How Do We Think About the Future?
Three Essays in Computational Economics

Dissertation
submitted to the
Faculty of Business, Economics and Informatics
of the University of Zurich

to obtain the degree of
Doktorin der Wirtschaftswissenschaften, Dr. oec.
(corresponds to Doctor of Philosophy, PhD)

presented by

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approved in February 2024 at the request of
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The Faculty of Business, Economics and Informatics of the University of Zurich hereby authorizes the printing of this dissertation, without indicating an opinion of the views expressed in the work.

Zurich, 14.02.2024
The Chairman of the Doctoral Board: Prof. Dr. Steven Ongena

Abstract

This thesis explores the critical role of time discounting in economic decision-making and examines different estimation methods and their implications. Essay One provides a comprehensive review of existing methodologies for estimating discounting behavior in economic models. By comparing the advantages and disadvantages of experimental and structural approaches, the essay suggests that structural models offer a more accurate estimation of discounting and advocates for allowing discounting to evolve over time. Essay Two introduces a novel approach to modeling and estimating discounting within dynamic structural models. This approach treats discounting as a stochastic process, offering a more realistic representation of agents' time preferences. The proposed estimation method effectively handles unobserved discounting states and allows for the estimation of a per-period discounting state. The empirical application reveals an intriguing relationship between discounting states and real interest rates, challenging the conventional assumption of constant discount factors. Essay Three builds upon the findings from Essay Two and investigates the joint movement of interest rates and discounting states in dynamic structural models. Two discounting models, the level model and the error model, are introduced to incorporate this relationship. Our empirical analysis suggests that the level model outperforms the error model, highlighting the potential of directly incorporating interest rate levels into the discounting process of structural models. In summary, this thesis contributes to our understanding of discounting behavior in intertemporal decision-making, advocates for dynamic structural models to estimate discounting, and demonstrates the impact of real interest rates on discounting. Ultimately, it offers valuable insights into the intricate relationship between discounting and economic factors, enhancing our understanding of this phenomenon.

Acknowledgements

I am deeply indebted and very grateful to my supervisors Prof. Dr. Karl Schmedders and Prof. Dr. José Parra Moyano, as well as, Dr. Gregor Reich, and Dr. Kenneth Judd for their constant support, teaching, and guidance. I also gratefully acknowledge the financial support of the Swiss National Science Foundation. I thank my family and close friends for their encouragement and moral support. Finally, I thank my colleagues at the Chair of Quantitative Business Administration and, in particular, Phillip Müller and Robert Erbe for countless discussions and valuable inputs.

Contents

Part I

Introduction

Introduction (“Rahmenpapier”)

In our everyday lives, we frequently have to make decisions that not only influence our current situation but also have an impact on our future. These intertemporal decisions range from the seemingly trivial, such as watching one more episode of a favorite TV show instead of prioritizing a good night’s sleep, to more life-altering choices, like leaving our current job to pursue further education to improve our job prospects. In economic models, when decision-making extends beyond immediate utility and involves considering the potential consequences and implications of future outcomes a fundamental assumption is that future utilities do not hold the same value as the same nominal utilities in the present. Consequently, these future utilities need to be adjusted and weighted in terms of their present value. This weighting is known as time discounting and plays a crucial role in all economic models involving decisions over time. This thesis comprises three essays, each delving into different aspects of discounting in economic models, including the current most popular approaches to its estimation, a novel suggestion regarding its treatment and estimation within structural models, and the use of economic data to further explain individuals’ discounting behavior in structural models.

Essay One begins with an exploration of how discounting is commonly estimated in behavioral economics. At the core of economic models addressing individuals’ decision-making over time is the discounted utility (DU) model, initially introduced by Samuelson (1937). In this model, agents are treated as utility maximizers who discount future rewards using an exponential discount function with a constant discount rate. The DU model’s appeal lies in its mathematical tractability and its simple yet convincing representation of people’s complex decision-making processes. Empirical evidence has, however, shown disparities between the behavior predicted by the DU model and the actual intertemporal decision-making behavior observed in reality. These disparities include, but are not limited to, time-inconsistencies and variations in the discount factor for different types of rewards. Experimental approaches are usually employed to estimate discounting and examine these disparities. However, these approaches have inherent limitations due to design flaws and the hypothetical nature of their setups. In order to address these limitations, a structural estimation approach that leverages real-life data to model individuals’ intertemporal decision problems more accurately can be adopted. Structural models have the potential to estimate the discount factor more effectively; often, however, these models treat it as a nuisance parameter rather than a variable of primary interest due to an identification problem. This problem lies in the fact that, without strong restrictions on the model’s primitives—such as the agent’s utility function, the law of motion, and the discount function—the observed decision rule can be explained by an infinite number of combinations of these primitives. Since the focus of structural models is typically on identifying and estimating utility parameters, the discount factor is usually fixed to a constant value. Recently there has been renewed interest in the field of structural modeling with the aim of addressing the identification and estimation challenges associated with the discount factor. The prevailing view in such papers is that structural models should, whenever feasible, estimate the discount factor and explore discount functions beyond the traditional exponential one. Despite this renewed interest, research into estimating discounting with a structural model remains influenced by findings from experimental studies. Many models estimate discounting as a constant factor or, alternatively, allow for variation over time by adopting a hyperbolic function. Overall, both experimental and structural approaches typically estimate discounting for a specific decision problem at a single time point, overlooking the opportunity to explore time preferences as an evolving state. Essay One ends on the notion that structural models be used to estimate discounting, as they are a powerful tool with which to examine discounting behavior in real-world settings. However, rather than restricting discounting to a constant factor it is recommended to leverage the fact that the data in such models contains recurring intertemporal decisions and to allow discount factors to evolve over time.

In summary, Essay One serves as a comprehensive literature review, surveying the methodologies employed in behavioral economics to examine and estimate individuals' discounting behavior. It sheds light on the theoretical foundations of modeling intertemporal decisions and explores the disparities that empirical research has revealed exist between predicted and observed discounting behavior. By examining the strengths and limitations of commonly employed approaches to estimating individuals' discounting behavior, this literature review lays the groundwork for the subsequent essays.

Building upon the insights from Essay One, Essay Two presents a novel approach to modeling and estimating discounting in dynamic structural models. Conventionally, in dynamic structural models discounting has been assumed to be a constant factor, which simplifies the estimation of structural parameters. However, this assumption raises questions from an economic standpoint as it implies that agents' time preferences remain constant over time and are uniform across all decision-makers. To address this limitation, Essay Two proposes a more flexible and realistic approach by modeling discounting as a stochastic process. This approach has several advantages as it allows discounting to evolve as a state variable described by its distributional parameters. The trade-off, however, is that discounting, being a subjective preference parameter known only to the agent, becomes an unobserved state variable to the modeler. This introduces a potentially large integral over this unobserved state variable into the likelihood function, the size of which is proportional to the time horizon of the data. To overcome this issue, the essay uses recursive likelihood integration, as introduced by Reich (2018), to split this large integral up into a series of low-dimensional integrals and interpolation problems. The structural parameters of the model, including those defining the stochastic discounting process, can then be estimated using either the nested fix point algorithm of Rust (1987) or mathematical programming with equilibrium constraints following Su and Judd (2012). After estimating the distributional parameters of the process, we can use maximum likelihood to estimate the discounting state with the highest probability of occurrence at each time point in the data set. These estimated states are referred to as the *most likely discounting process*. To illustrate the practical implementation of the proposed method, in the empirical part of the essay we model discounting as a process in the seminal bus engine replacement model of Rust (1987). By assuming an autoregressive process of order 1, AR(1), for discounting, the discounting behavior of the agent in the model, Zurcher, can be fully described using three parameters: the stationary mean, the stationary variance, and the persistence of the process. The estimated parameters reveal that Zurcher's discounting behavior has a stationary mean above one, low variance, and high persistence. Using these values to derive the most likely discounting process discloses a noteworthy pattern, indicating that Zurcher's discounting states per period are consistently above one and exhibit a decreasing trend from December 1974 to May 1985. These findings challenge the commonly held belief that discount factors should fall within the range of $(0, 1)$ and, consequently, that discount rates should be positive. From a behavioral perspective, however, a plausible interpretation emerges: Under the assumption of an exponential discount function, and considering that Zurcher's utility is predominantly influenced by monetary values, it is reasonable to infer that the discount rate should follow a positive linear relationship with interest rates. Thus, discounting states greater than one imply negative interest rates. Indeed, real interest rates at the time were negative for an extended period, and their values exhibit a significant and strong correlation with the estimated most likely discounting process. This observation underpins the link between individuals' intertemporal decision-making and prevailing economic conditions, providing valuable insights into the joint movement of discounting and real interest rates.

In conclusion, Essay Two presents a novel approach to modeling and estimating discounting in dynamic structural models. The incorporation of discounting as a stochastic process allows for a more realistic representation of agents' time preferences over time. We develop a theoretical framework that models discounting as an evolving state described by its distributional parameters. Further, the estimation method can effectively handle the unobserved nature of the discounting process and provides reliable estimates for the discounting state in each period. The empirical application to the bus engine replacement model demonstrates that the agent's discounting behavior is closely related to prevailing real interest rate levels, offering new insights into the interpretation of discounting in economic models. In the subsequent essay, the implications of the findings in Essay Two are further explored.

Essay Three delves into an investigation of whether incorporating the joint movement between interest rates and discounting states can enhance the fit of a dynamic structural model. Motivated by the intriguing findings from Essay Two, this essay proposes two discounting models that allow for the inclusion of interest rate movements in the evolution of the discounting states. Using the bus engine replacement model of Rust (1987) as a basis, two discounting models are introduced: the level model and the error model. In the level model, Zurcher's beliefs about the next period's discounting state are influenced by his current discounting state and the observed values of the interest rate. The evolution of the real interest rate is assumed to be independent of Zurcher's discounting states. While conceptually straightforward, this model introduces two additional state variables—the discounting state and real interest rate levels—to the model, and requires the modeler to estimate four additional parameters compared to a model with AR(1) discounting. In the error model, the discounting state is a function of its own past value and the covariance between the innovations in the interest rate and those in the discounting states. In this model, Zurcher is assumed to incorporate shocks to real interest rates rather than the level of the interest rate into his discounting behavior. Again, the evolution of the real interest rate is assumed to be independent of Zurcher's discounting process. In comparison to an AR(1) discounting process, the error model requires the estimation of four additional parameters. The empirical part of Essay Three extensively examines the relationship between Zurcher's per-period discounting states and the levels and innovations of various real interest rate candidates. These per-period discounting states are derived from the most likely discounting process, which was estimated in Essay Two for an AR(1) discounting process. Through a reduced form analysis of both the level and the error model, our empirical results reveal that the level model demonstrates a superior fit to the data. Depending on the selected real interest rate values, the level model's reduced form reveals that the variation in real interest rate levels can account for almost 70 percent of the variation in the estimated discounting states. The results of the error model analysis, meanwhile, do not offer conclusive evidence that this model specification would better explain Zurcher's discounting behavior than would the AR(1) process. Based on these findings, we propose further exploration of Zurcher's discounting behavior using a level model and a real interest rate derived from Moody's Seasoned Baa Corporate Bond Yield and the Personal Consumption Expenditures Excluding Food and Energy index. Although the empirical implementation of the level model in the bus engine replacement model remains an avenue for future research, the results presented in this essay highlight the potential of incorporating the co-movement of interest rates and discounting into the discounting process of dynamic structural models.

In sum, Essay Three presents a methodological approach to modeling the discounting behavior of an agent in a seminal dynamic discrete choice model by exploring the potential co-movement of unobserved discounting states and observed real interest rates. The essay introduces two specifications of discounting models that enable the incorporation of this co-movement into the discounting process. Through empirical analysis of the reduced forms of these specifications, we gain a better understanding of their effectiveness in improving the discounting process of the bus engine replacement model. Overall, the essay offers valuable insights into the relationship between discounting and economic factors, particularly the real interest rate, and demonstrates how leveraging this relationship can enhance the modeling of intertemporal decisions with dynamic structural models.

Part II

Three Essays in Computational Economics

Essay One
Estimating Discounting

Estimating Discounting

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October 2023

Abstract

In economic models, it is a ubiquitous assumption that future payoffs do not hold the same value for individuals as the equivalent nominal payoff in the present due to time preferences. The concept of incorporating these time preferences by discounting future utilities gained popularity with the work of Samuelson (1937). In his seminal note, the author proposes the discounted utility (DU) model as a general framework for addressing intertemporal decision-making. The DU model assumes exponential discounting with a constant discount rate, aligning with the principles of rational choice theory but often falling short of capturing real-world behavior. Experimental studies have consistently revealed that the discounting behavior predicted by the DU model does not align with the discounting behavior observed in experiments. However, these experiments are inherently flawed in design and limited in their ability to replicate intertemporal decision problems realistically. An alternative approach, structural estimation, seeks to estimate deep economic structures such as the discount factor by modeling real-world decision problems. Despite offering a solution to the limitations of experimental estimation, structural estimation encounters challenges in identifying the discount factor. This paper examines discounting within the context of economic modeling and delves into the field of estimating the discount factor in structural models.

1.1 Introduction

On a daily basis individuals are faced with decisions that require an evaluation of expected utilities at different points in time. Should you spend CHF 5 and get a sandwich now or save the money and eat the leftover pasta in your fridge? Is it cheaper to buy that concert ticket in advance or to wait until the day of the event in the hope of buying a discounted ticket from someone unable to attend? Do you study for an additional degree that qualifies you to take on that higher-paid position or do you continue in your current role since you might get promoted at some point anyway? Examples of so-called *intertemporal decision* problems are manifold, encompassing various aspects and areas of one's life (Mas-Colell et al., 1995). A common characteristic of all these decisions is the need to assign a weight to the utility in the distant future relative to the utility of immediate or near-future outcomes to account for *time preferences* (Frederick et al., 2002). Positive time preferences refer to the observed tendency of individuals to value near-future outcomes more highly than equivalent distant-future outcomes of the same nominal amount. Economic models typically incorporate this weighting, or *discounting*, through the use of a discount factor (Frederick et al., 2002). For example, to decide whether an investment is profitable a company will transform the expected payoffs to present value terms by discounting the cash flows in the future with the weighted average cost of capital (WACC) and comparing the sum of the discounted payoffs with the investment costs. The WACC is the average rate at which the company can raise capital. If the sum of the expected payoffs is larger than the costs, the company should proceed with the investment, otherwise not (Volkart, 2006). This financial concept of discounting future payoffs by using an appropriate interest rate and subsequently summing up the discounted cash flows to get the final value of a cash flow stream gained popularity in the early twentieth century, particularly through the work of Fisher (1930). Later, the approach was adopted for people's intertemporal decision-making processes in the work of Samuelson (1937): In the discounted utility (DU) model, a decision-maker or agent chooses between different streams of utilities by discounting each utility in the stream to its present value using a discount factor. The decision-maker then adds up all the discounted utilities of each stream, and the utility stream that yields the highest sum of discounted utilities is selected as the preferred choice. This approach soon established itself as the keystone of modeling individuals' decision-making over time, not only because of its simplicity but also because of its mathematical tractability. One central assumption of the DU model is that discounting is based on an exponential function with a constant discount rate. Samuelson (1937) suggests that this discount rate should be equal to the interest rate at the time of the decision. Assuming an exponential discount function with constant time preferences leads to stationary and time-consistent behavior, which aligns with the principles of rational choice theory. However, the DU model lacks descriptive legitimacy. Experimental studies examining time preferences of participants have revealed behavior that is not in line with a constant discount rate (Frederick et al., 2002). In fact, the discount rate is shown to be influenced by many factors, such as the size of the payoffs, whether the outcome is a cost or a gain, and other characteristics of the payoff itself or of the decision-maker. It is, however, debatable whether such experimental studies are even suitable for estimating the discount rate. It can be argued that the experiments have inherent design flaws that may introduce biases into the resulting discount rates. For example, the wording of survey questions has been shown to result in biased estimates. Besides the design flaws potentially inherent in experiments, they have limitations in terms of the time horizon they can examine and the size of the payoffs involved. Thus, an alternative approach to studying the discounting behavior of decision-makers involves estimating and examining the discount rate based on observed decisions made in the real world. This is exactly the approach adopted in structural estimation (Mas-Colell et al., 1995). Dynamic structural estimation aims to retrieve the “deep” structure of economic models—which includes, among other elements, the discount factor—by modeling, for example, observed behavior as a result of a dynamic decision problem (Magnac and Thesmar, 2002, p. 801). However, despite the theoretical capacity of these models to estimate discounting behavior they are rarely used for this purpose. The reason for this is that the discount factor is difficult to estimate due to an identification problem. Without strong restrictions on the model primitives, the discount factor cannot be identified in dynamic structural models. Only in recent years, encouraged by the findings of experimental studies, has a growing body of literature arguing for identifying and estimating the discount factor in structural models emerged.

In the following section, Section 1.2, the historical background of discounting as well as the general framework of the DU model will be discussed briefly before elaborating on the estimation of the discount factor. Section 1.3 delves into the field of experimentally estimating the discount factor. It examines the disparities between the discounting behavior observed in these experiments and the behavior expected from theoretical models, offering insights into the challenges associated with using an experimental approach to estimate discounting. Section 1.4 discusses the alternative approach in economics, which involves estimating discounting with structural models, and outlines the latest contributions in this field. Finally, the concluding section of this paper, Section 1.5, summarizes the wide range of papers discussed and provides an outlook on possible avenues for future research.

1.2 The Origins of Discounting

Neoclassical economists such as John Rae and Adam Smith recognized early that intertemporal decisions play an important role not only in the personal wealth of an individual but also in the wealth of a nation (Rae, 1905). Intertemporal choices are made when people have to decide between outcomes at different points in time, an example being the decision as to how much one should spend of one's salary now to enjoy an immediate positive utility from it and how much one should save for later consumption. This decision influences how much personal wealth people accrue over time and, on a broader scale, how much wealth their nation accrues (Rae, 1905; Frederick et al., 2002). Saving for the future is, though, only one example among many. Intertemporal decision problems are ubiquitous in situations where a decision incorporates a temporal dimension. It comes as no surprise that the study of how individuals make choices over time has emerged as a central topic in behavioral economics. In intertemporal choice theory, the general consensus is that utilities occurring at a later date in time are discounted to their present value terms. This discounting can be understood as a weight given to future utilities (Mas-Colell et al., 1995). The extent to which individuals choose to assign weight to the future is influenced by numerous factors, including the age of the decision-maker, the economic state of the country, and personal characteristics such as impatience or myopia. In the early years of behavioral economics, all these factors that affect people's intertemporal decision-making were summarized under the broad concept of *time preferences*. Time preferences can be either positive or negative. Positive time preferences describe people's tendency to prefer utility in the present over utility of the same nominal amount at a later date. They are called positive time preferences because individuals are assumed to mentally discount future utilities with a positive discount rate (Frederick et al., 2002). Vice versa, when individuals prefer later outcomes over the same nominal amount now they are assumed to have negative time preferences. Given that positive time preferences are more commonly observed in reality than are negative ones, it has become customary to use the term *time preferences* to refer specifically to positive time preferences. Unless explicitly stated otherwise, the same convention is employed in the following. The reasons for time preferences, and consequently for discounting in the classical sense, are manifold and can vary across schools of thought within intertemporal choice theory. In the following, frequently presented reasons for discounting are summarized.

1.2.1 Rationales for Discounting

In work by Goodin (1982), the commonly mentioned arguments for discounting are summarized into the following four key motives: **(A)** psychological reasons, **(B)** uncertainty and risk, **(C)** opportunity cost, and **(D)** diminishing marginal utility. The following paragraph explores these motives based on the discussion presented by Goodin (1982).

Psychological motives are an attempt to explain discounting that takes place without any apparent rational reason. Discounting occurs as a result of people's intrinsic tendency to prefer immediate gratification, which in turn leads to discomfort when utility has to be delayed (Sidgwick, 1907). For example, if someone were to offer you CHF 10 now or CHF 15 in a month, most people would likely choose the immediate CHF 10 now. Under the intrinsic tendency motive, the preference for the immediate outcome over the outcome at a later date is driven by the desire to obtain the CHF 10 instantaneously; obtaining the CHF 15, meanwhile, requires waiting. In economics, this so-called pure time discounting motive is generally seen as untenable since it reflects irrational behavior: "Rationality implies an impartial concern for all parts of our life. The mere difference of location in time, of something's being earlier or later, is not in itself a rational ground for having more or less regard for it" (Rawls, 1999, p. 259). Ramsey (1928, p. 543) goes so far as to dismiss pure time preferences as nothing more than a "weakness of the imagination". As the focus in the following section is on discounting in economic models, the arguments for and against pure time discounting are not elaborated further. The remaining three motives—uncertainty and risk, diminishing utility, and opportunity costs—are considered to be rational explanations for discounting.

The actual realization of future payoffs often involves a degree of uncertainty and risk, a consideration accounted for in the second of the four arguments Goodin (1982) identifies. Due to the presence of uncertainty or risk, outcomes that occur at a later date are perceived to have a lower utility compared to the same outcomes occurring earlier. Returning to the example of choosing between CHF 10 now and CHF 15 in a month's time, there is an inherent element of uncertainty as there is no guarantee that the person offering the money will fulfill their promise in a month's time. The person might forget about it or, in a more unfortunate scenario, experience some event that prevents them from following through on the promise. In fact, Sozou and Seymour (2003) note that death is the ultimate risk factor of the future. Thus, individuals tend to prefer immediate utilities because their realization is typically free from (or at least accompanied by less) uncertainty or risk. This argument for discounting further implies that people living in riskier or more unstable environments, such as countries experiencing political unrest or impoverished rural areas, are expected to discount the future more. Indeed, there is a notable amount of literature showing that individuals in poorer areas generally act more myopically than those in areas with a higher median household income (see, for example, Tanaka et al. (2010) and Haushofer and Fehr (2014)). Discounting future payoffs due to the uncertainty and risk involved has a sensible financial interpretation. In the context of financial markets, the principle of the time value of money explains why any monetary amount received today is considered to have a higher value than the same nominal amount received in the future: the amount received in the present can be invested and thus has the potential to accrue interest that is at least equal to the risk-free rate. Consequently, stocks of companies are traded with a risk premium that aims to fully compensate investors for the risk they face when they choose to invest in a company instead of opting for a risk-free treasury bill with a fixed payoff. This risk premium accounts for the inherent uncertainty and volatility associated with investing in stocks. As a result, any expected return an investor anticipates from an investment in a company is discounted using this risk-adjusted interest rate. While the uncertainty and risk argument provides a rational economic explanation for why people tend to assign a lower value to future outcomes, it is not without its critics. Discounting is often conceptualized as the weighting of future utilities. Thus, as argued, for example, by Frederick et al. (2002), the uncertainty and risk of future outcomes lowers the expected value of these outcomes at the time of occurrence. It does not, however, impact the weight given to these expected future outcomes in order to transform them into present value terms.

The opportunity cost argument attempts to explain the preference for immediate utility by considering the opportunity cost associated with delayed payment. For instance, if one chooses to accept CHF 10 now, one has the opportunity to invest that amount in a stock that exhibits a significant upward trend. Within a month this investment could potentially grow to an amount exceeding CHF 15. Thus, by not choosing the CHF 10 now, individuals forego the opportunity to invest it, resulting in an opportunity cost. This argument is often considered the most financially motivated justification for discounting. It suggests that any immediate payment can be invested to generate returns over time. Opting to spend the money, meanwhile, involves a trade-off, as this requires sacrificing the potential gains that could have been attained through investment. Furthermore, the opportunity cost argument for discounting justifies the use of at least a risk-free interest rate for the weighting of future outcomes (Eckstein, 1961; Arrow, 1976). The risk-free interest rate accounts for the lowest opportunity cost associated with not investing the amount in question. Because of this direct link to the interest rate, opportunity costs are certainly a valid rationale for time preferences in intertemporal decisions encompassing monetary outcomes or easily tradable goods that can be invested without constraints. However, some goods are simply not tradable or cannot be interpreted in monetary terms. For example, on a hot summer's day, preferring one ice cream immediately over ten ice creams in a week is hard to explain in terms of opportunity costs. Additionally, it is important to consider that not all individuals may be aware of alternative investment opportunities.

Lastly, the rationale for applying the concept of diminishing marginal utility to explain discounting is grounded in the presumption of a concave utility function. With this particular utility shape, the additional utility derived from consuming an additional unit of a good decreases as more of that good is consumed. Thus, goods occurring at a later point in time have a lower value, because one expects to have already consumed a certain amount of the good by then. Proponents of the diminishing marginal utility argument specifically use this reasoning to explain the intertemporal consumption choices made by society (Arrow, 1976). According to this perspective, in emerging economies the satisfaction derived from additional consumption in the present is expected to be higher than the satisfaction derived in the future. This is because society is expected to experience a better economic environment in the future, implying being farther along the utility curve and consequently having a lower marginal utility in comparison to the present. The diminishing marginal utility argument faces three main criticisms, particularly in the context of its application to intertemporal consumption within a society. Firstly, it is debatable whether comparing the utility function of two different economic states is appropriate. Economic states at different points in time may involve distinct generations of entirely different individuals, leading to variations in their utility functions. Hence, the assumption that utility functions remain constant across different economic states is questionable. Secondly, there is no guarantee of economic growth or an expectation of growth in all economies. In reality, economies can experience periods of stagnation or decline. Consequently, following the diminishing marginal utility argument, in declining economies time preferences should be negative. This is because society transitions from a state with lower marginal utility to a state with higher marginal utility (Goodin, 1982). Furthermore, regardless of whether one is examining an individual's or a society's intertemporal decision problem, the diminishing marginal utility argument focuses on the shape of the utility function. It does not, however, directly address the specific weights assigned to utilities in different time periods (Frederick et al., 2002).

While there is no consensus on the exact reasons why individuals have time preferences, economists generally agree that in models involving intertemporal choices and rational expectations future outcomes should be discounted. This notion was popularized by Samuelson (1937) with the DU model. In the following section, the DU model, its key features, and its underlying axioms are described.

1.2.2 The DU Model

Samuelson (1937) introduces a general approach to analyzing utility streams over time, the DU model. In this model, all factors that influence a person's time preferences are summarized into a single parameter known as the discount rate. Due to its simplicity and mathematical tractability, this approach quickly became the standard method for addressing intertemporal decisions in economic models. This section provides a brief explanation of the model of Samuelson (1937), based on the axioms in the work of Koopmans (1960).

The consumption of a good c at time t results in the utility $u(c_t)$ at that time. When consumption of the good is in the future but the decision regarding its consumption is in the present, utility must be discounted to present value terms using the discount function $b(r, \Delta_t)$. Here, r represents the discount rate, and Δ_t denotes the time difference between when the good is consumed and when the decision is made—that is, when the utility stream is evaluated. The utility streams at different points in time are additively separable. Thus, when a person evaluates a sequence of consumable goods at different points in time $(c_t, c_{t+1}, \ldots, c_T)$, the intertemporal utility function is expressed as the sum of the discounted utilities at the time of evaluation:

$$U(c_t, c_{t+1}, \ldots, c_T) = \sum_{\Delta_t=0}^{T-t} b(r, \Delta_t) u(c_{t+\Delta_t}), \tag{1.1}$$

where $b(r, \Delta_t)$ is considered to be the discounting part of the model and $u(c_{t+\Delta_t})$ the utility part. When confronted with several consumption streams, individuals simply calculate for each the sum of the discounted utilities and choose the option with the highest value. In the case of discrete time intervals, the discount function has the exponential form

$$b(r, \Delta_t) = \left(\frac{1}{1+r}\right)^{\Delta_t}. \tag{1.2}$$

For a specific value of r, the term $(1+r)^{-1}$ is commonly referred to as the discount factor. In experimental studies discounting is often assumed to be continuous, in which case the discount function is $b(r, \Delta_t) = \exp(-r\Delta_t)$.[1] The DU model's concept of the intertemporal decision-making of individuals can be linked to how companies make choices between different investments. For instance, when a company faces multiple project options it typically discounts the cash flow streams of each project using the WACC. Subsequently, the company selects the project with the highest net present value as the preferred investment option (Volkart, 2006). Similar to how the WACC of a company incorporates the various costs the company encounters in investment decisions, an individual's discount rate encompasses all aspects of their time preferences. It is important to note that, generally, the discount rate is not equal to the interest rate, although the comparison between the WACC and time preferences might suggest so. Discount rates typically capture the trade-off between utilities at different time points, whereas interest rates reflect the returns or costs associated with monetary investments. There are, however, situations—particularly in purely monetary intertemporal decision problems—where these two concepts can be used interchangeably (Goodin, 1982; Cohen et al., 2020). In fact, Samuelson (1937) proposes using the interest rate of the market as a reference value for the discount rate.

The DU model makes several assumptions regarding the discounting as well as the utility part of the model (Koopmans, 1960). Firstly, discounting is assumed to remain constant over time and goods. This implies that both the discount rate and the discount factor maintain a constant value regardless of when the consumption stream is evaluated or what type of goods are under consideration. Secondly, agents have positive time preferences, indicating a discount rate of $r > 0$. Thus, individuals exhibit a preference for present consumption over future consumption (of the same amount). Furthermore, the level of impatience can be

[1] In a continuous time setting, the intertemporal utility function takes the form of an integral rather than a sum (Samuelson, 1937). That is to say, the discounted utility for a consumption stream $C = (c_1, c_2, \ldots, c_T)$ reads $U(C) = \int_{t=1}^{T} \exp(-r(t-1))u(c_t)$.

determined by the discount rate: the higher the discount rate, or the lower the discount factor, the greater the impatience, and vice versa. Thirdly, both the utility and consumption are considered to be independent of time. Under the assumption of utility independence, the utility in one period is not affected by the utility in another period. This allows for the discounted utilities to be separated and summed up over time and ensures that only the total sum of discounted utilities is the determining factor for the maximum. As a result, when comparing two consumption streams with an equal total of discounted utilities—one with goods evenly distributed over time and the other with all goods concentrated in a single period—individuals are indifferent between the two options. Consumption independence implies that the utility derived from consuming a specific good in one period is unaffected by the consumption choices made in another period. For instance, if an individual finds the highest utility in eating cornflakes for breakfast chosen from among several breakfast options, their preference for cornflakes remains unchanged when facing the same decision the following day. Thus, regardless of whether they had cornflakes for breakfast on the previous day their preference for cornflakes remains consistent. Lastly, the utility function is assumed to be constant. This means that the instantaneous utility derived from consuming a certain good does not change with time. As an example, if an agent experiences an immediate utility of 5 from eating cornflakes at time t_1, that agent will also have an immediate utility of 5 from eating cornflakes at time t_2 with $t_1 \neq t_2$.

The DU model was well received in the field of economics and quickly became the standard for modeling individuals' intertemporal decision-making problems. Even though Samuelson (1937) did not make any claims about its empirical validity, the simplicity of the DU model was convincing: Individuals are considered to be utility maximizers who aim to maximize their overall well-being and consumption throughout their lives. A potentially complex decision-making process is simplified to a straightforward trade-off between present and future utilities. In addition to its behavioral implications, the DU model has advantageous economic and mathematical properties. The assumptions regarding the utility function and the exponential form of the discounting part result in stationary and time-consistent decision-making behavior over time. Both attributes make the model well-suited for representing rational behavior, which is a fundamental assumption in most economic models. In fact, the DU model with an exponential discount function is the only model that allows for stationary and time-consistent modeling of intertemporal decision problems (Cohen et al., 2020). Intertemporal choices are considered to be stationary when preferences over a set of options remain unchanged regardless of whether the options are delayed or accelerated. That is to say, an agent who prefers option A at time t over option B at time t will also prefer option A at time $t+\tau$ over option B at time $t+\tau$ with $\tau > 0$. The notion of time-consistency describes the fact that when an agent decides at time t_1 to consume a certain good that arrives at time t_2 with $t_1 < t_2$, this decision does not change as time progresses. Time-consistency ensures that as long as no new information arrives, agents stick to their decisions.[2] Furthermore, from a mathematical standpoint an exponential discount function has the advantage that as the time horizon approaches infinity, $T \to \infty$, the discount function simplifies to a single fraction. That is to say, the sum over the discounting part converges to

$$\lim_{T\to\infty} \sum_{\Delta_t=0}^{T} \left(\frac{1}{1+r}\right)^{\Delta_t} = 1 + \frac{1}{r}. \tag{1.3}$$

This property is advantageous in models where the consumed good is constant over time.

Despite the aforementioned advantages of the DU model with exponential discounting, its descriptive validity is often questioned. For example, the assumption of consumption independence implies that a person would not mind eating cornflakes every single morning of their entire life if eating cornflakes brought them the highest utility. Even Samuelson (1937, p. 159) himself noted that his proposal for how to treat people's intertemporal

[2] Note that stationarity and time-consistency are often used interchangeably. The abovementioned definitions are based on the work of Strotz (1955) and Koopmans (1960), and on clarifications by Drouhin (2020).

decision-making process was “completely arbitrary”. Alternative models that try to explain the intertemporal behavior of individuals include models of multiple selves, domain specific models, and models that allow for intertemporal substitution. For a discussion of these approaches, see, for example, the work of Cohen et al. (2020). Regardless of the existence of alternative models, the DU model continues to serve as the standard framework for modeling intertemporal behavior. The following discussion highlights the frequently observed disparities between the intertemporal decision behavior predicted by the DU model and the actual behavior people display, specifically with regard to the assumption of a constant discount rate. It is important to note that many of these anomalies have been identified through experimental studies. Such studies are, however, themselves subject to certain biases and limitations, which are addressed in a later subsection, Section 1.3.3.

1.3 Estimating Discounting

The DU model assumes exponential discounting with a constant discount rate. Additionally, both the discounting part and the utility part of the model are considered to be independent of the timing of both the decision and consumption, as well as the specific good being consumed. These assumptions play a crucial role in the DU model and have contributed to its popularity in economic modeling. However, the empirical validity of the model is subject to criticism. In reality, people’s intertemporal decision-making often exhibits inconsistencies that do not align with the behavior predicted by the model. This section discusses well-documented disparities between the assumptions of the DU model and the actual behavior exhibited by individuals. The main focus is on the observed anomalies regarding the discount function. Criticisms concerning the utility part—such as the concavity of the utility function—are not exclusive to the DU model and are thus not part of this discussion. Readers interested in a discussion of the utility function are referred to, for example, the works of Mas-Colell et al. (1995) and Lehr (2022). As the majority of the discrepancies regarding the discounting part of the DU model are revealed within experimental setups, the following subsection summarizes the conventional design of such experiments.

1.3.1 Estimating Discounting with Experiments

Experimentally estimating individuals’ time preferences usually involves finding the indifference point between a sooner but lower utility and a higher but later utility. Ideally such experiments should focus only on the discounting part of the DU model by separating it from the utility part. Thus, study participants should be presented with actual utilities occurring on different dates. However, utilities are an economic concept that is difficult to replicate in experiments since they are a theoretical construct, making direct quantification unfeasible. Thus, in experiments researchers usually resort to food or monetary payoffs as a proxy for utilities. For example, in the well-known Stanford marshmallow experiment conducted by Mischel and Ebbesen (1970), children between the ages of three and five were offered a marshmallow and informed that if they waited for an undisclosed period of time, they would receive an additional marshmallow as a reward. This study aimed to investigate the concept of delayed gratification and self-control. While studies involving food rewards can offer valuable insights into aspects such as patience, it is important to note that such experimental designs are not well-suited to accurately estimating a discount function, let alone a discount rate. Thus, the majority of studies that estimate people’s time preferences experimentally rely on monetary payoffs. Assuming a specific functional form for the discount function and the utility function, by utilizing monetary payoffs it becomes feasible to compute the discount rate.

Usually experiments with monetary payoffs set up the decision problem either as a choice task or as a matching task (Cohen et al., 2020). In choice tasks study participants have to decide between a payment now and a payment later. Based on the preferred payoff it is possible to directly calculate either an upper or a lower bound for the discount rate from the assumed underlying discount and utility functions. When a study presents many such choices with slight variations in the outcomes, it allows the narrowing down of the discount rate to an interval. For example, in a study conducted by Kirby and Maraković (1996) a questionnaire containing 21 choice tasks was administered to over 600 students. In each of these choice tasks, the participants were presented with the option of choosing an immediate reward or a slightly higher reward that would be received after a varying number of days. The choices in the questionnaire comprised a wide range of reward amounts and time delays between the payoffs. This variation in rewards and delays enabled the researchers to narrow down the potential range of discount rates and establish both an upper and a lower bound for their estimated values. Under the assumption of a linear utility function and an exponential discount function, the interval for the discount rate r can be inferred from the sooner payment p_{t_1}, the later payment p_{t_2}, and the delay between the payments, $\Delta_t = t_2 - t_1$. As an example, one of the 21 choice tasks was to decide between USD 15 tonight and USD 35 in 10 days' time. A participant who prefers the USD 15 has a minimum daily discount rate of approximately 8.47 percent, since

$$\begin{aligned} p_{t_1} &> p_{t_2} \cdot \exp(-r\Delta_t) \\ r &> -\frac{\log\left(p_{t_1}/p_{t_2}\right)}{\Delta_t}. \end{aligned}$$

In the study conducted by Kirby and Maraković (1996) the order of these choice tasks was randomized. In other studies, choices are, however, often presented either in descending or ascending order of the implied discount rate, with the aim of simplifying identification of the upper and lower bounds of the discount rate. As an example, Coller and Williams (1999) examine the time preferences of a group of 35 students. Each student is presented with a list of choices between receiving USD 500 in one month's time and USD 500 plus some positive amount, x, in three months' time. The choices are presented in ascending order of x. Assuming a linear utility function and exponential discounting, the sequence of the choice tasks corresponds to ascending annualized discount rates (for indifference between the choices).

While in choice tasks participants are presented with predefined options (and thus predefined implicit discount rates), in matching tasks they have to explicitly specify their indifference point. That is to say, a matching task requires study participants to match an outcome sooner to an outcome later. The matching medium is usually money or time. For example, in matching tasks with money participants are asked to state the monetary amount that would make them indifferent between a fixed payoff in the present and the stated amount in the future, or vice versa. In time matching tasks, participants have to specify how long they are willing to wait to accept a later but larger payoff over a smaller but sooner reward, or how much earlier a smaller, sooner reward has to be paid out to be preferred over a larger, later one. With such matching tasks, the discount rate can be directly inferred from the underlying functional form of the utility and the discounting parts. For example, in a study by Thaler (1981) the matching medium was the price. A group of around 20 university students were asked to give the smallest monetary amount that would make them equally satisfied by a given payoff in the present and the stated amount in the future. Ainslie and Haendel (1983), meanwhile, asked patients from a drug rehabilitation center to state the longest time they were willing to wait in order to receive a prize that was worth double the amount of an immediate reward. Both studies allowed the researchers to directly calculate the discount rate as a function of payment sooner, p_{t_1}, payment later, p_{t_2}, and the time between the payments, $\Delta_t = t_2 - t_1$. That is to say, with linear utilities and exponential discounting, the discount rate is

$$r = -\frac{\log(p_{t_1}/p_{t_2})}{\Delta_t}. \tag{1.4}$$

Fewer study designs use rating or pricing tasks (Frederick et al., 2002). These tasks are used when researchers aim to examine individuals' time preferences in relation to outcomes that are not directly monetary in nature. In rating tasks, participants are presented with different outcomes at different times and have to rate them on a scale. As an example, in work by Redelmeier and Heller (1993) the authors challenge a group of medical students and professionals to score three different health scenarios that could occur on five different dates in the future. The implicit discount rates are calculated by comparing the ratings of a specific scenario for different delays. Thus, the given ratings are considered to be the utilities of the event happening on that date. The discount function is assumed to be exponential and the utilities linear, and the estimated discount rates are transformed into effective annual rates in order to compare them to the annual interest rates at that time. The rating scale ranges from zero to 100, with zero representing the lowest utility and 100 the highest.[3] For illustration, one health scenario is a severe episode of depression occurring one day, six months, one year, five years, or ten years into the future. Suppose the rating for the scenario happening in one year's time is 20 and that for the scenario happening in five years' time is 40, then the nominal discount rate can be calculated as $r = -\log(20/40)/(5-1) \approx 0.17$.

In pricing tasks, study participants are required to assign a monetary value or price to certain non-monetary outcomes under different conditions. These conditions include receiving the outcomes immediately, at a later time, or not receiving them at all. The stated prices can be interpreted as the utility of the event happening under these conditions. For instance, in work by Loewenstein (1987) 30 students have to state how much they are willing to pay to experience or avoid a certain event. One task is to avoid a small electric shock. This shock could either occur immediately or with a delay of between three hours and 10 years. With linear utilities, the discount rates are calculated as a function of the price paid to avoid the immediate electric shock, p_{t_1}, the price paid to avoid a shock with a time delay, p_{t_2}, and the difference in time between the options, Δ_t, as $-\log(p_{t_1}/p_{t_2})/\Delta_t$. Effectively, rating tasks and pricing tasks are the same but with different utility measures. In rating tasks, individuals assign subjective ratings from a predefined scale to non-monetary outcomes, while in pricing tasks individuals assign monetary values to these outcomes.

Which type of task is chosen to estimate the discount rate—that is, how the experiment is designed—has been demonstrated to have a non-negligible influence on the resulting discount rates. This topic will be discussed in a later section, Section 1.3.3.

1.3.2 Inconsistencies with the DU Model

The two main critiques of the DU model revolve around the assumption of a constant discount factor or rate and the assumption of independence between discounting and utilities. These two aspects often fail to align with the actual decision-making behavior of individuals. The phenomenon of discount rates changing with the time span between the decision and the actual payoff is a well-documented anomaly in the field of behavioral economics. This observation not only challenges the DU model's implied assumption of stationary and time-consistent time preferences but also raises doubts about the validity of using an exponential discount function. Already in the work of Strotz (1955, p. 177) the author questions an exponential discount function, instead proposing that most individuals possess a discount function that "over-values" immediate positive utilities or satisfaction compared to those that are more distant. Recall the example where you were asked to decide between CHF 10 now and CHF 15 in a month's time. Assume you are additionally offered CHF 10 in one year's time or CHF 15 in one year and one month's time. Most people would probably prefer CHF 10 now to CHF 15 in a month's time, but at the same time they would not mind waiting 13 instead of 12 months to get the higher payoff of CHF 15.

[3]In this work by Redelmeier and Heller (1993), the authors first transform the utilities to a categorical scale to calculate the implicit nominal annual discount rates, r. They subsequently transform these rates into an effective annual rate using the formula $\exp(r) - 1$. These steps are omitted in the calculation example presented above.

This thought experiment suggests that the discount rate varies with time.

Early empirical evidence hinting at time-inconsistent and nonstationary discounting behavior is presented in a study by Thaler (1981). Based on a matching task, the participating students have to state the monetary amount that would make them indifferent between receiving a fixed, immediate payment and receiving the stated amount with a delay. Depending on the questionnaire form used, the immediate payments vary between USD 15 and USD 3,000, and the waiting time between three months and 10 years. One option even includes immediate fines of USD 15 and USD 100. All payments are hypothetical, meaning no money is paid out (or claimed). By way of an example, in one questionnaire the students are presented with an immediate payment of USD 15 and the delays are three months, one year, and three years. The median response of matching USD 15 now is USD 30 in three months, USD 60 in a year, and USD 100 in three years. Under the assumption of a linear utility function and an exponential discount function, these responses imply annualized discount rates of 277, 139, and 63 percent, respectively. Thus, participants showed implicitly more patience for longer waiting times. The other questionnaire types, which involved different immediate payment offers and waiting times—excluding the one with the fine—also showed the same declining effect of the implicit discount rates as the waiting time increased. Thaler (1981) has further noted that all estimated discount rates are relatively high when compared to actual interest rates. However, this disparity was attributed to the hypothetical nature of the study in question. Similar results to those of Thaler (1981) are found, for example, in work by Benzion et al. (1989). Over 200 Economics and Finance students participated in a study with matching tasks. They were asked to match, in different hypothetical scenarios, a payment or debt in the near future with an appropriate payment in the further future; and vice versa to match a further-in-the-future payment or debt to one in the nearer future. The authors assumed linear utilities and exponential discounting (with discrete time). One scenario involved, for example, an employee of a public institution who was told that a salary payment would be postponed, and the students had to state which amount the institution would have to pay the employee such that the employee would agree to the later payment. For this specific scenario, matching a payment of USD 40 now to one in half a year, one year, two years, and four years, the stated amounts implied average annualized discount rates of 59.8, 39.3, 26.3, and 21.9 percent, respectively. In almost every scenario the authors found declining discount rates over time. Other experimental studies demonstrating the effect of discount rates changing with time include those of Redelmeier and Heller (1993), Chapman and Elstein (1995), Chapman (1996), Pender (1996), Kirby (1997), and Green et al. (1997).

Interestingly, the anomaly of discount rates declining with time is not only observed in individual experiments but also across studies. In the literature survey by Frederick et al. (2002), the longer the average time horizon is in an experiment, the larger the average estimated discount factor becomes. This trend is, however, only apparent in studies with an average time horizon of less than one year. For studies with a horizon of more than one year, the positive correlation between the time horizon and the discount factor is insignificant and even slightly negative.

The finding that discount rates decrease as the time horizon extends results in time-inconsistent behavior and also leads to changing preferences over time. Such nonstationary behavior is revealed, for example, in a study by Green et al. (1994). A group of 24 students are presented with various choice tasks requiring a choice between a smaller, immediate reward and a larger, delayed reward. All payments are hypothetical. The study includes three pairs of rewards: USD 20 versus USD 50, USD 100 versus USD 250, and USD 500 versus USD 1,250, with the time intervals between the options ranging from one week to 20 years. For example, participants have to decide between receiving USD 20 in one week's time and receiving USD 50 in three months and one week's time, USD 20 in two weeks' time and USD 50 in three months and two weeks's time, USD 20 in one month's time and USD 50 in four months' time and so on. Note that the time interval between the rewards is always three months. The researchers record the percentage of participants who preferred the smaller, sooner reward over the larger, delayed reward. When the payment date for the smaller reward is closer to the present, a higher percentage of participants prefer that option. However, as the delay for the smaller reward increases more participants switch their preference to the larger reward. This effect is observed for all of the three reward pairs. Additional studies that support this finding of preference reversal include those of Kirby and Herrnstein (1995), Read and van Leeuwen (1998), Read et al. (1999), and Augenblick et al. (2015).

The presence of nonstationary and time-inconsistent time preferences contradicts the notion of an exponential discount function and challenges the DU model's assumption of a constant discount rate over time. In fact, the results from the aforementioned experiments reveal intertemporal behavior that aligns with so-called *hyperbolic discounting* (Laibson, 1997). Hyperbolic discounting refers to the phenomenon by which individuals exhibit greater impatience when it comes to payoffs that are closer to the decision time compared to payoffs that occur further into the future. Consequently, people tend to assign a higher discount rate to intertemporal decisions in the near future while choices of equal magnitude but in the distant future are better reflected by a lower discount rate. As a substantial body of experimental evidence contradicts the exponential discount function, economists were prompted to incorporate such nonstationary and time-inconsistent behavior of agents by modeling discounting, for example, as a quasi-hyperbolic discount function (Phelps and Pollak, 1968; Laibson, 1997). With quasi-hyperbolic discounting, the discount function in (1.1) reads

$$b(r, \Delta_t) = \begin{cases} 1 & \text{if } \Delta_t = 0 \\ \alpha \left(\frac{1}{1+r}\right)^{\Delta_t} & \text{if } \Delta_t > 0. \end{cases} \tag{1.5}$$

As in the exponential formula, the time difference between the utility's realization and the decision is denoted as Δ_t, while the discount rate is represented by r. Additionally, the discount factor is multiplied by a parameter, $\alpha \in (0, 1]$, that indicates the *present bias*. When the present bias factor, α, is set to one, the discount function simplifies to the exponential form. Consumption in the present is not discounted because $b(r, \Delta_t) = 1$ for $\Delta_t = 0$. Thus, such a discount function can account for greater impatience in the short run and also explain preference reversals. Hyperbolic discounting found widespread acclaim in economic applications and is nowadays used to explain a variety of market anomalies. Even in emerging fields such as neuroeconomics, evidence for hyperbolic discounting has been found. In a study by McClure et al. (2004) it is demonstrated that the reward region of the brain exhibits a hyperbolic or quasi-hyperbolic decline in activation as the time horizon for receiving a reward is extended.

Time-inconsistencies and non-stationarity resulting from decreasing discount rates over time are not the only anomaly that challenges the discount function of the DU model. Another widely acknowledged discrepancy is the observation that people discount outcomes differently depending on the outcome's *sign* or *size*. The sign effect refers to the observation that intertemporal choices involving the same absolute monetary amounts and time span result in different discount factors depending on whether the outcomes considered are positive or negative. Generally, people have been shown to be less patient when it comes to positive payoffs than they are with regard to negative ones of the same size. For example, in the aforementioned study by Thaler (1981), the author shows that people have lower discount rates for losses in the form of a fine than for cash prizes of comparable sizes. One of the tasks in this particular work involves matching the amount of a fine, which is due in three months' time, to an immediate fine of USD 15. The median answer for this task was USD 16. Assuming again a linear utility function, this would imply an annual discount rate of 26 percent. For the same matching task, but with a positive amount of USD 15, the students were willing to wait three months if the later payment was, on average, USD 30. This would imply an annual discount rate of 277 percent. The same effect is shown by Benzion et al. (1989). In the scenario where the students had to decide what amount an employee should be willing to accept to have a salary payment of USD 40 paid out only in half a year's time instead of immediately, the stated amounts implied an average annual discount rate of 59.8 percent. In another scenario, the students had to decide what amount of money the same employee should offer a public institution in order to have the due date of a debt of USD 40 to the institution pushed out by half a year. For this *negative* case, the average discount rate was only 33.4 percent.

The size effect refers to the phenomenon by which individuals exhibit greater myopia, or shortsightedness, when it comes to smaller positive payoffs compared to larger ones. In other words, while keeping the time span between two outcomes fixed, individuals tend to display higher implicit discount rates when making choices involving smaller rewards than when making choices involving larger rewards. This effect is also prevalent in work by Thaler (1981): When the students were tasked with matching an immediate cash sum to a payoff one year later, the implied discount rate decreased as the size of the immediate payment increased. For example, the students stated indifference between a payoff of USD 15 now and USD 60 in a year's time, and between USD 3,000 now and USD 4,000 in a year's time. These values imply a discount rate of 139 percent and one of 29 percent, respectively. The same observation is documented in the work of Benzion et al. (1989). In the scenario with the salary delay, the salary amount ranged from USD 40 to USD 5,000. According to the students' answers, the employee should be willing to accept a delay of half a year to the otherwise immediate payment of the salary of USD 40 if the delayed salary is USD 49.80. This would imply an average discount rate of 59.8 percent. When the immediate salary was USD 5,000, the delayed salary demanded was, on average, USD 5,426.30, which implies a discount rate of 18.4 percent. Other studies that also revealed a sign or size effect in estimated discount rates include those of Holcomb and Nelson (1992), Redelmeier and Heller (1993), Raineri and Rachlin (1993), Chapman and Elstein (1995), and Kirby (1997).

In summary, experimental studies have uncovered several disparities between people's actual discounting behavior and the behavior predicted by the DU model. The most prominent anomaly observed in intertemporal decision-making is the presence of time-inconsistencies and non-stationarity in choices. This is evident in the finding that estimated discount rates decrease as the time horizon between rewards or between the reward and the decision is extended. One plausible explanation for this observation is that the discount function follows a hyperbolic form instead of an exponential one. Other discrepancies between the behavior described by the DU model and people's intertemporal decision-making include the discount factor not being constant for choices involving negative rewards or choices involving high monetary gains. It is, however, important to note that most of the studies that have uncovered and examined these discrepancies were conducted within an experimental setup. The design of the experiment itself impacts the estimated rates. This is discussed in the subsequent section, along with other criticisms related to the experimental estimation of discount rates.

1.3.3 Inconsistencies or False Estimation?

The previous subsection highlighted studies that have identified anomalies in the DU model by estimating discount rates with data on intertemporal decisions gathered from experiments. How this data is retrieved—that is, how the experiment is designed—can bias the estimated discount rates. This section will delve into a discussion of these design biases as well as other issues that arise when using experiments to examine individuals' discounting behavior.

As previously noted, in Section 1.3.1, experimental designs commonly used to examine discounting encompass matching tasks, choice tasks, and, less frequently, pricing and rating tasks. Depending on the specific study design employed to find the indifference point, the estimated discount rate can be affected by different biases. When presented with a choice task that contains multiple offers, the sequence in which these choices are presented plays a crucial role in the decision-making. Often, the estimated discount rates are biased toward the first choice that was offered in the sequence. This so-called anchoring effect is a well-known problem of experiments (Green et al., 1998). It refers to the phenomenon by which individuals' judgments or decisions are influenced by the first piece of information presented to them, as this initial piece of information serves as the reference point for their subsequent decisions. The presence of the anchoring effect has also been observed in experimental discounting studies. For example, in Hermann and Musshoff (2016) participants were presented with a sequence of choices between lower payment now (A) and higher payment later (B). Payment (A) served as the baseline and remained the same throughout all offers. The sequence of choices either reflected low discount rates or high discount rates. That is to say, the low-rate sequence implied a discount rate increasing from zero to approximately 16 percent, while the value for the high-rate sequence ranged from zero to 50 percent. At which offer people switched from option (A) to option (B) indicated in what range a person's discount rate was. The earlier the switch, the lower the inferred discount rate, or, more precisely, the lower the values of the discount rate interval. The authors show that participants who started with the low-rate sequence switched earlier from (A) to (B) in the high-rate sequence than did participants who started with the high-rate sequence.

When an experiment is based on matching tasks, the discount rates differ depending on what is used for matching—so, payoff now, payoff later, or time. As an example, in work by Roelofsma (1994, cited in Roelofsma (1996)), discount rates from matching on time were compared to ones matched on money. One group of participants were asked to state the lowest reimbursement they would accept for the delivery of an expensive bike (900 Dutch guilders (NLG)) being postponed by nine months. The median amount was NLG 250. A second group were then asked how long a delay they would accept to the delivery of a bike (again worth NLG 900) in exchange for compensation of NLG 250. The median waiting time was only three weeks. Thus, matching on time resulted in much more myopic behavior than matching on money.

The fact that rating and pricing tasks usually require participants to evaluate each option by itself and not in comparison to other options or another set of options (Frederick et al., 2002) has also drawn criticism. Since the time factor of the option might not be perceived as important when the options have to be rated individually, this might also affect the estimated discount rates. In fact, Loewenstein (1988) has demonstrated this phenomenon. A group of over 100 Economics students was split into two. In the first round, one group were asked how much they would be willing to pay for a USD 100 restaurant gift card that was immediately valid. The other group were asked the same question, but their gift cards only became valid in six months' time. The median stated amounts in each group—USD 78 for the immediately valid gift card and USD 86 for the gift card with the delay—were not significantly different from each other. In the second round, the group with the immediately valid gift card were asked to state the amount of compensation they would accept in exchange for delaying the validity of the gift card by six months. The group with the valid-in-six-months gift card, meanwhile, were asked how much they would pay for the gift card to be valid immediately. In this second round, the answers differed significantly: the group that was asked to delay the validity asked for compensation of around USD 23.85, while the other group, the one asked to speed up the validity, would pay on average only USD 10.17. Thus, when the two groups were asked to individually price the gift cards, the factor of time did not play a significant role. Both gift cards, despite having different validity thresholds, were valued similarly by the participants. However, once the participants were asked to consider delaying or speeding up the validity of the cards, they became aware of the time aspect, and their time preferences started to influence their chosen amounts. Loewenstein (1988) considers this result a type of framing effect, by which the way information about time is presented or framed impacts participants' decision-making.[4]

The design of the questionnaire itself is not the only reason that discount rates estimated with experiments can be subject to biases. Another crucial factor is whether the outcomes that are being considered are hypothetical or real. Usually, the choice or matching questions in these experimental studies center around some

[4] Other researchers view this phenomenon as indicative of a delay-speed-up effect. See, for example, the work of Cohen et al. (2020).

monetary gain or loss. Whether these rewards are actually paid out, or if the fines are truly imposed at the given time, may play a role in how participants make their decisions. For instance, in work by Kirby and Maraković (1995) two experiments were conducted. In the first, around 20 students were tasked with stating the minimum amount they would accept as an immediate payment compared to a larger payment with a delay. The experiment had several rounds with different delayed payments. For one random round, the students would actually receive either their stated immediate reward or the delayed reward at the given date. The largest delayed reward was USD 28.50 with a delay of three to 29 days. In the second experiment, a different group of 18 students took part in the same matching tasks with the same larger-later payments. In this second round, however, the students would not actually receive any money. Kirby and Maraković (1995) found that in the first experiment with real rewards the discount rates were higher than they were for the hypothetical second experiment. A similar result is observed by Coller and Williams (1999). The study participants, students of a business school, are divided into six groups of 35 students. For a sequence of choice tasks, the students in a group have to decide between a payment of USD 500 in a month's time and a payment, in three months' time, ranging from USD 501.67 to USD 590.54. In five of the six groups, the students are informed that one of them would be randomly selected and would actually receive one of the rewards. For the sixth group, the payments were overtly hypothetical. Initially, the authors found higher discount rates for the groups that entered the experiment with real payments. However, after controlling for sociodemographic factors they discovered that the discount rates of the hypothetical experiment were higher than those of the real ones. Other studies did not identify any significant difference in participants' discounting behavior between hypothetical and real payoffs, including those of Madden et al. (2003) and Madden et al. (2004). In Madden et al. (2003), 20 study participants—students from the University of Wisconsin—had to decide between an immediate but smaller and a larger but later payoff for a series of choice tasks. Two questionnaires were administered to the participants. In the first, they were informed that there would be no payout at the end of the experiment. In the second they had an opportunity to win one of the selected payouts. The maximum amount a student could win was USD 10 with a delay of at least six hours and up to one year. A similar approach was adopted in Madden et al. (2004), which involved a larger sample group of 40 students and a maximum payoff of approximately USD 18 with a delay of around half a year. Both studies demonstrated no significant difference in choice patterns between the tasks with hypothetical outcomes and those with real outcomes. In summary, there is currently no clear evidence as to whether discount rates are lower or higher for actual payoffs compared to hypothetical ones. It is, however, worth noting that even if the outcomes in experiments are real, there is usually a limitation to the amount of money that can be offered or the time horizon that can be incorporated, which might contribute to the absence of consensus on this matter. In the aforementioned studies the actual payments were relatively low compared to the monetary consequences that real-life intertemporal decisions can entail. In the work by Kirby and Maraković (1995) the actual payment was at most USD 28.50—in Madden et al. (2003) it was USD 10, and in Madden et al. (2004) it was around USD 18. Madden et al. (2003) also emphasize that in the real experiment the actual expected maximum monetary amount was lower because only one of the choice tasks' payments was selected to be paid out. The same holds true for the findings of Kirby and Maraković (1995), Madden et al. (2004), and Coller and Williams (1999). While the Coller and Williams (1999) study had the highest real payment, only one person of a group of 35 participants received the maximal reward of USD 590.54, which lowers the expected (maximal) payoff to approximately USD 17.

Another reason for biases in the experimental estimation of discounting can result from the study participants' backgrounds. Many of the studies in behavioral economics are conducted with students, sometimes even with Economics and Finance students. Cohen et al. (2020) review 220 papers on discounting and find that around a third have students as participants. Out of the 27 experimental studies mentioned thus far, only six were carried out with non-student participants; for two the background of the participants is unspecified. The remaining studies involved students, most of whom were majoring in Psychology or Economics and Finance. The financial knowledge of a participant plays a crucial role in their intertemporal decision-making process, especially when monetary outcomes are involved. Students with an economics or finance background are better informed regarding, for example, opportunity costs, or are more aware of their potential for irrational behavior. The latter assertion likely extends to psychology students as well. When participants act in a way that they think is expected of them, we call this participant bias (Duignan, 2016). Furthermore, when participants are not truly representative of the population, additional errors may occur in the form of selection bias. Students generally fall within a younger age range, tend to have higher levels of education, and often have limited financial resources (Slonim et al., 2013). It is questionable whether such a group truly represents the wide range of people in the world who make intertemporal decisions on a daily basis. To investigate this, Andersen et al. (2010) compare the risk and time preferences of students to those of a group representing the Danish population. Discount rates are calculated based on choice tasks involving a sequence of smaller-sooner versus larger-later payoffs. The authors do not find a statistically significant difference in the average discount rate, with 25 percent for the group representing the average population and 27.9 percent for the student group. It is, however, important to note that the average population group consisted of 253 participants, which was more than twice the size of the student group, which contained only 90 participants. Furthermore, the questionnaire for the student group was limited to three sequences of choice tasks with time delays between the payoffs of 1, 4, and 6 months. In contrast, the population group was presented with six sequences, involving time delays of 1, 4, 6, 12, 18, and 24 months. These differences in the tasks might have biased the results. At the same time, even when study participants are recruited from the general population it is widely known that a specific type of individual is more likely to volunteer and participate in experiments. This is referred to as self-selection bias. For example, Slonim et al. (2013, p. 68) found that "recruited lab participants were not representative of the target population they were recruited from; participants had lower income, more leisure time, more interests in economics and lab activities, and were more pro-social on the dimension of volunteering time."

Finally, one issue with regard to experimental studies that we are yet to address is the prevailing assumption of a (locally) linear utility function. A linear utility function simplifies the calculation of the discount rate, as monetary values can be directly inserted into the discount rate formula—see, for example, (1.4)—without considering any curvature in the utility function. Recall the work of Thaler (1981), in which participants had to state the amount they would accept, for example, a year from now to make them indifferent between receiving that amount and receiving USD 15 immediately. In this case, the annual discount rate is calculated by dividing the USD 15 by the stated amount and taking the logarithm (times minus one). This calculation does not consider the possibility that marginal utility might decrease with higher monetary rewards, as would be the case with a concave utility function. Linear utility functions serve as the standard functional form in experiments aimed at estimating the discount rate from monetary choices. However, when the true utility function is concave and a linear form is assumed, this can result in an overestimation of the discount rate. In the experiment by Thaler (1981), assume, for example, that one of the study participants states that they are indifferent between USD 15 now and USD 60 in a year's time. Further assume that this person's true utility function is $u(x) = \log_{10}(x)$. With such a utility function, the annual discount rate is $-\log(u(15)/u(60))$ instead of $-\log(15/60)$. This results in an annual discount rate of approximately 41 percent, as opposed to the 139 percent estimated with linear utilities. Recognizing this limitation, recent research has sought to address this issue by introducing study designs that aim to estimate both the shape of the utility function and an individual's time preference simultaneously. Andersen et al. (2008) introduce a two-step estimation procedure to estimate both the time preferences and the risk preferences of 253 Danes. In a first step, the participants are presented with multiple choice tasks that involve two payment options: a payment of 3,000 Danish krone (DKK) in a month's time and a larger payment in seven months' time. The larger payment increases with each choice, which, assuming a linear utility function holds, reflects choice tasks with increasing discount rates. The point at which a participant switches from earlier payment to later payment serves as an indicator of their discount rate interval. The second step of the experiment is designed to measure the shape of the utility function by estimating the participants' level of risk aversion. According to the authors, a risk premium is required for later payments to make a participant indifferent between the earlier and the later payment, even in the absence of discounting. Thus, in the second experiment the same participants are repeatedly tasked with choosing between a safe lottery and a risky one from a series of lottery combinations. The safe lottery offers payments of DKK 2,000 with a probability of a and DKK 1,600 with a probability of $1 - a$, with a increasing by 1 percent with each choice. The risky lottery has winnings of DKK 3,850 with a probability of b and DKK 100 with a probability of $1 - b$, with b increasing by 1 percent with each choice. Risk-neutral people would switch from playing the safe lottery to playing the risky one once the expected values of the two lotteries were equal. Risk-averse people would switch later, and risk-seeking people earlier in comparison to a risk-neutral person. The switching point allows the calculation of a parameter that indicates the degree of risk aversion. In the first experiment, assuming a risk-neutral, linear utility function the average estimated discount rate is 25.2 percent. When incorporating the risk aversion parameter from the second experiment the estimated average discount rate significantly decreases, to 10.1 percent. On a different note, Andreoni and Sprenger (2012) propose a study design in which the discount rate and the curvature of the utility function can be identified in one step. A group of 97 students are asked to allocate a fixed amount of tokens between a sooner payment date and a later payment date. The sooner payment dates are today, one week from today, and 35 days from today, while the later payment dates are 35 days, 70 days, and 98 days from today. Allocating tokens to the earlier date, t_1, results in a payment of a_{t_1} per token, whereas allocating tokens to the later date, t_2, yields a_{t_2} per token, with $a_{t_1} \leq a_{t_2}$. The relation between the two values is $a_{t_2}/a_{t_1} = 1 + r$, where r is the interest rate for $(t_2 - t_1)$ days. Each participant encounters a total of 45 allocation problems and the annualized gross interest rates range from 0 percent to over 1,000 percent. Analyzing the responses for a fixed combination of sooner and later dates while varying the interest rate r provides information on the curvature of the utility function. Similarly, when the interest rate r is fixed, and the participants' choices for different combinations of sooner and later dates are examined, the authors gain insights into the discounting parameters. With this study design, Andreoni and Sprenger (2012) estimate a utility function on the aggregate level that exhibits more linearity compared to the findings of Andersen et al. (2008). As a result, their estimated average discount rate is relatively high, with a rate of 30 percent. In conclusion, both studies show that accounting for concavity in the utility function results in a decrease in the estimated discount rates.

As outlined in this section, experimental studies aimed at estimating individuals' discounting behavior are subject to various biases and limitations. Additionally, the prevalent study designs often assume a linear utility function, leading to an overestimation of the discount rate. Despite these limitations, experimental studies have become the standard methodology for estimating discount factors or rates, even though viable alternatives exist. A different approach is to estimate discounting behavior based on recorded micro- or macroeconomic data involving intertemporal decisions. This approach helps overcome some of the limitations associated with experiments as it allows for modeling real-world intertemporal decision-making without study design biases. Depending on the available data set, this approach can also incorporate longer time horizons and decisions that involve larger monetary consequences or even intangible goods. As an example of this, in work by Viscusi and Moore (1989) the authors analyze workers' job choices. Based on the level of risk associated with a job and the corresponding salary, workers evaluate different job offers and choose the job that maximizes their lifetime utility. The job choices are derived from panel data comprising household income and employment records, while the characteristics of a job come from a job riskiness survey. The authors use a structural approach to estimate the discount factor and find an average discount rate of 11 percent. Such structural estimation approaches are often used to estimate behavioral preferences at both the micro- and the macroeconomic level. Examples encompass various areas, from explaining laundry detergent purchases to examining a country's birth rates.

The next section discusses the use of structural models as an alternative approach to estimating discounting behavior.

1.4 Estimating Discounting with Structural Models

The previous section focused on experimental approaches to estimating people's time preferences. These approaches offer valuable insights into human behavior and are frequently referenced in discussions concerning the disparity between people's actual discounting behavior and the behavior predicted by theoretical economic models. It is, however, crucial to acknowledge that experimental setups can be prone to various biases. As highlighted in Section 1.3.3, the estimated discount rate can be influenced not only by the design of the study but also by factors such as whether the experiment yields real outcomes or not, among others. One criticism of such experimental setups is their predominantly hypothetical nature, which limits their ability to accurately replicate real intertemporal decision-making scenarios. Even when the outcomes of these experiments are real, they can only replicate problems within certain predefined boundaries. Firstly, only a short time span between alternative payoffs can be tested. The largest time span in the experiments with real outcomes discussed in Section 1.3 was one year. In reality, decision problems can span several years, recurring periodically and influencing long-term outcomes. For instance, consider the decision of a young adult to pursue additional schooling, resulting in a lower income during schooling while potentially leading to higher earnings in the future, as opposed to entering the workforce immediately, which yields a higher initial income but may limit long-term income growth. Realistically replicating such extended time frames in experimental setups is challenging as few participants are willing to engage in experiments that run for more than a few weeks, which makes it difficult to capture the full scope of long-term decision-making dynamics. Secondly, in experiments typically only smaller payments are made, whereas real-life decision problems can involve monetary amounts well into the thousands. Again in the example of pursuing additional education, the difference between the median annual earnings of adults aged 25 to 34 with a Master's degree and those of adults with a Bachelor's degree in the US amounts to approximately USD 13,000 for the year 2021 (National Center for Education Statistics, 2023). Lastly, real-life decision problems often involve intangible goods that cannot be easily substituted with monetary payoffs alone. For instance, obtaining a higher degree can be perceived as a status symbol, resulting in utility beyond any potential increase in salary. In summary, estimating people's time preferences through experiments is prone to biases and limited in its capacity to replicate long time spans, recurring problems, larger outcomes, and utilities from intangible goods.

An alternative approach to estimating people's subjective beliefs about the future is to employ a structural model built upon economic data. The objective of structural estimation is to uncover the underlying deep structural parameters that drive people's behavior (Rust, 1994; Magnac and Thesmar, 2002). This involves modeling their decision-making processes as a function of unobserved structural parameters and subsequently estimating these parameters using observed data on people's choices. These structural parameters encompass various elements, including individuals' utility preferences, their beliefs regarding the evolution of certain states, and, in the case of dynamic models, their time preferences. Going back to the schooling example, one could model the decision with regard to obtaining another degree versus entering the workforce as an annual decision problem. In this model, the agent makes their annual decisions by comparing the current and expected utility from working or being in school to the current and expected utility from switching. As the decision-maker operates as a utility maximizer, they opt for the choice that offers the highest discounted utility. The structural parameters are the implicit and explicit costs and gains for each decision, beliefs about the evolution of the economic state, and the parameters of the agent's discount function. These parameters are then estimated using, for example, documented application rates of universities as an indicator for what decision was made, and information on the economic state such as the gross domestic product. See, for example, the work of Keane and Wolpin (1997) for a similar model.

While structural models theoretically have the means to estimate the discount function, they are typically not employed for such a purpose. One reason for this is the presence of an identification problem, which makes the estimation of the discount factor challenging. The following subsection offers a general outline of structural models based on dynamic discrete choice (DDC) models. The identification issue surrounding the discount factor is briefly summarized. Recent studies that address this concern are presented in Section 1.4.2.

1.4.1 Structural Model Basics

DDC models are a class of structural models that are used to examine people's decision-making processes over time. The agent's intertemporal decision problem is modeled as a function of their beliefs and preferences and available information at the time of making the decision. The aim is to retrieve the deep structure of the model by analyzing data on actual decisions and the outcome of these decisions (Rust, 1994; Magnac and Thesmar, 2002). The structural parameters of the model describe the agent's unobserved beliefs and preferences, including but not limited to the discount factor. DDC models can be used to study and explain a wide range of decision-making problems, covering areas in labor market dynamics, investment decisions, and consumer behavior. One advantage of DDC models is their ability to predict how a change in the parameters affects decision-making. This makes such models especially useful for evaluating the impact of counterfactual policies without actually implementing any policy changes. In the following, the basics of DDC models are explained using a single-agent DDC model. For more details, see, for example, works by Rust (1994), Rust (2006), Keane et al. (2011), and Aguirregabiria and Mira (2010). The following explanations draw from these sources.

In single-agent DDC models, an individual's intertemporal choice problem is approximated by discrete time intervals and for a discrete set of choice options. That is to say, over a finite or infinite time period T, at every point in time t, the agent makes a decision, d_t, between a discrete set of choices $d_t \in D$ with $D = \{0, 1, \dots N\}$. The agent evaluates each choice option at time t based on:

(A) the value of the state variables s_t,

(B) their beliefs about the evolution of the state variables or *state transition probabilities*, $p(s_{t+1} \mid d_t, s_t)$, which are also dependent on their current choices,

(C) their subjective preferences about the decision and states at time t, which are fully reflected in the agent's utility function $u(d_t, s_t)$, and

(D) their time preference, summarized in the discount function b.

The agent's decision at a point in time not only influences their immediate utility but also the evolution of the state variables and thus the expected utility. The agent's intertemporal utility function is given as

$$U(d, s) = \sum_{t=0}^{T} b(r, t) u(d_t, s_t), \tag{1.6}$$

where the decision d_t at time t leads to the utility $u(d_t, s_t)$, and s_t contains all the information about the state variables at time t. The discount function $b(r, t)$ is dependent on the discount rate r and the time period t and is usually assumed to have the exponential functional form

$$b(r, t) = \left(\frac{1}{1 + r} \right)^t. \tag{1.7}$$

Thus, discounting each period is reduced to the discount factor $\beta = \frac{1}{1+r}$.

Following the principles of the DU model, the agent is a rational utility maximizer: At every point in time, they try to make the one decision that maximizes their immediate and expected utility. Their goal is to find a decision rule, $\delta = (\delta_t, \delta_{t+1}, \dots, \delta_T)$, that maximizes the expected sum of the discounted utilities,

$$\max_{\delta} \ \mathbb{E}_\delta \left[U(d,s) \right]. \tag{1.8}$$

The decision rule is Markovian, meaning that only the state at the time of the decision matters for decision-making. Consequently, when the time horizon is finite, $T < \infty$, the agent finds the optimal decision rule, $\hat{\delta}$, with backward induction. In problems with infinite horizons, $T = \infty$, the optimal decision rule can either be approximated with the optimal decision rule for the same problem with $T \to \infty$ or—if stationarity is assumed—the optimization problem of the agent reduces to a two-stage problem. In the following, stationarity is assumed. For more detailed information on the nonstationary case, the reader is referred to the work of Rust (1994) and Rust (2006).

Assuming the decision problem is stationary implies constant state transition probabilities, a constant utility function, and an exponential discount function as described in (1.7), with a constant discount factor, $\beta \in (0,1)$. Thus, only the current state (today) and the next state (tomorrow) matter for decision-making. In the following, tomorrow's states are indicated with a prime. Under Bellman's principle of optimality, the agent's value function, $V(s)$, is defined by the recursive expression

$$V(s) = \max_{d \in D} \left[u(d,s) + \beta \int V(s') p(s'|d,s) \ \mathrm{d}s' \right]. \tag{1.9}$$

Furthermore, defining the choice-specific value function for every d as

$$v(d,s) = u(d,s) + \beta \int V(s') p(s'|d,s) \ \mathrm{d}s', \tag{1.10}$$

the agent's optimal decision rule $\hat{\delta}$ is the solution to $\arg\max_{d \in D}\{v(d,s)\}$.

From a researcher's perspective, the goal is to find the primitives that influence the agent's decision-making process. These include the state transition probabilities $p(s'|d,s)$, the parameters that define the utility function $u(d,s)$, and the discount factor β. These structural parameters are summarized into the vector θ. Finding θ requires gathering data about the agent's decision, d_t, and the observable state variables, x_t, for every point in time. Note that the state variables, $s_t = \{x_t, \epsilon_t\}$, are divided into a part that both the researcher and the agent observe, x_t, and a part that only the agent observes, ϵ_t. The latter is an unobserved state variable, also called the model error, that "rationalizes" all the deviations of the agent's decisions from those predicted by the model (Rust, 1994, p. 3101). Such deviations may arise from the inability to capture all the information influencing an agent's decision-making process. Alternatively, when introducing additional agents to the model heterogeneity among agents may also be reflected in the unobserved state variables. The relationship between the unobserved and the observed state variables is crucial when deciding on an appropriate estimation method. For example, in work by Rust (1994) eight assumptions are discussed for estimating the structural parameters of DDC models with maximum likelihood estimation. The important assumptions that define the relationship between the unobserved and the observed state variables are

Additive separability (AS): the utility function can be additively separated into the unobserved and the observed parts; that is, $u(d,s) = u(d,x) + \epsilon(d)$.

Conditional independence (CI): the evolution of the observed state variables is independent of the unobserved state variables; that is, $p(s'|d,s) = p(\epsilon'|x')p(x'|d,x)$.

Furthermore, the error term follows an IID extreme value type I distribution. These two assumptions, along with the error term's distributional form, are crucial to the estimatation of many (infinite horizon) DDC models as they enable the derivation of closed-form solutions for the conditional choice probabilities (CCPs). Firstly, the choice-specific value function can be decomposed as $v(d,s) = v(d,x) + \epsilon(d)$, where

$$v(d,x) = u(d,x) + \beta \int V(x')p(x'|d,x)\ \mathrm{d}x'. \tag{1.11}$$

The function $V(x)$ is the expected value function $V(x) \equiv \int V(x,\epsilon)p(\epsilon|x)d\epsilon$, which is a unique solution to the integrated Bellman equation

$$V(x) = \int \max_{d\in D} \left[v(d,x) + \epsilon(d)\right] p(\epsilon|x)\mathrm{d}\epsilon. \tag{1.12}$$

Replacing $V(x)$ in (1.11) with (1.12) defines the functional equation $v(d,x) = \Gamma(v)(d,x)$. The CCPs are then derived by integrating the optimal decision rule, $\arg\max_{d\in D}\{v(d,x) + \epsilon(d)\}$, with respect to the error term ϵ. Since the error term is defined to be IID and follows an extreme value type I distribution, the closed-form solution for the CCP for choice $d = n$ reads

$$P(n|x) = \frac{\exp(v(n,x))}{\sum_{d=1}^{N} \exp(v(d,x))}. \tag{1.13}$$

This allows for the definition of the likelihood function as the product of the decision and state transition probabilities. Thus, to find an optimal estimate for the structural parameters of the model, one maximizes the likelihood function,

$$L(\theta) = \prod_{t=0}^{T} P(d_t|x_t;\theta)p(x_{t+1}|d_t,x_t;\theta) \tag{1.14}$$

with respect to θ, while simultaneously solving for the fixed point of the operator equation $v(d_t,x_t) = \Gamma(v)(d_t,x_t)$.

Rust (1987) proposes using a nested fixed point (NFXP) algorithm to find the maximum likelihood estimates. For an initial guess of the structural parameters, θ, the value function is solved as a fixed point of the operator equation $v(d_t,x_t) = \Gamma(v)(d_t,x_t)$. This fixed point is used to maximize the likelihood, leading to an updated estimate for the parameters. The process iterates, with each iteration refining the guess for the parameters and finding the corresponding fixed point of the value function. The steps continue until the estimates converge to their maximum likelihood values. Another solution approach is the method of mathematical programming with equilibrium constraints (MPEC) of Su and Judd (2012). In this estimation method, $v(d_t,x_t) = \Gamma(v)(d_t,x_t)$ is defined as the constraints to the likelihood function, and the likelihood is maximized as a constrained optimization problem. Both the NFXP algorithm and MPEC are full solution approaches, which are considered computationally intensive to solve. Thus, Hotz and Miller (1993) propose solving the model by estimating the CCPs in equation (1.13) directly from the data. Their multiple-step CCP approach makes use of the fact that the difference in the CCPs only depends on the difference in the choice-specific value function. This allows for the non-parametric estimation of the CCPs in equation (1.13) as a mapping from states to the normalized choice-specific value function. The normalized choice-specific value function is defined as $\Delta v(d,x) = v(d,x) - v(1,x)$ and represents the difference between the value of choosing alternative d and the value of choosing a reference alternative (here, $d = 1$) given the state x. Assuming, as previously stated, that the distribution of the error term is IID and follows an extreme value type I distribution, consistent estimates for the normalized value function can be computed by inverting the non-parametric estimates for the CCPs, $\hat{P}(d \mid x)$,

$$\Delta\hat{v}(d,x) = \log\left(\frac{\hat{P}(d \mid x)}{\hat{P}(1 \mid x)}\right).$$

The estimated normalized value function can then be used to obtain a consistent estimate of the decision rule $\hat{\delta}$. With the estimated decision rule, knowledge of the error term distribution, and an estimate of the transition probabilities (usually derived directly from the data) one can simulate realizations for the states $\{x_t, \epsilon_t\}$ and corresponding decisions for any initializing value d_0 and x_0 from the sample. In a second step, using these simulated data sets the normalized simulated value function, $\Delta\breve{v}(d,x)$, can be computed. The best estimate for the structural parameters is the one that minimizes the deviation between the normalized simulated value function and the estimated value function, $\Delta\breve{v}(d,x) - \Delta\hat{v}(d,x)$. For a more detailed description of this estimation approach, see, for example, work by Hotz and Miller (1993) and Rust (1994). Other estimation methods that avoid explicitly maximizing the likelihood function use Bayesian approaches with Markov chain Monte Carlo simulation, and include that of Imai et al. (2009).

In general, without imposing strong restrictions on the model primitives structural models are typically non-parametrically unidentified. This implies that, for an observed decision rule, there are infinitely many sets of primitives that can explain the decision rule equally well. Formally, a DDC model is considered to be identified if the mapping $(p, u, \beta) \mapsto d$ is invertible. In empirical applications, the assumptions necessary to identify and estimate the parameters of a DDC model depend on the model itself, as well as on the estimation method intended. Most estimation approaches rely on the general assumptions of AS and CI, define the distribution of the error term, parameterize the utility function, and further assume the discount factor to be known by fixing the discount rate, for example, to the prevailing interest rate level (Rust, 1994; Magnac and Thesmar, 2002). The reason for fixing the discount factor is that the primary focus in such models has typically been on identifying and estimating the utility function. The discount function and the utility function are often only jointly identifiable as a result of their multiplicative interconnection. Thus, by assuming the discount factor to be known, the identification of the utility function is facilitated (Komarova et al., 2018). However, recent experimental studies that have demonstrated a wide range of values for the discount rate have sparked interest among structural modelers in using structural modeling to investigate the identification and estimation of the discount factor. These latest approaches to estimating the discount factor with structural models are discussed in the next subsection.

1.4.2 Recent Advances in Structural Modeling

Dynamic structural models offer a promising way of addressing the challenges and biases that emerge when estimating the discount factor through experimental setups. However, the discount factor is often not estimated in these models—despite it being one of the main primitives that represent the agent's subjective beliefs. Instead, it is often set to a predetermined value, typically reflecting the prevailing interest rate level (Magnac and Thesmar, 2002; Dubé et al., 2014). There are two reasons for this practice. Firstly, dynamic structural models are usually employed to assess the implications of counterfactual policies. In this context, the focus lies on identifying and estimating the utility parameters, as they directly represent the factors that policy interventions can manipulate to examine decision-making under these changes (Rust, 2006). The discount factor, meanwhile, is considered less crucial for policy analysis and is thus treated as a nuisance parameter. Secondly, estimating the discount factor is considered a challenging task as it is "poorly identified" (Aguirregabiria and Mira, 2010, p. 39). That is to say, without strong restrictions on the primitives any discount factor within the interval $(0, 1)$ can explain the observed states and corresponding decisions equally well. In light of the increasing evidence from experimental studies revealing significant variations in the discount factor across different situations, in recent years researchers have started redirecting their attention toward the identification and estimation of the discount factor within structural models. The following paragraphs discuss relevant contributions, each addressing specific aspects of estimating discounting with structural models. Of course, the literature extends beyond the examples provided below. Unless specified otherwise, all the models presented assume that the error distribution is known and make the assumptions of AS and CI.

An early work that examines general identification assumptions is that of Magnac and Thesmar (2002). To identify and estimate the discount factor, the authors propose so-called exclusion restrictions. These are defined as a known pair of states with the same current utility but different expected discounted utilities—that is, given a decision $d = n$, the states x_1 and x_2 with $x_1 \neq x_2$ lead to $u(n, x_1) = u(n, x_2)$, but $v(n, x_1) \neq v(n, x_2)$. Following the CCP approach of Hotz and Miller (1993), the authors show that the existence of these states allows for the point estimation of the discount factor as a function of the difference in the normalized choice-specific value functions. The rationale behind exclusion variables can be explained as follows: If an agent is completely myopic, then the existence of such an exclusion restriction should not affect their current decisions as exclusion variables only impact expected utility. Thus, the degree to which such exclusion restrictions affect current choices provides insight into the agent's discounting behavior. How such exclusion variables can be incorporated in an empirical context remains, however, an unanswered question, as the contribution of Magnac and Thesmar (2002) is of a theoretical nature. In fact, Abbring and Daljord (2020, p. 473) criticize the fact that "the existence of a pair of states that affects, in some specific way, expected discounted future utilities, but not the 'current value', [...] is a high level exclusion restriction that is difficult to interpret and hard to verify in applications". Thus, Abbring and Daljord (2020) propose an extension of the exclusion restrictions introduced in Magnac and Thesmar (2002). They suggest directly constraining the primitive utility by normalizing it to some reference utility. Using this approach, the exclusion variables impact the expected utility while having no effect on the normalized current utility. The authors further note that while the exclusion restriction of Magnac and Thesmar (2002) and their proposed exclusion restriction on primitive utility can set identify the discount factor, they do not guarantee point identification. However, when multiple exclusion restrictions are available their combination can lead to point identification. The approach adopted by Abbring and Daljord (2020) was influenced by the works of Wang (2014) and Fang and Wang (2015), which will be discussed later in this subsection. For a comprehensive understanding of the theoretical underpinnings of exclusion restrictions, readers are referred to the works of Magnac and Thesmar (2002), as well as to the extension by Abbring and Daljord (2020).

Wang (2014) imposes an exclusion restriction by allowing for variables that affect the state transition probabilities but not the current utility function. The author examines the decision as to whether to quit or continue smoking, using data from a biennial health survey conducted among approximately 7,700 households covering the years 1992 to 2010. The author models the agent's utility as a function of the choice to quit or continue smoking, their current health status, and household income. The transition of the state variables—that is, the health and income transition probabilities—is dependent not only on current health status and income but also on certain "relatively" exogenous variables (Wang, 2014, p. 3). These exogenous variables comprise biological factors such as the agent's age, gender, and race, and the longevity of their parents. The last of these variables is assumed to affect the agent's beliefs about their expected health status in the future. It does not, however, affect the agent's current choice to quit or continue smoking, and thus has no impact on current utility. Hence, parental longevity serves as the exclusion restriction, as proposed by Magnac and Thesmar (2002). Following the CCP approach of Hotz and Miller (1993), the author estimates the model for two different specifications. One where the agents are assumed to be rational and health status is based on the reported exogenous variables and one where a self-reported health variable is incorporated, the latter being considered the subjective case.

In the rational case, the estimated discount factor is found to be 0.86, while in the subjective case the discount factor is 0.98. Although the primary focus of this paper is not specifically on estimating the discount factor, it does offer a first valuable insight into how to interpret and incorporate the exclusion restriction proposed by Magnac and Thesmar (2002) within an empirical framework.

A different approach to incorporating exclusion restrictions is chosen by Chung et al. (2014). In this paper, the effect of different compensation plans on a salesperson's effort is examined. The agent's monthly utility is dependent on the effort they put into sales and on their compensation plan. The compensation plan consists of a monthly salary as the base component supplemented by either quarterly or annual bonuses. The payment of these bonuses is based on whether a certain sales quota is reached. The authors argue that the existence of bonuses provides an exclusion restriction that can be used to identify and estimate the discount factor. The bonuses are paid out at the end of the measurement period. Thus, the probability of receiving the bonuses—which is measured by how close sales are to the quota—does not affect current utility but rather expected future utilities. Secondly, the model under consideration is of a finite horizon due to the recalibration of the quotas at the end of each measurement period. Consequently, in the last decision period the salesperson's problem becomes static, which allows for the separate identification of the utility function and of the discount factor. Thus, the presence of bonuses and the finite horizon allows for point identification of the discount factor. The structural parameters are estimated with a simulation-based approach based on the work of Hotz and Miller (1993). Based on monthly sales data from 348 agents over a three-year period, the estimated discount factor has a value of 0.9 and is significant at the 5 percent level. The authors regard the estimated discount factor as relatively low for a purely monetary decision problem. However, they note that discount factors for effort-related decisions tend to fall within the lower range. The findings presented by Chung et al. (2014) underscore the significance of estimating a discount factor, even in scenarios where the predominantly monetary nature of the utility might imply a discount rate equal to the interest rate.

Similarly to Wang (2014), Fang and Wang (2015) also interpret the exclusion restrictions for identification as a variable that shifts agents' beliefs about state transition probabilities but does not affect (normalized) current utilities. However, the authors estimate the discount factor based on a quasi-hyperbolic discount function rather than on an exponential one as used by Wang (2014). In quasi-hyperbolic discounting, an additional parameter is introduced to account for the present bias when considering future utilities. That is to say, besides the traditional discount factor, future utilities are additionally multiplied by a parameter $\alpha \in (0, 1]$; see equation (1.5). In their model, the authors study women's decisions with regard to mammography screening. The data pertaining to the decision-making of around 6,500 women is derived from a biennial health survey spanning the years 1996 to 2010. The agent's choice-specific utilities are modeled as a function of household income, health, and other factors. The normalized utility function is then defined as the difference between the utility of undergoing a mammogram and the utility of not undergoing one, given the same states. The state variables can be categorized into two groups: those that are included in the normalized utility function—namely, health status and income—and those that do not contribute to the normalized utilities as they yield identical instantaneous utilities regardless of the decision. Since the latter still affect the transition of the state variables—and consequently the agent's expectations—they are considered to be the exclusion variables. There are nine distinct factors that qualify as exclusion variables, including the woman's age, ethnicity, and education. Using a simulation-based approach, the authors estimate the discounting parameters and utility parameters for different sets of exclusion variables. Depending on the set of exclusion restrictions, the estimated discount factor ranges from 0.681 to 0.947, while the present bias factor ranges from 0.508 to 0.791. Note that all of the estimated discounting parameters are statistically significant at the 1 percent significance level. Furthermore, through a counterfactual experiment the authors provide evidence that the present bias parameter significantly influences an agent's decision to undergo a mammography. This finding highlights not only the importance of estimating the discount factor but also that of exploring other discount functions than exponential ones.

De Groote and Verboven (2019) examine the influence of subsidies on the adoption of new green technologies. When analyzing policy changes regarding such technologies, it is particularly crucial to obtain an accurate estimate of the discount factor as there is often no existing reference value to rely on. New technologies are usually more expensive to begin with and exhibit relatively poor quality, but quickly decrease in price and increase in quality. Thus, for such decisions the discount factor has to account not only for the weight given to costs and benefits in the future but also for the agent's willingness to delay their adoption of the new technology until a better price-quality ratio materializes. The authors' model concerns households' adoption of solar photovoltaic systems in the region of Flanders (Belgium). The data is at a monthly frequency for the years 2006 to 2012. A household's decision whether to adopt a new technology or not depends on the quality of the technology at the time of adoption as well as the immediate investment costs and future cost savings. The authors incorporate exclusion restrictions by allowing for subsidies in the form of certificates that have future monetary benefits but do not affect immediate investment costs. Furthermore, adoption also lowers future energy costs and provides tax benefits, both of which directly influence future cash savings. Following the estimation approaches in the work of Hotz and Miller (1993) and Scott (2014), the authors find the annualized discount rate to be 15.09 percent. This is significantly higher than average mortgage rates during this time, which were between 3.6 percent and 5.3 percent. Additionally the authors examine the effect of household heterogeneity on the discount factor by dividing their data set into different local markets. At the local level, most of the discount rates are between 13 percent and 17 percent. In sum, when it comes to the adoption of new technologies households seem to underestimate future cost benefits in comparison to what an average market interest rate would suggest.

Ching and Osborne (2020) argue that in durable goods models exclusion restrictions are naturally occurring in the data. Durable goods models try to predict consumers' reactions to prices for goods that are easily storable. Usually, the price policies of such goods exhibit a high-low development. That is to say, they are low for a short period of time during promotions and then high again until the next promotion. Consumers anticipate future price drops and, since the good is storable, buy in bulk. Typically these models are estimated using a fixed discount rate equal to the prevailing interest rate. However, given that such models are often built on weekly data and encompass goods that constitute only a minor portion of household budgets, it is questionable whether consumers truly discount with an interest rate that concerns investments in more expensive, long-term goods. Thus, the authors argue for estimating the discount factor, especially considering that the storable goods market provides several exclusion restrictions that can be reasonably interpreted. Their model focuses on determining the optimal pricing strategy for laundry detergents using weekly purchase data from 312 households spanning the years 2005 to 2007. The consumer's purchasing decisions are influenced by the price of the laundry detergent and the costs associated with their current inventory level. The exclusion restrictions are directly related to the inventory. In contrast to other stockpiling models, the authors depart from representing the inventory cost as a continuously increasing function of the inventory level and instead model it as a function of the number of packages stored. For a given quantity of packages stored, the inventory cost remains constant, regardless of the actual amount of detergent left in the bottle(s). Consequently, current utility is not affected by a decreasing inventory level, while expected utility is: as the amount of detergent stored continues to diminish, the consumer's willingness to repurchase increases, particularly when there is a price promotion. In this setup, the definition of various inventory levels with constant costs provides the basis for several exclusion restrictions. By having data available for at least three different inventory levels, it becomes possible to point identify the discount factor. The authors estimate the parameters of the model with a modified Bayesian Markov chain Monte Carlo algorithm; see work by Imai et al. (2009). For 312 households, they estimate an average weekly discount factor of 0.71, which corresponds to an annual discount rate of approximately 5.43E+09 percent. This rate is notably higher than the commonly assumed interest rate, of 5 percent. However, the authors recognize a wide range in their discount factor estimates, ranging from the lower tertile, of 0.62, to the upper tertile, of 0.99. Finally, in a counterfactual analysis they show that a model with a discount factor fixed to the interest rate leads to an overprediction of sales in comparison to a model with an estimated discount factor. While acknowledging that exclusion restrictions may not be feasible for all types of data, the authors reach the conclusion that when such restrictions are available, it is crucial to estimate discount factors instead of fixing them. This is because misspecifications of the time preference have a significant impact on counterfactual analysis.

Bajari et al. (2016) propose a semiparametric approach to estimating the discount factor. In their finite horizon model, the authors analyze the decision-making process of subprime mortgage borrowers in terms of whether to default, make mortgage payments, or repay their loans. The borrower's instantaneous utility is a function of their action, as well as of specific characteristics of the mortgage including factors such as home value and monthly payments. Rather than introducing an additional variable to shift the expected utility—as, for example, done by Wang (2014)—the authors demonstrate that the discount factor can be point identified when the utility function is assumed to be time homogeneous and data from at least two consecutive periods is observed. In the context of their model, the time homogeneity of the utility function is simply defined as the per-period utility being independent of the age of the loan. Thus, the time to maturity does not affect current utility but expected value.[5] In line with the CCP approach introduced by Hotz and Miller (1993), the model estimation follows a multiple-step procedure. Based on monthly observations spanning from 2000 to the end of 2007, using a sample of approximately 11,000 30-year fixed-rate mortgages, the authors find an estimated monthly discount factor of 0.953. This monthly discount factor of 0.953 is considerably lower than the discount factor derived from the average monthly interest rate paid by the borrowers on their mortgages, which is 0.993. Fixing the discount factor to the latter value would lead to a misspecification of the model: "We demonstrate that (incorrectly) imposing a discount factor that corresponds to the market interest rate and using the resulting structural estimates for counterfactual analysis could lead to significantly biased predictions. This highlights the importance of the ability to recover the discount factor, one of the key identification results of the paper" (Bajari et al., 2016, p. 274). In work by Daljord et al. (2019), the authors confirm the findings of Bajari et al. (2016). Using the same data set and model assumptions, but employing a different estimation approach, the authors find the discount factor to be 0.982.

The papers discussed thus far have primarily employed simulation-based methods to estimate the discount factor by extending the CCP method of Hotz and Miller (1993). In contrast, Müller and Reich (2022) propose a homotopy continuation method for estimating the discount factor. This approach allows the solution and primitives of a structural model to be traced as a function of an unknown parameter—in this case, the discount factor. Such an estimation method is a full solution approach. The authors apply this method to the bus engine replacement model originally introduced by Rust (1987). In this model, the agent is the head of maintenance at the Madison (Wisconsin) Metropolitan Bus Company, and is responsible for making decisions about whether to maintain or replace the engines of each bus in a fleet based on its respective mileage state. The data covers the monthly replacement decisions and mileage states over the time span December 1974 to May 1985. In the original paper by Rust (1987), the discount factor is set at a fixed value of 0.9999. Müller and Reich (2022) estimate the discount factor to be 1.028, and this estimate shows a statistically significant difference from the original value. Moreover, their estimation approach allows the modeling of the discount factor with a structural break in its level and, further, the tracing of the specific time period when the discount factor underwent a change. The authors find that the structural break occurred around September 1979. Prior to this break, the estimated discount factor is 1.024, whereas after the break it is estimated to be 1.036. In the structural

[5] See the work of Bajari et al. (2016, p. 279 et seq.) for details and that of Daljord et al. (2019) for an intuition behind the identification assumption.

break model, the authors can also reject the hypothesis that the discount factor is equal to the original value of 0.9999. Additionally, the estimated discount factors in both models—with or without a structural break—are significantly larger than one. At first glance this result may seem counterintuitive as it implies negative time preferences. However, the authors offer an economic explanation for these estimates. They argue that during the time period of the data, the real interest rate level remained negative for an extended duration. As a result, considering the assumption that the discount rate is a positive linear function of the real interest rate their estimates for the discount factor have a reasonable economic interpretation.

To sum up this section, structural models offer an opportunity to study (recurring) intertemporal decision problems spanning long time periods, as well as to examine choices involving large monetary amounts or utilities from intangible goods. The papers discussed cover empirical applications that model intertemporal decisions across a wide range, from simpler choices like determining the amount of laundry detergent to bulk purchase to more profound decisions with life-altering consequences, such as the choice of whether to quit smoking. Consistent with the broad range of decision problems presented in these papers, the estimated discount rates also exhibit a substantial range, typically exceeding the prevailing interest rate at the time. This further underscores the common notion of estimating the discount rate within structural models rather than fixing it to a current interest rate.

Despite the advantages of structural approaches to studying human behavior, they often face criticism from the experimentalist camp of economics. Critics argue that the estimates from such models depend heavily on the numerous assumptions embedded in the structural model (Frederick et al., 2002). Indeed, to identify and estimate the discount factor, additional restrictions must be imposed on the model, which may not always be practically feasible with the data available. It is, however, important to note that discount rates derived from experimental studies are also based on multiple assumptions. Keane (2010, p. 4) argues that "atheoretical 'experimentalist' approaches do not rely on fewer or weaker assumptions than do structural approaches. The real distinction is that, in a structural approach, one's a priori assumptions about behavior must be laid out explicitly, while in an experimentalist approach, key assumptions are left implicit". Thus, dismissing the structural literature solely based on the number of assumptions made is unjustified, especially in light of recent contributions regarding the identification and estimation of the discount factor. Not only do the identifying assumptions involve sensible behavioral interpretations, the estimation results also offer valuable insights into time preferences in domains that are not readily replicable through experiments. This highlights the significance of employing both structural and experimental approaches in our attempts to advance our understanding of intertemporal decision-making processes.

1.5 Summary and Discussion

Economic models commonly assume that future utilities do not hold the same value as present utilities of the same nominal amount. The assumption that the time of the utility's realization influences its value is critical to our evaluation of the expected outcomes of decisions with a temporal dimension. Such intertemporal decision problems occur in different areas of life, and require individuals to weigh the utility of immediate or near-future outcomes against that of those in the distant future. Several factors influence the extent to which individuals give weight to future utilities. This weighting, referred to as time preferences, can be influenced by aspects such as age, economic conditions, personal preferences (such as impatience or myopia), and more. While there is no consensus on the exact reasons for time preferences, economists generally agree to summarize this weighting into a discount factor. This notion emerged from the DU model, introduced by Samuelson (1937). The DU model simplifies complex intertemporal decision-making processes by considering individuals as utility maximizers who trade off present and future utilities using a discount function. The DU model quickly gained popularity due to its simplicity and mathematical tractability. It assumes that discounting follows an exponential function with a constant discount rate and treats the discount function and the utility function as independent of the timing and the specific good(s) consumed. However, the empirical validity of the DU model has been subject to criticism. Studies have revealed inconsistencies between people's actual discounting behavior and the behavior predicted by the model. These anomalies include phenomena such as time-inconsistent and nonstationary preferences, where estimated discount rates decline as the time horizon between rewards or between the reward and the decision increases. In other words, people tend to discount utilities less strongly when they occur further in the future. Other discrepancies involve the discount rate not remaining constant for choices involving negative rewards or choices that involve high monetary gains. It is important to note that the majority of studies revealing these discrepancies were conducted within experimental settings. The design of the experiment can significantly impact the estimated rates. Furthermore, experiments are often criticized for their hypothetical nature and limited ability to replicate real decision-making scenarios, especially when dealing with larger payments, or with intangible goods that cannot be easily substituted with monetary payoffs alone. An alternative approach to estimating people's beliefs about the future is the use of a structural model. In this approach, the agent's intertemporal decision problem is modeled as a function of their beliefs and preferences and the available information regarding the state variables at the time of the decision. The aim is to examine data on real-life decisions and the resulting outcomes in order to uncover the underlying structure of the model, including the discount factor. These models have the ability to encompass longer time periods, handle recurring decisions, and address both monetary and non-monetary utilities. Although structural models theoretically have the potential to estimate the discount factor, they are often not used for this purpose. This is because the estimation of the discount factor is of less importance in such models and, in addition, poses challenges due to an identification problem. In recent years, there has been fresh interest in the field of structural modeling in investigating the identification and estimation of the discount factor. This renewed attention has led to promising results, opening up the possibility of studying discounting behavior with structural models.

The future of examining individuals' discounting behavior remains uncertain as the field is currently exploring various approaches to investigating intertemporal decision-making. The experimental camp continues to take the lead in establishing sophisticated setups to study time preferences. Methods employed by researchers such as Andersen et al. (2008) and Andreoni and Sprenger (2012) aim to distinguish the utility function from the discount function. Others are employing neuroeconomic methods to gain insights into intertemporal behavior, as seen in the work of McClure et al. (2004). Some even question the concept of time preferences, attributing discounting partially to factors such as trust issues or confusion about future rewards. Discussions on this topic can be found in the contributions of Gabaix and Laibson (2017) and Imas et al. (2022). Structural models, meanwhile, are incorporating insights from experimental studies when estimating the discount factor. Researchers such as Fang and Wang (2015) have introduced a hyperbolic discount function to account for the experimentally observed non-constancy of discount rates. No matter the approach, a common limitation of all is that they typically examine the discount factor as if it were a constant at specific decision points in time. Whether it is experimental studies that capture participants' time preferences only at the moment of each isolated decision or structural models that assume a classical exponential or hyperbolic discount function, the discount rate is treated as a constant value for each decision instance. Instead of continuing to examine discount rates as static values, it would be more compelling to investigate how discounting evolves over time. Goodin (1982) has already questioned the assumption that the discount rate remains constant over an individual's lifetime, and this is a valid concern. Why would someone have the same discount rate throughout their entire life? It seems reasonable to assume that a person becomes, over time, more myopic with regard to decision-making, given the ultimate risk factor of the future—death (Sozou and Seymour, 2003). Similarly, the opportunity cost argument of time preferences, which links discounting to the interest rate level, also suggests that discount rates may not be constant, as interest rates fluctuate. The latter reasoning for non-constant discount rates is especially intriguing given the results of Müller and Reich (2022), who demonstrate an economically reasonable connection between their estimated discount factor and real interest rate levels during the observation period. While experimental studies could be used to explore the evolution of the discount factor over time, continuous examination of participants over an extended period would be necessary. DDC models already track decisions over time, making it a natural extension to incorporate the discount factor as another state that can evolve over time. Admittedly, adding another state to a structural model presents challenges, given the inherent difficulty of estimating these models. Nonetheless, the necessary tools are available, and researchers from the structural modeling camp should consider pushing the boundaries by incorporating evolving discount factors, instead of merely adopting the latest insights from experiments.

Essay Two
Estimating Stochastic Processes for Discounting in Dynamic Structural Models

Estimating Stochastic Processes for Discounting in Dynamic Structural Models

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October 2023

Abstract

In this paper, we relax the assumption of a constant discount factor for dynamic structural models and assume discounting to be a parametric stochastic process to allow for time-variant and heterogeneous discounting. From an economic perspective, these properties are desirable. However, such a generalization makes the estimation of the parameters of the dynamic structural model computationally more challenging. The likelihood function of the model becomes a high-dimensional integral, a problem that has recently been addressed in Reich (2018) by recursively approximating this integral to high precision. Our objective is to extend one seminal dynamic discrete choice (DDC) model—the bus engine replacement model of Rust (1987)—with a time-variant discounting process. Once the process-determining parameters are estimated, we use these estimates to further determine the most likely evolution of the discounting states. By comparing this estimated process to the prevailing interest rates at the time, we find compelling evidence that in the context of the bus engine replacement model discounting is indeed associated with the interest rate.

2.1 Introduction

In dynamic discrete choice (DDC) models, the aim is to estimate an individual's unobserved preference parameters —referred to as *structural parameters*—by using data on the choices made and on the outcomes of these choices over time. Given that these decisions are made over time, DDC models must consider that an individual's utility derived from future outcomes does not carry the same weight as the immediate utility. Consequently, future utilities have to be discounted to incorporate the agent's time preferences (Mas-Colell et al., 1995). In many DDC models, discounting is commonly assumed to be a factor that remains constant over time and across different agents. However, an explicit economic rationale for this assumption is often not provided. For example, in work by Aguirregabiria and Mira (2010), the latest contributions to the literature on discrete choice modeling are discussed. The authors acknowledge the prevalent assumption of a constant discount factor, highlighting that it is often assumed to be "constant over time, e.g., hyperbolic discounting is ruled out" and further that the "discount factor is assumed constant across agents", with the remark that "in most applications this parameter is not estimated because it is poorly identified" (Aguirregabiria and Mira, 2010, p. 39). From an economic standpoint, limiting discounting to be a constant parameter for all agents and decision periods is arguable. For example, in the labor market model of Keane and Wolpin (1997) agents repeatedly decide between continuing their education and entering the workforce. It is reasonable to assume that an individual's time preference varies with changes in the economic or sociopolitical environment and differs from that of other agents. Yet the discount factor is assumed to be constant. In recent years there has been growing interest among researchers in investigating the identification of discount factors and exploring DDC models that allow for non-constant discounting. For example, Fang and Wang (2015) propose a model with hyperbolic discounting. Using data on mammography screenings, they find a significant present bias in the discount function. They emphasize the importance of estimating the discount factor and extending DDC models to include discount functions that go beyond the conventional exponential form. In a recent work by Müller and Reich (2022), the authors not only successfully identify the discount factor of a seminal DDC model but also provide compelling evidence in support of the notion that the discount factor is not constant and related to the interest rate. For the bus engine replacement model introduced by Rust (1987), the authors estimate a discount factor that exhibits a structural break. Interestingly, the timing of this structural break aligns with a date of historical importance for interest rate levels.

Building upon recent approaches that allow for non-constancy in the discount factor, we adopt a modeling framework that treats discounting as a parametric stochastic process.[1] More precisely, we allow discounting to follow an autoregressive process of order one, AR(1). Modeling discounting as an AR(1) process has some economically favorable traits. Such a process models an individual's time preference in a persistent way. That is to say, when the agent's discount factor is high today, it is likely to be high tomorrow too, and vice versa. Further, an AR(1) process is fully defined by its mean, variance, and persistence. This limits the number of process parameters to be estimated to three. While discounting as a process comes with many desirable characteristics, it also has one major inconvenience: From the agent's perspective, discounting becomes an additional state variable that has to be considered when forming expectations about the future. As this state variable is unobserved by the modeler, modeling discounting as a process introduces a potentially large integral to the likelihood function. Reich (2018) introduces the recursive likelihood integration (RLI) method as a numerical approximation technique for addressing such integrals. The RLI method breaks down high-dimensional integrals into a series of lower-dimensional integrals and interpolation problems, which can then be calculated through recursive approximation. We apply the approach of modeling discounting as an AR(1) process to the bus

[1] Non-constant discount factors have been pursued in finance using more standard methodological approaches such as simulated method of moments. See, for example, the work of Albuquerque et al. (2016).

engine replacement model introduced by Rust (1987), and estimate the distributional parameters of the AR(1) process—as well as the remaining structural parameters of the model—using maximum likelihood. Once the distributional parameters of the discounting process have been identified, we can estimate the most likely realizations of the process and compare them to the economic conditions during the time period covered by the data. More specifically, we examine the connection between the discounting process and interest rates, a relationship previously highlighted by Müller and Reich (2022). Generally, our approach to modeling discounting can be applied to both single-agent and multiple-agent models. In the case of a multiple-agent model, individuals would have homogeneous beliefs about the distributional parameters of the discounting process, but the evolution of the process could vary across agents. This allows for the integration of heterogeneous discounting without any additional costs in comparison to a single-agent model.

In the following section, Section 2.2, we explain in detail the implementation of stochastic discounting and how we use RLI to deal with the inferred large integral in the likelihood function. We present our approach to modeling discounting as a process, using the seminal bus engine replacement model of Rust (1987). After verifying the efficiency of our estimation method with a simulation study in Section 2.3, the maximum likelihood results are presented in Section 2.4. In Section 2.5, we elaborate on the methodology employed to use the estimated discounting parameters for estimating the most likely discounting states in each period. The concluding discussion, in Section 2.6, draws a comparison between the estimated discounting process and real interest rate levels during the observation period.

2.2 Idea and Methodology: Stochastic Discounting Process

While the assumption of a constant discount factor is prevalent in many DDC models due to its computational convenience, its economic implications are questionable. Regardless of the economic or sociopolitical climate at the time of the decision, the agent's time preference remains the same for all decision periods. By relaxing the assumption of a constant discount factor and allowing it to be a stochastic discounting process, we can enhance the model's ability to capture the complex dynamics inherent in intertemporal decision-making. On the downside, introducing an unobserved serially correlated state variable to the model extends the likelihood function by a potentially large integral, the dimension of which is proportional to the time horizon of the data set. One approach to maximizing the likelihood function would be to simulate its outcome using Monte Carlo. However, Monte Carlo simulations have low convergence rates and, in addition, introduce simulation noise into the likelihood function, making it harder to optimize over (Reich, 2018). Thus, Reich (2018) proposes a method with which it is possible to split up integrals of high dimensionality into a series of low-dimensional integrals and interpolation problems. This allows us to compute the integral in the likelihood function by recursive approximation at potentially high convergence rates. In the following subsections, we first explain the original bus engine replacement model of Rust (1987) to establish a common notation and provide an outline of the model framework. The model is then adjusted to incorporate a generic stochastic discounting process. For the empirical work, we then define discounting as an AR(1) process.

2.2.1 The Bus Engine Replacement Model

In the bus engine replacement model of Rust (1987), the agent—Harold Zurcher, superintendent of maintenance at the Madison (Wisconsin) Metropolitan Bus Company—inspects a fleet of buses every month and has to decide for each bus whether regular maintenance or replacement of the engine should be carried out. His decision is based on the bus's age, measured by its odometer reading, and other information about its condition. If Zurcher decides to perform regular maintenance, which includes tasks such as repairing components and adjusting brakes, the costs that occur mainly depend on the mileage of the bus at the time of inspection. If the decision is to replace the engine, the bus undergoes a complete overhaul. In this case, the costs are factually fixed and thus independent of the bus's mileage. After replacement, the bus is considered to be new, and the odometer reading is reset to zero. Generally, the immediate cost of regular maintenance is lower than that of a complete overhaul. It results, however, in the bus accumulating more mileage by the next check, resulting in potentially higher maintenance costs in the next period. Replacement, meanwhile, is initially more expensive, but—since the odometer is reset to zero—the mileage traveled until the next check is usually lower, leading to lower maintenance costs in the next period. This intertemporal trade-off, which Zurcher faces each period when making his decision, is described in the following in more detail.

Formally, Zurcher's decision each period t leads to monthly costs, or negative utility, and can be summarized as

$$U_\theta(i_t, x_t) + \epsilon_t(i_t) \text{ with } U_\theta(i_t, x_t) = \begin{cases} -RC & \text{if } i_t = 1 \\ -\theta_1 x_t & \text{if } i_t = 0 \end{cases}, \tag{2.1}$$

where i_t denotes the decision at month t of replacing the engine ($i_t = 1$) or not ($i_t = 0$), x_t is the mileage state based on the odometer reading, RC are the fixed costs of replacement, θ_1 is the cost parameter of regular maintenance, and $\epsilon_t(i_t)$ is a choice-specific random utility component. The model's dependence on structural parameters is denoted by θ. In Rust (1987), ϵ_t is assumed to be IID and follows an extreme value type I distribution. The decision of replacement or regular maintenance not only influences the immediate costs that Zurcher faces every month but also the expected costs in the future. As the decision to replace the engine sets the mileage state back to zero, in the next period the accumulated mileage will be lower than if the decision were to perform regular maintenance. With regular maintenance, the mileage continues to accrue until the next check. Thus, both decisions affect the future mileage states and, consequently, the expected costs in the future. Under the assumption that Zurcher's decisions are dynamically optimal, he tries to minimize the expected sum of his costs—or, to put it differently, maximize the sum of his expected utility—for an infinite time horizon by finding the optimal decision for each time period. The value function reads

$$V_\theta(x_t, \epsilon_t) = \sup_\pi \mathbb{E}\left[\sum_{j=t}^{\infty} \beta^{j-t} U_\theta(\pi(x_j, \epsilon_j), x_j) + \epsilon_j(\pi(x_j, \epsilon_j)) \middle| x_t, \epsilon_t; \theta\right]. \tag{2.2}$$

Following the notation of Rust (1987), the decision rule $\pi : x, \epsilon \mapsto i$ maps the mileage states and error to Zurcher's decisions. Further, β is the factor that is used to discount the expected utilities to present values. Under the condition that $\beta \in (0, 1)$, Bellman's principle of optimality can be used to form Zurcher's optimal decision policy over time:

$$V_\theta(x_t, \epsilon_t) = \max_{i \in \{0,1\}} \{U_\theta(i, x_t) + \epsilon_t(i) + \beta \mathbb{E}[V_\theta(x_{t+1}, \epsilon_{t+1}) \mid i, x_t; \theta]\}. \tag{2.3}$$

Note that the *conditional independence* assumption holds, meaning that the mileage state is conditionally independent of the error state. Thus, the conditionality of the value function on the error term can be omitted. The conditional expectation over future values in the Bellman equation (2.3) is defined as

$$EV_\theta(i_t, x_t) \equiv \mathbb{E}[V_\theta(x_{t+1}, \epsilon_{t+1}) \mid i_t, x_t; \theta] \tag{2.4}$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} V_\theta(x_{t+1}, \epsilon_{t+1}) p_{x\epsilon}(x_{t+1}, \epsilon_{t+1} \mid i_t, x_t; \theta) dx_{t+1} d\epsilon_{t+1}. \tag{2.5}$$

As the conditional independence assumption holds, the joint probability $p_{x\epsilon}(x_{t+1}, \epsilon_{t+1} \mid i_t, x_t; \theta)$ factors to

$$p_{x\epsilon}(x_{t+1}, \epsilon_{t+1} \mid i_t, x_t; \theta) = p_{\epsilon|x}(\epsilon_{t+1} \mid x_{t+1}; \theta) p_x(x_{t+1} \mid i_t, x_t; \theta) \tag{2.6}$$

with $p_x(x_{t+1} \mid i_t, x_t; \theta)$ denoting the so-called mileage transition probabilities—that is, the probability of a bus traveling a specific mileage until the next check at time $t+1$ given the mileage state and decision at time t. With this, the expected value function can be rewritten as

$$EV_\theta(i_t, x_t) = \int_{-\infty}^{\infty} \int_0^{\infty} V_\theta(x_{t+1}, \epsilon_{t+1}) p_{\epsilon|x}(\epsilon_{t+1} \mid x_{t+1}; \theta) p_x(x_{t+1} \mid i_t, x_t; \theta) dx_{t+1} d\epsilon_{t+1} \tag{2.7}$$

$$= \int_0^{\infty} \log \sum_{j \in \{0,1\}} \exp \left[U(j, x_{t+1}) + \beta EV_\theta(j, x_{t+1}) \right] p_x(x_{t+1} \mid i_t, x_t; \theta) dx_{t+1}. \tag{2.8}$$

Equation (2.8) follows from the error term being IID and of extreme value type I, which allows for a closed-form solution for the integral over ϵ_{t+1}. Further, $EV_\theta(i_t, x_t)$ defines a fixed point of the non-linear operator T such that the contraction mapping

$$EV_\theta(i_t, x_t) = T(EV_\theta)(i_t, x_t) \tag{2.9}$$

holds. In the original model by Rust (1987), mileage states are discretized into 90 bins, each spanning 5,000 miles, with a maximum mileage state of 450,000 miles; that is, $x_t \in \{1, \ldots, 90\}$. Within each period, a bus can change its mileage bin by a maximum of two bins. Therefore, the discretized mileage transition probabilities can be summarized by the vector $\theta_3 \equiv (\theta_{30}, \theta_{31}, \theta_{32})$ and follow the Markov process

$$\theta_{3\Delta_x} = Pr\left(x_{t+1} = \Delta_x + (1 - i_t) x_t \mid i_t, x_t; \theta\right) \text{ with } \Delta_x \in \{0, 1, 2\}. \tag{2.10}$$

The agent, Zurcher, observes each period the mileage state of the buses, x_t, and the error, ϵ_t, and knows his cost parameters, RC and θ_1. We the modelers, on the other hand, only observe the mileage state, x_t, and the agent's decision, i_t, every period. Thus, our objective is to estimate the parameter $\theta = \{RC, \theta_1, \theta_3\}$. Estimating θ can be done by maximizing the likelihood of observing the history of the buses' mileage states and Zurcher's replacement decisions for each period. The likelihood for a fleet of buses of size N is

$$L(\theta) = \prod_{t=1}^{T-1} \prod_{n=1}^{N} Pr(x_{t+1,n} \mid i_{t,n}, x_{t,n}; \theta)\; Pr(i_{t+1,n} \mid x_{t+1,n}; \theta). \tag{2.11}$$

In this section, as well as in the following sections, we omit the likelihood's dependence on the states for brevity in notation. Note that the likelihood function can be separated into likelihood contributions for each bus $n \in \{1, \ldots, N\}$. The transition probabilities θ_3 can be directly inferred from the mileage records of the buses. For the decision probability $Pr(i_{t,n} \mid x_{t,n}; \theta)$, a closed-form solution follows from the extreme value type I error distribution

$$Pr(i_{t,n} \mid x_{t,n}; \theta) = \frac{\exp\left[U_\theta(i_{t,n}, x_{t,n}) + \beta EV_\theta(i_{t,n}, x_{t,n})\right]}{\sum_{j \in \{0,1\}} \exp\left[U_\theta(j, x_{t,n}) + \beta EV_\theta(j, x_{t,n})\right]}. \tag{2.12}$$

Rust (1987) estimates the parameter vector θ by maximizing the likelihood function with the nested fixed point algorithm (NFXP). Alternatively, mathematical programming with equilibrium constraints (MPEC) can be used; see the contribution of Su and Judd (2012).

2.2.2 Adjusting the Original Model

In the original model, the discount factor is not part of the parameters to be estimated since it is assumed to have a fixed and constant value of 0.9999. While constant discount factors are a ubiquitous assumption in many DDC models due to their numerical tractability, from an economic and behavioral standpoint such an assumption has its limitations. In the bus engine replacement model, a constant discount factor implies that Zurcher's time preference remains constant across periods, irrespective of the economic or sociopolitical environment. To overcome this limitation, we now relax the assumption of β being a constant factor and assume discounting to be a time-dependent process, $\{\beta_t\}$. Introducing time-dependent discounting allows us to account for variations in Zurcher's subjective valuation of future outcomes. This favorable trait comes with the downside of our having to adjust the original model for an additional unobserved state variable, β_t, which introduces an integral to the likelihood function (2.11). This subsection explains these adjustments for a generic discounting process, where the only assumption the process has to fulfill is that it follows the Markov property. For our empirical analysis, we then adopt an AR(1) process to model discounting.

Zurcher's decision-making process at time t still takes into account the current cost of his decision and the expected costs in the future. While the instantaneous costs from maintenance or replacement are defined by the same utility function in (2.1), with discounting as a process the expected future costs now depend not only on the future mileage states but also on the discounting state in the future. Consequently, Zurcher needs to anticipate both the future mileage states of the buses and the future discounting states when making his decision. He therefore aims to find the decision rule $\pi : x, \epsilon, \beta \mapsto i$ that maximizes his expected utility over an infinite time period. The adjusted value function reads

$$V_\theta(x_t, \epsilon_t, \beta_t) = \sup_\pi \mathbb{E}\left[\sum_{j=t}^{\infty} \beta_j^{j-t} U_\theta(\pi(x_j, \epsilon_j, \beta_j), x_j) + \epsilon_j(\pi(x_j, \epsilon_j, \beta_j)) \middle| x_t, \epsilon_t, \beta_t; \theta\right]. \tag{2.13}$$

With a constant discount factor in the interval $(0, 1)$, Bellman's principle of optimality can be used to form Zurcher's optimal decision policy over time. Stachurski and Zhang (2021) show that for dynamic programming models with stochastic discounting the Bellman equation remains a sufficient rule for the agent to behave dynamically optimally as long as *eventual discounting* holds. That is to say, $\{\beta_t\}$ is a measurable and bounded Markov process with a stochastic mean smaller than one. Thus, following Stachurski and Zhang (2021), the sufficient optimality condition for (2.13) reads

$$V_\theta(x_t, \epsilon_t, \beta_t) = \max_{i \in \{0,1\}} \{U_\theta(i, x_t) + \epsilon_t(i) + \beta_t \mathbb{E}[V_\theta(x_{t+1}, \epsilon_{t+1}, \beta_{t+1}) \mid i, x_t, \beta_t; \theta]\}. \tag{2.14}$$

As in the original model, the conditional independence assumption holds, and the conditionality of the value function on the error term can be omitted. The expected value function is defined as

$$EV_\theta(i_t, x_t, \beta_t) \equiv \mathbb{E}[V_\theta(x_{t+1}, \epsilon_{t+1}, \beta_{t+1}) \mid i_t, x_t, \beta_t; \theta] \tag{2.15}$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} \int V_\theta(x_{t+1}, \epsilon_{t+1}, \beta_{t+1}) p_{\epsilon|x}(\epsilon_{t+1} \mid x_{t+1}; \theta) p_x(x_{t+1} \mid i_t, x_t; \theta) p_\beta(\beta_{t+1} \mid \beta_t; \theta) d\beta_{t+1} dx_{t+1} d\epsilon_{t+1} \tag{2.16}$$

$$= \int_0^{\infty} \int \log \sum_{j \in \{0,1\}} \exp\left[U(j, x_{t+1}) + \beta_{t+1} EV_\theta(j, x_{t+1}, \beta_{t+1})\right] p_x(x_{t+1} \mid i_t, x_t; \theta) p_\beta(\beta_{t+1} \mid \beta_t; \theta) d\beta_{t+1} dx_{t+1}, \tag{2.17}$$

where $p_\beta(\beta_{t+1} \mid \beta_t; \theta)$ denotes the (conditional) probability function of the discounting process and, as previously, $p_x(x_{t+1} \mid i_t, x_t; \theta)$ denotes the mileage transition probabilities. Note that we omit the integration limits of the discounting process for the moment. Equation (2.16), above, follows from the conditional independence assumption and the assumption that the values of the discounting process are fully independent of the mileage states and the error term, respectively. Further, since the error term is IID and has an extreme value type I distribution, a closed-form solution for the integral over ϵ_t exists, such that (2.17) can be defined. Finally, the expected value function defines the solution to the functional equation $EV_\theta(i_t, x_t, \beta_t) = T(EV_\theta)(i_t, x_t, \beta_t)$.

In the original bus engine replacement model, Zurcher knows the cost parameters RC and θ_1 and his discount factor, β, and observes the mileage states x_t as well as the random utility component ϵ_t. Based on these values, he makes a decision, i_t. With discounting as a process, we assume that he also observes the value of β_t at time t and that he knows the distributional parameters of the discounting process in order to form an expectation for the next period's discounting state. We the modelers, on the other hand, only observe Zurcher's decisions, i_t, and the mileage of the buses each period, x_t. The cost parameters and, particularly, the realized values of the discounting process and its distributional parameters are unknown to us. Consequently, the cost parameters, the mileage transition probabilities, and, in comparison to the original model, the distributional parameters of the discounting process have to be estimated. The parameter of interest is $\theta = \{RC, \theta_1, \theta_3, B\}$, where θ_3 are the discretized mileage transition probabilities according to (2.10), and B denotes a set of distributional parameters that describe the discounting process.

2.2.3 Estimation of the Structural Parameters

We use maximum likelihood to estimate the parameter vector θ. The unobservability of the discounting process introduces an integral over the discounting states β_t to the likelihood function. We can separate the likelihood function into two parts, denoted as $L_1(\theta)$ and $L_2(\theta)$. In the following we will refer to $L_1(\theta)$ as the *mileage transition part* of the likelihood and $L_2(\theta)$ as the *discounting part* of the likelihood. The likelihood for a set of N buses reads

$$L(\theta) = L_1(\theta) L_2(\theta) \tag{2.18}$$

with

$$L_1(\theta) = \prod_{t=1}^{T-1} \prod_{n=1}^{N} Pr(x_{t+1,n} \mid i_{t,n}, x_{t,n}; \theta) \tag{2.19}$$

and

$$\begin{aligned} L_2(\theta) = \int \cdots \int & \left[\prod_{n=1}^{N} Pr(i_{1,n} \mid x_{1,n}, \beta_1; \theta) \right] p_\beta(\beta_1; \theta) \\ \cdot \prod_{t=1}^{T-1} & \left[\prod_{n=1}^{N} Pr(i_{t+1,n} \mid x_{t+1,n}, \beta_{t+1}; \theta) \right] p_\beta(\beta_{t+1} \mid \beta_t; \theta)\, d\beta_1 \ldots d\beta_T, \end{aligned} \tag{2.20}$$

where the conditional decision probabilities are computed analogously to the decision probabilities in the original model as

$$Pr(i_{t,n} \mid x_{t,n}, \beta_t; \theta) = \frac{\exp\left[U_\theta(i_{t,n}, x_{t,n}) + \beta_t EV_\theta(i_{t,n}, x_{t,n}, \beta_t)\right]}{\sum_{j \in \{0,1\}} \exp\left[U_\theta(j, x_{t,n}) + \beta_t EV_\theta(j, x_{t,n}, \beta_t)\right]}. \tag{2.21}$$

Note that the discounting process $\{\beta_t\}$ is not indexed by n since its state at time t affects all buses that are being checked in this period simultaneously. Thus, the likelihood function (2.18) cannot be split up into separate likelihood contributions for each bus as it could in the original model. In every period, the integral over the next period's discounting state is built. Depending on the time horizon, the integrals over the discounting state can potentially be quite large and thus computationally challenging to calculate. However, the RLI method of

Reich (2018) allows us to separate these integrals into a sequence of low-dimensional integrals and interpolation problems. To briefly explain the RLI algorithm for our specific case, consider only the *last sub-integral* in the likelihood part $L_2(\theta)$, the integral over β_T, which is

$$\int \dots \left[\prod_{n=1}^{N} Pr(i_{T,n} \mid x_{T,n}, \beta_T; \theta) \right] p_\beta(\beta_T \mid \beta_{T-1}; \theta) d\beta_T.$$

With RLI this last sub-integral is calculated for a set of possible values of the discounting process in period $(T-1)$, β_{T-1}, and the results are defined by the function $f_{T-1}(\beta_{T-1})$. Then, in a second step, the previous-to-last integral—that is, the integral over the discounting states in $(T-1)$—

$$\int \dots \left[\prod_{n=1}^{N} Pr(i_{T-1,n} \mid x_{T-1,n}, \beta_{T-1}; \theta) \right] p_\beta(\beta_{T-1} \mid \beta_{T-2}; \theta) f_{T-1}(\beta_{T-1}) d\beta_{T-1}$$

is calculated for a set of possible β_{T-2} values, and the results are defined by the function $f_{T-2}(\beta_{T-2})$, and so on. The algorithm recursively approximates the integrals until we can summarize our original integrals with the function $f_1(\beta_1)$. Formally this process is described by the recurrence relation

$$f_t(\beta_t) = \begin{cases} 1, & \text{if } t \geq T \\ \int f_{t+1}(\beta_{t+1}) p_\beta(\beta_{t+1} \mid \beta_t; \theta) \prod_{n=1}^{N} Pr(i_{t+1,n} \mid x_{t+1,n}, \beta_{t+1}; \theta) d\beta_{t+1} & \text{if } 0 < t < T \end{cases}. \tag{2.22}$$

Thus, the discounting part of the likelihood function (2.20) takes the form

$$L_2(\theta) = \int f_1(\beta_1) p_\beta(\beta_1; \theta) \prod_{n=1}^{N} Pr(i_{1,n} \mid x_{1,n}, \beta_1; \theta) d\beta_1 \tag{2.23}$$

with $f_1(\beta_1)$ according to the recurrence relation (2.22). With RLI the discounting process $\{\beta_t\}$ can now efficiently be integrated out. The integral in the recurrence relation can be approximated, for example, with Gauss quadrature rule, and the function $f_t(\beta_t)$ is approximated by an interpolation function I_{f_t} with finitely many evaluations of $f_t(\beta_t)$.

Before carrying out the empirical work in subsequent sections we first have to define a discounting process. We assume β_t to follow a stationary AR(1) process. An AR(1) process limits the number of discounting parameters to be estimated to three and allows us to model the time-dependent conditional mean in a simple manner. Our AR(1) process has the form

$$\beta_{t+1} = \mu_{\beta_1}(1-\rho) + \rho\beta_t + \varepsilon_t \text{ with } \varepsilon_t \sim N(0, \sigma_\varepsilon^2), \tag{2.24}$$

where μ_{β_1} is the stationary mean, and ρ denotes a parameter for the persistence of the process with $|\rho| < 1$ to ensure stationarity. The innovations ε_t follow a Gaussian white noise process with mean zero and variance σ_ε^2. The conditional probability distribution function in (2.22), $p_\beta(\beta_{t+1} \mid \beta_t; \theta)$, follows

$$N\left(\mu_{\beta_1}(1-\rho) + \rho\beta_t, \sigma_\varepsilon^2\right), \tag{2.25}$$

and the stationary probability distribution function in (2.23), $p_\beta(\beta_1; \theta)$, follows

$$N\left(\mu_{\beta_1}, \frac{\sigma_\varepsilon^2}{1-\rho^2}\right). \tag{2.26}$$

In summary, the parameters that describe the distribution of this AR(1) process—and which we want to estimate—are $B = \{\mu_{\beta_1}, \rho, \sigma_\varepsilon\}$. Thus, the parameter of interest reads $\theta = \{RC, \theta_1, \theta_3, \mu_{\beta_1}, \rho, \sigma_\varepsilon\}$. We use maximum likelihood to estimate the parameter vector θ and *relative values* to solve the expected value function (White, 1963; Müller and Reich, 2022). More details regarding the relative value approach can be found in Appendix 2.A.2.

Before presenting and analyzing the results, we want to ensure that our model is able to retrieve the true parameters. Thus, the following section summarizes the outcomes of a simulation study.

2.3 Simulation Study

To validate whether the estimation algorithm proposed in Section 2.2.3 can retrieve the true parameters, we conduct a simulation study. In a first step, we use Monte Carlo simulation to generate 600 data sets for the mileage states and decisions. Each of these data sets contains the mileage records of 200 buses throughout a lifespan of 400 months, along with the corresponding simulated monthly decisions. We decide that the size of the bus fleet should be within the range of the original data set in the work of Rust (1987), and the relatively long lifespan of the buses should guarantee an adequate number of replacement decisions. The true cost parameters and mileage transition probabilities are derived from the original model for (homogeneous) bus groups 1 to 4 of Rust (1987). For the discounting process we opt for a relatively stable process with a small yet positive level of variation. As noted by Rust (1987), the discount factor tends to approach one. Despite our algorithm being able to deal with discounting states greater than one, for the purpose of this simulation study we want to ensure that the simulated discounting states remain within the conventional bounds of $(0, 1)$ with high probability. The true stationary distribution of the discounting process follows $N(0.95, 1.0\text{E-}4)$, and the persistence parameter is set to 0.95. The data sets simulated, in a second step we use the maximum likelihood approach outlined in Section 2.2.3 to estimate the parameter vector θ. The parameters are estimated with MPEC. For the interpolation function I_{f_t}, we use cubic splines with 11 interpolation points, and to approximate the integral over the discounting states in both the likelihood function and the expected value function we employ Gauss–Hermite quadrature with 7 nodes. Out of the total of 600 simulated data sets, a solution within the specified constraint tolerances was successfully found for 459. Table 2.1 presents a statistical summary of the estimated parameters, including their true values. Additionally, Figure 2.1 displays kernel smoothing density plots as a solid blue line and the normal density plots of the estimated values as a dotted blue line. The density function values are on the left vertical axis. The red plot is a vertical box plot, placed on the right vertical axis. The true values are depicted by a vertical black line. Note that we exclude the mileage transition probabilities as the estimation focuses on the cost and discounting parameters.[2]

	True	Mean	Median	SD
RC	9.7558E+00	1.0349E+01	9.8605E+00	2.0754E+00
θ_1	2.6275E+00	8.9706E+00	2.6736E+00	1.3797E+01
μ_{β_1}	9.5000E−01	7.9830E−01	9.5020E−01	3.3202E−01
σ_ε	3.1225E−03	1.9733E−02	6.4580E−03	3.2134E−02
ρ	9.5000E−01	8.0536E−01	9.1543E−01	3.0991E−01

Table 2.1: Statistical summary of the estimated parameters from the 459 successful solver runs, including the true value, mean, median, and standard deviation (SD). Source: Own calculation.

[2]The mileage transition probabilities can be estimated independently of the cost and discounting parameters and are also independent of the serial correlation in the unobserved state variables. Therefore, θ_3 is treated as a nuisance parameter.

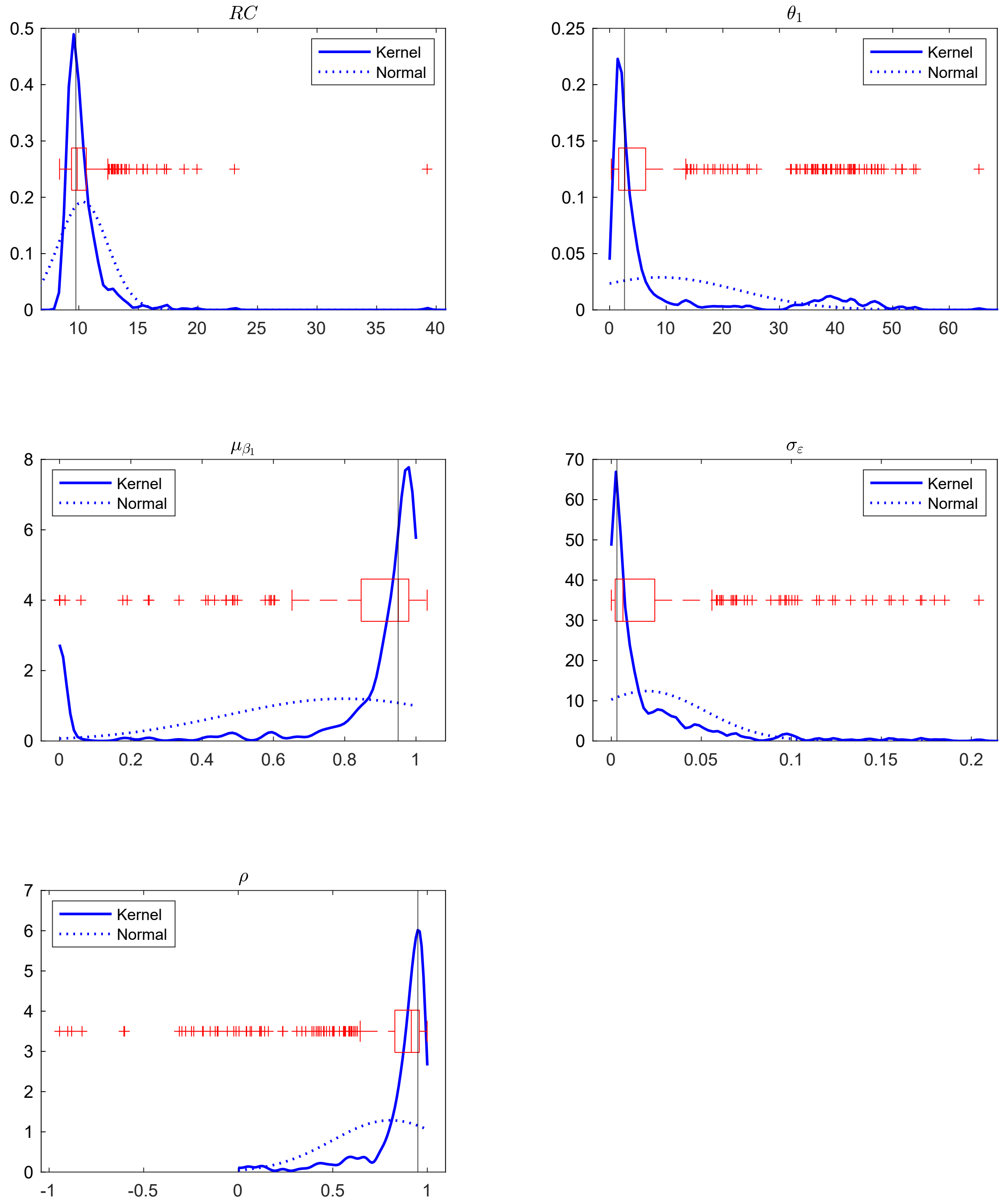

Figure 2.1: Distribution overview of the estimated parameters from the 459 successful solver runs. The kernel density plots are depicted by the solid blue line and the normal density plots by the dotted blue line; the box plot is in red. The vertical black line marks the true value. Source: Own representation.

Based on the results of our simulation study, we draw the following conclusions: Firstly, our estimation algorithm has demonstrated consistent and reliable performance. Of the 600 simulated data sets, we have successfully obtained solutions within the specified constraint tolerances for 459. Of the 141 solver runs that did not converge, approximately one-third ran out of iterations, while the rest did not meet the specified constraint tolerances. Secondly, our estimation algorithm accurately retrieves the true parameter values. The kernel density plots indicate that a significant portion of the estimates cluster closely around their true values. The only exception is the stationary mean of the discounting process, which shows an additional mass at zero. This is likely due to zero being the lower bound of the stationary mean. In maximum likelihood estimation there is a possibility of local maxima occurring at the boundaries of the parameter space (Pawitan, 2013). Upon examining the statistical summary, we observe that both cost parameters are overestimated, with the mean of the estimated maintenance cost parameters being more than three times greater than their true value. As for the discounting parameters, the stationary mean and the persistence are slightly underestimated, while the standard deviation of the innovations is overestimated. All parameters exhibit relatively large standard deviations, which is also reflected in the flat normal density plots. However, the statistical summary and the box plots reveal that the median of all parameter estimates is close to the true values. Additionally, the interquartile ranges are relatively narrow. Thus, we attribute the large standard deviations to a few outliers.

In conclusion, despite the high standard deviations the concentration of estimates around their true values, as depicted in the kernel density plots, demonstrates the effectiveness of the estimation algorithm in accurately retrieving the true parameters. Only the estimated stationary means show an additional peak near their lower boundary. This observation highlights that estimates near the boundaries of the parameter space may be subject to biases and thus should be interpreted with caution.

2.4 Application and Results

The simulation study has provided reassuring evidence that our estimation method is capable of retrieving the true values of the structural parameters. In this section, we present the estimation results for the original data set introduced by Rust (1987). We start with an overview of the data and highlight some distinctive features that are crucial for the estimation process. The estimation results are presented in Section 2.4.2. In Section 2.4.3 we further examine the robustness of our estimation results with regard to the chosen level of accuracy.

2.4.1 Data and Estimation Technicalities

The data set used by Rust (1987) comprises monthly bus fleet data for the Madison (Wisconsin) Metropolitan Bus Company, spanning from December 1974 to May 1985. For each month, the data set contains information about all buses that underwent inspection during that month. This includes details regarding the bus's identification number, the odometer reading at the time of inspection, and Zurcher's decision regarding the type of maintenance operation that needs to be performed. These maintenance operations can be categorized into three types: **(A)** general maintenance, such as brake adjustments, **(B)** replacement or fixing of certain parts, and **(C)** the complete overhaul or replacement of the engine. Rust (1987) considers the first two types of maintenance operations to be *normal maintenance*, and they are denoted with the decision variable $i_t = 0$. The last category, which involves the complete overhaul or replacement of the engine, is considered to be a fresh start or *replacement* and is assigned a decision variable of $i_t = 1$. During the ten-year period covered by the data, a total of 162 buses were recorded, including some that were purchased after December 1974. These buses can be categorized into eight groups, G_m with $m \in \{1, \dots, 8\}$, based on factors such as bus model, engine type, and time of purchase. The eight groups can be further divided into three partitions, as per the classification used by Rust (1987) and Müller and Reich (2022). Partition 1 comprises bus groups 1 to 3. Partition 2 includes buses of model 5308A and engine type 8V71, which corresponds to bus groups 4, 5, and 7. Finally, partition 3 consists of buses from groups 6 and 8, which share the model 4523A and engine type 6V71. To summarize, the three partitions are $P_1 = \{G_1, G_2, G_3\}$, $P_2 = \{G_4, G_5, G_7\}$, and $P_3 = \{G_6, G_8\}$. Table 2.2 provides an overview of the bus types, groups, and partitions.

Partition	Group	No. of Buses	Manufacturer	Model	Engine	Year
1	1	15	Grumman	870	V6-92 series	1983
1	2	4	Chance	RT-50	3208 CAT	1981
1	3	48	GMC	T8H203	8V71	1979
2	4	37	GMC	5308A	8V71	1975
2	5	12	GMC	5308A	8V71	1974
2	7	18	GMC	5308A	8V71	1972
3	6	10	GMC	4523A	6V71	1974
3	8	18	GMC	4523A	6V71	1972

Table 2.2: Overview of the bus partitions, groups, number of buses in each group, and types of bus, including manufacturer, bus model, engine type, and year of purchase. Source: Own representation based on work by Rust (1987) and Müller and Reich (2022).

The focus in Rust (1987) is placed on investigating and estimating the parameters of bus groups 1 to 4, which joined the fleet after December 1974. Thus, these buses either had no engine replacement or not enough replacement decisions made in the first years of the sampling period. Bus groups 1 to 4 saw no engine replacements in the years 1975 and 1976, and only one and two in the years 1977 and 1978, respectively. Identification of the structural parameters relies on having a sufficient number of replacement decisions (Rust, 1987; Müller and Reich, 2022). Therefore, to ensure identification, we used the complete data set and estimated the model by including data from all bus groups, 1 to 8. Given the similarity in driving schedules within each group, it is reasonable to assume that the mileages of the buses within a specific group accumulate according to the same underlying probability function. In other words, we restricted the mileage transition probabilities of a bus in a specific group to be equal to those of the other buses in the same group. Furthermore, the buses within a particular partition have the same or similar engine type, which leads to similar operational costs. Hence, within a single partition all the buses belonging to that partition are assumed to share the same cost parameters. Consequently, we had to estimate eight different transition probability vectors, θ_{3,G_m} with $m \in \{1, \ldots, 8\}$, and six different cost parameters, denoted as RC_{P_n} and θ_{1,P_n} with $n \in \{1, 2, 3\}$.

We end this subsection with additional details regarding the actual estimation process and some computational technicalities: Firstly, the fact that there is only one discounting state per period introduces some complexity to the system of equations that needs to be solved for the expected value. In the original model, a separate Bellman equation can be defined for each bus group, which allows the likelihood to be divided into individual *likelihood contributions* for each group. The full likelihood is computed by multiplying the individually maximized likelihoods, or in the case of log-likelihoods, by summing them up. In our case, such a separation into likelihood contributions is not possible. The discounting state for a specific period is shared by all bus groups and partitions. Thus, at time t all (active) Bellman equations have to be solved simultaneously—that is, for the approximation of the function $f_t(\beta_t)$ in the recurrence relation (2.22), all buses that underwent an inspection during that month t have to be considered when approximating the integral over the next period's discounting state β_{t+1}. Secondly, to find the maximum of the likelihood function we used two optimization approaches: NFXP by Rust (1987) and MPEC by Su and Judd (2012). Further, the integral each period t over the next period's value of the discounting process, both in the expected value function and in the likelihood function, is calculated using Gauss–Hermite quadrature based on 7 nodes.[3] To calculate the interpolation function I_{f_t} that approximates $f_t(\beta_t)$, we used cubic spline interpolation with 11 grid nodes for β_t. The interpolation grid for β_t, denoted as $\overrightarrow{b}$, is a uniformly spaced interval around the stationary mean of the discounting process, with upper and lower bounds limited to three standard deviations. Lastly, in line with the findings of Müller and Reich (2022) we expected the stationary mean of our discounting process to be greater than one. This can result in the expected values approaching infinity. To address this, Müller and Reich (2022) propose a solution that involves calculating the expected values relative to a reference state. We adopted this approach and solved the expected values relative to the reference value $\overrightarrow{EV}$, with $\overrightarrow{EV}$ being a vector of expected values for the decision $i = 0$, mileage state $x = 1$, and the discounting grid $\overrightarrow{b}$. Additional details regarding the RLI algorithm and the relative expected value approach can be found in Appendix 2.A.1 and 2.A.2, respectively. The estimation results are presented in the next section.

2.4.2 Results for Heterogenous Bus Groups

We estimated the model using three different specifications for the discounting process. In the first specification, we estimated the model with a constant discount factor of 0.9999. This involves fixing the stationary mean of the discounting process to 0.9999 and setting both the standard deviation of the innovations and the persistence to zero. This model represents the original discounting case of Rust (1987). In a second specification we estimated a constant discounting process. Thus, the stationary mean of the process is estimated but the standard deviation of the error and the persistence are fixed to a value of zero. This corresponds to the case of Müller and Reich (2022) without a structural break, where discounting is a constant factor but is estimated rather than fixed. Finally, we estimated a model where all three discounting parameters B are free to be estimated. In sum, we estimated the bus engine replacement model with the following three discounting specifications:[4]

Case 1 The original Rust (1987) case, where $\mu_{\beta_1} = 0.9999$, $\rho = 0$, and $\sigma_\varepsilon = 0$.

Case 2 The case of a constant process, where μ_{β_1} is estimated, and $\rho = 0$ and $\sigma_\varepsilon = 0$.

Case 3 The stochastic case, where μ_{β_1}, ρ, and σ_ε are estimated.

The estimation results of these three specifications are summarized in Table 2.3. Note that, in line with Rust (1987), the reported maximum likelihood estimates (MLEs) were derived in a two-step estimation process. In the first step, we estimated the mileage transition probabilities by maximizing the mileage part of the likelihood $L_1(\theta)$ in (2.19). In the second step, we fixed the values for θ_{3,G_m} to the estimates from step one, $\theta_{3,G_m} = \hat{\theta}_{3,G_m}$, and estimated $\theta_{-\theta_3} = \{\theta \backslash \theta_{3,G_m}\}$ by maximizing the discounting part of the likelihood $L_2(\theta_{-\theta_3}, \hat{\theta}_{3,G_m})$ in (2.20). The results for the full likelihood estimation can be found in Appendix 2.A.3. In Table 2.3, the values in brackets below the structural parameters are the 95 percent confidence intervals, which were calculated using the constrained optimization approach of Reich and Judd (2020). This approach involves minimizing or maximizing

[3] We used a change of variables that allows for the approximation of the integrals with Gauss–Hermite quadrature. Since Gauss–Hermite quadrature approximates integrals with the limits $(-\infty, +\infty)$, but the values of the discounting state have to be strictly positive, this may seem unusual. However, since we anticipated that the discounting process will be relatively persistent with a large stationary mean and minimal variation in the innovations, it is unlikely that the lower Gauss nodes will lead to negative values.

[4] In Cases 1 and 2, the estimation does not rely on a recursive integration of the discounting process because the evolution of the discounting state is constant. However, our estimation algorithm requires an initialization grid $\overrightarrow{b}$ for the interpolation function I_{f_t}. This initialization grid is uniformly spaced around μ_{β_1} with upper and lower bounds limited to three standard deviations. For the constant cases, we simply fixed the standard deviation to a very small value. Comparing the likelihood value for a model with a truly constant discounting process to one with a negligible variance showed no significant difference.

each parameter θ_p, a component of the parameter vector $\{RC_{P_n}, \theta_{1,P_n}, \mu_{\beta_1}, \rho, \sigma_\varepsilon\}$, according to

$$\begin{aligned} \text{min or max} \quad & \theta_p \\ \text{s.t.} \quad & L(\theta) \geq L(\hat{\theta}) - 0.5\chi^2_{df}(\kappa) \\ & EV_\theta(i,x,\beta) = T(EV_\theta)(i,x,\beta) \quad \forall i,x,\beta, \end{aligned} \tag{2.27}$$

where $\hat{\theta}$ denotes the MLE. The term $\chi^2_{df}(\kappa)$ denotes the $100 \cdot \kappa^{\text{th}}$-percentile of the χ^2-distribution with $df = 1$ degrees of freedom and (in this context) $\kappa = 0.95$. For a more comprehensive understanding of this approach, see the work of Reich and Judd (2020).

Referring to Table 2.3, we can state that our estimation method successfully recovers the original log-likelihood value for a fixed discount factor of 0.9999. Furthermore, it accurately estimates the log-likelihood under the assumption of a constant, but estimated, discount factor as in the work of Müller and Reich (2022). While we observe some minor differences in the parameter estimates for Cases 1 and 2 compared to those documented by Müller and Reich (2022), these discrepancies can be attributed to differences in the computational approaches employed. In the case of a stochastic discounting process, Case 3, the estimated parameters of the discounting process are $\mu_{\beta_1} = 1.028$, $\sigma_\varepsilon = 6.410\text{E-}04$, and $\rho = 9.806\text{E-}01$. These estimates indicate that the discounting process is relatively persistent with minimal variance in the innovations. The stationary mean is above one and in the same range as in Case 2: the estimate for the stationary mean for Case 2 is 1.026, and for Case 3 it is 1.028. Based on a likelihood ratio test with a test statistic of approximately 8.828 and a p value of 0.032, we can reject the null hypothesis that the model from Case 1 provides a better fit to the data than the model of Case 3 at a 5 percent significance level. When comparing the log-likelihood value of Case 3 to the one of Case 2, the likelihood of the model with a stochastic discounting process is larger than the likelihood for a constant discounting process, with a difference in the log-likelihood of approximately 1.333. However, with a test statistic of approximately 2.665 and a p value of 0.264 for the likelihood ratio test, we cannot reject the null hypothesis that the model in Case 2 provides a better fit than that in Case 3. This finding is further confirmed by the fact that the 95 percent confidence intervals for the MLE of the standard deviation of the innovations and for the MLE of the persistence both include zero. In contrast, the stationary mean is significantly larger than one, with the lower bound of the 95 percent confidence interval being 1.002.[5]

Additionally, we examined the interdependence between the long-run mean of the discounting process and the structural parameters. Thus, we fixed μ_{β_1} to values on the interval $(0.970, 1.028)$ and estimated the remaining structural parameters $\{\theta_{-\theta_3} \backslash \mu_{\beta_1}\}$ by maximizing the discounting part of the likelihood. The profile log-likelihood and the estimates for $\{RC_{P_n}, \theta_{1,P_n}, \rho, \sigma_\varepsilon\}$ are plotted as a function of μ_{β_1} in Figure 2.2. We observe that the replacement costs increase and the maintenance costs decrease, with an increase in the value of the stationary mean, μ_{β_1}. This is in line with the findings of Müller and Reich (2022): with an increase in the discount factor—or in our case, the long-run mean of the discounting process—future costs are being discounted less. Thus, replacement has to become more expensive relative to regular maintenance in order for the model to explain the same number of replacement decisions. Further, the estimated values for σ_ε and ρ increase when the stationary mean is restricted to values lower than its MLE, especially when setting μ_{β_1} to values below one. If the true stationary mean is indeed around 1.028, fixing it to smaller values would require a higher stationary variance to achieve the same level of discounting in the long run. We will discuss the implications of these observations in more detail in Section 2.5, where we estimate the discounting states for each month, and in this essay's concluding discussion, in Section 2.6.

[5] At this point, it is important to note that the finding of a stationary mean being greater than one may be a result of model misspecifications. Already Müller and Reich (2022) note that misspecifications of the cost function and mileage transition probabilities could bias the level of the discount factor. We cannot exclude this possibility either.

	Case 1	Case 2	Case 3
$\log L(\theta)$	-11,101.5	-11,098.4	-11,097.1
μ_{β_1}	0.9999	1.026	1.028
	(fixed)	[1.006 , 1.048]	[1.002 , 1.051]
σ_ε	0	0	6.410E-04
	(fixed)	*(fixed)*	[0 , 4.595E-02]
ρ	0	0	9.806E-01
	(fixed)	*(fixed)*	[0 , 1)
RC_{P_1}	12.056	15.146	15.139
	[8.863 , 16.841]	[10.000 , 25.200]	[9.915 , 25.812]
RC_{P_2}	11.087	16.128	16.713
	[9.102 , 13.642]	[10.813 , 28.929]	[10.765 , 74.815]
RC_{P_3}	9.673	14.372	17.576
	[7.365 , 12.965]	[8.659 , 32.588]	[9.501 , 153.369]
θ_{1,P_1}	4.790	2.728	2.900
	[2.633 , 7.986]	[0.883 , 5.822]	[0.867 , 6.451]
θ_{1,P_2}	2.830	1.318	1.310
	[1.995 , 3.881]	[0.459 , 2.760]	[0.439 , 2.871]
θ_{1,P_3}	2.154	0.493	0.541
	[1.141 , 3.557]	[0.018 , 2.102]	[0.019 , 2.415]

Table 2.3: Summary of the estimation results for the three discounting cases. Case 1, where $\mu_{\beta_1} = 0.9999$, $\rho = 0$, and $\sigma_\varepsilon = 0$ are fixed. Case 2, where μ_{β_1} is estimated, and $\rho = 0$ and $\sigma_\varepsilon = 0$ are fixed. Case 3, where μ_{β_1}, ρ, and σ_ε are estimated. The values are the MLEs for the profile likelihood function with fixed θ_{3,G_m}. The intervals below the structural parameters are the 95 percent confidence intervals. Source: Own calculation based on data from Rust (1987).

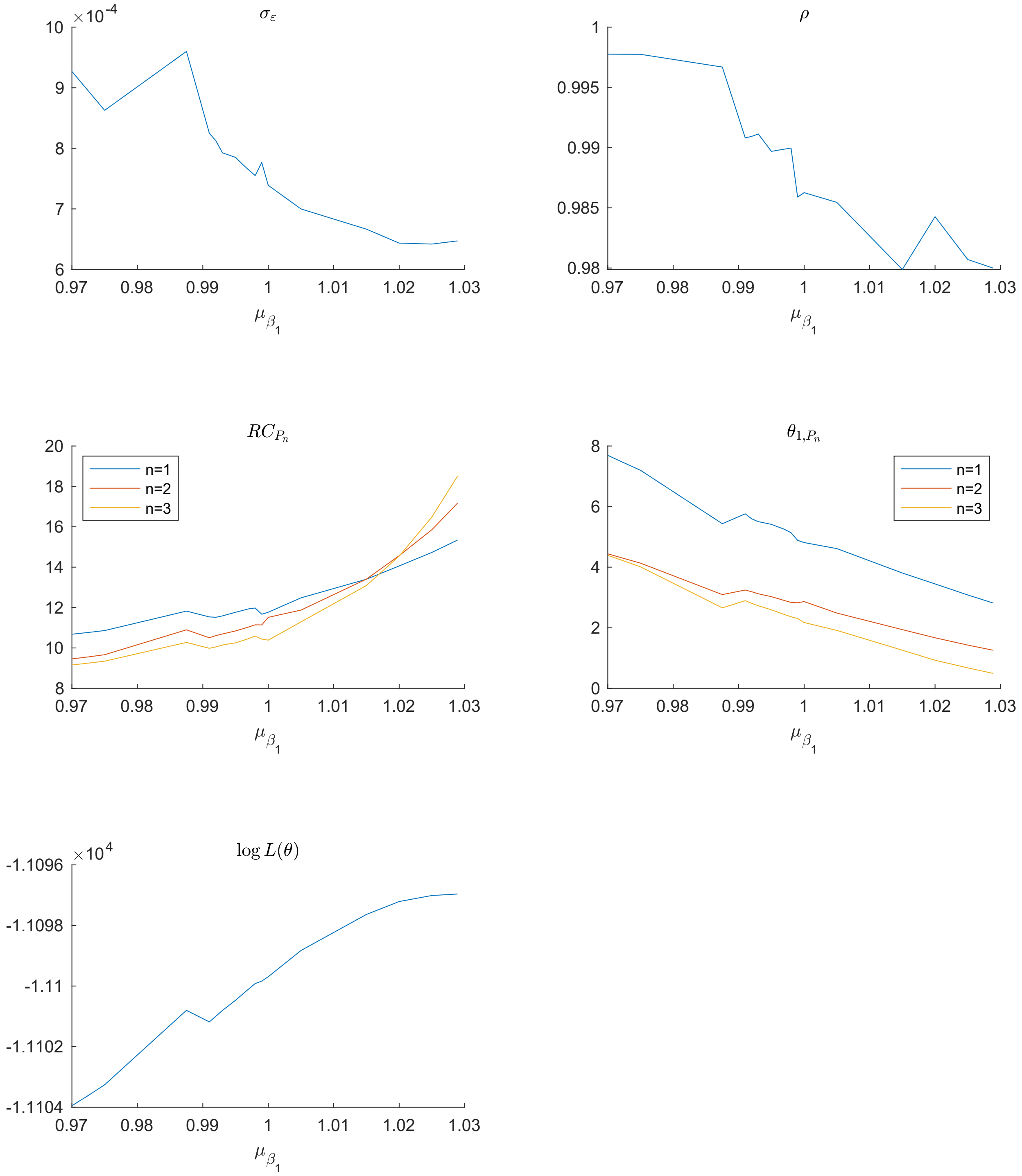

Figure 2.2: Plot for the profile likelihood estimates for the parameters $\{RC_{P_n}, \theta_{1,P_n}, \rho, \sigma_\varepsilon\}$ and the log-likelihood $\log L(\theta)$ as a function of μ_{β_1}. The horizontal axis represents μ_{β_1}, and the vertical axis represents the profile likelihood estimates. Source: Own representation based on data from Rust (1987).

To summarize our estimation results, we cannot reject the hypothesis that Zurcher's discounting process is constant: both the innovation variance and the persistence parameter have lower boundaries of zero for the 95 percent confidence interval, and the likelihood ratio test does not allow us to reject the discounting model in Case 2 in favor of the model in Case 3. It is, however, important to note that the identification of the discounting parameters relies on having a sufficient number of replacement decisions. In the ten years of available data, we observe a total of 128 replacement decisions. Considering this relatively low number, we can interpret the inclusion of zero within the 95 percent confidence interval for both the error standard deviation and the persistence parameter with less cause for concern.

2.4.3 Robustness Check

Before we elaborate on the estimation of the most likely discounting states, we would like to provide some final remarks regarding the level of accuracy we have chosen for our numerical approximation of the integrals over the discounting state. As detailed in Section 2.2.3, the RLI method separates the integrals over the discounting state in the likelihood (2.20) into a series of low-dimensional integrals and interpolation problems. To approximate these integrals, we employed Gauss–Hermite quadrature rule, which requires specifying a particular number of nodes. In Case 3 we used 7 Gauss nodes. Moreover, we fitted the interpolation function I_{f_t} using cubic splines with 11 grid points. The error in Gaussian integration and the error in the interpolation function depend on the number of nodes and grid points selected. Typically, increasing the number of nodes and grid points reduces the approximation error and enhances accuracy (Judd, 1998). To examine the sensitivity of our MLE in Case 3 to the number of Gauss nodes and the number of interpolation points, we calculated the $L_2(\theta)$-likelihood of the MLE for an increasing number of nodes and points, respectively. The log-likelihood values are summarized in Table 2.4. The values in italics below the log-likelihood values are the average runtime in seconds for 100 evaluations of the likelihood.

$m \backslash k$	7	15	31
11	-603.449460	-603.449285	-603.448845
	0.4328	*0.5105*	*0.5547*
23	-603.457803	-603.457222	-603.457222
	0.4949	*0.5791*	*0.6542*
47	-603.457964	-603.457631	-603.457638
	0.5325	*0.5898*	*0.7097*
95	-603.458009	-603.457704	-603.457682
	0.5954	*0.6875*	*0.7581*

Table 2.4: Log-likelihood values, $\log L_2(\theta)$, of the MLE for an increasing number of interpolation grid points and Gauss nodes, where m is the number of interpolation points and k the number of Gauss nodes. The values in italics below the $\log L_2(\theta)$ values are the average wall time in seconds over 100 iterations. Source: Own calculation based on data from Rust (1987).

Given the values in Table 2.4, doubling the number of interpolation grid points from 11 to 23 results in a decrease in the log-likelihood value of approximately 8E-03 while maintaining a fixed number of Gauss nodes. Doubling the interpolation points again from 23 to 47 leads to a further decrease in the order of magnitude 1E-04, with around -1.6E-04 for $k = 7$ and approximately -4E-04 for $k = 15$ and $k = 31$. Additionally, for a fixed number of interpolation points doubling the number of Gauss nodes from 7 to 15 results in a log-likelihood increase of approximately 1.7E-04, up to 5.8E-04. Doubling the number from 15 to 31 nodes leads to a maximum increase in the likelihood of 4.4E-04. Furthermore, for each combination of the number of Gauss nodes and interpolation points, we recorded the average wall time in seconds based on 100 evaluations of the likelihood function for that specific combination. The average wall time does not exhibit a consistent pattern. For instance, with 7 Gauss nodes, increasing the number of interpolation grid points from 11 to 23 results in a runtime increase of approximately 14.3 percent, and further increasing it from 23 to 47 grid points leads to an increase in runtime of around 7.6 percent. For a fixed number of interpolation points, the increase in runtime when doubling the number of Gauss nodes from 7 to 15 ranges from approximately 10.8 to 17.9 percent. Further increasing the number of Gauss nodes from 15 to 31 leads to an increase in runtime of around 8.6 to 20.3 percent. The non-linear increase in wall time is noteworthy, as we would expect the runtime to at least double with a doubling in either the number of Gauss nodes or the number of interpolation points. Given, however, that the likelihood function's fixed costs (in terms of runtime) are relatively high, the non-linear increase might be just the result of a measurement error. In summary, assuming that the log-likelihood value for 31 Gauss nodes and 95 interpolation points represents the likelihood value of the highest precision, we can state the following: as expected, increasing the number of interpolation grid points or Gauss nodes improves the accuracy of the likelihood estimate, but this improvement is accompanied by an increase in computational time. In this context, the rather small increase in accuracy achieved by doubling the number of Gauss nodes does not justify the corresponding loss in computational efficiency. However, doubling the number of interpolation points from 11 to 23 for a fixed number of 7 Gauss nodes would lead to a two-digit improvement, with only a marginal increase in runtime, of 14 percent. Based on this analysis, we can conclude that the chosen combination of 11 interpolation points and 7 Gauss nodes provides sufficient accuracy. For future runs, however, an increase in the number of interpolation points might be considered.

2.5 Estimation of the Most Likely Discounting Process

The finding that the stationary mean of the discounting process is above one contradicts the conventional belief that discount factors fall within the range $(0, 1)$. Müller and Reich (2022) attribute their estimated discount factors greater than one to the occurrence of negative real interest rates during the data period. Their observation raises an intriguing question about how the states of our discounting process in Case 3 compare to the documented (real) interest rate levels from December 1974 to May 1985. Indeed, our approach to modeling discounting allows us to estimate the most likely realization of the discounting process for each month, which we can then compare to the historical interest rate values for that month. This subsection elaborates on the estimation of these discounting states. A comprehensive discussion of the implications of a stationary mean above one and its relation to real interest rate values can be found in the concluding section of this essay, Section 2.6.

The actual discounting states are known only to Zurcher, while they remain unobserved by us as modelers. However, since an AR(1) process is fully described by its distributional parameters we can estimate for each time point the discounting state with the highest likelihood contribution based on the observed decisions and mileage states in that month. We refer to these estimated discounting states as the *most likely discounting process*. To estimate the most likely discounting process, we first set the structural parameters to our MLE, $\theta = \hat{\theta}$. Then we maximize the estimated likelihood:

$$\begin{aligned} L_\theta(\{\beta_t\}_{t=1}^T) = \quad & \left[p_\beta(\beta_1;\theta) \prod_{n=1}^{N} Pr(i_{1,n} \mid x_{1,n}, \beta_1; \theta) \right] \\ & \cdot \left[\prod_{t=1}^{T-1} p_\beta(\beta_{t+1} \mid \beta_t; \theta) \prod_{n=1}^{N} Pr(i_{t+1,n} \mid x_{t+1,n}, \beta_{t+1}; \theta) \right]. \end{aligned} \tag{2.28}$$

The vector $\{\beta_t\}_{t=1}^T$ that maximizes (2.28) is the most likely discounting process for our MLE.

Figure 2.3 depicts the most likely discounting process for the MLE in blue. Further, we used the same

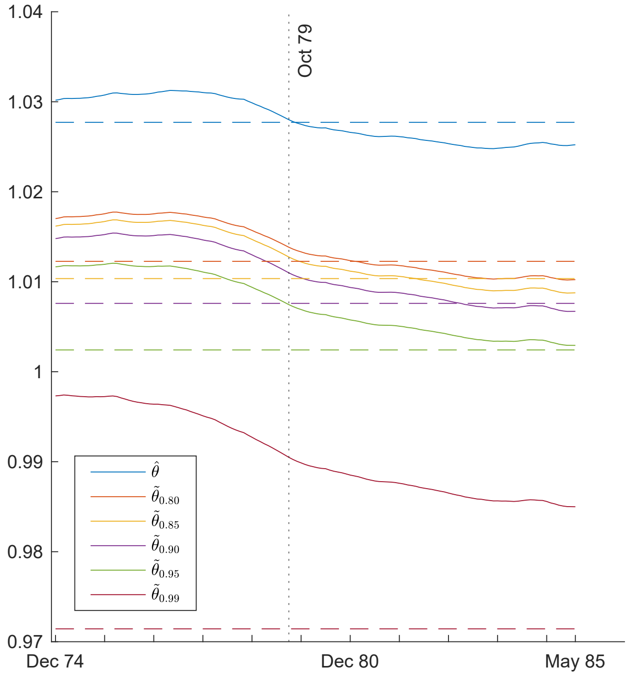

Figure 2.3: Plots of the most likely discounting processes for the MLE, $\hat{\theta}$, and for the profile likelihood estimates, $\tilde{\theta}_\kappa$, for the lower $\kappa \cdot 100$ percent confidence band of the stationary mean, where $\kappa \in \{0.80, 0.85, 0.90, 0.95, 0.99\}$. The dashed horizontal lines illustrate the corresponding stationary mean of the processes. The vertical black dotted line shows the month in which the most likely discounting process for the MLE falls below its stationary mean. Source: Own representation based on data from Rust (1987).

	μ_{β_1}	σ_ε	ρ
$\hat{\theta}$	1.028	6.410E-04	9.806E-01
$\tilde{\theta}_{0.80}$	1.012	6.664E-04	9.883E-01
$\tilde{\theta}_{0.85}$	1.010	6.813E-04	9.909E-01
$\tilde{\theta}_{0.90}$	1.008	6.927E-04	9.934E-01
$\tilde{\theta}_{0.95}$	1.002	6.534E-04	9.965E-01
$\tilde{\theta}_{0.99}$	0.971	7.350E-04	9.995E-01

Table 2.5: Estimated discounting parameters for the MLE, $\hat{\theta}$, and for the profile likelihood estimates, $\tilde{\theta}_\kappa$, for the lower $\kappa \cdot 100$ percent confidence band of the stationary mean, where $\kappa \in \{0.80, 0.85, 0.90, 0.95, 0.99\}$. Source: Own calculation based on data from Rust (1987).

approach to estimate the most likely discounting processes for the profile likelihood estimates from (2.27) for the lower $\kappa \cdot 100$ percent confidence band of the stationary mean, where $\kappa \in \{0.80, 0.85, 0.90, 0.95, 0.99\}$. We denote these profile estimates as $\tilde{\theta}_\kappa$ with $\kappa \in \{0.80, 0.85, 0.90, 0.95, 0.99\}$. The most likely discounting processes for the MLE, $\hat{\theta}$, as well as for the profile likelihood estimates, $\tilde{\theta}_\kappa$, are plotted as solid lines in Figure 2.3. The dashed horizontal lines represent the corresponding stationary mean of each process. The month at which the most likely discounting process for the MLE falls below its stationary mean is visually indicated by the vertical black dotted line. Further, the values of the distributional parameters of the discounting processes for $\hat{\theta}$ and each $\tilde{\theta}_\kappa$ can be found in Table 2.5.

As depicted in Figure 2.3, for all estimated discounting parameters the respective most likely discounting process exhibits a decreasing trend throughout the observation period. The most likely discounting process for the MLE shows a relatively stable evolution during the initial three years of the observation period, followed by a gradual decline, with the discounting states falling below the stationary mean in October 1979. When the stationary mean is fixed at lower values than its MLE, the estimated discounting states tend to fall below their corresponding stationary mean at later time points. The most likely discounting processes for the profile likelihood estimates $\tilde{\theta}_{0.99}$ and $\tilde{\theta}_{0.95}$ even remain above their stationary means throughout the entire observation period. Further, only for $\tilde{\theta}_{0.99}$ are the estimated discounting states below one for all months t. In contrast, the estimated discounting states for the other profile likelihood estimates (and for the MLE) consistently remain above one.

In summary, when μ_{β_1} is restricted to values below its MLE the estimated most likely discounting states tend to have values above their stationary mean. This finding corresponds with the fact that when the stationary mean is constrained to values below its MLE the stationary variance has to increase for the process to reach the same level of discounting in the long run; see Table 2.5. Finally, we want to emphasize the noticeable downward trend prevalent in all processes. A possible explanation for this trend is discussed in the next section.

2.6 Concluding Discussion

The results in sections 2.4.2 and 2.5 showed that both the stationary mean and the states of the most likely discounting process for the MLE of Case 3 are above one. This finding challenges the conventional notion that discount rates are expected to be positive. Under the assumption of an exponential discount function, the discount factor is equal to

$$\beta_t = \frac{1}{1 + d_t}, \tag{2.29}$$

where d_t is the discount rate at time t. Thus, when the discount factor is larger than one the discount rate is negative. From a behavioral perspective, this suggests that Zurcher has *negative time preferences* for the time period covered by the data. Negative time preferences suggest that agents are inclined to prefer utility at a later time even if it has the same value as utility received sooner (Frederick et al., 2002). In the context of Zurcher's decision-making between engine maintenance and replacement, his utility function is primarily determined by monetary costs. Therefore, a negative discount rate implies that Zurcher would prefer to make payments of the same nominal amount immediately instead of deferring them to a later date. Furthermore, from an economic standpoint, when utility is equated to mainly monetary values it is reasonable to assume that an agent's discount rate should be at least equal to the prevailing risk-free interest rate (Goodin, 1982). Consequently, a negative discount rate implies that the agent's economic environment is characterized by negative interest rates. In a financial context, this may appear counterintuitive, suggesting a scenario in which a lender pays interest for lending money, or—conversely—a borrower earns interest by borrowing money. However, negative nominal as well as real interest rates do exist in the economy. Negative nominal interest rates have been implemented by central banks in the past as a monetary policy tool during periods of severe recession. The aim is to encourage borrowing, particularly for corporations, in order to stimulate economic activity and investment. For example, the Swiss National Bank set the central bank rate to -0.25 percent in 2015 (Canetg, 2022). Although negative nominal interest rates exist, they are considered a rather unconventional monetary policy. Negative real interest rates are a more common observation. They occur at times when the nominal interest rate falls below the rate of inflation. Real interest rates can simply be approximated with

$$r_{real,t} \approx r_{nominal,t} - \iota_t,$$

where $r_{nominal,t}$ is the nominal interest rate at time t and ι_t the rate of inflation. Indeed, from December 1974 to May 1985 there were periods of negative real interest rates. Before we further elaborate on the real interest rate levels during the data period, it is worth considering whether it is even reasonable to assume that Zurcher incorporates real interest rates when discounting his future costs. The general discounting principle states that nominal cash flows should be discounted using nominal interest rates, while real cash flows should be discounted using real interest rates (Volkart, 2006). In the bus engine replacement model, the monthly costs are modeled to be constant over time without any adjustments for inflation. Considering the substantial inflation that occurred during the observation period, it is reasonable to interpret these costs in real rather than nominal terms and to examine Zurcher's discount rate as a function of the real interest rate (Müller and Reich, 2022). In the following, we provide some historical background in order to facilitate an understanding of the evolution of real interest rates during the time period covered by the bus engine replacement model's data.

The period known as *the Great Inflation*, which occurred between 1965 and 1982 in the United States, was characterized by significant inflationary pressure resulting from questionable monetary policies (Bryan, 2013). Preceding this era, there were years of so-called *easy money* policies that involved low interest rates designed to encourage borrowing. However, these policies resulted in increased consumer prices as demand exceeded the supply side of the economy. Additionally, an oil crisis occurred, exacerbating the situation by causing companies to raise their prices even further. In 1974, the annualized inflation rate nearly doubled, reaching approximately 11 percent compared to the previous year's 6.2 percent. Despite an increase in the Federal Fund Effective rate during the mid-1970s, the key interest rate was unable to match the rapid rise in the inflation rate. As a result, negative real interest rates prevailed throughout the mid to late 1970s. In August 1979, Paul Volcker was appointed to the position of chairman of the Federal Reserve System (the Fed). During his tenure, Volcker implemented a strategy to tackle high inflation rates by employing equally high interest rates. In March 1980, inflation reached an all-time high of over 14 percent, followed by a peak in the Federal Fund Effective rate of nearly 18 percent in April 1980 (US Bureau of Labor Statistics, 2023b; Federal Reserve Economic Data, 2023c). Despite facing significant criticism, the tight monetary policy implemented by the Fed under Volcker's leadership began to yield results. Inflation gradually decreased, and economic growth resumed (Bryan, 2013).

In the first half of the data period December 1974 to May 1985, the real interest rate was below zero for most months, especially from late 1974 to early 1981. This can be seen in Figure 2.4. The blue line in Figure 2.4 is the annualized monthly real interest rate, calculated as the difference between the Federal Fund Effective rate and the rate of the Consumer Price Index for All Urban Consumers (US Bureau of Labor Statistics, 2023b; Federal Reserve Economic Data, 2023c). The solid black line corresponds to the discount rate derived from the estimated most likely discounting process for our MLE. We assume an exponential discount function to hold. The monthly discount rates can be derived from (2.29) as $d_t = (1/\hat{\beta}_t) - 1$, where $\hat{\beta}_t$ is the estimated most likely discounting state for month t. The vertical black dotted line for October 1979 marks the month in which the most likely discounting process of the MLE falls below its stationary mean. Note that the percentage values of the discount rate and the real interest rate are not on the same scale: the left vertical axis is for the discount rates and the right vertical axis is for the real interest rates. As depicted in Figure 2.4, both the discount rate and the real interest rate show an upward trend from December 1974 until Volcker's appointment as chairman of the Fed in August 1979. In the months following August 1979, the real interest rate experienced a significant decline, followed by a sharp rise with fluctuations throughout the years 1980 to 1982. This trend is consistent with Volcker's policy announcement at a press conference in October 1979, where he outlined the objective of targeting inflation—a policy shift that was expected to result in interest rate fluctuations (Medley, 2013). In comparison, the estimated discount rate shows a stable increase, which can be attributed to the estimated parameters of the AR(1) process: The MLE of the discounting parameters indicates a low level of variation and high persistence. Processes with such characteristics tend to exhibit a relatively stable pattern

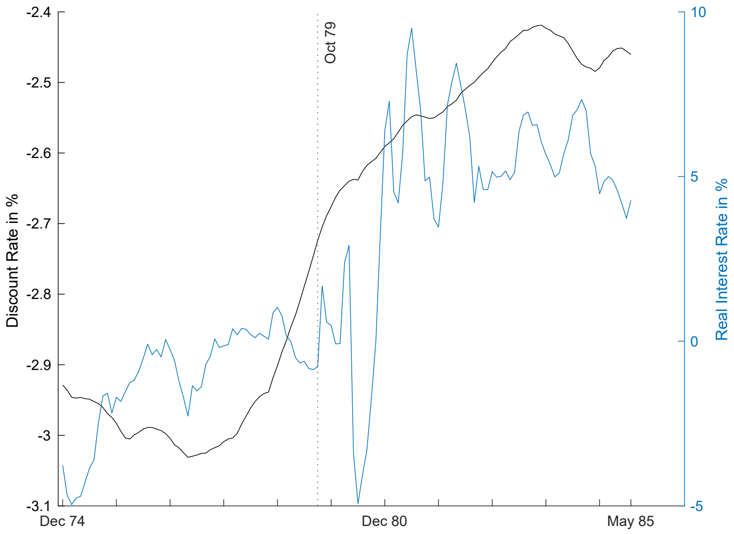

Figure 2.4: Evolution of monthly discount rates—calculated from the most likely discounting process for the MLE—in black (left vertical axis) and annualized monthly real interest rates in blue (right vertical axis), calculated as the difference between the Federal Fund Effective rate and the rate of the Consumer Price Index for All Urban Consumers. The vertical black dotted line indicates the month in which the most likely discounting process for the MLE falls below its stationary mean. Source: Own representation based on data from the US Bureau of Labor Statistics (2023b) and Federal Reserve Economic Data (2023c).

with little random variation. Consequently, any realization of such an AR(1) process will most likely not show extreme and sudden fluctuations in its development. Further, we observed in Figure 2.3 that the most likely discounting process derived from our MLE remains above its stationary mean for the months preceding Volcker's appointment. The process falls below its stationary mean in October 1979. This change in the states' levels between before and after Volcker's appointment suggests a potential shift in discounting behavior during that period. In fact, Müller and Reich (2022) argue for a structural break in the discount factor around September 1979. If a model with a structural break describes Zurcher's true discounting behavior, this would lead to a shift in the underlying process-generating parameters, and could explain the estimated most likely discounting process for the MLE falling below the stationary mean in the same time frame. Finally, although the values of the real interest rate and the derived discount rate are not on the same scale, both rates exhibit a consistent trend in the same direction. This observation is further supported by the strong correlation between the monthly real interest rate and our estimated most likely discounting process. The Pearson correlation coefficient has a value of approximately -0.71 with a p value of less than 6.8E-21.

In summary, our findings align with the observed macroeconomic conditions during the data period. The discount rates derived from the estimated most likely discounting process for the MLE exhibit a similar trend to historical real interest rates. The significant and strong negative correlation between the most likely discounting states and the real interest rates further supports the assumption that Zurcher's discounting behavior is affected by the real interest rate. This correlation finding raises the question of whether incorporating the real interest rate into the discounting process could improve the fit of the bus engine replacement model. Such a model extension would be an interesting avenue for future research as it could provide a more comprehensive understanding of the relationship between discounting behavior and prevailing macroeconomic factors. Finally, we encourage researchers to incorporate stochastic discounting processes into their DDC models. In particular, it would be interesting to examine how such processes behave in settings where utility extends beyond monetary values and whether the strong correlation between the (real) interest rate and the discounting states still holds in these cases. Moreover, our approach to estimating discounting can also be applied to models with multiple agents. An AR(1) process is fully described by its mean, variance, and persistence. By assuming homogeneity in the distributional parameters of the AR(1) process, discounting can be modeled independently across individuals. This opens up the possibility of applying our discounting approach to models that usually involve panel data, such as labor-market models.

2.A Appendix

2.A.1 RLI in Detail

This section describes in detail the RLI estimation approach outlined in Section 2.2.3, specifically the Gauss–Hermite approximation used.

Recall the likelihood function for the discounting part

$$L_2(\theta) = \int f_1(\beta_1) p_\beta(\beta_1;\theta) \prod_{n=1}^{N} Pr(i_{1,n} \mid x_{1,n}, \beta_1; \theta) d\beta_1 \tag{2.30}$$

and the recurrence relation

$$f_t(\beta_t) = \begin{cases} 1, & \text{if } t \geq T \\ \int f_{t+1}(\beta_{t+1}) p_\beta(\beta_{t+1} \mid \beta_t; \theta) \prod_{n=1}^{N} Pr(i_{t+1,n} \mid x_{t+1,n}, \beta_{t+1}; \theta) d\beta_{t+1} & \text{if } 0 < t < T \end{cases}. \tag{2.31}$$

We defined discounting to be an AR(1) process:

$$\beta_{t+1} = \mu_{\beta_1}(1-\rho) + \rho\beta_t + \varepsilon_t \text{ with } \varepsilon_t \sim N(0, \sigma_\varepsilon^2). \tag{2.32}$$

Thus, $p_\beta(\beta_1;\theta)$ is distributed according to

$$N\left(\mu_{\beta_1}, \frac{\sigma_\varepsilon^2}{1-\rho^2}\right), \tag{2.33}$$

and $p_\beta(\beta_{t+1} \mid \beta_t; \theta)$ according to

$$N\left(\mu_{\beta_1}(1-\rho) + \rho\beta_t, \sigma_\varepsilon^2\right). \tag{2.34}$$

To approximate the integral over the discounting states with Gauss–Hermite quadrature rule, we first have to define the following changes of variables (COV). For the integral over β_1 in (2.30), we define the COV

$$\gamma = \frac{1}{\sqrt{2}}\left(\frac{\beta_1 - \mu_{\beta_1}}{\sigma_\varepsilon/\sqrt{1-\rho^2}}\right) \Leftrightarrow \beta_1 = \mu_{\beta_1} + \frac{\sqrt{2}\sigma_\varepsilon}{\sqrt{1-\rho^2}}\gamma. \tag{2.35}$$

With this COV, we can rewrite (2.30) to obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} f_1(\beta_1) p_\beta(\beta_1;\theta) \prod_{n=1}^{N} Pr\left(i_{1,n} \mid x_{1,n}, \beta_1; \theta\right) d\beta_1 \\ =&\int_{-\infty}^{\infty} \frac{e^{-\gamma^2}}{\sqrt{\pi}} f_1\left(\mu_{\beta_1} + \frac{\sqrt{2}\sigma_\varepsilon}{\sqrt{1-\rho^2}}\gamma\right) \prod_{n=1}^{N} Pr\left(i_{1,n} \mid x_{1,n}, \mu_{\beta_1} + \frac{\sqrt{2}\sigma_\varepsilon}{\sqrt{1-\rho^2}}\gamma; \theta\right) d\gamma \\ \approx&\frac{1}{\sqrt{\pi}} \sum_{k=1}^{K} \omega_k f_1\left(\mu_{\beta_1} + \frac{\sqrt{2}\sigma_\varepsilon}{\sqrt{1-\rho^2}}\gamma_k\right) \prod_{n=1}^{N} Pr\left(i_{1,n} \mid x_{1,n}, \mu_{\beta_1} + \frac{\sqrt{2}\sigma_\varepsilon}{\sqrt{1-\rho^2}}\gamma_k; \theta\right), \end{aligned} \tag{2.36}$$

where ω_k are the Gauss weights, and γ_k the corresponding Gauss nodes. Furthermore, for the integral over β_{t+1} in the recurrence relation (2.31) we define the COV

$$\alpha = \frac{1}{\sqrt{2}}\left(\frac{\varepsilon_t}{\sigma_\varepsilon}\right) \Leftrightarrow \varepsilon_t = \sqrt{2}\sigma_\varepsilon\alpha, \tag{2.37}$$

which allows us to approximate the integral in (2.31) for $0 < t < T$ as

$$\begin{aligned}
&\int_{-\infty}^{\infty} f_{t+1}\left(\mu_{\beta_1}(1-\rho)+\rho\beta_t+\varepsilon_t\right) p_\beta(\mu_{\beta_1}(1-\rho)+\rho\beta_t+\varepsilon_t \mid \beta_t;\theta) \\
&\quad \cdot \prod_{n=1}^{N} Pr\left(i_{t+1,n} \mid x_{t+1,n}, \mu_{\beta_1}(1-\rho)+\rho\beta_t+\varepsilon_t;\theta\right) d(\mu_{\beta_1}(1-\rho)+\rho\beta_t+\varepsilon_t) \\
&= \int_{-\infty}^{\infty} \frac{e^{-\alpha^2}}{\sqrt{\pi}} f_{t+1}\left(\mu_{\beta_1}(1-\rho)+\rho\beta_t+\sqrt{2}\sigma_\varepsilon\alpha\right) \prod_{n=1}^{N} Pr\left(i_{t+1,n} \mid x_{t+1,n}, \mu_{\beta_1}(1-\rho)+\rho\beta_t+\sqrt{2}\sigma_\varepsilon\alpha;\theta\right) d\alpha \\
&\approx \frac{1}{\sqrt{\pi}} \sum_{k=1}^{K} v_k f_{t+1}\left(\mu_{\beta_1}(1-\rho)+\rho\beta_t+\sqrt{2}\sigma_\varepsilon\alpha_k\right) \prod_{n=1}^{N} Pr\left(i_{t+1,n} \mid x_{t+1,n}, \mu_{\beta_1}(1-\rho)+\rho\beta_t+\sqrt{2}\sigma_\varepsilon\alpha_k;\theta\right). \qquad (2.38)
\end{aligned}$$

The Gauss weights are v_k, and the corresponding Gauss nodes are α_k.

Discounting as a process not only impacts the likelihood function but also affects the expected value function. At time t, Zurcher forms expectations about the future mileage state of a bus as well as the discount state. First, recall the expected value function

$$\begin{aligned}
EV_\theta(i_t, x_t, \beta_t) = \int_0^\infty \int \log \sum_{j\in\{0,1\}} \exp\left[U(j, x_{t+1}) + \beta_{t+1} EV_\theta(j, x_{t+1}, \beta_{t+1})\right] \\
\cdot\, p_x(x_{t+1} \mid i_t, x_t;\theta) p_\beta(\beta_{t+1} \mid \beta_t;\theta) d\beta_{t+1} dx_{t+1}. \qquad (2.39)
\end{aligned}$$

The integral over β_{t+1} in the expected value function can also be approximated with the Gauss–Hermite quadrature rule. Focusing only on the integral over β_{t+1}—and ignoring the integral over the mileage state—we define the COV

$$q = \frac{1}{\sqrt{2}}\left(\frac{\varepsilon_t}{\sigma_\varepsilon}\right) \Leftrightarrow \varepsilon_t = \sqrt{2}\sigma_\varepsilon q, \qquad (2.40)$$

which leads to

$$\begin{aligned}
&\int_{-\infty}^{\infty} \log \sum_{j\in\{0,1\}} \exp\left[U(j,x_{t+1}) + \left(\mu_{\beta_1}(1-\rho)+\rho\beta_t+\varepsilon_t\right) EV_\theta\left(j, x_{t+1}, \mu_{\beta_1}(1-\rho)+\rho\beta_t+\varepsilon_t\right)\right] \\
&\cdot\, p_x(x_{t+1} \mid i_t, x_t;\theta) p_\beta(\mu_{\beta_1}(1-\rho)+\rho\beta_t+\varepsilon_t \mid \beta_t;\theta) d(\mu_{\beta_1}(1-\rho)+\rho\beta_t+\varepsilon_t) \\
&= \int_{-\infty}^{\infty} \log \sum_{j\in\{0,1\}} \exp\left[U(j,x_{t+1}) + \left(\mu_{\beta_1}(1-\rho)+\rho\beta_t+\sqrt{2}\sigma_\varepsilon q\right) EV_\theta\left(j, x_{t+1}, \mu_{\beta_1}(1-\rho)+\rho\beta_t+\sqrt{2}\sigma_\varepsilon q\right)\right] \\
&\cdot\, p_x(x_{t+1} \mid i_t, x_t;\theta) \frac{e^{-q^2}}{\sqrt{\pi}} dq \\
&\approx \frac{1}{\sqrt{\pi}} \sum_{s=1}^{S} \log \sum_{j\in\{0,1\}} \exp\left[U(j,x_{t+1}) + \left(\mu_{\beta_1}(1-\rho)+\rho\beta_t+\sqrt{2}\sigma_\varepsilon q_s\right) EV_\theta\left(j, x_{t+1}, \mu_{\beta_1}(1-\rho)+\rho\beta_t+\sqrt{2}\sigma_\varepsilon q_s\right)\right] \\
&\cdot\, p_x(x_{t+1} \mid i_t, x_t;\theta) w_s, \qquad (2.41)
\end{aligned}$$

where w_s are the Gauss weights, and q_s the corresponding Gauss nodes. For simplicity, we used the same number of Gauss–Hermite weights and nodes for all approximations of the integrals over the discounting state.

2.A.2 Relative Expected Value Function

In infinite horizon problems, the present value of the utility stream for discount factors greater than one might have a value equal to infinity. Indeed, the field of operations research has been at the forefront of developing concepts for such cases. One notable contribution is by White (1963), who introduced the concept of *relative* *value iteration.* Instead of solving for the expected values, this method focuses on computing the expected values relative to a reference state. In Müller and Reich (2022), the authors demonstrate that such an approach to solving the expected value function converges for discount factors greater than one. As we anticipated our discounting states to surpass a value of one, we also used a relative expected value function to compute the expected values. That is, we replace the functional equation

$$EV_\theta(i_t, x_t, \beta_t) = T(EV_\theta)(i_t, x_t, \beta_t) \tag{2.42}$$

with

$$ev_\theta(i_t, x_t, \beta_t) = T(ev_\theta)(i_t, x_t, \beta_t) - T(ev_\theta)(i_k, x_k, \beta_k), \tag{2.43}$$

where (i_k, x_k, β_k) defines the reference state k which relative value, $ev_\theta(i_t, x_t, \beta_t)$, is equal to zero. For the reference state we set $i_k = 0$, $x_k = 1$, and $\beta_k = \overrightarrow{b}$, and redefine the conditional choice probabilities (2.21) to be

$$Pr(i_{t,n} \mid x_{t,n}, \beta_t; \theta) = \frac{\exp\left[U_\theta(i_{t,n}, x_{t,n}) + \beta_t ev_\theta(i_{t,n}, x_{t,n}, \beta_t)\right]}{\sum_{j\in\{0,1\}} \exp\left[U_\theta(j, x_{t,n}) + \beta_t ev_\theta(j, x_{t,n}, \beta_t)\right]} . \tag{2.44}$$

2.A.3 Full Likelihood Estimates

Table 2.6 summarizes the full likelihood estimates for the three specifications of the discounting process defined in Section 2.4.2—that is,

Case 1 The original Rust (1987) case, where $\mu_{\beta_1} = 0.9999$, $\rho = 0$, and $\sigma_\varepsilon = 0$.

Case 2 The case of a constant process, where μ_{β_1} is estimated, and $\rho = 0$ and $\sigma_\varepsilon = 0$.

Case 3 The stochastic case, where μ_{β_1}, ρ, and σ_ε are estimated.

Note that confidence intervals are not provided as they were too computationally demanding to calculate.

Case 1					
$\log L(\theta)$	-11,101.41		θ_{30}	θ_{31}	θ_{32}
μ_{β_1}	0.9999	θ_{3,G_1}	1.971E−01	7.890E−01	1.390E−02
σ_ε	0	θ_{3,G_2}	3.891E−01	6.004E−01	1.050E−02
ρ	0	θ_{3,G_3}	3.072E−01	6.826E−01	1.026E−02
RC_{P_1}	12.056	θ_{3,G_4}	3.913E−01	5.959E−01	1.286E−02
RC_{P_2}	11.112	θ_{3,G_5}	4.869E−01	5.083E−01	4.714E−03
RC_{P_3}	9.675	θ_{3,G_6}	6.177E−01	3.823E−01	3.742E−23
θ_{1,P_1}	4.789	θ_{3,G_7}	6.032E−01	3.941E−01	2.611E−03
θ_{1,P_2}	2.838	θ_{3,G_8}	7.224E−01	2.776E−01	6.037E−28
θ_{1,P_3}	2.153				

Case 2					
$\log L(\theta)$	-11,098.33		θ_{30}	θ_{31}	θ_{32}
μ_{β_1}	1.026	θ_{3,G_1}	1.971E−01	7.890E−01	1.390E−02
σ_ε	0	θ_{3,G_2}	3.878E−01	6.016E−01	1.058E−02
ρ	0	θ_{3,G_3}	3.073E−01	6.825E−01	1.025E−02
RC_{P_1}	15.195	θ_{3,G_4}	3.921E−01	5.951E−01	1.280E−02
RC_{P_2}	16.246	θ_{3,G_5}	4.857E−01	5.095E−01	4.745E−03
RC_{P_3}	14.505	θ_{3,G_6}	6.202E−01	3.798E−01	5.444E−10
θ_{1,P_1}	2.708	θ_{3,G_7}	6.018E−01	3.956E−01	2.637E−03
θ_{1,P_2}	1.307	θ_{3,G_8}	7.203E−01	2.797E−01	3.095E−10
θ_{1,P_3}	0.486				

Case 3					
$\log L(\theta)$	-11,096.96		θ_{30}	θ_{31}	θ_{32}
μ_{β_1}	1.029	θ_{3,G_1}	1.971E−01	7.890E−01	1.390E−02
σ_ε	6.472E−04	θ_{3,G_2}	3.878E−01	6.016E−01	1.058E−02
ρ	9.800E−01	θ_{3,G_3}	3.072E−01	6.825E−01	1.025E−02
RC_{P_1}	15.343	θ_{3,G_4}	3.923E−01	5.949E−01	1.278E−02
RC_{P_2}	17.169	θ_{3,G_5}	4.852E−01	5.101E−01	4.760E−03
RC_{P_3}	18.502	θ_{3,G_6}	6.226E−01	3.774E−01	3.584E−22
θ_{1,P_1}	2.814	θ_{3,G_7}	6.005E−01	3.968E−01	2.657E−03
θ_{1,P_2}	1.256	θ_{3,G_8}	7.203E−01	2.797E−01	9.263E−23
θ_{1,P_3}	0.492				

Table 2.6: Summary of the estimation results for the three discounting cases. Case 1, where $\mu_{\beta_1} = 0.9999$, $\rho = 0$, and $\sigma_\varepsilon = 0$ are fixed. Case 2, where μ_{β_1} is estimated, and $\rho = 0$ and $\sigma_\varepsilon = 0$ are fixed. Case 3, where μ_{β_1}, ρ, and σ_ε are estimated. The values are the MLEs for the full likelihood estimation. Source: Own calculation based on data from Rust (1987).

Essay Three
Estimating Stochastic Processes for Discounting in Dynamic Structural Models: An Extension

Estimating Stochastic Processes for Discounting in Dynamic Structural Models: An Extension

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October 2023

Abstract

Maag and Reich (2023) have proposed treating the discount factor in dynamic discrete choice (DDC) models as a stochastic process and demonstrated a significant and economically sound correlation between real interest rates and discounting in the seminal bus engine replacement model of Rust (1987). Expanding on this idea, we advance the discourse and further challenge the prevailing assumption of constant discount factors within DDC models. Our main proposition is the integration of real interest rates into the discounting process. We introduce two distinct model frameworks designed to incorporate the co-movement of real interest rates and the discounting process into dynamic structural models. Our research includes a comprehensive empirical analysis to assess the fit of the two model frameworks and evaluate the joint distribution of various real interest rate candidates and the estimated discounting process. The empirical evaluation of these two discounting models provides valuable insights for future research and serves as a reference for selecting the most suitable model framework and real interest rate to incorporate into DDC models.

3.1 Introduction

In dynamic discrete choice (DDC) models, the agent's decisions over time are modeled as a function of state variables and primitives. The latter describe the agent's beliefs and preferences and can be categorized into the following three points: **(A)** the utility function, which defines the decision-maker's preferences for certain outcomes at a specific time, **(B)** the agent's beliefs regarding the evolution of the state variables, and **(C)** the discount factor, which represents the subjective time preference of an individual. The discount factor is commonly treated as a constant parameter in DDC models (Aguirregabiria and Mira, 2010). Consequently, the agent's time preferences remain the same over time and are assumed to be uniform across all agents within the model. The assumption of constant discounting has been subject to criticism. It suggests that regardless of the economic or sociopolitical circumstances influencing an agent, their preferences regarding future utilities remain unaffected over time and are identical to those of other individuals. For example, in the fertility model of Wolpin (1984) parents face the decision of whether to have another child, considering factors such as the potential risks of giving birth and child rearing, as well as the financial expenses involved. The model is then used to examine how changes in the healthcare system and in the costs of childbearing and rearing influence parents' decisions regarding having another child. It is reasonable to assume that, for example, improving healthcare quality for childbirth can impact time preferences. Enhanced healthcare services may result in increased life expectancy for women, potentially leading to a reduction in short-term focus and a shift toward a more long-term perspective in decision-making. Further, young parents are unlikely to share the same time preferences as a middle-aged couple who already have children. Yet in the work of Wolpin (1984), the discount factor is assumed to be constant.

Thus far, researchers have employed different approaches to address the shortcomings of constant discount factors and to incorporate more realistic discounting behavior into DDC models. One such approach involves deviating from the commonly used exponential discounting form and modeling time preferences as a hyperbolic discount function. Hyperbolic discounting, unlike exponential discounting, allows discount rates to decrease when the time delay between the decision and the receipt of the utility increases. This dynamic element can account for present biases—or the overvaluing of immediate rewards—in decision-making behavior (O'Donoghue and Rabin, 1999). Recently, Fang and Wang (2015) have estimated a DDC model with a quasi-hyperbolic discount factor modeled after that of O'Donoghue and Rabin (1999). Using data on women's choices regarding mammography screenings, they find that the agents not only exhibit a significant present bias but are also aware of their biases. A different approach is taken by Müller and Reich (2022), who model a potential change in discounting behavior by allowing for a structural break in the discount factor's level. For the well-known bus engine replacement model of Rust (1987), the authors demonstrate that a shift in the discount factor occurs during a time in which the real interest rate level experienced a significant drop. Building on the findings presented by Müller and Reich (2022), Maag and Reich (2023) further investigate the agent's discounting behavior in the bus engine replacement model. In their work, they extend the original model by assuming that discounting follows an autoregressive process of order one (AR(1)). Their results further support the notion that, in the context of the bus engine replacement model, the discounting process appears to be influenced by the prevailing real interest rate levels: By assuming an AR(1) process for discounting, it is possible to estimate the most likely state of the discounting process for each month. This estimated discounting process exhibits a strong negative correlation with the empirically derived real interest rate, and this correlation is statistically significant. Overall, these recent contributions to DDC modeling aim to challenge the assumption of constant discount factors through different approaches, introducing dynamic elements that capture the complexities of discounting behavior.

This paper extends the model proposed by Maag and Reich (2023). We introduce two distinct model frameworks for integrating interest rates into the discounting process of the bus engine replacement model of Rust (1987): the level model and the error model. Both specifications enable us to address the constraints of constant discount factors and, consequently, constant time preferences, by incorporating changes in the agent's macroeconomic environment into their intertemporal decision-making.

In the following section, we revisit the general concept of discounting as a stochastic process based on the bus engine replacement model of Rust (1987). Subsequently, in Section 3.3, two model frameworks are presented, allowing for the incorporation of (real) interest rates into the discounting process. Additionally, we discuss the challenges associated with the proposed approaches to discounting and examine various nominal interest rates and inflation measures that can be used as proxies for the real interest rate, in Section 3.4. In the empirical part, Section 3.5, we provide preliminary evidence for the appropriate choice of interest rate and aim to identify a suitable model for its incorporation into the discounting process.

3.2 The Bus Engine Replacement Model with a Stochastic Discounting Process

This section provides an overview of the bus engine replacement model with a stochastic discounting process. For details regarding the derivation of the likelihood function, readers are referred to Maag and Reich (2023).

The bus engine replacement model introduced by Rust (1987) addresses the decision-making problem of Harold Zurcher, the superintendent of maintenance at the Madison (Wisconsin) Metropolitan Bus Company. Zurcher's responsibilities involve conducting monthly inspections of the company's bus fleet and deciding whether each bus should undergo *normal maintenance* or have its engine replaced. His decisions are influenced by the bus's age, which is determined by the odometer reading. When Zurcher opts for normal maintenance, including tasks like component repairs and brake adjustments, the costs incurred depend largely on the bus's mileage at the time of inspection. Choosing *replacement*, meanwhile, involves a complete overhaul of the bus's engine. When chosen, this option leads to fixed costs and a reset of the odometer reading to zero. Zurcher's immediate utility each month t can be formalized as

$$U_\theta(i_t, x_t) + \epsilon_t(i_t) \text{ with } U_\theta(i_t, x_t) = \begin{cases} -RC & \text{if } i_t = 1 \\ -\theta_1 x_t & \text{if } i_t = 0 \end{cases}. \tag{3.1}$$

The decisions are denoted as i_t, and the mileage state as x_t. The model's dependence on structural parameters is denoted by the subscript θ. A value of one for the decision, $i_t = 1$, corresponds to *replacement*, resulting in fixed costs RC. Conversely, *normal maintenance* is equal to a value of zero, $i_t = 0$, and leads to costs θ_1 times the mileage state. Note that these decisions not only result in direct costs from $U_\theta(i_t, x_t)$ but also include a choice-specific random cost component $\epsilon_t(i_t)$. This random cost component is assumed to be IID and to follow an extreme value type I distribution.

Zurcher's decision-making process is dynamic since his decision at the time of inspection affects not only immediate costs but also future costs. With normal maintenance, the bus continues to accrue mileage, potentially resulting in higher costs in the future. In contrast, choosing replacement usually incurs higher immediate fixed costs than maintenance does. However, the odometer is reset to zero, likely leading to lower costs in the next month. Thus, each month Zurcher aims to minimize both his current costs and expected future costs. Following the extended optimality condition of Stachurski and Zhang (2021) and assuming Zurcher behaves dynamically optimally, the Bellman equation defines a solution to his infinite horizon decision problem as

$$V_\theta(x_t, \epsilon_t, \beta_t) = \max_{i \in \{0,1\}} \{U_\theta(i, x_t) + \epsilon_t(i) + \beta_t \mathbb{E}[V_\theta(x_{t+1}, \epsilon_{t+1}, \beta_{t+1}) \mid i, x_t, \beta_t; \theta]\}. \tag{3.2}$$

In the original model, discounting is a factor, β. In the extension by Maag and Reich (2023), discounting is treated as a Markov process, $\{\beta_t\}$. Consequently, when Zurcher forms his expectations about future costs he has to consider not only the expected future mileage states but also the expected discounting state. Thus, beliefs about the next period's discounting state, β_{t+1}, are part of Zurcher's expectation-building process. The error term is assumed to be conditionally independent of the mileage states and the discounting state is assumed to be independent of both the error term and the mileage states. These assumptions, along with the distributional form assumption regarding the error term, allow for the definition of the expected value function. The expected value function is defined as

$$\begin{aligned} EV_\theta(i_t, x_t, \beta_t) \equiv& \mathbb{E}[V_\theta(x_{t+1}, \epsilon_{t+1}, \beta_{t+1}) \mid i_t, x_t, \beta_t; \theta] && (3.3)\\ =& \int_0^\infty \int \log \sum_{j \in \{0,1\}} \exp\left[U(j, x_{t+1}) + \beta_{t+1} EV_\theta(j, x_{t+1}, \beta_{t+1})\right] && (3.4)\\ & \cdot p_x(x_{t+1} \mid i_t, x_t; \theta) p_\beta(\beta_{t+1} \mid \beta_t; \theta) d\beta_{t+1} dx_{t+1} && (3.5) \end{aligned}$$

and establishes a fixed point for the functional equation $EV_\theta(i_t, x_t, \beta_t) = T(EV_\theta)(i_t, x_t, \beta_t)$.

Zurcher knows his discounting state at time t and the parameters defining the distribution of the process. Additionally, he has knowledge of the cost parameters and forms beliefs about the transition of the mileage states. For the modeler, meanwhile, these aspects remain unobserved. The modeler's observations are limited to the mileage states and the agent's monthly decisions. Therefore, in addition to estimating the original parameters—that is, the cost parameters and the mileage transition probabilities—with discounting as a process the modeler is also required to estimate the distributional parameters of the discounting process. Maag and Reich (2023) estimate the structural parameters $\theta = \{RC, \theta_1, \theta_3, B\}$ using maximum likelihood. Note that θ_3 denotes the discretized mileage transition probabilities, and B refers to a set of distributional parameters characterizing the discounting process. In the work of Rust (1987), mileage states are discretized into bins of 5,000 miles, with a maximum mileage state of 450,000 miles—that is, $x_t \in \{1, \ldots, 90\}$. The discretized mileage transition probabilities $\theta_3 \equiv \theta_{3\Delta_x}$ follow the Markov process

$$\theta_{3\Delta_x} = Pr\left(x_{t+1} = \Delta_x + (1 - i_t)x_t \mid i_t, x_t; \theta\right) \text{ with } \Delta_x \in \{0, 1, 2\}. \tag{3.6}$$

Thus, during each period a bus transitions a maximum of two mileage bins, $\Delta_x \in \{0, 1, 2\}$. The discounting states are unobserved, which introduces an integral over the discounting process $\{\beta_t\}$ into the likelihood function. This integral can be separated into a series of low-dimensional integrals and interpolation problems using the recursive likelihood integration (RLI) method proposed by Reich (2018). The likelihood function $L(\theta)$, considering all buses $n \in N$, can be derived as

$$L(\theta) = L_1(\theta) L_2(\theta) \tag{3.7}$$

with

$$L_1(\theta) = \prod_{t=1}^{T-1} \prod_{n=1}^{N} Pr(x_{t+1,n} \mid i_{t,n}, x_{t,n}; \theta) \tag{3.8}$$

and

$$L_2(\theta) = \int f_1(\beta_1) p_\beta(\beta_1;\theta) \prod_{n=1}^{N} Pr(i_{1,n} \mid x_{1,n}, \beta_1;\theta) d\beta_1. \tag{3.9}$$

The function $f_1(\beta_1)$ is defined according to the recurrence relation

$$f_t(\beta_t) = \begin{cases} 1, & \text{if } t \geq T \\ \int f_{t+1}(\beta_{t+1}) p_\beta(\beta_{t+1} \mid \beta_t;\theta) \prod_{n=1}^{N} Pr(i_{t+1,n} \mid x_{t+1,n}, \beta_{t+1};\theta) d\beta_{t+1} & \text{if } 0 < t < T \end{cases}, \tag{3.10}$$

and the conditional decision probabilities are computed analogously to the decision probabilities in the original model by Rust (1987) as

$$Pr(i_{t,n} \mid x_{t,n}, \beta_t;\theta) = \frac{\exp\left[U_\theta(i_{t,n}, x_{t,n}) + \beta_t EV_\theta(i_{t,n}, x_{t,n}, \beta_t)\right]}{\sum_{j\in\{0,1\}} \exp\left[U_\theta(j, x_{t,n}) + \beta_t EV_\theta(j, x_{t,n}, \beta_t)\right]}. \tag{3.11}$$

Note that for notational brevity, the likelihood's dependence on the states is omitted.

Thus far, the discounting process is described as a general stochastic process. In the work of Maag and Reich (2023), discounting is assumed to follow an AR(1) process,

$$\beta_{t+1} = \mu_{\beta_1}(1-\rho) + \rho\beta_t + \varepsilon_t \text{ with } \varepsilon_t \sim N(0, \sigma_\varepsilon^2). \tag{3.12}$$

Consequently, the discounting parameters to be estimated are reduced to three: the stationary mean μ_{β_1}, the standard deviation of the innovations σ_ε, and the persistence ρ. Maag and Reich (2023) estimate the maximum likelihood estimates (MLEs) of the discounting parameters to be $\hat{\mu}_{\beta_1} = 1.028$, $\hat{\sigma}_\varepsilon = 6.410\text{E-}04$, and $\hat{\rho} = 9.806\text{E-}01$. These values suggest that the discounting process exhibits relatively high persistence, small variance in the innovations, and a stationary mean above one. Once the parameters of the discounting process are found, they further estimate a so-called *most likely discounting process*. That is to say, fixing the structural parameters to the MLE $\theta = \hat{\theta}$, for each month, the discounting state with the largest density contribution to the estimated likelihood is estimated with

$$\begin{aligned} \max_{\{\beta_t\}_{t=1}^T} L_\theta(\{\beta_t\}_{t=1}^T) = \quad & \left[p_\beta(\beta_1;\theta) \prod_{n=1}^{N} Pr(i_{1,n} \mid x_{1,n}, \beta_1;\theta) \right] \\ & \cdot \left[\prod_{t=1}^{T-1} p_\beta(\beta_{t+1} \mid \beta_t;\theta) \prod_{n=1}^{N} Pr(i_{t+1,n} \mid x_{t+1,n}, \beta_{t+1};\theta) \right]. \end{aligned} \tag{3.13}$$

In the work of Maag and Reich (2023), the estimated most likely discounting process for the MLE, $\{\hat{\beta}_t\}$, shows an overall decreasing trend over the observation period from December 1974 to May 1985. Additionally, the discounting states exhibit a statistically significant correlation with prevailing real interest rates. Maag and Reich (2023) calculate the monthly real interest rate as the difference between the Federal Fund Effective rate and an inflation rate based on the Consumer Price Index for All Urban Consumers. The Pearson correlation coefficient is approximately -0.71 with a p value of less than 1 percent. This strongly negative and significant correlation between historical real interest rate levels and the estimated most likely discounting states suggests that incorporating a discounting process that accounts for this correlation could potentially enhance the model's fit. In the following, we propose two specifications of the discounting process that would allow us to incorporate the relationship with real interest rates.

3.3 Model Extensions

To capture the joint evolution of the discounting state and the interest rate, we can treat the two as a multivariate time series and model them using a vector autoregressive (VAR) process. VAR models allow a time series to be modeled not only as a function of its own lagged values but also as a function of the lagged values of a correlated time series. Following the notation of Lütkepohl (2005), we define the discounting process $\{\beta_t\}$ and the interest rate process $\{r_t\}$ as the two-dimensional vector of variables $\begin{bmatrix}\beta_t & r_t\end{bmatrix}^T$. A general VAR model of lag order p can be expressed as

$$\begin{bmatrix}\beta_t \\ r_t\end{bmatrix} = \begin{bmatrix}c_\beta \\ c_r\end{bmatrix} + A_1 \begin{bmatrix}\beta_{t-1} \\ r_{t-1}\end{bmatrix} + A_2 \begin{bmatrix}\beta_{t-2} \\ r_{t-2}\end{bmatrix} + \cdots + A_p \begin{bmatrix}\beta_{t-p} \\ r_{t-p}\end{bmatrix} + \begin{bmatrix}e_{\beta,t-1} \\ e_{r,t-1}\end{bmatrix} \text{ for } t = 1, 2, \ldots, T. \tag{3.14}$$

For this two-dimensional case the coefficient matrices are $A_p = \begin{bmatrix}a_{p,11} & a_{p,12} \\ a_{p,21} & a_{p,22}\end{bmatrix}$. The covariance matrix for the error term reads $\Sigma_e = \begin{bmatrix}\sigma^2_{e_\beta} & \sigma_{e_\beta,e_r} \\ \sigma_{e_r,e_\beta} & \sigma^2_{e_r}\end{bmatrix}$, where $\sigma^2_{e_\beta}$ and $\sigma^2_{e_r}$ are the error variances of the discounting process and the interest rate, respectively, and $\sigma_{e_\beta,e_r} = \sigma_{e_r,e_\beta}$ denotes their covariance. The innovations are further assumed to be serially uncorrelated.

Finding a specific formulation for the model in (3.14) poses a challenge. The model should capture the (potentially) strong correlation between the discounting state and the interest rate in a realistic manner. At the same time, choosing an overly complex model increases the risk of encountering numerical complications. Ideally, the determination of the number of lags should follow a top-down approach, starting with a higher number of lags and subsequently reducing it based on the significance of the coefficients (Lütkepohl, 2005). However, due to the discounting process not being observable we have to adopt a more practical approach. We decide on the number of lags based on computational feasibility while aiming to ensure an accurate representation of the assumed real-world relations. In the following subsections, we provide two model specifications for the joint evolution of $\{\beta_t\}$ and $\{r_t\}$.

3.3.1 The Level Model

For the first specification, we reduce the number of lags to one and restrict the coefficient $a_{1,21}$ to zero. We refer to this specification as the *level model*. The two processes can be separated into the equations

$$\beta_{t+1} = c_\beta + a_{1,11}\beta_t + a_{1,12}r_t + e_{\beta,t} \tag{3.15}$$

and

$$r_{t+1} = c_r + a_{1,22}r_t + e_{r,t}. \tag{3.16}$$

Further, we define the innovations to follow the multivariate normal distribution $N(0, \Sigma_e)$ with $\Sigma_e = \begin{bmatrix}\sigma^2_{e_\beta} & 0 \\ 0 & \sigma^2_{e_r}\end{bmatrix}$. With this specification, the next period's discounting state is expressed as a function of its own lagged value and the lagged interest rate value. The interest rate process, meanwhile, is solely dependent on its own lagged values, as we do not expect the real interest rate to account for Zurcher's discounting behavior. Zurcher now incorporates not only the current level of β_t when building his expectation regarding the next period's discounting state but also the real interest rate level at time t. Such a specification of the discounting process requires Zurcher's value function to be adjusted for another state variable, r_t (in comparison to an AR(1) discounting process). Thus, the Bellman equation reads

$$V_\theta(x_t, \epsilon_t, \beta_t, r_t) = \max_{i\in\{0,1\}} \{U_\theta(i, x_t) + \epsilon_t(i) + \beta_t \mathbb{E}\left[V_\theta(x_{t+1}, \epsilon_{t+1}, \beta_{t+1}, r_{t+1}) \mid i, x_t, \beta_t, r_t; \theta\right]\} \tag{3.17}$$

with Zurcher's utility function defined as before, in (3.1). Again, the conditional independence assumption regarding the mileage states and the error holds. Furthermore, we assume that the discounting state and the interest rate state are independent of both the mileage state and the error. The expected value function can be defined as

$$\begin{aligned} EV_\theta(i_t, x_t, \beta_t, r_t) &\equiv \mathbb{E}[V_\theta(x_{t+1}, \epsilon_{t+1}, \beta_{t+1}, r_{t+1}) \mid i_t, x_t, \beta_t, r_t; \theta] \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \int \log \sum_{j \in \{0,1\}} \exp\left[U(j, x_{t+1}) + \beta_{t+1} EV_\theta(j, x_{t+1}, \beta_{t+1}, r_{t+1})\right] \\ &\quad \cdot p_x(x_{t+1} \mid i_t, x_t; \theta) p_\beta(\beta_{t+1} \mid \beta_t, r_t; \theta) p_r(r_{t+1} \mid r_t; \theta) d\beta_{t+1} dx_{t+1} dr_{t+1}. \end{aligned} \tag{3.18}$$

Similar to discounting modeled as an AR(1) process, Zurcher knows—besides the cost parameters—his discounting state at time t, the distributional parameters of his discounting process, including the parameters defining the relation to the real interest rate, and the distributional parameters of the real interest rate.[1] Further, he has access to real interest rate data at time t. To the modeler, the discounting states remain unobserved, whereas the real interest rates are observed. The parameter vector to be estimated now reads $\theta = \{RC, \theta_1, \theta_3, B\}$ with $B = \{c_\beta, c_r, a_{1,11}, a_{1,12}, a_{1,22}, \Sigma_e\}$. The likelihood function part $L_2(\theta)$ in (3.7) changes to

$$L_2(\theta) = \int f_1(\beta_1) p_\beta(\beta_1; \theta) p_r(r_1 \mid r_0; \theta) \prod_{n=1}^{N} Pr(i_{1,n} \mid x_{1,n}, \beta_1, r_1; \theta) d\beta_1 \tag{3.19}$$

with

$$f_t(\beta_t) = \begin{cases} 1, & \text{if } t \geq T \\ \int f_{t+1}(\beta_{t+1}) p_\beta(\beta_{t+1} \mid \beta_t, r_t; \theta) p_r(r_{t+1} \mid r_t; \theta) & \\ \prod_{n=1}^{N} Pr(i_{t+1,n} \mid x_{t+1,n}, \beta_{t+1}, r_{t+1}; \theta) d\beta_{t+1} & \text{if } 0 < t < T \end{cases}. \tag{3.20}$$

The decision probabilities are defined according to the closed-form solution provided in (3.11) but adjusted for the real interest rate level,

$$Pr(i_{t,n} \mid x_{t,n}, \beta_t, r_t; \theta) = \frac{\exp\left[U_\theta(i_{t,n}, x_{t,n}) + \beta_t EV_\theta(i_{t,n}, x_{t,n}, \beta_t, r_t)\right]}{\sum_{j \in \{0,1\}} \exp\left[U_\theta(j, x_{t,n}) + \beta_t EV_\theta(j, x_{t,n}, \beta_t, r_t)\right]}. \tag{3.21}$$

While incorporating the level of the interest rate at time t seems reasonable—especially considering the strong correlation observed by Maag and Reich (2023) between the most likely discounting process for the MLE and the empirically derived real interest rate—it creates some computational challenges. Such a specification requires adjusting the expected value function (3.18) to account for an additional state variable and increases the number of parameters to be estimated by four compared to an AR(1) discounting process. This expanded parameter space results in increased computational complexity and may pose challenges in the estimation. However, since the evolution of the interest rate process is independent of the discounting state, the parameters defining its distribution, including the standard deviation of its innovations, can be estimated separately from the discounting process. More precisely, we propose a two-step approach to estimating θ. First, estimate c_r, $a_{1,22}$, and σ_{e_r} based on the historical real interest rates by approximating (3.16) with an AR(1) model. Then, in a second step, fix the distributional parameters of the interest rate to the estimates from the first step, $\{c_r,\ a_{1,22}, \sigma_{e_r}\} = \{\bar{c}_r, \bar{a}_{1,22}, \bar{\sigma}_{e_r}\}$, to estimate $\theta_{-r} = \{\theta \backslash \{c_r, a_{1,22}, \sigma_{e_r}\}\}$ by maximizing the profile likelihood $L(\theta_{-r}, \{\bar{c}_r, \bar{a}_{1,22}, \bar{\sigma}_{e_r}\}) = L_P(\theta_{-r})$.

[1] Admittedly, the assumption that Zucher knows the distributional parameters of the real interest rate is rather strong, implying that he has perfect foresight about future rates. Given, however, that central banks typically announce changes to their monetary policy, including changes to their target interest rates, in advance, the strength of this assumption is weakened to some extent.

3.3.2 The Error Model

The second model, referred to as the *error model*, addresses the relationship between the discount factor and the interest rate by incorporating shocks to the interest rate into the discounting behavior. That is to say, the correlation between the real interest rate and the discounting states is included by introducing the innovations in the real interest rate into the definition of the discounting process. To be more precise, we define discounting as the (degenerate) VAR process

$$\beta_{t+1} = c_\beta + a_{1,11}\beta_t + \sigma_{e_\beta} e_{\beta,t} + \sigma_{e_r,e_\beta} e_{r,t-1}. \tag{3.22}$$

The real interest rate process is, as in the level model, independent of the discounting process with

$$r_{t+1} = c_r + a_{1,22} r_t + e_{r,t}. \tag{3.23}$$

The innovations follow the multivariate normal distribution $N(0, \Sigma_e)$, where $\Sigma_e = \begin{bmatrix} \sigma^2_{e_\beta} & \sigma_{e_\beta,e_r} \\ \sigma_{e_\beta,e_r} & \sigma^2_{e_r} \end{bmatrix}$. Thus, the discounting state at time $t+1$ is a function of its own lagged value, its innovation $e_{\beta,t}$, and the lagged innovation of the interest rate process $e_{r,t-1}$. The evolution of the interest rate is only dependent on its own past value and innovation. With such an error model, from an economic standpoint Zurcher's expectation regarding his next period's discounting state depends on his discounting state today and the shock to real interest rates yesterday.[2] This association has a rational explanation as media outlets frequently highlight substantial deviations from interest or inflation rate norms rather than absolute levels. Therefore, for Zurcher to behave dynamically optimally he has to adjust his value function not only by the discounting state but also by the last period's interest rate shock,

$$V_\theta(x_t, \epsilon_t, \beta_t, e_{r,t-1}) = \max_{i\in\{0,1\}} \{U_\theta(i, x_t) + \epsilon_t(i) + \beta_t \mathbb{E}[V_\theta(x_{t+1}, \epsilon_{t+1}, \beta_{t+1}, e_{r,t}) \mid i, x_t, \beta_t, e_{r,t-1}; \theta]\}. \tag{3.24}$$

Consequently, the expected value function has to be adjusted for another state variable and reads

$$\begin{aligned} EV_\theta(i_t, x_t, \beta_t, e_{r,t-1}) \equiv& \mathbb{E}[V_\theta(x_{t+1}, \epsilon_{t+1}, \beta_{t+1}, e_{r,t}) \mid i_t, x_t, \beta_t, e_{r,t-1}; \theta] \\ =& \int_{-\infty}^{\infty} \int_0^{\infty} \int \log \sum_{j\in\{0,1\}} \exp\left[U(j, x_{t+1}) + \beta_{t+1} EV_\theta(j, x_{t+1}, \beta_{t+1}, e_{r,t})\right] \\ & \cdot p_x(x_{t+1} \mid i_t, x_t; \theta) p_\beta(\beta_{t+1} \mid \beta_t, e_{r,t-1}; \theta) d\beta_{t+1} dx_{t+1} de_{r,t}. \end{aligned} \tag{3.25}$$

Again, the model error ϵ_t is conditionally independent of the mileage states. Additionally, both the discounting state and the interest rate shock are independent of the mileage states and the model error.

As in the simple AR(1) discounting model and the level model, Zurcher knows his discounting state at time t and the distributional parameters characterizing the discounting process, including those that describe its relationship to the interest rate, and observes the shock to the interest rate. Conversely, from the modeler's standpoint values for the first two elements remain unobserved, while the real interest rate innovations are observed. Compared to discounting as an AR(1) process, the parameter vector to be estimated is $\theta = \{RC, \theta_1, \theta_3, B\}$ with $B = \{c_\beta, c_r, a_{1,11}, a_{1,22}, \Sigma_e\}$. The $L_2(\theta)$-part of the likelihood function (3.7) now reads

$$L_2(\theta) = \int f_1(\beta_1) p_\beta(\beta_1, e_{r,0}; \theta) \prod_{n=1}^{N} Pr(i_{1,n} \mid x_{1,n}, \beta_1, e_{r,0}; \theta) d\beta_1, \tag{3.26}$$

[2]Or, in this context, "yestermonth".

with the recurrence relation

$$f_t(\beta_t) = \begin{cases} 1, & \text{if } t \geq T \\ \int f_{t+1}(\beta_{t+1}) p_\beta(\beta_{t+1} \mid \beta_t, e_{r,t-1}; \theta) \prod_{n=1}^{N} Pr(i_{t+1,n} \mid x_{t+1,n}, \beta_{t+1}, e_{r,t}; \theta) d\beta_{t+1} & \text{if } 0 < t < T \end{cases}. \quad (3.27)$$

The decision probabilities are defined as

$$Pr(i_{t,n} \mid x_{t,n}, \beta_t, e_{r,t-1}; \theta) = \frac{\exp\left[U_\theta(i_{t,n}, x_{t,n}) + \beta_t EV_\theta(i_{t,n}, x_{t,n}, \beta_t, e_{r,t-1})\right]}{\sum_{j \in \{0,1\}} \exp\left[U_\theta(j, x_{t,n}) + \beta_t EV_\theta(j, x_{t,n}, \beta_t, e_{r,t-1})\right]}. \quad (3.28)$$

The error model specification requires the estimation of four additional parameters in comparison to discounting as an AR(1) process: c_r, $a_{1,22}$, σ_{e_β,e_r}, and σ_{e_r}. Since the evolution of the interest rate process is independent of the discounting state, we propose the same two-step estimation process as in the level model to estimate the structural parameters. That is to say, in a first step estimate $\{c_r, a_{1,22}, \sigma_{e_r}\}$ by fitting an AR(1) model to the historical real interest rates. Then, fix the values of these parameters, to estimate in a second step $\theta_{-r} = \{\theta \backslash \{c_r, a_{1,22}, \sigma_{e_r}\}\}$ by maximizing the profile likelihood function $L_P(\theta_{-r})$.

We have introduced two model specifications that allow for the incorporation of the (potentially) strong correlation between discounting states and real interest rates into the discounting process: the level model and the error model. For both models, we decided to restrict the lag order to one. Note that the lag does not necessarily have to be the previous month's value. In the level model, r_t in (3.15) can be generalized to r_{t-p}, where $p \in \mathbb{N}_0$. In the error model, the same holds true for $e_{r,t-1}$ in (3.22), where the generalized form reads $e_{r,t-p}$ with $p \in \mathbb{N}_1$. Before we empirically examine which of the two model specifications best captures the relation between discounting states and real interest rates, the following section discusses appropriate real interest rate candidates that can be used in evaluating these models.

3.4 Discussion of the Real Interest Rate

In Maag and Reich (2023), the real interest rate used for calculating the correlation with the most likely discounting process is derived from the difference between the Federal Fund Effective rate and an inflation rate based on the Consumer Price Index for All Urban Consumers. This specific real interest rate is just one of several potential choices available, as the optimal selection of the nominal interest rate and the inflation indicator depends on the context of the model. The following subsections discuss the various nominal interest rates and inflation metrics commonly used in economic applications and provide further details on potential candidates that could be used for the bus engine replacement model. Before we proceed, it is important to clarify whether, within the context of the bus engine replacement model, it is appropriate to incorporate real interest rates or if nominal interest rates should be used instead. Real interest rates are simply defined as nominal interest rates adjusted for inflation, for example, by subtracting the monthly inflation rate from the monthly nominal interest rate. As a general rule, nominal cash flows are discounted using nominal interest rates, while real cash flows are discounted using real interest rates (Volkart, 2006). The original data set of Rust (1987) covers the months between December 1974 and May 1985. These years were characterized by periods of high inflation, also known as *the Great Inflation*. The Great Inflation lasted from 1965 to 1982, marking almost two decades of extreme inflation in the United States. As a result of questionable monetary policies and an energy crisis during this era inflation soared, reaching nearly 15 percent in March 1980 (Bryan, 2013). Although inflation reached a peak during the period covered by the original data from the bus engine replacement model, the cost parameters in the model are assumed to remain constant: neither the replacement costs nor the maintenance costs are adjusted over time, and thus they cannot account for any inflation-related changes (Müller and Reich, 2022; Maag and Reich, 2023). Hence, it is reasonable to assume that the costs in the model are expressed in real values, unaffected by inflation. Consequently, the goal of the following subsections is to find an appropriate real interest rate that can be used for the empirical analysis.

3.4.1 Nominal Interest Rate Candidates

Nominal interest rates reflect stated interest rates without accounting for the impact of inflation. According to Hull (2022), there are three main factors that need to be considered when deciding on an appropriate nominal interest rate: the reference risk-free rate, the risk premium, and the time horizon. As implied by the term, a risk-free rate is the rate of return for an investment without any risk. While it is a theoretical concept—since truly risk-free investments do not exist in reality—certain bonds are considered practically risk free and are employed as benchmarks for the risk-free rate. In the realm of economic applications, three reference rates have emerged as commonly used benchmarks for the risk-free rate; these will be discussed shortly. The risk premium or credit spread is any add-on to the risk-free interest rate, and should account for the inherent risk associated with a particular investment. For example, when a bank issues a loan to a company, the interest rate charged is determined by conducting a thorough assessment of the company's financial situation. This assessment takes into consideration various factors, including the company's default risk. Finally, the maturity of an investment, which is the time until the principal is due, also influences the interest rate. There are three types of maturities: short-term investments with a due date under a year, medium-term investments with a due date ranging between one and ten years, and long-term investments with a maturity of over ten years. Typically, longer investment terms are associated with higher risk, leading to higher interest rates. Since the time horizon affects the riskiness of an investment, it can be summarized under the concept of the risk premium. The following discussion regarding the risk-free rates draws from the work of Volkart (2006) and Hull (2022).

Government bonds and bills serve as financial instruments issued by countries to raise funds through borrowing in their respective currencies. Given the prevailing notion that developed countries possess minimal default risk, the interest rates associated with government bonds are widely acknowledged to be practically risk free. In financial applications, however, government bonds are usually excluded when modeling a risk-free portfolio. This is primarily due to the fact that income derived from government bonds is tax-exempt in many countries, resulting in lower interest rates compared to taxable low-risk investments. Additionally, various regulatory factors can also contribute to the lower rates associated with government securities. The next closest alternative to a risk-free rate is overnight rates. These rates are frequently employed as a reference for the risk-free rate in financial applications. Overnight rates pertain to the interest rates used by financial institutions for borrowing and lending money among themselves to maintain mandatory reserve requirements at the central bank. Since central banks manage and administer these rates, they serve as key policy instruments for influencing the economic activities of a country. Another interest rate with minimal to negligible risk is the repurchase agreement (repo) rate. Repos are financial instruments used for short-term borrowing by financial institutions. In a typical scenario, a bank borrows cash from another bank and offers government securities as collateral, with an agreement to repurchase the securities at a higher price after a short period of time. The repo rate is determined by the difference between the borrowed funds and the repurchase price. Since repos are backed by the underlying securities, they are generally considered to be relatively risk free (Hull, 2022).

The database maintained by the Research Department at the Federal Reserve System (the Fed), known as Federal Reserve Economic Data (FRED, 2023b), serves as our primary source for (real) interest rate candidates. The original data set used by Rust (1987) spans from December 1974 to May 1985 and includes monthly records of odometer readings and maintenance decisions. Consequently, candidate nominal interest rates have to correspond to the same frequency and encompass the same years. Table 3.1 provides an overview of the interest rates we have added to our list of candidates for the reference nominal interest rate, along with descriptive statistics for their monthly values from December 1974 to May 1985.

For the risk-free rate, we gathered monthly rates for US Treasury bills and US Treasury bonds. The short-term bills have maturities of three months, six months, and one year, while US Treasury bonds' maturities range from one year to up to 20 years. The tickers for the short-term bills are TB3MS, TB6MS, and TB1YR, and the tickers for the bonds are GS1, G3, GS5, GS10, and GS20 (see Table 3.1 for a description). Additionally, there is a discontinued series available for the rate of the unweighted average of bonds with a maturity of over ten years: the Long-Term US Government Securities rate (LTGOVTBD). Furthermore, we included the Federal Fund Effective rate (FEDFUNDS) in our selection of reference risk-free rates. FEDFUNDS is the overnight interbank rate that serves as both the target rate and the benchmark rate set by the Fed (FRED, 2023c). Regarding a potential candidate for the repo rate, it is worth noting that FRED does not provide monthly repo rate data prior to the year 2018, precluding it from being a potential candidate for our analysis.[3] The rates selected so far are considered to be relatively risk free. Since there is no monthly data on risk premiums available for before 1982, in order to incorporate a credit spread into the nominal interest rate we expanded our selection of interest rate options by including Moody's Corporate Bond rates for both Aaa and Baa ratings, as well as the Bank Prime Loan rate—see tickers AAA, BAA, and MPRIME, respectively, in Table 3.1. The rates of AAA and BAA are regarded as benchmark rates for evaluating the performance of bonds with high credit ratings (FRED, 2023d,e). The MPRIME rate is the interest rate at which banks issue loans to the most creditworthy corporations (FRED, 2023a). Both of these rates have a relatively small credit spread. However, within the context of the bus engine replacement model the agent is employed by a city-owned corporation, and the negative utility or costs he incurs are solely related to his work (Rust, 1987). It is thus reasonable to assume that if the agent's discount factor were influenced by a (real) interest rate, it would be an interest rate with a relatively conservative credit spread.

Referring to the statistical measures in Table 3.1, we observe the following: Overall, the interest rate candidates demonstrate values in the mean and median that align with theoretical expectations. The interest rates associated with higher risk premiums, such as AAA, BAA, and MPRIME, exhibit both higher mean and median monthly values. In comparison, treasury bill rates with shorter maturities have lower interest rates in both the mean and median, with the three-month rate TB3MS showing the lowest mean and median monthly interest rate. This pattern is consistent with the lower risk associated with rates of a shorter term. Additionally, the highest variance is observed for the MPRIME rate. This observation aligns with the inherent volatility of commercial lending influenced by economic factors and borrower risk. Conversely, the lower variance in the interest rates for government-backed securities, such as GS20, reflects the stability and low-risk nature of these instruments. In summary, the candidates for the nominal interest rate exhibit a considerable range, spanning from the overall smallest monthly value of approximately 3.55E-03 to the overall largest value of 1.57E-02. We plot the candidate nominal interest rates from December 1974 to May 1985 in Figure 3.1. Generally, over time, the interest rates exhibit a similar trend. From December 1976 onward, there is an overall increase in the interest rate values, followed by a drop around July 1980. The peak around the end of 1980 and the beginning of 1981 is particularly notable for the MPRIME and the FEDFUNDS rates. From December 1981 onward, the rates display a similar pattern again.

To summarize this subsection, both the statistical summary and the plot of the nominal interest rates emphasize the importance of considering various interest rates for our empirical analysis. Apart from selecting an appropriate nominal interest rate, it is equally important to explore suitable inflation metrics. This aspect

[3] There is a discontinued time series for the aggregated volume of repos on the commercial bank level that dates back to 1974 (FRED, 2023f). However, it cannot be used to calculate an interest rate since there is no information available on the interest paid or the duration of these repos.

Ticker	Description	Mean	Median	Min	Max	Variance
AAA	*Moody's Seasoned Aaa Corporate Bond Yield*	8.62E−03	8.72E−03	6.37E−03	1.21E−02	2.94E−06
BAA	*Moody's Seasoned Baa Corporate Bond Yield*	9.72E−03	9.89E−03	7.05E−03	1.33E−02	3.72E−06
FEDFUNDS	*Federal Funds Effective Rate*	7.62E−03	7.41E−03	3.76E−03	1.47E−02	8.27E−06
GS1	*Market Yield on U.S. Treasury Securities at 1-Year Constant Maturity*	7.68E−03	7.48E−03	3.99E−03	1.30E−02	5.25E−06
GS10	*Market Yield on U.S. Treasury Securities at 10-Year Constant Maturity*	8.19E−03	8.24E−03	5.55E−03	1.19E−02	3.20E−06
GS20	*Market Yield on U.S. Treasury Securities at 20-Year Constant Maturity*	8.24E−03	8.36E−03	5.89E−03	1.18E−02	2.88E−06
GS3	*Market Yield on U.S. Treasury Securities at 3-Year Constant Maturity*	7.95E−03	7.73E−03	4.61E−03	1.26E−02	4.14E−06
GS5	*Market Yield on U.S. Treasury Securities at 5-Year Constant Maturity*	8.07E−03	7.99E−03	4.95E−03	1.24E−02	3.72E−06
LTGOVTBD	*Long-Term U.S. Government Securities*	7.71E−03	7.92E−03	5.18E−03	1.11E−02	3.16E−06
MPRIME	*Bank Prime Loan Rate*	9.01E−03	8.73E−03	5.06E−03	1.57E−02	8.75E−06
TB1YR	*1-Year Treasury Bill Secondary Market Rate*	7.05E−03	6.93E−03	3.79E−03	1.15E−02	3.94E−06
TB3MS	*3-Month Treasury Bill Secondary Market Rate*	6.93E−03	6.67E−03	3.55E−03	1.27E−02	5.41E−06
TB6MS	*6-Month Treasury Bill Secondary Market Rate*	7.07E−03	6.90E−03	3.68E−03	1.21E−02	4.85E−06

Table 3.1: Description and statistical summary of the nominal interest rate candidates. The rates are on a monthly basis for the period from December 1974 to May 1985. Source: Own calculation based on data from FRED (2023b).

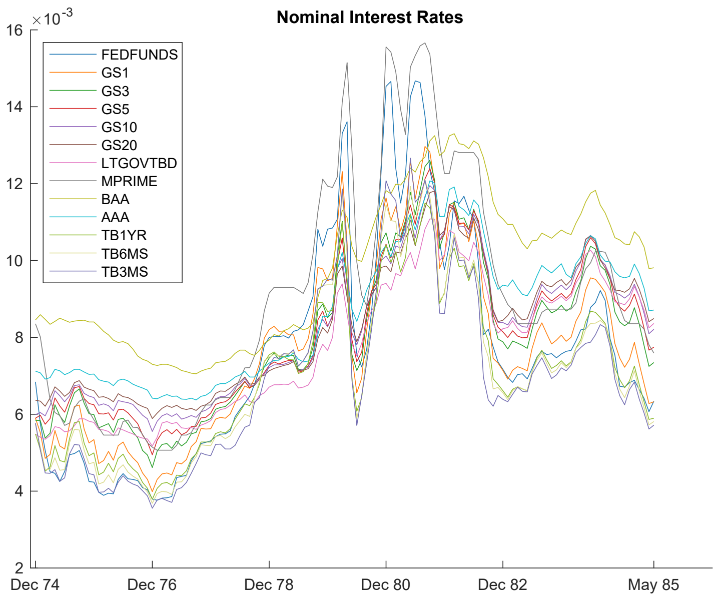


Figure 3.1: Overview of the monthly nominal interest rate candidates for the period from December 1974 to May 1985. Source: Own representation based on data from FRED (2023b).

will be addressed in the following subsection.

3.4.2 Inflation Rate Candidates

Inflation measures the increase in the price levels of goods and services and is typically expressed as a percentage change. Usually, price indices, which monitor the prices of a basket of goods and services, serve as a reference point for assessing the overall price level (Blanchard, 2013). The following paragraphs discuss the most commonly used price indices and other measures of inflation.

Consumer price indices (CPIs) and wholesale/producer price indices are widely used as reference indicators for measuring inflation. The CPI of a country represents a periodic record of the prices for various goods and services typically consumed by the average person. The CPI basket is subject to regular revisions, with discontinued items being removed, new items added, and the items' weights adjusted as needed. Given that there is no single *average* consumer, the CPI is often divided into subindices that differ based on the region they cover or the characteristics of the goods and the consumer groups they represent. This allows for a more detailed analysis of price trends and inflationary impacts specific to different segments of the population. For example, the US Bureau of Labor Statistics (BLS, 2023a) distinguishes three main CPI series:

(A) the CPI for all urban consumers CPI-U,

(B) the CPI for urban wage earners and clerical workers CPI-W, and

(C) the chained CPI for all urban consumers C-CPI-U.

While the CPI-U is representative of the consumption habits of all consumers in urban areas, the CPU-W covers only a subgroup of urban consumers, including clerical workers, construction workers, and laborers. For both indices, the weights of the items are set at the beginning of the year and then kept constant. The C-CPI-U also contains the prices of goods consumed by the urban population, but the weights are adjusted more regularly to provide a more up-to-date representation of consumers' habits. Furthermore, the BLS publishes monthly CPIs for the four major census regions: Northeast, South, West, and Midwest. Additionally, average prices of items are available for 211 categories and 38 regional areas (BLS, 2023a). In many countries, central banks typically target a modified version of the CPI known as the core CPI. The core CPI is calculated by excluding certain goods with volatile or highly seasonal prices from the CPI calculation. For example, the Fed adopts the CPI-U excluding food and energy prices as one of the key indicators for guiding its inflation policies (BLS, 2023a).

While CPIs are commonly used as a key metric for measuring inflation, they face criticism regarding how accurately they represent actual changes in price levels (Kliesen, 1997). One notable criticism is the limited consideration of changes in the quality of goods. For instance, computer prices have significantly decreased in the last decades, while their computing power has increased almost exponentially, resulting in computers becoming comparatively cheaper in comparison to their processing capabilities (Roser et al., 2023). Such price-versus-performance improvements are not adequately captured in CPI calculations, leading to a so-called *quality bias* (Kliesen, 1997). Further, CPIs are also prone to substitution and new product biases. *Substitution bias* arises from fixing the weights and the items in the consumer basket at the beginning of the year (Kliesen, 1997). Thus, the index fails to account for consumers' tendency to switch to alternative products when certain goods become too expensive. As a result, price changes in these alternative goods may not be included in the CPI, even though they reflect a part of consumers' spending. To mitigate the biases arising from changes in consumers' behavior and preferences over time, chained CPIs reduce the substitution effect by regularly adjusting the weights assigned to specific goods and services. For instance, the weights of the C-CPI-U are updated on a monthly basis (BLS, 2023a). *New product biases* occur because new goods and services need to establish their significance before being incorporated into the market basket, leading to a delay in capturing their price developments (Kliesen, 1997).

Another price index that can serve as a reference measure for inflation is the Producer Price Index (PPI), which was previously known as the Wholesale Price Index in the United States before it was renamed in 1978 (BLS, 2023c). The PPI measures the average prices of goods at the producer or wholesale level—that is, before they reach the retail market. These prices are considered to capture changes in the overall economy more quickly than consumer prices because increases in prices first reach the producer level before eventually being passed on to consumers. However, using a PPI as an inflation metric has two main drawbacks (BLS, 2023d). Firstly, if the PPI basket includes a higher share of products that rely on imported raw materials, it can be more affected by global shocks whose impacts may not be passed on to the national level. Secondly, the PPI mainly covers the prices of goods and thus disregards a large part of the service economy.

The Gross Domestic Product (GDP) deflator is the third most used indicator for measuring inflation (Blanchard, 2013). It measures price changes by relating changes in nominal GDP to real GDP. Unlike the CPI and the PPI, the GDP deflator considers price changes for all goods and services produced within a country's borders. As it is not based on a fixed basket of goods, the GDP deflator offers advantages over price indices when it comes to handling substitution effects and the introduction of new products. However, the GDP deflator also has its limitations (Church, 2016). Since GDP includes all goods and services produced in an economy, it may not accurately reflect the purchasing power of the average consumer as the production of consumable goods represents only a portion of total GDP. Additionally, the GDP deflator does not include imported goods. Imported goods make up a large part of people's consumption, which limits the accuracy of the GDP deflator when it comes to reflecting people's purchasing power.

Table 3.2 contains an overview of the indices we used as candidates for measuring inflation in our analysis. The data is sourced from FRED (2023b), and the monthly inflation rates, ι_t, are calculated as

$$\iota_t = \frac{I_t - I_{t-1}}{I_{t-1}}, \tag{3.29}$$

where I_t is the index value at month t. Before examining the statistical summary of the calculated rates, we will first provide some remarks regarding the selected inflation measures. Firstly, the CPI is available for various subindices that vary by region, consumption group, or specific goods and services. In our analysis, we included two main CPI indices published by the BLS: CPI-U and CPI-W at the state level. The chained CPI-U is not available for before 1990. We, however, added a chain-type consumption index to our inflation metrics, the Personal Consumption Expenditures Excluding Food and Energy (PCEPILFE) index. Similar to CPIs, personal consumption indices also assess consumer prices, but the weights for adjustment are derived from business surveys (BLS, 2011). Additionally, we attempted to select CPI subindices that cover industries that have an impact on the agent in our model, particularly the transportation service industry. Most of these CPI (sub)indices are available in both seasonally adjusted and non–seasonally adjusted forms. Overall, the CPI (sub)indices we have added to our selection of inflation rate candidates comprise CPIAUCNS, CPIAUCSL, CPIULFNS, CPIULFSL, CUSR0000SAS4, CUUR0000SAS4, CWSR0000SA0, CWSR0000SA0L1, CWSR0000SAS4, CWUR0000SA0, CWUR0000SA0L1, and CWUR0000SAS4. For a description of these indices, please refer to Table 3.2. Secondly, while the BLS provides PPI subindices for different industries and sectors, only the main index for all commodities—that is, the Producer Price Index by Commodity All Commodities (PPIACO)—is available for the full period from December 1974 to May 1985. Thirdly, the GDP deflator is only available on a quarterly basis and cannot be incorporated into our analysis. Finally, the costs Zurcher incurs on a monthly basis are not his personal out-of-pocket expenses. Therefore, using consumer prices as a measure of inflation may be questionable. However, considering that the Fed also uses a subindex of the CPI to target inflation, it

Ticker	Description	SA	Mean	Median	Min	Max	Variance
CPIAUCNS	*Consumer Price Index for All Urban Consumers: All Items in U.S. City Average*	no	5.79E−03	5.15E−03	-4.08E−03	1.52E−02	1.32E−05
CPIAUCSL	*Consumer Price Index for All Urban Consumers: All Items in U.S. City Average*	yes	5.79E−03	5.25E−03	-3.06E−03	1.43E−02	1.14E−05
CPIULFNS	*Consumer Price Index for All Urban Consumers: All Items Less Food in U.S. City Average*	no	6.03E−03	5.64E−03	-5.10E−03	1.59E−02	1.45E−05
CPIULFSL	*Consumer Price Index for All Urban Consumers: All Items Less Food in U.S. City Average*	yes	6.04E−03	5.37E−03	-4.09E−03	1.67E−02	1.31E−05
CUSR0000SAS4	*Consumer Price Index for All Urban Consumers: Transportation Services in U.S. City Average*	yes	6.63E−03	5.94E−03	-2.04E−03	2.92E−02	2.51E−05
CUUR0000SAS4	*Consumer Price Index for All Urban Consumers: Transportation Services in U.S. City Average*	no	6.63E−03	5.80E−03	-2.04E−03	3.12E−02	2.61E−05
CWSR0000SA0	*Consumer Price Index for All Urban Wage Earners and Clerical Workers: All Items in U.S. City Average*	yes	5.69E−03	5.22E−03	-3.05E−03	1.55E−02	1.25E−05
CWSR0000SA0L1	*Consumer Price Index for All Urban Wage Earners and Clerical Workers: All Items Less Food in U.S. City Average*	yes	5.93E−03	5.37E−03	-5.08E−03	1.71E−02	1.46E−05
CWSR0000SAS4	*Consumer Price Index for All Urban Wage Earners and Clerical Workers: Transportation Services in U.S. City Average*	yes	6.52E−03	5.87E−03	-3.06E−03	2.90E−02	2.61E−05
CWUR0000SA0	*Consumer Price Index for All Urban Wage Earners and Clerical Workers: All Items in U.S. City Average*	no	5.70E−03	5.20E−03	-4.07E−03	1.42E−02	1.43E−05
CWUR0000SA0L1	*Consumer Price Index for All Urban Wage Earners and Clerical Workers: All Items Less Food in U.S. City Average*	no	5.94E−03	5.58E−03	-5.07E−03	1.68E−02	1.60E−05
CWUR0000SAS4	*Consumer Price Index for All Urban Wage Earners and Clerical Workers: Transportation Services in U.S. City Average*	no	6.51E−03	5.84E−03	-3.06E−03	2.90E−02	2.70E−05
PCEPILFE	*Personal Consumption Expenditures Excluding Food and Energy (Chain-Type Price Index)*	yes	5.21E−03	5.22E−03	7.32E−04	9.98E−03	3.21E−06
PPIACO	*Producer Price Index by Commodity: All Commodities*	no	4.61E−03	3.39E−03	-5.24E−03	2.16E−02	3.08E−05

Table 3.2: Description and statistical summary of the inflation rate candidates. The rates are on a monthly basis for the period from December 1974 to May 1985 and include seasonally adjusted (SA) figures. Source: Own calculation based on data from FRED (2023b).

can be argued that price changes within the CPI basket do have an impact on the overall economy.

Referring to the statistical summary in Table 3.2, the inflation rates derived from consumer price (sub)indices exhibit similar ranges in both the mean and median values. As anticipated, the non–seasonally adjusted CPI inflation rates—CPIAUCNS, CPIULFNS, CUUR0000SAS4, CWUR0000SA0, CWUR0000SA0L1, and CWUR0000SAS4—display greater variation compared to their seasonally adjusted counterparts. Notably, the inflation rate based on the PPIACO shows the highest variance among the inflation rates. This is consistent with the notion that changes in prices have a greater impact on producer price indices. Businesses are often hesitant to immediately pass on increased production costs to consumers. Instead, they may absorb some of these cost increases themselves before eventually adjusting retail prices (Clark, 1995). The chain-type index PCEPILFE has the lowest variance among the inflation rate candidates. In comparison to the other inflation rates, its values also show noticeable differences in the extremes. While most inflation rates have minimum values below zero, indicating periods of deflation, the PCEPILFE has a minimum value of approximately 7.32E-04. PCEPILFE also exhibits the smallest maximum value, of approximately 9.98E-03, whereas the other indices have maxima ranging from 1.42E-02 to 3.12E-02. Chain-type indices typically exhibit less extreme inflation values due to the regular adjustment of the weightings for goods and services—in this case, quarterly (BLS, 2011, 2023a). Figure 3.2 displays the evolution of the monthly inflation rates from December 1974 to May 1985. In general, the rates

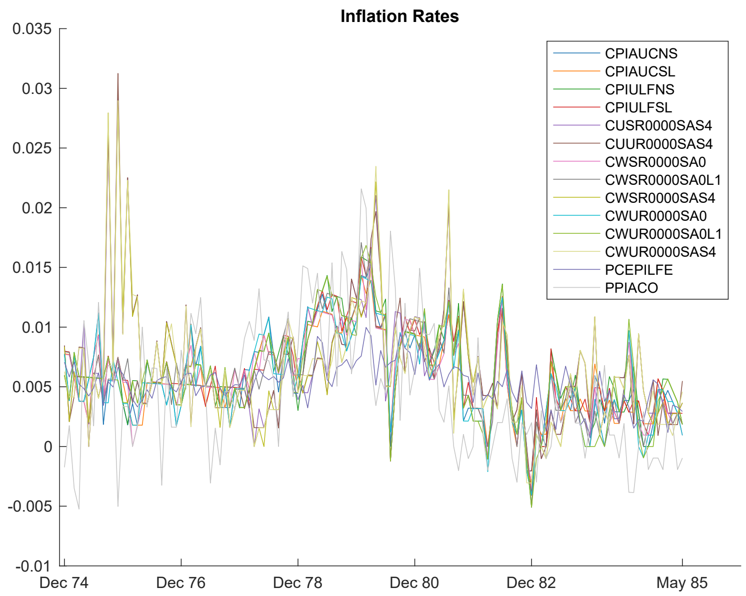


Figure 3.2: Overview of the monthly inflation rate candidates for the period from December 1974 to May 1985. Source: Own representation based on data from FRED (2023b).

exhibit a similar pattern over time. During the initial three years, there is a relatively stable, slightly decreasing trend in the overall levels. The majority of indices reach their highest points around March 1980. Only the inflation rates related to the transportation service industry—CUSR0000SAS4, CUUR0000SAS4, CWSR0000SAS4, and CWUR0000SAS4—reach their maximum values already in November 1975. From March 1980 onward, the inflation rate candidates appear to align again and show an overall decreasing trend.

In summary, this subsection highlights the variability observed among various inflation metrics and stresses the importance of considering different inflation measures, as they can have an impact on the model outcomes.

3.4.3 Real Interest Rate Candidates

The real interest rates are simply calculated by subtracting the monthly inflation rates from the monthly nominal interest rates. By combining each of the 13 nominal interest rate candidates with each of the 14 inflation rate candidates, we obtain a total of 182 candidates for the monthly real interest rates. In Appendix 3.A.1, to examine the similarities between the real interest rate candidates we provide a visual overview that displays, for each nominal interest rate, the corresponding real interest rates. We further compare the correlations among the different real interest rates and observe the following: While all real interest rates exhibit positive correlations with one another, the strength of the correlations varies from weak to almost one. The highest Pearson correlation coefficient is found between the real interest rate based on GS10 and PPIACO and the real interest rate based on GS20 and PPIACO. These rates have a Pearson correlation coefficient of 0.997 and a p value of below 1 percent. The smallest Pearson correlation coefficient, meanwhile, is observed between the real interest rate based on TB3MS and CUUR0000SAS4 and the rate based on BAA and PPIACO, with a correlation of 0.166 and a p value of 0.061.

Having discussed all important factors that need to be considered when selecting an appropriate real interest rate for the level or the error model, we examine—in the next section—the relation between the real interest rate candidates and the most likely discounting process as estimated by Maag and Reich (2023).

3.5 Empirical Analysis

In this section, our focus is on investigating the relationship between the real interest rate candidates, $\{r_t\}$, and the most likely discounting process of the MLE, $\{\hat{\beta}_t\}$. The most likely discounting process is estimated from equation (3.13) and based on the MLE of Maag and Reich (2023). The goal is to find a suitable real interest rate to incorporate into the discounting process of the bus engine replacement model and to determine which of the two presented models might lead to an improved and more efficient estimation of the discounting process. In Section 3.5.2, we examine the correlation between the levels of the most likely discounting process and the real interest rate values, as well as the correlation between the errors of the two data series. Furthermore, we examine a reduced form of the level and the error model using ordinary least squared (OLS) linear regression. By examining the simplified reduced form of the two models first, we can avoid unnecessary re-estimations when implementing the discounting process for the more complex structural model. The following subsection provides a description of the regression models.

3.5.1 Regression Models

To investigate the relationship between the most likely discounting process and the real interest rate through a regression model, we have decided to include more lags of r_t and $e_{r,t}$ compared to what we proposed in the level model (3.15) in Section 3.3.1 and the error model (3.22) in Section 3.3.2, respectively. By including additional lags in the reduced form model, we leverage the available data to potentially reveal a more accurate representation of the relationship between the real interest rate and the most likely discounting process than what was proposed in those earlier sections. While we still prioritize computational simplicity by including only one lag of interest rate or interest error in the structural model, the lagged value does not necessarily have to be the previous month's value. For example, in the level model (3.15), the next month's discounting state is defined as a function of this month's discounting state and real interest rate. To incorporate the real interest rate from the previous month only, one may substitute r_t with r_{t-1}.

For the reduced form of the level model, the linear regression equation of the most likely discounting states,

$\hat{\beta}_t$, on the real interest rates, r_t, reads

$$\hat{\beta}_t = b_0 + b_1 r_t + b_2 r_{t-1} + b_3 r_{t-2} + b_4 r_{t-3} + b_5 r_{t-4} + v_t, \tag{3.30}$$

where b_0 is the intercept of the model and $\{b_1, b_2, b_3, b_4, b_5\}$ are the coefficients, all of which are estimated using OLS. The residuals are denoted as v_t.

To examine a reduced form of the error model, we first need to approximate the innovations in the discounting process and the real interest rate time series. The error of the discounting process at time t, $e_{\hat{\beta},t}$, is directly derived from the estimated most likely discounting process,

$$e_{\hat{\beta},t} \approx \hat{\beta}_{t+1} - \hat{\mu}_{\beta_1}(1-\hat{\rho}) - \hat{\rho}\hat{\beta}_t, \tag{3.31}$$

where $\hat{\mu}_{\beta_1} = 1.028$ and $\hat{\rho} = 9.806\text{E-}01$. For the error term of the real interest rate at time t, $e_{r,t}$, we first fit an AR(1) model to the real interest rate candidates according to (3.16) or (3.23). That is to say,

$$r_{t+1} = c_r + a_{1,22} r_t + e_{r,t}. \tag{3.32}$$

The errors, $e_{r,t}$, are then derived as the residuals from the fitted AR(1) model. We then estimate the regression model

$$e_{\hat{\beta},t} = a_0 + a_1 e_{r,t} + a_2 e_{r,t-1} + a_3 e_{r,t-2} + a_4 e_{r,t-3} + a_5 e_{r,t-4} + w_t, \tag{3.33}$$

where a_0 is the intercept of the model, $\{a_1, a_2, a_3, a_4, a_5\}$ are the estimated OLS coefficients, and w_t are the residuals.

3.5.2 Correlation and Regression Results

We conduct a correlation study to examine the relationship between the most likely discounting process and the real interest rate candidates. The Pearson correlation coefficients are calculated for the levels of $\{\hat{\beta}_t\}$ and $\{r_t\}$ as well as for the errors of the two data series—that is $\{e_{\hat{\beta},t}\}$ and $\{e_{r,t}\}$. The correlation values, along with their corresponding p values, are detailed in Appendix 3.A.2. To provide a visual representation of these correlation values, Figure 3.3 displays a plot of the Pearson correlation coefficients on the horizontal axis against the corresponding p values on the vertical axis. The correlations between the levels of $\{\hat{\beta}_t\}$ and $\{r_t\}$ are plotted in blue, while the correlations involving the errors of the two data series, $\{e_{\hat{\beta},t}\}$ and $\{e_{r,t}\}$, are shown in red.

Overall, the correlation values between $\{\hat{\beta}_t\}$ and $\{r_t\}$ indicate a negative relationship between the most likely discounting process and the real interest rates. Under the assumption that an exponential discount function holds—that is, the discount factor at time t is $\beta_t = 1/(1+d_t)$, and the discount rate d_t is (positively) linearly related to the interest rate—this finding aligns with the expectation that higher real interest rates are associated with lower discount factors. While there is variation in the values of the correlation coefficients, it is noteworthy that all of them demonstrate statistical significance at the 1 percent level. The smallest Pearson correlation coefficient observed between the most likely discounting process and the real interest rate is -0.771, with a corresponding p value of approximately 2.07E-26. This correlation is based on the real interest rate derived from the nominal interest rate BAA and the inflation rate from PCEPILFE. The largest correlation observed, meanwhile, is approximately -0.418, with a p value of 9.13E-07 for a real interest rate based on TB1YR and CUUR0000SAS4.

Regarding the correlation between $\{e_{\hat{\beta},t}\}$ and $\{e_{r,t}\}$, the majority of correlation values are closely clustered around zero, with corresponding p values exceeding 0.5. This indicates that there is no significant pattern of association, whether positive or negative, between the errors of the interest rate and the discounting process for most real interest rates. In general, we would expect that positive shocks to real interest rate levels would have

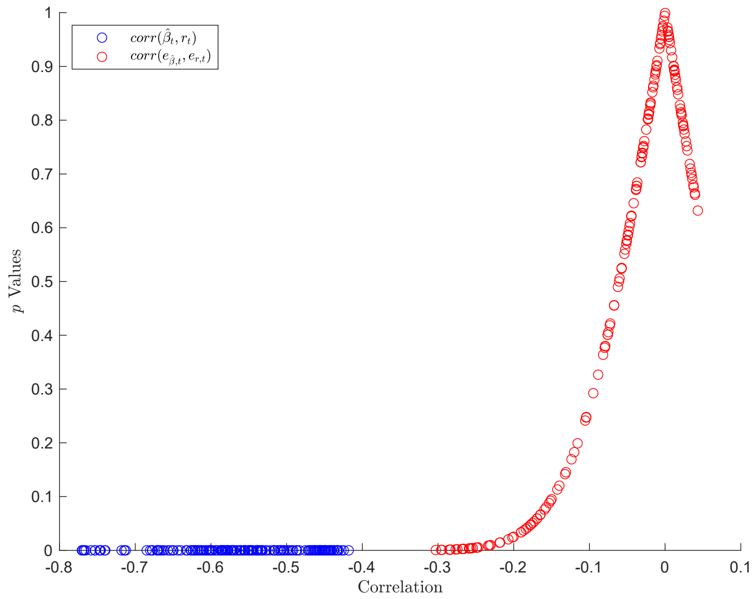


Figure 3.3: Overview of the Pearson correlation coefficients between the most likely discounting process and the real interest rate candidates in blue and the innovations of the two data series in red. The horizontal axis is the correlation value and the vertical axis is the corresponding p value. Source: Own representation based on data from FRED (2023b).

a negative impact on Zurcher's expectations regarding his discounting state. Notably, certain real interest rate errors exhibit a statistically significant negative correlation with the discounting errors: the smallest Pearson correlation coefficient is approximately -0.303 with a p value of 5.95E-04, observed for the errors of a real interest rate derived from FEDFUNDS and CWUR0000SAS4.

Table 3.3 presents the output for the three level regression models with the highest R^2 values. A summary of the remaining regression outputs can be found in Appendix 3.A.3. The regression outputs reveal a similar pattern in the coefficients across all three models. The intercepts in all three models have coefficients of approximately 1.03, with p values indicating statistical significance at the 1 percent level. This suggests a significant impact of the estimated mean of the discounting process on the discounting levels when all real interest rate levels are assumed to be zero. The coefficients of the lagged real interest rate levels are all negative and of similar magnitude: in all three models, the coefficients for r_t and r_{t-4} have values in the range of -2.24E-01 to -2.81E-01, with statistical significance at the 1 percent level. The coefficients for the other lags are not statistically significant at the 5 percent level, with values for the coefficients around -1E-01. Overall, the three level models demonstrate a good fit, with R^2 values of around 0.75 and p values for the F test below 1E-32, indicating significance at the 1 percent level.

The regression output for the three best-fitted error regression models is presented in Table 3.4, and a summary of the remaining regression models can be found in Appendix 3.A.3. Overall, we observe that all of the three error models presented provide a better fit than a simple intercept model. The obtained p values of the F tests indicate that the models have significant explanatory power, with values consistently below 0.05. The highest R^2 value among the models is 0.366, implying that more than 36 percent of the variation in the discounting error can be explained by the variation in the errors of the real interest rates. When examining the estimated coefficients, several notable patterns emerge: The intercepts for all three models exhibit relatively small values, below -2E-05. Furthermore, the coefficients of the lagged real interest rate errors are consistently

TB3MS and PCEPILFE				
	Coefficient	SE	t Statistic	p Value
Intercept	1.03E+00	1.45E−04	7.12E+03	<1 E−50
r_t	-2.71E−01	7.29E−02	-3.71E+00	3.13E−04
r_{t-1}	-1.23E−01	8.63E−02	-1.42E+00	1.57E−01
r_{t-2}	-1.52E−01	8.51E−02	-1.79E+00	7.67E−02
r_{t-3}	-1.30E−01	8.56E−02	-1.52E+00	1.31E−01
r_{t-4}	-2.81E−01	7.28E−02	-3.87E+00	1.80E−04
RMSE	1.26E−03			
R^2	0.746			
R^2_{adj}	0.735			
F Test				
Statistic	69.4			
p Value	1.70E−33			
MPRIME and PCEPILFE				
	Coefficient	SE	t Statistic	p Value
Intercept	1.03E+00	1.95E−04	5.29E+03	<1 E−50
r_t	-2.24E−01	6.86E−02	-3.26E+00	1.45E−03
r_{t-1}	-8.34E−02	8.46E−02	-9.85E−01	3.26E−01
r_{t-2}	-1.19E−01	8.38E−02	-1.42E+00	1.58E−01
r_{t-3}	-9.41E−02	8.40E−02	-1.12E+00	2.65E−01
r_{t-4}	-2.49E−01	6.86E−02	-3.64E+00	4.13E−04
RMSE	1.26E−03			
R^2	0.745			
R^2_{adj}	0.735			
F Test				
Statistic	69.1			
p Value	1.98E−33			
TB6MS and PCEPILFE				
	Coefficient	SE	t Statistic	p Value
Intercept	1.03E+00	1.51E−04	6.80E+03	<1 E−50
r_t	-2.72E−01	7.42E−02	-3.67E+00	3.65E−04
r_{t-1}	-1.32E−01	8.69E−02	-1.52E+00	1.30E−01
r_{t-2}	-1.61E−01	8.56E−02	-1.88E+00	6.27E−02
r_{t-3}	-1.39E−01	8.61E−02	-1.61E+00	1.10E−01
r_{t-4}	-2.79E−01	7.41E−02	-3.77E+00	2.57E−04
RMSE	1.26E−03			
R^2	0.745			
R^2_{adj}	0.734			
F Test				
Statistic	69.0			
p Value	2.13E−33			

Table 3.3: Regression output for the three level regressions with the highest R^2. The regression equation reads $\hat{\beta}_t = b_0 + b_1 r_t + b_2 r_{t-1} + b_3 r_{t-2} + b_4 r_{t-3} + b_5 r_{t-4} + v_t$. Source: Own calculation based on data from FRED (2023b).

FEDFUNDS and CWSR0000SAS4				
	Coefficient	SE	t Statistic	p Value
Intercept	-2.44E−05	7.15E−06	-3.41E+00	8.89E−04
$e_{r,t}$	-4.60E−03	1.68E−03	-2.73E+00	7.24E−03
$e_{r,t-1}$	-7.96E−03	1.73E−03	-4.60E+00	1.07E−05
$e_{r,t-2}$	-5.75E−03	1.76E−03	-3.27E+00	1.40E−03
$e_{r,t-3}$	-3.48E−03	1.73E−03	-2.01E+00	4.63E−02
$e_{r,t-4}$	-2.34E−03	1.67E−03	-1.40E+00	1.65E−01
RMSE	7.69E−05			
R^2	0.366			
R^2_{adj}	0.339			
F Test				
Statistic	13.5			
p Value	2.08E−10			

FEDFUNDS and CWUR0000SAS4				
	Coefficient	SE	t Statistic	p Value
Intercept	-2.41E−05	7.20E−06	-3.35E+00	1.08E−03
$e_{r,t}$	-4.64E−03	1.64E−03	-2.84E+00	5.36E−03
$e_{r,t-1}$	-7.42E−03	1.69E−03	-4.39E+00	2.55E−05
$e_{r,t-2}$	-5.48E−03	1.68E−03	-3.26E+00	1.46E−03
$e_{r,t-3}$	-3.25E−03	1.69E−03	-1.92E+00	5.68E−02
$e_{r,t-4}$	-2.02E−03	1.63E−03	-1.24E+00	2.18E−01
RMSE	7.73E−05			
R^2	0.359			
R^2_{adj}	0.332			
F Test				
Statistic	13.1			
p Value	3.84E−10			

MPRIME and CWSR0000SAS4				
	Coefficient	SE	t Statistic	p Value
Intercept	-8.30E−06	7.95E−06	-1.04E+00	2.99E−01
$e_{r,t}$	-4.44E−03	1.70E−03	-2.62E+00	9.94E−03
$e_{r,t-1}$	-8.27E−03	1.78E−03	-4.65E+00	8.89E−06
$e_{r,t-2}$	-6.96E−03	1.77E−03	-3.93E+00	1.43E−04
$e_{r,t-3}$	-4.72E−03	1.78E−03	-2.65E+00	9.22E−03
$e_{r,t-4}$	-2.81E−03	1.68E−03	-1.67E+00	9.73E−02
RMSE	7.74E−05			
R^2	0.358			
R^2_{adj}	0.33			
F Test				
Statistic	13.0			
p Value	4.40E−10			

Table 3.4: Regression output for the three error regressions with the highest R^2. The regression equation reads $e_{\hat{\beta},t} = a_0 + a_1 e_{r,t} + a_2 e_{r,t-1} + a_3 e_{r,t-2} + a_4 e_{r,t-3} + a_5 e_{r,t-4} + w_t$. Source: Own calculation based on data from FRED (2023b).

negative, with the smallest coefficients found for the error $e_{r,t-1}$. Except for the error terms $e_{r,t-4}$, and $e_{r,t-3}$ in the second regression output, the coefficients in all models are statistically significant at the 5 percent level. Notably, only in the case of the third model does the p value for the coefficient of e_{t-4} fall below the 10 percent threshold.

In summary, the empirical analysis suggests that both the level model and the error model are promising approaches to integrating the correlation between discounting and real interest rates into the discounting process. Of the two, the level model exhibits superior performance: all correlation coefficients between $\{\hat{\beta}_t\}$ and $\{r_t\}$ exhibit negative values and achieve statistical significance. Additionally, the regression models consistently reveal a coherent pattern with high R^2 values across all real interest rate candidates.

3.6 Summary and Outlook

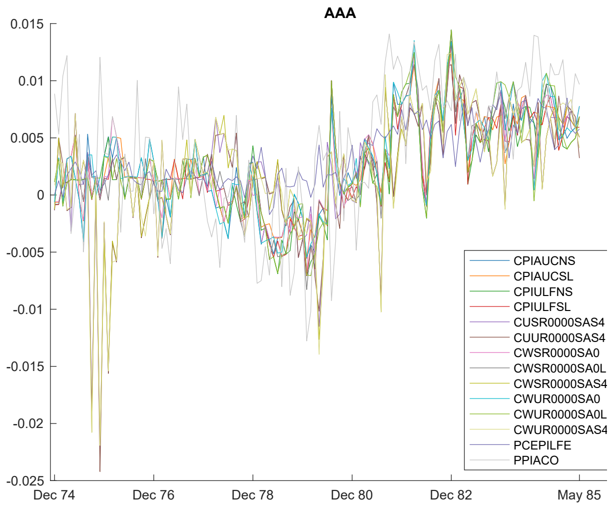
In recent years, there has been growing interest in exploring alternative discount functions for DDC models, departing from the conventional exponential form. Building upon this development, Maag and Reich (2023) introduce the modeling of discounting as an AR(1) process in the seminal bus engine replacement model of Rust (1987). This modeling approach allows the estimation of the most likely discounting process—that is, the most probable evolution of the discounting states between December 1974 and May 1985. Their estimated most likely discounting process exhibits a strong and statistically significant correlation with the prevailing real interest rate values during that period. This finding has an economically sensible interpretation as the discount rate is commonly assumed to be a linear function of the interest rate (Goodin, 1982). Based on the results presented by Maag and Reich (2023), this paper suggests a potential improvement to the discounting process in the bus engine replacement model by incorporating its relation to the real interest rate into the evolution of the process. We introduce two model specifications: **(A)** the level model, which defines the discounting state as a function of its own past values and the values of the real interest rate, and **(B)** the error model, where the discounting state is a function of its own past values and the innovations in the real interest rate. We evaluate both model specifications for their fit based on various real interest rate candidates. The results indicate that the level model provides a good fit to the most likely discounting process for most real interest rates, with a median R^2 of approximately 0.41 for the level regressions. Further, in the majority of these level regressions we observe significant coefficients for the real interest rate lags of r_t and r_{t-4}. The level model with the highest R^2 is for the real interest rate based on the short-term treasury bill TB3MS and the personal consumption inflation indicator PCEPILFE. When only one level of the real interest rate is included, the best fit is for a real interest rate based on Moody's BAA and the PCEPILFE inflation rate, which can be inferred from the correlation results. The R^2 values of the error regression models are, on the other hand, notably smaller, indicating less explanatory power for this model type. In the error regression with the best fit, approximately 37 percent of the variation in the discounting state innovations is explained by the variation in real interest rate innovations. The real interest rate used in this specific error regression is calculated based on the Fed's target rate, FEDFUNDS, and the CPI subindex specific to workers in the transportation service industry, CWSR0000SAS4.

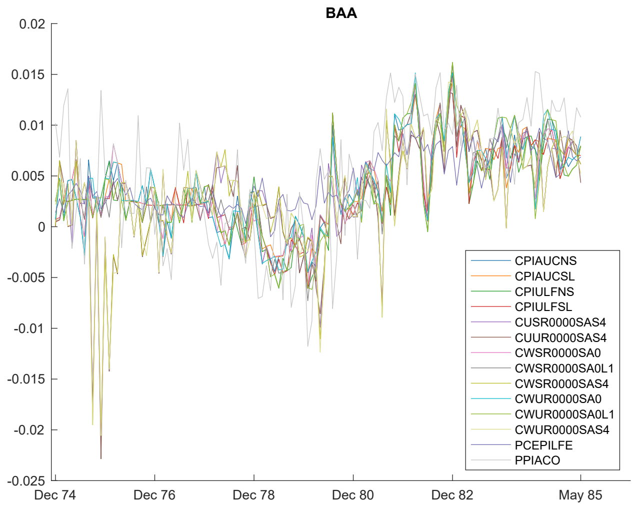
In summary, the two models specified in Section 3.3 offer a straightforward approach to incorporating the real interest rate's evolution into the discounting process. For both model specifications, we have proposed a two-step estimation procedure. In the first step, we estimate the distributional parameters of the real interest rate process by fitting an AR(1) model to $\{r_t\}$. Then, in the second step, we set the real interest rate parameters to the values from the first step and estimate the remaining model parameters with maximum likelihood. This two-step approach reduces the number of parameters to be estimated when maximizing the (profile) likelihood of the problem to just one additional parameter compared to an AR(1) model—so, in the level model the coefficient of the real interest rate in the discounting formulation, and in the error model the covariance between the innovations in the real interest rate and the innovations in the discounting states. Upon comparing the correlation and regression findings from the level model with those from the error model, we conclude that the results from the level model provide a more favorable fit, supported by robust and strong correlations. Thus, based on the findings in Section 3.5.2, we recommend incorporating the correlation between the discounting states and the real interest rate using a level model with a single lag of order one for the real interest rate, as specified in equation (3.15). For such a level model, we propose using the real interest rate derived from the nominal interest rate BAA and the inflation indicator PCEPILFE. This real interest rate candidate shows the strongest Pearson correlation coefficient with a value of approximately -0.77 and a p value of below 1 percent. Given the complexity of the task, a comprehensive estimation of the structural model featuring a level discounting process exceeds the scope of this work. We plan to explore this in an upcoming research project.

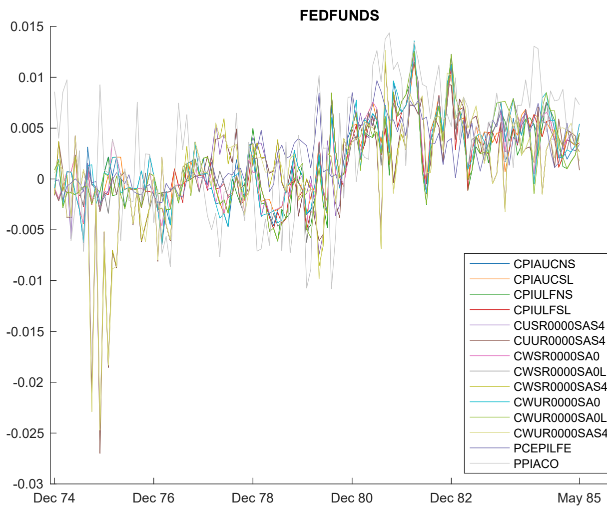
3.A Appendix

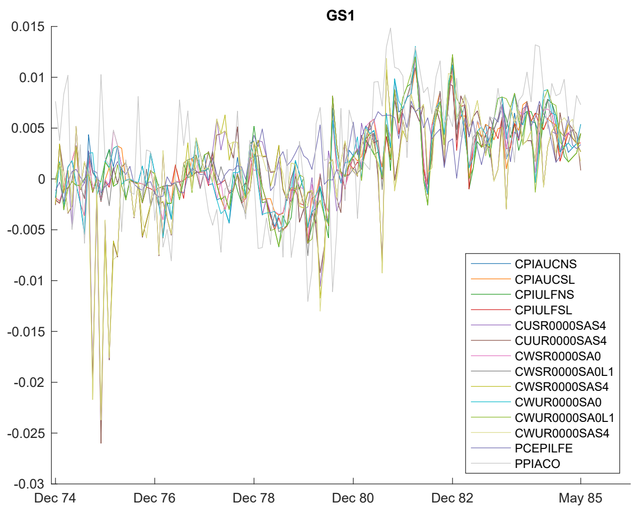
3.A.1 Real Interest Rate Plots

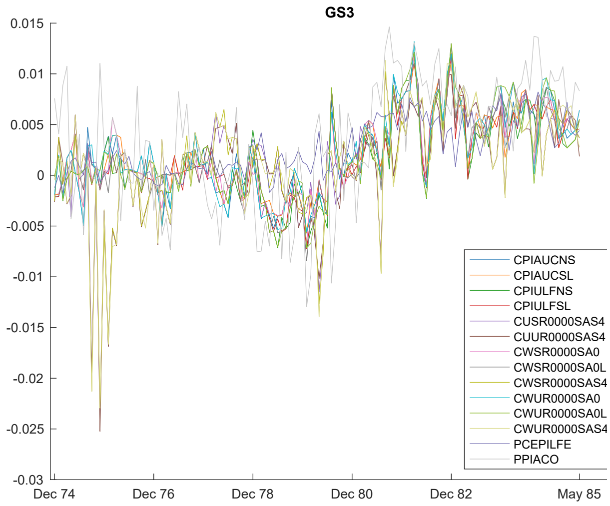
The following figures show the monthly real interest rate candidates for each nominal interest rate candidate.







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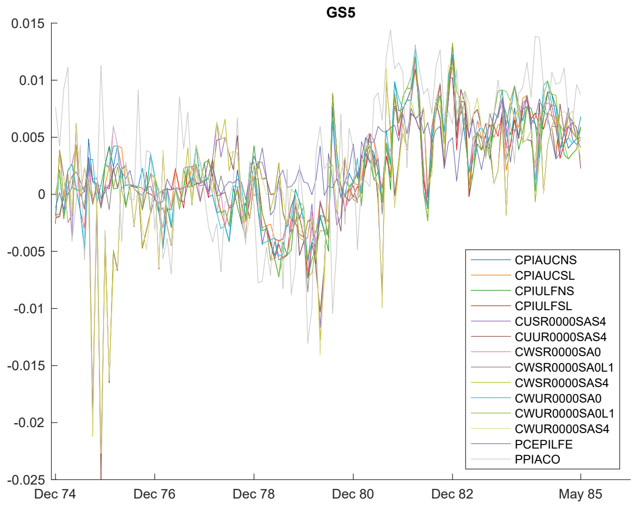
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
-0.03
Dec 74
Dec 76
Dec 78
Dec 80
Dec 82
May 85
CPIAUCNS
CPIAUCSL
CPIULFNS
CPIULFSL
CUSR0000SAS4
CUUR0000SAS4
CWSR0000SA0
CWSR0000SA0L
CWSR0000SAS4
CWUR0000SA0
CWUR0000SA0L
CWUR0000SAS4
PCEPILFE
PPIACO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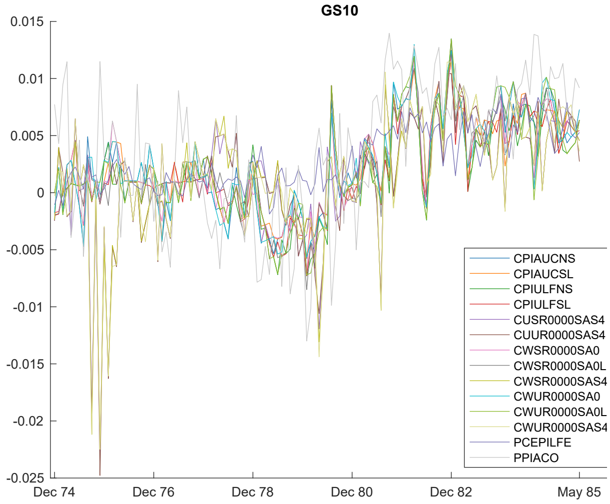
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
Dec 74
Dec 76
Dec 78
Dec 80
Dec 82
May 85
CPIAUCNS
CPIAUCSL
CPIULFNS
CPIULFSL
CUSR0000SAS4
CUUR0000SAS4
CWSR0000SA0
CWSR0000SA0L1
CWSR0000SAS4
CWUR0000SA0
CWUR0000SA0L1
CWUR0000SAS4
PCEPILFE
PPIACO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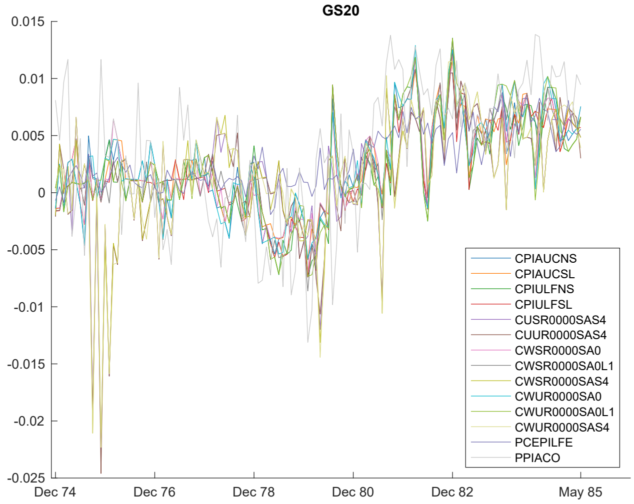
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
Dec 74
Dec 76
Dec 78
Dec 80
Dec 82
May 85
CPIAUCNS
CPIAUCSL
CPIULFNS
CPIULFSL
CUSR0000SAS4
CUUR0000SAS4
CWSR0000SA0
CWSR0000SA0L
CWSR0000SAS4
CWUR0000SA0
CWUR0000SA0L
CWUR0000SAS4
PCEPILFE
PPIACO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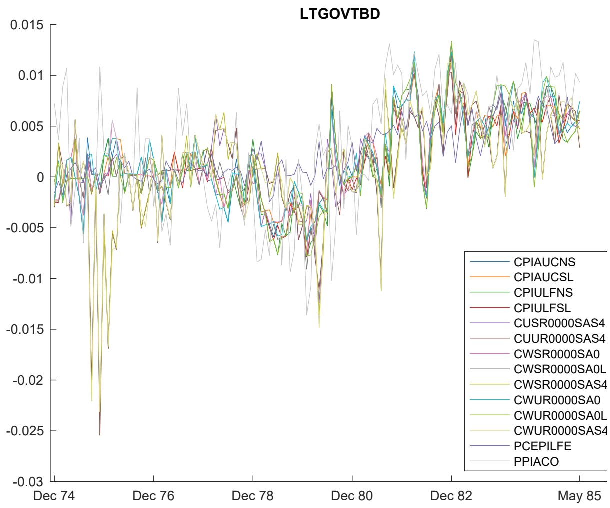
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
Dec 74
Dec 76
Dec 78
Dec 80
Dec 82
May 85
CPIAUCNS
CPIAUCSL
CPIULFNS
CPIULFSL
CUSR0000SAS4
CUUR0000SAS4
CWSR0000SA0
CWSR0000SA0L1
CWSR0000SAS4
CWUR0000SA0
CWUR0000SA0L1
CWUR0000SAS4
PCEPILFE
PPIACO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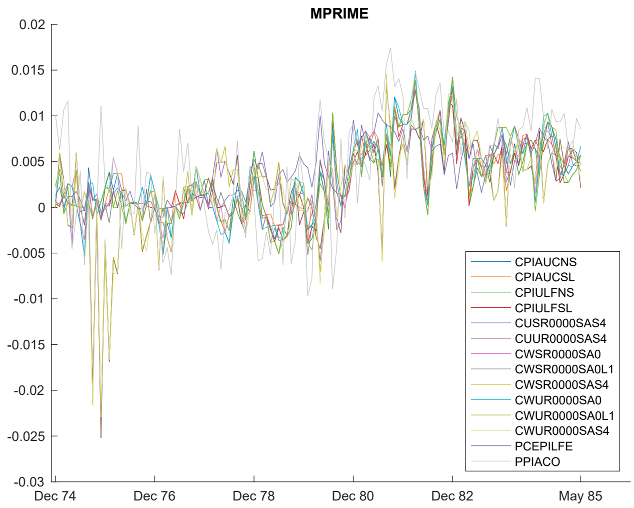
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
-0.03
Dec 74
Dec 76
Dec 78
Dec 80
Dec 82
May 85
CPIAUCNS
CPIAUCSL
CPIULFNS
CPIULFSL
CUSR0000SAS4
CUUR0000SAS4
CWSR0000SA0
CWSR0000SA0L
CWSR0000SAS4
CWUR0000SA0
CWUR0000SA0L
CWUR0000SAS4
PCEPILFE
PPIACO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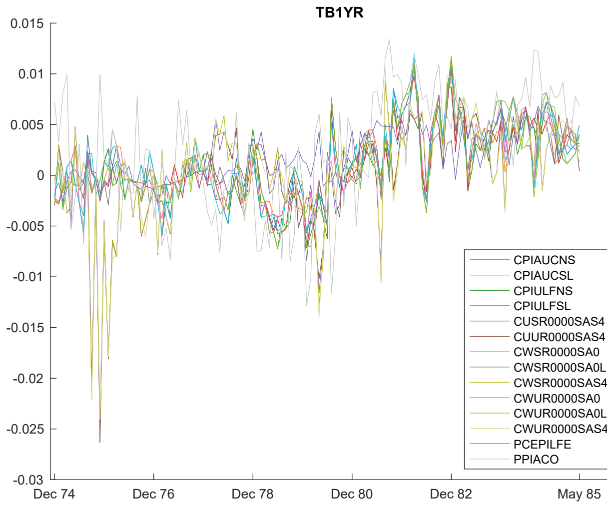
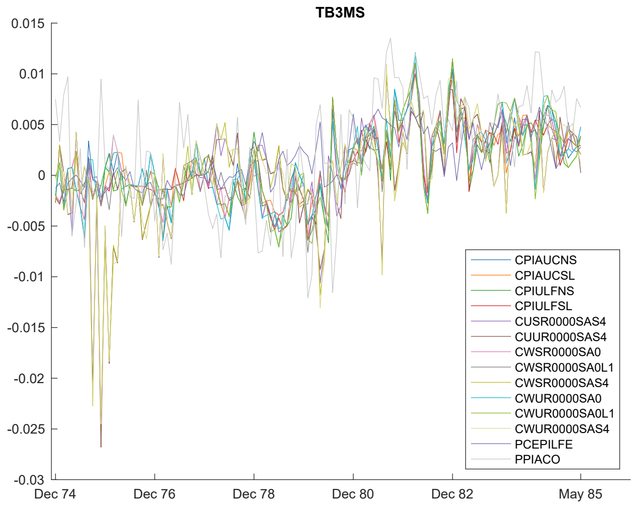
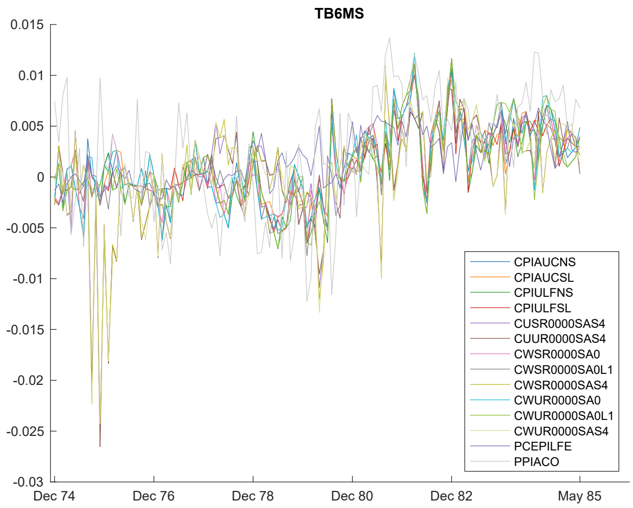
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
-0.03
Dec 74
Dec 76
Dec 78
Dec 80
Dec 82
May 85
CPIAUCNS
CPIAUCSL
CPIULFNS
CPIULFSL
CUSR0000SAS4
CUUR0000SAS4
CWSR0000SA0
CWSR0000SA0L1
CWSR0000SAS4
CWUR0000SA0
CWUR0000SA0L1
CWUR0000SAS4
PCEPILFE
PPIACO







3.A.2 Correlation Results

The following table provides an overview of the Pearson correlation coefficients between the most likely discounting process and the real interest rate candidates, $corr(\hat{\beta}_t, r_t)$, and the innovations of the two data series, $corr(e_{\hat{\beta},t}, e_{r,t})$. Note that the p values are to the right of the correlation values.

Table 3.5: Summary of the Pearson correlation coefficients between $\{\hat{\beta}_t\}$ and $\{r_t\}$, and between $\{e_{\hat{\beta},t}\}$ and $\{e_{r,t}\}$. Source: Own calculation based on data from FRED (2023b).

Interest Rate	Inflation	corr$(\hat{\beta}_t, r_t)$	p Value	corr$(e_{\hat{\beta},t}, e_{r,t})$	p Value
FEDFUNDS	CPIAUCNS	−6.64E−01	1.29E−17	−7.30E−02	4.18E−01
FEDFUNDS	CPIAUCSL	−7.12E−01	4.12E−21	−5.74E−02	5.25E−01
FEDFUNDS	CPIULFNS	−6.33E−01	1.12E−15	−5.75E−02	5.24E−01
FEDFUNDS	CPIULFSL	−6.71E−01	4.30E−18	−7.58E−02	4.01E−01
FEDFUNDS	CUSR0000SAS4	−4.91E−01	3.90E−09	−2.68E−01	2.49E−03
FEDFUNDS	CUUR0000SAS4	−4.83E−01	7.67E−09	−2.96E−01	8.12E−04
FEDFUNDS	CWSR0000SA0	−7.14E−01	2.87E−21	−9.49E−02	2.92E−01
FEDFUNDS	CWSR0000SA0L1	−6.77E−01	1.64E−18	−1.04E−01	2.48E−01
FEDFUNDS	CWSR0000SAS4	−5.03E−01	1.45E−09	−2.84E−01	1.35E−03
FEDFUNDS	CWUR0000SA0	−6.70E−01	5.22E−18	−8.19E−02	3.64E−01
FEDFUNDS	CWUR0000SA0L1	−6.40E−01	4.17E−16	−7.25E−02	4.22E−01
FEDFUNDS	CWUR0000SAS4	−4.98E−01	2.29E−09	−3.03E−01	5.95E−04
FEDFUNDS	PCEPILFE	−7.40E−01	1.72E−23	−1.16E−01	1.99E−01

Continued on next page

Interest Rate	Inflation	corr($\hat{\beta}_t, r_t$)	p Value	corr($e_{\hat{\beta},t}, e_{r,t}$)	p Value
FEDFUNDS	PPIACO	−5.44E−01	3.18E−11	−5.27E−02	5.59E−01
GS1	CPIAUCNS	−6.14E−01	1.28E−14	−3.01E−02	7.39E−01
GS1	CPIAUCSL	−6.57E−01	3.68E−17	−1.55E−02	8.64E−01
GS1	CPIULFNS	−5.79E−01	7.81E−13	−1.26E−02	8.89E−01
GS1	CPIULFSL	−6.11E−01	1.80E−14	−2.88E−02	7.50E−01
GS1	CUSR0000SAS4	−4.56E−01	6.27E−08	−2.32E−01	9.17E−03
GS1	CUUR0000SAS4	−4.49E−01	1.04E−07	−2.58E−01	3.64E−03
GS1	CWSR0000SA0	−6.62E−01	1.74E−17	−5.38E−02	5.51E−01
GS1	CWSR0000SA0L1	−6.16E−01	9.88E−15	−6.09E−02	5.00E−01
GS1	CWSR0000SAS4	−4.69E−01	2.42E−08	−2.49E−01	5.15E−03
GS1	CWUR0000SA0	−6.21E−01	5.47E−15	−4.15E−02	6.46E−01
GS1	CWUR0000SA0L1	−5.85E−01	4.05E−13	−3.22E−02	7.22E−01
GS1	CWUR0000SAS4	−4.63E−01	3.64E−08	−2.67E−01	2.59E−03
GS1	PCEPILFE	−7.53E−01	1.14E−24	−7.93E−02	3.79E−01
GS1	PPIACO	−5.17E−01	4.11E−10	−2.26E−02	8.03E−01
GS3	CPIAUCNS	−5.91E−01	2.03E−13	3.12E−03	9.72E−01
GS3	CPIAUCSL	−6.26E−01	2.75E−15	1.63E−02	8.57E−01
GS3	CPIULFNS	−5.56E−01	9.22E−12	1.96E−02	8.28E−01
GS3	CPIULFSL	−5.82E−01	5.93E−13	5.11E−03	9.55E−01
GS3	CUSR0000SAS4	−4.60E−01	4.86E−08	−1.82E−01	4.26E−02
GS3	CUUR0000SAS4	−4.53E−01	8.07E−08	−2.07E−01	2.05E−02
GS3	CWSR0000SA0	−6.31E−01	1.38E−15	−2.27E−02	8.02E−01
GS3	CWSR0000SA0L1	−5.86E−01	3.50E−13	−3.02E−02	7.38E−01
GS3	CWSR0000SAS4	−4.71E−01	1.94E−08	−2.00E−01	2.53E−02
GS3	CWUR0000SA0	−5.98E−01	9.17E−14	−1.13E−02	9.00E−01
GS3	CWUR0000SA0L1	−5.61E−01	5.43E−12	−3.92E−03	9.65E−01
GS3	CWUR0000SAS4	−4.66E−01	2.92E−08	−2.18E−01	1.46E−02
GS3	PCEPILFE	−7.68E−01	3.68E−26	−4.70E−02	6.02E−01
GS3	PPIACO	−5.09E−01	8.33E−10	−1.69E−03	9.85E−01
GS5	CPIAUCNS	−5.80E−01	7.13E−13	1.24E−02	8.91E−01
GS5	CPIAUCSL	−6.13E−01	1.48E−14	2.49E−02	7.83E−01
GS5	CPIULFNS	−5.45E−01	2.78E−11	2.86E−02	7.52E−01
GS5	CPIULFSL	−5.69E−01	2.32E−12	1.44E−02	8.74E−01
GS5	CUSR0000SAS4	−4.56E−01	6.28E−08	−1.65E−01	6.57E−02
GS5	CUUR0000SAS4	−4.49E−01	1.03E−07	−1.90E−01	3.35E−02
GS5	CWSR0000SA0	−6.18E−01	7.71E−15	−1.41E−02	8.76E−01
GS5	CWSR0000SA0L1	−5.74E−01	1.44E−12	−2.17E−02	8.10E−01
GS5	CWSR0000SAS4	−4.68E−01	2.50E−08	−1.84E−01	4.00E−02
GS5	CWUR0000SA0	−5.87E−01	3.30E−13	−2.83E−03	9.75E−01
GS5	CWUR0000SA0L1	−5.50E−01	1.71E−11	3.97E−03	9.65E−01
GS5	CWUR0000SAS4	−4.63E−01	3.74E−08	−2.02E−01	2.40E−02
GS5	PCEPILFE	−7.67E−01	4.46E−26	−3.75E−02	6.78E−01
GS5	PPIACO	−5.04E−01	1.30E−09	4.68E−03	9.59E−01
GS10	CPIAUCNS	−5.62E−01	5.06E−12	2.15E−02	8.12E−01
GS10	CPIAUCSL	−5.93E−01	1.56E−13	3.36E−02	7.10E−01
GS10	CPIULFNS	−5.27E−01	1.72E−10	3.74E−02	6.79E−01
GS10	CPIULFSL	−5.49E−01	1.86E−11	2.35E−02	7.95E−01
GS10	CUSR0000SAS4	−4.44E−01	1.55E−07	−1.50E−01	9.51E−02
GS10	CUUR0000SAS4	−4.37E−01	2.45E−07	−1.75E−01	5.10E−02
GS10	CWSR0000SA0	−5.99E−01	8.03E−14	−5.47E−03	9.52E−01
GS10	CWSR0000SA0L1	−5.54E−01	1.12E−11	−1.31E−02	8.85E−01
GS10	CWSR0000SAS4	−4.56E−01	6.32E−08	−1.69E−01	5.93E−02
GS10	CWUR0000SA0	−5.70E−01	2.28E−12	5.75E−03	9.49E−01
GS10	CWUR0000SA0L1	−5.32E−01	1.06E−10	1.21E−02	8.94E−01
GS10	CWUR0000SAS4	−4.51E−01	9.10E−08	−1.87E−01	3.69E−02
GS10	PCEPILFE	−7.59E−01	3.42E−25	−2.75E−02	7.61E−01
GS10	PPIACO	−4.95E−01	2.75E−09	1.12E−02	9.01E−01
GS20	CPIAUCNS	−5.48E−01	2.26E−11	2.75E−02	7.61E−01
GS20	CPIAUCSL	−5.78E−01	9.21E−13	3.92E−02	6.64E−01
GS20	CPIULFNS	−5.12E−01	6.66E−10	4.33E−02	6.32E−01
GS20	CPIULFSL	−5.34E−01	8.37E−11	2.95E−02	7.44E−01
GS20	CUSR0000SAS4	−4.34E−01	3.12E−07	−1.40E−01	1.20E−01
GS20	CUUR0000SAS4	−4.28E−01	4.80E−07	−1.65E−01	6.67E−02
GS20	CWSR0000SA0	−5.84E−01	4.63E−13	7.57E−05	9.99E−01
GS20	CWSR0000SA0L1	−5.40E−01	4.99E−11	−7.55E−03	9.33E−01
GS20	CWSR0000SAS4	−4.46E−01	1.28E−07	−1.59E−01	7.58E−02
GS20	CWUR0000SA0	−5.55E−01	1.01E−11	1.13E−02	9.00E−01
GS20	CWUR0000SA0L1	−5.17E−01	4.09E−10	1.73E−02	8.48E−01
GS20	CWUR0000SAS4	−4.42E−01	1.80E−07	−1.77E−01	4.82E−02
GS20	PCEPILFE	−7.47E−01	4.24E−24	−2.15E−02	8.12E−01
GS20	PPIACO	−4.86E−01	5.91E−09	1.58E−02	8.61E−01
LTGOVTBD	CPIAUCNS	−5.70E−01	2.11E−12	2.18E−02	8.09E−01
LTGOVTBD	CPIAUCSL	−6.01E−01	6.58E−14	3.45E−02	7.02E−01
LTGOVTBD	CPIULFNS	−5.33E−01	9.21E−11	3.79E−02	6.75E−01
LTGOVTBD	CPIULFSL	−5.56E−01	9.86E−12	2.40E−02	7.91E−01
LTGOVTBD	CUSR0000SAS4	−4.49E−01	1.08E−07	−1.53E−01	8.78E−02
LTGOVTBD	CUUR0000SAS4	−4.42E−01	1.73E−07	−1.78E−01	4.68E−02
LTGOVTBD	CWSR0000SA0	−6.06E−01	3.59E−14	−4.77E−03	9.58E−01
LTGOVTBD	CWSR0000SA0L1	−5.60E−01	6.35E−12	−1.22E−02	8.93E−01
LTGOVTBD	CWSR0000SAS4	−4.61E−01	4.48E−08	−1.73E−01	5.40E−02
LTGOVTBD	CWUR0000SA0	−5.77E−01	1.01E−12	6.41E−03	9.43E−01
LTGOVTBD	CWUR0000SA0L1	−5.38E−01	5.99E−11	1.31E−02	8.85E−01
LTGOVTBD	CWUR0000SAS4	−4.56E−01	6.55E−08	−1.90E−01	3.35E−02
LTGOVTBD	PCEPILFE	−7.70E−01	2.39E−26	−2.89E−02	7.49E−01
LTGOVTBD	PPIACO	−5.04E−01	1.34E−09	1.22E−02	8.93E−01
MPRIME	CPIAUCNS	−6.71E−01	4.50E−18	−4.46E−02	6.21E−01
MPRIME	CPIAUCSL	−7.14E−01	2.90E−21	−2.85E−02	7.52E−01

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Interest Rate	Inflation	corr($\hat{\beta}_t, r_t$)	p Value	corr($e_{\hat{\beta},t}, e_{r,t}$)	p Value
MPRIME	CPIULFNS	−6.44E−01	2.42E−16	−3.09E−02	7.32E−01
MPRIME	CPIULFSL	−6.80E−01	1.03E−18	−4.63E−02	6.09E−01
MPRIME	CUSR0000SAS4	−5.16E−01	4.74E−10	−2.19E−01	1.43E−02
MPRIME	CUUR0000SAS4	−5.08E−01	9.57E−10	−2.47E−01	5.49E−03
MPRIME	CWSR0000SA0	−7.18E−01	1.43E−21	−6.74E−02	4.55E−01
MPRIME	CWSR0000SA0L1	−6.85E−01	4.86E−19	−7.92E−02	3.80E−01
MPRIME	CWSR0000SAS4	−5.28E−01	1.53E−10	−2.34E−01	8.72E−03
MPRIME	CWUR0000SA0	−6.78E−01	1.54E−18	−5.74E−02	5.25E−01
MPRIME	CWUR0000SA0L1	−6.50E−01	9.67E−17	−5.06E−02	5.75E−01
MPRIME	CWUR0000SAS4	−5.22E−01	2.61E−10	−2.54E−01	4.23E−03
MPRIME	PCEPILFE	−7.65E−01	7.02E−26	−7.58E−02	4.01E−01
MPRIME	PPIACO	−5.51E−01	1.56E−11	−4.45E−02	6.22E−01
BAA	CPIAUCNS	−5.65E−01	3.55E−12	2.04E−02	8.21E−01
BAA	CPIAUCSL	−5.94E−01	1.46E−13	3.26E−02	7.19E−01
BAA	CPIULFNS	−5.32E−01	1.02E−10	3.51E−02	6.98E−01
BAA	CPIULFSL	−5.55E−01	1.08E−11	2.12E−02	8.15E−01
BAA	CUSR0000SAS4	−4.65E−01	3.29E−08	−1.06E−01	2.41E−01
BAA	CUUR0000SAS4	−4.58E−01	5.51E−08	−1.31E−01	1.45E−01
BAA	CWSR0000SA0	−6.01E−01	6.56E−14	−6.64E−03	9.41E−01
BAA	CWSR0000SA0L1	−5.60E−01	6.13E−12	−1.58E−02	8.61E−01
BAA	CWSR0000SAS4	−4.75E−01	1.42E−08	−1.24E−01	1.69E−01
BAA	CWUR0000SA0	−5.74E−01	1.45E−12	3.82E−03	9.66E−01
BAA	CWUR0000SA0L1	−5.38E−01	5.95E−11	9.43E−03	9.17E−01
BAA	CWUR0000SAS4	−4.70E−01	2.11E−08	−1.42E−01	1.13E−01
BAA	PCEPILFE	−7.71E−01	2.07E−26	−1.91E−02	8.33E−01
BAA	PPIACO	−5.00E−01	1.85E−09	5.07E−03	9.55E−01
AAA	CPIAUCNS	−5.47E−01	2.32E−11	2.42E−02	7.89E−01
AAA	CPIAUCSL	−5.78E−01	9.36E−13	3.59E−02	6.91E−01
AAA	CPIULFNS	−5.12E−01	6.63E−10	3.96E−02	6.61E−01
AAA	CPIULFSL	−5.35E−01	7.78E−11	2.58E−02	7.75E−01
AAA	CUSR0000SAS4	−4.38E−01	2.37E−07	−1.32E−01	1.42E−01
AAA	CUUR0000SAS4	−4.31E−01	3.72E−07	−1.57E−01	7.99E−02
AAA	CWSR0000SA0	−5.84E−01	4.66E−13	−3.17E−03	9.72E−01
AAA	CWSR0000SA0L1	−5.40E−01	4.65E−11	−1.11E−02	9.02E−01
AAA	CWSR0000SAS4	−4.50E−01	1.01E−07	−1.51E−01	9.22E−02
AAA	CWUR0000SA0	−5.55E−01	1.02E−11	7.93E−03	9.30E−01
AAA	CWUR0000SA0L1	−5.17E−01	4.09E−10	1.38E−02	8.79E−01
AAA	CWUR0000SAS4	−4.45E−01	1.43E−07	−1.69E−01	5.92E−02
AAA	PCEPILFE	−7.53E−01	1.19E−24	−2.26E−02	8.02E−01
AAA	PPIACO	−4.87E−01	5.52E−09	1.21E−02	8.94E−01
TB1YR	CPIAUCNS	−5.77E−01	9.70E−13	−1.97E−02	8.28E−01
TB1YR	CPIAUCSL	−6.21E−01	5.43E−15	−6.38E−03	9.44E−01
TB1YR	CPIULFNS	−5.40E−01	4.77E−11	−6.77E−04	9.94E−01
TB1YR	CPIULFSL	−5.72E−01	1.80E−12	−1.69E−02	8.52E−01
TB1YR	CUSR0000SAS4	−4.25E−01	5.89E−07	−2.31E−01	9.65E−03
TB1YR	CUUR0000SAS4	−4.18E−01	9.13E−07	−2.56E−01	3.92E−03
TB1YR	CWSR0000SA0	−6.27E−01	2.54E−15	−4.45E−02	6.22E−01
TB1YR	CWSR0000SA0L1	−5.77E−01	9.69E−13	−4.92E−02	5.86E−01
TB1YR	CWSR0000SAS4	−4.38E−01	2.38E−07	−2.49E−01	5.09E−03
TB1YR	CWUR0000SA0	−5.85E−01	3.97E−13	−3.10E−02	7.31E−01
TB1YR	CWUR0000SA0L1	−5.47E−01	2.48E−11	−2.10E−02	8.17E−01
TB1YR	CWUR0000SAS4	−4.33E−01	3.37E−07	−2.67E−01	2.63E−03
TB1YR	PCEPILFE	−7.40E−01	2.07E−23	−8.84E−02	3.27E−01
TB1YR	PPIACO	−4.93E−01	3.38E−09	−1.02E−02	9.10E−01
TB6MS	CPIAUCNS	−6.03E−01	5.24E−14	−3.82E−02	6.72E−01
TB6MS	CPIAUCSL	−6.50E−01	1.09E−16	−2.49E−02	7.82E−01
TB6MS	CPIULFNS	−5.66E−01	3.35E−12	−1.93E−02	8.31E−01
TB6MS	CPIULFSL	−6.02E−01	5.59E−14	−3.67E−02	6.84E−01
TB6MS	CUSR0000SAS4	−4.40E−01	1.98E−07	−2.49E−01	5.19E−03
TB6MS	CUUR0000SAS4	−4.33E−01	3.22E−07	−2.74E−01	1.95E−03
TB6MS	CWSR0000SA0	−6.54E−01	6.07E−17	−6.23E−02	4.90E−01
TB6MS	CWSR0000SA0L1	−6.06E−01	3.27E−14	−6.73E−02	4.56E−01
TB6MS	CWSR0000SAS4	−4.53E−01	7.80E−08	−2.66E−01	2.70E−03
TB6MS	CWUR0000SA0	−6.09E−01	2.34E−14	−4.82E−02	5.94E−01
TB6MS	CWUR0000SA0L1	−5.72E−01	1.77E−12	−3.75E−02	6.78E−01
TB6MS	CWUR0000SAS4	−4.48E−01	1.14E−07	−2.84E−01	1.32E−03
TB6MS	PCEPILFE	−7.46E−01	5.87E−24	−1.04E−01	2.47E−01
TB6MS	PPIACO	−5.07E−01	1.01E−09	−2.25E−02	8.03E−01
TB3MS	CPIAUCNS	−6.18E−01	8.07E−15	−5.13E−02	5.70E−01
TB3MS	CPIAUCSL	−6.67E−01	8.79E−18	−3.85E−02	6.70E−01
TB3MS	CPIULFNS	−5.82E−01	5.94E−13	−3.21E−02	7.22E−01
TB3MS	CPIULFSL	−6.21E−01	5.62E−15	−5.04E−02	5.77E−01
TB3MS	CUSR0000SAS4	−4.49E−01	1.05E−07	−2.59E−01	3.54E−03
TB3MS	CUUR0000SAS4	−4.42E−01	1.77E−07	−2.85E−01	1.27E−03
TB3MS	CWSR0000SA0	−6.69E−01	6.43E−18	−7.50E−02	4.06E−01
TB3MS	CWSR0000SA0L1	−6.24E−01	3.62E−15	−7.97E−02	3.77E−01
TB3MS	CWSR0000SAS4	−4.62E−01	4.06E−08	−2.77E−01	1.79E−03
TB3MS	CWUR0000SA0	−6.23E−01	3.83E−15	−6.01E−02	5.06E−01
TB3MS	CWUR0000SA0L1	−5.87E−01	3.20E−13	−4.90E−02	5.87E−01
TB3MS	CWUR0000SAS4	−4.57E−01	6.06E−08	−2.95E−01	8.45E−04
TB3MS	PCEPILFE	−7.46E−01	4.90E−24	−1.20E−01	1.83E−01
TB3MS	PPIACO	−5.16E−01	4.60E−10	−3.01E−02	7.39E−01

3.A.3 Regression Results

The following tables show the regression results for the level regression model

$$\hat{\beta}_t = b_0 + b_1 r_t + b_2 r_{t-1} + b_3 r_{t-2} + b_4 r_{t-3} + b_5 r_{t-4} + v_t$$

and the error regression model

$$e_{\hat{\beta},t} = a_0 + a_1 e_{r,t} + a_2 e_{r,t-1} + a_3 e_{r,t-2} + a_4 e_{r,t-3} + a_5 e_{r,t-4} + w_t.$$

Note that the levels and innovations of the discounting states are derived from the most likely discounting process for the MLE from Maag and Reich (2023). The level and innovations for the real interest rates are based on the nominal interest rate and inflation rate candidates. In the first column, the first row provides the ticker of the nominal interest rate, and the second row the ticker of the inflation rate. For the coefficients of the regression models, the first value in a row is the estimated OLS coefficient, and the second row is the corresponding p value. In the last column, the first row provides the F statistic, and the second row the corresponding p value. Note that the intercept is abbreviated as Int.

Table 3.6: Regression results for all level models. Source: Own calculation based on data from FRED (2023b).

Interest Rate **Inflation**	**Int**	r_t	r_{t-1}	r_{t-2}	r_{t-3}	r_{t-4}	R^2	**F Statistic** **p Value**
FEDFUNDS CPIAUCNS	1.029 <1E−50	−1.60E−01 3.46E−03	−9.75E−02 1.02E−01	−1.06E−01 7.25E−02	−9.77E−02 9.56E−02	−1.42E−01 8.13E−03	6.20E−01	3.85E+01 2.79E−23
FEDFUNDS CPIAUCSL	1.029 <1E−50	−1.76E−01 3.37E−03	−1.04E−01 1.15E−01	−1.14E−01 8.21E−02	−9.92E−02 1.33E−01	−1.48E−01 1.33E−02	6.56E−01	4.51E+01 8.12E−26
FEDFUNDS CPIULFNS	1.029 <1E−50	−1.66E−01 3.04E−03	−7.95E−02 2.10E−01	−9.57E−02 1.28E−01	−8.07E−02 1.98E−01	−1.56E−01 5.09E−03	5.76E−01	3.21E+01 1.53E−20
FEDFUNDS CPIULFSL	1.029 <1E−50	−1.60E−01 5.83E−03	−1.03E−01 9.81E−02	−1.16E−01 5.75E−02	−9.46E−02 1.27E−01	−1.41E−01 1.50E−02	6.08E−01	3.66E+01 1.80E−22
FEDFUNDS CUSR0000SAS4	1.028 <1E−50	−7.92E−02 5.96E−02	−8.07E−02 5.67E−02	−5.64E−02 2.22E−01	−8.72E−02 3.92E−02	−9.54E−02 2.32E−02	4.39E−01	1.85E+01 1.65E−13
FEDFUNDS CUUR0000SAS4	1.028 <1E−50	−7.63E−02 6.21E−02	−8.27E−02 4.35E−02	−6.04E−02 1.77E−01	−8.53E−02 3.70E−02	−9.23E−02 2.37E−02	4.38E−01	1.84E+01 1.78E−13
FEDFUNDS CWSR0000SA0	1.029 <1E−50	−1.70E−01 2.21E−03	−1.17E−01 4.77E−02	−8.64E−02 1.49E−01	−1.15E−01 5.30E−02	−1.37E−01 1.28E−02	6.68E−01	4.76E+01 1.01E−26
FEDFUNDS CWSR0000SA0L1	1.029 <1E−50	−1.64E−01 3.41E−03	−9.90E−02 1.06E−01	−9.62E−02 1.16E−01	−9.16E−02 1.35E−01	−1.39E−01 1.24E−02	6.12E−01	3.73E+01 8.94E−23
FEDFUNDS CWSR0000SAS4	1.028 <1E−50	−8.36E−02 3.96E−02	−7.80E−02 5.80E−02	−5.18E−02 2.49E−01	−8.45E−02 3.96E−02	−9.87E−02 1.52E−02	4.56E−01	1.98E+01 2.91E−14
FEDFUNDS CWUR0000SA0	1.029 <1E−50	−1.69E−01 1.31E−03	−8.18E−02 1.69E−01	−1.06E−01 7.21E−02	−8.53E−02 1.44E−01	−1.45E−01 4.85E−03	6.31E−01	4.03E+01 5.16E−24
FEDFUNDS CWUR0000SA0L1	1.029 <1E−50	−1.79E−01 1.96E−03	−5.35E−02 4.44E−01	−9.74E−02 1.59E−01	−5.70E−02 4.07E−01	−1.65E−01 3.99E−03	5.77E−01	3.21E+01 1.50E−20
FEDFUNDS CWUR0000SAS4	1.028 <1E−50	−7.89E−02 4.39E−02	−7.95E−02 4.34E−02	−6.06E−02 1.54E−01	−8.48E−02 3.17E−02	−9.41E−02 1.66E−02	4.59E−01	2.00E+01 2.06E−14
FEDFUNDS PCEPILFE	1.030 <1E−50	−2.31E−01 1.18E−03	−8.77E−02 3.03E−01	−1.22E−01 1.50E−01	−9.92E−02 2.42E−01	−2.63E−01 2.42E−04	7.25E−01	6.22E+01 1.85E−31
FEDFUNDS PPIACO	1.029 <1E−50	−1.33E−01 1.28E−04	−5.14E−02 1.68E−01	−7.07E−02 4.70E−02	−3.47E−02 3.49E−01	−8.97E−02 8.53E−03	5.15E−01	2.51E+01 3.87E−17
GS1 CPIAUCNS	1.029 <1E−50	−1.56E−01 1.09E−02	−9.11E−02 1.77E−01	−9.76E−02 1.43E−01	−8.95E−02 1.75E−01	−1.27E−01 3.53E−02	5.21E−01	2.57E+01 1.88E−17
GS1 CPIAUCSL	1.029 <1E−50	−1.76E−01 9.70E−03	−9.74E−02 1.96E−01	−1.03E−01 1.70E−01	−8.95E−02 2.34E−01	−1.33E−01 4.84E−02	5.53E−01	2.91E+01 3.66E−19
GS1 CPIULFNS	1.029 <1E−50	−1.61E−01 1.07E−02	−7.06E−02 3.28E−01	−7.97E−02 2.65E−01	−7.09E−02 3.19E−01	−1.41E−01 2.39E−02	4.70E−01	2.09E+01 6.64E−15
GS1 CPIULFSL	1.029 <1E−50	−1.59E−01 1.52E−02	−9.12E−02 2.00E−01	−9.71E−02 1.67E−01	−7.86E−02 2.68E−01	−1.29E−01 4.80E−02	4.93E−01	2.29E+01 5.13E−16
GS1 CUSR0000SAS4	1.028 <1E−50	−7.73E−02 7.13E−02	−8.04E−02 6.21E−02	−5.88E−02 2.02E−01	−8.36E−02 5.18E−02	−8.87E−02 3.82E−02	3.89E−01	1.50E+01 2.20E−11
GS1 CUUR0000SAS4	1.028 <1E−50	−7.52E−02 7.21E−02	−8.21E−02 5.03E−02	−6.09E−02 1.77E−01	−8.18E−02 5.07E−02	−8.68E−02 3.76E−02	3.88E−01	1.50E+01 2.31E−11
GS1 CWSR0000SA0	1.029 <1E−50	−1.69E−01 7.64E−03	−1.09E−01 1.07E−01	−8.06E−02 2.39E−01	−1.04E−01 1.25E−01	−1.22E−01 5.13E−02	5.67E−01	3.09E+01 5.75E−20
GS1 CWSR0000SA0L1	1.029 <1E−50	−1.59E−01 1.12E−02	−8.71E−02 2.13E−01	−8.10E−02 2.47E−01	−7.82E−02 2.63E−01	−1.25E−01 4.45E−02	4.98E−01	2.34E+01 2.83E−16
GS1 CWSR0000SAS4	1.028 <1E−50	−8.32E−02 4.41E−02	−7.81E−02 6.23E−02	−5.26E−02 2.43E−01	−8.13E−02 5.17E−02	−9.38E−02 2.33E−02	4.07E−01	1.62E+01 3.70E−12
GS1 CWUR0000SA0	1.029 <1E−50	−1.65E−01 5.29E−03	−7.51E−02 2.67E−01	−9.80E−02 1.42E−01	−7.83E−02 2.37E−01	−1.31E−01 2.40E−02	5.33E−01	2.70E+01 4.18E−18
GS1	1.029	−1.71E−01	−4.69E−02	−8.17E−02	−5.11E−02	−1.47E−01	4.70E−01	2.10E+01

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Interest Rate Inflation	Int	r_t	r_{t-1}	r_{t-2}	r_{t-3}	r_{t-4}	R^2	F Statistic p Value
CWUR0000SA0L1	<1E−50	8.15E−03	5.52E−01	2.96E−01	5.10E−01	2.08E−02		5.99E−15
GS1	1.028	−7.87E−02	−7.93E−02	−6.08E−02	−8.14E−02	−8.95E−02	4.10E−01	1.64E+01
CWUR0000SAS4	<1E−50	4.75E−02	4.76E−02	1.53E−01	4.22E−02	2.47E−02		2.80E−12
GS1	1.030	−2.58E−01	−1.23E−01	−1.47E−01	−1.30E−01	−2.62E−01	7.39E−01	6.67E+01
PCEPILFE	<1E−50	7.82E−04	1.64E−01	9.26E−02	1.37E−01	6.28E−04		9.08E−33
GS1	1.029	−1.29E−01	−5.13E−02	−6.81E−02	−2.88E−02	−7.92E−02	4.48E−01	1.92E+01
PPIACO	<1E−50	7.28E−04	2.05E−01	7.41E−02	4.74E−01	3.39E−02		6.42E−14
GS3	1.029	−1.50E−01	−7.71E−02	−7.99E−02	−7.18E−02	−9.94E−02	4.49E−01	1.92E+01
CPIAUCNS	<1E−50	2.20E−02	2.88E−01	2.66E−01	3.12E−01	1.20E−01		6.13E−14
GS3	1.029	−1.70E−01	−8.11E−02	−7.97E−02	−6.90E−02	−1.04E−01	4.70E−01	2.09E+01
CPIAUCSL	<1E−50	2.05E−02	3.24E−01	3.32E−01	4.02E−01	1.53E−01		6.47E−15
GS3	1.029	−1.51E−01	−5.86E−02	−6.39E−02	−5.56E−02	−1.13E−01	4.01E−01	1.58E+01
CPIULFNS	<1E−50	2.29E−02	4.46E−01	4.03E−01	4.63E−01	8.37E−02		6.66E−12
GS3	1.029	−1.51E−01	−7.72E−02	−7.38E−02	−5.76E−02	−1.03E−01	4.15E−01	1.68E+01
CPIULFSL	<1E−50	2.91E−02	3.11E−01	3.29E−01	4.49E−01	1.34E−01		1.70E−12
GS3	1.028	−7.99E−02	−8.01E−02	−5.54E−02	−7.73E−02	−8.16E−02	3.77E−01	1.43E+01
CUSR0000SAS4	<1E−50	6.38E−02	6.47E−02	2.33E−01	7.41E−02	5.79E−02		6.58E−11
GS3	1.028	−7.78E−02	−8.19E−02	−5.71E−02	−7.53E−02	−8.03E−02	3.76E−01	1.42E+01
CUUR0000SAS4	<1E−50	6.44E−02	5.25E−02	2.10E−01	7.39E−02	5.56E−02		7.01E−11
GS3	1.029	−1.62E−01	−9.17E−02	−6.13E−02	−8.36E−02	−9.47E−02	4.84E−01	2.21E+01
CWSR0000SA0	<1E−50	1.70E−02	2.15E−01	4.13E−01	2.59E−01	1.60E−01		1.42E−15
GS3	1.029	−1.50E−01	−7.25E−02	−6.17E−02	−6.02E−02	−1.00E−01	4.22E−01	1.72E+01
CWSR0000SA0L1	<1E−50	2.34E−02	3.33E−01	4.12E−01	4.22E−01	1.28E−01		9.16E−13
GS3	1.028	−8.52E−02	−7.76E−02	−4.93E−02	−7.53E−02	−8.72E−02	3.95E−01	1.54E+01
CWSR0000SAS4	<1E−50	4.02E−02	6.59E−02	2.77E−01	7.34E−02	3.56E−02		1.20E−11
GS3	1.029	−1.56E−01	−6.25E−02	−8.18E−02	−6.33E−02	−1.04E−01	4.61E−01	2.02E+01
CWUR0000SA0	<1E−50	1.31E−02	3.91E−01	2.56E−01	3.74E−01	9.07E−02		1.64E−14
GS3	1.029	−1.60E−01	−3.66E−02	−6.70E−02	−3.99E−02	−1.20E−01	4.04E−01	1.60E+01
CWUR0000SA0L1	<1E−50	1.81E−02	6.63E−01	4.21E−01	6.29E−01	7.20E−02		4.94E−12
GS3	1.028	−8.06E−02	−7.87E−02	−5.71E−02	−7.54E−02	−8.33E−02	3.97E−01	1.56E+01
CWUR0000SAS4	<1E−50	4.34E−02	5.08E−02	1.83E−01	6.16E−02	3.74E−02		9.58E−12
GS3	1.030	−2.65E−01	−1.33E−01	−1.48E−01	−1.27E−01	−2.18E−01	7.22E−01	6.13E+01
PCEPILFE	<1E−50	1.29E−03	1.54E−01	1.09E−01	1.69E−01	7.85E−03		3.39E−31
GS3	1.029	−1.22E−01	−4.92E−02	−6.05E−02	−2.18E−02	−6.51E−02	4.06E−01	1.61E+01
PPIACO	<1E−50	2.02E−03	2.44E−01	1.27E−01	6.04E−01	9.33E−02		4.23E−12
GS5	1.029	−1.50E−01	−7.29E−02	−7.48E−02	−6.61E−02	−9.29E−02	4.25E−01	1.74E+01
CPIAUCNS	<1E−50	2.42E−02	3.26E−01	3.08E−01	3.63E−01	1.53E−01		6.77E−13
GS5	1.029	−1.70E−01	−7.62E−02	−7.30E−02	−6.24E−02	−9.74E−02	4.44E−01	1.88E+01
CPIAUCSL	<1E−50	2.25E−02	3.66E−01	3.86E−01	4.59E−01	1.87E−01		9.97E−14
GS5	1.029	−1.50E−01	−5.51E−02	−5.92E−02	−5.09E−02	−1.06E−01	3.79E−01	1.44E+01
CPIULFNS	<1E−50	2.61E−02	4.83E−01	4.48E−01	5.11E−01	1.10E−01		5.34E−11
GS5	1.029	−1.51E−01	−7.32E−02	−6.75E−02	−5.11E−02	−9.67E−02	3.91E−01	1.52E+01
CPIULFSL	<1E−50	3.22E−02	3.47E−01	3.81E−01	5.11E−01	1.66E−01		1.71E−11
GS5	1.028	−8.08E−02	−7.97E−02	−5.37E−02	−7.45E−02	−7.89E−02	3.67E−01	1.37E+01
CUSR0000SAS4	<1E−50	6.26E−02	6.83E−02	2.52E−01	8.72E−02	6.85E−02		1.62E−10
GS5	1.028	−7.85E−02	−8.14E−02	−5.52E−02	−7.25E−02	−7.78E−02	3.66E−01	1.36E+01
CUUR0000SAS4	<1E−50	6.36E−02	5.55E−02	2.29E−01	8.76E−02	6.56E−02		1.73E−10
GS5	1.029	−1.62E−01	−8.65E−02	−5.48E−02	−7.72E−02	−8.89E−02	4.57E−01	1.99E+01
CWSR0000SA0	<1E−50	1.89E−02	2.55E−01	4.76E−01	3.10E−01	1.94E−01		2.49E−14
GS5	1.029	−1.49E−01	−6.82E−02	−5.58E−02	−5.46E−02	−9.47E−02	3.98E−01	1.56E+01
CWSR0000SA0L1	<1E−50	2.63E−02	3.73E−01	4.67E−01	4.76E−01	1.57E−01		9.50E−12
GS5	1.028	−8.59E−02	−7.71E−02	−4.76E−02	−7.28E−02	−8.48E−02	3.85E−01	1.48E+01
CWSR0000SAS4	<1E−50	4.01E−02	6.99E−02	2.98E−01	8.63E−02	4.27E−02		3.04E−11
GS5	1.029	−1.56E−01	−5.83E−02	−7.67E−02	−5.81E−02	−9.82E−02	4.37E−01	1.83E+01
CWUR0000SA0	<1E−50	1.51E−02	4.34E−01	2.98E−01	4.25E−01	1.17E−01		1.96E−13
GS5	1.029	−1.59E−01	−3.32E−02	−6.22E−02	−3.63E−02	−1.13E−01	3.82E−01	1.46E+01
CWUR0000SA0L1	<1E−50	2.10E−02	6.99E−01	4.65E−01	6.67E−01	9.45E−02		3.88E−11
GS5	1.028	−8.12E−02	−7.82E−02	−5.54E−02	−7.28E−02	−8.09E−02	3.87E−01	1.49E+01
CWUR0000SAS4	<1E−50	4.35E−02	5.42E−02	2.01E−01	7.31E−02	4.48E−02		2.49E−11
GS5	1.030	−2.69E−01	−1.36E−01	−1.51E−01	−1.24E−01	−2.03E−01	7.08E−01	5.72E+01
PCEPILFE	<1E−50	1.69E−03	1.63E−01	1.15E−01	1.99E−01	1.72E−02		6.38E−30
GS5	1.029	−1.21E−01	−4.89E−02	−5.84E−02	−1.97E−02	−6.11E−02	3.92E−01	1.52E+01
PPIACO	<1E−50	2.47E−03	2.53E−01	1.45E−01	6.43E−01	1.20E−01		1.53E−11
GS10	1.029	−1.49E−01	−6.87E−02	−6.97E−02	−6.14E−02	−8.87E−02	3.96E−01	1.55E+01
CPIAUCNS	<1E−50	2.69E−02	3.65E−01	3.53E−01	4.08E−01	1.79E−01		1.12E−11
GS10	1.029	−1.70E−01	−7.18E−02	−6.68E−02	−5.74E−02	−9.34E−02	4.13E−01	1.66E+01
CPIAUCSL	<1E−50	2.51E−02	4.05E−01	4.38E−01	5.06E−01	2.14E−01		2.12E−12
GS10	1.029	−1.48E−01	−5.15E−02	−5.39E−02	−4.69E−02	−1.01E−01	3.50E−01	1.27E+01
CPIULFNS	<1E−50	2.99E−02	5.19E−01	4.98E−01	5.52E−01	1.33E−01		6.78E−10
GS10	1.029	−1.49E−01	−6.93E−02	−6.12E−02	−4.59E−02	−9.25E−02	3.61E−01	1.34E+01
CPIULFSL	<1E−50	3.61E−02	3.81E−01	4.36E−01	5.61E−01	1.91E−01		2.59E−10
GS10	1.028	−8.03E−02	−7.78E−02	−5.10E−02	−7.12E−02	−7.58E−02	3.45E−01	1.24E+01
CUSR0000SAS4	<1E−50	6.69E−02	7.85E−02	2.81E−01	1.06E−01	8.31E−02		1.05E−09
GS10	1.028	−7.80E−02	−7.96E−02	−5.25E−02	−6.92E−02	−7.49E−02	3.44E−01	1.24E+01
CUUR0000SAS4	<1E−50	6.83E−02	6.41E−02	2.58E−01	1.07E−01	7.94E−02		1.13E−09
GS10	1.029	−1.62E−01	−8.19E−02	−4.85E−02	−7.24E−02	−8.58E−02	4.27E−01	1.76E+01
CWSR0000SA0	<1E−50	2.12E−02	2.93E−01	5.39E−01	3.53E−01	2.19E−01		5.64E−13
GS10	1.029	−1.48E−01	−6.41E−02	−4.99E−02	−4.99E−02	−9.10E−02	3.68E−01	1.37E+01
CWSR0000SA0L1	<1E−50	2.99E−02	4.11E−01	5.24E−01	5.23E−01	1.80E−01		1.42E−10
GS10	1.028	−8.53E−02	−7.53E−02	−4.51E−02	−6.97E−02	−8.18E−02	3.63E−01	1.35E+01
CWSR0000SAS4	<1E−50	4.40E−02	8.07E−02	3.31E−01	1.05E−01	5.31E−02		2.15E−10
GS10	1.029	−1.55E−01	−5.43E−02	−7.22E−02	−5.37E−02	−9.43E−02	4.08E−01	1.63E+01
CWUR0000SA0	<1E−50	1.75E−02	4.77E−01	3.38E−01	4.72E−01	1.40E−01		3.36E−12

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Interest Rate Inflation	Int	r_t	r_{t-1}	r_{t-2}	r_{t-3}	r_{t-4}	R^2	F Statistic p Value
GS10	1.029	−1.57E−01	−2.99E−02	−5.70E−02	−3.32E−02	−1.08E−01	3.55E−01	1.30E+01
CWUR0000SA0L1	<1E−50	2.43E−02	7.33E−01	5.11E−01	6.99E−01	1.14E−01		4.66E−10
GS10	1.028	−8.05E−02	−7.64E−02	−5.26E−02	−6.97E−02	−7.82E−02	3.65E−01	1.36E+01
CWUR0000SAS4	<1E−50	4.79E−02	6.34E−02	2.31E−01	9.06E−02	5.56E−02		1.80E−10
GS10	1.031	−2.75E−01	−1.42E−01	−1.53E−01	−1.23E−01	−1.94E−01	6.87E−01	5.18E+01
PCEPILFE	<1E−50	2.19E−03	1.60E−01	1.24E−01	2.17E−01	2.94E−02		3.56E−28
GS10	1.029	−1.21E−01	−4.90E−02	−5.68E−02	−1.79E−02	−5.78E−02	3.76E−01	1.42E+01
PPIACO	<1E−50	3.08E−03	2.61E−01	1.63E−01	6.79E−01	1.49E−01		7.07E−11
GS20	1.029	−1.49E−01	−6.56E−02	−6.64E−02	−5.76E−02	−8.42E−02	3.73E−01	1.40E+01
CPIAUCNS	<1E−50	2.98E−02	3.96E−01	3.86E−01	4.47E−01	2.10E−01		9.19E−11
GS20	1.029	−1.70E−01	−6.84E−02	−6.22E−02	−5.37E−02	−8.88E−02	3.89E−01	1.50E+01
CPIAUCSL	<1E−50	2.79E−02	4.38E−01	4.81E−01	5.43E−01	2.46E−01		2.10E−11
GS20	1.029	−1.47E−01	−4.89E−02	−5.06E−02	−4.36E−02	−9.63E−02	3.29E−01	1.15E+01
CPIULFNS	<1E−50	3.39E−02	5.48E−01	5.32E−01	5.87E−01	1.60E−01		4.34E−09
GS20	1.029	−1.48E−01	−6.69E−02	−5.67E−02	−4.20E−02	−8.78E−02	3.39E−01	1.21E+01
CPIULFSL	<1E−50	4.05E−02	4.06E−01	4.79E−01	6.01E−01	2.22E−01		1.87E−09
GS20	1.028	−8.01E−02	−7.67E−02	−4.94E−02	−6.84E−02	−7.30E−02	3.29E−01	1.16E+01
CUSR0000SAS4	<1E−50	7.08E−02	8.66E−02	3.04E−01	1.25E−01	9.93E−02		4.17E−09
GS20	1.028	−7.76E−02	−7.87E−02	−5.08E−02	−6.63E−02	−7.21E−02	3.28E−01	1.15E+01
CUUR0000SAS4	<1E−50	7.29E−02	7.06E−02	2.80E−01	1.27E−01	9.51E−02		4.46E−09
GS20	1.029	−1.62E−01	−7.84E−02	−4.40E−02	−6.88E−02	−8.17E−02	4.03E−01	1.59E+01
CWSR0000SA0	<1E−50	2.38E−02	3.26E−01	5.86E−01	3.89E−01	2.51E−01		5.89E−12
GS20	1.029	−1.47E−01	−6.15E−02	−4.55E−02	−4.66E−02	−8.67E−02	3.46E−01	1.25E+01
CWSR0000SA0L1	<1E−50	3.38E−02	4.39E−01	5.69E−01	5.58E−01	2.09E−01		1.02E−09
GS20	1.028	−8.49E−02	−7.43E−02	−4.36E−02	−6.71E−02	−7.91E−02	3.47E−01	1.26E+01
CWSR0000SAS4	<1E−50	4.75E−02	8.87E−02	3.53E−01	1.23E−01	6.46E−02		8.82E−10
GS20	1.029	−1.55E−01	−5.12E−02	−6.91E−02	−5.03E−02	−9.00E−02	3.86E−01	1.48E+01
CWUR0000SA0	<1E−50	2.00E−02	5.11E−01	3.69E−01	5.09E−01	1.66E−01		2.85E−11
GS20	1.029	−1.56E−01	−2.76E−02	−5.37E−02	−3.11E−02	−1.03E−01	3.33E−01	1.18E+01
CWUR0000SA0L1	<1E−50	2.80E−02	7.57E−01	5.44E−01	7.23E−01	1.39E−01		2.89E−09
GS20	1.028	−8.00E−02	−7.55E−02	−5.10E−02	−6.72E−02	−7.55E−02	3.49E−01	1.27E+01
CWUR0000SAS4	<1E−50	5.20E−02	7.00E−02	2.52E−01	1.07E−01	6.78E−02		7.47E−10
GS20	1.031	−2.78E−01	−1.41E−01	−1.51E−01	−1.20E−01	−1.89E−01	6.62E−01	4.61E+01
PCEPILFE	<1E−50	3.06E−03	1.81E−01	1.46E−01	2.51E−01	4.36E−02		3.36E−26
GS20	1.029	−1.20E−01	−4.86E−02	−5.54E−02	−1.65E−02	−5.50E−02	3.62E−01	1.34E+01
PPIACO	<1E−50	3.58E−03	2.70E−01	1.79E−01	7.07E−01	1.74E−01		2.50E−10
LTGOVTBD	1.029	−1.50E−01	−6.77E−02	−6.72E−02	−6.00E−02	−8.92E−02	4.03E−01	1.59E+01
CPIAUCNS	<1E−50	2.50E−02	3.70E−01	3.70E−01	4.18E−01	1.74E−01		5.70E−12
LTGOVTBD	1.029	−1.70E−01	−7.00E−02	−6.34E−02	−5.54E−02	−9.53E−02	4.20E−01	1.71E+01
CPIAUCSL	<1E−50	2.37E−02	4.17E−01	4.62E−01	5.21E−01	2.02E−01		1.10E−12
LTGOVTBD	1.029	−1.48E−01	−5.05E−02	−5.07E−02	−4.53E−02	−1.01E−01	3.55E−01	1.30E+01
CPIULFNS	<1E−50	2.82E−02	5.26E−01	5.22E−01	5.65E−01	1.30E−01		4.41E−10
LTGOVTBD	1.029	−1.50E−01	−6.75E−02	−5.78E−02	−4.41E−02	−9.33E−02	3.66E−01	1.36E+01
CPIULFSL	<1E−50	3.42E−02	3.92E−01	4.61E−01	5.75E−01	1.84E−01		1.68E−10
LTGOVTBD	1.028	−7.85E−02	−7.56E−02	−4.92E−02	−6.88E−02	−7.36E−02	3.43E−01	1.23E+01
CUSR0000SAS4	<1E−50	7.27E−02	8.66E−02	2.99E−01	1.18E−01	9.24E−02		1.23E−09
LTGOVTBD	1.028	−7.62E−02	−7.73E−02	−5.06E−02	−6.69E−02	−7.27E−02	3.42E−01	1.23E+01
CUUR0000SAS4	<1E−50	7.41E−02	7.14E−02	2.76E−01	1.19E−01	8.81E−02		1.34E−09
LTGOVTBD	1.029	−1.62E−01	−8.05E−02	−4.49E−02	−7.09E−02	−8.71E−02	4.33E−01	1.80E+01
CWSR0000SA0	<1E−50	2.02E−02	3.01E−01	5.69E−01	3.63E−01	2.09E−01		3.08E−13
LTGOVTBD	1.029	−1.48E−01	−6.26E−02	−4.62E−02	−4.86E−02	−9.14E−02	3.72E−01	1.40E+01
CWSR0000SA0L1	<1E−50	2.87E−02	4.20E−01	5.54E−01	5.32E−01	1.75E−01		9.77E−11
LTGOVTBD	1.028	−8.33E−02	−7.32E−02	−4.34E−02	−6.73E−02	−7.94E−02	3.61E−01	1.33E+01
CWSR0000SAS4	<1E−50	4.87E−02	8.92E−02	3.50E−01	1.17E−01	6.01E−02		2.67E−10
LTGOVTBD	1.029	−1.55E−01	−5.32E−02	−6.98E−02	−5.22E−02	−9.48E−02	4.15E−01	1.67E+01
CWUR0000SA0	<1E−50	1.65E−02	4.85E−01	3.54E−01	4.84E−01	1.35E−01		1.80E−12
LTGOVTBD	1.029	−1.57E−01	−2.90E−02	−5.38E−02	−3.20E−02	−1.08E−01	3.59E−01	1.32E+01
CWUR0000SA0L1	<1E−50	2.30E−02	7.40E−01	5.34E−01	7.09E−01	1.13E−01		3.15E−10
LTGOVTBD	1.028	−7.86E−02	−7.41E−02	−5.05E−02	−6.73E−02	−7.59E−02	3.63E−01	1.34E+01
CWUR0000SAS4	<1E−50	5.28E−02	7.11E−02	2.50E−01	1.02E−01	6.27E−02		2.31E−10
LTGOVTBD	1.030	−2.71E−01	−1.42E−01	−1.50E−01	−1.20E−01	−1.91E−01	6.98E−01	5.45E+01
PCEPILFE	<1E−50	2.32E−03	1.54E−01	1.28E−01	2.26E−01	3.15E−02		4.34E−29
LTGOVTBD	1.029	−1.21E−01	−4.91E−02	−5.75E−02	−1.82E−02	−5.78E−02	3.85E−01	1.48E+01
PPIACO	<1E−50	2.88E−03	2.57E−01	1.56E−01	6.74E−01	1.46E−01		3.13E−11
MPRIME	1.030	−1.59E−01	−8.78E−02	−9.37E−02	−8.57E−02	−1.35E−01	6.14E−01	3.75E+01
CPIAUCNS	<1E−50	3.56E−03	1.47E−01	1.18E−01	1.48E−01	1.16E−02		7.51E−23
MPRIME	1.030	−1.75E−01	−9.21E−02	−9.80E−02	−8.55E−02	−1.41E−01	6.43E−01	4.26E+01
CPIAUCSL	<1E−50	3.67E−03	1.70E−01	1.43E−01	2.03E−01	1.87E−02		7.15E−25
MPRIME	1.029	−1.66E−01	−7.27E−02	−8.36E−02	−7.15E−02	−1.49E−01	5.78E−01	3.23E+01
CPIULFNS	<1E−50	2.98E−03	2.56E−01	1.89E−01	2.58E−01	6.92E−03		1.30E−20
MPRIME	1.030	−1.61E−01	−9.36E−02	−1.01E−01	−8.23E−02	−1.36E−01	6.05E−01	3.61E+01
CPIULFSL	<1E−50	5.63E−03	1.36E−01	1.02E−01	1.90E−01	1.84E−02		2.81E−22
MPRIME	1.029	−8.21E−02	−8.53E−02	−6.10E−02	−8.78E−02	−9.41E−02	4.78E−01	2.16E+01
CUSR0000SAS4	<1E−50	4.23E−02	3.61E−02	1.67E−01	3.05E−02	2.00E−02		2.63E−15
MPRIME	1.029	−7.91E−02	−8.76E−02	−6.47E−02	−8.62E−02	−9.13E−02	4.78E−01	2.16E+01
CUUR0000SAS4	<1E−50	4.53E−02	2.67E−02	1.32E−01	2.90E−02	2.05E−02		2.70E−15
MPRIME	1.030	−1.69E−01	−1.05E−01	−7.50E−02	−1.01E−01	−1.32E−01	6.60E−01	4.57E+01
CWSR0000SA0	<1E−50	2.20E−03	8.08E−02	2.16E−01	9.34E−02	1.63E−02		4.63E−26
MPRIME	1.030	−1.64E−01	−9.07E−02	−8.54E−02	−8.16E−02	−1.35E−01	6.14E−01	3.75E+01
CWSR0000SA0L1	<1E−50	2.96E−03	1.37E−01	1.61E−01	1.81E−01	1.35E−02		7.37E−23
MPRIME	1.029	−8.54E−02	−8.25E−02	−5.68E−02	−8.55E−02	−9.69E−02	4.94E−01	2.30E+01
CWSR0000SAS4	<1E−50	3.02E−02	3.84E−02	1.90E−01	3.14E−02	1.40E−02		4.48E−16
MPRIME	1.030	−1.69E−01	−7.19E−02	−9.60E−02	−7.43E−02	−1.40E−01	6.28E−01	3.98E+01

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Interest Rate Inflation	Int	r_t	r_{t-1}	r_{t-2}	r_{t-3}	r_{t-4}	R^2	F Statistic p Value
CWUR0000SA0	<1E−50	1.32E−03	2.35E−01	1.09E−01	2.10E−01	6.65E−03		8.59E−24
MPRIME	1.029	−1.79E−01	−4.94E−02	−8.53E−02	−5.17E−02	−1.59E−01	5.82E−01	3.29E+01
CWUR0000SA0L1	<1E−50	1.78E−03	4.81E−01	2.20E−01	4.53E−01	4.94E−03		6.88E−21
MPRIME	1.029	−8.01E−02	−8.41E−02	−6.61E−02	−8.62E−02	−9.21E−02	4.98E−01	2.34E+01
CWUR0000SAS4	<1E−50	3.42E−02	2.68E−02	1.05E−01	2.36E−02	1.53E−02		2.80E−16
MPRIME	1.031	−2.24E−01	−8.34E−02	−1.19E−01	−9.41E−02	−2.49E−01	7.45E−01	6.91E+01
PCEPILFE	<1E−50	1.45E−03	3.26E−01	1.58E−01	2.65E−01	4.13E−04		1.98E−33
MPRIME	1.029	−1.31E−01	−4.77E−02	−6.84E−02	−2.89E−02	−8.44E−02	5.13E−01	2.48E+01
PPIACO	<1E−50	1.65E−04	2.03E−01	5.49E−02	4.38E−01	1.31E−02		5.09E−17
BAA	1.029	−1.50E−01	−6.09E−02	−6.35E−02	−5.09E−02	−7.90E−02	3.91E−01	1.52E+01
CPIAUCNS	<1E−50	2.58E−02	4.25E−01	4.02E−01	4.96E−01	2.29E−01		1.72E−11
BAA	1.030	−1.71E−01	−6.32E−02	−5.76E−02	−4.64E−02	−8.23E−02	4.05E−01	1.61E+01
CPIAUCSL	<1E−50	2.44E−02	4.70E−01	5.10E−01	5.96E−01	2.73E−01		4.47E−12
BAA	1.029	−1.48E−01	−4.60E−02	−4.99E−02	−3.91E−02	−9.25E−02	3.52E−01	1.28E+01
CPIULFNS	<1E−50	2.84E−02	5.65E−01	5.30E−01	6.21E−01	1.65E−01		6.12E−10
BAA	1.029	−1.50E−01	−6.45E−02	−5.37E−02	−3.70E−02	−8.40E−02	3.61E−01	1.33E+01
CPIULFSL	<1E−50	3.43E−02	4.17E−01	4.96E−01	6.41E−01	2.32E−01		2.63E−10
BAA	1.029	−8.58E−02	−8.13E−02	−5.12E−02	−6.97E−02	−7.43E−02	3.74E−01	1.41E+01
CUSR0000SAS4	<1E−50	4.55E−02	6.04E−02	2.71E−01	1.06E−01	8.23E−02		8.66E−11
BAA	1.029	−8.30E−02	−8.36E−02	−5.27E−02	−6.74E−02	−7.35E−02	3.73E−01	1.40E+01
CUUR0000SAS4	<1E−50	4.75E−02	4.72E−02	2.47E−01	1.09E−01	7.80E−02		9.46E−11
BAA	1.030	−1.63E−01	−7.41E−02	−3.91E−02	−6.28E−02	−7.60E−02	4.20E−01	1.71E+01
CWSR0000SA0	<1E−50	2.04E−02	3.47E−01	6.24E−01	4.26E−01	2.76E−01		1.10E−12
BAA	1.029	−1.49E−01	−5.97E−02	−4.30E−02	−4.27E−02	−8.34E−02	3.69E−01	1.38E+01
CWSR0000SA0L1	<1E−50	2.83E−02	4.45E−01	5.83E−01	5.85E−01	2.16E−01		1.30E−10
BAA	1.029	−8.97E−02	−7.79E−02	−4.49E−02	−6.75E−02	−8.00E−02	3.89E−01	1.50E+01
CWSR0000SAS4	<1E−50	3.08E−02	6.55E−02	3.25E−01	1.09E−01	5.35E−02		2.19E−11
BAA	1.029	−1.55E−01	−4.68E−02	−6.69E−02	−4.44E−02	−8.53E−02	4.05E−01	1.60E+01
CWUR0000SA0	<1E−50	1.71E−02	5.44E−01	3.80E−01	5.56E−01	1.80E−01		4.79E−12
BAA	1.029	−1.57E−01	−2.44E−02	−5.43E−02	−2.64E−02	−1.01E−01	3.57E−01	1.31E+01
CWUR0000SA0L1	<1E−50	2.27E−02	7.81E−01	5.32E−01	7.59E−01	1.38E−01		3.66E−10
BAA	1.029	−8.44E−02	−7.92E−02	−5.25E−02	−6.78E−02	−7.63E−02	3.91E−01	1.51E+01
CWUR0000SAS4	<1E−50	3.44E−02	4.97E−02	2.23E−01	9.31E−02	5.64E−02		1.80E−11
BAA	1.032	−2.71E−01	−1.28E−01	−1.45E−01	−1.08E−01	−1.69E−01	6.88E−01	5.20E+01
PCEPILFE	<1E−50	2.30E−03	2.02E−01	1.43E−01	2.79E−01	5.64E−02		3.04E−28
BAA	1.029	−1.19E−01	−4.83E−02	−5.36E−02	−1.50E−02	−5.09E−02	3.77E−01	1.43E+01
PPIACO	<1E−50	3.07E−03	2.65E−01	1.86E−01	7.27E−01	1.97E−01		6.41E−11
AAA	1.029	−1.48E−01	−6.32E−02	−6.57E−02	−5.50E−02	−8.19E−02	3.72E−01	1.40E+01
CPIAUCNS	<1E−50	2.98E−02	4.14E−01	3.92E−01	4.69E−01	2.21E−01		1.00E−10
AAA	1.029	−1.69E−01	−6.62E−02	−6.13E−02	−5.14E−02	−8.54E−02	3.88E−01	1.49E+01
CPIAUCSL	<1E−50	2.84E−02	4.54E−01	4.89E−01	5.62E−01	2.65E−01		2.38E−11
AAA	1.029	−1.46E−01	−4.71E−02	−5.10E−02	−4.14E−02	−9.42E−02	3.29E−01	1.16E+01
CPIULFNS	<1E−50	3.38E−02	5.63E−01	5.28E−01	6.06E−01	1.66E−01		4.20E−09
AAA	1.029	−1.47E−01	−6.57E−02	−5.60E−02	−4.04E−02	−8.54E−02	3.39E−01	1.21E+01
CPIULFSL	<1E−50	4.07E−02	4.15E−01	4.84E−01	6.15E−01	2.33E−01		1.79E−09
AAA	1.029	−8.12E−02	−7.72E−02	−4.91E−02	−6.78E−02	−7.25E−02	3.35E−01	1.19E+01
CUSR0000SAS4	<1E−50	6.55E−02	8.30E−02	3.04E−01	1.27E−01	9.95E−02		2.60E−09
AAA	1.029	−7.85E−02	−7.94E−02	−5.07E−02	−6.56E−02	−7.16E−02	3.34E−01	1.18E+01
CUUR0000SAS4	<1E−50	6.80E−02	6.66E−02	2.79E−01	1.29E−01	9.52E−02		2.78E−09
AAA	1.029	−1.62E−01	−7.62E−02	−4.30E−02	−6.69E−02	−7.90E−02	4.02E−01	1.58E+01
CWSR0000SA0	<1E−50	2.39E−02	3.40E−01	5.95E−01	4.03E−01	2.66E−01		6.45E−12
AAA	1.029	−1.46E−01	−6.04E−02	−4.52E−02	−4.52E−02	−8.46E−02	3.46E−01	1.25E+01
CWSR0000SA0L1	<1E−50	3.38E−02	4.46E−01	5.70E−01	5.69E−01	2.18E−01		9.48E−10
AAA	1.029	−8.57E−02	−7.45E−02	−4.33E−02	−6.62E−02	−7.85E−02	3.52E−01	1.28E+01
CWSR0000SAS4	<1E−50	4.45E−02	8.64E−02	3.55E−01	1.26E−01	6.54E−02		5.99E−10
AAA	1.029	−1.54E−01	−4.87E−02	−6.92E−02	−4.77E−02	−8.79E−02	3.85E−01	1.48E+01
CWUR0000SA0	<1E−50	2.00E−02	5.33E−01	3.70E−01	5.33E−01	1.75E−01		3.01E−11
AAA	1.029	−1.55E−01	−2.54E−02	−5.52E−02	−2.86E−02	−1.02E−01	3.34E−01	1.18E+01
CWUR0000SA0L1	<1E−50	2.75E−02	7.75E−01	5.32E−01	7.44E−01	1.42E−01		2.69E−09
AAA	1.029	−8.07E−02	−7.58E−02	−5.07E−02	−6.64E−02	−7.49E−02	3.54E−01	1.29E+01
CWUR0000SAS4	<1E−50	4.91E−02	6.76E−02	2.53E−01	1.10E−01	6.87E−02		5.03E−10
AAA	1.031	−2.76E−01	−1.39E−01	−1.55E−01	−1.20E−01	−1.84E−01	6.71E−01	4.81E+01
PCEPILFE	<1E−50	2.76E−03	1.81E−01	1.30E−01	2.45E−01	4.57E−02		6.56E−27
AAA	1.029	−1.20E−01	−4.86E−02	−5.48E−02	−1.58E−02	−5.32E−02	3.63E−01	1.34E+01
PPIACO	<1E−50	3.56E−03	2.70E−01	1.83E−01	7.18E−01	1.87E−01		2.31E−10
TB1YR	1.029	−1.59E−01	−8.86E−02	−9.41E−02	−8.63E−02	−1.25E−01	4.67E−01	2.07E+01
CPIAUCNS	<1E−50	1.40E−02	2.14E−01	1.82E−01	2.17E−01	4.94E−02		8.47E−15
TB1YR	1.029	−1.80E−01	−9.55E−02	−9.87E−02	−8.67E−02	−1.32E−01	4.98E−01	2.34E+01
CPIAUCSL	<1E−50	1.26E−02	2.33E−01	2.17E−01	2.79E−01	6.50E−02		2.78E−16
TB1YR	1.028	−1.61E−01	−6.72E−02	−7.50E−02	−6.67E−02	−1.37E−01	4.13E−01	1.66E+01
CPIULFNS	<1E−50	1.49E−02	3.77E−01	3.20E−01	3.73E−01	3.67E−02		2.14E−12
TB1YR	1.028	−1.60E−01	−8.80E−02	−9.10E−02	−7.39E−02	−1.26E−01	4.34E−01	1.81E+01
CPIULFSL	<1E−50	2.04E−02	2.42E−01	2.22E−01	3.25E−01	6.60E−02		2.58E−13
TB1YR	1.028	−7.65E−02	−7.83E−02	−5.60E−02	−7.96E−02	−8.50E−02	3.44E−01	1.24E+01
CUSR0000SAS4	<1E−50	8.48E−02	7.96E−02	2.42E−01	7.40E−02	5.53E−02		1.19E−09
TB1YR	1.028	−7.43E−02	−8.02E−02	−5.80E−02	−7.77E−02	−8.32E−02	3.43E−01	1.23E+01
CUUR0000SAS4	<1E−50	8.65E−02	6.51E−02	2.16E−01	7.33E−02	5.44E−02		1.24E−09
TB1YR	1.029	−1.73E−01	−1.07E−01	−7.56E−02	−1.02E−01	−1.22E−01	5.13E−01	2.49E+01
CWSR0000SA0	<1E−50	9.84E−03	1.37E−01	3.00E−01	1.59E−01	6.63E−02		4.66E−17
TB1YR	1.028	−1.60E−01	−8.34E−02	−7.48E−02	−7.38E−02	−1.23E−01	4.41E−01	1.86E+01
CWSR0000SA0L1	<1E−50	1.53E−02	2.59E−01	3.12E−01	3.18E−01	6.07E−02		1.37E−13
TB1YR	1.028	−8.24E−02	−7.62E−02	−4.99E−02	−7.77E−02	−9.04E−02	3.63E−01	1.35E+01
CWSR0000SAS4	<1E−50	5.44E−02	7.97E−02	2.86E−01	7.33E−02	3.49E−02		2.22E−10

Continued on next page

Interest Rate Inflation	Int	r_t	r_{t-1}	r_{t-2}	r_{t-3}	r_{t-4}	R^2	F Statistic p Value
TB1YR	1.029	−1.68E−01	−7.24E−02	−9.49E−02	−7.50E−02	−1.29E−01	4.81E−01	2.19E+01
CWUR0000SA0	<1E−50	7.20E−03	3.11E−01	1.79E−01	2.83E−01	3.42E−02		1.90E−15
TB1YR	1.028	−1.71E−01	−4.31E−02	−7.70E−02	−4.72E−02	−1.44E−01	4.16E−01	1.68E+01
CWUR0000SA0L1	<1E−50	1.15E−02	6.04E−01	3.50E−01	5.63E−01	3.15E−02		1.68E−12
TB1YR	1.028	−7.78E−02	−7.74E−02	−5.80E−02	−7.78E−02	−8.63E−02	3.66E−01	1.36E+01
CWUR0000SAS4	<1E−50	5.89E−02	6.21E−02	1.90E−01	6.12E−02	3.69E−02		1.75E−10
TB1YR	1.030	−2.85E−01	−1.43E−01	−1.68E−01	−1.46E−01	−2.78E−01	7.33E−01	6.49E+01
PCEPILFE	<1E−50	3.76E−04	1.16E−01	6.13E−02	1.06E−01	4.90E−04		3.00E−32
TB1YR	1.029	−1.29E−01	−5.20E−02	−6.74E−02	−2.78E−02	−7.59E−02	4.14E−01	1.67E+01
PPIACO	<1E−50	1.10E−03	2.14E−01	8.71E−02	5.04E−01	4.95E−02		1.89E−12
TB6MS	1.029	−1.63E−01	−9.45E−02	−1.01E−01	−9.28E−02	−1.32E−01	5.15E−01	2.51E+01
CPIAUCNS	<1E−50	8.71E−03	1.64E−01	1.31E−01	1.64E−01	2.97E−02		3.68E−17
TB6MS	1.029	−1.83E−01	−1.03E−01	−1.09E−01	−9.46E−02	−1.39E−01	5.51E−01	2.90E+01
CPIAUCSL	<1E−50	7.94E−03	1.73E−01	1.48E−01	2.11E−01	4.21E−02		4.41E−19
TB6MS	1.028	−1.66E−01	−7.32E−02	−8.38E−02	−7.32E−02	−1.45E−01	4.61E−01	2.02E+01
CPIULFNS	<1E−50	9.02E−03	3.12E−01	2.44E−01	3.06E−01	2.11E−02		1.57E−14
TB6MS	1.028	−1.64E−01	−9.68E−02	−1.04E−01	−8.34E−02	−1.32E−01	4.89E−01	2.26E+01
CPIULFSL	<1E−50	1.37E−02	1.77E−01	1.43E−01	2.43E−01	4.51E−02		7.94E−16
TB6MS	1.028	−7.84E−02	−8.04E−02	−5.77E−02	−8.30E−02	−8.89E−02	3.71E−01	1.39E+01
CUSR0000SAS4	<1E−50	7.26E−02	6.75E−02	2.22E−01	5.84E−02	4.16E−02		1.14E−10
TB6MS	1.028	−7.60E−02	−8.22E−02	−6.01E−02	−8.11E−02	−8.68E−02	3.70E−01	1.39E+01
CUUR0000SAS4	<1E−50	7.44E−02	5.43E−02	1.93E−01	5.73E−02	4.13E−02		1.20E−10
TB6MS	1.029	−1.76E−01	−1.14E−01	−8.35E−02	−1.09E−01	−1.29E−01	5.64E−01	3.05E+01
CWSR0000SA0	<1E−50	5.91E−03	9.55E−02	2.27E−01	1.12E−01	4.20E−02		7.97E−20
TB6MS	1.028	−1.65E−01	−9.10E−02	−8.46E−02	−8.17E−02	−1.30E−01	4.93E−01	2.29E+01
CWSR0000SA0L1	<1E−50	9.53E−03	1.96E−01	2.30E−01	2.46E−01	3.95E−02		5.09E−16
TB6MS	1.028	−8.42E−02	−7.81E−02	−5.15E−02	−8.07E−02	−9.40E−02	3.89E−01	1.50E+01
CWSR0000SAS4	<1E−50	4.59E−02	6.80E−02	2.64E−01	5.84E−02	2.59E−02		2.05E−11
TB6MS	1.029	−1.71E−01	−7.72E−02	−1.01E−01	−8.02E−02	−1.37E−01	5.27E−01	2.63E+01
CWUR0000SA0	<1E−50	4.10E−03	2.58E−01	1.32E−01	2.30E−01	1.95E−02		8.91E−18
TB6MS	1.028	−1.76E−01	−4.80E−02	−8.45E−02	−5.21E−02	−1.52E−01	4.62E−01	2.02E+01
CWUR0000SA0L1	<1E−50	6.81E−03	5.45E−01	2.83E−01	5.05E−01	1.81E−02		1.53E−14
TB6MS	1.028	−7.95E−02	−7.93E−02	−5.98E−02	−8.10E−02	−8.96E−02	3.92E−01	1.52E+01
CWUR0000SAS4	<1E−50	5.00E−02	5.21E−02	1.70E−01	4.79E−02	2.76E−02		1.56E−11
TB6MS	1.030	−2.72E−01	−1.32E−01	−1.61E−01	−1.39E−01	−2.79E−01	7.45E−01	6.90E+01
PCEPILFE	<1E−50	3.65E−04	1.30E−01	6.27E−02	1.10E−01	2.57E−04		2.13E−33
TB6MS	1.029	−1.32E−01	−5.32E−02	−7.03E−02	−3.04E−02	−8.04E−02	4.44E−01	1.89E+01
PPIACO	<1E−50	5.94E−04	1.91E−01	6.69E−02	4.53E−01	3.23E−02		9.59E−14
TB3MS	1.029	−1.66E−01	−9.82E−02	−1.05E−01	−9.65E−02	−1.36E−01	5.45E−01	2.82E+01
CPIAUCNS	<1E−50	5.81E−03	1.35E−01	1.06E−01	1.35E−01	2.11E−02		9.87E−19
TB3MS	1.029	−1.86E−01	−1.07E−01	−1.14E−01	−9.85E−02	−1.43E−01	5.83E−01	3.30E+01
CPIAUCSL	<1E−50	5.23E−03	1.41E−01	1.17E−01	1.76E−01	3.05E−02		6.18E−21
TB3MS	1.028	−1.70E−01	−7.69E−02	−8.82E−02	−7.67E−02	−1.50E−01	4.92E−01	2.28E+01
CPIULFNS	<1E−50	5.80E−03	2.73E−01	2.04E−01	2.67E−01	1.42E−02		5.69E−16
TB3MS	1.028	−1.67E−01	−1.02E−01	−1.11E−01	−8.86E−02	−1.36E−01	5.23E−01	2.58E+01
CPIULFSL	<1E−50	9.63E−03	1.42E−01	1.05E−01	2.00E−01	3.45E−02		1.51E−17
TB3MS	1.028	−8.04E−02	−8.16E−02	−5.79E−02	−8.44E−02	−9.11E−02	3.86E−01	1.48E+01
CUSR0000SAS4	<1E−50	6.22E−02	6.04E−02	2.15E−01	5.16E−02	3.46E−02		2.84E−11
TB3MS	1.028	−7.78E−02	−8.35E−02	−6.07E−02	−8.25E−02	−8.88E−02	3.85E−01	1.48E+01
CUUR0000SAS4	<1E−50	6.45E−02	4.78E−02	1.83E−01	5.02E−02	3.46E−02		3.02E−11
TB3MS	1.029	−1.79E−01	−1.17E−01	−8.58E−02	−1.12E−01	−1.33E−01	5.94E−01	3.45E+01
CWSR0000SA0	<1E−50	3.71E−03	7.57E−02	1.98E−01	8.95E−02	2.93E−02		1.38E−21
TB3MS	1.028	−1.69E−01	−9.52E−02	−8.93E−02	−8.57E−02	−1.35E−01	5.25E−01	2.61E+01
CWSR0000SA0L1	<1E−50	6.26E−03	1.63E−01	1.91E−01	2.09E−01	2.85E−02		1.20E−17
TB3MS	1.028	−8.59E−02	−7.92E−02	−5.19E−02	−8.20E−02	−9.59E−02	4.04E−01	1.60E+01
CWSR0000SAS4	<1E−50	3.92E−02	6.11E−02	2.55E−01	5.17E−02	2.14E−02		4.92E−12
TB3MS	1.029	−1.75E−01	−7.99E−02	−1.04E−01	−8.29E−02	−1.41E−01	5.55E−01	2.94E+01
CWUR0000SA0	<1E−50	2.57E−03	2.27E−01	1.10E−01	2.01E−01	1.32E−02		2.69E−19
TB3MS	1.028	−1.82E−01	−5.04E−02	−8.77E−02	−5.43E−02	−1.58E−01	4.90E−01	2.27E+01
CWUR0000SA0L1	<1E−50	4.25E−03	5.14E−01	2.51E−01	4.74E−01	1.18E−02		6.68E−16
TB3MS	1.028	−8.12E−02	−8.04E−02	−6.04E−02	−8.22E−02	−9.16E−02	4.07E−01	1.62E+01
CWUR0000SAS4	<1E−50	4.28E−02	4.64E−02	1.62E−01	4.20E−02	2.28E−02		3.71E−12
TB3MS	1.030	−2.71E−01	−1.23E−01	−1.52E−01	−1.30E−01	−2.81E−01	7.46E−01	6.94E+01
PCEPILFE	<1E−50	3.13E−04	1.57E−01	7.67E−02	1.31E−01	1.80E−04		1.70E−33
TB3MS	1.029	−1.34E−01	−5.40E−02	−7.22E−02	−3.17E−02	−8.31E−02	4.63E−01	2.04E+01
PPIACO	<1E−50	3.67E−04	1.76E−01	5.56E−02	4.25E−01	2.42E−02		1.27E−14

Table 3.7: Regression results for all error models. Source: Own calculation based on data from FRED (2023b).

Interest Rate Inflation	Int	$e_{r,t}$	$e_{r,t-1}$	$e_{r,t-2}$	$e_{r,t-3}$	$e_{r,t-4}$	R^2	F Statistic p Value
FEDFUNDS	−2.92E−05	−2.72E−03	−3.98E−03	−3.33E−03	−4.82E−03	−5.11E−03	5.09E−02	1.26E+00
CPIAUCNS	2.32E−03	3.99E−01	2.20E−01	3.06E−01	1.35E−01	1.11E−01		2.88E−01
FEDFUNDS	−2.88E−05	−3.58E−03	−5.57E−03	−4.79E−03	−4.76E−03	−6.41E−03	5.12E−02	1.26E+00
CPIAUCSL	2.86E−03	3.33E−01	1.45E−01	2.14E−01	2.13E−01	8.59E−02		2.85E−01
FEDFUNDS	−3.18E−05	−2.08E−03	−2.78E−03	−2.72E−03	−3.73E−03	−5.26E−03	4.12E−02	1.01E+00
CPIULFNS	7.48E−04	5.12E−01	3.86E−01	3.98E−01	2.41E−01	9.55E−02		4.18E−01
FEDFUNDS	−2.98E−05	−2.96E−03	−4.11E−03	−3.79E−03	−3.98E−03	−5.19E−03	4.71E−02	1.16E+00
CPIULFSL	1.90E−03	3.79E−01	2.37E−01	2.69E−01	2.51E−01	1.24E−01		3.35E−01

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Interest Rate Inflation	Int	$e_{r,t}$	$e_{r,t-1}$	$e_{r,t-2}$	$e_{r,t-3}$	$e_{r,t-4}$	R^2	F Statistic p Value
FEDFUNDS	−2.57E−05	−3.89E−03	−7.45E−03	−5.78E−03	−3.82E−03	−2.37E−03	3.45E−01	1.23E+01
CUSR0000SAS4	5.54E−04	2.86E−02	5.59E−05	1.83E−03	3.42E−02	1.75E−01		1.30E−09
FEDFUNDS	−2.55E−05	−4.12E−03	−6.90E−03	−5.17E−03	−3.63E−03	−2.21E−03	3.45E−01	1.23E+01
CUUR0000SAS4	6.10E−04	1.72E−02	1.48E−04	3.93E−03	4.10E−02	1.92E−01		1.34E−09
FEDFUNDS	−2.68E−05	−4.74E−03	−5.43E−03	−4.58E−03	−6.09E−03	−5.79E−03	6.40E−02	1.60E+00
CWSR0000SA0	5.50E−03	1.67E−01	1.28E−01	1.97E−01	8.85E−02	9.32E−02		1.66E−01
FEDFUNDS	−2.94E−05	−4.13E−03	−4.06E−03	−3.66E−03	−4.37E−03	−5.17E−03	5.19E−02	1.28E+00
CWSR0000SA0L1	2.05E−03	2.06E−01	2.26E−01	2.76E−01	1.93E−01	1.15E−01		2.76E−01
FEDFUNDS	−2.44E−05	−4.60E−03	−7.96E−03	−5.75E−03	−3.48E−03	−2.34E−03	3.66E−01	1.35E+01
CWSR0000SAS4	8.89E−04	7.24E−03	1.07E−05	1.40E−03	4.63E−02	1.65E−01		2.08E−10
FEDFUNDS	−2.96E−05	−3.30E−03	−2.97E−03	−3.67E−03	−4.48E−03	−4.92E−03	4.83E−02	1.19E+00
CWUR0000SA0	1.96E−03	2.95E−01	3.49E−01	2.51E−01	1.56E−01	1.15E−01		3.20E−01
FEDFUNDS	−3.28E−05	−2.53E−03	−2.50E−03	−2.21E−03	−3.59E−03	−4.99E−03	3.69E−02	8.96E−01
CWUR0000SA0L1	4.52E−04	4.44E−01	4.50E−01	5.09E−01	2.73E−01	1.30E−01		4.86E−01
FEDFUNDS	−2.41E−05	−4.64E−03	−7.42E−03	−5.48E−03	−3.25E−03	−2.02E−03	3.59E−01	1.31E+01
CWUR0000SAS4	1.08E−03	5.36E−03	2.55E−05	1.46E−03	5.68E−02	2.18E−01		3.84E−10
FEDFUNDS	−2.42E−05	−1.05E−02	−1.09E−02	−6.82E−03	−7.81E−03	−1.10E−02	8.96E−02	2.30E+00
PCEPILFE	1.20E−02	3.63E−02	3.15E−02	1.83E−01	1.19E−01	2.90E−02		4.91E−02
FEDFUNDS	−3.47E−05	−1.11E−03	−1.03E−03	−1.80E−04	−5.01E−04	−1.18E−03	9.90E−03	2.34E−01
PPIACO	5.69E−04	5.66E−01	5.83E−01	9.22E−01	7.90E−01	5.29E−01		9.47E−01
GS1	−3.32E−05	−1.18E−03	−1.99E−03	−1.43E−03	−2.77E−03	−3.68E−03	1.81E−02	4.31E−01
CPIAUCNS	7.81E−04	7.21E−01	5.49E−01	6.67E−01	4.00E−01	2.60E−01		8.26E−01
GS1	−3.32E−05	−1.57E−03	−3.05E−03	−2.25E−03	−2.06E−03	−4.36E−03	1.73E−02	4.11E−01
CPIAUCSL	8.43E−04	6.77E−01	4.31E−01	5.64E−01	5.93E−01	2.48E−01		8.40E−01
GS1	−3.54E−05	−5.24E−04	−9.31E−04	−8.51E−04	−1.81E−03	−3.82E−03	1.40E−02	3.33E−01
CPIULFNS	2.44E−04	8.72E−01	7.77E−01	7.96E−01	5.79E−01	2.37E−01		8.92E−01
GS1	−3.44E−05	−1.26E−03	−1.83E−03	−1.31E−03	−1.58E−03	−3.53E−03	1.37E−02	3.25E−01
CPIULFSL	4.70E−04	7.13E−01	6.05E−01	7.08E−01	6.55E−01	3.04E−01		8.97E−01
GS1	−2.59E−05	−3.29E−03	−6.57E−03	−5.13E−03	−3.35E−03	−2.13E−03	2.70E−01	8.66E+00
CUSR0000SAS4	1.01E−03	6.99E−02	4.28E−04	6.28E−03	6.70E−02	2.36E−01		5.26E−07
GS1	−2.58E−05	−3.56E−03	−6.10E−03	−4.61E−03	−3.22E−03	−2.00E−03	2.69E−01	8.63E+00
CUUR0000SAS4	1.10E−03	4.54E−02	9.64E−04	1.22E−02	7.54E−02	2.55E−01		5.53E−07
GS1	−3.13E−05	−2.89E−03	−3.05E−03	−2.61E−03	−3.60E−03	−3.93E−03	2.36E−02	5.66E−01
CWSR0000SA0	1.74E−03	4.10E−01	4.01E−01	4.71E−01	3.22E−01	2.64E−01		7.26E−01
GS1	−3.36E−05	−2.42E−03	−1.92E−03	−1.58E−03	−2.13E−03	−3.49E−03	1.73E−02	4.11E−01
CWSR0000SA0L1	5.87E−04	4.63E−01	5.69E−01	6.40E−01	5.30E−01	2.92E−01		8.40E−01
GS1	−2.47E−05	−4.01E−03	−7.05E−03	−5.08E−03	−3.09E−03	−2.18E−03	2.90E−01	9.58E+00
CWSR0000SAS4	1.57E−03	2.22E−02	1.11E−04	5.64E−03	8.19E−02	2.08E−01		1.11E−07
GS1	−3.33E−05	−1.85E−03	−1.18E−03	−1.96E−03	−2.61E−03	−3.48E−03	1.79E−02	4.26E−01
CWUR0000SA0	7.06E−04	5.65E−01	7.16E−01	5.47E−01	4.17E−01	2.73E−01		8.30E−01
GS1	−3.56E−05	−1.14E−03	−8.93E−04	−5.81E−04	−1.89E−03	−3.63E−03	1.38E−02	3.26E−01
CWUR0000SA0L1	1.83E−04	7.34E−01	7.91E−01	8.64E−01	5.69E−01	2.76E−01		8.96E−01
GS1	−2.45E−05	−4.05E−03	−6.51E−03	−4.84E−03	−2.90E−03	−1.88E−03	2.84E−01	9.26E+00
CWUR0000SAS4	1.81E−03	1.71E−02	2.40E−04	5.73E−03	9.40E−02	2.62E−01		1.88E−07
GS1	−2.64E−05	−8.94E−03	−9.96E−03	−5.73E−03	−6.23E−03	−9.14E−03	5.06E−02	1.25E+00
PCEPILFE	1.02E−02	1.12E−01	8.46E−02	3.30E−01	2.79E−01	1.05E−01		2.92E−01
GS1	−3.81E−05	−4.37E−04	−3.55E−04	6.27E−04	1.88E−04	−5.47E−04	3.22E−03	7.57E−02
PPIACO	1.97E−04	8.26E−01	8.56E−01	7.38E−01	9.23E−01	7.77E−01		9.96E−01
GS3	−3.71E−05	7.36E−05	−2.51E−04	3.02E−04	−1.13E−03	−2.37E−03	5.23E−03	1.23E−01
CPIAUCNS	2.12E−04	9.82E−01	9.40E−01	9.28E−01	7.35E−01	4.69E−01		9.87E−01
GS3	−3.73E−05	1.56E−04	−9.12E−04	−7.69E−05	−7.75E−05	−2.57E−03	4.73E−03	1.11E−01
CPIAUCSL	1.98E−04	9.67E−01	8.15E−01	9.84E−01	9.84E−01	4.97E−01		9.90E−01
GS3	−3.84E−05	7.07E−04	5.92E−04	6.71E−04	−3.72E−04	−2.45E−03	5.93E−03	1.40E−01
CPIULFNS	8.06E−05	8.27E−01	8.57E−01	8.39E−01	9.09E−01	4.46E−01		9.83E−01
GS3	−3.82E−05	1.24E−04	−3.58E−05	6.39E−04	1.35E−04	−2.03E−03	3.87E−03	9.10E−02
CPIULFSL	1.22E−04	9.71E−01	9.92E−01	8.56E−01	9.69E−01	5.52E−01		9.94E−01
GS3	−2.58E−05	−2.65E−03	−5.78E−03	−4.62E−03	−2.85E−03	−1.71E−03	1.93E−01	5.59E+00
CUSR0000SAS4	2.15E−03	1.62E−01	2.90E−03	1.84E−02	1.38E−01	3.62E−01		1.17E−04
GS3	−2.56E−05	−2.93E−03	−5.33E−03	−4.12E−03	−2.76E−03	−1.61E−03	1.92E−01	5.55E+00
CUUR0000SAS4	2.31E−03	1.14E−01	5.54E−03	3.18E−02	1.47E−01	3.81E−01		1.27E−04
GS3	−3.53E−05	−1.34E−03	−1.03E−03	−8.66E−04	−1.81E−03	−2.41E−03	6.14E−03	1.45E−01
CWSR0000SA0	4.57E−04	7.03E−01	7.78E−01	8.12E−01	6.19E−01	4.93E−01		9.81E−01
GS3	−3.68E−05	−1.11E−03	−3.27E−04	−5.20E−05	−7.03E−04	−2.20E−03	4.82E−03	1.13E−01
CWSR0000SA0L1	1.89E−04	7.36E−01	9.23E−01	9.88E−01	8.35E−01	5.05E−01		9.89E−01
GS3	−2.46E−05	−3.36E−03	−6.27E−03	−4.59E−03	−2.60E−03	−1.75E−03	2.12E−01	6.31E+00
CWSR0000SAS4	3.13E−03	6.61E−02	9.47E−04	1.65E−02	1.62E−01	3.34E−01		3.23E−05
GS3	−3.67E−05	−6.17E−04	3.28E−04	−4.46E−04	−1.23E−03	−2.23E−03	5.64E−03	1.33E−01
CWUR0000SA0	2.21E−04	8.48E−01	9.20E−01	8.92E−01	7.03E−01	4.83E−01		9.85E−01
GS3	−3.78E−05	−8.07E−05	3.45E−04	6.56E−04	−7.80E−04	−2.49E−03	6.08E−03	1.43E−01
CWUR0000SA0L1	7.92E−05	9.81E−01	9.18E−01	8.46E−01	8.14E−01	4.53E−01		9.82E−01
GS3	−2.45E−05	−3.42E−03	−5.77E−03	−4.35E−03	−2.44E−03	−1.48E−03	2.06E−01	6.05E+00
CWUR0000SAS4	3.43E−03	5.33E−02	1.77E−03	1.72E−02	1.78E−01	3.99E−01		5.09E−05
GS3	−3.13E−05	−6.15E−03	−6.95E−03	−2.49E−03	−3.46E−03	−6.24E−03	2.13E−02	5.10E−01
PCEPILFE	3.20E−03	3.01E−01	2.61E−01	6.94E−01	5.76E−01	2.96E−01		7.68E−01
GS3	−4.10E−05	1.15E−04	1.76E−04	1.34E−03	7.36E−04	4.86E−05	4.94E−03	1.16E−01
PPIACO	6.80E−05	9.54E−01	9.29E−01	4.80E−01	7.07E−01	9.80E−01		9.89E−01
GS5	−3.81E−05	4.42E−04	2.63E−04	7.59E−04	−7.22E−04	−2.01E−03	4.16E−03	9.78E−02
CPIAUCNS	1.51E−04	8.93E−01	9.37E−01	8.21E−01	8.28E−01	5.37E−01		9.92E−01
GS5	−3.83E−05	6.27E−04	−3.22E−04	4.43E−04	3.42E−04	−2.11E−03	3.77E−03	8.85E−02
CPIAUCSL	1.40E−04	8.68E−01	9.34E−01	9.10E−01	9.30E−01	5.76E−01		9.94E−01
GS5	−3.92E−05	1.06E−03	1.03E−03	1.06E−03	−3.72E−05	−2.08E−03	5.91E−03	1.39E−01
CPIULFNS	6.05E−05	7.43E−01	7.55E−01	7.48E−01	9.91E−01	5.17E−01		9.83E−01
GS5	−3.92E−05	5.36E−04	4.79E−04	1.13E−03	5.21E−04	−1.64E−03	3.67E−03	8.61E−02

Continued on next page

Interest Rate Inflation	Int	$e_{r,t}$	$e_{r,t-1}$	$e_{r,t-2}$	$e_{r,t-3}$	$e_{r,t-4}$	R^2	F Statistic p Value
CPIULFSL	8.65E−05	8.75E−01	8.92E−01	7.47E−01	8.83E−01	6.30E−01		9.94E−01
GS5	−2.56E−05	−2.44E−03	−5.53E−03	−4.47E−03	−2.69E−03	−1.56E−03	1.71E−01	4.82E+00
CUSR0000SAS4	2.77E−03	2.03E−01	4.86E−03	2.42E−02	1.67E−01	4.11E−01		4.82E−04
GS5	−2.55E−05	−2.73E−03	−5.09E−03	−3.98E−03	−2.61E−03	−1.46E−03	1.69E−01	4.77E+00
CUUR0000SAS4	2.97E−03	1.46E−01	8.85E−03	4.06E−02	1.76E−01	4.30E−01		5.24E−04
GS5	−3.63E−05	−9.01E−04	−4.55E−04	−4.27E−04	−1.41E−03	−2.03E−03	3.81E−03	8.94E−02
CWSR0000SA0	3.27E−04	7.97E−01	9.01E−01	9.06E−01	6.99E−01	5.63E−01		9.94E−01
GS5	−3.76E−05	−7.38E−04	1.21E−04	3.27E−04	−3.90E−04	−1.88E−03	3.41E−03	8.00E−02
CWSR0000SA0L1	1.44E−04	8.22E−01	9.71E−01	9.23E−01	9.08E−01	5.69E−01		9.95E−01
GS5	−2.44E−05	−3.15E−03	−6.04E−03	−4.45E−03	−2.46E−03	−1.60E−03	1.90E−01	5.48E+00
CWSR0000SAS4	3.98E−03	8.84E−02	1.68E−03	2.17E−02	1.94E−01	3.83E−01		1.44E−04
GS5	−3.76E−05	−2.67E−04	7.66E−04	−4.52E−05	−9.05E−04	−1.91E−03	4.38E−03	1.03E−01
CWUR0000SA0	1.64E−04	9.34E−01	8.14E−01	9.89E−01	7.80E−01	5.48E−01		9.91E−01
GS5	−3.84E−05	2.07E−04	6.79E−04	9.62E−04	−5.48E−04	−2.20E−03	5.52E−03	1.30E−01
CWUR0000SA0L1	6.46E−05	9.51E−01	8.39E−01	7.75E−01	8.69E−01	5.06E−01		9.85E−01
GS5	−2.44E−05	−3.22E−03	−5.54E−03	−4.21E−03	−2.30E−03	−1.34E−03	1.83E−01	5.25E+00
CWUR0000SAS4	4.31E−03	7.21E−02	3.01E−03	2.29E−02	2.12E−01	4.52E−01		2.21E−04
GS5	−3.26E−05	−5.28E−03	−6.00E−03	−1.66E−03	−2.84E−03	−5.55E−03	1.59E−02	3.79E−01
PCEPILFE	2.38E−03	3.86E−01	3.44E−01	8.00E−01	6.55E−01	3.64E−01		8.63E−01
GS5	−4.20E−05	3.06E−04	3.53E−04	1.57E−03	8.97E−04	2.37E−04	6.70E−03	1.58E−01
PPIACO	4.94E−05	8.78E−01	8.57E−01	4.09E−01	6.47E−01	9.02E−01		9.77E−01
GS10	−3.90E−05	7.78E−04	6.62E−04	1.14E−03	−3.23E−04	−1.65E−03	3.83E−03	9.00E−02
CPIAUCNS	1.13E−04	8.13E−01	8.43E−01	7.33E−01	9.22E−01	6.11E−01		9.94E−01
GS10	−3.92E−05	1.06E−03	1.20E−04	8.62E−04	7.44E−04	−1.65E−03	3.59E−03	8.43E−02
CPIAUCSL	1.05E−04	7.77E−01	9.75E−01	8.24E−01	8.47E−01	6.60E−01		9.95E−01
GS10	−3.99E−05	1.38E−03	1.34E−03	1.37E−03	2.86E−04	−1.70E−03	6.36E−03	1.50E−01
CPIULFNS	4.72E−05	6.68E−01	6.82E−01	6.76E−01	9.30E−01	5.94E−01		9.80E−01
GS10	−4.01E−05	9.03E−04	8.59E−04	1.53E−03	8.83E−04	−1.24E−03	4.17E−03	9.80E−02
CPIULFSL	6.45E−05	7.89E−01	8.06E−01	6.61E−01	8.01E−01	7.14E−01		9.92E−01
GS10	−2.56E−05	−2.25E−03	−5.29E−03	−4.29E−03	−2.53E−03	−1.44E−03	1.52E−01	4.19E+00
CUSR0000SAS4	3.37E−03	2.42E−01	7.24E−03	3.12E−02	1.96E−01	4.51E−01		1.55E−03
GS10	−2.54E−05	−2.55E−03	−4.87E−03	−3.82E−03	−2.46E−03	−1.35E−03	1.50E−01	4.14E+00
CUUR0000SAS4	3.60E−03	1.76E−01	1.26E−02	5.06E−02	2.04E−01	4.70E−01		1.70E−03
GS10	−3.72E−05	−4.85E−04	1.40E−06	−5.75E−05	−9.89E−04	−1.63E−03	2.26E−03	5.30E−02
CWSR0000SA0	2.47E−04	8.89E−01	1.00E+00	9.87E−01	7.85E−01	6.41E−01		9.98E−01
GS10	−3.83E−05	−3.81E−04	4.70E−04	6.51E−04	−6.94E−05	−1.53E−03	2.56E−03	6.01E−02
CWSR0000SA0L1	1.13E−04	9.07E−01	8.88E−01	8.46E−01	9.83E−01	6.40E−01		9.98E−01
GS10	−2.44E−05	−2.96E−03	−5.81E−03	−4.29E−03	−2.31E−03	−1.49E−03	1.70E−01	4.80E+00
CWSR0000SAS4	4.78E−03	1.12E−01	2.66E−03	2.80E−02	2.26E−01	4.22E−01		5.02E−04
GS10	−3.84E−05	7.05E−05	1.11E−03	2.94E−04	−5.75E−04	−1.57E−03	3.67E−03	8.62E−02
CWUR0000SA0	1.27E−04	9.82E−01	7.33E−01	9.28E−01	8.58E−01	6.19E−01		9.94E−01
GS10	−3.89E−05	4.80E−04	9.16E−04	1.20E−03	−3.11E−04	−1.89E−03	5.18E−03	1.22E−01
CWUR0000SA0L1	5.43E−05	8.85E−01	7.83E−01	7.19E−01	9.25E−01	5.65E−01		9.87E−01
GS10	−2.43E−05	−3.04E−03	−5.32E−03	−4.05E−03	−2.16E−03	−1.23E−03	1.64E−01	4.58E+00
CWUR0000SAS4	5.11E−03	9.20E−02	4.60E−03	2.98E−02	2.44E−01	4.94E−01		7.53E−04
GS10	−3.36E−05	−4.42E−03	−5.29E−03	−9.20E−04	−2.18E−03	−4.79E−03	1.20E−02	2.85E−01
PCEPILFE	1.98E−03	4.75E−01	4.14E−01	8.90E−01	7.37E−01	4.41E−01		9.21E−01
GS10	−4.30E−05	4.90E−04	5.17E−04	1.80E−03	1.07E−03	4.31E−04	9.02E−03	2.13E−01
PPIACO	3.65E−05	8.07E−01	7.93E−01	3.47E−01	5.86E−01	8.24E−01		9.56E−01
GS20	−3.96E−05	1.04E−03	1.00E−03	1.43E−03	−7.74E−05	−1.45E−03	4.30E−03	1.01E−01
CPIAUCNS	8.75E−05	7.52E−01	7.64E−01	6.69E−01	9.81E−01	6.55E−01		9.92E−01
GS20	−3.98E−05	1.38E−03	4.94E−04	1.17E−03	9.88E−04	−1.41E−03	4.02E−03	9.45E−02
CPIAUCSL	8.17E−05	7.12E−01	8.98E−01	7.62E−01	7.97E−01	7.07E−01		9.93E−01
GS20	−4.04E−05	1.62E−03	1.63E−03	1.60E−03	4.81E−04	−1.50E−03	7.39E−03	1.74E−01
CPIULFNS	3.79E−05	6.14E−01	6.18E−01	6.24E−01	8.82E−01	6.39E−01		9.72E−01
GS20	−4.07E−05	1.19E−03	1.20E−03	1.83E−03	1.11E−03	−1.03E−03	5.19E−03	1.22E−01
CPIULFSL	4.99E−05	7.24E−01	7.31E−01	5.98E−01	7.52E−01	7.60E−01		9.87E−01
GS20	−2.58E−05	−2.11E−03	−5.11E−03	−4.18E−03	−2.41E−03	−1.35E−03	1.39E−01	3.77E+00
CUSR0000SAS4	3.42E−03	2.77E−01	1.00E−02	3.75E−02	2.21E−01	4.83E−01		3.37E−03
GS20	−2.57E−05	−2.40E−03	−4.70E−03	−3.71E−03	−2.36E−03	−1.26E−03	1.37E−01	3.72E+00
CUUR0000SAS4	3.65E−03	2.04E−01	1.70E−02	5.97E−02	2.28E−01	5.02E−01		3.70E−03
GS20	−3.79E−05	−1.86E−04	3.82E−04	2.15E−04	−7.54E−04	−1.42E−03	1.79E−03	4.20E−02
CWSR0000SA0	1.93E−04	9.58E−01	9.17E−01	9.53E−01	8.35E−01	6.85E−01		9.99E−01
GS20	−3.88E−05	−1.30E−04	7.69E−04	8.92E−04	1.12E−04	−1.35E−03	2.54E−03	5.96E−02
CWSR0000SA0L1	9.19E−05	9.68E−01	8.18E−01	7.90E−01	9.73E−01	6.79E−01		9.98E−01
GS20	−2.46E−05	−2.81E−03	−5.63E−03	−4.18E−03	−2.20E−03	−1.40E−03	1.57E−01	4.35E+00
CWSR0000SAS4	4.87E−03	1.34E−01	3.80E−03	3.37E−02	2.52E−01	4.54E−01		1.14E−03
GS20	−3.89E−05	3.14E−04	1.41E−03	5.41E−04	−3.74E−04	−1.39E−03	3.83E−03	8.99E−02
CWUR0000SA0	1.02E−04	9.22E−01	6.66E−01	8.68E−01	9.08E−01	6.61E−01		9.94E−01
GS20	−3.92E−05	6.78E−04	1.14E−03	1.39E−03	−1.77E−04	−1.74E−03	5.51E−03	1.30E−01
CWUR0000SA0L1	4.64E−05	8.38E−01	7.31E−01	6.78E−01	9.57E−01	5.96E−01		9.85E−01
GS20	−2.46E−05	−2.90E−03	−5.16E−03	−3.94E−03	−2.05E−03	−1.14E−03	1.50E−01	4.14E+00
CWUR0000SAS4	5.17E−03	1.10E−01	6.41E−03	3.61E−02	2.72E−01	5.29E−01		1.68E−03
GS20	−3.46E−05	−3.73E−03	−4.52E−03	−2.02E−04	−1.69E−03	−4.37E−03	9.72E−03	2.30E−01
PCEPILFE	1.53E−03	5.50E−01	4.91E−01	9.76E−01	7.96E−01	4.87E−01		9.49E−01
GS20	−4.37E−05	6.36E−04	6.54E−04	1.97E−03	1.18E−03	5.50E−04	1.10E−02	2.60E−01
PPIACO	2.84E−05	7.51E−01	7.40E−01	3.05E−01	5.50E−01	7.77E−01		9.34E−01
LTGOVTBD	−3.88E−05	8.18E−04	6.42E−04	1.02E−03	−5.25E−04	−1.82E−03	4.18E−03	9.82E−02
CPIAUCNS	7.49E−05	8.02E−01	8.46E−01	7.58E−01	8.73E−01	5.73E−01		9.92E−01
LTGOVTBD	−3.90E−05	1.12E−03	1.13E−04	7.42E−04	5.50E−04	−1.84E−03	3.78E−03	8.88E−02
CPIAUCSL	7.12E−05	7.64E−01	9.76E−01	8.46E−01	8.85E−01	6.24E−01		9.94E−01
LTGOVTBD	−3.95E−05	1.38E−03	1.30E−03	1.23E−03	7.75E−05	−1.87E−03	6.56E−03	1.54E−01
CPIULFNS	3.48E−05	6.65E−01	6.88E−01	7.06E−01	9.81E−01	5.56E−01		9.78E−01

Continued on next page

Interest Rate Inflation	Int	$e_{r,t}$	$e_{r,t-1}$	$e_{r,t-2}$	$e_{r,t-3}$	$e_{r,t-4}$	R^2	F Statistic p Value
LTGOVTBD	−3.96E−05	9.34E−04	8.25E−04	1.36E−03	6.28E−04	−1.44E−03	4.12E−03	9.69E−02
CPIULFSL	4.46E−05	7.81E−01	8.11E−01	6.93E−01	8.56E−01	6.68E−01		9.93E−01
LTGOVTBD	−2.98E−05	−2.08E−03	−5.09E−03	−4.16E−03	−2.45E−03	−1.41E−03	1.50E−01	4.13E+00
CUSR0000SAS4	4.73E−04	2.78E−01	9.38E−03	3.61E−02	2.08E−01	4.58E−01		1.73E−03
LTGOVTBD	−2.96E−05	−2.38E−03	−4.66E−03	−3.68E−03	−2.39E−03	−1.33E−03	1.48E−01	4.07E+00
CUUR0000SAS4	5.05E−04	2.04E−01	1.63E−02	5.83E−02	2.14E−01	4.76E−01		1.91E−03
LTGOVTBD	−3.74E−05	−3.81E−04	9.96E−05	−7.10E−05	−1.08E−03	−1.71E−03	2.51E−03	5.89E−02
CWSR0000SA0	1.47E−04	9.12E−01	9.78E−01	9.84E−01	7.63E−01	6.24E−01		9.98E−01
LTGOVTBD	−3.83E−05	−3.12E−04	5.20E−04	6.13E−04	−1.83E−04	−1.62E−03	2.70E−03	6.33E−02
CWSR0000SA0L1	7.20E−05	9.23E−01	8.75E−01	8.53E−01	9.56E−01	6.19E−01		9.97E−01
LTGOVTBD	−2.87E−05	−2.79E−03	−5.59E−03	−4.14E−03	−2.21E−03	−1.44E−03	1.68E−01	4.71E+00
CWSR0000SAS4	6.89E−04	1.34E−01	3.60E−03	3.34E−02	2.45E−01	4.35E−01		5.89E−04
LTGOVTBD	−3.83E−05	1.18E−04	1.13E−03	2.18E−04	−7.05E−04	−1.69E−03	4.06E−03	9.54E−02
CWUR0000SA0	8.40E−05	9.70E−01	7.25E−01	9.46E−01	8.26E−01	5.92E−01		9.93E−01
LTGOVTBD	−3.88E−05	5.24E−04	9.37E−04	1.15E−03	−4.21E−04	−1.99E−03	5.47E−03	1.29E−01
CWUR0000SA0L1	4.04E−05	8.74E−01	7.77E−01	7.30E−01	8.98E−01	5.44E−01		9.86E−01
LTGOVTBD	−2.86E−05	−2.87E−03	−5.11E−03	−3.90E−03	−2.07E−03	−1.19E−03	1.61E−01	4.49E+00
CWUR0000SAS4	7.53E−04	1.10E−01	6.16E−03	3.59E−02	2.62E−01	5.05E−01		8.82E−04
LTGOVTBD	−3.30E−05	−4.56E−03	−5.56E−03	−1.40E−03	−2.84E−03	−5.49E−03	1.35E−02	3.21E−01
PCEPILFE	2.17E−03	4.61E−01	3.90E−01	8.33E−01	6.60E−01	3.79E−01		8.99E−01
LTGOVTBD	−4.22E−05	4.78E−04	4.73E−04	1.68E−03	9.19E−04	2.97E−04	8.03E−03	1.89E−01
PPIACO	3.06E−05	8.11E−01	8.10E−01	3.74E−01	6.40E−01	8.79E−01		9.66E−01
MPRIME	−2.89E−05	−2.30E−03	−3.06E−03	−2.83E−03	−4.52E−03	−4.89E−03	3.34E−02	8.08E−01
CPIAUCNS	4.93E−03	4.79E−01	3.63E−01	4.03E−01	1.77E−01	1.29E−01		5.46E−01
MPRIME	−2.94E−05	−2.84E−03	−4.07E−03	−3.91E−03	−4.20E−03	−5.93E−03	3.03E−02	7.32E−01
CPIAUCSL	4.32E−03	4.51E−01	3.03E−01	3.31E−01	2.88E−01	1.18E−01		6.01E−01
MPRIME	−3.15E−05	−1.64E−03	−1.84E−03	−2.12E−03	−3.36E−03	−4.92E−03	2.65E−02	6.38E−01
CPIULFNS	1.82E−03	6.10E−01	5.78E−01	5.25E−01	3.05E−01	1.25E−01		6.71E−01
MPRIME	−2.89E−05	−2.51E−03	−3.19E−03	−3.41E−03	−3.85E−03	−5.10E−03	3.00E−02	7.23E−01
CPIULFSL	5.70E−03	4.64E−01	3.78E−01	3.52E−01	2.88E−01	1.38E−01		6.07E−01
MPRIME	−9.42E−06	−3.71E−03	−7.65E−03	−6.88E−03	−5.03E−03	−2.89E−03	3.37E−01	1.19E+01
CUSR0000SAS4	2.45E−01	3.63E−02	4.89E−05	2.25E−04	6.53E−03	9.81E−02		2.68E−09
MPRIME	−8.97E−06	−3.91E−03	−7.13E−03	−6.28E−03	−4.78E−03	−2.69E−03	3.38E−01	1.19E+01
CUUR0000SAS4	2.68E−01	2.42E−02	1.32E−04	5.11E−04	8.91E−03	1.14E−01		2.37E−09
MPRIME	−2.59E−05	−4.39E−03	−4.55E−03	−4.44E−03	−6.18E−03	−5.95E−03	4.69E−02	1.15E+00
CWSR0000SA0	1.21E−02	2.08E−01	2.19E−01	2.27E−01	9.56E−02	8.97E−02		3.37E−01
MPRIME	−2.76E−05	−3.86E−03	−3.43E−03	−3.63E−03	−4.56E−03	−5.39E−03	4.04E−02	9.84E−01
CWSR0000SA0L1	7.12E−03	2.40E−01	3.18E−01	2.94E−01	1.85E−01	1.03E−01		4.31E−01
MPRIME	−8.30E−06	−4.44E−03	−8.27E−03	−6.96E−03	−4.72E−03	−2.81E−03	3.58E−01	1.30E+01
CWSR0000SAS4	2.99E−01	9.94E−03	8.89E−06	1.43E−04	9.22E−03	9.73E−02		4.40E−10
MPRIME	−2.92E−05	−2.95E−03	−2.16E−03	−3.27E−03	−4.27E−03	−4.82E−03	3.45E−02	8.37E−01
CWUR0000SA0	3.95E−03	3.54E−01	5.08E−01	3.23E−01	1.88E−01	1.27E−01		5.26E−01
MPRIME	−3.25E−05	−2.17E−03	−1.64E−03	−1.67E−03	−3.32E−03	−4.78E−03	2.69E−02	6.46E−01
CWUR0000SA0L1	9.19E−04	5.14E−01	6.25E−01	6.22E−01	3.16E−01	1.48E−01		6.65E−01
MPRIME	−7.88E−06	−4.48E−03	−7.73E−03	−6.64E−03	−4.40E−03	−2.45E−03	3.52E−01	1.27E+01
CWUR0000SAS4	3.28E−01	7.43E−03	1.95E−05	1.42E−04	1.24E−02	1.37E−01		7.24E−10
MPRIME	−2.93E−05	−7.81E−03	−7.22E−03	−4.01E−03	−5.81E−03	−9.03E−03	4.22E−02	1.03E+00
PCEPILFE	3.13E−03	1.44E−01	1.75E−01	4.62E−01	2.74E−01	9.33E−02		4.03E−01
MPRIME	−3.45E−05	−9.68E−04	−8.89E−04	−2.02E−04	−4.83E−04	−1.06E−03	6.98E−03	1.64E−01
PPIACO	1.29E−03	6.20E−01	6.39E−01	9.15E−01	7.99E−01	5.73E−01		9.75E−01
BAA	−3.88E−05	7.40E−04	6.19E−04	1.03E−03	−4.81E−04	−1.71E−03	4.05E−03	9.51E−02
CPIAUCNS	2.01E−04	8.21E−01	8.55E−01	7.64E−01	8.87E−01	5.96E−01		9.93E−01
BAA	−3.87E−05	9.93E−04	3.64E−05	6.03E−04	3.81E−04	−1.85E−03	3.62E−03	8.49E−02
CPIAUCSL	1.66E−04	7.92E−01	9.93E−01	8.79E−01	9.22E−01	6.24E−01		9.94E−01
BAA	−4.00E−05	1.33E−03	1.34E−03	1.32E−03	2.13E−04	−1.67E−03	6.22E−03	1.46E−01
CPIULFNS	8.01E−05	6.78E−01	6.84E−01	6.89E−01	9.48E−01	6.00E−01		9.81E−01
BAA	−3.99E−05	8.91E−04	8.22E−04	1.37E−03	6.87E−04	−1.29E−03	3.78E−03	8.87E−02
CPIULFSL	1.31E−04	7.92E−01	8.17E−01	7.03E−01	8.47E−01	7.04E−01		9.94E−01
BAA	−1.55E−05	−2.24E−03	−5.56E−03	−5.04E−03	−3.29E−03	−1.76E−03	1.40E−01	3.81E+00
CUSR0000SAS4	1.14E−01	2.46E−01	6.33E−03	1.31E−02	1.04E−01	3.57E−01		3.13E−03
BAA	−1.52E−05	−2.50E−03	−5.15E−03	−4.56E−03	−3.17E−03	−1.65E−03	1.39E−01	3.77E+00
CUUR0000SAS4	1.23E−01	1.86E−01	1.09E−02	2.18E−02	1.14E−01	3.80E−01		3.33E−03
BAA	−3.62E−05	−7.11E−04	−3.28E−04	−5.10E−04	−1.64E−03	−2.14E−03	3.90E−03	9.15E−02
CWSR0000SA0	5.10E−04	8.40E−01	9.30E−01	8.90E−01	6.59E−01	5.44E−01		9.93E−01
BAA	−3.73E−05	−5.73E−04	1.82E−04	2.22E−04	−6.09E−04	−1.94E−03	3.44E−03	8.09E−02
CWSR0000SA0L1	2.69E−04	8.61E−01	9.57E−01	9.48E−01	8.58E−01	5.54E−01		9.95E−01
BAA	−1.45E−05	−2.91E−03	−6.09E−03	−5.08E−03	−3.08E−03	−1.79E−03	1.57E−01	4.35E+00
CWSR0000SAS4	1.34E−01	1.20E−01	2.34E−03	1.09E−02	1.19E−01	3.35E−01		1.14E−03
BAA	−3.78E−05	−4.26E−05	9.61E−04	8.43E−05	−8.77E−04	−1.81E−03	4.43E−03	1.04E−01
CWUR0000SA0	2.32E−04	9.89E−01	7.70E−01	9.80E−01	7.89E−01	5.67E−01		9.91E−01
BAA	−3.85E−05	3.52E−04	7.78E−04	9.81E−04	−5.87E−04	−2.10E−03	5.64E−03	1.33E−01
CWUR0000SA0L1	9.07E−05	9.15E−01	8.14E−01	7.69E−01	8.58E−01	5.21E−01		9.85E−01
BAA	−1.44E−05	−2.99E−03	−5.62E−03	−4.80E−03	−2.86E−03	−1.48E−03	1.51E−01	4.15E+00
CWUR0000SAS4	1.40E−01	9.92E−02	3.88E−03	1.18E−02	1.37E−01	4.10E−01		1.66E−03
BAA	−3.49E−05	−3.62E−03	−4.54E−03	−6.12E−04	−2.26E−03	−4.66E−03	9.70E−03	2.29E−01
PCEPILFE	9.26E−04	5.61E−01	4.87E−01	9.28E−01	7.31E−01	4.57E−01		9.49E−01
BAA	−4.40E−05	5.76E−04	5.52E−04	1.88E−03	1.18E−03	6.19E−04	8.71E−03	2.06E−01
PPIACO	6.89E−05	7.74E−01	7.81E−01	3.47E−01	5.53E−01	7.50E−01		9.60E−01
AAA	−3.93E−05	9.10E−04	8.27E−04	1.24E−03	−2.48E−04	−1.58E−03	4.12E−03	9.68E−02
CPIAUCNS	1.22E−04	7.81E−01	8.04E−01	7.11E−01	9.40E−01	6.24E−01		9.93E−01
AAA	−3.93E−05	1.21E−03	2.83E−04	9.05E−04	7.20E−04	−1.65E−03	3.80E−03	8.93E−02
CPIAUCSL	1.13E−04	7.48E−01	9.42E−01	8.17E−01	8.52E−01	6.60E−01		9.94E−01
AAA	−4.02E−05	1.49E−03	1.48E−03	1.45E−03	3.53E−04	−1.61E−03	6.83E−03	1.61E−01

Continued on next page

Interest Rate Inflation	Int	$e_{r,t}$	$e_{r,t-1}$	$e_{r,t-2}$	$e_{r,t-3}$	$e_{r,t-4}$	R^2	F Statistic p Value
CPIULFNS	5.05E−05	6.42E−01	6.50E−01	6.58E−01	9.13E−01	6.13E−01		9.76E−01
AAA	−4.04E−05	1.06E−03	1.04E−03	1.63E−03	9.26E−04	−1.17E−03	4.52E−03	1.06E−01
CPIULFSL	7.21E−05	7.53E−01	7.67E−01	6.43E−01	7.91E−01	7.29E−01		9.91E−01
AAA	−2.31E−05	−2.15E−03	−5.22E−03	−4.36E−03	−2.58E−03	−1.41E−03	1.39E−01	3.78E+00
CUSR0000SAS4	1.04E−02	2.67E−01	8.87E−03	3.00E−02	1.92E−01	4.61E−01		3.27E−03
AAA	−2.29E−05	−2.44E−03	−4.80E−03	−3.88E−03	−2.51E−03	−1.32E−03	1.38E−01	3.73E+00
CUUR0000SAS4	1.11E−02	1.97E−01	1.51E−02	4.83E−02	2.01E−01	4.81E−01		3.58E−03
AAA	−3.72E−05	−3.90E−04	1.05E−04	−6.82E−05	−1.08E−03	−1.71E−03	2.47E−03	5.80E−02
CWSR0000SA0	2.86E−04	9.11E−01	9.77E−01	9.85E−01	7.67E−01	6.25E−01		9.98E−01
AAA	−3.83E−05	−2.90E−04	5.55E−04	6.38E−04	−1.41E−04	−1.58E−03	2.71E−03	6.35E−02
CWSR0000SA0L1	1.36E−04	9.29E−01	8.68E−01	8.49E−01	9.66E−01	6.29E−01		9.97E−01
AAA	−2.20E−05	−2.84E−03	−5.73E−03	−4.38E−03	−2.37E−03	−1.46E−03	1.57E−01	4.36E+00
CWSR0000SAS4	1.40E−02	1.29E−01	3.36E−03	2.64E−02	2.19E−01	4.32E−01		1.13E−03
AAA	−3.85E−05	1.69E−04	1.22E−03	3.38E−04	−5.59E−04	−1.57E−03	3.96E−03	9.31E−02
CWUR0000SA0	1.40E−04	9.58E−01	7.07E−01	9.18E−01	8.63E−01	6.19E−01		9.93E−01
AAA	−3.89E−05	5.47E−04	9.80E−04	1.19E−03	−3.33E−04	−1.90E−03	5.45E−03	1.28E−01
CWUR0000SA0L1	6.00E−05	8.68E−01	7.67E−01	7.20E−01	9.19E−01	5.61E−01		9.86E−01
AAA	−2.19E−05	−2.92E−03	−5.26E−03	−4.13E−03	−2.21E−03	−1.19E−03	1.51E−01	4.15E+00
CWUR0000SAS4	1.48E−02	1.07E−01	5.64E−03	2.84E−02	2.40E−01	5.08E−01		1.67E−03
AAA	−3.39E−05	−4.07E−03	−4.97E−03	−9.54E−04	−2.43E−03	−5.03E−03	1.10E−02	2.61E−01
PCEPILFE	1.83E−03	5.16E−01	4.48E−01	8.88E−01	7.11E−01	4.25E−01		9.33E−01
AAA	−4.38E−05	6.04E−04	6.14E−04	1.93E−03	1.17E−03	5.57E−04	1.01E−02	2.38E−01
PPIACO	3.74E−05	7.63E−01	7.56E−01	3.21E−01	5.53E−01	7.74E−01		9.45E−01
TB1YR	−3.56E−05	−5.71E−04	−1.23E−03	−7.34E−04	−2.23E−03	−3.30E−03	1.30E−02	3.07E−01
CPIAUCNS	1.83E−04	8.64E−01	7.10E−01	8.24E−01	4.95E−01	3.16E−01		9.08E−01
TB1YR	−3.55E−05	−7.89E−04	−2.19E−03	−1.42E−03	−1.44E−03	−3.89E−03	1.25E−02	2.96E−01
CPIAUCSL	2.08E−04	8.34E−01	5.68E−01	7.11E−01	7.07E−01	3.05E−01		9.14E−01
TB1YR	−3.71E−05	5.67E−05	−3.01E−04	−3.04E−04	−1.41E−03	−3.46E−03	1.07E−02	2.53E−01
CPIULFNS	6.88E−05	9.86E−01	9.27E−01	9.26E−01	6.65E−01	2.85E−01		9.38E−01
TB1YR	−3.67E−05	−5.60E−04	−9.92E−04	−5.01E−04	−1.04E−03	−3.13E−03	9.06E−03	2.14E−01
CPIULFSL	1.01E−04	8.71E−01	7.77E−01	8.85E−01	7.68E−01	3.64E−01		9.56E−01
TB1YR	−3.36E−05	−3.10E−03	−6.23E−03	−4.72E−03	−2.97E−03	−1.93E−03	2.47E−01	7.66E+00
CUSR0000SAS4	2.25E−05	9.30E−02	9.48E−04	1.38E−02	1.09E−01	2.92E−01		2.93E−06
TB1YR	−3.35E−05	−3.39E−03	−5.75E−03	−4.20E−03	−2.88E−03	−1.82E−03	2.45E−01	7.61E+00
CUUR0000SAS4	2.44E−05	6.10E−02	2.03E−03	2.52E−02	1.17E−01	3.09E−01		3.20E−06
TB1YR	−3.42E−05	−2.09E−03	−1.99E−03	−1.70E−03	−2.74E−03	−3.23E−03	1.54E−02	3.66E−01
CWSR0000SA0	3.59E−04	5.52E−01	5.80E−01	6.35E−01	4.46E−01	3.60E−01		8.71E−01
TB1YR	−3.60E−05	−1.74E−03	−1.06E−03	−7.66E−04	−1.44E−03	−2.90E−03	1.06E−02	2.50E−01
CWSR0000SA0L1	1.30E−04	5.99E−01	7.52E−01	8.19E−01	6.68E−01	3.81E−01		9.39E−01
TB1YR	−3.23E−05	−3.83E−03	−6.69E−03	−4.63E−03	−2.69E−03	−1.98E−03	2.67E−01	8.53E+00
CWSR0000SAS4	3.68E−05	3.17E−02	2.78E−04	1.32E−02	1.34E−01	2.63E−01		6.48E−07
TB1YR	−3.55E−05	−1.26E−03	−4.65E−04	−1.32E−03	−2.08E−03	−3.02E−03	1.25E−02	2.96E−01
CWUR0000SA0	1.84E−04	6.97E−01	8.86E−01	6.83E−01	5.16E−01	3.44E−01		9.14E−01
TB1YR	−3.71E−05	−6.40E−04	−3.33E−04	−6.46E−05	−1.52E−03	−3.24E−03	1.02E−02	2.42E−01
CWUR0000SA0L1	6.49E−05	8.49E−01	9.21E−01	9.85E−01	6.49E−01	3.31E−01		9.43E−01
TB1YR	−3.21E−05	−3.88E−03	−6.15E−03	−4.41E−03	−2.54E−03	−1.71E−03	2.60E−01	8.22E+00
CWUR0000SAS4	4.41E−05	2.49E−02	5.81E−04	1.35E−02	1.47E−01	3.17E−01		1.10E−06
TB1YR	−2.44E−05	−9.77E−03	−1.10E−02	−7.05E−03	−7.66E−03	−1.04E−02	6.45E−02	1.61E+00
PCEPILFE	1.64E−02	8.66E−02	6.02E−02	2.35E−01	1.88E−01	6.81E−02		1.62E−01
TB1YR	−3.92E−05	−1.44E−04	−7.58E−05	9.33E−04	3.21E−04	−4.07E−04	3.17E−03	7.45E−02
PPIACO	7.55E−05	9.43E−01	9.69E−01	6.17E−01	8.69E−01	8.34E−01		9.96E−01
TB6MS	−3.41E−05	−1.18E−03	−2.05E−03	−1.52E−03	−2.95E−03	−3.85E−03	2.18E−02	5.20E−01
CPIAUCNS	3.12E−04	7.21E−01	5.34E−01	6.44E−01	3.66E−01	2.41E−01		7.60E−01
TB6MS	−3.38E−05	−1.63E−03	−3.24E−03	−2.49E−03	−2.39E−03	−4.69E−03	2.21E−02	5.30E−01
CPIAUCSL	3.90E−04	6.65E−01	3.98E−01	5.16E−01	5.31E−01	2.15E−01		7.54E−01
TB6MS	−3.60E−05	−5.37E−04	−1.03E−03	−1.03E−03	−2.05E−03	−4.03E−03	1.70E−02	4.05E−01
CPIULFNS	1.07E−04	8.69E−01	7.52E−01	7.53E−01	5.27E−01	2.11E−01		8.45E−01
TB6MS	−3.52E−05	−1.23E−03	−1.88E−03	−1.45E−03	−1.82E−03	−3.76E−03	1.68E−02	4.00E−01
CPIULFSL	1.80E−04	7.20E−01	5.94E−01	6.75E−01	6.04E−01	2.76E−01		8.48E−01
TB6MS	−3.31E−05	−3.39E−03	−6.60E−03	−4.96E−03	−3.14E−03	−2.03E−03	2.78E−01	8.99E+00
CUSR0000SAS4	2.06E−05	6.24E−02	3.97E−04	8.62E−03	8.60E−02	2.59E−01		2.97E−07
TB6MS	−3.29E−05	−3.67E−03	−6.11E−03	−4.43E−03	−3.02E−03	−1.91E−03	2.76E−01	8.94E+00
CUUR0000SAS4	2.25E−05	3.94E−02	9.04E−04	1.65E−02	9.42E−02	2.76E−01		3.24E−07
TB6MS	−3.26E−05	−2.86E−03	−3.00E−03	−2.56E−03	−3.60E−03	−3.94E−03	2.68E−02	6.43E−01
CWSR0000SA0	6.49E−04	4.15E−01	4.04E−01	4.75E−01	3.16E−01	2.63E−01		6.67E−01
TB6MS	−3.47E−05	−2.40E−03	−1.89E−03	−1.56E−03	−2.16E−03	−3.51E−03	1.90E−02	4.54E−01
CWSR0000SA0L1	2.19E−04	4.68E−01	5.74E−01	6.43E−01	5.22E−01	2.90E−01		8.10E−01
TB6MS	−3.17E−05	−4.11E−03	−7.06E−03	−4.87E−03	−2.85E−03	−2.08E−03	2.99E−01	9.96E+00
CWSR0000SAS4	3.46E−05	1.92E−02	1.05E−04	8.16E−03	1.08E−01	2.31E−01		5.90E−08
TB6MS	−3.42E−05	−1.85E−03	−1.20E−03	−2.03E−03	−2.71E−03	−3.58E−03	2.06E−02	4.92E−01
CWUR0000SA0	2.99E−04	5.65E−01	7.10E−01	5.29E−01	3.96E−01	2.61E−01		7.81E−01
TB6MS	−3.62E−05	−1.17E−03	−9.83E−04	−7.04E−04	−2.06E−03	−3.78E−03	1.58E−02	3.77E−01
CWUR0000SA0L1	9.32E−05	7.28E−01	7.70E−01	8.34E−01	5.34E−01	2.57E−01		8.64E−01
TB6MS	−3.15E−05	−4.16E−03	−6.52E−03	−4.64E−03	−2.67E−03	−1.80E−03	2.91E−01	9.61E+00
CWUR0000SAS4	4.23E−05	1.48E−02	2.32E−04	8.38E−03	1.21E−01	2.85E−01		1.05E−07
TB6MS	−2.30E−05	−1.07E−02	−1.20E−02	−8.17E−03	−8.55E−03	−1.13E−02	8.33E−02	2.13E+00
PCEPILFE	2.01E−02	5.27E−02	3.31E−02	1.53E−01	1.27E−01	4.03E−02		6.70E−02
TB6MS	−3.80E−05	−4.24E−04	−3.43E−04	6.16E−04	1.11E−04	−6.33E−04	3.33E−03	7.83E−02
PPIACO	1.21E−04	8.32E−01	8.60E−01	7.41E−01	9.54E−01	7.44E−01		9.95E−01
TB3MS	−3.38E−05	−1.59E−03	−2.47E−03	−1.93E−03	−3.34E−03	−4.10E−03	2.83E−02	6.83E−01
CPIAUCNS	2.94E−04	6.31E−01	4.51E−01	5.55E−01	3.03E−01	2.10E−01		6.38E−01
TB3MS	−3.33E−05	−2.17E−03	−3.80E−03	−3.07E−03	−2.94E−03	−5.10E−03	2.97E−02	7.16E−01
CPIAUCSL	3.86E−04	5.63E−01	3.19E−01	4.21E−01	4.41E−01	1.76E−01		6.13E−01

Continued on next page

Interest Rate **Inflation**	**Int**	$e_{r,t}$	$e_{r,t-1}$	$e_{r,t-2}$	$e_{r,t-3}$	$e_{r,t-4}$	R^2	F **Statistic** p **Value**
TB3MS	−3.58E−05	−9.30E−04	−1.42E−03	−1.43E−03	−2.43E−03	−4.32E−03	2.21E−02	5.29E−01
CPIULFNS	1.01E−04	7.74E−01	6.63E−01	6.59E−01	4.51E−01	1.80E−01		7.54E−01
TB3MS	−3.49E−05	−1.65E−03	−2.36E−03	−1.97E−03	−2.26E−03	−4.07E−03	2.29E−02	5.50E−01
CPIULFSL	1.71E−04	6.31E−01	5.01E−01	5.68E−01	5.21E−01	2.39E−01		7.38E−01
TB3MS	−3.47E−05	−3.57E−03	−6.77E−03	−5.04E−03	−3.18E−03	−2.05E−03	2.93E−01	9.68E+00
CUSR0000SAS4	6.62E−06	4.79E−02	2.51E−04	7.17E−03	7.89E−02	2.49E−01		9.29E−08
TB3MS	−3.46E−05	−3.83E−03	−6.27E−03	−4.50E−03	−3.05E−03	−1.93E−03	2.92E−01	9.64E+00
CUUR0000SAS4	7.23E−06	2.96E−02	5.86E−04	1.39E−02	8.74E−02	2.67E−01		1.00E−07
TB3MS	−3.22E−05	−3.32E−03	−3.47E−03	−2.95E−03	−4.03E−03	−4.25E−03	3.47E−02	8.42E−01
CWSR0000SA0	6.03E−04	3.42E−01	3.31E−01	4.08E−01	2.59E−01	2.25E−01		5.23E−01
TB3MS	−3.44E−05	−2.83E−03	−2.31E−03	−1.96E−03	−2.54E−03	−3.81E−03	2.53E−02	6.08E−01
CWSR0000SA0L1	2.06E−04	3.92E−01	4.91E−01	5.58E−01	4.50E−01	2.51E−01		6.94E−01
TB3MS	−3.33E−05	−4.28E−03	−7.22E−03	−4.94E−03	−2.88E−03	−2.09E−03	3.14E−01	1.07E+01
CWSR0000SAS4	1.12E−05	1.40E−02	6.33E−05	6.73E−03	1.01E−01	2.23E−01		1.70E−08
TB3MS	−3.39E−05	−2.23E−03	−1.57E−03	−2.40E−03	−3.07E−03	−3.85E−03	2.66E−02	6.40E−01
CWUR0000SA0	2.84E−04	4.86E−01	6.25E−01	4.55E−01	3.35E−01	2.26E−01		6.69E−01
TB3MS	−3.60E−05	−1.56E−03	−1.32E−03	−1.06E−03	−2.40E−03	−4.06E−03	2.03E−02	4.85E−01
CWUR0000SA0L1	8.95E−05	6.41E−01	6.93E−01	7.52E−01	4.68E−01	2.23E−01		7.87E−01
TB3MS	−3.31E−05	−4.32E−03	−6.68E−03	−4.71E−03	−2.70E−03	−1.81E−03	3.07E−01	1.04E+01
CWUR0000SAS4	1.41E−05	1.07E−02	1.44E−04	6.96E−03	1.15E−01	2.77E−01		3.08E−08
TB3MS	−2.29E−05	−1.13E−02	−1.24E−02	−8.54E−03	−8.96E−03	−1.19E−02	9.92E−02	2.58E+00
PCEPILFE	1.70E−02	3.57E−02	2.33E−02	1.23E−01	9.92E−02	2.78E−02		3.00E−02
TB3MS	−3.75E−05	−5.90E−04	−4.83E−04	4.51E−04	−4.46E−06	−7.53E−04	4.03E−03	9.47E−02
PPIACO	1.32E−04	7.67E−01	8.04E−01	8.08E−01	9.98E−01	6.97E−01		9.93E−01

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Curriculum Vitae

Personal Information

Name, Surname	Jasmin Maag
Citizenship	Switzerland
Date of Birth	March 22nd, 1990

Education

08/2018 – 02/2024	PhD Student in Business Administration, funded by Swiss National Science Foundation University of Zurich
08/2014 – 04/2018	Master of Arts in Business Administration, with distinction (summa cum laude) University of Zurich
08/2010 – 04/2014	Bachelor of Arts in Business Administration, with distinction (summa cum laude) University of Zurich

Professional Experience

09/2015 – 12/2023	Assistant, Chair of Quantitative Business Administration University of Zurich, 8006 Zurich