On the orthogonality of zero-mean Gaussian measures: Sufficiently dense sampling

Furrer, Reinhard ; Hediger, Michael

DOI: https://doi.org/10.1016/j.spa.2024.104356

Originally published at:
Furrer, Reinhard; Hediger, Michael (2024). On the orthogonality of zero-mean Gaussian measures: Sufficiently dense sampling. Stochastic Processes and their Applications, 173:104356.
DOI: https://doi.org/10.1016/j.spa.2024.104356

Stochastic Processes and their Applications 173 (2024) 104356

Contents lists available at ScienceDirect

Stochastic Processes and their Applications

journal homepage: www.elsevier.com/locate/spa

On the orthogonality of zero-mean Gaussian measures: Sufficiently dense sampling

Reinhard Furrer *, Michael Hediger

Department of Mathematical Modeling and Machine Learning, Department of Mathematics, Winterthurerstrasse 190, 8057 Zurich, Switzerland

ARTICLE INFO

MSC:
60G10
60G15
60G17
60G30
60G60

Keywords:
Gaussian random fields
Stationarity
Isotropy
Equivalence of Gaussian measures
Orthogonality of Gaussian measures

ABSTRACT

For a stationary random function ξ, sampled on a subset D of $\mathbb{R}^d$, we examine the equivalence and orthogonality of two zero-mean Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ associated with ξ. We give the isotropic analog to the result that the equivalence of $\mathbb{P}_1$ and $\mathbb{P}_2$ is linked with the existence of a square-integrable extension of the difference between the covariance functions of $\mathbb{P}_1$ and $\mathbb{P}_2$ from D to $\mathbb{R}^d$. We show that the orthogonality of $\mathbb{P}_1$ and $\mathbb{P}_2$ can be recovered when the set of distances from points of D to the origin is dense in the set of non-negative real numbers.

1. Introduction

1.1. Primary notation

We use the notation $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and $\mathbb{R}_+ := [0, \infty)$ for the set of non-zero natural numbers and non-negative real numbers, respectively. Given $x \in \mathbb{R}^d$, the Euclidean norm of x is identified with $\|x\| := \sqrt{\langle x, x\rangle}$, where $\langle x, y\rangle := x^{\mathrm{t}} y$, $x, y \in \mathbb{R}^d$, is the dot product on $\mathbb{R}^d$. An open ball of radius $r \in \mathbb{R}_+$, centered at $x \in \mathbb{R}^d$, is denoted with $B_r(x)$. Finally, the Borel σ-algebra on $\mathbb{R}^d$ is written as $\mathfrak{B}(\mathbb{R}^d)$.

1.2. General framework

Let $\mathbb{X}$ denote the space of real valued functions defined on $\mathbb{R}^d$, i.e., $\mathbb{X} := \{s : s(x) \in \mathbb{R}, x \in \mathbb{R}^d\}$. $\mathbb{X}$ shall be equipped with the σ-algebra $\mathcal{U}$, the smallest σ-algebra which contains the algebra of sets

$$\mathcal{A} := \bigcup_{n=1}^{\infty} \bigcup_{\substack{x_i \in \mathbb{R}^d \\ 1 \le i \le n}} \sigma\big(\{C_{x_1,\ldots,x_n}(B_n) : B_n \in \mathfrak{B}(\mathbb{R}^n)\}\big), \tag{1}$$

with $C_{x_1,\ldots,x_n}(B_n) := \{s \in \mathbb{X} : (s(x_1), \ldots, s(x_n)) \in B_n\}$, $B_n \in \mathfrak{B}(\mathbb{R}^n)$, being the cylinder sets on $\mathbb{X}$ over the coordinates $x_1, \ldots, x_n$. The σ-algebra $\mathcal{U}$ is referred to as the σ-algebra generated by the cylinder sets. On a measure space $(\Omega, \mathcal{F})$ we consider a random function

* Corresponding author.
E-mail addresses: reinhard.furrer@uzh.ch (R. Furrer), michael.hediger@math.uzh.ch (M. Hediger).

https://doi.org/10.1016/j.spa.2024.104356
Received 21 December 2022; Received in revised form 15 October 2023; Accepted 7 April 2024
Available online 15 April 2024

$\xi : \Omega \to \mathbb{X}$ which is $\mathcal{F}/\mathcal{U}$ measurable. That is, we consider a random field $\{\xi_x : x \in \mathbb{R}^d\}$ which has, for fixed $\omega \in \Omega$, real valued sample paths $\xi(\omega) = [x \mapsto \xi_x(\omega)]$ defined on $\mathbb{R}^d$. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be two probability measures, defined on $\mathcal{F}$, under which ξ is Gaussian, i.e., for $\ell = 1, 2$, $(\xi_{x_1}, \ldots, \xi_{x_n})$ is a Gauss vector under $\mathbb{P}_\ell$ for any $n \in \mathbb{N}^*$ and $x_1, \ldots, x_n \in \mathbb{R}^d$. In particular, $\mathbb{P}_1$ and $\mathbb{P}_2$ are said to be Gaussian measures on $\sigma(\xi) := \{\xi^{-1}(U) : U \in \mathcal{U}\}$, the σ-algebra generated by ξ. Given a real valued random variable Y, defined on $(\Omega, \mathcal{F})$, we write $\mathbb{E}_\ell[Y] := \int_\Omega Y(\omega)\mathbb{P}_\ell(d\omega)$ for the mean of Y under $\mathbb{P}_\ell$, $\ell = 1, 2$. The mean and covariance functions of ξ under $\mathbb{P}_\ell$ are denoted with $\mu_\ell(x) := \mathbb{E}_\ell[\xi_x]$ and $c_\ell(x, y) := \mathbb{E}_\ell[(\xi_x - \mu_\ell(x))(\xi_y - \mu_\ell(y))]$, $\ell = 1, 2$. Given a subset $D \subset \mathbb{R}^d$, we define the sub σ-algebra $\mathcal{U}_D \subset \mathcal{U}$ upon taking the union in (1) over D instead of $\mathbb{R}^d$. We then denote $\sigma_D(\xi) := \{\xi^{-1}(U), U \in \mathcal{U}_D\}$ as the sub σ-algebra ($\sigma_D(\xi) \subset \sigma(\xi)$) generated by ξ with sample paths $x \mapsto \xi_x(\omega)$, $\omega \in \Omega$, which are restricted to D. We call D the sampling domain for ξ.

1.3. On the equivalence and orthogonality of Gaussian measures with different covariance functions

Conditions for the equivalence or orthogonality of the Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on $\sigma_D(\xi)$ are discussed in Chapter III of [5]. Other common references are [3] (Chapter VII) or also [12] (Chapter III). Let us recall some terminology. $\mathbb{P}_2$ is said to be absolutely continuous with respect to $\mathbb{P}_1$ on $\sigma_D(\xi)$ if $\mathbb{P}_1(A) = 0$ implies $\mathbb{P}_2(A) = 0$, $A \in \sigma_D(\xi)$. $\mathbb{P}_1$ and $\mathbb{P}_2$ are termed equivalent if they are mutually absolutely continuous. On the other hand, $\mathbb{P}_1$ and $\mathbb{P}_2$ are referred to as orthogonal on $\sigma_D(\xi)$ if there exist $A \in \sigma_D(\xi)$ for which $\mathbb{P}_1(A) = 0$ but $\mathbb{P}_2(A) = 1$. Orthogonal measures are denoted by $\mathbb{P}_1 \perp \mathbb{P}_2$. Using Lebesgue's decomposition theorem, it can be shown that $\mathbb{P}_1 \perp \mathbb{P}_2$ on $\sigma_D(\xi)$ if there exists $(A_n) \subset \sigma_D(\xi)$ such that $\mathbb{P}_1(A_n) \to 0$ but $\mathbb{P}_2(A_n) \to 1$ as $n \to \infty$ (compare to p. 64 in [5]). Further, it is well known that Gaussian measures are either equivalent or orthogonal (see for instance Theorem 1 in Chapter III of [5]).

Throughout this article we assume that ξ is stationary (homogeneous) under $\mathbb{P}_1$ and $\mathbb{P}_2$. That is, for $\ell = 1, 2$, ξ has constant mean function μ_ℓ and covariance function c_ℓ that depends only on the difference $x - y$, $x, y \in \mathbb{R}^d$. Furthermore, the following two items are assumed to be satisfied:

(i) $\mu_\ell(x) = 0$, $\ell = 1, 2$;
(ii) for $\ell = 1, 2$, there exists a finite measure F_ℓ, uniquely defined on $\mathfrak{B}(\mathbb{R}^d)$, such that

$$c_\ell(x, y) = \int_{\mathbb{R}^d} \mathrm{e}^{i\langle\lambda, x-y\rangle}\, F_\ell(d\lambda).$$

Recall that if for any $x_0 \in \mathbb{R}^d$, $\mathbb{E}_\ell[|\xi_x - \xi_{x_0}|^2] \to 0$ as $\|x - x_0\| \to 0$, ξ is said to be continuous in mean-square (m.s. continuous) under $\mathbb{P}_\ell$. We remark that under the assumption of stationarity, (ii) is satisfied when ξ is m.s. continuous under both $\mathbb{P}_1$ and $\mathbb{P}_2$ (see Theorem 2 in Section 2 of Chapter IV in [3]). Further, since for $\ell = 1, 2$, ξ is stationary under $\mathbb{P}_\ell$, there exists $k_\ell : \mathbb{R}^d \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}^d$, $c_\ell(x, y) = k_\ell(x - y)$. Then, see p. 208 in the latter reference, ξ is m.s. continuous under $\mathbb{P}_\ell$ if and only if k_ℓ is continuous at zero.

Chapter III.4.2 of [5] gives an overview of the case where Gaussian measures differ only in terms of their covariance function. In particular, the equivalence or orthogonality of $\mathbb{P}_1$ and $\mathbb{P}_2$ on $\sigma_D(\xi)$ is linked to the difference between the covariance functions c_1 and c_2 restricted to $D \times D$,

$$\delta(x, y) := c_1(x, y) - c_2(x, y), \quad x, y \in D.$$

For further reference, the following assumption is needed.

Assumption 1.1 (*Absolute Continuity of Spectral Measures*)**.** For $\ell = 1, 2$, the spectral measure F_ℓ is absolutely continuous with respect to the Lebesgue measure on $\mathfrak{B}(\mathbb{R}^d)$ with spectral density $f_\ell(\lambda) = F_\ell(d\lambda)/d\lambda$.

Remark 1.1. It follows from (ii) that for $\ell = 1, 2$, the spectral density f_ℓ must be non-negative a.e. with respect to the Lebesgue measure on $\mathbb{R}^d$. To see it, we recall that because of (ii) there exists a stochastic orthogonal measure ζ_ℓ, defined on $\mathfrak{B}(\mathbb{R}^d)$, with structure function F_ℓ, s.t. for any $x \in \mathbb{R}^d$ we have that $\mathbb{P}_\ell$ a.s.,

$$\xi_x = \int_{\mathbb{R}^d} \mathrm{e}^{i\langle\lambda, x\rangle}\, \zeta_\ell(d\lambda).$$

See for instance Theorem 1 in Section 5 of Chapter IV in [3]. Then, if we let $A = f_\ell^{-1}\big((-\infty, 0)\big)$, by Assumption 1.1, f_ℓ is Borel measurable and hence $A \in \mathfrak{B}(\mathbb{R}^d)$. In particular,

$$\int_A f_\ell(\lambda) d\lambda = F_\ell(A) = \mathbb{E}_\ell\left[|\zeta_\ell(A)|^2\right],$$

which shows that A must have Lebesgue measure zero.

The following result serves as an anchor point for our study. For a proof, we refer to Section III.4.2 in [5] (see Theorem 11).

Theorem 1.1. *Suppose that Assumption 1.1 is satisfied where f_1 and f_2 are bounded on $\mathbb{R}^d$. Then, the Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent on $\sigma_D(\xi)$ if and only if the restriction δ satisfies the following properties:*

(a) There exists extension $^\dagger\delta$ of δ to $\mathbb{R}^d \times \mathbb{R}^d$ which is square-integrable, i.e.,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |^\dagger\delta(x,y)|^2 dx dy < \infty\,;$$

(b) The Fourier transform φ of $^\dagger\delta$ satisfies

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\varphi(\lambda,\mu)|^2}{f_1(\lambda)f_2(\mu)} d\lambda d\mu < \infty.$$

Note that the proof given in [5] is based on random functions that have sample paths defined on $\mathbb{R}$ instead of $\mathbb{R}^d$. Nevertheless, the arguments proposed for the case $d = 1$ can be recycled to prove the case where $d > 1$. We also remark that the existence of the spectral density f_ℓ is guaranteed if $k_\ell(x)$ is absolutely integrable on $\mathbb{R}^d$ (see for instance p. 211 in [3]).

Theorem 1.1 allows us to easily deduce the orthogonality of $\mathbb{P}_1$ and $\mathbb{P}_2$ when D is chosen to be the entire $\mathbb{R}^d$. In particular, under the assumption of bounded spectral densities, Theorem 1.1 shows that if f_1 and f_2 differ on a set of positive Lebesgue measure, $\mathbb{P}_1$ and $\mathbb{P}_2$ must be orthogonal on $\sigma(\xi)$ (see for instance p. 95 in [5]). Using the two-dimensional Hankel transform [8] we will give the analog of Theorem 1.1 when ξ is isotropic under $\mathbb{P}_1$ and $\mathbb{P}_2$ (Theorem 2.3). Recall that ξ is isotropic under $\mathbb{P}_\ell$ if $k_\ell(x)$ is a function of $\|x\|$ only. In a further step, we aim to recover the orthogonality of $\mathbb{P}_1$ and $\mathbb{P}_2$ when D is dense in $\mathbb{R}^d$ (respectively $D_+ := \{\|x\| : x \in D\}$ is dense in $\mathbb{R}_+$). Specifically, we will prove that if the covariance functions c_1 and c_2 are uniformly continuous on $\mathbb{R}^d \times \mathbb{R}^d$, the density of D in $\mathbb{R}^d$ is enough to obtain orthogonal measures $\mathbb{P}_1$ and $\mathbb{P}_2$ (Theorem 2.5) — this under the assumption that f_1 and f_2 are bounded and different on a set of positive Lebesgue measure. If ξ is isotropic, we show that one arrives at the same conclusion if D_+ is dense in $\mathbb{R}_+$ (Theorem 2.6). The latter result allows us to easily deduce the orthogonality of $\mathbb{P}_1$ and $\mathbb{P}_2$ when ξ is sampled along a continuous and unbounded path which starts at zero. As an example, we will deduce the almost sure orthogonality of $\mathbb{P}_1$ and $\mathbb{P}_2$ when ξ is sampled along a d-dimensional Brownian motion which starts at the origin (Example 3.1). In particular, Theorems 2.5 and 2.6 complement the results given in [5] by deducing the orthogonality of two Gaussian measures based on a countable and unbounded collection of points sampled in $\mathbb{R}^d$. In terms of an illustration, we will revisit the relationship between orthogonal families of Gaussian distributions and covariance parameter estimation [1] and discuss conditions under which consistent estimators can be obtained (Theorem 4.1).

2. Main results

2.1. Continuous extension

We use Theorem 1.1 as a starting point and note that with regard to the necessity of the imposed conditions, a slight modification is possible. In particular, the extension $^\dagger\delta$ of item (a) in Theorem 1.1 can be chosen to be continuous. This follows from the fact that the measures F_1 and F_2 are finite. More explicitly, from Theorem 8 in Section III.3 of [5] we can see that the Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent on $\sigma_D(\xi)$ if and only if the restriction δ permits a representation

$$\delta(x,y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{-i(\langle\lambda,x\rangle-\langle\mu,y\rangle)}\Psi(\lambda,\mu)F_1(d\lambda)F_2(d\mu),$$

with Ψ satisfying $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\Psi(\lambda,\mu)|^2 F_1(d\lambda)F_2(d\mu) < \infty$. Then, since F_1 and F_2 are finite, by Hölder's inequality,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\Psi(\lambda,\mu)|F_1(d\lambda)F_2(d\mu) < \infty.$$

Thus, since f_1 and f_2 are non-negative a.e. (see Remark 1.1), the function $\Psi(\lambda,\mu)f_1(\lambda)f_2(\mu)$ is absolutely integrable on $\mathbb{R}^d \times \mathbb{R}^d$. Therefore, its Fourier transform $^\dagger\delta$ is continuous and absolutely integrable on $\mathbb{R}^d \times \mathbb{R}^d$. But, since f_1 and f_2 are also bounded, we have that $^\dagger\delta$ is also square-integrable on $\mathbb{R}^d \times \mathbb{R}^d$. Further, on $D \times D$, $^\dagger\delta$ agrees with δ. Hence, we have proven the following result:

Theorem 2.1. *Suppose that Assumption 1.1 is satisfied where f_1 and f_2 are bounded on $\mathbb{R}^d$. Then, if the Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent on $\sigma_D(\xi)$, there exists a continuous extension $^\dagger\delta$ of δ to the entire $\mathbb{R}^d \times \mathbb{R}^d$ which is absolutely and square-integrable on $\mathbb{R}^d \times \mathbb{R}^d$.*

We will see later that the above relation between equivalent measures and continuous extensions of δ, will allow us to easily read the orthogonality of $\mathbb{P}_1$ and $\mathbb{P}_2$ from the uniform continuity of c_1 and c_2 if D is dense in $\mathbb{R}^d$. Before we arrive there, we establish analogous versions of Theorems 1.1 and 2.1, when ξ is not only stationary but also isotropic.

2.2. Isotropic random fields

Given $x \in \mathbb{R}^d$, we adopt a polar coordinate system $x = (r_x, \theta_x)$, $r_x \in \mathbb{R}_+$, $\theta_x \in \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$. On $L^2(\mathbb{S}^{d-1})$ we consider a real orthonormal basis composed of spherical (surface) harmonics S_m^l, $l = 1,\dots,h(m,d)$ of degree $m \in \mathbb{N}$ (see

Chapter XI, Section 11.3 in [2] or also Chapter IV, Section 2 of [9]). We recall that

$$h(m,d)=\begin{cases}1, & m=0,\\ d, & m=1,\\ \binom{d+m-1}{m}-\binom{d+m-3}{m-2}, & m\geq 2.\end{cases}$$

Assumption 2.1 (*Isotropy*). ξ is isotropic under $\mathbb{P}_\ell$, $\ell=1,2$.

Under Assumption 2.1, since (ii) of Section 1.3 is satisfied, we have, for $\ell=1,2$, and any $x,y\in\mathbb{R}^d$ (see (4.145) of [13]),

$$c_\ell(x,y)=K_d^2\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,d)}\int_0^\infty S_m^l(\theta_x)\frac{J_{m+\frac{d-2}{2}}(\kappa r_x)}{(\kappa r_x)^{\frac{d-2}{2}}}S_m^l(\theta_y)\frac{J_{m+\frac{d-2}{2}}(\kappa r_y)}{(\kappa r_y)^{\frac{d-2}{2}}}\Phi_\ell(d\kappa), \tag{2}$$

where $K_d^2=2^{d-1}\Gamma(d/2)\pi^{d/2}$, Γ is the Gamma function, $J_{m+(d-2)/2}$ is the Bessel function of the first kind of order $m+(d-2)/2$ and Φ_ℓ is a finite measure on $\mathbb{R}_+$ defined upon $\Phi_\ell([a,b))=F_\ell(B_b(0)\setminus B_a(0))$.

We consider the real vector space of sequences of functions $a(\kappa):=(a_m^l(\kappa))$, $\kappa\in\mathbb{R}_+$, $l=1,\dots,h(m,d)$, $m\in\mathbb{N}$. On the latter vector space, we introduce the inner product

$$\langle a,b\rangle_{\Phi_\ell}:=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,d)}\int_0^\infty a_m^l(\kappa)b_m^l(\kappa)\Phi_\ell(d\kappa),\quad \ell=1,2.$$

Further, we define $\mathcal{L}_D^0$ as the linear span over $\mathbb{R}$ of the set of sequences of functions

$$\left\{a: a(\kappa)=\left(K_d S_m^l(\theta_x)\frac{J_{m+\frac{d-2}{2}}(\kappa r_x)}{(\kappa r_x)^{\frac{d-2}{2}}}\right),\ x=(r_x,\theta_x)\in D\right\},$$

and introduce the correspondence

$$\eta(a):=\sum_{k=1}^{N}\beta_k\xi_{x_k},\quad a=\sum_{k=1}^{N}\beta_k a_k\in\mathcal{L}_D^0. \tag{3}$$

Let $L_D(\mathbb{P}_\ell)$ be the linear span of $\{\xi_x : x\in D\}$ over $\mathbb{R}$ under $\mathbb{P}_\ell$, $\ell=1,2$. Given $\ell=1,2$, we view $L_D(\mathbb{P}_\ell)$ as a subspace of the inner product space $L^2(\Omega,\mathcal{F},\mathbb{P}_\ell)$. Then, we readily see that for any $a,b\in\mathcal{L}_D^0$,

$$\langle a,b\rangle_{\Phi_\ell}=\mathbb{E}_\ell[\eta(a)\eta(b)],\quad \ell=1,2. \tag{4}$$

Thus, for $\ell=1,2$, (3) provides an isometric correspondence between the inner product space $\mathcal{L}_D^0$ (equipped with $\langle\cdot,\cdot\rangle_{\Phi_\ell}$) and $L_D(\mathbb{P}_\ell)$. Note that if we define $\mathcal{L}_D(\Phi_\ell)$, $\ell=1,2$, as the closure of $\mathcal{L}_D^0$ with respect to $\langle\cdot,\cdot\rangle_{\Phi_\ell}$, the isometric correspondence (3) can be extended to an isometric correspondence between $\mathcal{L}_D(\Phi_\ell)$ and the closure of $L_D(\mathbb{P}_\ell)$.

Suppose that there exists $a\in\mathcal{L}_D^0$ such that $\|a\|_{\Phi_1}\neq 0$ but $\|a\|_{\Phi_2}=0$. Then, the Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ are orthogonal on $\sigma_D(\xi)$. This is because ξ is Gaussian with a zero-mean function. Hence the latter assumption allows us to conclude that

$$\mathbb{P}_1\big(\eta(a)=0\big)=0\quad\text{but}\quad\mathbb{P}_2\big(\eta(a)=0\big)=1.$$

We recall that the two norms $\|\cdot\|_{\Phi_1}$ and $\|\cdot\|_{\Phi_2}$ are termed equivalent on $\mathcal{L}_D^0$ if

$$\exists\ C_1,C_2>0\text{ s.t. }\forall\ a\in\mathcal{L}_D^0,\ 0<C_1\|a\|_{\Phi_2}\leq\|a\|_{\Phi_1}\leq C_2\|a\|_{\Phi_2}<\infty. \tag{5}$$

Then, it can be shown that the Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ are orthogonal on $\sigma_D(\xi)$ if the condition (5) is violated (compare to Section III.1.3 in [5]). Notice that if (5) is satisfied, then by construction of $\mathcal{L}_D(\Phi_1)$ and $\mathcal{L}_D(\Phi_2)$, (5) remains true with $\mathcal{L}_D^0$ replaced with either $\mathcal{L}_D(\Phi_1)$ or $\mathcal{L}_D(\Phi_2)$. The following lemma is of central importance.

Lemma 2.2. *Suppose that Assumption 2.1 is satisfied. Then, the Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent on $\sigma_D(\xi)$ if and only if for any pair $m,i\in\mathbb{N}$, $l=1,\dots,h(m,d)$, $j=1,\dots,h(i,d)$, the difference*

$$\delta_{m,i}^{l,j}(r_x,r_y):=\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\delta(x,y)S_m^l(\theta_x)S_i^j(\theta_y)d\theta_x d\theta_y,\quad x,y\in D, \tag{6}$$

is representable as

$$\delta_{m,i}^{l,j}(r_x,r_y)=K_d^2\int_0^\infty\int_0^\infty\frac{J_{m+\frac{d-2}{2}}(r_x\kappa)}{(r_x\kappa)^{\frac{d-2}{2}}}\frac{J_{i+\frac{d-2}{2}}(r_y\iota)}{(r_y\iota)^{\frac{d-2}{2}}}\psi_{m,i}^{l,j}(\kappa,\iota)\Phi_1(d\kappa)\Phi_2(d\iota),$$

with

$$\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,d)}\int_0^\infty\sum_{i=0}^{\infty}\sum_{j=1}^{h(i,d)}\int_0^\infty|\psi_{m,i}^{l,j}(\kappa,\iota)|^2\Phi_1(d\kappa)\Phi_2(d\iota)<\infty. \tag{7}$$

Proof. We define the functions

$$\mathbb{R}^d \times \mathbb{R}^d \ni (\lambda, \mu) \mapsto \mathfrak{e}_{x,y}(\lambda, \mu) := \mathrm{e}^{i(\langle \lambda, x\rangle - \langle \mu, y\rangle)}, \quad x, y \in D.$$

Then, as on p. 82 of [5], we let $L^0_{D\times D}$ denote the linear space of functions of the form

$$u(\lambda, \mu) = \sum_{p,q} \beta_{pq} \mathfrak{e}_{x_p, y_q}(\lambda, \mu),$$

where $x_p, y_q \in D$ and β_{pq} are real coefficients. Further, the Hilbert space $L_{D\times D}(F_1 \times F_2)$ is defined as the closure of $L^0_{D\times D}$ with respect to the inner product

$$\langle u, v\rangle_{F_1\times F_2} := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(\lambda, \mu) \overline{v(\lambda, \mu)} F_1(d\lambda) F_2(d\mu).$$

Using polar coordinates, we introduce the class of sequences of functions

$$\mathbb{R}_+ \times \mathbb{R}_+ \ni (\kappa, \iota) \mapsto \mathfrak{a}_{x,y}(\kappa, \iota) := \left(\mathfrak{a}^{l,j}_{m,i}(\kappa, \iota)\right), \quad x = (r_x, \theta_x),\ y = (r_y, \theta_y) \in D,$$

where for $m, i \in \mathbb{N}$, $l = 1, \dots, h(m, d)$, $j = 1, \dots, h(i, d)$, the entries of $\mathfrak{a}_{x,y}(\kappa, \iota)$ are given by

$$\mathfrak{a}^{l,j}_{m,i}(\kappa, \iota) = K_d^2 S^l_m(\theta_x) \frac{J_{m+\frac{d-2}{2}}(r_x\kappa)}{(r_x\kappa)^{\frac{d-2}{2}}} S^j_i(\theta_y) \frac{J_{i+\frac{d-2}{2}}(r_y\iota)}{(r_y\iota)^{\frac{d-2}{2}}}.$$

Then, similar to $L^0_{D\times D}$, we define $\mathcal{L}^0_{D\times D}$ as the space of sequences of functions which are of the form

$$a(\kappa, \iota) = \sum_{p,q} \beta_{pq} \mathfrak{a}_{x_p, y_q}(\kappa, \iota),$$

where $x_p, y_q \in D$ and β_{pq} are real coefficients. Finally, we define $\mathcal{L}_{D\times D}(\Phi_1 \times \Phi_2)$ as the closure of $\mathcal{L}^0_{D\times D}$ with respect to the inner product

$$\langle a, b\rangle_{\Phi_1\times\Phi_2} := \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,d)} \int_0^{\infty} \sum_{i=0}^{\infty} \sum_{j=1}^{h(i,d)} \int_0^{\infty} a^{l,j}_{m,i}(\kappa, \iota) b^{l,j}_{m,i}(\kappa, \iota) \Phi_1(d\kappa) \Phi_2(d\iota).$$

Using (4), we observe that

$$\langle \mathfrak{e}_{x_p, y_q}, \mathfrak{e}_{x,y}\rangle_{F_1\times F_2} = \langle \mathfrak{a}_{x_p, y_q}, \mathfrak{a}_{x,y}\rangle_{\Phi_1\times\Phi_2}. \tag{8}$$

According to Theorem 8 in Section III.3 of [5], the Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent on $\sigma_D(\xi)$ if and only if the restriction δ permits a representation

$$\delta(x, y) = \langle \Psi, \mathfrak{e}_{x,y}\rangle_{F_1\times F_2},$$

with $\Psi \in L_{D\times D}(F_1 \times F_2)$. Thus, by definition of $L_{D\times D}(F_1 \times F_2)$, we write $\Psi = \lim_{n\to\infty} \Psi_n$, where $(\Psi_n) \subset L^0_{D\times D}$. Using (8), we have that

$$\begin{aligned}
\delta(x, y) &= \langle \Psi, \mathfrak{e}_{x,y}\rangle_{F_1\times F_2} \\
&= \lim_{n\to\infty} \langle \Psi_n, \mathfrak{e}_{x,y}\rangle_{F_1\times F_2} \\
&= \lim_{n\to\infty} \langle \psi_n, \mathfrak{a}_{x,y}\rangle_{\Phi_1\times\Phi_2} \\
&= \langle \psi, \mathfrak{a}_{x,y}\rangle_{\Phi_1\times\Phi_2},
\end{aligned}$$

with $(\psi_n) \subset \mathcal{L}^0_{D\times D}$ such that $\lim_{n\to\infty} \psi_n = \psi \in \mathcal{L}_{D\times D}(\Phi_1 \times \Phi_2)$. Hence, we have shown that $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent on $\sigma_D(\xi)$ if and only if δ is representable as

$$\delta(x, y) = K_d^2 \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,d)} \int_0^{\infty} \sum_{i=0}^{\infty} \sum_{j=1}^{h(i,d)} \int_0^{\infty} S^l_m(\theta_x) \frac{J_{m+\frac{d-2}{2}}(r_x\kappa)}{(r_x\kappa)^{\frac{d-2}{2}}} S^j_i(\theta_y) \frac{J_{i+\frac{d-2}{2}}(r_y\iota)}{(r_y\iota)^{\frac{d-2}{2}}} \psi^{l,j}_{m,i}(\kappa, \iota) \Phi_1(d\kappa) \Phi_2(d\iota),$$

where

$$\sum_{m=0}^{\infty} \sum_{l=1}^{h(m,d)} \int_0^{\infty} \sum_{i=0}^{\infty} \sum_{j=1}^{h(i,d)} \int_0^{\infty} |\psi^{l,j}_{m,i}(\kappa, \iota)|^2 \Phi_1(d\kappa) \Phi_2(d\iota) < \infty.$$

Then, we can conclude the proof using the orthonormality property of the spherical harmonics. □

Notice, if Assumptions 1.1 and 2.1 are satisfied, then, for $\ell = 1, 2$, since k_ℓ is the Fourier transform of f_ℓ, and k_ℓ is assumed to be radial, we must conclude that f_ℓ is radial itself. We denote its radial version with g_ℓ, i.e., $f_\ell(x) = g_\ell(\|x\|)$. The analog to Theorem 1.1 for isotropic random functions reads as follows:

Theorem 2.3. *Suppose that Assumptions 1.1 and 2.1 are satisfied where f_1 and f_2 are bounded on $\mathbb{R}^d$. Then, the Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent on $\sigma_D(\xi)$ if and only if, for any pair $m, i \in \mathbb{N}$, $l = 1, \dots, h(m,d)$, $j = 1, \dots, h(i,d)$, the scaled difference*

$$\eth^{l,j}_{m,i}(r_1, r_2) := r_1^{\frac{d-1}{2}} r_2^{\frac{d-1}{2}} \delta^{l,j}_{m,i}(r_1, r_2), \quad r_1, r_2 \in D_+,$$

satisfies the following properties:

(a) There exists extension ${}^{\uparrow}\eth^{l,j}_{m,i}$ of $\eth^{l,j}_{m,i}$ to $\mathbb{R}_+ \times \mathbb{R}_+$ which is square-integrable, i.e.,

$$\int_0^\infty \int_0^\infty |{}^{\uparrow}\eth^{l,j}_{m,i}(r_1, r_2)|^2 dr_1 dr_2 < \infty\,;$$

(b) The two-dimensional Hankel transform

$$h^{l,j}_{m,i}(\kappa, \iota) = \int_0^\infty \int_0^\infty {}^{\uparrow}\eth^{l,j}_{m,i}(r_1, r_2) \sqrt{r_1 \kappa} \sqrt{r_2 \iota} J_{m+\frac{d-2}{2}}(r_1 \kappa) J_{i+\frac{d-2}{2}}(r_2 \iota) dr_1 dr_2,$$

of ${}^{\uparrow}\eth^{l,j}_{m,i}$ satisfies

$$\sum_{m=0}^{\infty} \sum_{l=1}^{h(m,d)} \int_0^\infty \sum_{i=0}^{\infty} \sum_{j=1}^{h(i,d)} \int_0^\infty \frac{|h^{l,j}_{m,i}(\kappa, \iota)|^2}{g_1(\kappa) g_2(\iota)} d\kappa d\iota < \infty.$$

Proof. We first notice that for any $b \geq 0$, by Assumption 1.1,

$$\Phi_\ell([0, b)) = F_\ell(B_b(0)) = \int_{\{x:\, \|x\| < b\}} f_\ell(x) dx, \quad \ell = 1, 2.$$

Since for $\ell = 1, 2$, f_ℓ is radial, we use spherical coordinates and get

$$\Phi_\ell([0, b)) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^b \kappa^{d-1} g_\ell(\kappa) d\kappa, \quad \ell = 1, 2. \tag{9}$$

Assume that $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent on $\sigma_D(\xi)$. We apply Lemma 2.2 and conclude that

$$\eth^{l,j}_{m,i}(r_1, r_2) = \frac{2^{d+1} \pi^{\frac{3d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty \int_0^\infty \kappa^{\frac{d-1}{2}} g_1(\kappa) \iota^{\frac{d-1}{2}} g_2(\iota) \psi^{l,j}_{m,i}(\kappa, \iota) \sqrt{r_1 \kappa} \sqrt{r_2 \iota} J_{m+\frac{d-2}{2}}(\kappa r_1) J_{i+\frac{d-2}{2}}(\iota r_2) d\kappa d\iota,$$

whenever $r_1, r_2 \in D_+$, for some $\psi^{l,j}_{m,i}$ which satisfies (7). Notice that since f_1 and f_2 are real valued and bounded on $\mathbb{R}^d$, the respective radial versions g_1 and g_2 must be real valued and bounded on $\mathbb{R}_+$. Thus, if we define

$$h^{l,j}_{m,i}(\kappa, \iota) := \frac{2^{d+1} \pi^{\frac{3d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \kappa^{\frac{d-1}{2}} g_1(\kappa) \iota^{\frac{d-1}{2}} g_2(\iota) \psi^{l,j}_{m,i}(\kappa, \iota), \quad \kappa, \iota \in \mathbb{R}_+,$$

using (7), together with (9), we have that

$$\int_0^\infty \int_0^\infty |h^{l,j}_{m,i}(\kappa, \iota)|^2 d\kappa d\iota < \infty.$$

Therefore, we define ${}^{\uparrow}\eth^{l,j}_{m,i}$ on $\mathbb{R}_+ \times \mathbb{R}_+$, as the two-dimensional Hankel transform of $h^{l,j}_{m,i}(\kappa, \iota)$, which is square-integrable (see Corollary 6.1 in [8]). This then proves (a) of Theorem 2.3. In addition, by (7), also (b) of Theorem 2.3 must be satisfied. To prove the other direction, suppose that (a) and (b) of Theorem 2.3 are satisfied. Then by (a), since ${}^{\uparrow}\eth^{l,j}_{m,i}$ is square-integrable on $\mathbb{R}_+ \times \mathbb{R}_+$, the two-dimensional Hankel transform

$$\tilde{h}^{l,j}_{m,i} \quad \text{of} \quad \left(\frac{2^{d+1} \pi^{\frac{3d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\right)^{-1} {}^{\uparrow}\eth^{l,j}_{m,i}$$

exists and is square-integrable on $\mathbb{R}_+ \times \mathbb{R}_+$. Therefore, on $D_+ \times D_+$, we have that

$$r_1^{\frac{d-1}{2}} r_2^{\frac{d-1}{2}} \delta^{l,j}_{m,i}(r_1, r_2) = \frac{2^{d+1} \pi^{\frac{3d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty \int_0^\infty \tilde{h}^{l,j}_{m,i}(\kappa, \iota) \sqrt{r_1 \kappa} \sqrt{r_2 \iota} J_{m+\frac{d-2}{2}}(r_1 \kappa) J_{i+\frac{d-2}{2}}(r_2 \iota) d\kappa d\iota.$$

We set

$$\psi^{l,j}_{m,i}(\kappa, \iota) := \frac{\tilde{h}^{l,j}_{m,i}(\kappa, \iota)}{\kappa^{\frac{d-1}{2}} g_1(\kappa) \iota^{\frac{d-1}{2}} g_2(\iota)},$$

and obtain

$$r_1^{\frac{d-1}{2}} r_2^{\frac{d-1}{2}} \delta^{l,j}_{m,i}(r_1, r_2) = K_d^2 \int_0^\infty \int_0^\infty \sqrt{r_1 r_2} \frac{J_{m+\frac{d-2}{2}}(r_1 \kappa)}{\kappa^{\frac{d-2}{2}}} \frac{J_{i+\frac{d-2}{2}}(r_2 \iota)}{\iota^{\frac{d-2}{2}}} \psi^{l,j}_{m,i}(\kappa, \iota) \Phi_1(d\kappa) \Phi_2(d\iota),$$

with $r_1, r_2 \in D_+$. Further, by (b) of Theorem 2.3, we have that

$$\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,d)}\int_0^{\infty}\sum_{i=0}^{\infty}\sum_{j=1}^{h(i,d)}\int_0^{\infty}|\psi_{m,i}^{l,j}(\kappa,\iota)|^2\Phi_1(d\kappa)\Phi_2(d\iota)<\infty.$$

Finally, we can conclude that $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent on $\sigma_D(\xi)$ under application of Lemma 2.2. □

Similar to Theorem 2.1, the equivalence of $\mathbb{P}_1$ and $\mathbb{P}_2$ allows for an extension of $\delta_{m,i}^{l,j}$ that is continuous. To arrive there, we introduce the set

$$N := \{(r_1,r_2)\in\mathbb{R}_+\times\mathbb{R}_+ : (r_1,r_2)\in\{0\}\times\mathbb{R}_+\cup\mathbb{R}_+\times\{0\}\}.$$

Theorem 2.4. *Suppose that Assumptions 1.1 and 2.1 are satisfied where f_1 and f_2 are bounded on $\mathbb{R}^d$. Then, if the Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent on $\sigma_D(\xi)$, for any pair $m,i\in\mathbb{N}$, $l=1,\dots,h(m,d)$, $j=1,\dots,h(i,d)$, there exists a continuous extension ${}^{\dagger}\delta_{m,i}^{l,j}$ of $\delta_{m,i}^{l,j}$, from $D_+\times D_+\setminus N$ to $\mathbb{R}_+\times\mathbb{R}_+\setminus N$, which is such that*

$$r_1^{\frac{d-1}{2}} r_2^{\frac{d-1}{2}}\,{}^{\dagger}\delta_{m,i}^{l,j}(r_1,r_2)$$

is square-integrable on $\mathbb{R}_+\times\mathbb{R}_+$.

Proof. Let us fix a pair $m,i\in\mathbb{N}$, $l=1,\dots,h(m,d)$, $j=1,\dots,h(i,d)$. From the proof of Theorem 2.3 we see that the equivalence of $\mathbb{P}_1$ and $\mathbb{P}_2$ on $\sigma_D(\xi)$ implies that

$$\delta_{m,i}^{l,j}(r_1,r_2)=\frac{2^{d+1}\pi^{\frac{3d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^{\infty}\int_0^{\infty}{}_{m,i}^{l,j}w_{r_1,r_2}(\kappa,\iota)d\kappa d\iota,\quad (r_1,r_2)\in D_+\times D_+\setminus N,$$

with

$${}_{m,i}^{l,j}w_{r_1,r_2}(\kappa,\iota)=\frac{\kappa^{\frac{d-1}{2}}g_1(\kappa)\iota^{\frac{d-1}{2}}g_2(\iota)}{r_1^{\frac{d-1}{2}}r_2^{\frac{d-1}{2}}}\psi_{m,i}^{l,j}(\kappa,\iota)\sqrt{r_1\kappa}\sqrt{r_2\iota}J_{m+\frac{d-2}{2}}(\kappa r_1)J_{i+\frac{d-2}{2}}(\iota r_2).$$

We recall that $\psi_{m,i}^{l,j}$ is square-integrable with respect to

$$\Phi_1(d\kappa)\Phi_2(d\iota)=\left(\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\right)^2\kappa^{d-1}g_1(\kappa)\iota^{d-1}g_2(\iota)d\kappa d\iota.$$

As with the proof of Theorem 2.1, since Φ_1 and Φ_2 are bounded, we conclude that

$$\int_0^{\infty}\int_0^{\infty}|\psi_{m,i}^{l,j}(\kappa,\iota)|\Phi_1(d\kappa)\Phi_2(d\iota)<\infty, \tag{10}$$

as well. Then, for any pair $(r_1,r_2)\in\mathbb{R}_+\times\mathbb{R}_+\setminus N$, we define, for any $\kappa,\iota\in\mathbb{R}_+$,

$${}_{m,i}^{l,j}W_{r_1,r_2}(\kappa,\iota):=\frac{\kappa^{\frac{d-1}{2}}g_1(\kappa)\iota^{\frac{d-1}{2}}g_2(\iota)}{r_1^{\frac{d-1}{2}}r_2^{\frac{d-1}{2}}}\psi_{m,i}^{l,j}(\kappa,\iota)\sqrt{r_1\kappa}\sqrt{r_2\iota}J_{m+\frac{d-2}{2}}(\kappa r_1)J_{i+\frac{d-2}{2}}(\iota r_2).$$

We remark that for fixed $\kappa,\iota\in\mathbb{R}_+$, $(r_1,r_2)\mapsto{}_{m,i}^{l,j}W_{r_1,r_2}(\kappa,\iota)$ is continuous on $\mathbb{R}_+\times\mathbb{R}_+\setminus N$. Then, using Lommel's expression for the Bessel function of the first kind (see Section 3.3 of [11]), we estimate

$$\begin{aligned}
J_{m+\frac{d-2}{2}}(\kappa r_1)&=\frac{\left(\frac{1}{2}\kappa r_1\right)^{m+\frac{d-2}{2}}}{\Gamma\left(m+\frac{d-2}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi}\cos(\kappa r_1\cos(\theta))\sin^{2\left(m+\frac{d-2}{2}\right)}(\theta)d\theta\\
&\le\frac{\left(\frac{1}{2}\kappa r_1\right)^{\frac{d-1}{2}}\Gamma\left(m\right)}{\Gamma\left(m+\frac{d-2}{2}+\frac{1}{2}\right)}\frac{\left(\frac{1}{2}\kappa r_1\right)^{m-\frac{1}{2}}}{\Gamma\left(m\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi}\cos(\kappa r_1\cos(\theta))\sin^{2\left(m-\frac{1}{2}\right)}(\theta)d\theta\\
&=\frac{\left(\frac{1}{2}\kappa r_1\right)^{\frac{d-1}{2}}\Gamma\left(m\right)}{\Gamma\left(m+\frac{d-2}{2}+\frac{1}{2}\right)}J_{m-\frac{1}{2}}(\kappa r_1).
\end{aligned}$$

Hence, if one applies a similar estimate to $J_{i+(d-2)/2}(\iota r_2)$, we obtain that

$${}_{m,i}^{l,j}W_{r_1,r_2}(\kappa,\iota)\le C\kappa^{d-1}g_1(\kappa)\iota^{d-1}g_2(\iota)\psi_{m,i}^{l,j}(\kappa,\iota)\sqrt{r_1\kappa}\sqrt{r_2\iota}J_{m-\frac{1}{2}}(\kappa r_1)J_{i-\frac{1}{2}}(\iota r_2),$$

where C is some fixed constant, independent of r_1,r_2 and κ,ι. In addition, since the functions $z\mapsto\sqrt{z}J_{m-1/2}(z)$ and $z\mapsto\sqrt{z}J_{i-1/2}(z)$ are bounded on $(0,\infty)$ (see [4]), we get for any $(r_1,r_2)\in\mathbb{R}_+\times\mathbb{R}_+\setminus N$,

$$\frac{2^{d+1}\pi^{\frac{3d}{2}}}{\Gamma\left(\frac{d}{2}\right)}|{}_{m,i}^{l,j}W_{r_1,r_2}(\kappa,\iota)|\le\widetilde{C}\frac{2^{d+1}\pi^{\frac{3d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\kappa^{d-1}g_1(\kappa)\iota^{d-1}g_2(\iota)\psi_{m,i}^{l,j}(\kappa,\iota),$$

where $\widetilde{C}$ is independent of r_1, r_2 and κ, ι. The function on the right-hand side of the latter equation is absolutely integrable because of (10). Hence, we set

$$^{\dagger}\delta^{l,j}_{m,i}(r_1, r_2) := \frac{2^{d+1}\pi^{\frac{3d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty \int_0^\infty {}^{l,j}_{m,i}W_{r_1,r_2}(\kappa, \iota) d\kappa d\iota, \quad (r_1, r_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \setminus N,$$

and obtain an extension of $\delta^{l,j}_{m,i}$ from $D_+ \times D_+ \setminus N$ to $\mathbb{R}_+ \times \mathbb{R}_+ \setminus N$, which is continuous by Lebesgue's dominated convergence theorem. Then, using the same reasoning as in the proof of Theorem 2.3, since f_1 and f_2 are assumed to be bounded, we extend

$$r_1^{\frac{d-1}{2}} r_2^{\frac{d-1}{2}} \, {}^{\dagger}\delta^{l,j}_{m,i}(r_1, r_2) \tag{11}$$

to the entire $\mathbb{R}_+ \times \mathbb{R}_+$ by means of the two-dimensional Hankel transform of a square-integrable function. Finally, since N has Lebesgue measure zero, (11) is square-integrable on $\mathbb{R}_+ \times \mathbb{R}_+$. □

2.3. Sufficiently dense sampling

Upon Theorem 2.1, the next result relates the uniform continuity of the covariance functions c_1 and c_2 with the orthogonality of the Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ when D is dense in $\mathbb{R}^d$.

Theorem 2.5. *Suppose that Assumption 1.1 is satisfied where f_1 and f_2 are bounded on $\mathbb{R}^d$ and such that the set $\{\lambda \in \mathbb{R}^d : f_1(\lambda) \neq f_2(\lambda)\}$ has positive Lebesgue measure. Then, if c_1 and c_2 are uniformly continuous on $\mathbb{R}^d \times \mathbb{R}^d$ and D is dense in $\mathbb{R}^d$, the Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ are orthogonal on $\sigma_D(\xi)$.*

Proof. We recall that on $\mathbb{R}^d \times \mathbb{R}^d$, the difference $c_1 - c_2$ is given by

$$c_1(x, y) - c_2(x, y) = \int_{\mathbb{R}^d} e^{i\langle\lambda, x-y\rangle}\left(f_1(\lambda) - f_2(\lambda)\right) d\lambda. \tag{12}$$

Since $\{\lambda \in \mathbb{R}^d : f_1(\lambda) \neq f_2(\lambda)\}$ has positive Lebesgue measure, (12) cannot be square-integrable on $\mathbb{R}^d \times \mathbb{R}^d$. To see it, we recall that ξ is stationary and observe that

$$\int_{\mathbb{R}^d} |c_1(x, y) - c_2(x, y)|^2 dx = \int_{\mathbb{R}^d} |k_1(x - y) - k_2(x - y)|^2 dx.$$

Then, as the later integral is constant in y, we must have that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |k_1(x - y) - k_2(x - y)|^2 dx dy = \infty,$$

unless the L^2 norm of the difference $k_1 - k_2$ is zero. But, since the Fourier transform is an isometry on $L^2(\mathbb{R}^d)$, and $f_1 - f_2$ is assumed to have non-zero L^2 norm, this case is not possible. Thus, (12) is not square-integrable. Still, by assumption, it is continuous on $\mathbb{R}^d \times \mathbb{R}^d$. Furthermore, since δ is assumed to be uniformly continuous on $D \times D$ and D is dense in $\mathbb{R}^d$, any continuous extension of δ to $\mathbb{R}^d \times \mathbb{R}^d$ must be given by (12). This concludes the proof under application of Theorem 2.1. □

If ξ is also isotropic, the density of D_+ in $\mathbb{R}_+$ is sufficient to recover the orthogonality of the Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on $\sigma_D(\xi)$.

Theorem 2.6. *Suppose that Assumptions 1.1 and 2.1 are satisfied where f_1 and f_2 are bounded on $\mathbb{R}^d$ and such that the set $\{\lambda \in \mathbb{R}^d : f_1(\lambda) \neq f_2(\lambda)\}$ has positive Lebesgue measure. Then, if c_1 and c_2 are uniformly continuous on $\mathbb{R}^d \times \mathbb{R}^d$ and D_+ is dense in $\mathbb{R}_+$, the Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ are orthogonal on $\sigma_D(\xi)$.*

Proof. First of all, using (2), we see that

$$\begin{aligned} {}^{\dagger}_{*}\delta_m(r_x, r_y) &:= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \left(c_1(x, y) - c_2(x, y)\right) S^l_m(\theta_x) S^l_m(\theta_y) d\theta_x d\theta_y \\ &= (2\pi)^d \int_0^\infty \frac{J_{m+\frac{d-2}{2}}(\kappa r_x)}{(\kappa r_x)^{\frac{d-2}{2}}} \frac{J_{m+\frac{d-2}{2}}(\kappa r_y)}{(\kappa r_y)^{\frac{d-2}{2}}} \kappa^{d-1}\left(g_1(\kappa) - g_2(\kappa)\right) d\kappa, \end{aligned}$$

is one extension of $\delta^{l,j}_{m,i}$ to $\mathbb{R}_+ \times \mathbb{R}_+$ for the pair $m = i$. Since c_1 and c_2 are uniformly continuous on $\mathbb{R}^d \times \mathbb{R}^d$, ${}^{\dagger}_{*}\delta_m$ must be uniformly continuous on $\mathbb{R}_+ \times \mathbb{R}_+$. This follows from the fact that

$$\int_{\mathbb{S}^{d-1}} |S^l_m(\theta_x)| d\theta_x \leq \left(\frac{h(m, d) 2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\right)^{\frac{1}{2}}.$$

See for instance (b) of Corollary 2.9 in [9]. In particular, ${}^{\dagger}_{*}\delta_m$ is a continuous extension of $\delta^{l,l}_{m,m}$ from $D_+ \times D_+ \setminus N$ to $\mathbb{R}_+ \times \mathbb{R}_+ \setminus N$. But, because of the assumption that $\{\lambda \in \mathbb{R}^d : f_1(\lambda) \neq f_2(\lambda)\}$ has positive Lebesgue measure, it cannot be the case that

$$Q_m := \int_0^\infty \int_0^\infty r_x^{d-1} r_y^{d-1} |{}^{\dagger}_{*}\delta_m(r_x, r_y)|^2 dr_x dr_y < \infty.$$

To see this, we use the identity

$$c_1(x,y) - c_2(x,y) = \sum_{m=0}^{\infty}\sum_{l=1}^{h(m,d)} S_m^l(\theta_x)S_m^l(\theta_y)\,{}_*^{\dagger}\delta_m(r_x,r_y), \quad x,y \in \mathbb{R}^d,$$

and note that

$$\begin{aligned} Q_m &= \int_0^\infty \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_{\mathbb{S}^{d-1}} \left|c_1(x,y) - c_2(x,y)\right|^2 r_x^{d-1} r_y^{d-1} d\theta_x d\theta_y d_{r_x} d_{r_y} \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|c_1(x,y) - c_2(x,y)\right|^2 dx dy. \end{aligned}$$

Then, using the same reasoning as in the proof of Theorem 2.5, the latter integral is not finite. Therefore, we found a pair $m = i$ for which ${}_*^{\dagger}\delta_m$ is a continuous extension of $\delta_{m,m}^{l,l}$ from $D_+ \times D_+ \setminus N$ to $\mathbb{R}_+ \times \mathbb{R}_+ \setminus N$ which is such that $r_x^{(d-1)/2} r_y^{(d-1)/2}\,{}_*^{\dagger}\delta_m(r_x,r_y)$ is not square-integrable on $\mathbb{R}_+ \times \mathbb{R}_+$. Still, $\delta_{m,m}^{l,l}$ is uniformly continuous on $D_+ \times D_+$. In particular it is uniformly continuous on $D_+ \times D_+ \setminus N$. Since $D_+ \times D_+$ is dense in $\mathbb{R}_+ \times \mathbb{R}_+$, $D_+ \times D_+ \setminus N$ is dense in $\mathbb{R}_+ \times \mathbb{R}_+ \setminus N$. Hence, any continuous extension of $\delta_{m,m}^{l,l}$ from $D_+ \times D_+ \setminus N$ to $\mathbb{R}_+ \times \mathbb{R}_+ \setminus N$ must be given by ${}_*^{\dagger}\delta_m$. This concludes the proof under application of Theorem 2.4. □

From the proofs of Theorems 2.5 and 2.6, it becomes obvious that the assumption that $\{\lambda \in \mathbb{R}^d : f_1(\lambda) \neq f_2(\lambda)\}$ has positive Lebesgue measure can be replaced with the assumption that $\{x \in \mathbb{R}^d : k_1(x) \neq k_2(x)\}$ has positive Lebesgue measure. Notice also that if D is dense in $\mathbb{R}^d$, then clearly D_+ is dense in $\mathbb{R}_+$. Of course, the converse is not true. We will see a particular example in the next section. We further remark that if ξ is measurable with respect to some larger σ-algebra $\mathcal{G}$, i.e., $\mathcal{U} \subset \mathcal{G}$, then Theorems 2.5 and 2.6 give sufficient conditions to deduce the orthogonality of $\mathbb{P}_1$ and $\mathbb{P}_2$ on $\{\xi^{-1}(G) : G \in \mathcal{G}\}$. This is because the orthogonality of $\mathbb{P}_1$ and $\mathbb{P}_2$ on $\{\xi^{-1}(G) : G \in \mathcal{G}\}$ follows from the orthogonality of $\mathbb{P}_1$ and $\mathbb{P}_2$ on $\sigma_D(\xi)$. To finish this section, we have a closer look at a well-known family of covariance functions.

Example 2.1 (*Exponential Family of Covariance Functions*). The exponential family is defined by (see Example 1 on p. 115 of [13])

$$\phi_\theta(\tau) := \sigma^2 \mathrm{e}^{-\alpha\tau}, \quad \tau \in \mathbb{R}_+, \tag{13}$$

with $\theta = (\sigma^2, \alpha) \in (0,\infty)^2$. Clearly, the derivative of (13) with respect to τ is bounded. In particular, for any $\theta \in (0,\infty)^2$, $\tau \mapsto \phi_\theta(\tau)$ is Lipschitz continuous. Using the fact that the composition of Lipschitz continuous functions is again Lipschitz continuous, we conclude that for any $\theta \in (0,\infty)^2$, $c_\theta(x,y) := \phi_\theta(\|x-y\|)$, $x,y \in \mathbb{R}^d$, is Lipschitz continuous and thus uniformly continuous on $\mathbb{R}^d \times \mathbb{R}^d$. The family of radial spectral densities associated with (13) is given by (see [13]),

$$g_\theta(\kappa) = \sigma^2 \pi^{-1} \frac{\alpha}{\alpha^2 + \kappa^2}, \quad \kappa \in \mathbb{R}_+.$$

Hence, for any $\theta \in (0,\infty)^2$, $f_\theta(x) := g_\theta(\|x\|)$, $x \in \mathbb{R}^d$, is bounded. Take $\theta_1, \theta_2 \in (0,\infty)^2$ such that $\theta_1 \neq \theta_2$. It follows that $f_{\theta_1} \neq f_{\theta_2}$ on a set of positive Lebesgue measure. To see it, we can work with (13) and use the fact that the Fourier transform in an isometry $L^2(\mathbb{R}^d)$. Explicitly, if $\sigma_1^2 \neq \sigma_2^2$ and either $\alpha_1 = \alpha_2$ or $\alpha_1 \neq \alpha_2$, we observe that $\phi_{\theta_1}(0) \neq \phi_{\theta_2}(0)$. But, ϕ_θ is continuous at zero, thus, there exists an interval $[0,b)$, $b > 0$, on which $\phi_{\theta_1} \neq \phi_{\theta_2}$. In the remaining case, i.e., if $\alpha_1 \neq \alpha_2$ but $\sigma_1^2 = \sigma_2^2$, we can write $\alpha_1 = \alpha_2 + s$, with $s \neq 0$. Thus, in this case, we get

$$\phi_{\theta_1}(\tau) - \phi_{\theta_2}(\tau) = \sigma_1^2 \mathrm{e}^{-\alpha_2\tau}\left(\mathrm{e}^{-s\tau} - 1\right),$$

which is non-zero on $\mathbb{R}_+$. Hence, for any $\theta_1, \theta_2 \in (0,\infty)^2$ with $\theta_1 \neq \theta_2$ we have that $\phi_{\theta_1} \neq \phi_{\theta_2}$ on an interval $[0,b)$, $b > 0$. Therefore, with $k_\theta(x) := \phi_\theta(\|x\|)$, $x \in \mathbb{R}^d$, $k_{\theta_1} \neq k_{\theta_2}$ on $B_b(0)$. Clearly, k_θ is an element of $L^2(\mathbb{R}^d)$. Further, k_θ is the Fourier transform of f_θ. Thus, for $\theta_1 \neq \theta_2$, since $k_{\theta_1} \neq k_{\theta_2}$ on $B_b(0)$, we must conclude that the L^2 norm of $f_{\theta_1} - f_{\theta_2}$ on $\mathbb{R}^d$ is non-zero. Which implies that $f_{\theta_1} \neq f_{\theta_2}$ on a set of positive Lebesgue measure. In conclusion, two zero-mean Gaussian measures $\mathbb{P}_{\theta_1}$ and $\mathbb{P}_{\theta_2}$, $\theta_1, \theta_2 \in (0,\infty)^2$, $\theta_1 \neq \theta_2$, with exponential covariance functions c_{θ_1} and c_{θ_2}, respectively, are orthogonal on $\sigma_D(\xi)$ if D_+ is dense in $\mathbb{R}_+$.

3. Stochastic sampling

Let $\{X_t : t \in T\}$, $T \subset \mathbb{R}$, be a stochastic process defined on a probability space $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x)$, taking values in $\mathbb{R}^d$. That is, we consider a random function $\omega_x \mapsto X(\omega_x)$ with $\mathbb{R}^d$ valued sample paths defined on T. We assume that $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x)$ is complete in the measure theoretic sense. Further, X starts from the origin, i.e., $X_{t_0}(\omega_x) = 0$, $\omega_x \in \Omega_x$, and has continuous sample paths. For a given $\omega_x \in \Omega_x$, $X[T](\omega_x)$ denotes the image of $X(\omega_x)$, i.e., $x \in X[T](\omega_x)$ if and only if $x = X_t(\omega_x)$ for some $t \in T$. For now, we assume that ξ introduced in Section 1.2 is observed along sample paths restricted to $X[T](\omega_x)$, $\omega_x \in \Omega_x$. Explicitly, we consider two Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on $\sigma_{X[T](\omega_x)}(\xi)$ which differ only in their covariance functions c_1 and c_2. To adapt the notation of the previous sections, we put $X[T]_+(\omega_x) := \{\|x\| : x \in X[T](\omega_x)\}$, $\omega_x \in \Omega_x$. Following Theorem 2.5 we obtain:

Corollary 3.1. *Suppose that the assumptions of Theorem 2.5 are satisfied with c_1 and c_2 uniformly continuous on $\mathbb{R}^d \times \mathbb{R}^d$. Then, if $\mathbb{P}_x(X[T]$ is dense in $\mathbb{R}^d) = 1$, we have that*

$$\mathbb{P}_x\left(\mathbb{P}_1 \perp \mathbb{P}_2 \text{ on } \sigma_{X[T]}(\xi)\right) = 1, \tag{14}$$

i.e., the Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ are orthogonal on $\sigma_{X[T]}(\xi)$ $\mathbb{P}_x$ a.s.

We note that the set $\{X[T]$ is dense in $\mathbb{R}^d\}$ is a member of $\mathcal{F}_x$ since X takes values in $\mathbb{R}^d$ and has continuous sample paths on T. Further, since $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x)$ is assumed to be complete, under the assumptions of Corollary 3.1, $\{\mathbb{P}_1 \perp \mathbb{P}_2$ on $\sigma_{X[T]}(\xi)\}$ is a member of $\mathcal{F}_x$ as well. Similarly, for the isotropic case, the next result is deduced from Theorem 2.6.

Corollary 3.2. *Suppose that the assumptions of Theorem 2.6 are satisfied with c_1 and c_2 uniformly continuous on $\mathbb{R}^d \times \mathbb{R}^d$. Then, if $\mathbb{P}_x(X[T]_+$ is dense in $\mathbb{R}_+) = 1$, (14) is satisfied.*

As we have assumed that X starts from the origin, $X[T]_+$ is actually equal to $\mathbb{R}_+$, with $\mathbb{P}_x$ probability one, if X has sample paths that are almost surely unbounded. This is summarized in the following lemma:

Lemma 3.3. *Suppose that $X[T]$ is unbounded $\mathbb{P}_x$ a.s. Then $\mathbb{P}_x(X[T]_+ = \mathbb{R}_+) = 1$.*

Proof. Let $\omega_x \in \{X[T]$ is unbounded$\}$. Clearly $X[T]_+(\omega_x) \subset \mathbb{R}_+$. For the other direction, since the sample paths of X are continuous, we have that for any $\omega_x \in \Omega_x$, $X[T](\omega_x)$ is path-connected. Consider any $r \in \mathbb{R}_+$ and the neighborhood $B_r(0)$ around the origin. Since $\omega_x \in \{X[T]$ is unbounded$\}$, we must conclude that there exists v such that $v \in X[T](\omega_x)\setminus B_r(0)$. But since $X[T](\omega_x)$ is path-connected, there also exists v' such that $v' \in \partial B_r(0) \cap X[T](\omega_x)$. Therefore we have that $\|v'\| = r$, which shows that $\mathbb{R}_+ \subset X[T]_+(\omega_x)$. □

Using Lemma 3.3, we have proven the following theorem:

Theorem 3.4. *Suppose that the assumptions of Theorem 2.6 are satisfied with c_1 and c_2 uniformly continuous on $\mathbb{R}^d \times \mathbb{R}^d$. Then, if $X[T]$ is unbounded $\mathbb{P}_x$ a.s., we have*

$$\mathbb{P}_x\big(\mathbb{P}_1 \perp \mathbb{P}_2 \text{ on } \sigma_{X[T]}(\xi)\big) = 1.$$

Example 3.1 (*Gaussian Random Fields Sampled Along Brownian Paths*). It is well known that if we let $X = B$, with $T = \mathbb{R}_+$, a d-dimensional Brownian motion starting from the origin, then for $d \geq 2$, the Lebesgue measure of $B[\mathbb{R}_+]$ is zero with $\mathbb{P}_B$ probability one. This is shown in Théorème 53, p. 240, of [7] for the case where $d = 2$. A more recent proof, for the general case ($d \geq 2$), is given in the second paragraph of p. 197 in [6]. Still, it is true that for $d = 1, 2$, $B[\mathbb{R}_+]$ is dense in $\mathbb{R}$, $\mathbb{R}^2$, respectively, with $\mathbb{P}_B$ probability one (see Propositions 2.14 and 7.16 in [6]). For the case where $d \geq 3$, we know that $\mathbb{P}_B$ a.s. $\lim_{t\to\infty} \|B_t\| = \infty$ (see Theorem 7.17 in [6]). Further, for any $d \geq 1$, the sample paths of B are continuous (see Definitions 2.12 and 2.24 of [6]). In conclusion, given any $d \geq 1$, under sufficient conditions on c_1 and c_2 (see Corollary 3.1 and Theorem 3.4), with $\mathbb{P}_B$ probability one, Gaussian measures $\mathbb{P}_1$ and $\mathbb{P}_2$ are orthogonal on $\sigma_{B[\mathbb{R}_+]}(\xi)$, i.e., when ξ is sampled along the paths of B.

4. Inference on random fields

In this section, we let $D = \{x_i : i \in \mathbb{N}^*\}$ be a fixed sequence of coordinates in $\mathbb{R}^d$. That is, we consider

$$\sigma_D(\xi) = \big\{\xi^{-1}(U) : U \in \mathcal{U}\big\},$$

where now $\mathcal{U} = \sigma\big(\cup_{n=1}^{\infty} \mathcal{U}_n\big)$, with

$$\mathcal{U}_n = \sigma\big(\{C_{x_1,\ldots,x_n}(B_n) : B_n \in \mathfrak{B}(\mathbb{R}^n)\}\big).$$

Let $Y_n := (\xi_{x_1}, \ldots, \xi_{x_n})$, $n \in \mathbb{N}^*$. Then,

$$\sigma_D(\xi) = \sigma\big(\cup_{n=1}^{\infty} \sigma(Y_n)\big), \quad \sigma(Y_n) = \{Y_n^{-1}(B_n) : B_n \in \mathfrak{B}(\mathbb{R}^n)\}.$$

Let $\Theta \subset \mathbb{R}^p$. Suppose that $\mathbb{P}_\theta$, $\theta \in \Theta$, is a family of Gaussian measures defined on $\sigma_D(\xi)$. We remain in the setting of Section 1.3, i.e., for any two $\theta_1, \theta_2 \in \Theta$, under $\mathbb{P}_{\theta_1}$ and $\mathbb{P}_{\theta_2}$, ξ is stationary and items (i) and (ii) are satisfied. Suppose that there exists $\theta_0 \in \Theta$ such that the true distribution of ξ is obtained from $\mathbb{P}_{\theta_0}$. It is further assumed that there exists a neighborhood of θ_0 in Θ. In the framework of parameter estimation, θ_0 is treated as the unknown and Θ is regarded as the parameter space. A maximum likelihood (ML) estimator for θ_0 is defined to be any sequence of random variables $(\hat{\theta}_n)$, which is such that for any $n \in \mathbb{N}^*$,

$$\hat{\theta}_n(\omega) \in \arg\max_{\theta\in\Theta} p_n(\theta)(\omega), \quad \omega \in \Omega,$$

where

$$p_n(\theta) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma_n(\theta)}}\, \mathrm{e}^{-\frac{1}{2} Y_n^{\mathrm{t}} \Sigma_n(\theta)^{-1} Y_n},$$

with

$$\Sigma_n(\theta) = \big[c_\theta(x_i, x_j)\big]_{1\leq i,j\leq n}.$$

The function $\theta \mapsto p_n(\theta)(\omega)$ is called the likelihood function, the probability density function of Y_n, regarded as a function of θ. The sequence $(\hat{\theta}_n)$ is said to be strongly consistent for θ_0 if

$$\mathbb{P}_{\theta_0}\Big(\hat{\theta}_n \xrightarrow{n\to\infty} \theta_0\Big) = 1.$$

We say that the family $\mathbb{P}_\theta$, $\theta \in \Theta$, is a family of orthogonal Gaussian measures on $\sigma_D(\xi)$ if for any two $\theta_1, \theta_2 \in \Theta$, with $\theta_1 \neq \theta_2$, $\mathbb{P}_{\theta_1}$ and $\mathbb{P}_{\theta_2}$ are orthogonal on $\sigma_D(\xi)$. The next result is inspired by [14] (see the proof of Theorem 3).

Theorem 4.1. *Let Θ be closed and convex. Assume that a ML estimator for θ_0 exists and*

$$\theta \mapsto \varphi_n(\theta)(\omega) := \frac{p_n(\theta)(\omega)}{p_n(\theta_0)(\omega)}, \quad \omega \in \Omega,$$

is continuous on Θ. Suppose that there exists $N \in \mathbb{N}^*$ such that for any $n \geq N$ and $\omega \in \Omega$, $\theta \mapsto \varphi_n(\theta)(\omega)$ is log-concave on Θ. Then, if $\mathbb{P}_\theta$, $\theta \in \Theta$, is a family of orthogonal Gaussian measures on $\sigma_D(\xi)$, $(\hat{\theta}_n)$ is strongly consistent.

Proof. First of all, since $\mathbb{P}_\theta$, $\theta \in \Theta$, is a family of orthogonal Gaussian measures on $\sigma_D(\xi)$, we have that with $\mathbb{P}_{\theta_0}$ probability one, $(\varphi_n(\theta))$ converges pointwise to zero whenever $\theta \neq \theta_0$, i.e.,

$$\mathbb{P}_{\theta_0}\left(\varphi_n(\theta) \xrightarrow{n\to\infty} 0\right) = 1, \quad \theta \in \Theta,\ \theta \neq \theta_0.$$

This follows from the fact that the sequence $(\varphi_n(\theta))$ forms a martingale on $(\Omega, \sigma_D(\xi), \mathbb{P}_{\theta_0})$ with respect to the filtration $\{\sigma(Y_n) : n \in \mathbb{N}^*\}$ (see Theorem 1, p.442 in [3]). Given $\varepsilon > 0$, let $B_\varepsilon(\theta_0)$ be a neighborhood of θ_0 contained in Θ. We show that with $\mathbb{P}_{\theta_0}$ probability one,

$$\forall\ M > 0\ \exists\ N' \in \mathbb{N}^* \text{ s.t. } \forall\ \theta \in \Theta \setminus B_\varepsilon(\theta_0)\ \forall\ n \geq N' :\ \log(\varphi_n(\theta)) \leq -M. \tag{15}$$

In particular, (15) shows that

$$\mathbb{P}_{\theta_0}\left(\sup\{\varphi_n(\theta) : \theta \in \Theta \setminus B_\varepsilon(\theta_0)\} \xrightarrow{n\to\infty} 0\right) = 1. \tag{16}$$

To show (15), we let $\theta_m \in \partial B_\varepsilon(\theta_0)$ be such that for any $\theta \in \partial B_\varepsilon(\theta_0)$,

$$\log(\varphi_n(\theta_m))(\omega) \geq \log(\varphi_n(\theta))(\omega).$$

This follows from the assumption that for any $\omega \in \Omega$, $\theta \mapsto \varphi_n(\theta)(\omega)$ is continuous on Θ. Then, we choose $\omega \in \Omega$ as an element of the set

$$\{\omega \in \Omega : \forall\ M > 0\ \exists\ N_{\theta_m} \in \mathbb{N}^* \text{ s.t } \forall\ n \geq N_{\theta_m} :\ \log(\varphi_n(\theta_m))(\omega) \leq -M\}. \tag{17}$$

By assumption, there exists $N \in \mathbb{N}^*$ such that for any $n \geq N$, $\theta \mapsto \log(\varphi_n(\theta))(\omega)$ is concave on Θ. Hence, we put $N' := \max\{N, N_{\theta_m}\}$ and have that $\theta \mapsto \log(\varphi_n(\theta))(\omega)$ is concave on Θ for $n \geq N'$. Suppose, by contradiction, that there exists $n \geq N'$ and $\theta \in \Theta \setminus B_\varepsilon(\theta_0)$ such that $\log(\varphi_n(\theta))(\omega) > -M$. This implies that for $\lambda \in [0,1]$, with $(1-\lambda)\theta_0 + \lambda\theta \in \partial B_\varepsilon(\theta_0)$,

$$\begin{aligned}\log(\varphi_n(\theta_m))(\omega) &= (1-\lambda)\log(\varphi_n(\theta_m))(\omega) + \lambda \log(\varphi_n(\theta_m))(\omega)\\ &< (1-\lambda)\log(\varphi_n(\theta_0))(\omega) + \lambda\log(\varphi_n(\theta))(\omega), \quad n \geq N',\end{aligned}$$

since, for $n \geq N'$, $\log(\varphi_n(\theta_0))(\omega) = 0 > \log(\varphi_n(\theta_m))(\omega)$ and $\log(\varphi_n(\theta_m))(\omega) \leq -M$. Then, because of the fact that $\log(\varphi_n(\theta))(\omega)$ is concave on Θ for $n \geq N'$, we get,

$$\log(\varphi_n(\theta_m))(\omega) < \log(\varphi_n((1-\lambda)\theta_0 + \lambda\theta))(\omega) \leq \log(\varphi_n(\theta_m))(\omega), \quad n \geq N',$$

which is a contradiction. Thus, since under $\mathbb{P}_{\theta_0}$, (17) has measure one, (15) and hence (16) are shown. Following Wald's original consistency proof (Theorems 1 and 2 in [10]), we conclude from (16) that $\hat{\theta}_n \to \theta_0$, as $n \to \infty$, with $\mathbb{P}_{\theta_0}$ probability one. □

Given an ML estimator for θ_0, we remark that the assumptions on $\varphi_n(\theta)(\omega)$ given in Theorem 4.1 are satisfied if the Hessian matrix of the log-likelihood function $\theta \mapsto \log(p_n(\theta))(\omega)$ becomes negative semidefinite for n large enough. As a simple illustration we take $\Theta = [a,b]$, $0 < a < b < \infty$, and consider a family $\mathbb{P}_{\sigma^2}$, $\sigma^2 \in [a,b]$, defined upon the exponential family (13), with known scale parameter α_0 and unknown variance parameter σ^2. We readily see that the second derivative of the log-likelihood function is negative. Then, if the sequence of coordinates D is such that D_+ is dense in $\mathbb{R}_+$, we have seen in Example 2.1 that the family $\mathbb{P}_{\sigma^2}$, $\sigma^2 \in [a,b]$, is a family of orthogonal Gaussian measures on $\sigma_D(\xi)$. In this case, we can apply Theorem 4.1, and deduce that variance ML-estimators $(\hat{\sigma}_n^2)$ for the exponential family are strongly consistent. This adds to the results of [14], in which the sampling domain is assumed to be bounded. In particular, if we remain in the setting of Example 3.1 and consider a d-dimensional Brownian motion B starting from zero, we have that $B[\mathbb{R}_+ \cap \mathbb{Q}]_+$ is dense in $\mathbb{R}_+$ with $\mathbb{P}_B$ probability one. Thus, if we assume that ξ is sampled along $B[\mathbb{R}_+ \cap \mathbb{Q}]$, a sequence of variance ML-estimators $(\hat{\sigma}_n^2)$ is strongly consistent with $\mathbb{P}_B$ probability one.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors are grateful to Thomas Lehéricy for several helpful discussions. This work was supported by the Swiss National Science Foundation SNSF-175529.

References

[1] F. Bachoc, Asymptotic analysis of maximum likelihood estimation of covariance parameters for Gaussian processes: An introduction with proofs, in: A. Daouia, A. Ruiz-Gazen (Eds.), Advances in Contemporary Statistics and Econometrics, Springer, Cham, 2021, pp. 283–303, URL https://doi.org/10.1007/978-3-030-73249-3_15.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, Vol. II, Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981, Based on notes left by Harry Bateman, Reprint of the 1953 original.

[3] I.I. Gikhman, A.V. Skorokhod, The Theory of Stochastic Processes I, Springer, Berlin, 2004.

[4] H. Hankel, Die Cylinderfunctionen erster und zweiter Art, Math. Ann. 1 (3) (1869) 467–501, URL https://doi.org/10.1007/BF01445870.

[5] I.A. Ibragimov, Y.A. Rozanov, Gaussian Random Processes, Springer, New York, 1978.

[6] J.-F. Le Gall, Brownian Motion, Martingales, and Stochastic Calculus, Springer, Cham, 2016.

[7] P. Lévy, Processus Stochastiques et Mouvement Brownien, Gauthier-Villars, Paris, 1965.

[8] S. Molina, S.E. Trione, n-dimensional Hankel transform and complex powers of Bessel operator, Integral Transforms Spec. Funct. 18 (11–12) (2007) 897–911, URL https://doi.org/10.1080/10652460701511244.

[9] E.M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton, N.J., 1971.

[10] A. Wald, Note on the consistency of the maximum likelihood estimate, Ann. Math. Stat. 20 (1949) 595–601, URL https://doi.org/10.1214/aoms/1177729952.

[11] G.N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 1966, Reprint of the second (1944) edition.

[12] M.I. Yadrenko, Spectral Theory of Random Fields, Optimization Software, Inc., New York, 1983.

[13] A.M. Yaglom, Correlation Theory of Stationary and Related Random Functions, Vol. I: Basic Results, Springer, New York, 1987.

[14] H. Zhang, Inconsistent estimation and asymptotically equal interpolations in model-based geostatistics, J. Amer. Statist. Assoc. 99 (2004) 250–261, URL https://doi.org/10.1198/016214504000000241.