\min_{x=(x_1,x_2)} \; f(x,t) := -x_1 + x_1 x_2 + 2t^2 x_2^2 - 4t(1-t)x_2 \quad \text{s.t.} \;\; x = (x_1, x_2) \in M_\circ,$$

where $t \in \mathbb{R}$ is assumed to vary in the open interval $(-1, 1)$; it is a simple special case of the parametric model (1.1)∧(1.2). Obviously, $\Psi(t) = \{(1,0)\}$ and $\varphi(t) = -1$ if $t \le 0$. Now let $0 < t < 1$. When defining $x(t) := (0, \frac{1-t}{t})$ ($\in M_\circ$), we obtain

$$\varphi(t) \le f(x(t), t) = -2(1-t)^2, \;\; \text{from which } \liminf_{t \to +0} \varphi(t) \le -2 < \varphi(0),$$

i.e., φ is at $\bar{t} = 0$ not l.s.c., let alone calm from below. Of course, φ is at $\bar{t} = 0$ calm from above and u.s.c., in accordance with Theorem 3.1 (i).

It is worth to observe that the optimal set mapping Ψ has interesting properties. As noticed above, $\Psi(t)$ is a singleton for $t \le 0$. Given $0 < t < 1$, it is easy to verify that $x_2(t) = \frac{1-t}{t}$ is the unique solution of the (convex) quadratic program

$$\min_{x_2} \; t^2 x_2^2 - 2t(1-t)x_2 \quad \text{s.t.} \quad x_2 \ge 0,$$

with minimum value $v(t) = -(1-t)^2$. Therefore, since $x \in M_\circ$ implies $-x_1 + x_1x_2 \geq -1$, it follows $f(x,t) \geq -1 - 2(1-t)^2 > -3$, for all $x \in M_\circ$ whenever $t \in (0,1)$. Hence, the sets $\Psi(t)$ are also nonempty for all $t \in (0,1)$, by the Frank-Wolfe existence theorem of quadratic optimization; it is not difficult to see that these sets are compact. However, there is no compact set Q such that $\Psi(t) \subset Q$ for all t near 0. Indeed, assuming there is such a Q, we take any sequences $t_k \downarrow 0$ and $x^k \in \Psi(t_k) \subset M_\circ \cap Q$. Then, without loss of generality, there is some $\hat{x} \in M_\circ \cap Q$ such that $x^k \to \hat{x}$, by compactness of $M_\circ \cap Q$. We have seen above that $f(x^k, t_k) = \varphi(t_k) \leq -2(1-t_k)^2$. Then, by passing to the limit, we obtain $f(\hat{x}, 0) \leq -2$ which contradicts $\varphi(0) = -1$. □

In the remainder of this section, let us discuss the above assumptions imposed on the feasible set mapping M. There is a broad literature on Lipschitz stability analysis of M for various types of constraint systems, where in the abstract basis model (1.1) the sets $M(t)$ themselves could be also solution sets of optimization problems or variational inequalities. For this abstract setting, we refer exemplarily to the standard books [1, 5, 9, 23, 24] and the references therein.

In the context of (finite or semi-infinite) inequality constraints, Lipschitz properties of M are usually considered in the *continuous setting* of model (1.2), which means that I is a compact metric space, and $(i,x,t) \mapsto g_i(x,t)$ is continuous. For a discussion of the literature in this case, we restrict ourselves to the *continuous model under right-hand-side (RHS) perturbations*,

$$\begin{gathered} \widetilde{M}(b) := \{x \in \mathbb{R}^n \mid g_i(x) \leq b_i,\, i \in I\},\, b \in T = C(I) \ \text{ varies near } 0, \\ I \text{ is a compact metric space, } (i,x) \in I \times \mathbb{R}^n \mapsto g_i(x) \in \mathbb{R} \text{ is continuous,} \end{gathered} \tag{3.8}$$

i.e., a finite set I is not excluded. If the functions g_i, $i \in I$, are convex, we will speak of the *convex setting* of (3.8). If all g_i are C^1 functions and $(i,x) \mapsto Dg_i(x)$ is continuous, this will be called the C^1 *setting* of (3.8). It is known that in these settings, under suitable assumptions on the perturbations, properties like metric regularity or calmness carry over from $\widetilde{M}$ to the mapping M in the continuous setting of model (1.2); see, e.g., [5, 25, 26, 27, 28] and the references therein.

The inequality system in (3.8) can be rewritten as a cone constraint $g(x) - b \in C_-$, where $g(x)(i) := g_i(x)$, $i \in I$, and $C_- := \{\phi \in C(I) \mid \phi(i) \leq 0\ \forall i \in I\}$, or as a single inequality $h(x,b) \leq 0$ via the max-function $h(x,b) := \max_{i \in I}(g_i(x) - b_i)$. In the convex setting of (3.8), g is convex w.r. to the cone C_-, cf. [5, Prop. 2.174], and, obviously, h is convex, too. Further, in the C^1 setting of (3.8), g is also a C^1 function (cf. again [5, Prop. 2.174]), and h is locally Lipschitz (cf. [6]). So it is standard in the literature to use these reformulations for characterizing calmness and other properties of $\widetilde{M}$.

Let us first discuss calmness of $\widetilde{M}$ (3.8). A comprehensive study in the case of C^1 data can be found in [27], where sufficient conditions for detecting calmness are given, which are in the case of standard nonlinear programs stronger than Abadie's constraint qualification, but weaker than the Mangasarian-Fromovitz constraint qualification (MFCQ). Similar characterizations are proved in [29, 30], where also solution procedures are presented which converge (locally and of linear order) exactly if the mapping $\widetilde{M}$ is calm. For a general approach to calmness of mappings defined by cone constraints we refer to [31].

Calmness of $\widetilde{M}$ is closely related to the property that *h admits a local (linear) error bound* at $\bar{x}$, which means that there are positive numbers ρ and δ such that $\rho\,\mathrm{dist}(x, [h \leq 0]) \leq \max\{h(x,0), 0\}$ $\forall x \in B(\bar{x}, \delta)$, where $[h \leq 0] := \{\xi \mid h(\xi, 0) \leq 0\}$. So, the well-developed theory of error bounds applies to (3.8), which has been used for convex and linear constraint functions e.g. in [18, 32, 33]. For these and more general situations, we refer the interested reader also to surveys on error bounds in [34, 35] and to the references therein.

Characterizations of the Aubin property for the mapping $\widetilde{M}$ in terms of the problem data are known for long time. In the convex setting of (3.8), $\widetilde{M}$ has the Aubin property at $(0,\bar{x}) \in \operatorname{gph}\widetilde{M}$ if and only if the *Slater constraint qualification (SCQ)* holds for $\widetilde{M}(0)$, i.e., there is some $\tilde{x}$ such that $g_i(\tilde{x}) < 0$ for all $i \in I$, cf. e.g. [36]. In the C^1 setting of (3.8), the *extended Mangasarian-Fromovitz constraint qualification (EMFCQ)* [37, 38] is said to hold at $\bar{x} \in \widetilde{M}(0)$ if

$$\exists d \in \mathbb{R}^n : \ Dg_i(\bar{x})\, d < 0 \text{ for all } i \in I_0(\bar{x}) := \{ j \in I \,|\, g_j(\bar{x}) = 0\}, \tag{3.9}$$

which reduces to the usual MFCQ if I is finite. In the cone constraint formulation, we recall that *Robinson's constraint qualification (RCQ)* [28] is said to be satisfied at $\bar{x} \in \widetilde{M}(0)$ if

$$0 \in \operatorname{int}\{g(\bar{x}) + Dg(\bar{x})\mathbb{R}^n - C_-\}.$$

It is well-known that the following holds true (cf. [5, 37, 39]): Given $(0,\bar{x}) \in \operatorname{gph}\widetilde{M}$,

$$\widetilde{M} \text{ has the Aubin property at } (0,\bar{x}) \Leftrightarrow \text{EMFCQ holds at } \bar{x} \Leftrightarrow \text{RCQ is satisfied at } \bar{x}. \tag{3.10}$$

We finish this section by discussing the close relations and differences between the Aubin property and the Lipschitz lower semicontinuity for solution sets of inequality systems. For the standard model (1.1)$\wedge$(1.2), the Aubin property of M in general does not follow when M is Lipschitz l.s.c., we refer to the simple example (cf. [40])

$$M(t) = \{x \in \mathbb{R} \mid tx \le 0\}, \text{ where } t \in \mathbb{R} \text{ varies near } 0.$$

Obviously, M is Lipschitz l.s.c. and calm at $(0,0)$, but the Aubin property is violated at this point. In the case that only right-hand side perturbations of the constraints are allowed, there is a more involved example of that type [10]: Defining $M(t) := \{x \in \mathbb{R}^2 \mid x_2(x_2 - x_1^2) \ge 0, x_2 = t\}$, M is calm and Lipschitz l.s.c. at the origin, but has not the Aubin property there, see [10, Example 3.2] for details.

Let us go back to the multifunction $\widetilde{M}$ in the continuous model (3.8) with right-hand-side perturbations of the constraints. As shown in the next remark, Lipschitz lower semicontinuity and Aubin property coincide in special settings, but in general - even in the case of piecewise linear, quasiconvex functions - both properties may differ.

Remark 3.2. Consider the continuous model of a parametric semi-infinite inequality system (3.8). We discuss three different settings, let $(0,\bar{x}) \in \operatorname{gph}\widetilde{M}$. Note that in the cases 1. and 2., the perturbation of *all* parameters in the system (3.8) is crucial to get the equivalence.

1. *Convex setting.* If $\widetilde{M}$ is l.s.c. at $(0,\bar{x})$ then, in particular, $\widetilde{M}(b) \neq \emptyset$ for $b_i \equiv -\theta$ $(\forall i)$ and small $\theta > 0$, by definition of lower semicontinuity. This immediately implies SCQ, thus, by taking the above discussion into account, the statements (i) - (v) are equivalent: (i) $\widetilde{M}$ is Lipschitz l.s.c. at $(0,\bar{x})$, (ii) $\widetilde{M}$ is l.s.c. at $(0,\bar{x})$, (iii) SCQ holds for $\widetilde{M}(0)$, (iv) $\widetilde{M}$ has the Aubin property at $(0,\bar{x})$, (v) $\widetilde{M}$ is Lipschitz l.s.c. and calm at $(0,\bar{x})$. See e.g. [36, Lemma 3] for these and more equivalent properties.

2. C^1 *setting.* Following the methods of proof in [29, Lemma 1] or in [37, Thm. 2], which were given in other contexts, one gets

$$\text{(i) } \widetilde{M} \text{ is Lipschitz l.s.c. at } (0,\bar{x}) \in \operatorname{gph}\widetilde{M} \ \Rightarrow \ \text{(ii)' EMFCQ holds at } \bar{x} \in \widetilde{M}(0).$$

Indeed, take any sequence $\theta_k \downarrow 0$ and define $b^k \in C(I)$ by $b_i^k := -\theta_k$ for all $i \in I$. Then (i) implies that there are some $\rho > 0$ and points $x^k \in \widetilde{M}(b^k)$ such that $\|x^k - \bar{x}\| \leq \rho\, \theta_k$ for large k. If $g_i(\bar{x}) = 0$ then $-\rho^{-1}\|x^k - \bar{x}\| \geq -\theta_k \geq g_i(x^k) = Dg_i(\bar{x})(x^k - \bar{x}) + o(\|x^k - \bar{x}\|)$, if k is large enough. Division by $\|x^k - \bar{x}\|$ and passage to a cluster point d of $\|x^k - \bar{x}\|^{-1}(x^k - \bar{x})$ yields (3.9) (i.e. (ii)') which finishes the proof. So, by taking (3.10) into account, EMFCQ, RCQ, Aubin property and Lipschitz lower semicontinuity of $\widetilde{M}$ at the point under consideration are equivalent.

3. *Suppose in (3.8): g_i, $i \in I$, are quasiconvex and (locally) Lipschitz functions.* In general, the Aubin property and the Lipschitz lower semicontinuity of $\widetilde{M}$ differ in this framework. Consider the simple example of a single inequality defined by a concave, piecewise linear real function: $\widetilde{M}$ is defined by

$$\widetilde{M}(b) := \{x \in \mathbb{R} \mid \min\{x, 0\} \leq b\},\ b \textit{ near } 0.$$

Obviously, $\widetilde{M}(b) = \mathbb{R}$ if $b \geq 0$, and $\widetilde{M}(b) = (-\infty, b]$ if $b < 0$, i.e., $\widetilde{M}$ is calm and Lipschitz l.s.c. at $(0,0)$, but the Aubin property fails at this point. □

4. Concluding Remarks

Our paper has presented sufficient conditions for calmness from above and calmness from below for the optimal value function φ of parametric optimization problems with locally Lipschitz objective function, including semi-infinite programs and standard (non-)linear programs under perturbations. This has been related to classical results on upper and lower semicontinuity of φ. While calmness from above is a direct consequence of the Lipschitz lower semicontinuity of the feasible set mapping M, calmness from below is implied by the calmness of M, together with compactness and convexity requirements. It has been illustrated by examples that the assumptions imposed in the statements are essential even in the case of standard linear and quadratic programs. The question of computing calmness moduli for the optimal value function of the models under consideration has been outside the scope of this paper, however, we think that the knowledge of calmness conditions for φ can help to extend the known results (as e.g. in [11]) to more general settings.

References

[1] R.T. Rockafellar, R.J.-B. Wets, Variational Analysis, Springer, Berlin, Heidelberg, New York, 1998.
[2] F.H. Clarke, A new approach to Lagrange multipliers, Math. Oper. Res. 1 (1976), 165–174.
[3] J.P. Penot, Calmness and stability properties of marginal and performance functions, Numer. Funct. Anal. Optim. 25 (2004), 287–308.
[4] B. Bank, J. Guddat, D. Klatte, B. Kummer, K. Tammer, Non-Linear Parametric Optimization, Akademie-Verlag, Berlin, 1982.
[5] J.F. Bonnans, A. Shapiro, Perturbation Analysis of Optimization Problems, Springer, New York, 2000.
[6] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.
[7] A.V. Fiacco, Introduction to Sensitivity and Stability Analysis, Academic Press, New York, 1983.
[8] E.G. Golstein, Theory of Convex Programming, Trans. Math. Monographs 36, American Mathematical Society, Providence, 1972.
[9] D. Klatte, B. Kummer, Nonsmooth Equations in Optimization, Kluwer, Dordrecht, 2002.
[10] D. Klatte, B. Kummer, On calmness of the argmin mapping in parametric optimization problems, J. Optim. Theory Appl. 165 (2015), 708–719.
[11] M.J. Gisbert, M.J. Cánovas, J. Parra, F.J. Toledo, Calmness of the optimal value in linear programming, SIAM J. Optim. 28 (2018), 2201–2221.

[12] J. Renegar, Some perturbation theory for linear programming, Math. Program. 65A (1994), 73–91.
[13] J. Renegar, Linear programming, complexity theory and elementary functional analysis, Math. Program. 70 (1995), 279–769.
[14] S.M. Robinson, Some continuity properties of polyhedral multifunctions, Math. Progr. Study 14 (1981), 206-214.
[15] W. Alt, Lipschitzian perturbations in infinite dimensional optimization problems, In: A.V. Fiacco (ed.), Mathematical Programming with Data Perturbations, pp. 7–21, Marcel Dekker, New York, 1983.
[16] D. Klatte, On the stability of local and global optimal solutions in parametric problems of nonlinear optimization. Part I: Basic results, Seminarbericht Nr. 75 der Sektion Mathematik, Humboldt-Universität Berlin, pages 1–21, 1985.
[17] D. Klatte, B. Kummer, On Hölder calmness of minimizing sets, Optimization 71 (2022), 1055–1072.
[18] A.Y. Kruger, M.A. López, X.Q. Yang, J.X. Zhu, Hölder error bounds and Hölder calmness with applications to convex semi-infinite optimization, Set-Valued Var. Anal. 27 (2019), 995–1023.
[19] C. Berge, Topological Spaces, Macmillan, New York, 1963.
[20] W.W. Hogan, Point-to-set maps in mathematical programming, SIAM Rev. 15 (1973), 591–603.
[21] D. Klatte, Lower semicontinuity of the minimum in parametric convex programs, J. Optim. Theory Appl. 94 (1997), 511–517.
[22] J.P. Penot, Calculus without Derivatives, Springer, New York, 2013.
[23] A.D. Ioffe, Variational Analysis of Regular Mappings: Theory and Applications, Springer International Publisher, Cham, 2017.
[24] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation, I, II, Springer, Berlin, Heidelberg, 2006.
[25] R. Cominetti, Metric regularity, tangent sets, and second-order optimality conditions, Appl. Math. Optim. 21 (1990), 265–287.
[26] A.L. Dontchev, R.T. Rockafellar, Implicit Functions and Solution Mappings: A View from Variational Analysis, Springer, Dortrecht, Heidelberg, London, New York, 2014.
[27] R. Henrion, J. Outrata, Calmness of constraint systems with applications, Math. Program. B 104 (2005), 437–464.
[28] S.M. Robinson, Stability theorems for systems of inequalities. Part II: Differentiable nonlinear systems, SIAM J. Numer. Anal. 13 (1976), 497–513.
[29] D. Klatte, B. Kummer, Optimization methods and stability of inclusions in Banach spaces, Math. Program. B 117 (2009), 305–330.
[30] B. Kummer, Inclusions in general spaces: Hoelder stability, solution schemes and Ekeland's principle, J. Math. Anal. Appl. 358 (2009), 327–344.
[31] H. Gfrerer, First order and second order characterizations of metric subregularity and calmness of constraint set mappings, SIAM J. Optim. 21 (2011), 1439–1474.
[32] M.A. Goberna and M.A. López, Post-Optimal Analysis in Linear Semi-Infinite Optimization, Springer, New York, Heidelberg, Dordrecht, London, 2014.
[33] A.Y. Kruger, H.V. Ngai, M. Théra, Stability of error bounds for convex constraint systems in Banach spaces, SIAM J. Optim. 20 (2010), 3280-3296.
[34] D. Azé, A survey on error bounds for lower semicontinuous functions, In: Proceedings of 2003 MODE-SMAI Conference, ESAIM Proc., vol. 13, pp. 1–17 EDP Sci., Les Ulis, 2003.
[35] M.J. Fabian, R. Henrion, A.Y. Kruger, J.V. Outrata, Error bounds: Necessary and sufficient conditions, Set-Val. Anal. 18 (2010), 121–149.
[36] M.J. Cánovas, D. Klatte, M.A. López, J. Parra, Metric regularity in convex semi-infinite optimization under canonical perturbations, SIAM J. Optim. 18 (2007), 717–732.
[37] R. Henrion, D. Klatte, Metric regularity of the feasible set mapping in semi-infinite optimization, Appl. Math. Optim. 30 (1994), 103–109.
[38] H.T. Jongen, F. Twilt, G.W. Weber, Semi-infinite optimization: structure and stability of the feasible set, J. Optim. Theory Appl. 72 (1992), 529–552.
[39] A. Shapiro, On Lipschitz stability of optimal solutions of parametrized semi-infinite programs, Math. Oper. Res. 19 (1994), 743–753.

[40] M.J. Cánovas, M.J. Gisbert, R. Henrion, J. Parra, Lipschitz lower semicontinuity moduli for linear inequality systems, J. Math. Anal. Appl. 490 (2020), 124313.