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Nekhoroshev theorem for the periodic Toda lattice

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The periodic Toda lattice with N sites is globally symplectomorphic to a two parameter family of $N-1$ coupled harmonic oscillators. The action variables fill out the whole positive quadrant of $\mathbb{R}^{N-1}$. We prove that in the interior of the positive quadrant as well as in a neighborhood of the origin, the Toda Hamiltonian is strictly convex and therefore Nekhoroshev's theorem applies on (almost) all parts of phase space (2000 Mathematics Subject Classification: 37J35, 37J40, 70H06). © *2009 American Institute of Physics*. [DOI: 10.1063/1.3196783]

In this paper we consider Hamiltonian perturbations of the periodic Toda lattice. The Toda lattice is a dynamical system introduced by Morikazu Toda which describes the dynamics of a one-dimensional chain of particles with a nearest neighbor interaction governed by an exponential potential. It is a Fermi–Pasta–Ulam (FPU) lattice with the special property of being completely integrable. In view of the numerical experiments for FPU lattices by FPU exhibiting unexpected recurrence phenomena, experts in the field conjectured that results of the theory of perturbations of integrable systems such as the Kolmogorov–Arnold–Moser (KAM) or Nekhoroshev theorem might be applicable to these systems. The latter theorem says that under appropriate conditions such as convexity of the unperturbed Hamiltonian, the action variables slowly evolve for an exponentially long time interval under sufficiently small Hamiltonian perturbations. In practice the conditions of the Nekhoroshev theorem are difficult to verify as they are expressed in terms of the action variables and are *not* invariant under canonical transformations. In this paper we prove a convexity result of the periodic Toda lattice. First we explicitly compute the Birkhoff normal form of the Toda lattice near the elliptic fixed point and show that the Hessian of the Hamiltonian, when expressed in action variables, is positive definite at the fixed point. In a second step we show that the frequency map is nondegenerate on the open positive quadrant of the action variables. To this end we represent the frequencies of the Toda lattice as periods of an Abelian differential of the Riemann surface associated with an invariant torus of the periodic Toda lattice.

I. INTRODUCTION

Consider the periodic Toda lattice with period $N(N\geq 2)$,

$$\dot{q}_n = \partial_{p_n} H_{\text{Toda}}, \quad \dot{p}_n = -\partial_{q_n} H_{\text{Toda}}, \quad n \in \mathbb{Z}$$

where the (real) coordinates $(q_n,p_n)_{n\in\mathbb{Z}}$ satisfy $(q_{n+N},p_{n+N}) = (q_n,p_n)$ for any $n\in\mathbb{Z}$ and the Hamiltonian H_{Toda} is given by

$$H_{\text{Toda}} = \frac{1}{2}\sum_{n=1}^{N} p_n^2 + \sum_{n=1}^{N} V(q_n - q_{n+1})$$

with potential

$$V(x) = \gamma^2 e^{\delta x} + V_1 x + V_2 \tag{1}$$

and constants γ, δ, V_1, and $V_2 \in \mathbb{R}$ $(\gamma,\delta\neq 0)$. The Toda lattice has been introduced by Toda[22] and studied extensively in the sequel. It is a FPU lattice, i.e., a Hamiltonian system of particles in one space dimension with nearest neighbor interaction. Models of this type have been studied by FPU. In numerical experiments they found recurrent features for the lattices they considered. Despite an enormous effort from the physics and mathematics community in the last fifty years, by and large, these numerical experiments still defy an explanation. For a recent account of the fascinating history of the FPU problem, see, e.g., Ref. 1 or Ref. 4. At least in the case of the periodic Toda lattice, the recurrent features can be fully accounted for. In fact, Flaschka,[3] Hénon,[6] and Manakov[14] independently proved that the periodic Toda lattice is integrable. In this paper, we show that on the open dense subset of the phase space where all action variables are strictly positive, the Nekhoroshev theorem[17,18] applies. It means that the action variables of the Toda lattice vary slowly over an exponentially long time interval along solutions of a Hamiltonian system with Hamiltonian sufficiently close to H_{Toda}.

To continue, let us note that in Eq. (1), without loss of generality, we can assume that $V_1=V_2=0$. When expressed in the canonical coordinates $(\delta q_j,(1/\delta)p_j)_{1\leq j\leq N}$, the Hamiltonian H_{Toda} is up to a scaling factor δ^{-2} of the form

$$H_{\text{Toda}} = \frac{1}{2}\sum_{n=1}^{N} p_n^2 + \alpha^2 \sum_{n=1}^{N} e^{q_n - q_{n+1}}, \tag{2}$$

where $a=|\gamma\delta|$. Moreover, notice that the total momentum $\Sigma_{n=1}^{N} p_n$ is conserved. Hence the motion of the center of mass $(1/N)\Sigma_{n=1}^{N} q_n$ is linear and therefore unbounded. However, the orbits of the system relative to the center of mass all lie on tori. To describe these orbits, consider the relative coordinates $v_n := q_{n+1}-q_n (1\leq n\leq N-1)$ and their canonically conjugate ones, $u_n := n\beta - \Sigma_{j=1}^{n} p_k (1\leq n\leq N-1)$, where

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$\beta=(1/N)\Sigma_{j=1}^{N}p_n$. In the sequel, we view the Toda lattice as a two parameter family of integrable systems with the two parameters $\alpha>0$ and $\beta\in\mathbb{R}$. For $\alpha>0$ and $\beta\in\mathbb{R}$ arbitrary, denote by $H_{\beta,\alpha}$ the Toda Hamiltonian when expressed in the canonical coordinates $(v_k,u_k)_{1\le k\le N-1}\in\mathbb{R}^{2N-2}$ and the parameters α and β. In Ref. 8 we proved the following result.

Theorem 1.1: *The periodic Toda lattice admits Birkhoff coordinates. More precisely, there exist globally defined canonical coordinates* $(x_k,y_k)_{1\le k\le N-1}\in\mathbb{R}^{2N-2}$ so *that for any* $\beta\in\mathbb{R}$ *and* $\alpha>0$, *the Toda Hamiltonian* $H_{\beta,\alpha}$, *when expressed in these coordinates, takes the form* $N\beta^2/2+H_\alpha(I)$, *where* $H_\alpha(I)$ *is a real analytic function of the action variables* $I_k=(x_k^2+y_k^2)/2(1\le k\le N-1)$ *alone.*

In particular, Theorem 1.1 states that the action variables $(I_n)_{1\le n\le N-1}$ are independent of $\beta\in\mathbb{R}$ and $\alpha>0$. Note that each of the $N-1$ frequencies

$$\omega_i=\partial_{I_i}\left(\frac{N\beta^2}{2}+H_\alpha(I)\right)=\partial_{I_i}H_\alpha(I)$$

of the Toda lattice $H_{\beta,\alpha}$ is independent of the parameter β.

The main result of this paper says that the Hamiltonian H_α, introduced in Theorem 1.1, is a convex function of the action variables $(I_k)_{1\le k\le N-1}$.

Theorem 1.2: *In the open quadrant* $\mathbb{R}^{N-1}_{>0}$, *the Hamiltonian* H_α *is a strictly convex function of the action variables* $(I_k)_{1\le k\le N-1}$. *More precisely, for any compact subset* $U\subseteq\mathbb{R}^{N-1}_{>0}$ *and any compact interval* $[\alpha_1,\alpha_2]\subseteq\mathbb{R}_{>0}$, *there exists* $m>0$, *such that*

$$\langle\partial_I^2H_\alpha(I)\xi,\xi\rangle\ge m\|\xi\|^2,\quad\forall\,\xi\in\mathbb{R}^{N-1}\tag{3}$$

for any $I\in U$, *any* $\beta\in\mathbb{R}$, *and any* $\alpha_1\le\alpha\le\alpha_2$.

Theorem 1.2 implies that Nekhoroshev's theorem holds for the Toda lattice on

$$\mathcal{P}^\bullet:=\{(v,u)\in\mathbb{R}^{2N-2}|I_n(v,u)>0\;\forall\;1\le n\le N-1\},$$

an open and dense subset of $\mathbb{R}^{2N-2}$ by Theorem 1.1. In Corollary 2.3 below we characterize the set $\mathcal{P}^\bullet$ in terms of the coordinates used for the Lax pair formulation of the periodic Toda lattice.

Corollary 1.3: *For any* $\beta\in\mathbb{R}$ *and* $\alpha>0$, *Nekhoroshev's theorem applies to (sufficiently small) Hamiltonian perturbations of the Toda Hamiltonian* $H_{\beta,\alpha}$ *on all of* $\mathcal{P}^\bullet$. *(See Refs.* 12, 13, and 17–20 *for various versions of Nekhoroshev's theorem and their proofs.)*

Note that the exponents of stability in the Nekhoroshev theorem can be made independent of the number of particles for orbits in neighborhoods of resonant tori (see, e.g., Ref. 19).

In practice, it is difficult to verify for an integrable system with a given Hamiltonian H whether the convexity (or steepness) condition of Nekhoroshev's theorem is satisfied as this condition refers to H, when expressed in action variables, and is not invariant under canonical transformations. Typically one does not know the Hamiltonian as a function of the action variables explicitly enough to derive the convexity property.

To prove Theorem 1.2, we make use of the Birkhoff normal form of the Toda lattice $H_{\beta,\alpha}$ on $\mathbb{R}^{2N-2}$ near the elliptic fixed point $(v,u)=(0,0)$ established in Ref. 9.

Theorem 1.4: *Let* $\alpha>0$ *be arbitrary. Near* $I=0$, *the function* $H_\alpha(I)$ *introduced in Theorem 1.1 has an expansion of the form*

$$N\alpha^2+2\alpha\sum_{k=1}^{N-1}s_kI_k+\frac{1}{4N}\sum_{k=1}^{N-1}I_k^2+O(I^3)\tag{4}$$

with $s_k=\sin(k\pi/N)$ for $1\le k\le N-1$. *In particular, the Hessian of* $H_\alpha(I)$ *at* $I=0$ *is given by*

$$\partial_I^2H_\alpha|_{I=0}=\frac{1}{2N}Id_{N-1}.$$

As we have pointed out in Ref. 9 Theorem 1.1 and Theorem 1.4 allow to apply the KAM theorem (see, e.g., Ref. 10) to the periodic Toda lattice on an open dense subset of the phase space. Actually a stronger result holds in the case at hand. In the remarkable paper, Morbidelli and Giorgilli[16] show that the convexity property of Theorem 1.2 implies that tori near a KAM torus exhibit superexponential stability. Finally we remark that Nekhoroshev theory is not the only application of convexity in nearly integrable dynamics. As an example where convexity of the integrable Hamiltonian plays a role we mention results of Herman,[23] Sec. V, on the geometry of invariant Lagrangian tori for small perturbations.

As an immediate consequence of Theorem 1.4 we get

Corollary 1.5: *Near* $I=0$, $H_\alpha(I)$ *is strictly convex for any* $\alpha>0$.

Outside of $I=0$, we argue differently. For any $\alpha>0$, consider the frequency map

$$\mathbb{R}^{N-1}_{\ge0}\to\mathbb{R}^{N-1},\;I\mapsto\omega(I;\alpha):=\partial_IH_\alpha.$$

In view of Corollary 1.5, Theorem 1.2 follows once we can show that the frequency map is *nondegenerate* on all of $\mathbb{R}^{N-1}_{>0}$. Note that the property of being nondegenerate is *invariant* under coordinate transformations, an observation used in a crucial way in the sequel. In Sec. IV, we prove that on $\mathcal{P}^\bullet$, the frequencies can be expressed in terms of periods of a certain Abelian differential of the second kind. To show that the frequency map is nondegenerate on $\mathcal{P}^\bullet$ we use, in addition to Theorem 1.1, a version of Krichever's theorem (Theorem 3.3) suited for applications to the Toda frequencies. In Ref. 11 Krichever stated his result concerning the period map of certain Abelian differentials for Hill's curve. Bikbaev and Kuksin presented a proof of this result in Ref. 2. In Sec. III we apply their scheme of proof to prove the version of Krichever's theorem needed for our purposes. In Sec. V, we prove Theorem 1.2.

II. PRELIMINARIES

To prove the integrability of the Toda lattice, Flaschka introduced the (noncanonical) coordinates (cf. Ref. 3)

$$b_n:=-p_n\in\mathbb{R},\quad a_n:=\alpha e^{1/2(q_n-q_{n+1})}\in\mathbb{R}_{>0}\quad(n\in\mathbb{Z}).$$

These coordinates describe the motion of the Toda lattice relative to the center of mass. They are related to the relative coordinates defined in Sec. I as follows:

$$((u_n,v_n)_{1\le n\le N-1},\beta,\alpha)\mapsto(b_n,a_n)_{1\le n\le N}$$

with $a_n=\alpha\exp(-\frac{1}{2}v_n)(1\le n\le N-1)$, $a_N=\alpha\exp(\frac{1}{2}\Sigma_{k=1}^{N-1}v_k)$, $b_1=u_1-\beta$, $b_n=u_n-u_{n-1}-\beta(2\le n\le N-1)$, and $b_N=-u_{N-1}-\beta$. In the sequel we will work with the coordinates $(b_n,a_n)_{1\le n\le N}$ rather than the relative coordinates $(u_n,v_n)_{1\le n\le N-1}$. In these coordinates, the Hamiltonian H_{Toda} takes the simple form

$$H_{\text{Toda}}=\frac{1}{2}\sum_{n=1}^{N}b_n^2+\sum_{n=1}^{N}a_n^2, \tag{5}$$

and the equations of motion are

$$\dot{b}_n=a_n^2-a_{n-1}^2\quad(n\in\mathbb{Z})$$
$$\dot{a}_n=\tfrac{1}{2}a_n(b_{n+1}-b_n)\quad(n\in\mathbb{Z}). \tag{6}$$

Note that $(b_{n+N},a_{n+N})=(b_n,a_n)$ for any $n\in\mathbb{Z}$ and $\Pi_{n=1}^N a_n=\alpha^N$. Hence we can identify the sequences $(b_n)_{n\in\mathbb{Z}}$ and $(a_n)_{n\in\mathbb{Z}}$ with the vectors $(b_n)_{1\le n\le N}\in\mathbb{R}^N$ and $(a_n)_{1\le n\le N}\in\mathbb{R}^N_{>0}$. The phase space of the system (6) is then given by

$$\mathcal{M}:=\mathbb{R}^N\times\mathbb{R}^N_{>0},$$

and it turns out that Eq. (6) is a Hamiltonian system with a nonstandard Poisson structure J found by Flaschka[3] (cf. Ref. 7). This Poisson structure is degenerate and admits the two Casimir functions (a smooth function $C:\mathcal{M}\to\mathbb{R}$ is a Casimir function for J if $dC(x)\in\ker J(x)$ for any $x\in\mathcal{M}$),

$$C_1:=-\frac{1}{N}\sum_{n=1}^{N}b_n\quad\text{and}\quad C_2:=\left(\prod_{n=1}^{N}a_n\right)^{1/N}. \tag{7}$$

Let $\mathcal{M}_{\beta,\alpha}:=\{(b,a)\in\mathcal{M}:(C_1,C_2)=(\beta,\alpha)\}$ denote the level set of (C_1,C_2) at $(\beta,\alpha)\in\mathbb{R}\times\mathbb{R}_{>0}$. As C_1 and C_2 are real analytic on $\mathcal{M}$ and the gradients $\nabla_{b,a}C_1$ and $\nabla_{b,a}C_2$ are linearly independent everywhere on $\mathcal{M}$, the sets $\mathcal{M}_{\beta,\alpha}$ are real analytic submanifolds of $\mathcal{M}$ of (real) codimension two. Furthermore, the pullback of the Poisson structure J to $\mathcal{M}_{\beta,\alpha}$ is nondegenerate everywhere on $\mathcal{M}_{\beta,\alpha}$ and therefore induces a symplectic structure on $\mathcal{M}_{\beta,\alpha}$. In this way, we obtain a symplectic foliation of $\mathcal{M}$ with $\mathcal{M}_{\beta,\alpha}$ being the symplectic leaves. By a slight abuse of notation with respect to the definition made in Sec. I, we denote by $H_{\beta,\alpha}$ the restriction of the Hamiltonian H_{Toda} to $\mathcal{M}_{\beta,\alpha}$.

As a model space for the construction of canonical Cartesian coordinates on $\mathcal{M}$, we introduced in Ref. 8 the space $\mathcal{P}:=\mathbb{R}^{2(N-1)}\times\mathbb{R}\times\mathbb{R}_{>0}$ foliated by the leaves $\mathcal{P}_{\beta,\alpha}:=\mathbb{R}^{2(N-1)}\times\{\beta\}\times\{\alpha\}$, which are endowed with the standard symplectic structure. Denote by J_0 the degenerate Poisson structure on $\mathcal{P}$ having $\mathcal{P}_{\beta,\alpha}$ as its symplectic leaves with standard symplectic structure and the coordinates β and α as its Casimir. In Ref. 8 we proved the following theorem which describes in more detail the results stated in Theorem 1.1:

Theorem 2.1: *There exists a map*

$$\Phi:\quad(\mathcal{M},J)\to(\mathcal{P},J_0),$$

$$(b,a)\mapsto((x_n,y_n)_{1\le n\le N-1},C_1,C_2)$$

with the following properties:

(i) *Φ is a real analytic diffeomorphism.*
(ii) *Φ is canonical, i.e., it preserves the Poisson brackets. In particular, the symplectic foliation of $\mathcal{M}$ by $\mathcal{M}_{\beta,\alpha}$ is trivial.*
(iii) *The coordinates $(x_n,y_n)_{1\le n\le N-1},C_1,C_2$ are global Birkhoff coordinates for the periodic Toda lattice, i.e., the transformed Toda Hamiltonian $\hat{H}=H\circ\Phi^{-1}$ is a function of the actions $I_n:=(x_n^2+y_n^2)/2$ $(1\le n\le N-1)$ and C_1,C_2 alone. It is of the form $N\beta^2/2+H_\alpha(I)$.*

As an immediate consequence of Theorem 2.1 one gets

Corollary 2.2: *For any $\beta\in\mathbb{R}$, $\alpha>0$, the set*

$$\mathcal{M}^\bullet_{\beta,\alpha}=\{(b,a)\in\mathcal{M}_{\beta,\alpha}|I_n(b,a)\rangle 0\ \forall\ 1\le n\le N-1\}$$

is open and dense in $\mathcal{M}_{\beta,\alpha}$.

For later use we describe an important ingredient in the proof of Theorem 2.1. For any $(b,a)\in\mathcal{M}$ denote by $L^+(b,a)$ and $L^-(b,a)$ the symmetric $N\times N$-matrices defined by

$$L^\pm(b,a):=\begin{pmatrix} b_1 & a_1 & 0 & \dots & \pm a_N\\ a_1 & b_2 & a_2 & \ddots & \vdots\\ 0 & a_2 & b_3 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & a_{N-1}\\ \pm a_N & \dots & 0 & a_{N-1} & b_N\end{pmatrix} \tag{8}$$

and by B the skew-symmetric $N\times N$-matrix

$$B=\begin{pmatrix} 0 & a_1 & 0 & \dots & -a_N\\ -a_1 & 0 & a_2 & \ddots & \vdots\\ 0 & -a_2 & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & a_{N-1}\\ a_N & \dots & 0 & -a_{N-1} & 0\end{pmatrix}.$$

Flaschka[3] observed that system (6) admits the Lax pair formulation

$$\dot{L}^+=\partial_t L^+=[B,L^+].$$

As the flow of $\dot{L}^+=[B,L^+]$ is isospectral, the eigenvalues of L^+ are conserved quantities of the Toda lattice. We need some results about the spectral theory of periodic Jacobi matrices (8). For $(b,a)\in\mathcal{M}$, consider for any complex number λ the difference equation

$$a_{k-1}y(k-1)+b_ky(k)+a_ky(k+1)=\lambda y(k)\quad(k\in\mathbb{Z}) \tag{9}$$

associated with $L(b,a)$. The two fundamental solutions $y_1(\cdot,\lambda)$ and $y_2(\cdot,\lambda)$ of Eq. (9) are defined by the standard initial conditions $y_1(0,\lambda)=1$, $y_1(1,\lambda)=0$ and $y_2(0,\lambda)=0$, $y_2(1,\lambda)=1$. By solving Eq. (9) recursively, one sees that for any k, $y_i(k,\lambda)$ $(i=1,2)$ is a polynomial in λ. Denote by $\Delta(\lambda)\equiv\Delta(\lambda,b,a)$ the *discriminant* of Eq. (9) defined by

$$\Delta(\lambda):=y_1(N,\lambda)+y_2(N+1,\lambda). \tag{10}$$

By Floquet theory, for any $\lambda\in\mathbb{R}$, Eq. (9) admits a periodic or antiperiodic solution of period N if the discriminant $\Delta_\lambda\equiv\Delta(\lambda)$ satisfy $\Delta_\lambda=2$ or $\Delta_\lambda=-2$, respectively. It follows that $\Delta_\lambda\mp 2$ admits a product representation of the form

$$\Delta_\lambda \mp 2 = \alpha^{-N}\prod_{j=1}^{N}(\lambda - \lambda_j^{\pm}),$$

where $(\lambda_j^{\pm})_{1\le j\le N}$ are the eigenvalues of $L^{\pm}(b,a)$. They are real valued and we list them in increasing order and with their algebraic multiplicities. Hence

$$\Delta_\lambda^2 - 4 = \alpha^{-2N}\prod_{j=1}^{2N}(\lambda - \lambda_j), \tag{11}$$

where $(\lambda_j)_{1\le j\le 2N}$ is the combined sequence of the eigenvalues $(\lambda_j^{+})_{1\le j\le N}$ and $(\lambda_j^{-})_{1\le j\le N}$ listed in increasing order. One can show that

$$\lambda_N^+ > \lambda_N^- \ge \lambda_{N-1}^- > \lambda_{N-1}^+ \ge \lambda_{N-2}^+ > \lambda_{N-2}^- \ge \lambda_{N-3}^- > \lambda_{N-3}^+ \ge \cdots . \tag{12}$$

Again by Floquet theory, one sees that $(\lambda_j)_{1\le j\le 2N}$ are the eigenvalues of the $2N\times 2N$ Jacobi matrix $L^+((b,b),(a,a))$. Since Δ_λ is a polynomial of degree N, $\dot{\Delta}_\lambda = \partial_\lambda \Delta_\lambda$ is a polynomial of degree $N-1$. It admits a product representation of the form

$$\dot{\Delta}_\lambda = N\alpha^{-N}\prod_{k=1}^{N-1}(\lambda - \dot{\lambda}_k), \tag{13}$$

where the zeroes $(\dot{\lambda}_n)_{1\le n\le N-1}$ of $\dot{\Delta}_\lambda$ are all real valued and are listed in increasing order. They satisfy $\lambda_{2n}\le\dot{\lambda}_n\le\lambda_{2n+1}$ for any $1\le n\le N-1$. The open intervals $(\lambda_{2n},\lambda_{2n+1})$ are referred to as the nth *spectral gap* and $\gamma_n := \lambda_{2n+1}-\lambda_{2n}$ as the nth *gap length*. Note that $|\Delta_\lambda|>2$ on the spectral gaps. We say that the nth gap is *open* if $\gamma_n>0$ and *collapsed* otherwise. The set of elements $(b,a)\in\mathcal{M}$ for which the nth gap is collapsed is denoted by D_n,

$$D_n := \{(b,a)\in\mathcal{M} : \gamma_n = 0\}. \tag{14}$$

Using that γ_n^2 (unlike γ_n) is a real analytic function on $\mathcal{M}$, it can be shown that D_n is a real analytic submanifold of $\mathcal{M}$ of codimension 2 (cf. Ref. 10 for a similar statement in the case of Hill's operator). Moreover, one can show that for any $(b,a)\in\mathcal{M}$ and any $1\le n\le N-1$, $\gamma_n(b,a)=0$ if $I_n(b,a)=0$; see Ref. 7 for details.

Corollary 2.3: *For any $\beta\in\mathbb{R}$, $\alpha>0$, the set $\mathcal{M}^{\bullet}_{\beta,\alpha}$ satisfies*

$$\mathcal{M}^{\bullet}_{\beta,\alpha} = \mathcal{M}_{\beta,\alpha}\setminus \cup_{n=1}^{N-1} D_n = \{(b,a)\in\mathcal{M} \,|\, \gamma_n(b,a)\rangle 0 \quad \forall\, 1\le n\le N-1\}.$$

Finally, we remark that the zeros $(\lambda_j)_{1\le j\le 2N}$ and $(\dot{\lambda}_k)_{1\le k\le N-1}$ of Δ_λ^2-4 and $\dot{\Delta}_\lambda$, respectively, satisfy the following relation:

$$\sum_{k=1}^{N-1}\dot{\lambda}_k = \frac{N-1}{2N}\sum_{j=1}^{2N}\lambda_j. \tag{15}$$

To prove Eq. (15), one computes the λ-derivative of

$$\Delta_\lambda = \pm 2 + \alpha^{-N}\prod_{j=1}^{N}(\lambda - \lambda_j^{\pm})$$

and compares the coefficients of the expansions of $\dot{\Delta}_\lambda$ obtained in this way with Eq. (13).

III. KRICHEVER'S THEOREM

In this section, we present a version of Krichever's theorem suited to prove Theorem 1.2. Krichever's theorem concerns the period map of certain meromorphic differentials of a hyperelliptic Riemann surface. In Ref. 11 Krichever stated his theorem for a parameter family of hyperelliptic curves having the property that one of the ramification points is at infinity. In the version we need we have to consider a parameter family of hyperelliptic curves with no ramification points at infinity.

Let $E=(E_1,\ldots,E_{2N})$ be a sequence of distinct but otherwise arbitrary real numbers which we list in increasing order, $E_1<E_2<\cdots<E_{2N-1}<E_{2N}$. Introduce

$$R(\lambda) = \prod_{i=1}^{2N}(\lambda - E_i), \quad \lambda\in\mathbb{C}$$

and denote by $\mathcal{C}_E$ the affine curve

$$\mathcal{C}_E = \{(\lambda,w)\in\mathbb{C}^2 : w^2 = \sigma R(\lambda)\},$$

where $\sigma\in\mathbb{R}_{>0}$ is a scaling parameter. In our application to the Toda lattice it will be given by α^{-2N}. Then $\mathcal{C}_E$ is a two-sheeted curve with ramification points $(E_i,0)_{1\le i\le 2N}$ identified with E_i in the sequel. By Σ_E we denote the Riemann surface obtained from $\mathcal{C}_E$ by adding the two (unramified) points at infinity, ∞^+ and ∞^-, one on each of the two sheets. The sheet of Σ_E which contains ∞^- is also referred to as the canonical sheet and denoted by Σ_E^c. It is characterized by

$$w = \sqrt{R(\lambda - i0)} < 0 \quad \forall\,\lambda\in\mathbb{R} \quad \text{with } \lambda > E_{2N}.$$

The variable z around $z=0$ gives a complex chart in a neighborhood of ∞^+ or ∞^- of Σ_E via the substitution $\lambda=1/z$. By construction, these charts at ∞^+ and ∞^- are defined in a unique way and are referred to as standard charts of $\infty^{\pm}$.

It is convenient to introduce the projection $\pi\equiv\pi_E:\mathcal{C}_E\to\mathbb{C}$ onto the λ-plane, i.e., $\pi_E(\lambda,w)=\lambda$ and its extension to a map $\pi_E:\Sigma_E\to\mathbb{C}\cup\{\infty\}$, where $\pi_E(\infty^{\pm})=\infty$. Denote by $(c_k)_{1\le k\le N-1}$ the cycles on the canonical sheet of $\mathcal{C}_E$ so that $\pi(c_k)$ is a counterclockwise oriented closed curve in $\mathbb{C}$, containing in its interior the two ramification points E_{2k} and E_{2k+1}, whereas all other ramification points are outside of $\pi(c_k)$. The following result is straightforward to prove:

Lemma 3.1: *There exist Abelian differentials Ω_1 and Ω_2 on Σ_E uniquely determined by the following properties:*

(i) *Ω_1 and Ω_2 are holomorphic on Σ_E* except at ∞^+ and ∞^- where in the standard charts, *Ω_i admit an expansion of the form*

$$\Omega_1 = \mp\left(-\frac{1}{z}+e_1+O(z)\right)\times dz\left(=\mp\left(\frac{1}{\lambda}-\frac{e_1}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right)\right)d\lambda\right)$$

and

$$\Omega_2 = \mp\left(-\frac{1}{z^2}+f_1+O(z)\right)dz\left(=\mp\left(1+O\left(\frac{1}{\lambda^2}\right)\right)d\lambda\right).$$

(ii) *Ω_1 and Ω_2 satisfy the normalization conditions*

$$\int_{c_k}\Omega_i=0 \quad \forall\, 1\le k\le N-1,\ \ i=1,2. \tag{16}$$

On $\mathcal{C}_E\backslash E$, Ω_1 and Ω_2 take the form $\Omega_i=(\chi_i(\lambda)/\sqrt{R(\lambda)})d\lambda$ $(i=1,2)$, where $\chi_i(\lambda)$ are polynomials in λ of the form $\chi_1(\lambda)=\lambda^{N-1}+e\lambda^{N-2}+\cdots$ and $\chi_2(\lambda)=\lambda^N+f\lambda^{N-1}+\cdots$ with $f=-\frac{1}{2}\sum_{n=1}^{2N}E_n$. In particular, Ω_1 and Ω_2 do not depend on the scaling parameter σ.

Denote by $(d_k)_{1\le k\le N-1}$ pairwise disjoint cycles on $\mathcal{C}_E\backslash E$ so that for any $1\le n,\ k\le N-1$, the intersection indices with the cycles $(c_n)_{1\le n\le N-1}$ with respect to the orientation on Σ_E, induced by the complex structure, are given by $c_n\circ d_k=\delta_{nk}$. In order to be more precise, choose the cycles d_k in such a way that (i) the projection $\pi_E(d_k)$ of d_k is a smooth, convex counterclockwise oriented curve in $\mathbb{C}\backslash((E_1,E_{2k})\cup(E_{2k+1},\infty))$ and (ii) the points of d_k whose projection by π_E onto the λ-plane has a negative imaginary part lie on the canonical sheet of Σ_E. For any $1\le k\le N-1$, introduce the d_k-periods of Ω_1 and Ω_2,

$$U_k := \int_{d_k}\Omega_1;\quad V_k := \int_{d_k}\Omega_2, \tag{17}$$

and for any $p\in\mathcal{C}_E$ define the Abel integrals $(i=1,2)$,

$$J_i(p)=\frac{1}{2}\int_{\gamma_p}\Omega_i,$$

where γ_p is any path in $\mathcal{C}_E$ from p_* to p. The map $\iota:\mathcal{C}_E\to\mathcal{C}_E$, $p\mapsto p_*$ interchanges the two sheets of $\mathcal{C}_E$,

$$p_*=(\lambda,-w) \quad \forall\, p=(\lambda,w)\in\mathcal{C}_E.$$

Note that for any $i=1,2$, the function $p\mapsto J_i(p)$ is multivalued. Actually, $J_i(p)$ is well defined up to half periods of Ω_i. Hence locally it is a well defined smooth function. In particular, its differential dJ_i is well defined. Note that for $i=1,2$ and $1\le n\le 2N$, zero is one of the possible values of $J_i(E_n)$. For any $p\in\mathcal{C}_E$, denote by γ_p^0 a path on $\mathcal{C}_E$ from $E_{2N}\equiv(E_{2N},0)$ to p and define γ_p to be the path from p_* to p obtained by concatenating $-\iota(\gamma_p^0)$ and γ_p^0. Here $-\iota(\gamma_p^0)$ denotes the path from p_* to E_{2N} obtained by reversing the orientation of $\iota(\gamma_p^0)$ and $\iota(\gamma_p^0)$ is the path obtained by applying to γ_p^0 the map ι. In Lemma 3.2 we state the properties of Ω_i and J_i needed in the sequel.

Lemma 3.2:

(i) *The differential forms Ω_1 and Ω_2 are odd with respect to the map ι, i.e., the pullback $\iota^*\Omega_i$ of Ω_i satisfies $\iota^*\Omega_i=-\Omega_i$.*

(ii) *For $i=1,2$,*

$$\frac{1}{2}\int_{-\iota(\gamma_P^0)\circ\gamma_P^0}\Omega_i=\int_{\gamma_P^0}\Omega_i.$$

(iii) *When expressed in the local coordinate λ, on each of the two sheets, $\int_{E_{2N}}^{\lambda}\Omega_i$ admits an asymptotic expansion as $\lambda\to\infty$ (λ real) of the form*

$$\int_{E_{2N}}^{\lambda}\Omega_1=\mp\left(\log\lambda+e_0+e_1\frac{1}{\lambda}+\cdots\right) \tag{18}$$

and

$$\int_{E_{2N}}^{\lambda}\Omega_2=\mp(\lambda+f_0+\cdots), \tag{19}$$

where e_0 and e_1 are real valued.

Proof:

(i) Let $1\le i\le 2$. The claimed identity $\iota^*\Omega_i=-\Omega_i$ follows from the uniqueness of the differential Ω_i stated in Lemma 3.1, as $-\iota^*\Omega_i$ is a meromorphic differential which is holomorphic on $\mathcal{C}_E$ and satisfies the same asymptotics at $\infty^\pm$ and the same normalization condition (16) as the differential Ω_i.

(ii) In view of statement (i) we conclude that for any $p\in\mathcal{C}_E$,

$$\frac{1}{2}\int_{-\iota(\gamma_P^0)\circ\gamma_P^0}\Omega_i=\frac{1}{2}\left(-\int_{\iota(\gamma_P^0)}\Omega_i+\int_{\gamma_P^0}\Omega_i\right)=\int_{\gamma_P^0}\Omega_i.$$

(iii) The stated asymptotics follow from the asymptotics of Ω_i of Lemma 3.1. The claim of e_0 and e_1 being real follows from the assumption that $E_1,\ldots,E_{2N}$ are real and that for λ real with $\lambda>E_{2N}$, one has $R(\lambda)>0$.

To state the main result of this section, introduce the extended period map defined on the space of sequences $E=(E_1<\cdots<E_{2N})$ as follows:

$$\mathcal{F}:E\mapsto((U_i,V_i)_{1\le i\le N-1},e_1,e_0), \tag{20}$$

where e_1 and e_0 are the coefficients in the asymptotic expansion (18). It is straightforward to see that $\mathcal{F}$ is a smooth map with values in $\mathbb{R}^{2N}$. The version of Krichever's theorem needed for our purposes is the following one:

Theorem 3.3: *At each point $E=(E_1<\cdots<E_{2N})$, the map $\mathcal{F}$ is a local diffeomorphism, i.e., the differential $d_E\mathcal{F}:\mathbb{R}^{2N}\to\mathbb{R}^{2N}$ of $\mathcal{F}$ at E is a linear isomorphism.*

The Proof of Theorem 3.3 follows the scheme used in Ref. 2 to prove Krichever's theorem. First we need to derive some auxiliary results. For any $1\le i\le 2$, denote by N_{Ω_i} the set of zeroes of Ω_i and by N_{χ_i} the set of zeroes of the polynomials χ_i, where in both cases the zeroes are listed with their multiplicities. Note that $|N_{\chi_1}|=N-1$ and $|N_{\chi_2}|=N$, whereas for $i=1,2$,

$$|N_{\Omega_i}| \leq 2|N_{\chi_i}|.$$

The following result is due to Ref. 2:

Lemma 3.4. *The zero sets N_{χ_i} and N_{Ω_i} have the following properties:*

(i) *All zeroes of χ_1 are simple and real and $N_{\chi_1} \cap \{E_1, \ldots, E_{2N}\} = \emptyset$. Moreover, $N_{\Omega_1} = \pi_E^{-1}(N_{\chi_1})$ and $|N_{\Omega_1}| = 2N-2$.*

(ii) *All zeroes of χ_2 are simple except possibly one which then has multiplicity two. Furthermore,*

$$|N_{\chi_2} \setminus \{E_1, \ldots, E_{2N}\}| \geq N-1$$

and

$$|N_{\Omega_2} \setminus \{E_1, \ldots, E_{2N}\}| \geq 2N-2.$$

(iii) *$N_{\chi_1} \cap N_{\chi_2} = \emptyset$, and hence $N_{\Omega_1} \cap N_{\Omega_2} = \emptyset$* as well.

Proof of Lemma 3.4: The statements about the zero sets N_{Ω_i} of Ω_i are easily obtained from the ones about N_{χ_i} in view of the representation $\Omega_i = \chi_i(\lambda)/\sqrt{R(\lambda)}d\lambda$ and the property that Ω_i has a pole at ∞^+ and ∞^-. Hence we prove only the claimed statements for N_{χ_i}.

By the normalization condition (16), for any $1 \leq k \leq N-1$, $\chi_1(\lambda)$ has at least one real zero $\tau_{1,k}$ satisfying $E_{2k} < \tau_{1,k} < E_{2k+1}$. As $\chi_1(\lambda)$ is a polynomial of degree $N-1$, it follows that all zeroes $\tau_{1,k}$ are simple and that

$$N_{\chi_1} = \{\tau_{1,k} | 1 \leq k \leq N-1\}.$$

In particular $N_{\chi_1} \cap \{E_1, \ldots, E_{2N}\} = \emptyset$. Similarly, Eq. (16) implies that for any $1 \leq k \leq N-1$, $\chi_2(\lambda)$ has at least one real zero $\tau_{2,k}$ satisfying $E_{2k} < \tau_{2,k} < E_{2k+1}$. As $\chi_2(\lambda)$ is a polynomial of degree N,

$$N_{\chi_2} \setminus \{\tau_{2,k} | 1 \leq k \leq N-1\}$$

consists of one point $\tau_0 \in \mathbb{C}$. It is not excluded that τ_0 coincides with one of the zeroes $(\tau_{2,k})_{1 \leq k \leq N-1}$. In any case, $|N_{\chi_2} \cap \{E_1, \ldots, E_{2N}\}| \leq 1$. It remains to prove (iii). Assume that τ is a common zero of $\chi_1(\lambda)$ and $\chi_2(\lambda)$, i.e., $\tau \in N_{\chi_1} \cap N_{\chi_2}$. Then there exists $1 \leq k \leq N-1$ with $E_{2k} < \tau < E_{2k+1}$. As all roots of $\chi_1(\lambda)$ are *simple*, one has $\chi_1'(\tau) \neq 0$ ($' = d/d\lambda$). Hence we can choose the real parameter ξ in such a way that the polynomial $\chi_2 + \xi\chi_1$ has a double root at τ. Indeed, for $\xi = -\chi_2'(\tau)/\chi_1'(\tau)$ one has $\chi_2(\tau) + \xi\chi_1(\tau) = 0$ and $\chi_2'(\tau) + \xi\chi_1'(\tau) = 0$. As

$$\int_{c_j} (\chi_2(\lambda) + \xi\chi_1(\lambda))/\sqrt{R(\lambda)}d\lambda = 0 \qquad \forall\, 1 \leq j \leq N-1,$$

the N roots of $\chi_2 + \xi\chi_1$ are given by τ and $(\tau_{\xi,j})_{j \neq k}$, where τ is a double root and for any $j \neq k$, $E_{2j} < \tau_{\xi,j} < E_{2j+1}$ is simple. Therefore, $\chi_2(\lambda) + \xi\chi_1(\lambda)$ does not change sign in the interval $[E_{2k}, E_{2k+1}]$, contradicting the normalization condition $\int_{c_k}(\chi_2(\lambda) + \xi\chi_1(\lambda))d\lambda = 0$. Hence χ_1 and χ_2 have no zero in common as claimed.

Proof of Theorem 3.3: Assume that Theorem 3.3 does not hold. Then there exists a smooth one-parameter family $E(\tau)$, $-1 < \tau < 1$, so that for some $1 \leq n \leq 2N$, $\delta E_n \equiv \partial_\tau|_{\tau=0} E_n(\tau) \neq 0$, but

$$U(\tau) = U(0) + O(\tau^2), \quad V(\tau) = V(0) + O(\tau^2),$$

$$e_0(\tau) = e_0(0) + O(\tau^2), \quad e_1(\tau) = e_1(0) + O(\tau^2).$$

We will now prove that $\delta E_k = 0$ for any $1 \leq k \leq 2N$, leading to a contradiction. As above, we introduce for $p \in \mathcal{C}_{E(\tau)}$ the multivalued functions $J_i(p,\tau)$ defined up to half periods of $\Omega_i(\tau)$,

$$J_i(p,\tau) = \frac{1}{2}\int_{\gamma_p} \Omega_i(\tau),$$

where $\Omega_i(\tau)$ denote the Abelian differentials of Lemma 3.1, corresponding to the Riemann surface $\Sigma_{E(\tau)} = \mathcal{C}_{E(\tau)} \cup \{\infty^+, \infty^-\}$. By Lemma 3.2 (ii), $J_i(p,\tau) = \int_{\gamma_p^0} \Omega_i(\tau)$. In particular, the differential $dJ_i(p,\tau)$ is well defined and equals the restriction of $\Omega_i(\tau)$ to $\mathcal{C}_{E(\tau)}$. Near any point $p = (\lambda, w) \in \mathcal{C}_E \setminus E$, λ is a local coordinate. This remains true for τ sufficiently close to 0, and hence for any $p \in \mathcal{C}_E \setminus E$ we can define $(i = 1,2)$,

$$\delta J_i(p) := \partial_\tau|_{\tau=0} J_i(p,\tau). \tag{21}$$

By Lemma 3.5 below, δJ_1 is single-valued, extends to a meromorphic function on Σ_E, and is holomorphic on $\Sigma_E \setminus E$. At a ramification point E_k, the function δJ_1 might have a simple pole with residue of the form $r_1(k)\delta E_k$, where $r_1(k) \neq 0$. But by Proposition 3.6 below, $\delta J_1 \equiv 0$ and hence, in particular, $\delta E_k = 0$ for any $1 \leq k \leq 2N$. This contradicts the assumption made above that $\delta E_n \neq 0$.

It remains to prove Lemma 3.5 and Proposition 3.6 mentioned in the Proof of Theorem 3.3. Throughout the rest of this section we assume that the one-parameter family $E(\tau)$ satisfies the assumption made in the Proof of Theorem 3.3.

Lemma 3.5: *The functions δJ_1 and δJ_2 defined by Eq. (21) are single valued and extend to meromorphic functions on Σ_E. They are holomorphic on $\Sigma_E \setminus E$. At the ramification points $(E_n)_{1 \leq n \leq 2N}$, they might have poles of order 1 with residue of the form $(i = 1,2; 1 \leq n \leq 2N)$,*

$$\text{Res}_{\,p=E_n} \delta J_i = r_i(n)\delta E_n,$$

where for $i = 1$, $r_1(n) \neq 0$ for any $1 \leq n \leq 2N$. Moreover δJ_1 has a zero of order 2 at $\infty^\pm$.

Proof: Let $1 \leq i \leq 2$ be given. Although the integral $J_i(p,\tau)$ defined for $p \in \mathcal{C}_E$ is multivalued in the sense that it is defined only up to half-periods of $\Omega_i(\tau)$, the derivative $\partial_\tau|_{\tau=0} J_i(p,\tau)$ is single valued, since by assumption, the periods of $\Omega_i(\tau)$ are constant up to $O(\tau^2)$. To simplify the notation we write Ω_i instead of $\Omega_i(\tau)$ and E_n instead of $E_n(\tau)$. To see that $\delta J_i(\tau)$ extends meromorphically to any branching point E_n, note that near E_n, Ω_i admits an expansion in terms of $z = (\lambda - E_n)^{1/2}$,

$$\begin{aligned}\Omega_i(z,\tau) &= (x_0^i(E_n,\tau) + x_1^i(E_n,\tau)z + \cdots)dz \\ &= \tfrac{1}{2}(x_0^i(E_n,\tau)(\lambda - E_n)^{-1/2} + x_1^i(E_n,\tau) + \cdots)d\lambda,\end{aligned}$$

where we used that $dz = (1/2z)d\lambda$. Since by item (i) of Lemma 3.4, $\Omega_1(E_n,\tau) \neq 0$, it follows that $x_0^1(E_n,\tau) \neq 0$. We now integrate $\Omega_i(\tau)$ to get that

$$\int_{E_n}^{z}\Omega_i(\tau)=\int_{E_n}^{\lambda}\left(\frac{1}{2}\frac{x_0^i(E_n,\tau)}{(\lambda-E_n)^{1/2}}+\frac{1}{2}x_1^i(E_n,\tau)+\cdots\right)d\lambda$$
$$=x_0^i(E_n,\tau)(\lambda-E_n)^{1/2}+\frac{1}{2}x_1^i(E_n,\tau)(\lambda-E_n)+\cdots$$

is a value of the multivalued function $J_i(p,\tau)$. Then the τ-derivative of $J_i(p,\tau)$ at $\tau=0$ satisfies

$$\delta J_i(p)=-\frac{x_0^i(E_n,0)}{2}\delta E_n(\lambda-E_n)^{-1/2}+O(\lambda-E_n)^0.$$

Hence δJ_i admits at E_n a Laurent expansion and therefore is meromorphic near E_n. At E_n, it might have a pole of order 1 with residue $r_i(n)\delta E_n$ and $r_i(n)=-\frac{1}{2}x_0^i(E_n,0)$. Moreover $r_1(n)\neq 0$ as $x_0^1(E_n,0)\neq 0$ by the observation above.

To see that δJ_i extends meromorphically to ∞^+ and ∞^-, use the expansions (18) and (19) to conclude that for $\lambda\to\infty$,

$$J_1(\lambda,\tau)=\mp\left(\log\lambda+e_0(\tau)+e_1(\tau)\frac{1}{\lambda}+\cdots\right)$$

and

$$J_2(\lambda,\tau)=\mp(\lambda+f_0(\tau)+\cdots).$$

Hence, δJ_1 and δJ_2 are holomorphic near $\infty^{\pm}$ and in view of the assumption that $\delta e_0=0$ and $\delta e_1=0$ it follows that $\delta J_1(\lambda)=O(1/\lambda^2)$, and hence δJ_1 has a zero of order 2 at $\infty^{\pm}$.

The most important ingredient for the Proof of Theorem 3.3 is the following:

Proposition 3.6: $\delta J_1\equiv 0$.

To prove Proposition 3.6 we first need to introduce an auxiliary function. For $p\in\mathcal{C}_E\setminus N_{\Omega_1}$, $dJ_1(p)=\Omega_1(p)\neq 0$. Hence by the implicit function theorem, there exists a smooth curve $\tau\mapsto q(\tau):=q(\tau,p)$ with $q(0)=p$ defined for τ sufficiently close to zero so that $J_1(q(\tau),\tau)=J_1(p)$. In particular, for $p=E_n$ one has $q(\tau)=E_n(\tau)$. Then introduce for $p\in\mathcal{C}_E\setminus N_{\Omega_1}$

$$\delta K(p):=\left.\frac{d}{d\tau}\right|_{\tau=0}J_2(q(\tau),\tau).$$

As the periods of Ω_2 are constant up to $O(\tau^2)$ and $J_2(p,\tau)$ is well defined up to half periods of Ω_2, δK is single valued. Moreover, δK admits a meromorphic extension to Σ_E. Indeed, as $J_1(q(\tau),\tau)=J_1(p)$, one has for any $p\in\mathcal{C}_E\setminus N_{\Omega_1}$,

$$\delta J_1(p)+\langle\Omega_1(p),\delta q\rangle=0,$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $T_p^*\Sigma_E$ and $T_p\Sigma_E$. Hence

$$\delta K(p)=\left.\frac{d}{d\tau}\right|_{\tau=0}J_2(q(\tau),\tau)=\delta J_2(p)+\langle\Omega_2(p),\delta q\rangle$$

leads to

$$\delta K(p)=\delta J_2(p)-\frac{\Omega_2(p)}{\Omega_1(p)}\delta J_1(p). \quad (22)$$

By Lemma 3.4 we know that $\Omega_2(p)/\Omega_1(p)$ extends to a meromorphic function on Σ_E with poles of order 1 at $\infty^{\pm}$ and possible poles at the zeroes of Ω_1. In view of Lemma 3.5, δK admits a meromorphic extension to Σ_E.

Lemma 3.7: $\delta K\equiv 0$.

Proof of Lemma 3.7: We show that when counted with their orders, the number of poles of δK does not match the number of zeroes. First note that $\delta K(E_n)=0$ for any $1\leq n\leq 2N$. Indeed, if $p=E_n$ for some $1\leq n\leq 2N$, one has $q(\tau)=E_n(\tau)$ and hence for $i=1,2$, $J_i(E_n(\tau),\tau)$ contains zero for any τ, implying that $\delta K(E_n)=0$. On the other hand, by Eq. (22), the poles of δK in $\mathcal{C}_E$ are contained in the set N_{Ω_1} of zeroes of Ω_1. By Lemma 3.4, all these zeroes are simple and $|N_{\Omega_1}|=2N-2$. Now let us investigate the values of δK at ∞^+ and ∞^-. Using the standard charts $z=1/\lambda$ we have by Lemma 3.1,

$$\frac{\Omega_2(z)}{\Omega_1(z)}=O\left(\frac{1}{z}\right),$$

and by Lemma 3.5, $\delta J_1(z)=O(z^2)$. Hence

$$\frac{\Omega_2(z)}{\Omega_1(z)}\delta J_1(z)=O(z).$$

It means that $(\Omega_2(z)/\Omega_1(z))\delta J_1(z)$ vanishes at ∞^+ and ∞^-. In addition, again by Lemma 3.5, δJ_2 is holomorphic at ∞^+ and ∞^-. Altogether we have shown that the meromorphic function δK has at least $2N$ zeroes and at most $2N-2$ poles (counted with their multiplicities). As Σ_E is a compact surface it follows that $\delta K\equiv 0$.

Proof of Proposition 3.6: By Lemma 3.7, formula (22) implies that

$$\delta J_1\cdot\Omega_2\equiv\delta J_2\cdot\Omega_1. \quad (23)$$

By comparing poles and zeroes of $\delta J_2\cdot\Omega_1$ and $\delta J_1\cdot\Omega_2$ we want to conclude that $\delta J_1\equiv 0$ (and hence $\delta J_2\equiv 0$ as well). Indeed, by Lemma 3.5, any pole of δJ_1 has to be a ramification point of Σ_E and is of order 1. By Lemma 3.4, at least $2N-2$ zeroes of Ω_2 are contained in $\mathcal{C}_E\setminus E$. We now have to distinguish between two cases. If $\Omega_2(E_n)\neq 0$ for any $1\leq n\leq 2N$, then Ω_2 has $2N$ zeroes and they are all contained in $\Sigma_E\setminus(E\cup\{\infty^+,\infty^-\})$. By Lemma 3.4, the zeroes of Ω_2 cannot be zeroes of Ω_1, and hence Eq. (23) implies that they must be zeroes of δJ_2. In addition, by Lemma 3.5, δJ_1 vanishes at $\infty^{\pm}$ of order 2, whereas Ω_2 has a pole of order 2. Hence $\delta J_1\cdot\Omega_2$ is holomorphic at $\infty^{\pm}$. By Eq. (23), $\delta J_2\cdot\Omega_1$ is then holomorphic at $\infty^{\pm}$. As Ω_1 has a pole of order 1 at $\infty^{\pm}$ it follows that δJ_2 vanishes at $\infty^{\pm}$. Altogether, δJ_2 has at least $2N+2$ zeroes on Σ_E. On the other hand, by Lemma 1.5, δJ_2 has at most $2N$ poles (all of them simple). As Σ_E is a compact Riemann surface, the meromorphic function δJ_2 vanishes identically, and hence by Eq. (23), δJ_1 as well.

It remains to consider the case where there exists $1\leq n\leq 2N$ so that $\Omega_2(E_n)=0$. By Lemma 3.5, δJ_1 is either holomorphic near E_n or has a pole of order 1. Hence $\delta J_1\cdot\Omega_2$ is holomorphic near E_n. By Eq. (23), $\delta J_2\cdot\Omega_1$ is then holomorphic at E_n as well. By Lemma 3.4, $\Omega_1(E_n)\neq 0$, hence δJ_2 is holomorphic near E_n. Again by Lemma 3.5, we then see that δJ_2 has at most $2N-1$ poles in Σ_E. On the other hand, in view of Lemma 3.4, δJ_2 has at least $2N-2$ zeroes in $\mathcal{C}_E\setminus E$. We have already seen that δJ_2 vanishes at ∞^+ and ∞^-. Hence

δJ_2 has at least $2N$ zeroes and at most $2N-1$ poles in Σ_E. As Σ_E is a compact Riemann surface, the meromorphic function δJ_2 vanishes identically, and so does δJ_1.

IV. FORMULAS FOR THE TODA FREQUENCIES

In this section we derive formulas for the frequencies of the periodic Toda lattice in terms of periods of the Abelian differential Ω_2 introduced in Sec. III. These formulas will be used in an essential way to show that the frequency map is nondegenerate on $\mathbb{R}_{>0}^{N-1}$.

Introduce $\mathcal{M}^\bullet=\cup_{\alpha>0,\beta\in\mathbb{R}}\mathcal{M}^\bullet_{\beta,\alpha}$. As pointed out at the end of Sec. II, $\mathcal{M}^\bullet=\mathcal{M}\setminus\cup_{n=1}^{N-1}D_n$, i.e., for any $(b,a)\in\mathcal{M}^\bullet$, all the roots $(\lambda_i)_{1\le i\le 2N}$ of $\Delta_\lambda^2(b,a)-4$ are simple. As above, we list these roots in increasing order, $\lambda_1<\lambda_2<\cdots<\lambda_{2N}$. By Corollary 2.2, $\mathcal{M}^\bullet$ is open and dense in $\mathcal{M}$. Given $(b,a)\in\mathcal{M}^\bullet$, denote by $\Sigma_{b,a}$ the Riemann surface Σ_E with $E=(\lambda_1<\cdots<\lambda_{2N})$ and scaling factor $\sigma=\alpha^{-2N}$, where $\alpha=(\Pi_{i=1}^N a_i)^{1/N}$. In view of the product representation (11) of $\Delta_\lambda^2(b,a)-4$, $\Sigma_{b,a}=\mathcal{C}_{b,a}\cup\{\infty^+,\infty^-\}$, where

$$\mathcal{C}_{b,a}:=\{(\lambda,z)\in\mathbb{C}^2:z^2=\Delta_\lambda^2(b,a)-4\}. \tag{24}$$

To obtain a formula for the differential Ω_2 we first consider an auxiliary differential.

Lemma 4.1: *Assume that $(b,a)\in\mathcal{M}^\bullet_{\beta,\alpha}$ with $\beta=0$. Then the differential*

$$\tilde{\Omega}_2=\frac{\lambda\dot{\Delta}_\lambda}{\sqrt{\Delta_\lambda^2-4}}d\lambda$$

is holomorphic on $\mathcal{C}_{b,a}$ and has an expansion of the form $(\mp N/z^2+O(1))dz$ at $\infty^\pm$ when expressed in the standard chart $z=1/\lambda$ of $\infty^\pm$.

The proof of Lemma 4.1 is straightforward. For convenience of the reader it is given in Appendix A.

The Abelian differential $\tilde{\Omega}_2$ has to be appropriately normalized. For this purpose introduce the *ψ -functions*. Let $(b,a)\in\mathcal{M}^\bullet$ and $1\le n\le N-1$. Then there exists a unique polynomial $\psi_n(\lambda)$ of degree at most $N-2$ such that for any $1\le k\le N-1$,

$$\frac{1}{2\pi}\int_{c_k}\frac{\psi_n(\lambda)}{\sqrt{\Delta_\lambda^2-4}}d\lambda=\delta_{kn}. \tag{25}$$

Here $(c_k)_{1\le k\le N-1}$ denote the cycles on the canonical sheet $\Sigma_{b,a}^c$ of $\Sigma_{b,a}$ introduced at the beginning of Sec. III. For any $k\ne n$ it follows from Eq. (25) that

$$\frac{1}{\pi}\int_{\lambda_{2k}}^{\lambda_{2k+1}}\frac{\psi_n(\lambda)}{\sqrt[+]{\Delta_\lambda^2-4}}d\lambda=0. \tag{26}$$

As $(b,a)\in\mathcal{M}^\bullet$, $\gamma_k=\lambda_{2k+1}-\lambda_{2k}>0$ for any $1\le k\le N-1$, and hence in every gap $(\lambda_{2k},\lambda_{2k+1})$ with $k\ne n$ the polynomial ψ_n has a zero which we denote by σ_k^n. As $\psi_n(\lambda)$ is a polynomial of degree at most $N-2$, one has

$$\psi_n(\lambda)=M_n\prod_{\substack{1\le k\le N-1\\ k\ne n}}(\lambda-\sigma_k^n), \tag{27}$$

where $M_n\equiv M_n(b,a)\ne 0$. Clearly, the differential forms $(1\le n\le N-1)$

$$\zeta_n=\frac{\psi_n(\lambda)}{\sqrt{\Delta_\lambda^2-4}}d\lambda \tag{28}$$

are holomorphic on $\Sigma_{b,a}\setminus\{\infty^+,\infty^-\}$. As the ψ_n are polynomials in λ of degree at most $N-2$, they are also holomorphic at ∞^+ and ∞^-. Further, the action variables $I_n=I_n(b,a)$, $1\le n\le N-1$, introduced in Theorem 2.1 for any $(b,a)\in\mathcal{M}$, are given by

$$I_n=\frac{1}{2\pi}\int_{c_n}\lambda\frac{\dot{\Delta}_\lambda}{\sqrt{\Delta_\lambda^2-4}}d\lambda. \tag{29}$$

They can be interpreted as period integrals of $\tilde{\Omega}_2$,

$$I_n=\frac{1}{2\pi}\int_{c_n}\tilde{\Omega}_2\quad(1\le n\le N-1). \tag{30}$$

Now introduce the meromorphic differential

$$\Omega:=\tilde{\Omega}_2-\sum_{n=1}^{N-1}I_n\zeta_n$$

with $(\zeta_n)_{1\le n\le N-1}$ as given by Eq. (28).

Lemma 4.2: *Assume that $(b,a)\in\mathcal{M}^\bullet_{\beta,\alpha}$* with $\beta=0$ *but $\alpha>0$ arbitrary. Then the meromorphic differentials Ω_2 and Ω are related by $\Omega=-N\Omega_2$.*

Proof: In view of the uniqueness statement of Lemma 3.1 it suffices to show that Ω is a meromorphic differential so that (i) Ω is holomorphic on $\Sigma_{b,a}\setminus\{\infty^+,\infty^-\}$; (ii) when expressed in the standard chart $\lambda=1/z$ near ∞^+ and ∞^-, Ω has a Laurent expansion of the form $\Omega=(\mp N/z^2+O(1))dz$; (iii) $\int_{c_k}\Omega=0$ for any $1\le k\le N-1$.

Statements (i) and (ii) follow from Lemma 4.1 and the above mentioned fact that the ζ_n are holomorphic differentials on $\Sigma_{b,a}$. To see that the normalization conditions are satisfied, we use the identity (30) and the normalization conditions (25) to conclude that for any $1\le k\le N-1$,

$$\int_{c_k}\Omega=\int_{c_k}\tilde{\Omega}_2-\sum_{n=1}^{N-1}I_n\int_{c_k}\zeta_n=2\pi I_k-\sum_{n=1}^{N-1}I_n2\pi\delta_{kn}=0,$$

proving (iii).

Recall that the Toda frequencies are given by

$$\omega_n=\partial_{I_n}H_\alpha\quad(1\le n\le N-1), \tag{31}$$

where $H_\alpha=H_\alpha(I_1,\ldots,I_{N-1})$ is, up to an additive constant given in Theorem 2.1, the Hamiltonian of the Toda lattice expressed in terms of the action variables $I=(I_1,\ldots,I_{N-1})$ and the value α of the Casimir C_2. In particular, it follows that the frequencies $\omega_n(1\le n\le N-1)$ are independent of β. Without loss of generality we can therefore assume that $\beta=0$. Expressing the element $(b,a)\in\mathcal{M}^\bullet$ with $\beta=0$ in terms of the Birkhoff coordinates (x,y) of Theorem 2.1 we may view Δ_λ as an analytic function of λ, α, and (x,y). As Δ is a spectral invariant, it is indeed an analytic function of λ, α, and the action variables alone. Consider its gradient with respect to $I=(I_n)_{1\le n\le N-1}$ and introduce the one-forms

$$\eta_n := -\frac{\partial_{I_n}\Delta}{\sqrt{\Delta^2-4}}d\lambda. \tag{32}$$

These are holomorphic one forms on $\Sigma_{b,a}$ except possibly at $\infty^{\pm}$. As

$$\Delta_\lambda = 2 + \alpha^{-N}\prod_{j=1}^{N}(\lambda-\lambda_j^+)$$
$$= 2 + \alpha^{-N}\lambda^N + \alpha^{-N}\left(\sum_{j=1}^{N}\lambda_j^+\right)\lambda^{N-1} + O(\lambda^{N-2})$$

and $\Sigma_{j=1}^N\lambda_j^+=\Sigma_{n=1}^N b_n=-N\beta=0$ by assumption, $\partial_{I_n}\Delta$ is a polynomial in λ of degree at most $N-2$, and hence η_n is holomorphic at ∞^+ and ∞^- as well.

In view of the definition of η_n,

$$\eta_n = \partial_{I_n}\left(\operatorname{arccosh}\frac{\Delta_\lambda}{2}\right)d\lambda.$$

To analyze η_n near ∞^+ and ∞^-, we need to compute the asymptotic expansion of arccosh $\Delta_\lambda/2$ for $\lambda>\lambda_{2N}$ large. Denote by arccosh x the positive branch of arccosh, i.e., arccosh $x>0\,\forall x>1$. In Appendix B we prove

Proposition 4.3: *For any* $(b,a)\in\mathcal{M}_{\beta,\alpha}$, *arccosh* $\Delta_\lambda/2$ *admits the asymptotic expansion* $(\lambda\in\mathbb{R},\lambda\to\infty)$,

$$\operatorname{arccosh}\frac{\Delta_\lambda}{2} = N\log\lambda - N\log\alpha + \frac{N\beta}{\lambda} - \frac{H_{\text{Toda}}}{\lambda^2} + O(\lambda^{-3}).$$

Proposition 4.3 leads to the following asymptotic expansion of η_n:

$$\eta_n = \pm\left(\frac{\omega_n}{\lambda^2} + O(\lambda^{-3})\right)d\lambda, \tag{33}$$

with respect to the local coordinate λ near $\infty^{\pm}$.

Finally, we show that for any $1\le n\le N-1$, the holomorphic one-form η_n coincides with the one-form ζ_n introduced earlier.

Lemma 4.4: *For any* $(b,a)\in\mathcal{M}^\bullet$ *with* $\beta=0$ *and any* $1\le n\le N-1$,

$$\eta_n = \zeta_n. \tag{34}$$

Proof: Let $1\le n\le N-1$ be fixed. We already know that η_n is a holomorphic one form on $\Sigma_{b,a}$. To show that it coincides with ζ_n it suffices to prove that it satisfies the normalizing conditions (25)

$$\frac{1}{2\pi}\int_{c_k}\eta_n = \delta_{nk} \quad \forall\, 1\le k\le N-1$$

or

$$\int_{\pi(c_k)}\frac{\partial_{I_n}\Delta_\lambda}{\sqrt[c]{\Delta_\lambda^2-4}}d\lambda = -2\pi\delta_{nk} \quad \forall\, 1\le k\le N-1, \tag{35}$$

where $\pi:\Sigma_{b,a}\to\mathbb{C}\cup\{\infty\}$ is the projection introduced at the beginning of Sec. III. Note that the principal branch of the logarithm

$$\kappa(\lambda) = \log((-1)^{N-k}(\Delta_\lambda - \sqrt[c]{\Delta_\lambda^2-4}))$$

is well defined for λ near $\pi(c_k)$ and depends analytically on $(I_n)_{1\le n\le N-1}$. By a straightforward computation, for λ near $\pi(c_k)$,

$$\partial_{I_n}\kappa = \frac{\partial_{I_n}\Delta - \dfrac{\Delta\partial_{I_n}\Delta}{\sqrt[c]{\Delta^2-4}}}{\Delta-\sqrt[c]{\Delta^2-4}} = -\frac{\partial_{I_n}\Delta}{\sqrt[c]{\Delta^2-4}}.$$

Hence the left-hand side of the identity (35) can be computed to be

$$\int_{\pi(c_k)}\frac{\partial_{I_n}\Delta}{\sqrt[c]{\Delta^2-4}}d\lambda = -\int_{\pi(c_k)}\partial_{I_n}\kappa(\lambda)d\lambda$$
$$= -\partial_{I_n}\int_{\pi(c_k)}\kappa(\lambda)d\lambda. \tag{36}$$

On the other hand, for λ near $\pi(c_k)$,

$$\partial_\lambda\kappa = \frac{\dot\Delta - \dfrac{\Delta\dot\Delta}{\sqrt[c]{\Delta^2-4}}}{\Delta-\sqrt[c]{\Delta^2-4}} = -\frac{\dot\Delta}{\sqrt[c]{\Delta^2-4}},$$

and thus, by integration by parts,

$$2\pi I_k = \int_{\pi(c_k)}\lambda\frac{\dot\Delta}{\sqrt[c]{\Delta^2-4}} = \int_{\pi(c_k)}\lambda(-\partial_\lambda\kappa) = \int_{\pi(c_k)}\kappa(\lambda)d\lambda.$$

Combined with Eq. (36), we get the claimed identity (35).

Theorem 4.5: *For any* $(b,a)\in\mathcal{M}^\bullet$ *and any* $1\le n\le N-1$, *the Toda frequency* $\omega_n=\partial_{I_n}H_\alpha$ *satisfies*

$$\omega_n = \frac{i}{2}\int_{d_n}\Omega_2. \tag{37}$$

Proof: To prove Eq. (37) we use the Riemann bilinear relations. Fix $1\le n\le N-1$ and $(b,a)\in\mathcal{M}$. We have already observed that ω_n does not depend on β. Without loss of generality we therefore can assume that $\beta=0$ for the given element $(b,a)\in\mathcal{M}^\bullet$. Combining Eq. (33) with Lemma 4.4 we conclude that for λ near $\infty^{\pm}$,

$$\zeta_n = \pm\left(\frac{\omega_n}{\lambda^2}+O(\lambda^{-3})\right)d\lambda.$$

When expressed in the standard chart $\lambda=1/z$, we have $\zeta_n=f_n^{\pm}(z)dz$ for z near 0 with

$$f_n^{\pm}(z) = \mp\,\omega_n + O(z). \tag{38}$$

By the Riemann bilinear relations applied to Ω and ζ_n, we then get (cf., e.g., Ref. 5, p. 241)

$$\sum_{k=1}^{N-1}\left(\int_{c_k}\zeta_n\int_{d_k}\Omega - \int_{d_k}\zeta_n\int_{c_k}\Omega\right)$$
$$= 2\pi i(-Nf_n^+(0)+Nf_n^-(0)) = 4\pi i N\omega_n.$$

Using that $\int_{c_k}\Omega=0$ and $\int_{c_k}\zeta_n=2\pi\delta_{nk}$ for any $1\le k\le N-1$ the left hand side of the above identity equals $2\pi\int_{d_n}\Omega$. Hence

$$\int_{d_n} \Omega = 2Ni\omega_n. \tag{39}$$

By Lemma 4.2 and the assumption $\beta=0$, the one-forms Ω and Ω_2 are related by $\Omega=-N\Omega_2$. Together with Eq. (39) the claimed identity then follows.

We remark that in 1970's, Its and Matveev[24] obtained a formula for the frequencies of the Korteweg–de Vries equation (KdV) similar to Eq. (37), see, e.g., Ref. 10 for a detailed exposition. For the Toda lattice, computations similar to the ones in the proof of Theorem 4.5 can be found in Ref. 21.

Finally we note that $\int_{d_n}\Omega$ can be written as

$$\int_{d_n} \Omega = 2\int_{\lambda_1}^{\lambda_{2n}} \frac{\lambda\dot{\Delta}_\lambda}{\sqrt[c]{\Delta_{\lambda-i0}^2-4}}d\lambda - 2\sum_{k=1}^{N-1} I_k\int_{\lambda_1}^{\lambda_{2n}} \frac{\psi_k(\lambda)}{\sqrt[c]{\Delta_{\lambda-i0}^2-4}}d\lambda.$$

As $\mathcal{M}^\bullet$ is dense in $\mathcal{M}$ it then follows that for any $(b,a) \in \mathcal{M}$,

$$\omega(I;\alpha) = \frac{1}{Ni}\left(\left(\int_{\lambda_1}^{\lambda_{2n}} \frac{\lambda\dot{\Delta}_\lambda}{\sqrt[c]{\Delta_{\lambda-i0}^2-4}}d\lambda - \sum_{k=1}^{N-1} I_k\int_{\lambda_1}^{\lambda_{2n}} \frac{\psi_k(\lambda)}{\sqrt[c]{\Delta_{\lambda-i0}^2-4}}d\lambda\right)\right|_{(b+\beta 1_N,a)},$$

where $I=I(b,a)$ is given by Eq. (29), $\alpha=(\prod_{i=1}^N a_i)^{1/N}$, $\beta=(-1/N)\sum_{k=1}^N b_k$, and $1_N\in\mathbb{R}^N$ is the vector $1_N=(1,\ldots,1)$.

V. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2. The main ingredients are the Birkhoff normal form of the Toda lattice (Theorem 1.4 and Corollary 1.5) and Krichever's theorem (Theorem 3.3).

We begin by computing the components of the period map $\mathcal{F}$, defined by Eq. (20) in Sec. III, for sequences $\lambda_1 < \cdots < \lambda_{2N}$ given by the spectrum of the $2N\times 2N$-Jacobi matrix $L^+((b,b),(a,a))$ with $(b,a)\in\mathcal{M}^\bullet$. To compute the period $\int_{d_n}\Omega_1$ of Ω_1 we need

Lemma 5.1: *For any* $(b,a)\in\mathcal{M}^\bullet$,

$$\Omega_1 = -\frac{1}{N}\frac{\dot{\Delta}}{\sqrt{\Delta^2-4}}d\lambda. \tag{40}$$

Proof: Let $(b,a)\in\mathcal{M}^\bullet$ be given. Clearly, $-1/N(\dot{\Delta}/\sqrt{\Delta^2-4})d\lambda$ is a holomorphic one form on $\Sigma_{b,a}\backslash\{\infty^+,\infty^-\}$. We claim that it has poles of order 1 at $\infty^\pm$. Indeed, in the standard chart $z=1/\lambda$ near $\infty^\pm$ one has

$$\frac{\dot{\Delta}}{\sqrt{\Delta^2-4}}d\lambda = N\frac{z^{-(N-1)}\prod_{i=1}^{N-1}(1-\dot{\lambda}_i z)}{z^{-N}\sqrt{\prod_{i=1}^{2N}(1-\lambda_i z)}}\frac{dz}{-z^2} = \mp\left(\frac{N}{z}+O(1)\right)dz.$$

In view of the uniqueness statement of Lemma 3.1, it remains to show that the normalization conditions (16) are satisfied. One computes for any $1\le k\le N-1$,

$$\int_{c_k}\frac{\dot{\Delta}}{\sqrt{\Delta^2-4}}d\lambda = \int_{\pi(c_k)}\frac{\dot{\Delta}}{\sqrt[c]{\Delta^2-4}}d\lambda = 2\,\text{arccosh}\left((-1)^{N-k}\frac{\Delta_\lambda}{2}\right)\Bigg|_{\lambda=\lambda_{2k}}^{\lambda=\lambda_{2k+1}} = 0.$$

Now identity (40) follows from the uniqueness statement of Lemma 3.1.

Corollary 5.2: *For any* $(b,a)\in\mathcal{M}^\bullet$,

$$\int_{d_n}\Omega_1 = \frac{2\pi n}{N}i. \tag{41}$$

Proof: By Lemma 5.1 and the normalization conditions (16) one gets for any $1\le n\le N-1$,

$$\int_{d_n}\Omega_1 = -\frac{2}{N}\left(\int_{\lambda_1}^{\lambda_2}+\int_{\lambda_3}^{\lambda_4}+\cdots+\int_{\lambda_{2n-1}}^{\lambda_{2n}}\right)\frac{\dot{\Delta}_\lambda}{\sqrt[c]{\Delta_{\lambda-i0}^2-4}}d\lambda.$$

For any $\lambda_{2k-1}\le\lambda\le\lambda_{2k}$,

$$\frac{\dot{\Delta}_\lambda}{\sqrt[c]{\Delta_{\lambda-i0}^2-4}} = \frac{(-1)^{N-k}\dot{\Delta}/2}{i\sqrt[+]{1-(\Delta/2)^2}}d\lambda = \frac{1}{i}\partial_\lambda\left(\arcsin\left((-1)^{N-k}\frac{\Delta_\lambda}{2}\right)\right),$$

and thus

$$\int_{d_n}\Omega_1 = -\frac{2}{Ni}\sum_{k=1}^n\arcsin\left((-1)^{N-k}\frac{\Delta_\lambda}{2}\right)\Bigg|_{\lambda_{2k-1}}^{\lambda_{2k}} = \frac{2n\pi}{N}i$$

as claimed.

To obtain the last two components of the period map $\mathcal{F}$ we compute the asymptotics of $\int_{\lambda_{2N}}^\lambda\Omega_1$ with respect to the local coordinate λ near $\infty^\pm$. By Lemma 5.1, we get near $\infty^\pm$,

$$\int_{\lambda_{2N}}^{\lambda}\Omega_1 = -\frac{1}{N}\int_{\lambda_{2N}}^{\lambda}\frac{\dot{\Delta}_\lambda}{\sqrt{\Delta_\lambda^2-4}}d\lambda = \mp\frac{1}{N}\text{arccosh}\frac{\Delta_\lambda}{2},$$

and hence by Proposition 4.3,

$$\int_{\lambda_{2N}}^{\lambda}\Omega_1 = \mp\left(\log\lambda-\log\alpha+\frac{\beta}{\lambda}+O(\lambda^{-2})\right), \tag{42}$$

or, in the notation of Sec. III, $e_1=\beta$ and $e_0=-\log\alpha$. Taking into account that by Theorem 4.5,

$$\int_{d_n}\Omega_2 = -2i\omega_n \quad \forall\ 1\le n\le N-1$$

and that $\omega_n=\partial_{I_n}H_\alpha$, we therefore have proved

Proposition 5.3: *For any* $(b,a)\in\mathcal{M}^\bullet$,

$$\mathcal{F}(\lambda_1<\cdots<\lambda_{2N}) = \left(\left(\frac{2n\pi i}{N},-2i\partial_{I_n}H_\alpha\right)_{1\le n\le N-1},\beta,-\log\alpha\right).$$

Next we define the map

$$\Lambda:\quad \mathbb{R}_{>0}^{N-1}\times\mathbb{R}\times\mathbb{R}_{>0}\to\mathbb{R}^{2N},$$

$$((I_n)_{1\le n\le N-1},\beta,\alpha)\mapsto(\lambda_n)_{1\le n\le 2N},$$

where $(\lambda_n)_{1\le n\le 2N}$ is the spectrum of the Jacobi matrix $L^+((b,b),(a,a))$ and $(b,a)\in\mathcal{M}$ is determined by the Birkhoff map Φ (cf. Theorem 2.1),

$$(b,a)=\Phi^{-1}((\sqrt[+]{2I_n},0)_{1\le n\le N-1},\beta,\alpha).$$

Clearly, Λ is 1–1 and as $I_n>0$ for any $1\le n\le N-1$, Λ is smooth. On its image, the inverse Λ^{-1} of Λ can be explicitly computed. In view of Eqs. (29) and (42) one has for any $(\lambda_n)_{1\le n\le 2N}\in \mathrm{Im}\,\Lambda$,

$$\Lambda^{-1}((\lambda_n)_{1\le n\le 2N}) =\left(\left(\frac{1}{2\pi}\int_{c_n}\lambda\frac{\dot f_\lambda}{\sqrt{f_\lambda^2-4}}d\lambda\right)_{1\le n\le N-1},e_1,\exp(-e_0)\right),$$

where in view of Eq. (12), $f_\lambda=f(\lambda)$ is given by $f(\lambda)=2+\Pi_{i=1}^N(\lambda-\lambda_i^+)$ with $\lambda_N^+=\lambda_{2N}$, $\lambda_{N-1}^+=\lambda_{2N-3}$, $\lambda_{N-2}^+=\lambda_{2N-4}$, $\lambda_{N-3}^+=\lambda_{2N-7,\ldots}$, and e_1, e_0 are the coefficients in the expansion (18) of the differential form Ω_1 on the Riemann surface Σ_E with $E=(\lambda_1<\cdots<\lambda_{2N})$ and scaling parameter $\sigma=1$. (Note that by Lemma 3.1, Ω_1 is independent of the scaling factor σ.) Hence we have shown

Proposition 5.4: *The map*

$$\Lambda:\mathbb{R}_{>0}^{N-1}\times\mathbb{R}\times\mathbb{R}_{>0}\to\mathbb{R}^{2N}$$

is a smooth embedding.

With these preparations we are now ready to prove Theorem 1.2.

Proof of Theorem 1.2: In view of Proposition 5.3, the composition $\mathcal{F}\circ\Lambda:\mathbb{R}_{>0}^{N-1}\times\mathbb{R}\times\mathbb{R}_{>0}\to\mathbb{R}^{2N}$ is given by

$$\mathcal{F}\circ\Lambda((I_n)_{1\le n\le N-1},\beta,\alpha) =\left(\left(\frac{2\pi n i}{N},-2i\partial_{I_n}H_\alpha\right)_{1\le n\le N-1},\beta,-\log\alpha\right). \tag{43}$$

By Theorem 3.3, $\mathcal{F}$ is a local diffeomorphism, and by Proposition 5.4, Λ is a smooth embedding. Hence $\mathcal{F}\circ\Lambda$ is an embedding. Therefore, at each point $((I_n)_{1\le n\le N-1},\beta,\alpha)$ the differential $d(\mathcal{F}\circ\Lambda)$ has rank $N+1$. By Eq. (43) it is a $2N\times(N+1)$-matrix of the form

$$\begin{pmatrix} 0_{(N-1)\times(N-1)} & 0_{(N-1)\times 1} & 0_{(N-1)\times 1}\\ (-2i\partial_{I_n}\partial_{I_l}H_\alpha)_{1\le n,l\le N-1} & 0_{(N-1)\times 1} & 0_{(N-1)\times 1}\\ 0_{1\times(N-1)} & 1 & 0\\ \ldots & 0 & -\alpha^{-1}\end{pmatrix},$$

where $0_{N_1\times N_2}$ denotes the $N_1\times N_2$-matrix with all entries 0. Hence the rank of the $(N-1)\times(N-1)$-matrix $(\partial^2H_\alpha/\partial I_n\partial I_l)_{1\le n,l\le N-1}$ has to be $N-1$. This proves Theorem 1.2.

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APPENDIX A: PROOF OF LEMMA 4.1

By Eq. (13), $\dot\Delta_\lambda$ admits the product representation

$$\dot\Delta_\lambda=N\alpha^{-N}\prod_{n=1}^{N-1}(\lambda-\dot\lambda_n),$$

where the roots $(\dot\lambda_n)_{1\le n\le N-1}$ of $\dot\Delta$, when listed in increasing order, satisfy $\lambda_{2n}<\dot\lambda_n<\lambda_{2n+1}$ for any $1\le n\le N-1$. Hence

$$\tilde\Omega_2=N\frac{\lambda(\lambda-\dot\lambda_1)\cdots(\lambda-\dot\lambda_{N-1})}{\sqrt{(\lambda-\lambda_1)\cdots(\lambda-\lambda_{2N})}}d\lambda.$$

It is clear that $\tilde\Omega_2$ is holomorphic on the set $\Sigma_{b,a}\setminus\{\infty^+,\infty^-\}$. In the standard chart $\lambda=1/z$ at ∞^+ one has

$$\tilde\Omega_2=N\frac{(1-\dot\lambda_1z)\cdots(1-\dot\lambda_{N-1}z)}{\sqrt[+]{(1-\lambda_1z)\cdots(1-\lambda_{2N}z)}}\cdot\frac{dz}{-z^2}.$$

Using that $(1-\lambda_nz)^{-1/2}=1+\frac{1}{2}\lambda_nz+O(z^2)$ near $z=0$ one gets

$$\tilde\Omega_2=-\frac{N}{z^2}\left(1-\left(\sum_{n=1}^{N-1}\dot\lambda_n\right)z+O(z^2)\right) \times\left(1+\frac{1}{2}\left(\sum_{n=1}^{2N}\lambda_n\right)z+O(z^2)\right)dz =\left(-N\frac{1}{z^2}+N\left(\sum_{n=1}^{N-1}\dot\lambda_n-\frac{1}{2}\sum_{n=1}^{2N}\lambda_n\right)\frac{1}{z}+O(1)\right)dz.$$

By Eq. (15), one has

$$\sum_{n=1}^{N-1}\dot\lambda_n=\frac{N-1}{2N}\sum_{n=1}^{2N}\lambda_n=\frac{N-1}{N}\sum_{n=1}^{N}b_n=-(N-1)\beta.$$

Hence the coefficient of $1/z$ in the expansion above equals

$$N\left(\sum_{n=1}^{N-1}\dot\lambda_n-\frac{1}{2}\sum_{n=1}^{2N}\lambda_n\right)=N(-(N-1)\beta+N\beta)=N\beta,$$

which by assumption equals zero. Altogether we have proved that with respect to the standard chart $\lambda=1/z$ at ∞^+,

$$\tilde\Omega_2=\left(-\frac{N}{z^2}+O(1)\right)dz.$$

By a similar computation one sees that in the standard chart $\lambda=1/z$ at ∞^-, one has $\tilde\Omega_2=(N/z^2+O(1))dz$. This completes the proof of Lemma 4.1.

APPENDIX B: PROOF OF PROPOSITION 4.3

To prove Proposition 4.3 we first need to derive some auxiliary results. Let $(b,a)\in\mathcal{M}_{\beta,\alpha}$ and assume that $\lambda>\lambda_{2N}$ in the sequel. Recall that the Floquet multipliers associated with the difference Eq. (9) are defined as the eigenvalues of the monodromy matrix

$$\begin{pmatrix} y_1(N,\lambda) & y_2(N,\lambda) \\ y_1(N+1,\lambda) & y_2(N+1,\lambda) \end{pmatrix}.$$

Using the Wronskian identity, one sees that the characteristic polynomial of the monodromy matrix is given by $1-\Delta_\lambda\xi+\xi^2$, hence the Floquet multipliers are $\xi_\pm(\lambda)=\Delta_\lambda/2\pm\frac{1}{2}\sqrt{\Delta_\lambda^2-4}$. As $\Delta_\lambda>2$ for $\lambda>\lambda_{2N}$, $\xi_\pm(\lambda)$ are real valued and satisfy $\xi_+(\lambda)>1>\xi_-(\lambda)>0$ as well as $\xi_+(\lambda)\xi_-(\lambda)=1$. Solutions of Eq. (9) corresponding to the Floquet multiplier

$$w(\lambda)\equiv\xi_+(\lambda)=\frac{\Delta_\lambda}{2}+\frac{1}{2}\sqrt{\Delta_\lambda^2-4}$$

are thus expanding. On the other hand, as $\log(x+\sqrt{x^2-1})=\operatorname{arccosh} x$ for $x>1$ one has

$$\log\left(\frac{\Delta_\lambda}{2}+\frac{1}{2}\sqrt{\Delta_\lambda^2-4}\right)=\operatorname{arccosh}\frac{\Delta_\lambda}{2},$$

and therefore

$$\log w(\lambda)=\operatorname{arccosh}\frac{\Delta_\lambda}{2}.$$

For any $\lambda>\lambda_{2N}$ denote by $(u(n,\lambda))_{n\in\mathbb{Z}}$ a solution of Eq. (9) satisfying

$$u(n+N,\lambda)=w(\lambda)u(n,\lambda) \quad \forall\, n\in\mathbb{Z}.$$

We claim that

$$u(n,\lambda)\neq 0 \quad \forall\, n\in\mathbb{Z}. \tag{B1}$$

Indeed, if there were $k\in\mathbb{Z}$ with $u(k,\lambda)=0$, then λ would be an eigenvalue of $L_2(S^k(b,a))$, where $S^k(b,a)$ denotes the shifted element

$$S^k(b,a):=(b_{n+k},a_{n+k})_{1\le n\le N}\in\mathcal{M}$$

and $L_2(b,a)$ denotes the $(N-1)\times(N-1)$ Jacobi matrix given by

$$\begin{pmatrix} b_2 & a_2 & 0 & \dots & 0 \\ a_2 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ 0 & \dots & 0 & a_{N-1} & b_N \end{pmatrix}.$$

However,

$$\operatorname{spec} L^\pm(S^k(b,a))=\operatorname{spec} L^\pm(b,a),$$

and spec $(L_2(S^k(b,a)))$ is bounded by max $L^+(S^k(b,a))$ (cf. Ref. 15 or Ref. 7). This leads to a contradiction, and Eq. (B1) is proved. Hence the solution $(u(n,\lambda))_{n\in\mathbb{Z}}$ can always be normalized by $u(0,\lambda)=1$. Then $w(\lambda)=u(N,\lambda)/u(0,\lambda)=u(N,\lambda)$ or

$$\operatorname{arccosh}\frac{\Delta_\lambda}{2}=\log u(N,\lambda). \tag{B2}$$

Proof of Proposition 4.3: Let $\lambda>\lambda_{2N}$ and write $u(n)=u(n,\lambda)$. In view of Eq. (B1) we may define

$$\phi(n)\equiv\phi(n,\lambda):=\frac{u(n+1)}{u(n)}, \quad n\in\mathbb{Z}. \tag{B3}$$

One verifies that $\phi(n)$ satisfies the discrete Riccati equation (cf., e.g., Ref. 21)

$$a_n\phi(n)\phi(n-1)+(b_n-\lambda)\phi(n-1)+a_{n-1}=0. \tag{B4}$$

In the case $b_n=-\beta$ and $a_n=\alpha$ for any $n\in\mathbb{Z}$, $u(n,\lambda)_{n\in\mathbb{Z}}$ can be computed explicitly. Indeed, making the ansatz $u(n)=e^{\kappa n}$, one concludes that $\phi(n)\equiv e^\kappa$ is given by

$$e^\kappa=\frac{\lambda+\beta}{2\alpha}\left(1+\sqrt{1-\left(\frac{2\alpha}{\lambda+\beta}\right)^2}\right).$$

Hence for $\phi(n)$ one gets the expansion

$$\phi(n,\lambda)=\frac{\lambda}{\alpha}+\frac{\beta}{\alpha}+O(\lambda^{-1}), \quad \text{as } \lambda\to\infty. \tag{B5}$$

In the case of an arbitrary element $(b,a)\in\mathcal{M}_{\beta,\alpha}$, Eq. (B5) suggests to make the ansatz

$$\phi(n,\lambda)=\frac{\lambda}{a_n}-\frac{b_n}{a_n}+\frac{1}{a_n}\sum_{k=1}^{\infty}\frac{\phi_k(n)}{\lambda^k}. \tag{B6}$$

Substituting this ansatz into Eq. (B4) one gets by comparison of coefficients

$$\phi_1(n)=-a_{n-1}^2 \quad \forall\, n\in\mathbb{Z}. \tag{B7}$$

By Eq. (B3), we have $u(N,\lambda)=\Pi_{n=0}^{N-1}\phi(n,\lambda)$, and thus by Eq. (B2),

$$\operatorname{arccosh}\frac{\Delta_\lambda}{2}=\sum_{n=0}^{N-1}\log\phi(n,\lambda).$$

In view of the asymptotic expansion (B6) and the values of the coefficients $\phi_1(n)$ given by Eq. (B7) it then follows that

$$\operatorname{arccosh}\frac{\Delta_\lambda}{2}=\sum_{n=0}^{N-1}\log\frac{\lambda}{a_n}+\sum_{n=0}^{N-1}\log\left(1-\frac{b_n}{\lambda}-\frac{a_{n-1}^2}{\lambda^2}+O(\lambda^{-3})\right).$$

Note that

$$\sum_{n=0}^{N-1}\log\frac{\lambda}{a_n}=N\log\lambda-\log\prod_{n=0}^{N-1}a_n=N\log\lambda-N\log\alpha. \tag{B8}$$

Using $\log(1+x)=x-\frac{1}{2}x^2+O(x^3)$, one sees that

$$\sum_{n=0}^{N-1}\log\left(1-\frac{b_n}{\lambda}-\frac{a_{n-1}^2}{\lambda^2}+O(\lambda^{-3})\right)=\frac{N\beta}{\lambda}-\frac{1}{\lambda^2}\sum_{n=0}^{N-1}\left(a_{n-1}^2+\frac{1}{2}b_n^2\right)+O(\lambda^{-3}),$$

which by Eq. (5) equals

$$\frac{N\beta}{\lambda}-\frac{1}{\lambda^2}H_{\text{Toda}}(b,a)+O(\lambda^{-3}). \tag{B9}$$

Combining Eqs. (B8) and (B9) we get the claimed expansion

$$\operatorname{arccosh}\frac{\Delta_\lambda}{2} = N \log \lambda - N \log \alpha + \frac{N\beta}{\lambda} - \frac{1}{\lambda^2} H_{\mathrm{Toda}} + O(\lambda^{-3}).$$

This completes the proof of Proposition 4.3.

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