

Boundary layer solutions to functional elliptic equations

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Abstract. The goal of this paper is to study a class of nonlinear functional elliptic equations using very simple comparison principles. We first construct a nontrivial solution and then study its asymptotic behaviour when the diffusion coefficient goes to 0.

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Mathematical subject classification: 35A05, 35B25, 35B40, 35J60.

1 Introduction

Let Ω be a bounded open subset of \mathbb{R}^n . We denote by \mathcal{A} a function defined on $\Omega \times L^p(\Omega)$, $p \geq 1$, with values in \mathbb{R} such that

$$x \mapsto \mathcal{A}(x, u) \text{ is measurable} \quad \forall u \in L^p(\Omega), \quad (1.1)$$

$$u \mapsto \mathcal{A}(x, u) \text{ is continuous from } L^p(\Omega) \text{ into } \mathbb{R}, \text{ a.e. } x \in \Omega, \quad (1.2)$$

and there exist two constants a_0, a_∞ such that

$$0 < a_0 \leq \mathcal{A}(x, u) \leq a_\infty \quad \text{a.e. } x \in \Omega, \quad \forall u \in L^p(\Omega). \quad (1.3)$$

If f is a C^1 function vanishing at 0 and λ a positive parameter, we are interested in finding nontrivial solutions to the problem

$$\begin{cases} -\mathcal{A}(x, u)\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

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Here $\partial\Omega$ denotes the boundary of Ω . Of course problem (1.4) is understood in a weak sense as we will see below. We will also look at the behaviour of such solution when $\lambda \rightarrow +\infty$. The nature of (1.4) is obviously non-variational in general. Problems of this type in the local framework were considered in [1], [12], [13]–[15], [17].

The paper is divided as follows. In the next section we precise our assumptions in particular on the function f and give an existence result. Section 3 is devoted to examples of applications. Finally in the last section we let λ go to $+\infty$.

2 Existence of a nontrivial solution

Let θ be a positive number and $f : [0, \theta] \rightarrow \mathbb{R}$ a C^1 function with the following properties

$$f(0) = f(\theta) = 0, \quad (2.1)$$

$$f'(0) > 0, \quad (2.2)$$

$$f(t) > 0 \quad \forall t \in (0, \theta). \quad (2.3)$$

We denote by λ_1 the first eigenvalue for the Dirichlet problem in Ω – i.e.

$$\lambda_1 = \operatorname{Inf}_{H_0^1(\Omega)} \frac{\int_{\Omega} |\nabla v|^2 dx}{\int_{\Omega} v^2 dx}. \quad (2.4)$$

Then we have:

Theorem 2.1. *Under the assumptions above i.e. (1.1)–(1.3) and (2.1)–(2.4) and if*

$$\lambda > \frac{\lambda_1 a_{\infty}}{f'(0)} \quad (2.5)$$

there exists a nontrivial solution to

$$\begin{cases} -\Delta u = \frac{\lambda f(u)}{\mathcal{A}(x, u)} & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \quad (2.6)$$

such that $u(x) \in (0, \theta)$ for a.e. x . ((2.6) is of course understood in a weak sense).

Proof. Let us denote by φ_1 the first normalized eigenfunction to the Dirichlet problem that is to say the function such that

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\varphi_1 & \text{in } \Omega, \\ \varphi_1 \in H_0^1(\Omega), \varphi_1 \geq 0, \int_{\Omega} \varphi_1^2 dx = 1. \end{cases} \tag{2.7}$$

1. One can choose t_0 in such a way that $\underline{u} = t_0\varphi_1$ satisfies for every $w \in L^p(\Omega)$

$$-\Delta\underline{u} \leq \frac{\lambda f(\underline{u})}{\mathcal{A}(x, w)} \quad \text{in } \Omega. \tag{2.8}$$

Indeed, due to (2.7) we have

$$-\Delta\underline{u} = \lambda_1 t_0 \varphi_1 \quad \text{in } \Omega.$$

Assuming t_0 small enough in such a way that $0 \leq t_0\varphi_1 \leq \theta$ we have for any $w \in L^p(\Omega)$

$$\lambda \frac{f(t_0\varphi_1)}{a_{\infty}} \leq \frac{\lambda f(\underline{u})}{\mathcal{A}(x, w)}.$$

Then (2.8) will be fulfilled provided we choose t_0 such that

$$\lambda_1 a_{\infty} \leq \lambda \frac{f(t_0\varphi_1)}{t_0\varphi_1}$$

which is possible since

$$\lambda_1 a_{\infty} < \lambda f'(0).$$

From now on we fix t_0 in such a way that (2.8) holds. We set

$$K = \{ v \in L^2(\Omega) \mid t_0\varphi_1 \leq v \leq \theta \text{ a.e. } x \in \Omega \}. \tag{2.9}$$

It is clear that K is a closed convex subset of $L^2(\Omega)$.

2. We can choose μ in such a way that

$$g(u) = \lambda f(u) + \mu u \text{ is non decreasing on } (0, \theta). \tag{2.10}$$

Indeed it is enough to have

$$g'(u) = \lambda f'(u) + \mu \geq \lambda \operatorname{Inf}_{(0, \theta)} f' + \mu \geq 0$$

which is possible for $\mu > 0$ large.

We suppose in what follows that (2.10) holds. For $w \in K$ we consider then

$$u = Tw \tag{2.11}$$

the solution to

$$\begin{cases} -\Delta u + \frac{\mu u}{\mathcal{A}(x, w)} = \frac{g(w)}{\mathcal{A}(x, w)} & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{2.12}$$

Note that since $w \in K$, $w \in L^p(\Omega) \forall p \geq 1$ and everything makes sense, (2.12) being understood in terms of weak formulation. By the definition of g one remarks that a fixed point for T is a solution to (2.6).

3. $u = Tw \in K$.

Due to the monotonicity of g one has

$$\begin{aligned} -\Delta u + \frac{\mu u}{\mathcal{A}(x, w)} = \frac{g(w)}{\mathcal{A}(x, w)} &\leq \frac{g(\theta)}{\mathcal{A}(x, w)} = -\Delta \theta + \frac{\mu \theta}{\mathcal{A}(x, w)} && \text{in } \Omega, \\ -\Delta u + \frac{\mu u}{\mathcal{A}(x, w)} = \frac{g(w)}{\mathcal{A}(x, w)} &\geq \frac{g(\underline{u})}{\mathcal{A}(x, w)} \geq -\Delta \underline{u} + \frac{\mu \underline{u}}{\mathcal{A}(x, w)} && \text{in } \Omega. \end{aligned}$$

(The last inequality above follows from (2.8)). Since

$$\underline{u} \leq u \leq \theta \quad \text{on } \partial\Omega$$

we get by the weak maximum principle

$$\underline{u} \leq u \leq \theta \quad \text{in } \Omega$$

i.e. $u \in K$.

4. There exists a constant C independent of w such that

$$\|\nabla u\|_{2, \Omega} \leq C \tag{2.13}$$

i.e. u is bounded in $H_0^1(\Omega)$ independently of w . $\|\cdot\|_{2, \Omega}$ denotes the usual $L^2(\Omega)$ -norm, $|\cdot|$ the euclidean norm.

Indeed from the weak formulation of (2.12) we derive

$$\int_{\Omega} |\nabla u|^2 + \frac{\mu u^2}{\mathcal{A}(x, w)} dx = \int_{\Omega} \frac{g(w)}{\mathcal{A}(x, w)} u dx. \tag{2.14}$$

Denote by M the bound

$$M = \text{Sup}_{(0,\theta)} |g|.$$

From (2.14) we derive since μ, \mathcal{A} are nonnegative and by the Cauchy–Schwarz inequality

$$\int_{\Omega} |\nabla u|^2 dx \leq \frac{M}{a_0} \int_{\Omega} u dx \leq \frac{M}{a_0} \left(\int_{\Omega} u^2 dx \right)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \leq \frac{M}{a_0} \frac{|\Omega|^{\frac{1}{2}}}{\sqrt{\lambda_1}} \left(\int_{\Omega} |\nabla u|^2 \right)^{\frac{1}{2}}$$

(we used (2.4), $|\Omega|$ denotes the measure of Ω). It follows that

$$\|\nabla u\|_{2,\Omega} \leq \frac{M}{a_0} \frac{|\Omega|^{\frac{1}{2}}}{\sqrt{\lambda_1}}$$

which is (2.13).

5. The mapping $T : K \rightarrow K$ is continuous.

K is of course supposed to be endowed with the $L^2(\Omega)$ -topology. Let w_n be a sequence such that

$$w_n \rightarrow w \quad \text{in } L^2(\Omega), \quad w_n, w \in K. \tag{2.15}$$

One would like to show that

$$u_n = Tw_n \rightarrow Tw = u \quad \text{in } L^2(\Omega). \tag{2.16}$$

From (2.12) we easily derive

$$-\Delta(u - u_n) + \frac{\mu u}{\mathcal{A}(x, w)} - \frac{\mu u_n}{\mathcal{A}(x, w_n)} = \frac{g(w)}{\mathcal{A}(x, w)} - \frac{g(w_n)}{\mathcal{A}(x, w_n)}$$

i.e.

$$\begin{aligned} -\Delta(u - u_n) + \frac{\mu(u - u_n)}{\mathcal{A}(x, w)} &= \mu u_n \left\{ \frac{1}{\mathcal{A}(x, w_n)} - \frac{1}{\mathcal{A}(x, w)} \right\} \\ &+ \left\{ \frac{g(w)}{\mathcal{A}(x, w)} - \frac{g(w_n)}{\mathcal{A}(x, w_n)} \right\}. \end{aligned} \tag{2.17}$$

Up to a subsequence we deduce from (2.15) that

$$w_n \rightarrow w \quad \text{a.e. } x \in \Omega.$$

From the Lebesgue theorem since $w_n, w \in K$ it follows that

$$w_n \rightarrow w \quad \text{in } L^p(\Omega) \quad \forall p \geq 1. \quad (2.18)$$

Then the right hand side A_n from (2.17) is such that

$$|A_n| \leq \mu\theta \left| \frac{1}{\mathcal{A}(x, w_n)} - \frac{1}{\mathcal{A}(x, w)} \right| + \left| \frac{g(w)}{\mathcal{A}(x, w)} - \frac{g(w_n)}{\mathcal{A}(x, w_n)} \right|.$$

Due to (1.2) and (2.18) one has

$$\mathcal{A}(x, w_n) \rightarrow \mathcal{A}(x, w) \quad \text{a.e. } x \in \Omega$$

and by the Lebesgue theorem again it easily follows that

$$A_n \rightarrow 0 \quad \text{in } L^2(\Omega).$$

From (2.17) it follows that

$$u_n \rightarrow u \quad \text{in } H_0^1(\Omega).$$

This shows the continuity of T since the possible limit u is unique.

6. End of the proof.

Due to the compactness of the embedding from $H_0^1(\Omega)$ into $L^2(\Omega)$, T is a compact mapping from K into K . By the Schauder fixed point theorem it has a fixed point which is a solution to (2.6). Moreover – since $u \in K$ –

$$t_0\varphi_1 \leq u(x) \leq \theta \quad \text{a.e. } x \in \Omega. \quad (2.19)$$

This completes the proof of the theorem. \square

Remark 2.1. If h is a function satisfying (2.1)–(2.3) the existence of a non-trivial solution to

$$\begin{cases} -\Delta u = \frac{h(u)}{\mathcal{A}(x, u)}, \\ u \in H_0^1(\Omega), \end{cases} \quad (2.20)$$

is insured provided $h'(0) > \lambda_1 a_\infty$ (just set $h = \lambda f$).

3 Examples and applications

We consider $\theta' < 0 < \theta$ and $f : [\theta', \theta] \rightarrow \mathbb{R}$ a C^1 function satisfying

$$f(\theta') = f(0) = f(\theta) = 0, \tag{3.1}$$

$$f'(0) > 0, \tag{3.2}$$

$$f(t) < 0 \quad \forall t \in (\theta', 0), \quad f(t) > 0 \quad \forall t \in (0, \theta). \tag{3.3}$$

Under the conditions above we have

Theorem 3.1. *Assuming (1.1)–(1.3), (3.1)–(3.3) and*

$$\lambda > \frac{\lambda_1 a_\infty}{f'(0)}$$

the problem (2.6) possesses two nontrivial solutions.

Proof. There is one nontrivial solution between 0 and θ and if \tilde{u} is the nontrivial solution to

$$-\Delta u = -\frac{\lambda f(-u)}{\mathcal{A}(x, -u)}$$

between 0 and $-\theta'$ then clearly $u = -\tilde{u}$ is a nontrivial solution to (2.6) between θ' and 0. This completes the proof of the theorem. □

We turn now to examine what kind of function \mathcal{A} is suitable to fulfill our assumptions. First let us consider a well known local example. Let us denote by $a(x, u)$ a Carathéodory function from $\Omega \times \mathbb{R}$ into \mathbb{R} i.e. such that

$$x \mapsto a(x, u) \quad \text{is measurable} \quad \forall u \in \mathbb{R}, \tag{3.4}$$

$$u \mapsto a(x, u) \quad \text{is continuous a.e.} \quad x \in \Omega, \tag{3.5}$$

satisfying for some positive constants

$$0 < a_0 \leq a(x, u) \leq a_\infty \quad \text{a.e. } x \in \Omega, \quad \forall u \in \mathbb{R}. \tag{3.6}$$

Then if for $u \in L^2(\Omega)$ we define

$$\mathcal{A}(x, u) = a(x, u(x)) \tag{3.7}$$

it is clear that our assumptions (1.1)–(1.3) are fulfilled. To show (1.2) – the only perhaps non completely obvious assumption – if $u_n \rightarrow u$ in $L^2(\Omega)$ then, up to a subsequence, $u_n(x) \rightarrow u(x)$ a.e. x which implies that $a(x, u_n(x)) \rightarrow$

$a(x, u(x))$ a.e. $x \in \Omega$. Since the possible limit is unique this convergence is not up to a subsequence and this completes the proof of our claim. Problems involving (3.7) have been considered by many authors with different techniques (see [1], [23]). Our method is also well suited to attack nonlocal problems. The first interest for nonlocal problems goes perhaps back to Kirchhoff [18] (see also [2], [20]), where he considered a nonlinear wave equation. The topic was revisited recently in particular in the framework of asymptotic behaviour of parabolic equations. For an account to these issues we refer to [3]–[8], [9]–[11], [16], [19], [21] and [24]. To address a simple case consider a Carathéodory function a satisfying (3.4)–(3.6). Let j be a continuous function from $L^p(\Omega)$ into \mathbb{R} , $p \geq 1$. Then

$$\mathcal{A}(x, u) = a(x, j(u)) \quad (3.8)$$

fulfills the assumptions of our two preceding theorems. For j one can think for instance in the case $p = 1$ to

$$j(u) = \int_{\Omega} u(x) dx \quad (3.9)$$

if u is a density of population, $j(u)$ is then just the total population. One can restrict to a subpopulation by considering

$$j(u) = \int_{\Omega'} u(x) dx \quad \text{where } \Omega' \subset \Omega,$$

of course for some higher order p one can consider

$$j(u) = |u|_{p, \Omega}^p = \int_{\Omega} |u|^p dx$$

or variants of it. One can also mix the two dependences by setting

$$\mathcal{A}(x, u) = a(x, u(x), j(u)) \quad (3.10)$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is for instance a continuous function satisfying for some constants a_0, a_{∞}

$$0 < a_0 \leq a(x, u, v) \leq a_{\infty} \quad \forall x, u, v \in \Omega, \mathbb{R}, \mathbb{R}$$

and j a continuous mapping from $L^2(\Omega)$ into \mathbb{R} . Then, it is clear that $\mathcal{A}(x, u)$ defined by (3.10) satisfies our assumptions with $p = 2$. If φ is a one-to-one mapping from Ω into itself another nonlocal type of nonlinearity generalizing (3.7) could be

$$\mathcal{A}(x, u) = a(x, u(\varphi(x))).$$

This kind of problems have been addressed in [5]. The reader will of course be able to construct for himself further examples of applications.

We would like to address now the issue of asymptotic behaviour of these solutions to problem (2.6) when \mathcal{A} is given by one of the case above and when $\lambda \rightarrow +\infty$.

4 Asymptotic behaviour

We have first the following

Lemma 4.1. *Under the assumptions of Theorem 2 let $u = u_\lambda$ be a nontrivial solution to (2.6) as we constructed there. Then for every $p \geq 1$ one has*

$$u_\lambda \rightarrow \theta \quad \text{in } L^p_{\text{loc}}(\Omega). \tag{4.1}$$

Proof. We have to show that for every compact subset $S \subset \Omega$ one has when $\lambda \rightarrow +\infty$

$$u_\lambda \rightarrow \theta \quad \text{in } L^p(S).$$

1. We show that $\frac{f(u_\lambda)}{\mathcal{A}(x, u_\lambda)} \rightarrow 0$ in $\mathcal{D}'(\Omega)$.

As classical $\mathcal{D}'(\Omega)$ denotes the set of distributions on Ω , $\mathcal{D}(\Omega)$ the space of C^∞ -functions with compact support in Ω . By the weak formulation of (2.6) one has

$$\int_\Omega \frac{f(u_\lambda)}{\mathcal{A}(x, u_\lambda)} \varphi \, dx = \frac{1}{\lambda} \int_\Omega \nabla u_\lambda \nabla \varphi \, dx \quad \forall \varphi \in \mathcal{D}(\Omega).$$

By integration by parts we get

$$\int_\Omega \frac{f(u_\lambda)}{\mathcal{A}(x, u_\lambda)} \varphi \, dx = \frac{1}{\lambda} \int_\Omega u_\lambda \Delta \varphi \, dx \rightarrow 0 \tag{4.2}$$

since u_λ is uniformly bounded. This completes the proof of our claim.

Let S be a compact subset of Ω . We claim that

2. For any $\eta > 0$, $|\{x \in S \mid u_\lambda \leq \theta - \eta\}| \rightarrow 0$ as $\lambda \rightarrow +\infty$.

($|\cdot|$ denotes the measure of sets).

First some remarks are necessary. It is easy to see that the choice of t_0 in the proof of Theorem 2.1 can be made independently of λ large. Thus, since $u_\lambda \in K$, for every λ (see (2.9) for the definition of K) one has

$$\text{Inf}_S u_\lambda \geq \text{Inf}_S t_0 \varphi_1 = \gamma > 0. \tag{4.3}$$

(Indeed φ_1 as eigenfunction of the Laplace operator is smooth in Ω and since S is compact the infimum of $t_0\varphi_1$ on S is achieved and positive). Note here that the constant γ is independent of λ . Let us denote then by φ a nonnegative function such that

$$\varphi = 1 \quad \text{on } S, \quad \varphi \in \mathcal{D}(\Omega).$$

We have since f, \mathcal{A}, φ are nonnegative

$$\int_{\Omega} \frac{f(u_{\lambda})\varphi}{\mathcal{A}(x, u_{\lambda})} dx \geq \int_S \frac{f(u_{\lambda})}{\mathcal{A}(x, u_{\lambda})} dx \geq \int_{\{u_{\lambda} \leq \theta - \eta\}} \frac{f(u_{\lambda})}{a_{\infty}} dx \tag{4.4}$$

where we have set

$$\{u_{\lambda} \leq \theta - \eta\} = \{x \in S \mid u_{\lambda}(x) \leq \theta - \eta\}. \tag{4.5}$$

On this set above, by (4.3), we have (see (2.3))

$$f(u_{\lambda}) \geq \underset{(\gamma, \theta - \eta)}{\text{Inf}} f = c > 0.$$

Going back to (4.4) we deduce

$$\int_{\Omega} \frac{f(u_{\lambda})\varphi}{\mathcal{A}(x, u_{\lambda})} dx \geq \frac{c}{a_{\infty}} |\{u_{\lambda} \leq \theta - \eta\}|.$$

Claim 2 follows then from part 1.

3. End of proof.

We have for $p \geq 1$ and with the notation (4.5) for any $\eta > 0$

$$|u_{\lambda} - \theta|_{p,K}^p = \int_S |u_{\lambda} - \theta|^p dx = \int_{\{u_{\lambda} \leq \theta - \eta\}} |u_{\lambda} - \theta|^p dx + \int_{\{u_{\lambda} > \theta - \eta\}} |u_{\lambda} - \theta|^p dx$$

with an obvious notation for $\{u_{\lambda} > \theta - \eta\}$. It follows that

$$|u_{\lambda} - \theta|_{p,K}^p \leq \theta^p |\{u_{\lambda} \leq \theta - \eta\}| + \eta^p |\Omega|.$$

ε given, one can first choose η such that

$$\eta^p |\Omega| \leq \frac{\varepsilon^p}{2}$$

then for λ large enough one has

$$\theta^p |\{u_{\lambda} \leq \theta - \eta\}| \leq \frac{\varepsilon^p}{2}$$

by step 2. Combining these two last inequalities we have obtained for λ large enough

$$|u_{\lambda} - \theta|_{p,K}^p \leq \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} = \varepsilon^p$$

which completes the proof of the theorem. □

Remark 4.1. With the same proof one can show that every solution u_λ of (2.6) uniformly bounded from below on S converges toward θ in $L^p(S)$ for every $p \geq 1$.

As a consequence we have

Theorem 4.1. *Under the assumptions of Theorem 2.1 let $u = u_\lambda$ be the non-trivial solution to (2.6). Then for every $p \geq 1$ one has*

$$u_\lambda \rightarrow \theta \quad \text{in } L^p(\Omega). \quad (4.6)$$

Proof. Since Ω is bounded there is no loss of generality to assume $p > 1$. It is clear that u_λ is uniformly bounded in $L^p(\Omega)$. Thus, up to a subsequence, u_λ is converging in $L^p(\Omega)$ weakly. By Lemma 4.1 it follows that

$$u_\lambda \rightharpoonup \theta \quad \text{in } L^p(\Omega). \quad (4.7)$$

By the weak lower semi-continuity of the norm and the fact that $u_\lambda \in (0, \theta)$ we deduce

$$\liminf_{\lambda \rightarrow +\infty} \int_{\Omega} |u_\lambda|^p dx \geq \int_{\Omega} \theta^p dx \geq \overline{\lim}_{\lambda \rightarrow +\infty} \int_{\Omega} |u_\lambda|^p dx.$$

This implies that

$$\lim_{\lambda \rightarrow +\infty} \int_{\Omega} |u_\lambda|^p dx = \int_{\Omega} \theta^p dx$$

and (4.6) follows by (4.7) (see for instance [22]). \square

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