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## **Manin's conjecture for a quartic del Pezzo surface with A 4 singularity**

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# MANIN'S CONJECTURE FOR A QUARTIC DEL PEZZO SURFACE WITH $\mathbf{A}_4$ SINGULARITY

T.D. BROWNING AND U. DERENTHAL

ABSTRACT. The Manin conjecture is established for a split singular del Pezzo surface of degree four, with singularity type  $\mathbf{A}_4$ .

## CONTENTS

1. Introduction	1
2. Calculation of Peyre's constant	4
3. Arithmetic functions	6
4. The universal torsor	8
5. The main argument	11
5.1. Real-valued functions	12
5.2. Estimating $N_a(B)$ and $N_b(B)$ — first step	14
5.3. Estimating $N_a(B)$ — second step	16
5.4. Estimating $N_b(B)$ — second step	21
5.5. The final step	24
References	29

## 1. INTRODUCTION

The distribution of rational points on del Pezzo surfaces is a challenging topic that has enjoyed a surge of activity in recent years. Guided by the largely unverified conjectures of Manin [11] and his collaborators, the primary aim of this paper is to investigate further the situation for split singular del Pezzo surfaces of degree 4 in  $\mathbb{P}^4$ , that are defined over  $\mathbb{Q}$ . Our main achievement will be a proof of the Manin conjecture for the surface

$$x_0x_1 - x_2x_3 = x_0x_4 + x_1x_2 + x_3^2 = 0, \quad (1.1)$$

which we denote by  $S \subset \mathbb{P}^4$ . This surface contains a unique singularity of type  $\mathbf{A}_4$  and exactly three lines, all of which are defined over  $\mathbb{Q}$ .

Let  $U$  be the Zariski open subset formed by deleting the lines from  $S$ , and let

$$N_{U,H}(B) := \#\{x \in U(\mathbb{Q}) \mid H(x) \leq B\},$$

for any  $B \geq 1$ . Here  $H$  is the usual height on  $\mathbb{P}^4$ , in which the height  $H(x)$  is defined as  $\max\{|x_0|, \dots, |x_4|\}$  for a point  $x = (x_0 : \dots : x_4) \in U(\mathbb{Q})$ , provided that  $\mathbf{x} = (x_0, \dots, x_4)$  has integral coordinates that are relatively coprime. Bearing this in mind, the following is our principal result.

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**Theorem.** *We have*

$$N_{U,H}(B) = c_{S,H} B(\log B)^5 + O(B(\log B)^{5-2/7}),$$

where

$$c_{S,H} = \frac{1}{21600} \cdot \omega_\infty \cdot \prod_p \left(1 - \frac{1}{p}\right)^6 \left(1 + \frac{6}{p} + \frac{1}{p^2}\right)$$

and

$$\omega_\infty = \int_{|t_2|, |t_2 t_6 t_7|, |t_7(t_6^3 t_7 + t_2^2)|, |t_6^2 t_7| \leq 1, 0 < t_6 \leq 1} dt_2 dt_6 dt_7. \quad (1.2)$$

It is easily checked that the surface  $S$  is not toric, and we shall see in Lemma 3 that it is not an equivariant compactification of  $\mathbb{G}_a^2$ . Thus our result does not follow from the work of Tschinkel and his collaborators [1, 4].

As the minimal desingularisation  $\widetilde{X}_d$  of a split del Pezzo surface  $X_d \subset \mathbb{P}^d$  of degree  $d$  is the blow-up of  $\mathbb{P}^2$  in  $9 - d$  points, it has Picard group  $\text{Pic}(\widetilde{X}_d) \cong \mathbb{Z}^{10-d}$ . In the setting  $d = 4$ , Manin's conjecture [11] therefore predicts that

$$N_{U,H}(B) \sim \alpha(\widetilde{X}_4) \omega_H(\widetilde{X}_4) B(\log B)^5, \quad (1.3)$$

as  $B \rightarrow \infty$ , where the exponent of  $\log B$  is  $\text{rank Pic}(\widetilde{X}_4) - 1$ . Moreover, the constants  $\alpha(\widetilde{X}_4)$  and  $\omega_H(\widetilde{X}_4)$  are those predicted by Peyre [14]. Note that the exponent of  $\log B$  agrees with the statement of the theorem. We shall verify in § 2 that  $c_{S,H} = \alpha(\widetilde{S}) \omega_H(\widetilde{S})$  in this result.

An overview of progress relating to the Manin conjecture for arbitrary del Pezzo surfaces can be found in the first author's survey [3]. The present paper should be seen as a modest step on the path to its resolution for the singular del Pezzo surfaces of degree 4 that are split over  $\mathbb{Q}$ . According to the classification of such surfaces found in Coray and Tsfasman [6], it transpires that there are 15 possible singularity types for split singular del Pezzo surfaces of degree 4. It follows from the work of Batyrev and Tschinkel [1], la Bretèche and the first author [2], and the second author's joint work with Tschinkel [9], that the Manin conjecture is already known to hold for 5 explicit surfaces from this catalogue. In view of our theorem, which deals with a surface of singularity type  $\mathbf{A}_4$ , it remains to deal with the split quartic del Pezzo surfaces that have singularity types

$$\mathbf{A}_n \text{ for } n \in \{1, 2, 3\}, \quad 3\mathbf{A}_1, \quad \mathbf{A}_1 + \mathbf{A}_n \text{ for } n \in \{1, 2, 3\}. \quad (1.4)$$

Here one should note that there are two types of surfaces that have singularity type  $\mathbf{A}_3$ , one containing four lines and one containing five lines. Similarly, there are two types that have  $2\mathbf{A}_1$  singularities.

The surface that we have chosen to focus on in the present investigation satisfies the property that the cone of effective divisors associated to the minimal desingularisation  $\widetilde{S}$  is not merely generated by the divisors that form a basis for the Picard group  $\text{Pic}(\widetilde{S})$ , but requires one further divisor to generate it. This leads to some additional considerations in the proof, as we will see shortly.

The proof of the theorem uses a universal torsor. For each split del Pezzo surface of degree  $d$ , there is one (essentially unique) universal torsor, which is always an open subset of a  $(12 - d)$ -dimensional affine variety. For toric

varieties, universal torsors are open subsets of affine space. Salberger [15] has shown how to establish Manin's conjecture using universal torsors for split toric varieties defined over  $\mathbb{Q}$ . As a step towards handling non-toric del Pezzo surfaces that still have a relatively simple universal torsor, the second author [8] has determined which del Pezzo surfaces of degree at least 3 have a universal torsor that can be described as a hypersurface in  $\mathbb{A}^{13-d}$ . Out of the singularity types in (1.4), these include those surfaces of type  $\mathbf{A}_1 + \mathbf{A}_2$ ,  $\mathbf{A}_1 + \mathbf{A}_3$ ,  $3\mathbf{A}_1$  and the  $\mathbf{A}_3$  surface with five lines. The surfaces of type  $\mathbf{D}_5$  and  $\mathbf{D}_4$  considered in [2] and [9] also belong to this class, as does the  $\mathbf{A}_4$  surface  $S$  considered here. In fact we will see in § 4 that the universal torsor for the present problem is an open subset of the hypersurface

$$\eta_5\alpha_1 + \eta_1\alpha_2^2 + \eta_3\eta_4^2\eta_6^3\eta_7 = 0, \quad (1.5)$$

which is embedded in  $\mathbb{A}^9 \cong \text{Spec } \mathbb{Q}[\eta_1, \dots, \eta_7, \alpha_1, \alpha_2]$ . Note that one of the variables does not explicitly appear in the equation.

Our basic strategy is similar to the one used for the  $\mathbf{D}_5$  and  $\mathbf{D}_4$  quartic del Pezzo surfaces. The first step is to establish an explicit bijection between the rational points outside the lines on  $S$  and certain integral points on the universal torsor. We adopt the approach of Tschinkel and the second author [9] in order to obtain this bijection in an elementary way, motivated by the structure of the minimal desingularisation  $\tilde{S}$  as a blow-up of  $\mathbb{P}^2$  in five points. The integral points on the universal torsor are counted in § 5, using the method developed by la Bretèche and the first author [2]. The torsor variables  $\eta_1, \dots, \eta_7, \alpha_1, \alpha_2$  must satisfy (1.5), together with certain coprimality and height conditions. The first step is to fix the variables  $\eta_1, \dots, \eta_7$  and to estimate the relevant number of  $\alpha_1, \alpha_2$  by viewing the equation as a congruence modulo  $\eta_5$ . The resulting estimate is then summed over the remaining variables.

The order in which we handle the remaining variables is crucial and subtle. When it comes to summing over  $\eta_6$  and  $\eta_7$  we will run into trouble controlling the overall contribution from the error term each time, because both  $\eta_6$  and  $\eta_7$  can be rather big. Summing the number of  $\alpha_1, \alpha_2$  over  $\eta_7$ , for example, leads to an error term that we cannot estimate in a way that is sufficiently small when summed over  $\eta_1, \dots, \eta_5$  and large values of  $\eta_6$ . In line with this we shall let the order of summation depend on which of  $\eta_6$  or  $\eta_7$  has largest absolute value. When it comes to summing the integral points on the universal torsor that satisfy  $|\eta_6| \geq |\eta_7|$ , we sum first over  $\eta_6$  and then over  $\eta_7$ . For the alternative contribution we sum first over  $\eta_7$  and then over  $\eta_6$ . This process leads to two main terms that we put back together to get something of the general shape

$$M(\eta_1, \dots, \eta_5) := \omega_H(\tilde{S}) \cdot \frac{B}{\eta_1\eta_2\eta_3\eta_4\eta_5}, \quad (1.6)$$

where  $\omega_H(\tilde{S})$  is as in (1.3). The final task is to sum this quantity over the remaining variables  $\eta_1, \dots, \eta_5$ .

While essentially routine, it is in this final analysis that a further interesting feature of the proof of the theorem is revealed. For  $\mathbf{k} \in \mathbb{Z}_{>0}^5$ , define

the simplex

$$P_{\mathbf{k}} := \{(x_1, \dots, x_5) \in \mathbb{R}^5 \mid x_i \geq 0, \quad k_1 x_1 + \dots + k_5 x_5 \leq 1\}, \quad (1.7)$$

whose volume is easily determined as

$$\text{vol}(P_{\mathbf{k}}) = \frac{1}{5! \cdot k_1 \cdot k_2 \cdot k_3 \cdot k_4 \cdot k_5}.$$

In § 2 we will see that  $\alpha(\tilde{S}) = \text{vol}(P_{(2,4,3,2,3)}) - \text{vol}(P_{(3,6,4,2,5)})$ , whence

$$\alpha(\tilde{S}) = \frac{1}{5! \cdot 2 \cdot 4 \cdot 3 \cdot 2 \cdot 3} - \frac{1}{5! \cdot 3 \cdot 6 \cdot 4 \cdot 2 \cdot 5} = \frac{1}{21600}. \quad (1.8)$$

Returning to the summation of (1.6) over  $\eta_1, \dots, \eta_5 \in \mathbb{Z}_{>0}$ , which is subject to  $\eta_1^2 \eta_2^4 \eta_3^3 \eta_4^2 \eta_5^3 \leq B$ , it will transpire that there is a negligible contribution from those  $\eta_1, \dots, \eta_5$  for which  $\eta_1^3 \eta_2^6 \eta_3^4 \eta_4^2 \eta_5^5 > B$ . Summing over the  $\eta_1, \dots, \eta_5 \in \mathbb{Z}_{>0}$  that are remaining therefore leads to the final main term

$$(\text{vol}(P_{(2,4,3,2,3)}) - \text{vol}(P_{(3,6,4,2,5)})) \cdot \omega_H(\tilde{S}) B(\log B)^5,$$

as expected. Thus the main term in the asymptotic formula is really a difference of two main terms that conspire to give the predicted value for  $\alpha(\tilde{S})$ . It would be interesting to see whether the same sort of phenomenon occurs for other split del Pezzo surfaces of degree 4, with singularity type among the list (1.4).

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## 2. CALCULATION OF PEYRE'S CONSTANT

In this section we wish to show that the value of the constant  $c_{S,H}$  obtained in our theorem is in agreement with the prediction (1.3) of Peyre [14]. Beginning with the value of  $\omega_H(\tilde{S})$ , whose precise definition we will not include here but which corresponds to a product of local densities, we have

$$\omega_H(\tilde{S}) = \omega_\infty \prod_p \left(1 - \frac{1}{p}\right)^6 \omega_p, \quad (2.1)$$

where  $\omega_\infty$  and  $\omega_p$  are the real and  $p$ -adic densities, respectively. The calculation of  $\omega_p$  is routine and leads to the conclusion that

$$\omega_p = 1 + \frac{6}{p} + \frac{1}{p^2}.$$

The reader is referred to [2, § 2] for an analogous calculation. We now turn to the calculation of  $\omega_\infty$ , which needs to agree with (1.2).

Recall the equations (1.1) for the surface  $S$ , and write  $f_1(\mathbf{x}) = x_0 x_1 - x_2 x_3$  and  $f_2(\mathbf{x}) = x_0 x_4 + x_1 x_2 + x_3^2$ . To compute  $\omega_\infty$ , we parametrise the points by writing  $x_1, x_4$  as functions of  $x_0, x_2, x_3$ . Thus we have

$$x_1 = \frac{x_2 x_3}{x_0}, \quad x_4 = -\frac{x_1 x_2 + x_3^2}{x_0} = -\frac{x_2^2 x_3 + x_0 x_3^2}{x_0^2},$$

and furthermore,

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_1}{\partial x_4} & \frac{\partial f_2}{\partial x_4} \end{pmatrix} = \det \begin{pmatrix} x_0 & x_2 \\ 0 & x_0 \end{pmatrix} = x_0^2.$$

Since  $\mathbf{x}$  and  $-\mathbf{x}$  have the same image in  $\mathbb{P}^4$ , we have

$$\begin{aligned} \omega_\infty &= \frac{1}{2} \int_{|x_0|, \left| \frac{x_2 x_3}{x_0} \right|, |x_2|, |x_3|, \left| \frac{x_2^2 x_3 + x_0 x_3^2}{x_0^2} \right| \leq 1} x_0^{-2} dx_0 dx_2 dx_3 \\ &= \frac{1}{2} \int_{|t_6|, |t_2 t_6 t_7|, |t_2|, |t_6^2 t_7|, |t_2^2 t_7 + t_6^3 t_7^2| \leq 1} dt_2 dt_6 dt_7, \end{aligned}$$

on carrying out the change of variables  $x_0 = t_6$ ,  $x_2 = t_2$  and  $x_3 = t_6^2 t_7$ . But the range of integration is symmetric with respect to the transformation  $(t_2, t_6, t_7) \mapsto (t_2, -t_6, -t_7)$ , and so we may restrict to the range  $t_6 > 0$ . This therefore confirms the equality in (1.2).

It remains to deal with the constant  $\alpha(\tilde{S})$  that appears in (1.3). As we've already commented, the Picard group  $\text{Pic}(\tilde{S})$  of  $\tilde{S}$  has rank 6. Distinguished elements of  $\text{Pic}(\tilde{S})$  are the classes of irreducible curves with negative self intersection number. As described in § 4, these are the classes of four exceptional divisors  $E_1, \dots, E_4$  coming from the  $\mathbf{A}_4$ -singularity of  $S$  and the transforms  $E_5, E_6, E_7$  of the three lines on  $S$ . By the work of the second author [8, § 7],  $E_1, \dots, E_6$  form a basis of  $\text{Pic}(\tilde{S})$ . In terms of this basis we have  $E_7 = E_1 + 2E_2 + E_3 + 2E_5 - E_6$  and  $-K_{\tilde{S}} = 2E_1 + 4E_2 + 3E_3 + 2E_4 + 3E_5 + E_6$ .

The convex cone in  $\text{Pic}(\tilde{S})_{\mathbb{R}} := \text{Pic}(\tilde{S}) \otimes_{\mathbb{Z}} \mathbb{R}$  generated by classes of effective divisors is generated by  $E_1, \dots, E_7$  (see [10, Theorem 3.10]). The intersection of its dual with the hyperplane

$$\{x \in \text{Pic}(\tilde{S})_{\mathbb{R}} \mid (x, -K_{\tilde{S}}) = 1\}$$

is a polytope  $P$  whose volume is the constant  $\alpha(\tilde{S})$  defined by Peyre [14]. By definition

$$P = \left\{ (x_1, \dots, x_6) \in \text{Pic}(\tilde{S})_{\mathbb{R}} \mid \begin{array}{l} x_i \geq 0, \quad x_1 + 2x_2 + x_3 + 2x_5 - x_6 \geq 0, \\ 2x_1 + 4x_2 + 3x_3 + 2x_4 + 3x_5 + x_6 = 1 \end{array} \right\}.$$

Eliminating the last coordinate shows that  $P$  is isomorphic to

$$P' = \left\{ (x_1, \dots, x_5) \in \mathbb{R}^5 \mid \begin{array}{l} x_i \geq 0, \quad 2x_1 + 4x_2 + 3x_3 + 2x_4 + 3x_5 \leq 1, \\ 3x_1 + 6x_2 + 4x_3 + 2x_4 + 5x_5 \geq 1 \end{array} \right\}.$$

Analyzing the volume form with respect to which we must compute the volume of  $P$  in order to obtain  $\alpha(\tilde{S})$  (see [10, Section 2], for example), we see that

$$\alpha(\tilde{S}) = \text{vol}(P') = \text{vol}(P_{(2,4,3,2,3)}) - \text{vol}(P_{(3,6,4,2,5)}),$$

in the notation of (1.7). This therefore establishes (1.8).

An alternative approach to calculating  $\alpha(\tilde{S})$  is available to us through recent work of Joyce, Teitler and the second author [10]. Recall from [7, Table 1] that  $\alpha(S_0) = 1/180$  for any non-singular split del Pezzo surface  $S_0$  of degree 4. Since the order of the Weyl group associated to the root

system  $\mathbf{A}_n$  is  $(n+1)!$ , as recorded in [10, Table 2], so it follows from [10, Theorem 1.3] that

$$\alpha(\tilde{S}) = \frac{1}{180} \cdot \frac{1}{5!} = \frac{1}{21600}.$$

This completes the verification that our theorem confirms the Manin conjecture for the split  $\mathbf{A}_4$  surface (1.1).

### 3. ARITHMETIC FUNCTIONS

In this section we present some elementary facts about certain arithmetic functions and their average order, as required for our argument. Define the multiplicative arithmetic functions

$$\phi^*(n) := \prod_{p|n} \left(1 - \frac{1}{p}\right), \quad \phi^\dagger(n) := \prod_{p|n} \left(1 + \frac{1}{p}\right).$$

Both of these functions have average order  $O(1)$ , and one has

$$\sum_{n \leq x} \frac{\phi^\dagger(n)^j}{n} \ll_j \log x, \quad (3.1)$$

for any  $x > 1$  and any  $j \in \mathbb{Z}_{>0}$ . To see this we note that  $\phi^\dagger(n) \leq \sum_{d|n} 1/d$ , whence

$$\sum_{n \leq x} \frac{\phi^\dagger(n)^j}{n} \leq \sum_{n \leq x} \frac{1}{n} \sum_{d_1, \dots, d_j | n} \frac{1}{d_1 \dots d_j} \leq \sum_{d_1, \dots, d_j=1}^{\infty} \frac{1}{d_1 \dots d_j [d_1, \dots, d_j]} \sum_{e \leq x} \frac{1}{e},$$

where  $[d_1, \dots, d_j]$  denotes the least common multiple of  $d_1, \dots, d_j$ . The required bound (3.1) then follows from the estimate

$$\sum_{d_1, \dots, d_j=1}^{\infty} \frac{1}{d_1 \dots d_j [d_1, \dots, d_j]} \leq \sum_{d_1, \dots, d_j=1}^{\infty} \frac{1}{(d_1 \dots d_j)^{1+1/j}} \ll_j 1.$$

For given positive integers  $a, b$ , our work will lead us to work with the function

$$f_{a,b}(n) := \begin{cases} \phi^*(n)/\phi^*(\gcd(n, a)), & \text{if } \gcd(n, b) = 1, \\ 0, & \text{if } \gcd(n, b) > 1. \end{cases} \quad (3.2)$$

We begin by establishing the following result.

**Lemma 1.** *Let  $I = [t_1, t_2]$ , for  $t_1 < t_2$ . Let  $\alpha \in \mathbb{Z}$  such that  $\gcd(\alpha, q) = 1$ . Then we have*

$$\sum_{\substack{n \in I \cap \mathbb{Z} \\ n \equiv \alpha \pmod{q}}} f_{a,b}(n) = \frac{t_2 - t_1}{q} c_0 + O(2^{\omega(b)} \log |I|),$$

where  $|I| := 2 + \max\{|t_1|, |t_2|\}$  and

$$c_0 = \frac{\phi^*(b)}{\phi^*(\gcd(b, q))\zeta(2)} \prod_{p|abq} \left(1 - \frac{1}{p^2}\right)^{-1}. \quad (3.3)$$

*Proof.* We will follow the convention that  $\mu(-n) = \mu(n)$  and  $\mu(0) = 0$ . We begin by calculating the Dirichlet convolution

$$(f_{a,b} * \mu)(n) = \sum_{d|n} f_{a,b}(d)\mu(n/d) = \prod_{\substack{p^\nu || n \\ \nu \geq 1}} (f_{a,b}(p^\nu) - f_{a,b}(p^{\nu-1})).$$

It is clear that  $f_{a,b}(1) = 1$  and

$$f_{a,b}(p^j) = f_{a,b}(p) = \begin{cases} 1 - 1/p, & \text{if } p \nmid ab, \\ 1, & \text{if } p \nmid b \text{ and } p \mid a, \\ 0, & \text{if } p \mid b, \end{cases}$$

for any  $j \geq 1$ . Hence it follows that

$$(f_{a,b} * \mu)(n) = \begin{cases} \mu(n) \gcd(b, n)/|n|, & \text{if } \gcd(a, n) \mid b, \\ 0, & \text{otherwise.} \end{cases}$$

In particular

$$\sum_{n \leq N} |(f_{a,b} * \mu)(n)| \leq \sum_{n \leq N} \frac{\gcd(b, n)|\mu(n)|}{|n|} \ll 2^{\omega(b)} \log N,$$

for any  $N > 1$ . Since  $f_{a,b} = (f_{a,b} * \mu) * 1$ , we therefore deduce that

$$\begin{aligned} \sum_{\substack{n \in I \cap \mathbb{Z} \\ n \equiv \alpha \pmod{q}}} f_{a,b}(n) &= \sum_{\substack{d=1 \\ \gcd(d,q)=1}}^{\infty} (f_{a,b} * \mu)(d) \sum_{\substack{m \in d^{-1}I \cap \mathbb{Z} \\ md \equiv \alpha \pmod{q}}} 1 \\ &= \frac{t_2 - t_1}{q} \sum_{\substack{d=1 \\ \gcd(d,q)=1}}^{\infty} \frac{(f_{a,b} * \mu)(d)}{d} + O(2^{\omega(b)} \log |I|). \end{aligned}$$

Here we have observed that the outer sum in the first line is really a sum over  $d \leq |I|$ , making the previous bound applicable for dealing with the error term. We have then extended the summation over  $d$  to infinity, with acceptable error. Finally, it remains to observe that

$$\sum_{\substack{d=1 \\ \gcd(d,q)=1}}^{\infty} \frac{(f_{a,b} * \mu)(d)}{d} = \prod_p \left(1 - \frac{1}{p^2}\right) \prod_{p|abq} \left(1 - \frac{1}{p^2}\right)^{-1} \prod_{\substack{p|b \\ p \nmid q}} \left(1 - \frac{1}{p}\right) = c_0,$$

as required to complete the proof of the lemma.  $\square$

Rather than Lemma 1, we will actually need a corresponding estimate in which the summand is replaced by  $f_{a,b}(n)g(n)$ , for suitable real-valued functions  $g$ . This is supplied for us by the following result.

**Lemma 2.** *Let  $I = [t_1, t_2]$ , for  $t_1 < t_2$ , and let  $g : I \rightarrow \mathbb{R}$  be any function such that  $g$  has a continuous derivative on  $I$  which changes its sign only  $R_g(I) < \infty$  times on  $I$ . Let  $\alpha \in \mathbb{Z}$  such that  $\gcd(\alpha, q) = 1$ . Then we have*

$$\sum_{\substack{n \in I \cap \mathbb{Z} \\ n \equiv \alpha \pmod{q}}} f_{a,b}(n)g(n) = \frac{c_0}{q} \int_I g(t) dt + O(2^{\omega(b)} \cdot (\log |I|) \cdot M_I(g)),$$

with  $c_0$  given by (3.3) and  $M_I(g) := (1 + R_g(I)) \cdot \sup_{t \in I} |g(t)|$ .



*Proof.* Let  $S$  denote the sum that is to be estimated, and write

$$M(t) := \sum_{\substack{n \leq t \\ n \equiv \alpha \pmod{q}}} f_{a,b}(n),$$

for any  $t > 0$ . By partial summation,

$$S = M(t_2)g(t_2) - M(t_1)g(t_1) - \int_{t_1}^{t_2} M(t)g'(t) dt.$$

An application of Lemma 1 reveals that  $M(t) = c_0 t/q + O(2^{\omega(b)} \log(2 + |t|))$ . Hence partial integration yields

$$S = \frac{c_0}{q} \int_I g(t) dt + O(2^{\omega(b)} \cdot (\log |I|) \cdot (|g(t_2)| + |g(t_1)| + \int_{t_1}^{t_2} |g'(t)| dt)).$$

Splitting  $I$  into the  $R_g$  intervals where  $g'$  has constant sign therefore completes the proof of the lemma.  $\square$

#### 4. THE UNIVERSAL TORSOR

The purpose of this section is to establish a completely explicit bijection between the rational points on the open subset  $U$  of our  $\mathbf{A}_4$  quartic del Pezzo surface  $S$ , and the integral points on the universal torsor above  $\tilde{S}$  which are subject to a number of coprimality conditions. In doing so we shall follow the strategy of the second author's joint work with Tschinkel [9].

Along the way we will introduce new variables  $\eta_1, \dots, \eta_7$  and  $\alpha_1, \alpha_2$ . It will be convenient to henceforth write

$$\boldsymbol{\eta} = (\eta_1, \dots, \eta_5), \quad \boldsymbol{\eta}' = (\eta_1, \dots, \eta_7), \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2). \quad (4.1)$$

Furthermore, we will make frequent use of the notation

$$\boldsymbol{\eta}^{(k_1, k_2, k_3, k_4, k_5)} := \prod_{i=1}^5 \eta_i^{k_i}, \quad (4.2)$$

for any  $(k_1, \dots, k_5) \in \mathbb{Q}^5$ .

In order to derive the bijection alluded to above, we must begin by collecting together some useful information about the geometric structure of  $S$ , as defined by equations (1.1). By computing the Segre symbol of  $S$ , the definition of which can be found in Hodge and Pedoe [13], we see that  $S$  contains exactly one singularity. This has type  $\mathbf{A}_4$  and is easily determined as  $p = (0 : 0 : 0 : 0 : 1)$ . By the classification of singular quartic del Pezzo surfaces found in Coray and Tsfasman [6, Proposition 6.1],  $S$  contains exactly three lines. Let us call these lines  $E_5'', E_6''$  and  $E_7''$ , where  $E_5''$  and  $E_6''$  intersect in the singularity  $p$ , and  $E_7''$  intersects  $E_6''$  outside  $p$ . We easily determine these lines as  $E_5'' = \{x_0 = x_2 = x_3 = 0\}$ ,  $E_6'' = \{x_0 = x_1 = x_3 = 0\}$  and  $E_7'' = \{x_1 = x_3 = x_4 = 0\}$ .

The projection  $\mathbf{x} \mapsto (x_1 : x_3 : x_4)$  from  $E_7''$  is a birational map  $\phi : S \dashrightarrow \mathbb{P}^2$ , which maps

$$U := S \setminus (E_5'' \cup E_6'' \cup E_7'') = \{(x_0 : \dots : x_4) \in S \mid x_3 \neq 0\}$$

isomorphically to

$$\{(\alpha_2 : \eta_5 : \alpha_1) \in \mathbb{P}^2 \mid \eta_5 \neq 0, \alpha_1 \eta_5 + \alpha_2^2 \neq 0\} \subset \mathbb{P}^2.$$

The inverse map is  $\psi : \mathbb{P}^2 \dashrightarrow S$  given by

$$\psi : (\alpha_2 : \eta_5 : \alpha_1) \mapsto (\eta_5^3 : \alpha_2 \eta_7 : \alpha_2 \eta_5^2 : \eta_5 \eta_7 : \alpha_1 \eta_7), \quad (4.3)$$

where  $\eta_7 = -(\alpha_1 \eta_5 + \alpha_2^2)$ .

By [6, Proposition 6.1, Diagram 12], blowing up the singularity  $p$  leads to a minimal desingularisation  $\pi_0 : \tilde{S} \rightarrow S$  containing four  $(-2)$ -curves  $E_1, \dots, E_4$  (the four exceptional divisors obtained by blowing up  $p$ ) and three  $(-1)$ -curves  $E_5, E_6, E_7$  (the strict transforms of the lines  $E_5'', E_6'', E_7''$  on  $S$ ). The configuration of these  $(-1)$ - and  $(-2)$ -curves on  $\tilde{S}$  is described by Figure 1, where the number of edges between two curves is the intersection number, and self intersection numbers are given as upper indices. The divisors  $A_1, A_2$  will be introduced momentarily.

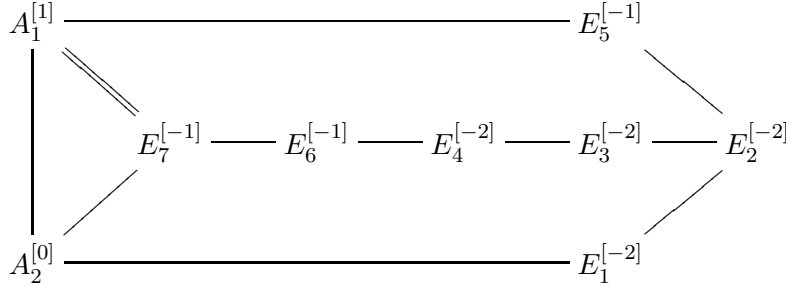


FIGURE 1. Configuration of curves on  $\tilde{S}$ .

The surface  $\tilde{S}$  is a blow-up  $\pi : \tilde{S} \rightarrow \mathbb{P}^2$  in five points. While there are several ways to construct  $\tilde{S}$  as such a blow-up of  $\mathbb{P}^2$ , we describe a map  $\pi$  that is compatible with the map  $\phi : S \dashrightarrow \mathbb{P}^2$  in the sense that  $\phi \circ \pi_0 : \tilde{S} \rightarrow S \dashrightarrow \mathbb{P}^2$  coincides with  $\pi$  where it is defined. Such a map  $\pi : \tilde{S} \rightarrow \mathbb{P}^2$  is obtained by contracting  $E_6, E_4, E_3, E_2, E_1$  on  $\tilde{S}$  in this order. We choose the same coordinates  $(\alpha_2 : \eta_5 : \alpha_1)$  on  $\mathbb{P}^2$  as before. Then  $\pi$  maps  $E_1, E_2, E_3, E_4, E_6$  to  $(0 : 0 : 1)$ . Furthermore,  $E_7$  is the strict transform of  $E_7' = \{\eta_7 = -(\alpha_1 \eta_5 + \alpha_2^2) = 0\} \subset \mathbb{P}^2$  and  $E_5$  is the strict transform of  $E_5' = \{\eta_5 = 0\} \subset \mathbb{P}^2$  under  $\pi$ .

To describe which points on  $\mathbb{P}^2$  we must blow up in order to recover  $\tilde{S}$ , we introduce  $A_1' = \{\alpha_1 = 0\} \subset \mathbb{P}^2$  and  $A_2' = \{\alpha_2 = 0\} \subset \mathbb{P}^2$ . We note that its strict transforms  $A_1, A_2$  under  $\pi$  on  $\tilde{S}$  intersect  $E_1, \dots, E_7$  as described by Figure 1, where  $A_1, A_2, E_7$  meet in one point which maps under  $\pi_0$  to  $(1 : 0 : 0 : 0 : 0) \in S$ . Given  $E_5', E_7', A_1', A_2' \subset \mathbb{P}^2$  as above, we may now perform the following sequence of five blow-ups to obtain  $\tilde{S}$ :

- blow up the intersection of  $E_5, E_7, A_2$  to obtain  $E_1$ ;
- blow up the intersection of  $E_1, E_5, E_7$  to obtain  $E_2$ ;
- blow up the intersection of  $E_2, E_7$  to obtain  $E_3$ ;
- blow up the intersection of  $E_3, E_7$  to obtain  $E_4$ ;
- blow up the intersection of  $E_4, E_7$  to obtain  $E_6$ .

Here we have renamed  $E_i'$  to  $E_i$  and  $A_j'$  to  $A_j$ , and we have used the same names for a divisor and its strict transform in each blow-up in the sequence. We proceed to establish the claim made in § 1.

**Lemma 3.** *The surface  $S$  is not an equivariant compactification of  $\mathbb{G}_a^2$ .*

*Proof.* To establish the lemma we assume for a contradiction that  $S$  is of this type and apply the work of Hassett and Tschinkel [12]. If  $S$  is an equivariant compactification of  $\mathbb{G}_a^2$  then the map  $\phi : S \dashrightarrow \mathbb{P}^2$  has to be  $\mathbb{G}_a^2$ -equivariant, resulting in an action of  $\mathbb{G}_a^2$  on  $\mathbb{P}^2$  which leaves  $E'_7 = \{\eta_7 = -(\alpha_1\eta_5 + \alpha_2^2) = 0\}$  invariant. However, we can check that the two distinct  $\mathbb{G}_a^2$ -structures on  $\mathbb{P}^2$  (see [12, Proposition 3.2]) do not leave any irreducible quadric curve invariant.  $\square$

We are now ready to derive the promised bijection between  $U(\mathbb{Q})$  and integral points on the universal torsor lying above  $\tilde{S}$ . The map  $\psi$  given by (4.3) induces a bijection

$$\psi_0 : (\alpha_1, \alpha_2, \eta_5, \eta_7) \mapsto (\eta_5^3, \alpha_2\eta_7, \alpha_2\eta_5^2, \eta_5\eta_7, \alpha_1\eta_7)$$

between

$$\{(\alpha, \eta_5, \eta_7) \in \mathbb{Z}^2 \times \mathbb{Z}_{>0} \times \mathbb{Z}_{\neq 0} \mid \alpha_1\eta_5 + \alpha_2^2 + \eta_7 = 0, \gcd(\alpha_1, \alpha_2, \eta_5) = 1\}$$

and

$$U(\mathbb{Q}) = \{(x_0 : \dots : x_4) \in S(\mathbb{Q}) \mid x_3 \neq 0\} \subset S(\mathbb{Q}).$$

Note that

$$H(\psi_0(\alpha_1, \alpha_2, \eta_5, \eta_7)) = \frac{\max_{0 \leq i \leq 4} |\psi_0(\alpha_1, \alpha_2, \eta_5, \eta_7)_i|}{\gcd(\{\psi_0(\alpha_1, \alpha_2, \eta_5, \eta_7)_i \mid 0 \leq i \leq 4\})}.$$

Motivated by the sequence of blow-ups above, we introduce new variables

$$\begin{aligned} \eta_1 &:= \gcd(\alpha_2, \eta_5, \eta_7), & \eta_2 &:= \gcd(\eta_1, \eta_5, \eta_7), & \eta_3 &:= \gcd(\eta_2, \eta_7), \\ \eta_4 &:= \gcd(\eta_3, \eta_7), & \eta_6 &:= \gcd(\eta_4, \eta_7), \end{aligned}$$

and in each step transform and rename the previous variables accordingly.

Observe that this gives a bijection

$$(\eta', \alpha) \mapsto (\eta^{(2,4,3,2,3)}\eta_6, \eta^{(1,1,1,1,0)}\eta_6\eta_7\alpha_2, \eta^{(2,3,2,1,2)}\alpha_2, \eta^{(1,2,2,2,1)}\eta_6^2\eta_7, \eta_7\alpha_1),$$

which we call  $\Psi$ , between

$$\mathcal{T} := \left\{ (\eta', \alpha) \in \mathbb{Z}_{>0}^6 \times \mathbb{Z}_{\neq 0} \times \mathbb{Z}^2 \mid \begin{array}{l} \eta_5\alpha_1 + \eta_1\alpha_2^2 + \eta_3\eta_4^2\eta_6^3\eta_7 = 0 \\ \text{coprimality conditions hold} \end{array} \right\}$$

and  $U(\mathbb{Q})$ . The coprimality conditions are described by the extended Dynkin diagram of  $E_1, \dots, E_7, A_1, A_2$  in Figure 1, following the rule that any of the variables  $\eta_i, \alpha_j$  are coprime if and only if there is no line connecting the divisors  $E_i, A_j$  in the Dynkin diagram. Once taken in conjunction with the equation

$$T(\eta', \alpha) = \eta_5\alpha_1 + \eta_1\alpha_2^2 + \eta_3\eta_4^2\eta_6^3\eta_7 = 0,$$

that is satisfied by the elements of  $\mathcal{T}$ , it is easily checked that the coprimality conditions can be rewritten as

$$\gcd(\alpha_1, \eta_2\eta_6) = 1 \tag{4.4}$$

$$\gcd(\alpha_2, \eta_2\eta_3\eta_4) = 1 \tag{4.5}$$

$$\gcd(\eta_6, \eta_1\eta_2\eta_3\eta_5) = 1 \tag{4.6}$$

$$\gcd(\eta_7, \eta_1\eta_2\eta_3\eta_4\eta_5) = 1 \tag{4.7}$$

$$\gcd(\eta_1, \eta_3\eta_4\eta_5) = 1, \gcd(\eta_2, \eta_4) = 1, \gcd(\eta_5, \eta_3\eta_4) = 1. \tag{4.8}$$

In particular it follows that  $H(\Psi(\boldsymbol{\eta}', \boldsymbol{\alpha})) = \max_{0 \leq i \leq 4} |\Psi(\boldsymbol{\eta}', \boldsymbol{\alpha})_i|$ , since the five coordinates of  $\Psi(\boldsymbol{\eta}', \boldsymbol{\alpha})$  are necessarily coprime for  $(\boldsymbol{\eta}', \boldsymbol{\alpha}) \in \mathcal{T}$ . The height conditions may therefore be written as

$$\max \left\{ \begin{array}{l} |\boldsymbol{\eta}^{(2,4,3,2,3)} \eta_6|, |\boldsymbol{\eta}^{(1,1,1,1,0)} \eta_6 \eta_7 \alpha_2|, \\ |\boldsymbol{\eta}^{(2,3,2,1,2)} \alpha_2|, |\boldsymbol{\eta}^{(1,2,2,2,1)} \eta_6^2 \eta_7|, |\eta_7 \alpha_1| \end{array} \right\} \leq B. \quad (4.9)$$

The equation  $T(\boldsymbol{\eta}', \boldsymbol{\alpha}) = 0$  is an embedding of the universal torsor over  $\tilde{S}$  in  $\mathbb{A}^9$ . Our argument so far has given us a parametrisation of rational points of bounded height in the complement  $U$  of the lines in  $S$ . This will play a pivotal role in our proof of the theorem.

## 5. THE MAIN ARGUMENT

In this section we give an overview of the proof of the theorem, and make our final preparations for its proof. Recall the notation introduced in (4.1) and (4.2) for  $\boldsymbol{\eta}, \boldsymbol{\alpha}$  and  $\boldsymbol{\eta}^{(k_1, k_2, k_3, k_4, k_5)}$ . We define the quantities

$$\begin{aligned} Y_0 &:= \left( \frac{\boldsymbol{\eta}^{(2,4,3,2,3)}}{B} \right)^{1/5}, \\ Y_2 &:= \left( \frac{B}{\boldsymbol{\eta}^{(2,-1,-2,-3,-2)}} \right)^{1/5}, \\ Y_6 &:= Y_0^{-1}, \\ Y_7 &:= \left( \frac{B}{\boldsymbol{\eta}^{(-3,-6,-2,2,-7)}} \right)^{1/5}, \end{aligned}$$

which clearly depend only on  $\boldsymbol{\eta}$  and  $B$ . Using the equation  $T(\boldsymbol{\eta}', \boldsymbol{\alpha}) = 0$ , a little thought reveals that we may write the height condition (4.9) as

$$|Y_0^4(\eta_6/Y_6)| \leq 1, \quad (5.1)$$

$$|Y_0^2(\eta_6/Y_6)(\eta_7/Y_7)(\alpha_2/Y_2)| \leq 1, \quad (5.2)$$

$$|Y_0^4(\alpha_2/Y_2)| \leq 1, \quad (5.3)$$

$$|Y_0^2(\eta_6/Y_6)^2(\eta_7/Y_7)| \leq 1, \quad (5.4)$$

$$|(\eta_7/Y_7)((\eta_6/Y_6)^3(\eta_7/Y_7) + Y_0^2(\alpha_2/Y_2)^2)| \leq 1, \quad (5.5)$$

with  $\eta_1, \dots, \eta_6 > 0$ . For example, eliminating  $\alpha_1$  from  $|\eta_7 \alpha_1| \leq B$  using  $T(\boldsymbol{\eta}', \boldsymbol{\alpha}) = 0$  gives (5.5). It follows from the contents of § 4 that  $N_{U,H}(B)$  is equal to the number of  $(\boldsymbol{\eta}', \boldsymbol{\alpha}) \in \mathbb{Z}_{>0}^6 \times \mathbb{Z}_{\neq 0} \times \mathbb{Z}^2$  such that (1.5) holds, with (4.4)–(4.8) and (5.1)–(5.5) all holding. As indicated in the introduction it will be necessary to follow different arguments according to which of  $\eta_6$  or  $|\eta_7|$  is biggest in the summation over the variables  $\boldsymbol{\eta}'$ . Accordingly, we write  $N_a(B)$  for the overall contribution to  $N_{U,H}(B)$  from  $(\boldsymbol{\eta}', \boldsymbol{\alpha})$  such that

$$\eta_6 \geq |\eta_7|, \quad (5.6)$$

and  $N_b(B)$  for the remaining contribution from  $(\boldsymbol{\eta}', \boldsymbol{\alpha})$  such that

$$\eta_6 < |\eta_7|. \quad (5.7)$$

These quantities will be estimated in § 5.3 and § 5.4, respectively.

Let us now recall the broad outlines of our approach to estimating  $N_a(B)$  and  $N_b(B)$ , as discussed in § 1. Thus the idea is to view the torsor equation (1.5) as a congruence modulo  $\eta_5$ , in order to take care of the summation over the variable  $\alpha_1$ . In § 5.2 we shall use this strategy to count the total number of permissible  $\alpha = (\alpha_1, \alpha_2)$ . This will lead to a preliminary estimate for both  $N_a(B)$  and  $N_b(B)$ , since it will make no difference whether (5.6) or (5.7) holds. It will then remain to sum this estimate over all of the remaining variables  $\eta'$ . We will estimate the overall contribution from the error term in § 5.2. For the treatment of the main term, however, we will need to treat the cases in which (5.6) or (5.7) holds differently. In estimating  $N_a(B)$ , we will sum the main term over  $\eta_6$  and then over  $\eta_7$ . This will be undertaken § 5.3. Alternatively, to estimate  $N_b(B)$ , we will sum the main term over  $\eta_7$  and then over  $\eta_6$ . This will be the object of § 5.4. Finally, in § 5.5 we will recombine our estimates and sum over the remaining variables  $\eta = (\eta_1, \dots, \eta_5)$ .

**5.1. Real-valued functions.** In estimating  $N_a(B)$  and  $N_b(B)$  we will meet a number of real-valued functions, whose basic properties it will be crucial to understand. Let

$$h(t_0, t_2, t_6, t_7) := \max\{|t_0^4 t_6|, |t_0^2 t_2 t_6 t_7|, |t_0^4 t_2|, |t_0^2 t_6^2 t_7|, |t_7(t_6^3 t_7 + t_0^2 t_2^2)|\} \quad (5.8)$$

Bearing this notation in mind, one notes that the height conditions in (5.1)–(5.5) are equivalent to  $h(Y_0, \alpha_2/Y_2, \eta_6/Y_6, \eta_7/Y_7) \leq 1$ . Finally, it is easy to see that

$$\omega_\infty = \int_{h(1, t_2, t_6, t_7) \leq 1, t_6 > 0} dt_2 dt_6 dt_7,$$

where  $\omega_\infty$  is given by (1.2).

We define the real-valued functions

$$g_0(t_0, t_6, t_7) := \int_{h(t_0, t_2, t_6, t_7) \leq 1} 1 dt_2 \quad (5.9)$$

$$g_1^a(t_0, t_7; \boldsymbol{\eta}; B) := \int_{Y_6 t_6 \geq |Y_7 t_7|, t_6 > 0} g_0(t_0, t_6, t_7) dt_6 \quad (5.10)$$

$$g_1^b(t_0, t_6; \boldsymbol{\eta}; B) := \int_{|Y_7 t_7| > \max\{Y_6 t_6, 1\}} g_0(t_0, t_6, t_7) dt_7 \quad (5.11)$$

$$\begin{aligned} g_2^a(t_0; \boldsymbol{\eta}; B) &:= \int_{h(t_0, t_2, t_6, t_7) \leq 1, Y_6 t_6 \geq |Y_7 t_7| > 1} dt_2 dt_6 dt_7 \\ &= \int_{|t_7| > 1/Y_7} g_1^a(t_0, t_7; \boldsymbol{\eta}; B) dt_7 \end{aligned} \quad (5.12)$$

$$\begin{aligned} g_2^b(t_0; \boldsymbol{\eta}; B) &:= \int_{h(t_0, t_2, t_6, t_7) \leq 1, |Y_7 t_7| > \max\{Y_6 t_6, 1\}, t_6 > 0} dt_2 dt_6 dt_7 \\ &= \int_0^\infty g_1^b(t_0, t_6; \boldsymbol{\eta}; B) dt_6 \end{aligned} \quad (5.13)$$

We clearly have

$$\begin{aligned} g_2(t_0; \boldsymbol{\eta}; B) &:= g_2^a(t_0; \boldsymbol{\eta}; B) + g_2^b(t_0; \boldsymbol{\eta}; B) \\ &= \int_{h(t_0, t_2, t_6, t_7) \leq 1, |Y_7 t_7| > 1, t_6 > 0} dt_2 dt_6 dt_7. \end{aligned} \quad (5.14)$$

Finally, we define

$$G_2(t_0) := \int_{h(t_0, t_2, t_6, t_7) \leq 1, t_6 > 0} dt_2 dt_6 dt_7.$$

The function  $G_2 : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  is intimately related to the real density  $\omega_\infty$ , as the following result shows.

**Lemma 4.** *We have*

$$G_2(t_0) = \frac{\omega_\infty}{t_0^2}.$$

*Proof.* This result follows on making the change of variables

$$t_2 = T_2 t_0^{-4}, \quad t_6 = T_6 t_0^{-4}, \quad t_7 = T_7 t_0^6.$$

Under this transformation one therefore obtains

$$G_2(t_0) = \frac{1}{t_0^2} \int_{h(t_0, T_2 t_0^{-4}, T_6 t_0^{-4}, T_7 t_0^6) \leq 1, T_6 > 0} dT_2 dT_6 dT_7,$$

where  $h(t_0, T_2 t_0^{-4}, T_6 t_0^{-4}, T_7 t_0^6) = h(1, T_2, T_6, T_7)$  is independent of  $t_0$ .  $\square$

During the course of our main argument it will be absolutely critical to control the size of the functions (5.9)–(5.11), as  $t_0, t_6, t_7$  vary. We may and shall assume that  $t_0, t_6, |t_7|$  take only positive values.

**Lemma 5.** *Let  $\boldsymbol{\eta} \in \mathbb{Z}_{>0}^5$  be given. Then the following hold:*

- (1)  $g_0(t_0, t_6, t_7) \ll \frac{1}{t_0 |t_7|^{1/2}}$ .
- (2)  $g_1^a(t_0, t_7; \boldsymbol{\eta}; B) \leq \int_0^\infty g_0(t_0, t_6, t_7) dt_6 \ll \min \left\{ \frac{1}{t_0 |t_7|^{7/6}}, \frac{1}{t_0^8} \right\}$ .
- (3)  $g_1^b(t_0, t_6; \boldsymbol{\eta}; B) \leq \int_{-\infty}^\infty g_0(t_0, t_6, t_7) dt_7 \ll \frac{1}{t_0 t_6^{3/4}}$ .

*Proof.* Recall the definition (5.8) of  $h$ . The upper bound  $O(t_0^{-8})$  that appears in (2) is easy. Indeed, it follows from the inequality  $h(t_0, t_2, t_6, t_7) \leq 1$  that  $|t_2| \leq 1/t_0^4$  and  $|t_6| \leq 1/t_0^4$ .

For the remaining statements, we distinguish the case  $|t_6^3 t_7^2| \leq 2$  and its opposite. Note that the inequality  $h(t_0, t_2, t_6, t_7) \leq 1$  implies

$$|t_6^3 t_7^2 + t_0^2 t_2^2 t_7| \leq 1. \tag{5.15}$$

Let us begin with the first case, in which case  $|t_0^2 t_2^2 t_7| \leq 3$ . We therefore obtain

$$t_2 \ll \frac{1}{t_0 |t_7|^{1/2}}, \quad t_6 \ll \frac{1}{|t_7|^{2/3}}, \quad t_7 \ll \frac{1}{t_6^{3/2}}.$$

The first of these inequalities implies statement (1), the first and second imply the first bound in statement (2), and finally, integrating the bound for  $g_0(t_0, t_6, t_7)$  from statement (1) over  $t_7 \ll 1/t_6^{3/2}$  gives statement (3).

In the second case  $|t_6^3 t_7^2| > 2$ , the inequality (5.15) implies  $t_7 < 0$  and

$$\frac{t_6^3 t_7^2 - 1}{t_0^2 |t_7|} \leq t_2^2 \leq \frac{t_6^3 t_7^2 + 1}{t_0^2 |t_7|}.$$

Note that the condition  $\sqrt{x} \leq t_2 \leq \sqrt{x+y}$  describes an interval for  $t_2$  of length  $O(y/x^{1/2})$ . Here,  $x = (t_6^3 t_7^2 - 1)/(t_0^2 |t_7|) \geq t_6^3 |t_7|/(2t_0^2)$  and  $y =$

$2/(t_0^2|t_7|)$ , whence

$$g_0(t_0, t_6, t_7) \ll \frac{1}{t_0 t_6^{3/2} |t_7|^{3/2}}.$$

The inequality  $t_6 > 2^{1/3}/|t_7|^{2/3}$  implies statement (1) and integrating over  $t_6 > 2^{1/3}/|t_7|^{2/3}$  results in the first bound in statement (2). Finally, integrating over  $|t_7| > 2^{1/2}/t_6^{3/2}$  gives statement (3).  $\square$

**5.2. Estimating  $N_a(B)$  and  $N_b(B)$  — first step.** We are now ready to begin our estimation of  $N_a(B)$  and  $N_b(B)$  in earnest. In what follows, we always have  $\eta_1, \dots, \eta_6 \in \mathbb{Z}_{>0}$  and  $\eta_7 \in \mathbb{Z}_{\neq 0}$ .

For fixed  $\boldsymbol{\eta}' = (\boldsymbol{\eta}, \eta_6, \eta_7)$  subject to the coprimality conditions (4.6), (4.7) and (4.8), we let  $N_0 := N_0(\boldsymbol{\eta}'; B)$  be the total number of  $\alpha_1, \alpha_2 \in \mathbb{Z}$  which satisfy the equation (1.5), subject to  $h(Y_0, \alpha_2/Y_2, \eta_6/Y_6, \eta_7/Y_7) \leq 1$  and the coprimality conditions (4.4) and (4.5). Employing a Möbius inversion for (4.4), we obtain

$$N_0 = \sum_{k_1 | \eta_2 \eta_6} \mu(k_1) \# \left\{ \alpha_2 \mid \begin{array}{l} \alpha_2^2 \eta_1 \equiv -\eta_3 \eta_4^2 \eta_6^3 \eta_7 \pmod{k_1 \eta_5}, \\ h(Y_0, \alpha_2/Y_2, \eta_6/Y_6, \eta_7/Y_7) \leq 1, \\ (4.5) \text{ holds} \end{array} \right\}.$$

It is easy to see that the summand vanishes unless  $\gcd(k_1, \eta_1 \eta_3 \eta_4) = 1$ . Indeed, if  $p \mid k_1, \eta_1$  then  $p \mid \eta_1, \eta_3 \eta_4 \eta_6 \eta_7$ , which is forbidden, and furthermore, if  $p \mid k_1, \eta_3 \eta_4$  then  $p \mid \eta_3 \eta_4, \alpha_2 \eta_1$ , which is also forbidden.

Let  $k_1$  be a squarefree divisor of  $\eta_2 \eta_6$ . Since  $\gcd(\eta_2, \eta_6) = 1$ , we can write  $k_1 = k_{12} k_{16}$  with  $k_{12} \mid \eta_2$  and  $k_{16} \mid \eta_6$ . Furthermore such a representation is unique. Writing  $\eta_6 = k_{16} \eta'_6$  we therefore obtain

$$N_0 = \sum_{\substack{k_{16} | \eta_6, k_{12} | \eta_2 \\ \gcd(k_{12} k_{16}, \eta_1 \eta_3 \eta_4) = 1}} \mu(k_{12}) \mu(k_{16}) N_0(k_{12}, k_{16})$$

where

$$N_0(k_{12}, k_{16}) := \# \left\{ \alpha_2 \mid \begin{array}{l} \alpha_2^2 \eta_1 \equiv -\eta_3 \eta_4^2 k_{16}^3 \eta_6^3 \eta_7 \pmod{k_{12} k_{16} \eta_5}, \\ h(Y_0, \alpha_2/Y_2, \eta_6/Y_6, \eta_7/Y_7) \leq 1, \\ (4.5) \text{ holds} \end{array} \right\}.$$

In view of the congruence we have  $k_{16} \mid \alpha_2^2 \eta_1$ , whence  $k_{16} \mid \alpha_2$  since  $\gcd(k_{16}, \eta_1) = 1$  and  $k_{16}$  is squarefree. Writing  $\alpha_2 = k_{16} \alpha'_2$ , we divide through the congruence by  $k_{16}$  to obtain

$$\alpha_2'^2 k_{16} \eta_1 \equiv -\eta_3 \eta_4^2 k_{16}^2 \eta_6^3 \eta_7 \pmod{k_{12} \eta_5}.$$

Using the relation  $\gcd(\eta_6, \eta_2 \eta_5) = 1$ , we see that  $\gcd(k_{16}, k_{12} \eta_5) = 1$ , whence we can remove a further factor of  $k_{16}$  in this congruence. It therefore follows that

$$N_0(k_{12}, k_{16}) = \# \left\{ \alpha_2' \mid \begin{array}{l} \alpha_2'^2 \eta_1 \equiv -\eta_3 \eta_4^2 k_{16} \eta_6^3 \eta_7 \pmod{k_{12} \eta_5}, \\ h(Y_0, \alpha_2' k_{16}/Y_2, \eta_6/Y_6, \eta_7/Y_7) \leq 1, \\ \gcd(\alpha_2', \eta_2 \eta_3 \eta_4) = 1 \end{array} \right\},$$

since  $\gcd(k_{16}, \eta_2 \eta_3 \eta_4) = 1$ .

Note that  $\gcd(k_{12}\eta_5, \eta_1) = 1$  and  $\gcd(k_{12}\eta_5, \eta_3\eta_4^2k_{16}\eta_6^3\eta_7) = 1$ . It therefore follows that for each  $\alpha'_2$  satisfying the congruence, there is a unique  $1 \leq \varrho \leq k_{12}\eta_5$ , with

$$\gcd(\varrho, k_{12}\eta_5) = 1, \quad \varrho^2\eta_1 \equiv -\eta_3\eta_6\eta_7 \pmod{k_{12}\eta_5}, \quad (5.16)$$

such that

$$\alpha'_2 \equiv \varrho\eta_4\eta'_6 \pmod{k_{12}\eta_5}.$$

Thus we obtain

$$N_0(k_{12}, k_{16}) = \sum_{\substack{1 \leq \varrho \leq k_{12}\eta_5 \\ (5.16) \text{ holds}}} \# \left\{ \alpha'_2 \mid \begin{array}{l} \alpha'_2 \equiv \varrho\eta_4\eta'_6 \pmod{k_{12}\eta_5}, \\ h(Y_0, \alpha'_2 k_{16}/Y_2, \eta_6/Y_6, \eta_7/Y_7) \leq 1, \\ \gcd(\alpha'_2, \eta_2\eta_3\eta_4) = 1 \end{array} \right\}.$$

We remove  $\gcd(\alpha'_2, \eta_2\eta_3\eta_4) = 1$  by a further application of Möbius inversion. Writing  $\alpha'_2 = k_2\alpha''_2$ , we see that  $N_0(k_{12}, k_{16})$  is equal to

$$\sum_{\substack{1 \leq \varrho \leq k_{12}\eta_5 \\ (5.16) \text{ holds}}} \sum_{k_2 \mid \eta_2\eta_3\eta_4} \mu(k_2) \# \left\{ \alpha''_2 \mid \begin{array}{l} k_2\alpha''_2 \equiv \varrho\eta_4\eta'_6 \pmod{k_{12}\eta_5}, \\ h(Y_0, \alpha''_2 k_{16}k_2/Y_2, \eta_6/Y_6, \eta_7/Y_7) \leq 1 \end{array} \right\}.$$

The summand vanished unless  $\gcd(k_2, k_{12}\eta_5) = 1$ , since  $p \mid k_2, k_{12}\eta_5$  implies  $p \mid k_{12}\eta_5, \varrho\eta_4\eta'_6$ , which is forbidden. Thus we may restrict our summation over  $k_2$  to  $\gcd(k_2, k_{12}\eta_5) = 1$ , and it therefore follows that the number of available  $\alpha''_2$  is

$$\frac{Y_2}{k_{12}k_{16}k_2\eta_5} g_0(Y_0, \eta_6/Y_6, \eta_7/Y_7) + O(1),$$

where  $g_0$  is given by (5.9). Recall the definition of the function  $\phi^*$  from § 3. We are now ready to establish the following result.

**Lemma 6.** *We have*

$$N_0 = \frac{Y_2}{\eta_5} g_0(Y_0, \eta_6/Y_6, \eta_7/Y_7) \vartheta_0(\boldsymbol{\eta}, \eta_6, \eta_7) + O(R_0(\boldsymbol{\eta}, \eta_6, \eta_7; B))$$

with

$$\vartheta_0(\boldsymbol{\eta}, \eta_6, \eta_7) := \frac{\phi^*(\eta_6)\phi^*(\eta_2\eta_3\eta_4)}{\phi^*(\gcd(\eta_6, \eta_4))} \sum_{\substack{k_{12} \mid \eta_2 \\ \gcd(k_{12}, \eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})}{k_{12}\phi^*(\gcd(\eta_2, k_{12}\eta_5))} \sum_{\substack{1 \leq \varrho \leq k_{12}\eta_5 \\ (5.16) \text{ holds}}} 1,$$

and

$$\sum_{\boldsymbol{\eta}, \eta_6, \eta_7} R_0(\boldsymbol{\eta}, \eta_6, \eta_7; B) \ll B(\log B)^3.$$

The final statement in Lemma 6 should be taken to mean that the overall contribution from the error term in the asymptotic formula for  $N_0$ , once summed over all of the available  $\boldsymbol{\eta}, \eta_6, \eta_7$ , is  $O(B(\log B)^3)$ . What is crucial here is that the exponent of  $\log B$  is strictly smaller than 5, so that this truly is an acceptable error term from the point of view of the main theorem. In the case of Lemma 6 we need to sum  $R_0(\boldsymbol{\eta}, \eta_6, \eta_7; B)$  over all  $\boldsymbol{\eta}, \eta_6, \eta_7$  which satisfy the height conditions (5.1)–(5.5), and the coprimality conditions (4.6)–(4.8). In the arguments to follow there will be several points at which the overall contribution from various error terms needs to



be estimated. In each case we will not stress the precise conditions on the variables to be summed over, these being invariably self-evident.

*Proof of Lemma 6.* Tracing through our argument above, it follows that

$$N_0 = \frac{Y_2}{\eta_5} g_0(Y_0, \eta_6/Y_6, \eta_7/Y_7) \vartheta_0(\boldsymbol{\eta}, \eta_6, \eta_7) + O(R_0(\boldsymbol{\eta}, \eta_6, \eta_7; B)),$$

with

$$\vartheta_0 = \sum_{\substack{k_{16} | \eta_6, k_{12} | \eta_2 \\ \gcd(k_{12} k_{16}, \eta_1 \eta_3 \eta_4) = 1}} \frac{\mu(k_{12}) \mu(k_{16})}{k_{12} k_{16}} \sum_{\substack{1 \leq \varrho \leq k_{12} \eta_5 \\ (5.16) \text{ holds}}} \sum_{\substack{k_2 | \eta_2 \eta_3 \eta_4 \\ \gcd(k_2, k_{12} \eta_5) = 1}} \frac{\mu(k_2)}{k_2},$$

and

$$\begin{aligned} R_0(\boldsymbol{\eta}, \eta_6, \eta_7; B) &\ll 2^{\omega(\eta_2 \eta_3 \eta_4) + \omega(\eta_6)} \sum_{k_{12} | \eta_2} |\mu(k_{12})| \sum_{\substack{1 \leq \varrho \leq k_{12} \eta_5 \\ (5.16) \text{ holds}}} 1 \\ &\ll 2^{\omega(\eta_2) + \omega(\eta_2 \eta_5)} 2^{\omega(\eta_2 \eta_3 \eta_4) + \omega(\eta_6)} \\ &\ll 8^{\omega(\eta_2)} 2^{\omega(\eta_3) + \omega(\eta_4) + \omega(\eta_5) + \omega(\eta_6)}. \end{aligned}$$

We have used here the fact that the congruence in (5.16) has at most  $2^{\omega(k_{12} \eta_5)} \leq 2^{\omega(\eta_2 \eta_5)}$  solutions  $\varrho$  modulo  $k_{12} \eta_5$ .

On noting that  $\gcd(\eta_6, \eta_1 \eta_3) = 1$  and  $\gcd(\eta_3 \eta_4, k_{12} \eta_5) = 1$ , we deduce that

$$\vartheta_0 = \sum_{\substack{k_{12} | \eta_2 \\ \gcd(k_{12}, \eta_1 \eta_3 \eta_4) = 1}} \frac{\mu(k_{12})}{k_{12}} \frac{\phi^*(\eta_6)}{\phi^*(\gcd(\eta_6, \eta_4))} \frac{\phi^*(\eta_2 \eta_3 \eta_4)}{\phi^*(\gcd(\eta_2, k_{12} \eta_5))} \sum_{\substack{1 \leq \varrho \leq k_{12} \eta_5 \\ (5.16) \text{ holds}}} 1.$$

This completes the proof of the main term in the lemma.

To show that  $R_0(\boldsymbol{\eta}, \eta_6, \eta_7; B)$  makes a satisfactory contribution once it is summed over all  $\boldsymbol{\eta}, \eta_6, \eta_7$  satisfying the height conditions in (4.9), we begin by summing over  $\eta_7$ . Thus it follows that

$$\begin{aligned} \sum_{\boldsymbol{\eta}, \eta_6, \eta_7} R_0(\boldsymbol{\eta}, \eta_6, \eta_7; B) &\ll \sum_{\boldsymbol{\eta}, \eta_6} \frac{8^{\omega(\eta_2)} 2^{\omega(\eta_3) + \omega(\eta_4) + \omega(\eta_5) + \omega(\eta_6)} B}{\boldsymbol{\eta}^{(1,2,2,2,1)} \eta_6^2} \\ &\ll B(\log B)^3, \end{aligned}$$

as required to complete the proof of the lemma.  $\square$

**5.3. Estimating  $N_a(B)$  — second step.** In this section our task is to sum the main term in Lemma 6 over all of the relevant  $\eta_6$  and  $\eta_7$ , such that (5.6) holds. As we've already indicated, we will begin by summing over the  $\eta_6$ . For fixed  $\boldsymbol{\eta}, \eta_7$  satisfying the coprimality conditions (4.7) and (4.8), define  $N_1^a := N_1^a(\boldsymbol{\eta}, \eta_7; B)$  to be the sum of the main term in Lemma 6 over all  $\eta_6 \in \mathbb{Z}_{>0}$  such that the coprimality condition (4.6) holds, and furthermore,  $\eta_6 \geq |\eta_7|$ .

We begin by noting that it is possible to remove  $\eta_5$  from (4.6), replacing this coprimality condition by  $\gcd(\eta_6, \eta_1 \eta_2 \eta_3) = 1$ . Indeed, if  $p \mid \eta_6, \eta_5$  then (5.16) implies that we must have  $p \mid \varrho^2 \eta_1$ , which is forbidden. Since  $\gcd(\eta_3 \eta_7, k_{12} \eta_5) = 1$ , so there exists a unique integer  $\beta \in [1, k_{12} \eta_5]$  such that

$$\varrho^2 \eta_1 \equiv -\eta_3 \eta_7 \beta \pmod{k_{12} \eta_5}.$$

It therefore follows that

$$N_1^a = \frac{Y_2}{\eta_5} \phi^*(\eta_2 \eta_3 \eta_4) \sum_{\substack{k_{12} | \eta_2 \\ \gcd(k_{12}, \eta_1 \eta_3 \eta_4) = 1}} \frac{\mu(k_{12})}{k_{12} \phi^*(\gcd(\eta_2, k_{12} \eta_5))} \sum_{\substack{1 \leq \varrho \leq k_{12} \eta_5 \\ \gcd(\varrho, k_{12} \eta_5) = 1}} A,$$

where

$$A = \sum_{\substack{\eta_6 \in \mathbb{Z}_{>0} \\ \eta_6 \geq |\eta_7| \\ \eta_6 \equiv \beta \pmod{k_{12} \eta_5}}} f_{\eta_4, \eta_1 \eta_2 \eta_3}(\eta_6) g_0(Y_0, \eta_6/Y_6, \eta_7/Y_7).$$

Here  $f_{\eta_4, \eta_1 \eta_2 \eta_3}$  is given by (3.2). Since  $g_0(Y_0, \eta_6/Y_6, \eta_7/Y_7) = 0$  for  $\eta_6 > B$ , we may restrict the summation to  $\eta_6$  in the range  $|\eta_7| \leq \eta_6 \leq B$ .

We will estimate  $A$  using Lemma 2. This produces a main term and an error term, the latter having size

$$\ll 2^{\omega(\eta_1 \eta_2 \eta_3)} (\log B) \sup_{t_6} g_0(Y_0, t_6, \eta_7/Y_7),$$

where the supremum is over all  $t_6 \in \mathbb{R}$  such that  $Y_6 t_6 \geq |\eta_7|$ . This therefore gives an overall contribution

$$\ll Y_2 2^{\omega(\eta_2) + \omega(\eta_1 \eta_2 \eta_3)} (\log B) \sup_{t_6} g_0(Y_0, t_6, \eta_7/Y_7), \quad (5.17)$$

to  $N_1^a$ , since

$$\frac{1}{k_{12} \eta_5} \sum_{\substack{1 \leq \varrho \leq k_{12} \eta_5 \\ \gcd(\varrho, k_{12} \eta_5) = 1}} 1 = \phi^*(k_{12} \eta_5).$$

The main term in our application of Lemma 2 to  $A$  is simply

$$\Theta(\boldsymbol{\eta}, k_{12}) \frac{Y_6}{k_{12} \eta_5} \int_{Y_6 t_6 \geq |\eta_7|, t_6 > 0} g_0(Y_0, t_6, \eta_7/Y_7) dt_6,$$

with

$$\begin{aligned} \Theta(\boldsymbol{\eta}, k_{12}) &= \frac{\phi^*(\eta_1 \eta_2 \eta_3)}{\zeta(2) \phi^*(\gcd(\eta_1 \eta_2 \eta_3, k_{12} \eta_5))} \prod_{p | \eta_1 \eta_2 \eta_3 \eta_4 \eta_5} \left(1 - \frac{1}{p^2}\right)^{-1} \\ &= \frac{\phi^*(\eta_1 \eta_2 \eta_3)}{\zeta(2) \phi^*(\gcd(\eta_2, k_{12} \eta_5))} \prod_{p | \eta_1 \eta_2 \eta_3 \eta_4 \eta_5} \left(1 - \frac{1}{p^2}\right)^{-1}. \end{aligned}$$

Here we have used the fact that  $\gcd(\eta_1 \eta_3, k_{12} \eta_5) = 1$ . Note for future reference that  $\Theta(\boldsymbol{\eta}, k_{12}) \ll 1$ . We are now ready to establish the following result.

**Lemma 7.** *We have*

$$N_1^a = \frac{Y_2 Y_6}{\eta_5} g_1^a(Y_0, \eta_7/Y_7, \boldsymbol{\eta}; B) \vartheta_1^a(\boldsymbol{\eta}) + O(R_1^a(\boldsymbol{\eta}, \eta_7; B))$$

with

$$\vartheta_1^a(\boldsymbol{\eta}) := \sum_{\substack{k_{12} | \eta_2 \\ \gcd(k_{12}, \eta_1 \eta_3 \eta_4) = 1}} \frac{\mu(k_{12})}{k_{12}} \Theta(\boldsymbol{\eta}, k_{12}) \phi^*(\eta_2 \eta_3 \eta_4 \eta_5),$$

and

$$\sum_{\boldsymbol{\eta}, \eta_7} R_1^a(\boldsymbol{\eta}, \eta_7; B) \ll B (\log B)^3.$$

*Proof.* It is clear from our calculations above that the main term in our estimate for  $N_1^a$  is equal to  $Y_2 Y_6 g_1^a(Y_0, \eta_7/Y_7; \boldsymbol{\eta}; B) \vartheta_1^a(\boldsymbol{\eta})/\eta_5$ , with

$$\begin{aligned} \vartheta_1^a(\boldsymbol{\eta}) &= \sum_{\substack{k_{12}|\eta_2 \\ \gcd(k_{12}, \eta_1 \eta_3 \eta_4)=1}} \frac{\mu(k_{12}) \phi^*(\eta_2 \eta_3 \eta_4) \phi^*(k_{12} \eta_5)}{k_{12} \phi^*(\gcd(\eta_2 \eta_3 \eta_4, k_{12} \eta_5))} \Theta(\boldsymbol{\eta}, k_{12}) \\ &= \phi^*(\eta_2 \eta_3 \eta_4 \eta_5) \sum_{\substack{k_{12}|\eta_2 \\ \gcd(k_{12}, \eta_1 \eta_3 \eta_4)=1}} \frac{\mu(k_{12})}{k_{12}} \Theta(\boldsymbol{\eta}, k_{12}), \end{aligned}$$

since  $k_{12} \eta_5$  is coprime to  $\eta_3 \eta_4$  and every divisor of  $k_{12}$  divides  $\eta_2$ . This completes the proof of the main term in the lemma.

Turning to the overall contribution from the error term  $R_1^a(\boldsymbol{\eta}, \eta_7; B)$ , which we have already seen has size (5.17), we conclude from (4.9) and (5.6) that

$$|\boldsymbol{\eta}^{(2,4,3,2,3)} \eta_7| \leq B,$$

for the  $\boldsymbol{\eta}, \eta_7$  that we need to sum over. We therefore deduce from Lemma 5(1) that

$$\begin{aligned} \sum_{\boldsymbol{\eta}, \eta_7} R_1^a(\boldsymbol{\eta}, \eta_7; B) &\ll \log B \sum_{\boldsymbol{\eta}, \eta_7} Y_2 4^{\omega(\eta_2)} 2^{\omega(\eta_1) + \omega(\eta_3)} \cdot \frac{Y_7^{1/2}}{Y_0 |\eta_7|^{1/2}} \\ &= \log B \sum_{\boldsymbol{\eta}, \eta_7} \frac{4^{\omega(\eta_2)} 2^{\omega(\eta_1) + \omega(\eta_3)} B^{1/2}}{\boldsymbol{\eta}^{(1/2, 0, 0, 0, -1/2)} |\eta_7|^{1/2}} \\ &\ll \log B \sum_{\boldsymbol{\eta}} \frac{4^{\omega(\eta_2)} 2^{\omega(\eta_1) + \omega(\eta_3)} B}{\boldsymbol{\eta}^{(3/2, 2, 3/2, 1, 1)}} \\ &\ll B(\log B)^3, \end{aligned}$$

as required to complete the proof of the lemma.  $\square$

Lemma 7 takes care of the summation of the main term in Lemma 6 over all of the relevant  $\eta_6$ . We proceed to sum the resulting main term over the  $\eta_7$ . Thus we let

$$N_2^a := N_2^a(\boldsymbol{\eta}; B) = \sum_{\substack{\eta_7 \in \mathbb{Z}_{\neq 0} \\ (4.7) \text{ holds}}} \frac{Y_2 Y_6}{\eta_5} g_1^a(Y_0, \eta_7/Y_7; \boldsymbol{\eta}; B) \vartheta_1^a(\boldsymbol{\eta})$$

We begin with an application of Möbius inversion to remove the coprimality condition (4.7). This gives

$$N_2^a = \frac{Y_2 Y_6}{\eta_5} \vartheta_1^a(\boldsymbol{\eta}) \sum_{k_7 | \eta_1 \eta_2 \eta_3 \eta_4 \eta_5} \mu(k_7) \sum_{|\eta_7'| \geq 1} g_1^a(Y_0, k_7 \eta_7'/Y_7; \boldsymbol{\eta}; B),$$

where we have written  $\eta_7 = k_7 \eta_7'$ . Partial summation now yields

$$\begin{aligned} N_2^a &= \frac{Y_2 Y_6 Y_7}{\eta_5} \vartheta_1^a(\boldsymbol{\eta}) \sum_{k_7 | \eta_1 \eta_2 \eta_3 \eta_4 \eta_5} \frac{\mu(k_7)}{k_7} \int_{|t_7| \geq k_7/Y_7} g_1^a(Y_0, t_7; \boldsymbol{\eta}; B) dt_7 \\ &\quad + O\left(\frac{Y_2 Y_6}{\eta_5} |\vartheta_1^a(\boldsymbol{\eta})| \sum_{k_7 | \eta_1 \eta_2 \eta_3 \eta_4 \eta_5} |\mu(k_7)| \sup_{|t_7| \geq k_7/Y_7} g_1^a(Y_0, t_7; \boldsymbol{\eta}; B)\right). \end{aligned}$$

The following result constitutes the final outcome of our summation over  $\eta_7$ .

**Lemma 8.** *We have*

$$N_2^a = \frac{Y_2 Y_6 Y_7}{\eta_5} g_2^a(Y_0, \boldsymbol{\eta}; B) \vartheta_2^a(\boldsymbol{\eta}) + O(R_2^a(\boldsymbol{\eta}; B))$$

with

$$\vartheta_2^a(\boldsymbol{\eta}) := \phi^*(\eta_1 \eta_2 \eta_3 \eta_4 \eta_5) \vartheta_1^a(\boldsymbol{\eta}),$$

and

$$\sum_{\boldsymbol{\eta}} R_2^a(\boldsymbol{\eta}; B) \ll B(\log B)^{5-2/7}.$$

*Proof.* The effect of replacing the integral  $\int_{|t_7| \geq k_7/Y_7} g_1^a(Y_0, t_7; \boldsymbol{\eta}; B) dt_7$  by  $g_2^a(Y_0, \boldsymbol{\eta}; B)$  in our estimate for  $N_2^a$ , is to create an additional term

$$\frac{Y_2 Y_6 Y_7}{\eta_5} |\vartheta_1^a(\boldsymbol{\eta})| \sum_{k_7 | \eta_1 \eta_2 \eta_3 \eta_4 \eta_5} \frac{|\mu(k_7)|}{k_7} \int_{1/Y_7 < |t_7| < k_7/Y_7} g_1^a(Y_0, t_7; \boldsymbol{\eta}; B) dt_7,$$

that must become part of  $R_2^a(\boldsymbol{\eta}; B)$ . Let us think of this as the first term in  $R_2^a(\boldsymbol{\eta}; B)$ . The second term that appears in  $R_2^a(\boldsymbol{\eta}; B)$  is the error appearing in the asymptotic formula for  $N_2^a$  that directly precedes the statement of the lemma.

We will need to estimate the overall contribution from both of these terms separately. It will be convenient to note that  $\vartheta_1^a(\boldsymbol{\eta}) = O(\phi^\dagger(\eta_2))$ , in the notation of § 3. Let  $\lambda > 0$  be a parameter to be selected in due course. Our argument will depend upon whether or not  $\boldsymbol{\eta}^{(3,6,4,2,5)} < \lambda B$  in the summation over the  $\boldsymbol{\eta}$ . Accordingly let  $E_1(\lambda)$  denote the overall contribution from the two errors terms once summed over  $\boldsymbol{\eta}$  such that

$$\boldsymbol{\eta}^{(3,6,4,2,5)} < \lambda B, \quad (5.18)$$

and let  $E_2(\lambda)$  denote the remaining contribution from  $\boldsymbol{\eta}$  such that

$$\boldsymbol{\eta}^{(3,6,4,2,5)} \geq \lambda B. \quad (5.19)$$

Beginning with the estimation of  $E_1(\lambda)$ , we employ Lemma 5(2) to conclude that

$$\int_{1/Y_7}^{k_7/Y_7} g_1^a(Y_0, t_7; \boldsymbol{\eta}; B) dt_7 \ll \int_{1/Y_7}^{k_7/Y_7} \frac{1}{Y_0 |t_7|^{7/6}} dt_7 \ll \frac{Y_7^{1/6}}{Y_0}.$$

Once summed over all  $\boldsymbol{\eta}$  such that (5.18) holds, we use (3.1) to estimate the overall contribution from the first term in  $R_2^a(\boldsymbol{\eta}; B)$  as

$$\begin{aligned} & \ll \sum_{\boldsymbol{\eta}} \sum_{k_7 | \eta_1 \eta_2 \eta_3 \eta_4 \eta_5} \frac{|\mu(k_7)|}{k_7} \frac{\phi^\dagger(\eta_2) Y_2 Y_6 Y_7^{7/6}}{\eta_5 Y_0} \\ & \ll \sum_{\boldsymbol{\eta}} \frac{\phi^\dagger(\eta_1 \eta_2 \eta_3 \eta_4 \eta_5) \phi^\dagger(\eta_2) B^{5/6}}{\boldsymbol{\eta}^{(1/2, 0, 1/3, 2/3, 1/6)}} \\ & \ll \sum_{\eta_1, \eta_2, \eta_3, \eta_4} \frac{\phi^\dagger(\eta_1) \phi^\dagger(\eta_3) \phi^\dagger(\eta_4) \phi^\dagger(\eta_2)^2 \lambda^{1/6} B}{\boldsymbol{\eta}^{(1, 1, 1, 1, 0)}} \\ & \ll \lambda^{1/6} B (\log B)^4. \end{aligned}$$

Turning to the overall contribution from the second term in  $R_2^a(\boldsymbol{\eta}; B)$ , we again deduce from Lemma 5(2) that

$$\sup_{|t_7| \geq k_7/Y_7} g_1^a(Y_0, t_7; \boldsymbol{\eta}; B) \ll \sup_{|t_7| \geq k_7/Y_7} \frac{1}{Y_0 |t_7|^{7/6}} \ll \frac{Y_7^{7/6}}{Y_0 k_7^{7/6}}.$$

Hence, in this case too, we obtain the overall contribution

$$\ll \sum_{\boldsymbol{\eta}} \frac{\phi^\dagger(\eta_2) Y_2 Y_6 Y_7^{7/6}}{\eta_5 Y_0} \ll \sum_{\eta_1, \eta_2, \eta_3, \eta_4} \frac{\phi^\dagger(\eta_2) \lambda^{1/6} B}{\boldsymbol{\eta}^{(1,1,1,1,0)}} \ll \lambda^{1/6} B (\log B)^4.$$

Thus far we have shown that  $E_1(\lambda) \ll \lambda^{1/6} B (\log B)^4$ .

It remains to produce a suitable upper bound for  $E_2(\lambda)$ . It will be convenient to record the estimates

$$\sum_{n \leq x} \frac{2^{\omega(n)} \phi^\dagger(n)}{n} \ll (\log x)^2, \quad \sum_{n > x} \frac{h^{\omega(n)}}{n^a} \ll x^{1-a} (\log x)^{h-1}.$$

The second inequality is valid for any  $h \in \mathbb{Z}_{>0}$  and any  $a > 1$ , and follows on combining partial summation with the bound

$$\sum_{n \leq x} h^{\omega(n)} \leq \sum_{n \leq x} \sum_{n=d_1 \dots d_h} 1 \ll \sum_{d_1, \dots, d_{h-1} \leq x} \frac{x}{d_1 \dots d_{h-1}} \ll x (\log x)^{h-1}.$$

Beginning with the first term in  $R_2^a(\boldsymbol{\eta}; B)$ , we deduce from Lemma 5(2) that

$$\int_{1/Y_7}^{k_7/Y_7} g_1^a(Y_0, t_7; \boldsymbol{\eta}; B) dt_7 \ll \int_{1/Y_7}^{k_7/Y_7} \frac{1}{Y_0^8} dt_7 \ll \frac{k_7}{Y_7 Y_0^8}.$$

Summing over  $\boldsymbol{\eta}$  such that (5.19) holds, we therefore obtain the overall contribution

$$\begin{aligned} &\ll \sum_{\boldsymbol{\eta}} \sum_{k_7 | \eta_1 \eta_2 \eta_3 \eta_4 \eta_5} \frac{|\mu(k_7)| \phi^\dagger(\eta_2) k_7 Y_2 Y_6}{k_7 \eta_5 Y_0^8} \\ &\ll \sum_{\boldsymbol{\eta}} \frac{2^{\omega(\eta_1 \eta_2 \eta_3 \eta_4 \eta_5)} \phi^\dagger(\eta_2) B^2}{\boldsymbol{\eta}^{(4,7,5,3,6)}} \\ &\ll \sum_{\eta_1, \dots, \eta_4} \frac{2^{\omega(\eta_1 \eta_2 \eta_3 \eta_4)} \phi^\dagger(\eta_2) B \log B}{\lambda \boldsymbol{\eta}^{(1,1,1,1,0)}} \\ &\ll \lambda^{-1} B (\log B)^9, \end{aligned}$$

by (3.1). Similarly, for the contribution from the second term in  $R_2^a(\boldsymbol{\eta}; B)$ , we may use Lemma 5(2) to deduce the overall contribution

$$\ll \sum_{\boldsymbol{\eta}} \frac{2^{\omega(\eta_1 \eta_2 \eta_3 \eta_4 \eta_5)} \phi^\dagger(\eta_2) Y_2 Y_6}{\eta_5 Y_0^8} \ll \lambda^{-1} B (\log B)^9.$$

Taken together this shows that  $E_2(\lambda) \ll \lambda^{-1} B (\log B)^9$ . We choose  $\lambda = (\log B)^{30/7}$ , which therefore gives the overall contribution

$$\sum_{\boldsymbol{\eta}} R_2^a(\boldsymbol{\eta}; B) \ll B (\log B)^{33/7},$$

as required.  $\square$

**5.4. Estimating  $N_b(B)$  — second step.** We must now return to the main term in Lemma 6, but this time reverse the order of summation for  $\eta_6$  and  $\eta_7$ . This will allow us to make use of the inequality (5.7) in our treatment of the error terms. We begin with the summation over  $\eta_7$ . For fixed  $\boldsymbol{\eta}$ ,  $\eta_6$  satisfying the coprimality conditions (4.6) and (4.8), define  $N_1^b := N_1^b(\boldsymbol{\eta}, \eta_6; B)$  to be the sum of the main term in Lemma 6 over all  $\eta_7 \in \mathbb{Z}_{\neq 0}$  such that the coprimality condition (4.7) holds, and furthermore,  $|\eta_7| > \eta_6 = \max\{\eta_6, 1\}$ .

Our argument is very similar in spirit to the preceding section. Removing (4.7) with an application of Möbius inversion, we find that

$$N_1^b = \frac{Y_2}{\eta_5} \frac{\phi^*(\eta_6)\phi^*(\eta_2\eta_3\eta_4)}{\phi^*(\gcd(\eta_6, \eta_4))} \sum_{\substack{k_{12}|\eta_2 \\ \gcd(k_{12}, \eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})}{k_{12}\phi^*(\gcd(\eta_2, k_{12}\eta_5))} \\ \times \sum_{\substack{1 \leq \varrho \leq k_{12}\eta_5 \\ \gcd(\varrho, k_{12}\eta_5)=1}} \sum_{\substack{k_7|\eta_1\eta_2\eta_3\eta_4\eta_5 \\ \gcd(k_7, k_{12}\eta_5)=1}} \mu(k_7)A,$$

where

$$A = \sum_{\substack{\eta'_7 \in \mathbb{Z}_{\neq 0} \\ \varrho^2 \eta_1 \equiv -\eta_3\eta_6k_7\eta'_7 \pmod{k_{12}\eta_5} \\ k_7|\eta'_7 > \eta_6}} g_0(Y_0, \eta_6/Y_6, k_7\eta'_7/Y_7),$$

and we have written  $\eta_7 = k_7\eta'_7$ . Note that we have been able to add the constraint  $\gcd(k_7, k_{12}\eta_5) = 1$  in the sum over  $k_7$ , since  $A = 0$  otherwise.

Since  $\gcd(\eta_3\eta_6k_7, k_{12}\eta_5) = 1$ , it follows from an easy application of partial summation that

$$A = \frac{Y_7}{k_{12}k_7\eta_5} g_1^b(Y_0, \eta_6/Y_6; \boldsymbol{\eta}; B) + O\left(\sup_{t_7} g_0(Y_0, \eta_6/Y_6, t_7)\right),$$

where the supremum is over  $t_7 \in \mathbb{R}$  such that  $|t_7| > \eta_6/Y_7$ . We may now establish the following result.

**Lemma 9.** *We have*

$$N_1^b = \frac{Y_2 Y_7}{\eta_5} g_1^b(Y_0, \eta_6/Y_6; \boldsymbol{\eta}; B) \vartheta_1^b(\boldsymbol{\eta}) \frac{\phi^*(\eta_6)}{\phi^*(\gcd(\eta_6, \eta_4))} + O(R_1^b(\boldsymbol{\eta}, \eta_6; B))$$

with

$$\vartheta_1^b(\boldsymbol{\eta}) := \phi^*(\eta_2\eta_3\eta_4\eta_5)\phi^*(\eta_1\eta_2\eta_3\eta_4) \sum_{\substack{k_{12}|\eta_2 \\ \gcd(k_{12}, \eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})}{k_{12}\phi^*(\gcd(\eta_2, k_{12}\eta_5))},$$

and

$$\sum_{\boldsymbol{\eta}, \eta_6} R_1^b(\boldsymbol{\eta}, \eta_6; B) \ll B(\log B)^3.$$

*Proof.* It is clear that the main term in the lemma is valid with

$$\begin{aligned}
\vartheta_1^b &= \sum_{\substack{k_{12}|\eta_2 \\ \gcd(k_{12}, \eta_1 \eta_3 \eta_4)=1}} \frac{\mu(k_{12})\phi^*(\eta_2 \eta_3 \eta_4)\phi^*(k_{12} \eta_5)}{k_{12}\phi^*(\gcd(\eta_2, k_{12} \eta_5))} \frac{\phi^*(\eta_1 \eta_2 \eta_3 \eta_4)}{\phi^*(\gcd(\eta_1 \eta_2 \eta_3 \eta_4, k_{12} \eta_5))} \\
&= \sum_{\substack{k_{12}|\eta_2 \\ \gcd(k_{12}, \eta_1 \eta_3 \eta_4)=1}} \frac{\mu(k_{12})}{k_{12}} \phi^*(k_{12} \eta_2 \eta_3 \eta_4 \eta_5) \frac{\phi^*(\eta_1 \eta_2 \eta_3 \eta_4)}{\phi^*(\gcd(\eta_2, k_{12} \eta_5))} \\
&= \phi^*(\eta_2 \eta_3 \eta_4 \eta_5) \phi^*(\eta_1 \eta_2 \eta_3 \eta_4) \sum_{\substack{k_{12}|\eta_2 \\ \gcd(k_{12}, \eta_1 \eta_3 \eta_4)=1}} \frac{\mu(k_{12})}{k_{12}\phi^*(\gcd(\eta_2, k_{12} \eta_5))},
\end{aligned}$$

as claimed. We have used here the fact that  $\gcd(k_{12} \eta_5, \eta_1 \eta_3 \eta_4) = 1$ .

For the error term, we deduce from (4.9) that  $\boldsymbol{\eta}^{(2,4,3,2,3)} \eta_6 \leq B$ , for the  $\boldsymbol{\eta}, \eta_6$  that we need to sum  $R_1^b(\boldsymbol{\eta}, \eta_6; B)$  over. Using Lemma 5(1) to bound  $g_0$ , we easily deduce that

$$\begin{aligned}
\sum_{\boldsymbol{\eta}, \eta_6} R_1^b(\boldsymbol{\eta}, \eta_6; B) &\ll \sum_{\boldsymbol{\eta}, \eta_6} Y_2 2^{\omega(\eta_2) + \omega(\eta_1 \eta_2 \eta_3 \eta_4)} \sup_{|t_7| > \eta_6 / Y_7} g_0(Y_0, \eta_6 / Y_6, t_7) \\
&\ll \sum_{\boldsymbol{\eta}, \eta_6} \frac{2^{\omega(\eta_2) + \omega(\eta_1 \eta_2 \eta_3 \eta_4)} Y_2 Y_7^{1/2}}{Y_0 \eta_6^{1/2}} \\
&= \sum_{\boldsymbol{\eta}, \eta_6} \frac{2^{\omega(\eta_2) + \omega(\eta_1 \eta_2 \eta_3 \eta_4)} B^{1/2}}{\boldsymbol{\eta}^{(1/2, 0, 0, 0, -1/2)} \eta_6^{1/2}} \\
&\ll \sum_{\boldsymbol{\eta}} \frac{2^{\omega(\eta_2) + \omega(\eta_1 \eta_2 \eta_3 \eta_4)} B}{\boldsymbol{\eta}^{(3/2, 2, 3/2, 1, 1)}} \\
&\ll B \sum_{\eta_4, \eta_5} \frac{2^{\omega(\eta_4)}}{\eta_4 \eta_5}.
\end{aligned}$$

But this is  $O(B(\log B)^3)$ , as required. This completes the proof of the lemma.  $\square$

We must now sum the main term in Lemma 9 over all of the relevant  $\eta_6$ , and then over  $\eta_1, \dots, \eta_5$ . In doing so it will be convenient distinguish between values of  $\boldsymbol{\eta}, \eta_6$  such that

$$\boldsymbol{\eta}^{(2,4,3,2,3)} \leq \frac{B}{(\log B)^A}, \tag{5.20}$$

for some  $A > 0$ , and those for which this inequality does not hold. We write  $N_{b_1}(B; A)$  and  $N_{b_2}(B; A)$  for the corresponding contributions. The following result shows that  $N_{b_2}(B; A)$  makes a negligible contribution to  $N_{U,H}(B)$ .

**Lemma 10.** *We have  $N_{b_2}(B; A) \ll_A B(\log B)^4(\log \log B)$ .*

*Proof.* Once taken in conjunction with the inequalities for  $\boldsymbol{\eta}, \eta_6$  in (4.9), the failure of (5.20) clearly implies that we must sum over  $\boldsymbol{\eta}, \eta_6$  for which

$$\eta_1^2 \eta_2^4 \eta_3^3 \eta_4^2 \eta_5^3 \eta_6 \leq B, \quad \eta_6 < (\log B)^A. \tag{5.21}$$

Recalling the definition of the main term from Lemma 9, we see that

$$\begin{aligned} N_{b_2}(B; A) &\ll \sum_{\substack{\boldsymbol{\eta}, \eta_6 \\ (5.21) \text{ holds}}} \frac{Y_2 Y_7}{\eta_5} g_1^b(Y_0, \eta_6/Y_6; \boldsymbol{\eta}; B) \vartheta_1^b(\boldsymbol{\eta}) \frac{\phi^*(\eta_6)}{\phi^*(\gcd(\eta_6, \eta_4))} \\ &\ll \sum_{\substack{\boldsymbol{\eta}, \eta_6 \\ (5.21) \text{ holds}}} \frac{Y_2 Y_6^{3/4} Y_7 \phi^\dagger(\eta_2)}{Y_0 \eta_5 \eta_6^{3/4}}, \end{aligned}$$

using Lemma 5(3). In view of the definitions of the  $Y_i$  we conclude that

$$\begin{aligned} N_{b_2}(B; A) &\ll B^{3/4} \sum_{\substack{\boldsymbol{\eta}, \eta_6 \\ (5.21) \text{ holds}}} \frac{\phi^\dagger(\eta_2)}{\boldsymbol{\eta}^{(1/2, 0, 1/4, 1/2, 1/4)} \eta_6^{3/4}} \\ &\ll B \sum_{\substack{\eta_2, \dots, \eta_6 \\ (5.21) \text{ holds}}} \frac{\phi^\dagger(\eta_2)}{\boldsymbol{\eta}^{(0, 1, 1, 1, 1)} \eta_6}. \end{aligned}$$

This last expression is clearly satisfactory for the lemma by (3.1) with  $j = 1$  and the fact that the  $\eta_6$  summation is over  $\eta_6 < (\log B)^A$ .  $\square$

Our focus now shifts to estimating  $N_{b_1}(B; A)$ , deemed to be the overall contribution from the main term in Lemma 9 that arises from  $\boldsymbol{\eta}, \eta_6$  for which (5.20) holds. For the moment let  $N_2^b := N_2^b(\boldsymbol{\eta}; B)$  be the quantity obtained by summing the main term in Lemma 9's estimate for  $N_1^b(\boldsymbol{\eta}, \eta_6; B)$ , over all  $\eta_6 \in \mathbb{Z}_{>0}$  such that (4.6) holds. An application of Lemma 2 with  $\alpha = 0$  and  $q = 1$  therefore reveals that

$$\begin{aligned} N_2^b &= \frac{Y_2 Y_7}{\eta_5} \vartheta_1^b(\boldsymbol{\eta}) \sum_{\eta_6 \geq 1} f_{\eta_4, \eta_1 \eta_2 \eta_3 \eta_5}(\eta_6) g_1^b(Y_0, \eta_6/Y_6; \boldsymbol{\eta}; B) \\ &= \frac{Y_2 Y_6 Y_7}{\eta_5} g_2^b(Y_0; \boldsymbol{\eta}; B) \vartheta_1^b(\boldsymbol{\eta}) \frac{\phi^*(\eta_1 \eta_2 \eta_3 \eta_5)}{\zeta(2)} \prod_{p|\eta_1 \eta_2 \eta_3 \eta_4 \eta_5} \left(1 - \frac{1}{p^2}\right)^{-1} \\ &\quad + O\left(\frac{Y_2 Y_7}{\eta_5} |\vartheta_1^b(\boldsymbol{\eta})| (\log B) 2^{\omega(\eta_1 \eta_2 \eta_3 \eta_5)} \sup_{t_6} g_1^b(Y_0, t_6; \boldsymbol{\eta}; B)\right) \\ &\quad + O\left(\frac{Y_2 Y_6 Y_7}{\eta_5} |\vartheta_1^b(\boldsymbol{\eta})| \int_{0 < t_6 < 1/Y_6} g_1^b(Y_0, t_6; \boldsymbol{\eta}; B) dt_6\right), \end{aligned}$$

where the supremum is over all  $t_6 \geq 1/Y_6$ . The following result is now straightforward.

**Lemma 11.** *We have*

$$N_2^b = \frac{Y_2 Y_6 Y_7}{\eta_5} g_2^b(Y_0; \boldsymbol{\eta}; B) \vartheta_2^b(\boldsymbol{\eta}) + O(R_2^b(\boldsymbol{\eta}; B))$$

with

$$\vartheta_2^b(\boldsymbol{\eta}) := \vartheta_1^b(\boldsymbol{\eta}) \frac{\phi^*(\eta_1 \eta_2 \eta_3 \eta_5)}{\zeta(2)} \prod_{p|\eta_1 \eta_2 \eta_3 \eta_4 \eta_5} \left(1 - \frac{1}{p^2}\right)^{-1},$$

and

$$\sum_{\substack{\boldsymbol{\eta} \\ (5.20) \text{ holds}}} R_2^b(\boldsymbol{\eta}; B) \ll B(\log B)^{9-A/4}.$$



*Proof.* The value of  $\vartheta_2^b(\boldsymbol{\eta})$  in the main term for  $N_2^b$  is a direct consequence of our manipulations above. In considering the overall contribution from the error term it will be convenient to note that  $\vartheta_2^b(\boldsymbol{\eta}) \ll \vartheta_1^b(\boldsymbol{\eta}) \ll \phi^\dagger(\eta_2)$ .

Once again the error  $R_2^b(\boldsymbol{\eta}; B)$  is comprised of two basic terms, the first one involving a supremum of  $g_1^b$  over  $t_6$  in an appropriate range, and the second involving an integration of  $g_1^b$ . We begin with dealing with the first term. It is here that we will make critical use of the inequality (5.20), that underpins our definition of  $N_{b_1}(B; A)$ . The first term in  $R_2^b(\boldsymbol{\eta}; B)$  clearly makes an overall contribution of

$$\ll \sum_{\boldsymbol{\eta}} \frac{2^{\omega(\eta_1 \eta_2 \eta_3 \eta_5)} \phi^\dagger(\eta_2) Y_2 Y_7 \log B}{\eta_5} \sup_{t_6 \geq 1/Y_6} g_1^b(Y_0, t_6, \boldsymbol{\eta}; B),$$

where the summation is restricted to  $\boldsymbol{\eta}$  for which (5.20) holds. Using Lemma 5(3) to estimate  $g_1^b$ , we may bound this as

$$\begin{aligned} &\ll \sum_{\boldsymbol{\eta}} \frac{2^{\omega(\eta_1 \eta_2 \eta_3 \eta_5)} \phi^\dagger(\eta_2) Y_2 Y_7 Y_6^{3/4} \log B}{\eta_5 Y_0} \\ &= \sum_{\boldsymbol{\eta}} \frac{2^{\omega(\eta_1 \eta_2 \eta_3 \eta_5)} \phi^\dagger(\eta_2) B^{3/4} \log B}{\boldsymbol{\eta}^{(1/2, 0, 1/4, 1/2, 1/4)}} \\ &\ll (\log B)^{1-A/4} \sum_{\eta_1, \eta_2, \eta_3, \eta_5} \frac{2^{\omega(\eta_1 \eta_2 \eta_3 \eta_5)} \phi^\dagger(\eta_2) B}{\boldsymbol{\eta}^{(1, 1, 1, 0, 1)}} \\ &\ll B(\log B)^{9-A/4}. \end{aligned}$$

Turning to the contribution from the second term in  $R_2^b(\boldsymbol{\eta}; B)$ , we employ Lemma 5(3) and (5.20) to derive the overall contribution

$$\begin{aligned} &\ll \sum_{\boldsymbol{\eta}} \frac{\phi^\dagger(\eta_2) Y_2 Y_6 Y_7}{\eta_5} \int_0^{1/Y_6} \frac{1}{Y_0 t_6^{3/4}} dt_6 \ll \sum_{\boldsymbol{\eta}} \frac{\phi^\dagger(\eta_2) Y_2 Y_6^{3/4} Y_7}{\eta_5 Y_0} \\ &= \sum_{\boldsymbol{\eta}} \frac{\phi^\dagger(\eta_2) B^{3/4}}{\boldsymbol{\eta}^{(1/2, 0, 1/4, 1/2, 1/4)}} \\ &\ll (\log B)^{-A/4} \sum_{\eta_1, \eta_2, \eta_3, \eta_5} \frac{\phi^\dagger(\eta_2) B}{\boldsymbol{\eta}^{(1, 1, 1, 0, 1)}} \\ &\ll B(\log B)^{4-A/4}. \end{aligned}$$

Together these two upper bounds complete the proof of the lemma.  $\square$

**5.5. The final step.** Let us take a moment to compile our work so far. We saw at the start of § 5 that

$$N_{U, H}(B) = N_a(B) + N_b(B).$$

It will be convenient to set  $B_0 = B/(\log B)^{36}$  in what follows.

The union of Lemmas 6, 7 and 8 shows that

$$N_a(B) = \sum_{\boldsymbol{\eta} \in \mathcal{E}(B)} \frac{Y_2 Y_6 Y_7}{\eta_5} \vartheta_2^a(\boldsymbol{\eta}) g_2^a(Y_0, \boldsymbol{\eta}; B) + O(B(\log B)^{5-2/7}),$$

where  $\vartheta_2^a(\boldsymbol{\eta})$  is as in the statement of Lemma 8, and

$$\mathcal{E}(B) := \{\boldsymbol{\eta} \in \mathbb{Z}_{>0}^5 : (4.8) \text{ holds and } \boldsymbol{\eta}^{(2,4,3,2,3)} \leq B\}.$$

Similarly, we can combine Lemmas 6, 9, 10 and 11, taking  $A = 36$  in the latter two results, to deduce that

$$N_b(B) = \sum_{\substack{\boldsymbol{\eta} \in \mathcal{E}(B) \\ \boldsymbol{\eta}^{(2,4,3,2,3)} \leq B_0}} \frac{Y_2 Y_6 Y_7}{\eta_5} \vartheta_2^b(\boldsymbol{\eta}) g_2^b(Y_0; \boldsymbol{\eta}; B) + O(B(\log B)^4 (\log \log B)),$$

where  $\vartheta_2^b(\boldsymbol{\eta})$  is as in the statement of Lemma 11.

We would now like to remove the constraint that  $\boldsymbol{\eta}^{(2,4,3,2,3)} \leq B_0$  in our estimate for  $N_b(B)$ . In view of the fact that  $\vartheta_2^b(\boldsymbol{\eta}) \ll \phi^\dagger(\eta_2)$ , it easily follows from (5.13) and Lemma 5(3) that

$$\begin{aligned} \sum_{\substack{\boldsymbol{\eta} \in \mathcal{E}(B) \\ B_0 < \boldsymbol{\eta}^{(2,4,3,2,3)} \leq B}} \frac{Y_2 Y_6 Y_7}{\eta_5} \vartheta_2^b(\boldsymbol{\eta}) g_2^b(Y_0; \boldsymbol{\eta}; B) &\ll \sum_{\boldsymbol{\eta}} \frac{Y_2 Y_6 Y_7 \phi^\dagger(\eta_2)}{\eta_5} \int_0^{1/Y_0^4} \frac{dt_6}{Y_0 t_6^{3/4}} \\ &\ll \sum_{\boldsymbol{\eta}} \frac{Y_2 Y_6^3 Y_7 \phi^\dagger(\eta_2)}{\eta_5} \\ &= \sum_{\boldsymbol{\eta}} \frac{B \phi^\dagger(\eta_2)}{\boldsymbol{\eta}^{(1,1,1,1,1)}} \\ &\ll B(\log B)^4 (\log \log B). \end{aligned}$$

In deducing the first bound we have used the fact that  $g_1^b(t_0, t_6; \boldsymbol{\eta}; B) = 0$  unless  $0 < t_6 \leq 1/t_0^4$ , which follows from the definition of (5.8). Thus we may replace the above formula for  $N_{b_1}(B; 36)$  by

$$\sum_{\boldsymbol{\eta} \in \mathcal{E}(B)} \frac{Y_2 Y_6 Y_7}{\eta_5} \vartheta_2^b(\boldsymbol{\eta}) g_2^b(Y_0; \boldsymbol{\eta}; B) + O(B(\log B)^4 (\log \log B)).$$

For given  $\boldsymbol{\eta} \in \mathcal{E}(B)$ , define

$$\begin{aligned} \vartheta(\boldsymbol{\eta}) &:= \frac{\phi^*(\eta_1 \eta_2 \eta_3) \phi^*(\eta_1 \eta_2 \eta_3 \eta_4 \eta_5) \phi^*(\eta_2 \eta_3 \eta_4 \eta_5)}{\zeta(2) \prod_{p|\eta_1 \eta_2 \eta_3 \eta_4 \eta_5} (1 - 1/p^2)} \\ &\quad \times \sum_{\substack{k_{12} | \eta_2 \\ \gcd(k_{12}, \eta_1 \eta_3 \eta_4) = 1}} \frac{\mu(k_{12})}{k_{12} \phi^*(\gcd(\eta_2, k_{12} \eta_5))}. \end{aligned}$$

It is easily seen that  $\vartheta(\boldsymbol{\eta}) = \vartheta_2^a(\boldsymbol{\eta})$ , in the notation of Lemmas 7 and 8. Furthermore, on noting that

$$\phi^*(\eta_1 \eta_2 \eta_3) \phi^*(\eta_1 \eta_2 \eta_3 \eta_4 \eta_5) = \phi^*(\eta_1 \eta_2 \eta_3 \eta_4) \phi^*(\eta_1 \eta_2 \eta_3 \eta_5),$$

since  $\gcd(\eta_4, \eta_5) = 1$ , we see that  $\vartheta(\boldsymbol{\eta}) = \vartheta_2^b(\boldsymbol{\eta})$  also. Thus we may draw together our argument so far to conclude that

$$N_{U,H}(B) = \sum_{\boldsymbol{\eta} \in \mathcal{E}(B)} \frac{Y_2 Y_6 Y_7}{\eta_5} \vartheta(\boldsymbol{\eta}) g_2(Y_0; \boldsymbol{\eta}; B) + O(B(\log B)^{5-2/7}),$$

where  $g_2(t_0; \boldsymbol{\eta}; B)$  is given by (5.14). It turns out that there is a negligible contribution to  $N_{U,H}(B)$  from summing  $Y_2 Y_6 Y_7 \vartheta(\boldsymbol{\eta}) g_2(t_0; \boldsymbol{\eta}; B) / \eta_5$  over

small values of  $\boldsymbol{\eta} \in \mathcal{E}(B)$ . The  $\boldsymbol{\eta}$  that give the dominant contribution belong to the set

$$\mathcal{E}^*(B) := \{\boldsymbol{\eta} \in \mathbb{Z}_{>0}^5 : (4.8) \text{ holds, } \boldsymbol{\eta}^{(2,4,3,2,3)} \leq B \text{ and } \boldsymbol{\eta}^{(3,6,4,2,5)} > B\}.$$

We also wish to remove the dependence on  $\boldsymbol{\eta}$  and  $B$  from the real-valued function  $g_2(Y_0; \boldsymbol{\eta}; B)$ . All of this will be achieved in the following result.

**Lemma 12.** *We have*

$$N_{U,H}(B) = \omega_\infty B \sum_{\boldsymbol{\eta} \in \mathcal{E}^*(B)} \frac{\vartheta(\boldsymbol{\eta})}{\boldsymbol{\eta}^{(1,1,1,1,1)}} + O(B(\log B)^{5-2/7}),$$

where  $\omega_\infty$  is given by (1.2).

*Proof.* We begin by showing that

$$M_1(B) := \sum_{\substack{\boldsymbol{\eta} \in \mathcal{E}(B) \\ \boldsymbol{\eta}^{(3,6,4,2,5)} \leq B}} \frac{Y_2 Y_6 Y_7}{\eta_5} \vartheta(\boldsymbol{\eta}) g_2(Y_0; \boldsymbol{\eta}; B) \ll B(\log B)^4.$$

Now it follows from (5.9) and (5.14) that

$$\begin{aligned} g_2(t_0; \boldsymbol{\eta}; B) &= \int_{h(t_0, t_2, t_6, t_7) \leq 1, |Y_7 t_7| > 1, t_6 > 0} dt_2 dt_6 dt_7 \\ &\leq \int_{|t_7| > 1/Y_7} \int_0^\infty g_0(t_0, t_6, t_7) dt_6 dt_7. \end{aligned}$$

Hence Lemma 5(2) yields

$$g_2(Y_0; \boldsymbol{\eta}; B) \ll \int_{|t_7| > 1/Y_7} \frac{1}{Y_0 |t_7|^{7/6}} dt_7 \ll \frac{Y_7^{-1/6}}{Y_0}.$$

Applying this we deduce that

$$\begin{aligned} M_1(B) &\ll \sum_{\boldsymbol{\eta}^{(3,6,4,2,5)} \leq B} \frac{\phi^\dagger(\eta_2) Y_2 Y_6 Y_7^{7/6}}{\eta_5 Y_0} \ll \sum_{\boldsymbol{\eta}^{(3,6,4,2,5)} \leq B} \frac{\phi^\dagger(\eta_2) B^{5/6}}{\boldsymbol{\eta}^{(1/2, 0, 1/3, 2/3, 1/6)}} \\ &\ll \sum_{\eta_1, \eta_2, \eta_3, \eta_4} \frac{\phi^\dagger(\eta_2) B}{\boldsymbol{\eta}^{(1, 1, 1, 1, 0)}} \\ &\ll B(\log B)^4, \end{aligned}$$

by (3.1), which therefore shows that

$$N_{U,H}(B) = \sum_{\boldsymbol{\eta} \in \mathcal{E}^*(B)} \frac{Y_2 Y_6 Y_7}{\eta_5} \vartheta(\boldsymbol{\eta}) g_2(Y_0; \boldsymbol{\eta}; B) + O(B(\log B)^{5-2/7}).$$

It remains to deal with the real-valued function  $g_2(Y_0; \boldsymbol{\eta}; B)$ .

We will show that

$$M_2(B) := \sum_{\boldsymbol{\eta} \in \mathcal{E}^*(B)} \frac{Y_2 Y_6 Y_7}{\eta_5} \vartheta(\boldsymbol{\eta}) \int_{\substack{h(Y_0, t_2, t_6, t_7) \leq 1 \\ |Y_7 t_7| \leq 1, t_6 > 0}} dt_2 dt_6 dt_7 \ll B(\log B)^4.$$

Once achieved, this will suffice to complete the proof of the lemma, since an application of Lemma 4 reveals that

$$\int_{h(Y_0, t_2, t_6, t_7) \leq 1, t_6 > 0} dt_2 dt_6 dt_7 = \frac{\omega_\infty}{Y_0^2},$$

and clearly

$$\frac{Y_2 Y_6 Y_7}{Y_0^2 \eta_5} = \frac{B}{\boldsymbol{\eta}^{(1,1,1,1,1)}}.$$

To establish the bound for  $M_2(B)$  we appeal to Lemma 5(2), which in a similar manner to our treatment of  $M_1(B)$ , implies that

$$\begin{aligned} M_2(B) &\ll \sum_{\boldsymbol{\eta} \in \mathcal{E}^*(B)} \frac{Y_2 Y_6 Y_7 \phi^\dagger(\eta_2)}{\eta_5} \int_{|t_7| \leq 1/Y_7} \frac{dt_7}{Y_0^8} \ll \sum_{\boldsymbol{\eta} \in \mathcal{E}^*(B)} \frac{Y_2 Y_6 \phi^\dagger(\eta_2)}{Y_0^8 \eta_5} \\ &= \sum_{\boldsymbol{\eta} \in \mathcal{E}^*(B)} \frac{\phi^\dagger(\eta_2) B^2}{\boldsymbol{\eta}^{(4,7,5,3,6)}} \\ &\ll \sum_{\eta_1, \eta_2, \eta_3, \eta_4} \frac{\phi^\dagger(\eta_2) B}{\boldsymbol{\eta}^{(1,1,1,1,0)}} \\ &\ll B(\log B)^4. \end{aligned}$$

This completes the proof of the lemma.  $\square$

Let us redefine the function  $\vartheta(\boldsymbol{\eta})$  so that it is equal to zero if  $\boldsymbol{\eta}$  fails to satisfy the coprimality relations in (4.8). For  $\mathbf{k} = (k_1, \dots, k_5) \in \mathbb{Z}_{>0}^5$ , let

$$\Delta_{\mathbf{k}}(n) := \sum_{\substack{\boldsymbol{\eta} \in \mathbb{Z}_{>0}^5 \\ \boldsymbol{\eta}^{(k_1, k_2, k_3, k_4, k_5)} = n}} \frac{\vartheta(\boldsymbol{\eta})}{\boldsymbol{\eta}^{(1,1,1,1,1)}}.$$

Then Lemma 12 implies that

$$N_{U,H}(B) = \omega_\infty B \sum_{n \leq B} (\Delta_{(2,4,3,2,3)}(n) - \Delta_{(3,6,4,2,5)}(n)) + O(B(\log B)^{5-2/7}). \quad (5.22)$$

We will want to establish an asymptotic formula for

$$M_{\mathbf{k}}(t) := \sum_{n \leq t} \Delta_{\mathbf{k}}(n),$$

as  $t \rightarrow \infty$ . We shall do so by studying the corresponding Dirichlet series

$$F_{\mathbf{k}}(s) = \sum_{n=1}^{\infty} \frac{\Delta_{\mathbf{k}}(n)}{n^s} = \sum_{\boldsymbol{\eta} \in \mathbb{Z}_{>0}^5} \frac{\vartheta(\boldsymbol{\eta})}{\boldsymbol{\eta}^{(k_1 s+1, k_2 s+1, k_3 s+1, k_4 s+1, k_5 s+1)}},$$

which is absolutely convergent for  $\Re(s) > 0$ .

By multiplicativity we clearly have an Euler product  $F_{\mathbf{k}}(s) = \prod_p F_{\mathbf{k},p}(s)$ , and a cumbersome computation reveals that the local factors  $F_{\mathbf{k},p}(s)$  are equal to

$$\begin{aligned} &(1 - 1/p) \cdot \left( (1 + 1/p) + \frac{1 - 1/p}{p^{k_1 s+1} - 1} \right) \\ &+ \frac{1 - 1/p}{p^{k_2 s+1} - 1} \left( (1 - 2/p) + \frac{1 - 1/p}{p^{k_1 s+1} - 1} + \frac{1 - 1/p}{p^{k_3 s+1} - 1} + \frac{1 - 1/p}{p^{k_5 s+1} - 1} \right) \\ &+ \frac{(1 - 1/p)^2}{p^{k_3 s+1} - 1} \left( 1 + \frac{1}{p^{k_4 s+1} - 1} \right) + \frac{1 - 1/p}{p^{k_4 s+1} - 1} + \frac{1 - 1/p}{p^{k_5 s+1} - 1}. \end{aligned}$$

Let  $\varepsilon > 0$  and assume that  $\mathbf{k} \in \{(2, 4, 3, 2, 3), (3, 6, 4, 2, 5)\}$ . Then it follows that for all  $s \in \mathbb{C}$  belonging to the half-plane  $\Re(s) \geq -1/12 + \varepsilon$ , we have

$$F_{\mathbf{k},p}(s) \prod_{j=1}^5 \left(1 - \frac{1}{p^{k_j s + 1}}\right) = 1 + O_\varepsilon(p^{-1-\varepsilon}).$$

Thus, on defining

$$E_{\mathbf{k}}(s) := \prod_{j=1}^5 \zeta(k_j s + 1), \quad G_{\mathbf{k}}(s) := \frac{F_{\mathbf{k}}(s)}{E_{\mathbf{k}}(s)},$$

we may conclude that  $F_{\mathbf{k}}(s)$  has a meromorphic continuation to the half-plane  $\Re(s) \geq -1/12 + \varepsilon$ , with a pole of order 5 at  $s = 0$ . It will be useful to note that

$$G_{\mathbf{k}}(0) = \prod_p \left(1 - \frac{1}{p}\right)^6 \left(1 + \frac{6}{p} + \frac{1}{p^2}\right). \quad (5.23)$$

To estimate  $M_{\mathbf{k}}(t)$  we now have everything in place to apply the following standard Tauberian theorem, which is recorded in work of Chambert-Loir and Tschinkel [5, Appendice A].

**Lemma 13.** *Let  $\{c_n\}_{n \in \mathbb{Z}_{>0}}$  be a sequence of positive real numbers, and let  $f(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ . Assume that:*

- (1) *the series defining  $f(s)$  converges for  $\Re(s) > 0$ ;*
- (2) *it admits a meromorphic continuation to  $\Re(s) > -\delta$  for some  $\delta > 0$ , with a unique pole at  $s = 0$  of order  $b \in \mathbb{Z}_{>0}$ ;*
- (3) *there exists  $\kappa > 0$  such that*

$$\left| \frac{f(s)s^b}{(s+2\delta)^b} \right| \ll (1 + \Im(s))^\kappa,$$

for  $\Re(s) > -\delta$ .

Then there exists a monic polynomial  $P$  of degree  $b$ , and a constant  $\delta' > 0$  such that

$$\sum_{n \leq t} c_n = \frac{\Theta}{b!} P(\log t) + O(t^{-\delta'}),$$

as  $t \rightarrow \infty$ , where  $\Theta = \lim_{s \rightarrow 0} s^b f(s)$ .

In fact [5, Appendice A] deals only with Dirichlet series possessing a unique pole at  $s = a > 0$ , but the extension to a pole at  $s = 0$  is straightforward. We apply Lemma 13 to estimate  $M_{\mathbf{k}}(t)$ , for

$$\mathbf{k} \in \{(2, 4, 3, 2, 3), (3, 6, 4, 2, 5)\}.$$

We have already seen that the corresponding Dirichlet series  $F_{\mathbf{k}}(s)$  satisfies parts (1) and (2) of the lemma, with  $b = 5$ . The third part follows from the boundedness of  $G_{\mathbf{k}}(s)$  on the half-plane  $\Re(s) \geq -1/12 + \varepsilon$ , and standard upper bounds for the size of the Riemann zeta function in the critical strip. In view of the fact that

$$\lim_{s \rightarrow 0} s^b F(s) = \frac{G_{\mathbf{k}}(0)}{\prod_{j=1}^5 k_j},$$

we therefore conclude that

$$M_{\mathbf{k}}(t) = \frac{G_{\mathbf{k}}(0)P(\log t)}{5! \cdot \prod_{j=1}^5 k_j} + O(t^{-\delta}), \quad (5.24)$$

for some  $\delta > 0$  and some monic polynomial  $P$  of degree 5.

We are now ready to complete the proof of our theorem. Recall the definition (2.1) of  $\omega_H(\tilde{S})$ . It therefore follows on combining (1.8), (5.22), (5.23) and (5.24) that

$$N_{U,H}(B) = \alpha(\tilde{S})\omega_H(\tilde{S})B(\log B)^5 + O(B(\log B)^{5-2/7}),$$

as required.

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