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## **On simplification of combinatorial link Floer homology**

Beliakova, A

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# A SIMPLIFICATION OF COMBINATORIAL LINK FLOER HOMOLOGY

ANNA BELIAKOVA

ABSTRACT. We define a new combinatorial complex computing the hat version of link Floer homology over  $\mathbb{Z}/2\mathbb{Z}$ , which turns out to be significantly smaller than the Manolescu–Ozsváth–Sarkar one.

## INTRODUCTION

Knot Floer homology is a powerful knot invariant constructed by Ozsváth–Szabó [15] and Rasmussen [18]. In its basic form, the knot Floer homology  $\widehat{\text{HF}}\widehat{\text{K}}(K)$  of a knot  $K \subset S^3$  is a finite-dimensional bigraded vector space over  $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$

$$\widehat{\text{HF}}\widehat{\text{K}}(K) = \bigoplus_{d \in \mathbb{Z}, i \in \mathbb{Z}} \widehat{\text{HF}}\widehat{\text{K}}_d(K, i),$$

where  $d$  is the Maslov and  $i$  is the Alexander grading. Its graded Euler characteristic

$$\sum_{d, i} (-1)^d \text{rank } \widehat{\text{HF}}\widehat{\text{K}}_d(K, i) t^i = \Delta_K(t)$$

is equal to the symmetrized Alexander polynomial  $\Delta_K(t)$ . The knot Floer homology enjoys the following symmetry extending that of the Alexander polynomial.

$$(1) \quad \widehat{\text{HF}}\widehat{\text{K}}_d(K, i) = \widehat{\text{HF}}\widehat{\text{K}}_{d-2i}(K, -i)$$

By the result of Ozsváth–Szabó [14], the maximal Alexander grading  $i$ , such that  $\widehat{\text{HF}}\widehat{\text{K}}_*(K, i) \neq 0$  is the Seifert genus  $g(K)$  of  $K$ . Moreover, Ghiggini showed for  $g(K) = 1$  [5] and Yi Ni in general [11], that the knot is fibered if and only if  $\text{rank } \widehat{\text{HF}}\widehat{\text{K}}_*(K, g(K)) = 1$ . A concordance invariant bounding from below the slice genus of the knot can also be extracted from knot Floer homology [13]. For torus knots the bound is sharp, providing a new proof of the Milnor conjecture. The first proof of the Milnor conjecture was given by Kronheimer and Mrowka [7], then Rasmussen [19] proved it combinatorially by using Khovanov homology [6].

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*Key words and phrases.* Heegaard Floer homology, link invariants, fibered knot, Seifert genus.

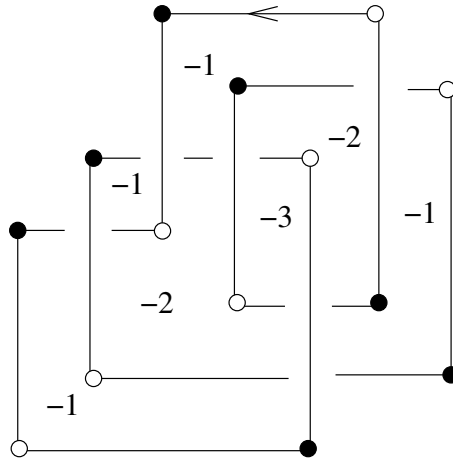


FIGURE 1. **A rectangular diagram for  $5_2$  knot.** The number associated to a domain is minus the winding number for its points. The sets  $\mathbb{X}$  and  $\mathbb{O}$  consist of black and white points, respectively.

Knot Floer homology was extended to links in [17]. The first combinatorial construction of the link Floer homology was given in [9] over  $\mathbb{F}$  and then in [10] over  $\mathbb{Z}$ . Both constructions use grid diagrams of links.

A *grid diagram* is a square grid on the plane with  $n \times n$  squares. Each square is decorated either with an  $X$ , an  $O$ , or nothing. Moreover, every row and every column contains exactly one  $X$  and one  $O$ . The number  $n$  is called *complexity* of the diagram. Following [10], we denote the set of all  $O$ 's and  $X$ 's by  $\mathbb{O}$  and  $\mathbb{X}$ , respectively.

Given a grid diagram, we construct an oriented, planar link projection by drawing horizontal segments from the  $O$ 's to the  $X$ 's in each row, and vertical segments from the  $X$ 's to the  $O$ 's in each column. We assume that at every intersection point the vertical segment overpasses the horizontal one. This produces a planar *rectangular diagram*  $D$  for an oriented link  $L$  in  $S^3$ . Any link in  $S^3$  admits a rectangular diagram (see e.g. [4]). An example is shown in Figure 1.

In [9], [10] the grid lies on the torus, obtained by gluing the top most segment of the grid to the bottommost one and the leftmost segment to the right most one. In the torus, the horizontal and vertical segments of the grid become circles. The MOS complex is then generated by  $n$ -tuples of intersection points between horizontal and vertical circles, such that exactly one point belongs to each horizontal (or vertical) circle. The differential is defined as follows:

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in S_{\mathbf{x}}} \sum_{r \in \text{Rect}^0(\mathbf{x}, \mathbf{y})} \mathbf{y},$$

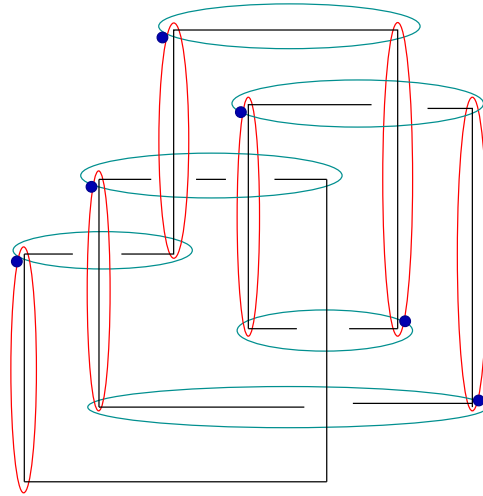


FIGURE 2. **Collection of short ovals for  $5_2$  knot.** The dots show a generator in Alexander grading 1.

where  $S_{\mathbf{x}}$  is the subset of generators that have  $n - 2$  points in common with  $\mathbf{x}$ . For  $\mathbf{y} \in S_{\mathbf{x}}$ ,  $\text{Rect}^0(\mathbf{x}, \mathbf{y})$  is the set of rectangles with vertices  $\mathbf{x} \setminus (\mathbf{x} \cap \mathbf{y})$  and  $\mathbf{y} \setminus (\mathbf{y} \cap \mathbf{x})$ , whose interior does not contain  $X$ 's and  $O$ 's or points among  $\mathbf{x}$  and  $\mathbf{y}$ . Moreover, a counterclockwise rotation along the arc of the horizontal oval, leads from the vertices in  $\mathbf{x}$  to the ones in  $\mathbf{y}$ .

The Alexander grading is given by formula (2) below, and the Maslov grading by (3) plus one. The MOS complex has  $n!$  generators. This number greatly exceeds the rank of its homology. For the trefoil, for example, the number of generators is 120, while the rank of  $\widehat{\text{HFK}}(3_1)$  is 3.

In this paper, we construct another combinatorial complex computing link Floer homology, which has significantly less generators. All knots with less than 6 crossings admit rectangular diagrams where all differentials in our complex are zero, and the rank of the homology group is equal to the number of generators.

**Main results.** Our construction also uses rectangular diagrams. Given an oriented link  $L$  in  $S^3$ , let  $D$  be its rectangular diagram in  $\mathbb{R}^2$ . Let us draw  $2n - 2$  narrow short ovals around all but one horizontal and all but one vertical segments of the rectangular diagram  $D$  in such a way, that the outside domain has at least one point among  $\mathbb{X}$  or  $\mathbb{O}$ . We denote by  $\mathbf{S}$  the set of unordered  $(n - 1)$ -tuples of intersection points between the horizontal and vertical ovals, such that exactly one point belongs to each horizontal (or vertical) oval. We assume throughout this paper that the ovals intersect transversely. An example is shown in Figure 2.

A chain complex  $(C_{\text{short}}(D), \partial_{\text{short}})$  computing the hat version of link Floer homology of  $L$  over  $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$  is defined as follows. The generators are elements of  $\mathbf{S}$ . The bigrading on  $\mathbf{S}$  can be constructed analogously to those in [9]. Suppose  $\ell$  is the number of components of  $L$ . Then the Alexander grading is a function  $A: \mathbf{S} \rightarrow (\frac{1}{2}\mathbb{Z})^\ell$ , defined as follows.

First, we define a function  $a: \mathbf{S} \rightarrow \mathbb{Z}^\ell$ . For a point  $p$ , the  $i^{\text{th}}$  component of  $a$  is minus the winding number of the projection of the  $i^{\text{th}}$  component of the oriented link around  $p$ . In the grid diagram, we have  $2n$  distinguished squares containing  $X$ 's or  $O$ 's. Let  $\{c_{i,j}\}$ ,  $i \in \{1, \dots, 2n\}$ ,  $j \in \{1, \dots, 4\}$ , be the vertices of these squares. Given  $\mathbf{x} \in \mathbf{S}$ , we set

$$(2) \quad A(\mathbf{x}) = \sum_{x \in \mathbf{x}} a(x) - \frac{1}{8} \left( \sum_{i,j} a(c_{i,j}) \right) - \left( \frac{n_1 - 1}{2}, \dots, \frac{n_\ell - 1}{2} \right),$$

where here  $n_i$  is the complexity of the  $i^{\text{th}}$  component of  $L$ , i.e. the number of horizontal segments belonging to this component.

The homological or Maslov grading is a function  $M: \mathbf{S} \rightarrow \mathbb{Z}$  defined as follows. Given two collections  $A, B$  of finitely many points in the plane, let  $\mathcal{I}(A, B)$  be the number of pairs  $(a_1, a_2) \in A$  and  $(b_1, b_2) \in B$  with  $a_1 < b_1$  and  $a_2 < b_2$ . Let  $J(A, B) := 1/2(\mathcal{I}(A, B) + \mathcal{I}(B, A))$ . Define

$$(3) \quad M(\mathbf{x}) = J(\mathbf{x}, \mathbf{x}) - 2J(\mathbf{x}, \mathbb{O}) + J(\mathbb{O}, \mathbb{O}).$$

To construct a differential  $\partial_{\text{short}}$  we first need to consider the complex  $(C_{\text{long}}(D), \partial_{\text{long}})$  defined in the same way as  $(C_{\text{short}}(D), \partial_{\text{short}})$  but where the ovals are as long as  $n \times n$  grid. An example is shown in Figure 3. The differential

$$\partial_{\text{long}}(\mathbf{x}) = \sum_{\mathbf{y} \in S_{\mathbf{x}} \cup S'_{\mathbf{x}}} \sum_{r \in \text{Rect}^0(\mathbf{x}, \mathbf{y}) \cup \text{Bigon}^0(\mathbf{x}, \mathbf{y})} \mathbf{y}$$

where  $S'_{\mathbf{x}}$  is the subset of generators that have  $n - 2$  points in common with  $\mathbf{x}$  and for  $\mathbf{y} \in S'_{\mathbf{x}}$ ,  $\text{Bigon}^0(\mathbf{x}, \mathbf{y})$  is the bigon with vertices  $\mathbf{x} \setminus (\mathbf{x} \cap \mathbf{y})$  and  $\mathbf{y} \setminus (\mathbf{y} \cap \mathbf{x})$ , whose interior does not contain  $X$ 's and  $O$ 's. Moreover, a counterclockwise rotation along the arc of the horizontal oval, leads from the vertex in  $\mathbf{x}$  to the one in  $\mathbf{y}$ .

Note that  $(C_{\text{long}}(D), \partial_{\text{long}})$  coincides with the complex  $(C(S^2, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbb{X}, \mathbb{O}), \partial)$  defined by Ozsváth–Szabó in [17], where the vertical and horizontal ovals are identified with  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  curves, respectively, and  $\mathbb{X}$  and  $\mathbb{O}$  are extra basepoints. Like the MOS complex, this complex is combinatorial, since all domains suitable for the differential are either rectangles or bigons (compare [20]). Following [17], we will call elements of  $\mathbb{X} \cup \mathbb{O}$  *basepoints* in what follows.

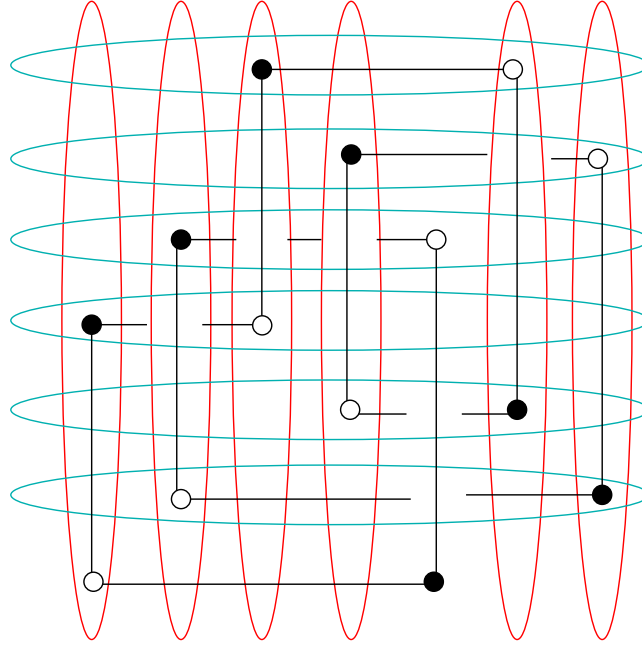


FIGURE 3. Collection of long ovals.

To this complex we further apply a simple lemma from homological algebra, that allows us to construct a homotopy equivalent complex  $(C_{\text{short}}(D), \partial_{\text{short}})$  with  $\mathbf{S}$  as a set of generators (or to shorten the ovals  $\alpha$  and  $\beta$  keeping track of the differential).

In Section 1, we describe an algorithm that for any domain connecting two generators decides whether it counts for the differential  $\partial_{\text{short}}$  or not. Furthermore, we distinguish a large class of domains that always count. In general, however the count depends on the order of shortening of ovals, which replace the choice of a complex structure in the analytic setting.

Let  $V_i$  be the two dimensional bigraded vector space over  $\mathbb{F}$  spanned by one generator in Alexander and Maslov gradings zero and another one in Maslov grading  $-1$  and Alexander grading minus the  $i$ -th basis vector.

**Theorem 1.** *Suppose  $D$  is a rectangular diagram of an oriented  $\ell$  component link  $L$ , where the  $i^{\text{th}}$  component of  $L$  has complexity  $n_i$ . Then the homology*

$$H_*(C_{\text{short}}(D), \partial_{\text{short}}) = \widehat{\text{HFL}}(L) \otimes \bigotimes_{i=1}^{\ell} V_i^{n_i-1}$$

*can be computed algorithmically.*

The complex  $(C_{\text{short}}(D), \partial_{\text{short}})$  has much fewer generators than the MOS complex (compare Section 3). Recently, Droz introduced signs in our construction [2] and wrote a computer program realizing our algorithm over  $\mathbb{Z}$  [3]. His program allows to determine the Seifert genus and fiberedness of knots until 16 crossings and also to study the torsion part of knot Floer homology.

The paper is organized as follows. Theorem 1 is proved in Section 1. In Section 2 we introduce a big class of domains that always count for the differential. In the last section we compute Floer homology of  $5_2$  knot and discuss further computations made by Droz.

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## 1. THE COMPLEX $(C_{\text{SHORT}}(D), \partial_{\text{SHORT}})$

**1.1. Intermediate complex  $(C, \partial)$ .** Suppose  $D$  is a rectangular diagram of complexity  $n$  for an oriented link  $L$ . Let  $(C, \partial)$  be the complex generated over  $\mathbb{F}$  by  $(n - 1)$ -tuples of intersection points between horizontal and vertical ovals, such that exactly one point belongs to each horizontal (or vertical) oval as defined in the Introduction. The length of the ovals can be intermediate between long and short ones. We also assume that the outside domain has at least one basepoint inside.

Given  $\mathbf{x}, \mathbf{y} \in S(C)$ , there is an oriented closed curve  $\gamma_{\mathbf{x}, \mathbf{y}}$  composed of arcs belonging to horizontal and vertical ovals, where each piece of a horizontal oval connects a point in  $\mathbf{x}$  to a point in  $\mathbf{y}$  (and hence each piece of the vertical one goes from a point in  $\mathbf{y}$  to a point in  $\mathbf{x}$ ). In  $S^2$ , there exists an oriented (immersed) domain  $D_{\mathbf{x}, \mathbf{y}}$  bounded by  $\gamma_{\mathbf{x}, \mathbf{y}}$ . The points in  $\mathbf{x}$  and  $\mathbf{y}$  are called *corners* of  $D_{\mathbf{x}, \mathbf{y}}$ .

Let  $D_i$  be the closures of the connected components of the complement of ovals in  $S^2$ . Suppose that the orientation of  $D_i$  is induced by the orientation of  $S^2$ . Then we say that a domain  $D = \sum_i n_i D_i$  connects two generators if for all  $i$ ,  $n_i \geq 0$  and  $D$  is connected. Let  $\mathfrak{D}$  be the set of all domains connecting two generators which contain neither corners nor points among  $\mathbb{X}$  and  $\mathbb{O}$  inside.

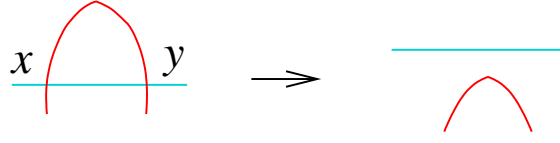


FIGURE 4. **Removing of a bigon without basepoints inside.**

We define

$$(4) \quad \partial \mathbf{x} := \sum_{M(\mathbf{y})=M(\mathbf{x})-1} \sum_{D_{\mathbf{x},\mathbf{y}} \in \mathfrak{D}} m(D_{\mathbf{x},\mathbf{y}}) \mathbf{y},$$

where  $M(\mathbf{x})$  is the Maslov grading defined by (3) and  $m(D_{\mathbf{x},\mathbf{y}}) \in \{0, 1\}$  is a multiplicity of  $D_{\mathbf{x},\mathbf{y}}$ . We set  $m(D_{\mathbf{x},\mathbf{y}}) = 1$  for rectangles and bigons without basepoints inside. This defines the differential for the long oval complex. In general,  $m(D_{\mathbf{x},\mathbf{y}})$  can be defined by using the procedure of shortening of ovals described in the next section inductively. In particular,  $\partial_{\text{short}}$  is defined in Section 1.3.

**1.2. Shortening of ovals.** Assume that one vertical and one horizontal oval used to define  $(C, \partial)$  form a bigon with corners  $x$  and  $y$  without basepoints inside. We further assume that a counterclockwise rotation along the arc of the horizontal oval, leads from  $x$  to  $y$ . An example is shown on the left of Figure 4. Suppose that  $\partial$  is given by (4) with known multiplicities.

Let  $(C', \partial')$  be a new complex obtained from  $(C, \partial)$  as follows. The set of generators  $S(C')$  is obtained from  $S(C)$  by removing all generators containing  $x$  or  $y$ . The differential

$$\partial' := P \circ (\partial + \partial \circ h \circ \partial) \circ I,$$

where  $I : C' \rightarrow C$  and  $P : C \rightarrow C'$  are the obvious inclusion and projection. Moreover, for any  $\mathbf{x} \in S(C)$ ,  $h(\mathbf{x})$  is zero whenever  $y \notin \mathbf{x}$ , otherwise  $h(\mathbf{x})$  is obtained from  $\mathbf{x}$  by replacing  $y$  by  $x$ .

**Proposition 1.1.**  *$(C', \partial')$  is a chain complex homotopic to  $(C, \partial)$ . The differential  $\partial'$  can be given by (4) for a new system of curves obtained by shortening of one oval as shown in Figure 4.*

For the proof of this proposition we will need the following lemma.

**Lemma 1.2.** *Whenever  $x \in \mathbf{x}$ , we have  $h \circ \partial \mathbf{x} = \mathbf{x}$*

*Proof.* We have to show that for  $\mathbf{x}$  with  $x \in \mathbf{x}$ ,  $\mathbf{z}$  with  $y \in \mathbf{z}$  occurs in  $\partial \mathbf{x}$  if and only if  $\mathbf{z}$  is obtained from  $\mathbf{x}$  by replacing  $x$  by  $y$ .



Indeed, if  $\mathbf{z} \in \partial\mathbf{x}$ , then there exists a domain  $D_{\mathbf{x},\mathbf{z}} \in \mathfrak{D}$  with  $m(D_{\mathbf{x},\mathbf{z}}) = 1$ . Let us construct its boundary  $\gamma_{\mathbf{x},\mathbf{z}} = \partial D_{\mathbf{x},\mathbf{z}}$ . We start at  $x$  and go to  $y$  along an arc of a horizontal oval. No other points in  $\mathbf{x}$  or  $\mathbf{z}$  belong to this arc, since only one point of each generator is on the same oval. There are two choices for this arc: the short or the long one. The domain having the long arc as a part of its boundary contains basepoints inside and will not be counted. Analogously, going back from  $y$  to  $x$  we have to take the short arc of the vertical oval, otherwise the domain will not be in  $\mathfrak{D}$ . Hence,  $D_{\mathbf{x},\mathbf{z}}$  contains a small bigon between  $x$  and  $y$  as a boundary component. Let us show that  $D_{\mathbf{x},\mathbf{z}}$  has only one boundary component. Clearly, there are no further boundary components inside of this bigon. On the other hand, the bigon could not be an inner boundary component, hence in this case, its orientation would be reversed, and then  $D_{\mathbf{z},\mathbf{x}}$  would be in  $\mathfrak{D}$ , but not  $D_{\mathbf{x},\mathbf{z}}$ . Since  $\mathfrak{D}$  contains connected domains only, we proved the claim.  $\square$

**Proof of Proposition 1.1.** Let us define the maps  $F : C \rightarrow C'$  and  $G : C' \rightarrow C$  as follows.

$$F = P \circ (\mathbb{I} + \partial \circ h) \quad G = (\mathbb{I} + h \circ \partial) \circ I$$

Here  $\mathbb{I}$  is the identity map. It is not difficult to see, that  $F \circ G$  is the identity map on  $C'$  since  $P \circ h$ ,  $h \circ I$  and  $h^2$  are zero maps.

The map  $G \circ F$  is homotopic to the identity on  $C$ , i.e.  $G \circ F + \mathbb{I} = \partial \circ h + h \circ \partial$ . This is easy to see for generators in  $S(C) \cap S(C')$  or for  $\mathbf{x}$  with  $x \in \mathbf{x}$ . Let us assume  $y \in \mathbf{y}$ , then  $\partial \circ h(\mathbf{y}) = \mathbf{y} + \tilde{\mathbf{x}} + \tilde{\mathbf{z}}$ , where  $\tilde{\mathbf{x}}$  is a linear combination of generators, such that any of them contains  $x$  and  $\tilde{\mathbf{z}}$  is a linear combination of generators in  $S(C) \cap S(C')$ . Then

$$G \circ F(\mathbf{y}) = (\mathbb{I} + h \circ \partial)(\tilde{\mathbf{z}}) = \tilde{\mathbf{z}} + h(\partial\mathbf{y}) + \tilde{\mathbf{x}}$$

by using  $\partial^2 = 0$  and the previous claim. On the other hand,

$$(\mathbb{I} + h \circ \partial + \partial \circ h)(\mathbf{y}) = \mathbf{y} + \mathbf{y} + \tilde{\mathbf{x}} + \tilde{\mathbf{z}} + h(\partial\mathbf{y}).$$

Now using Lemma 1.2 one can easily show that the differential  $\partial'$  coincides with  $F \circ \partial \circ G$ . Furthermore, using the homotopy to the identity, proved above, we derive  $\partial'^2 = F \circ \partial \circ G \circ F \circ \partial \circ G = 0$ . This shows that  $(C', \partial')$  is indeed a chain complex.

To show that  $F$  and  $G$  are chain maps, i.e.  $\partial' \circ F = F \circ \partial$  and  $\partial \circ G = G \circ \partial'$ , we again use  $\partial \circ h \circ \partial \circ h \circ \partial = \partial \circ h \circ \partial$ , which is a consequence of Lemma 1.2.

Finally, we would like to show that the new differential  $\partial' = P \circ (\partial + \partial \circ h \circ \partial) \circ I$  can be realized by counting of Maslov index one domains for a new system of curves. We write  $\mathbf{y} \in \partial\mathbf{x}$  if  $m(D_{\mathbf{x},\mathbf{y}}) = 1$ . Assume  $\mathbf{y} \in \partial\mathbf{x}$  and  $\mathbf{x}, \mathbf{y}$  do not contain the corners of

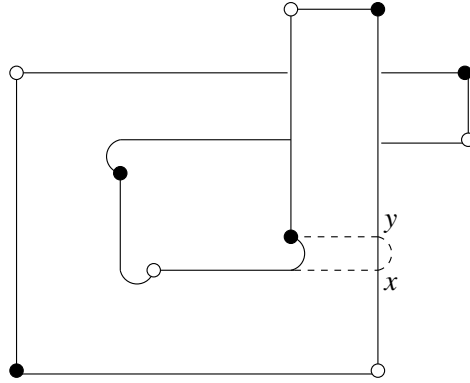


FIGURE 5. **Immersed polygon realizing a differential from black to white points.** The prolonged oval is shown by a dashed line.

the bigon  $x$  and  $y$ . For all such  $\mathbf{x}$  and  $\mathbf{y}$ , we also have  $\mathbf{y} \in \partial'\mathbf{x}$ , and they are connected by a Maslov index one domain  $D_{\mathbf{x},\mathbf{y}} \in \mathfrak{D}$ .

Furthermore, assume  $\mathbf{a}, \mathbf{b} \in \mathbf{S}(C)$  do not contain  $x$  and  $y$ , then for any  $\mathbf{x}, \mathbf{y} \in \mathbf{S}(C)$  with  $y \in \mathbf{y}$ ,  $x \in \mathbf{x}$ , such that  $h(\mathbf{y}) = \mathbf{x}$ ,  $\mathbf{y} \in \partial\mathbf{a}$ , and  $\mathbf{b} \in \partial\mathbf{x}$ ,  $\mathbf{b}$  occurs once in  $\partial\mathbf{a}$ . Note that  $\mathbf{b} \notin \partial\mathbf{a}$ , since any domain connecting  $\mathbf{a}$  to  $\mathbf{b}$  contains either the bigon with negative orientation or basepoints, hence they do not count for the differential. However, the new system of ovals contains a domain connecting  $\mathbf{a}$  to  $\mathbf{b}$  which is obtained from  $D_{\mathbf{a},\mathbf{y}} \cup D_{\mathbf{x},\mathbf{b}}$  by shortening the oval.

This process is illustrated in Figure 5, where  $\mathbf{a}$  and  $\mathbf{b}$  are given by black and white points respectively;  $\mathbf{y}$  is obtained from  $\mathbf{a}$  by switching the black point on the dashed oval to  $y$  and the upper black point to the white point on the same oval;  $\mathbf{x}$  is obtained from  $\mathbf{y}$  by switching  $y$  to  $x$ .

It remains to show that the domain, obtained from  $D_{\mathbf{a},\mathbf{y}} \cup D_{\mathbf{x},\mathbf{b}}$  by shortening the oval, belongs to  $\mathfrak{D}$ . The domain is obtained by connecting  $D_{\mathbf{a},\mathbf{y}}$  to  $D_{\mathbf{x},\mathbf{b}}$  by two arcs shown on the right of Figure 4, i.e. the domain is connected. It has no basepoints inside, since  $D_{\mathbf{a},\mathbf{y}}$  and  $D_{\mathbf{x},\mathbf{b}}$  do not have them. Analogously, it can be written as  $\sum_i n_i D_i$  with all  $n_i \geq 0$ . Finally, we show that our domain has no corners inside. If  $D_{\mathbf{a},\mathbf{y}} \cap D_{\mathbf{x},\mathbf{b}}$  is empty, it follows from the assumption that  $D_{\mathbf{x},\mathbf{b}}, D_{\mathbf{a},\mathbf{y}} \in \mathfrak{D}$ , i.e. have no corners inside. If the intersection is not empty, then its boundary either contains no corners or at least two corners, one of them in  $\mathbf{y} \setminus y = \mathbf{x} \setminus x$  (see Figure 6). The last is impossible since  $D_{\mathbf{x},\mathbf{b}} \in \mathfrak{D}$  has no corners of  $\mathbf{x}$  and  $D_{\mathbf{a},\mathbf{y}}$  no corners of  $\mathbf{y}$  inside.  $\square$

**Note.** The first statement of Proposition 1.1 is a particular case of the Gaussian elimination considered in e.g. [1, Lemma 4.2].

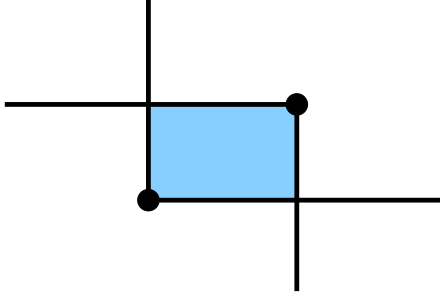


FIGURE 6. Intersection of two domains with two corners on its boundary.

1.3. **Definition**  $(C_{\text{short}}(D), \partial_{\text{short}})$ . The complex  $(C_{\text{short}}(D), \partial_{\text{short}})$  is obtained from the complex with long ovals by applying Lemma 1.1 several times, until the complex has  $\mathbf{S}$  as the set of generators. This subsection aims to give a recursive definition of the multiplicity  $m(D_{\mathbf{a}, \mathbf{b}})$  of  $\mathbf{b}$  in the differential  $\partial_{\text{short}}(\mathbf{a})$  for all  $\mathbf{a} \in \mathbf{S}$ . In general, it will depend on the order in which the ovals were shortened.

Let us fix this order. If  $D_{\mathbf{a}, \mathbf{b}} \in \mathfrak{D}$  is a polygon we count it with multiplicity one. For any other domain in  $\mathfrak{D}$  we prolongate the last oval that was shortened to obtain this domain, and show whether in the resulting complex  $(C'(D), \partial')$ , one can find  $\mathbf{x}$  and  $\mathbf{y}$  with  $\mathbf{y} \in \partial' \mathbf{a}$ ,  $h(\mathbf{y}) = \mathbf{x}$  and  $\mathbf{b} \in \partial' \mathbf{x}$  as in the proof of Lemma 1.1. If there is an odd number of such  $\mathbf{x}$  and  $\mathbf{y}$ , then  $m(D_{\mathbf{a}, \mathbf{b}}) = 1$ , otherwise the multiplicity is zero. To determine  $m(D_{\mathbf{a}, \mathbf{y}})$  and  $m(D_{\mathbf{x}, \mathbf{b}})$  in the new complex, we prolongate the next oval, and continue to do so until the domains in question are polygons which always count.

In fact, the algorithm terminates already when the domains in question are strongly indecomposable without bad components (as defined in the next section) since they all count for the differential (cf. Theorem 2.1 below).

1.4. **Proof of Theorem 1.** It remains to compute the homology of our complex.

By Lemma 1.1, the complex  $(C_{\text{short}}(D), \partial_{\text{short}})$  is homotopy equivalent to the complex with long ovals. The homotopy preserves both gradings, since  $\partial_{\text{short}}$  count only Maslov index one domains without basepoints inside. The complex  $(C_{\text{long}}(D), \partial_{\text{long}})$  coincides with the complex computing the hat version of link Floer homology from the genus zero Heegaard splitting of  $S^3$  with extra basepoints (see [9] for summary). The relative Maslov and Alexander of these two complexes are also the same. Moreover, the absolute Alexander and Maslov gradings in our complex are fixed in such a way, that

$$\chi(C_{\text{short}}(D), \partial_{\text{short}}) = \begin{cases} \prod_{i=1}^{\ell} (1 - t^{-1})^{n_i - 1} \Delta_L(t_1, \dots, t_{\ell}) & \ell > 1 \\ (1 - t^{-1})^{n_i - 1} \Delta_L(t) & \ell = 1 \end{cases}$$

This can be shown by comparing our and MOS complexes (see [2, Theorem 4.2] for more details).

Hence Proposition 2.4 in [9] computes the homology of our combinatorial complex.  $\square$

## 2. DOMAINS THAT ALWAYS COUNT

To run the algorithm defining  $\partial_{\text{short}}$  we have to decide at each step which domains count and which do not. In this Section we simplify the algorithm by selecting a large class of domains that always count for the differentials obtained from  $\partial_{\text{long}}$  by applying Lemma 1.1 several times. Let us start with some definitions.

**2.1. Maslov index.** Let  $e(S)$  be the Euler measure of a surface  $S$ , which for any surface  $S$  with  $k$  acute right-angled corners,  $l$  obtuse ones, and Euler characteristic  $\chi(S)$  is equal to  $\chi(S) - k/4 + l/4$ . Moreover, the Euler measure is additive under disjoint union and gluing along boundaries. In [8, Section 4], Lipshitz gave a formula computing the Maslov index  $M(D_{\mathbf{x},\mathbf{y}})$  of  $D_{\mathbf{x},\mathbf{y}}$  as follows.

$$(5) \quad M(D_{\mathbf{x},\mathbf{y}}) = e(D_{\mathbf{x},\mathbf{y}}) + n_{\mathbf{x}} + n_{\mathbf{y}},$$

where  $n_{\mathbf{x}} = \sum_{x \in \mathbf{x}} n_x$ . The number  $n_x$  is the local multiplicity of the domain at the corner  $x$ , e.g.  $n_x = 0$  for an isolated corner,  $n_x = 1/4$  for an acute (or  $\pi/2$ -angled) corner,  $n_x = 1/2$  for a straight (or  $\pi$ -angled) corner or  $n_x = 3/4$  for an obtuse (or  $3\pi/2$ -angled) one. For a composition of two domains  $D_{\mathbf{x},\mathbf{z}} = D_{\mathbf{x},\mathbf{y}} \circ D_{\mathbf{y},\mathbf{z}}$ , we have  $M(D_{\mathbf{x},\mathbf{z}}) = M(D_{\mathbf{x},\mathbf{y}}) + M(D_{\mathbf{y},\mathbf{z}})$ .

A path in a domain starting at an obtuse or straight corner and following a horizontal or vertical oval until the boundary of the domain will be called a *cut*. There are two cuts at any obtuse corner and one at any straight corner.

A domain  $D$  is called *decomposable* if it is a composition of Maslov index zero and one domains; any other Maslov index 1 domain is called *indecomposable*. A domain is called *strongly indecomposable* if the following conditions are satisfied:

- it is indecomposable;
- no prolongations of ovals inside this domain destroy its indecomposability;
- the cuts do not intersect inside the domain.

An example of an indecomposable, but not strongly indecomposable domain is shown in Figure 7.

**2.2. Count of domains.** In what follows any domain is assumed to belong to  $\mathfrak{D}$  and to have Maslov index one, i.e. our domains have no corners with negative multiplicities or with multiplicities bigger than  $3/4$  (since domains from  $\mathfrak{D}$  have no corners inside).

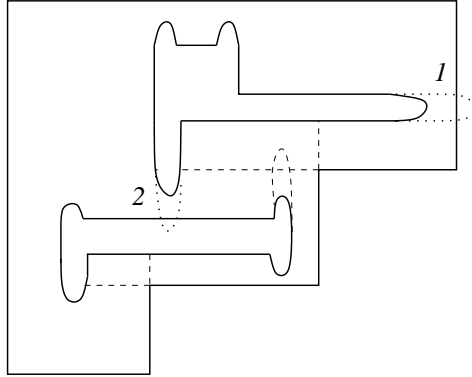


FIGURE 7. **Indecomposable, but not strongly indecomposable domain.**  
The oval destroying the indecomposability is shown by the dashed line without any number.

We say that a cut touches a boundary component  $A$  at an oval  $B$  if either the end point of this cut belongs to  $B$  or the cut leaves  $A$  along  $B$ . We define the distance between two cuts touching  $A$  to be odd, if one of them touches  $A$  at a vertical oval and another one at a horizontal oval; otherwise the distance is even.

A boundary component is called *special* if it is an oval with a common corner of two generators (see Figure 12 right for an example), otherwise the component is non-special. Let us call an inner boundary component *bad*, if it does not have obtuse corners.

**Theorem 2.1.** *Any strongly indecomposable domain without bad components counts for the differential.*

The proof is given in Section 2.4 after the detailed analysis of the structure of indecomposable domains.

**Remark.** In the long oval complex,  $\partial_{\text{long}}$  coincides with the differential of link Floer homology defined by counting of pseudo-holomorphic discs. The differential  $\partial_{\text{short}}$  at least for some order of shortening can not be realized by such count. This is because, according to the Gromov compactness theorem the count of indecomposable domains is independent of the complex structure, i.e. each of them either counts or not for any complex structure. In our setting, e.g. for the domain in Figure 7 one can always choose an order of shortening in such a way that our count differs from the analytic one. Indeed, if this domain is obtained by first shortening the dotted part of the oval labeled by 1 and then the dashed part, the domain does not count. However, if we get it by shortening the part 2, and then the dashed one, it counts.

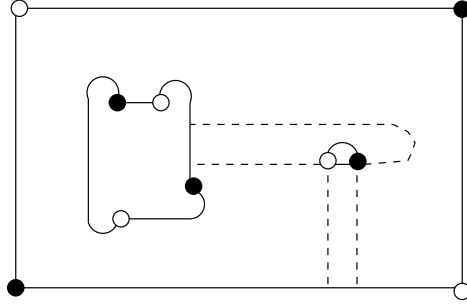


FIGURE 8. An indecomposable domain whose inner boundary components are not  $y$ -connected

2.3. **Structure of domains.** Here we provide some technical results needed for the proof of Theorem 2.1.

**Definition 2.2.** A boundary component  $C_1 \subset \partial D_{\mathbf{x},\mathbf{y}}$  is called  $y$ -connected with another component  $C_2 \subset \partial D_{\mathbf{x},\mathbf{y}}$  if for any point  $y \in C_1$  and  $q \in C_2$  disjoint from the corners, there exists a unique path without self-intersections starting at  $y$  and ending at  $q$ , such that

- 1) the path goes along cuts or  $\partial D_{\mathbf{x},\mathbf{y}}$ , where the arcs of horizontal and vertical ovals alternate along the path (an arc can consist of the union of a cut and some part of  $\partial D_{\mathbf{x},\mathbf{y}}$ , as long as they are on the same oval);
- 2) the corners of the path (i.e. intersection points of horizontal and vertical segments) come alternatively from  $\mathbf{x}$  and  $\tilde{\mathbf{y}}$ , where  $\tilde{\mathbf{y}}$  contains  $\mathbf{y}$  and intersection points of cuts with  $\partial D_{\mathbf{x},\mathbf{y}}$ ;
- 3) the intersection of the path with a boundary component is neither a point nor the whole component.
- 4) the first corner belongs to  $\mathbf{x}$ ;

As an example consider the domain shown in Figure 8. Let us denote the left inner boundary component by  $C_1$  and the right one by  $C_2$ . For any choice of a point  $y$  on a vertical oval of  $C_1$  (disjoint from the corners), there is no path  $y$ -connecting  $C_1$  with  $C_2$ . On the other hand, if we choose  $y$  on the horizontal oval of  $C_1$  there are two such paths. Hence,  $C_1$  and  $C_2$  are not  $y$ -connected.

**Lemma 2.3.** *In a strongly indecomposable domain without bad components, any two boundary components are  $y$ -connected.*

*Proof.* Let  $c$  be the total number of boundary components in our domain. The proof is by induction on  $c$ . Assume first that our domain has no special components.

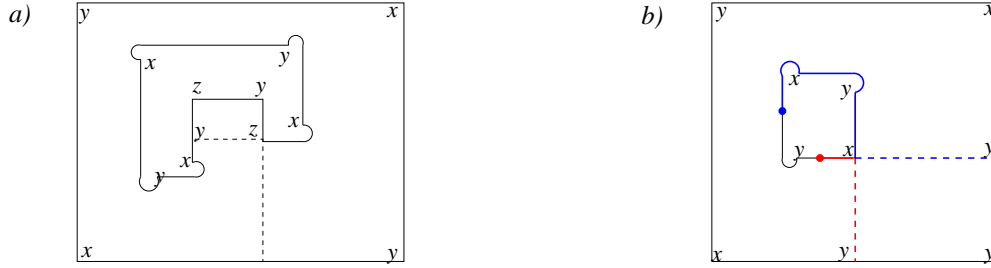


FIGURE 9. **Case**  $b = 0$ ,  $c = 2$ . *a)* The domain is the composition  $D_{x,y} \circ D_{y,z}$ , where  $\mathbf{z} \setminus (\mathbf{z} \cap \mathbf{y})$  are the two points marked by  $z$ . The cuts are shown by dashed lines. *b)* Indecomposable domain. The corners from  $\tilde{\mathbf{y}}$  are marked by  $y$ . For any choice of a point  $y$  on the vertical oval of the inner component, the path  $y$ -connecting both components uses a horizontal cut. Analogously, for  $y$  on a horizontal oval, we have to use the vertical cut. Examples are shown in blue and red, respectively.

Further, all corners of our domain have positive multiplicities not bigger than  $3/4$  and our domain has Maslov index one. Since we have no bad components, (5) implies that every inner boundary component has exactly one obtuse corner.

Suppose  $c = 2$ . If one of the cuts from the obtuse corner connects the inner boundary component with itself, the domain is decomposable (see Figure 9 (a)). If it is not the case, then an easy check verifies the claim (compare Figure 9 (b)).

Assume the claim holds for  $c = n - 1$ . Suppose  $c = n$ , and our domain is indecomposable. Let us denote by  $A$  the  $n$ -th component. Let us first assume that there are no cuts ending at  $A$ . In this case the two cuts from the obtuse corner  $y$ -connect  $A$  with some other components which are all  $y$ -connected by induction.

If there is a component connected with  $A$  by two cuts, then it is  $y$ -connected with the outside exactly in the case when  $A$  has this property. To check this, it is sufficient to find a required path for two choices of  $y$  (before and after one corner) on this component. An example is shown in Figure 10. Hence, when all cuts ending at  $A$  come from components connected with  $A$  by two cuts (as in Figure 10), then  $A$  is  $y$ -connected to the outside by the previous argument.

In the case, when  $C \subset \partial D_{x,y}$  is connected with  $A$  by just one cut or  $A$  and  $C$  exchange their cuts, all components are again  $y$ -connected to each other except when the following happens. The path described in Definition 2.2 after leaving  $A$  (along one of the cuts) comes back to  $A$  without visiting all other components. Since this path leaves and enter any component along cuts at odd distance (compare Figure 11), and

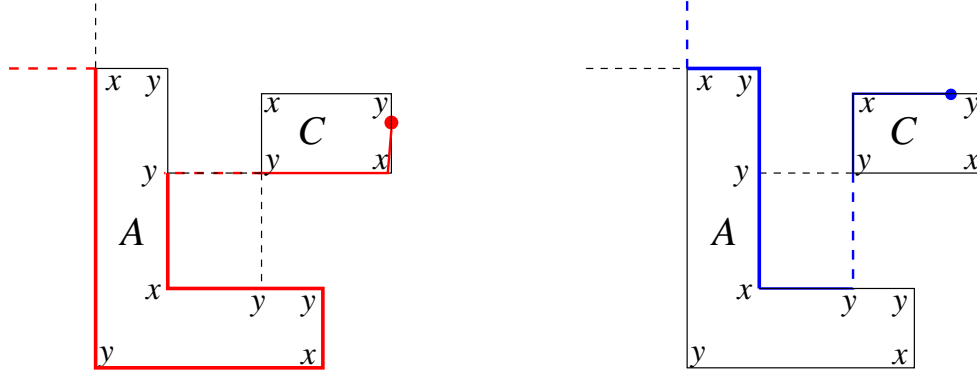


FIGURE 10.  $C$  is  $y$ -connected to  $A$ . The two choices of  $y$  are shown by red and blue dots. The connected paths have the corresponding colors. All corners without cuts are assumed to be acute.

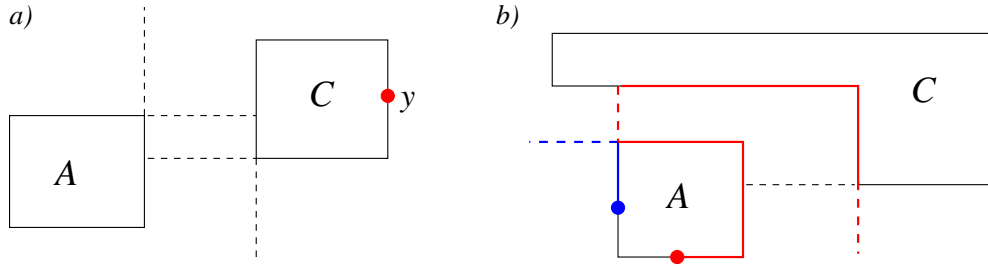


FIGURE 11.  $A$  and  $C$  exchange two cuts. *a)* Domain is decomposable. The point  $y \in C$  is not connected with  $A$  by a path described in Definition 2.2. *b)* Indecomposable domain.  $A$  and  $C$  are  $y$ -connected.

has no self intersections, it can be used to decompose the domain, which contradicts the assumption.

It remains to consider the case where  $D_{x,y}$  has straight corners, or special components, i.e. ovals with a common corner of two generators. We proceed by induction on the number of special components. Assume that we have one special component. Then the cut from the straight corner  $y$ -connects this component with any other one (otherwise not connected with the special one) by the previous argument. The obvious induction completes the proof.  $\square$

**2.4. Proof of Theorem 2.1.** The proof is again by induction on the number of boundary components  $c$  in the domain. If  $c = 1$ , it is easy to see by prolonging ovals that any immersed polygon counts.



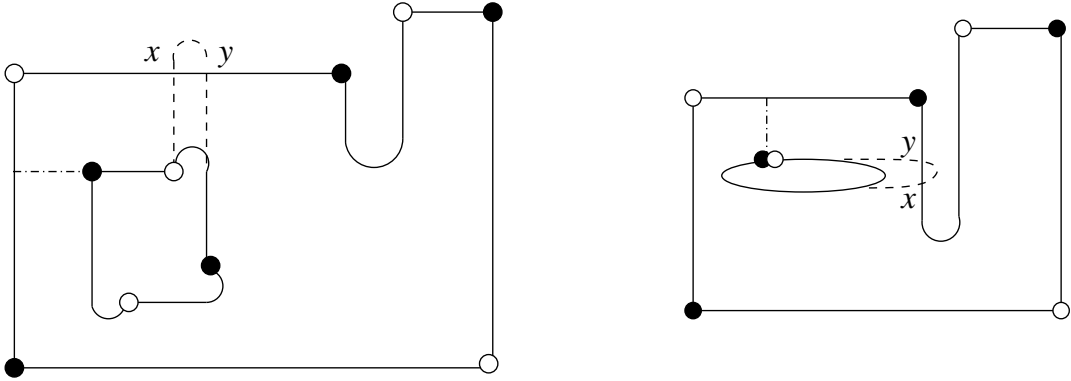


FIGURE 12. **Strongly indecomposable domains realizing differentials from black to white points.** The prolonged oval is shown by a dashed line.

Assume that for  $c = n - 1$ , the claim holds. Suppose our complex has a strongly indecomposable domain  $D \in \mathfrak{D}$  without bad components and  $c = n$ . Examples with  $c = 2$  are drawn in Figure 12.

Let us stretch one oval in  $D$  connecting two boundary components. The result is a domain  $D'$ . Let  $x$  and  $y$  be the corners of the bigon, obtained after stretching. The stretched oval connects  $y$  to some boundary component, say  $A$ . By Lemma 2.3,  $y$  can also be connected with  $A$  by a unique path inside  $D$ . This path is not affected by the prolongation, since otherwise the domain would not be strongly indecomposable (compare Figure 7).

Hence,  $D'$  can be represented as a union of two domains connecting some generators and having less boundary components. The unique path connecting  $y$  with  $A$  leaves any boundary component along a cut. Moreover, the path has no self-intersections. Therefore,  $D'$  is a union of two strongly indecomposable Maslov index one domains without bad components and with positive multiplicities at the corners not bigger than  $3/4$ . These both domains count for the differential by the induction hypothesis. We conclude that the domain  $D$  also counts for the differential.

□

### 3. COMPUTATIONS

In this section we show how  $\widehat{\text{HFK}}$  of small knots can be computed by hand and discuss the computer program written by Droz.

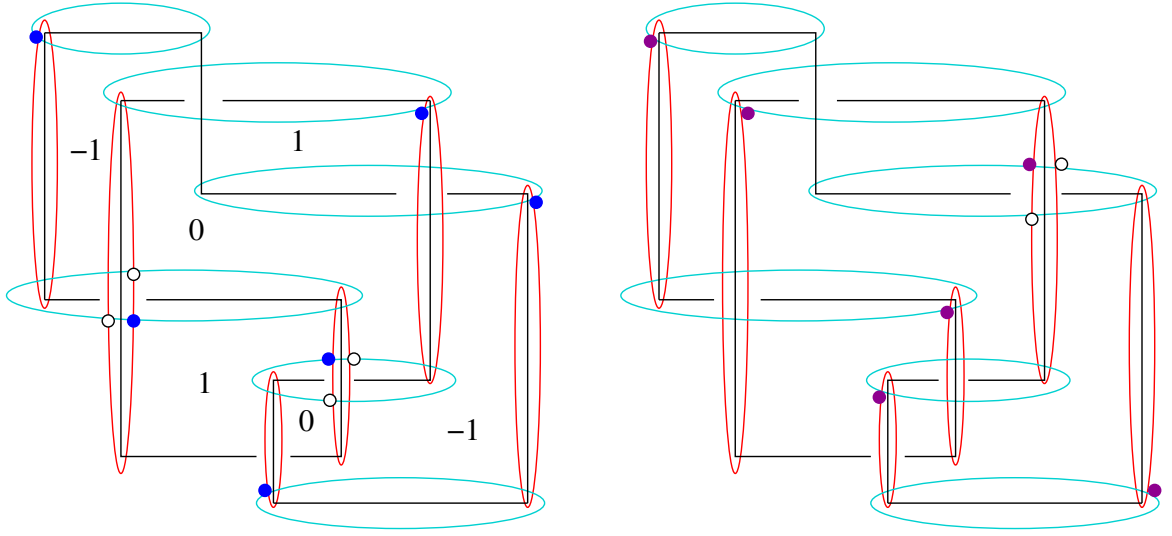


FIGURE 13. **A complex for  $5_2$  knot.** The colored dots show generators in the maximal Alexander grading equal to 1. A number assigned to a region is minus the winding number for its points.

3.1.  **$5_2$  knot.** Figure 13 shows a rectangular diagram for  $5_2$  knot of complexity  $n = 7$  obtained from the original diagram in Figure 1 by cyclic permutations (compare [4]). An advantage of this diagram is that there are no regions counted for the differential.

The Alexander grading of a generator is given by the formula  $A(\mathbf{x}) = \sum_{x \in \mathbf{x}} a(x) - 2$ . The maximal Alexander grading is equal to one. There are two generators in this grading shown by colored dots in Figure 13. Both of them have Maslov grading 2.

The homology of our complex is  $\widehat{\text{HFK}}(5_2) \otimes V^6$ . Hence in Alexander grading zero, we have 12 additional generators coming from the multiplication with  $V$ . Note that our complex has 15 generators in Alexander grading zero. Indeed, 12 of them can be obtained by moving one point of a generator in Alexander grading one to the other side of the oval. In three cases, depicted by white dots there are two possibilities to move a point. This gives 3 additional generators. Note that these moves drop Maslov index by one. To compute  $\widehat{\text{HFK}}$  in the negative Alexander gradings we use the symmetry (1).

Finally, we derive that  $\widehat{\text{HFK}}(5_2)$  has rank two in the Alexander–Maslov bigrading  $(1, 2)$ , rank three in  $(0, 1)$ , and rank two in the bigrading  $(-1, 0)$ . To compare, the Alexander polynomial is  $\Delta_{5_2}(t) = 2(t + t^{-1}) - 3$ . The knot  $5_2$  is not fibered and its Seifert genus is one.

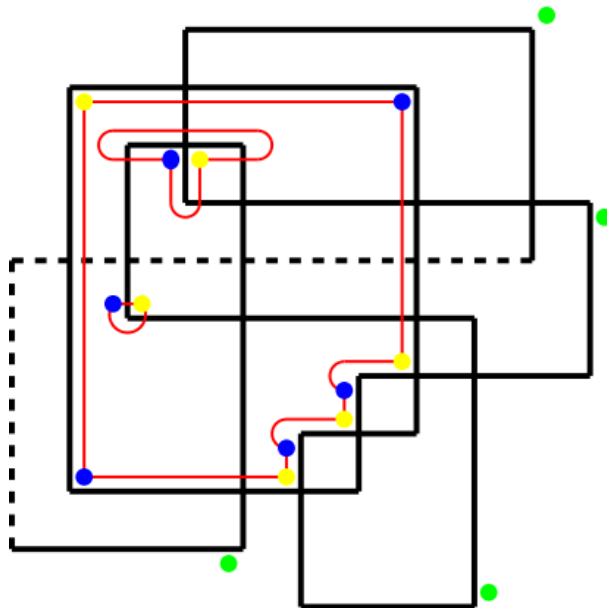


FIGURE 14. **Domain counted twice.** The knot  $11n3$  is drawn in black, the dashed lines have no ovals. The boundary of the domain from the blue to the yellow points is red. The common points of these two generators are green. If the domain is obtained by shortening of the smallest inner component, it counts twice over  $\mathbb{Z}$ .

3.2. **Droz's program.** In [2], Droz extend our construction over  $\mathbb{Z}$  and wrote a computer program calculating the homology of the resulting complex [3]. The program is installed on the Bar–Natan's Knot Atlas. As a byproduct, his program generates rectangular diagrams of knots and links and allows to change them by Cromwell–Dynniov moves. The program can be used to determine Seifert genus and fiberedness of knots until 16 crossings.

According to Droz's computations, the number of generators in our complex is significantly smaller than that in the MOS complex. Moreover, for small knots, almost all domains suitable for the differential are embedded polygons, so they always count for the differential. For example, for knots admitting rectangular diagrams of complexity 10, the number of generators in the MOS complex is  $10! = 3'628'800$ . Our complex has on average about 50'000 generators among them about 1'000 in the positive Alexander gradings. The knot  $12n2000$  admits a rectangular diagram of complexity 12, where  $12! = 479'001'600$ . Our complex has 1'411'072 generators with 16'065 of them in the positive Alexander gradings.

Furthermore, Droz's program produced examples of domains counted more than once over  $\mathbb{Z}$ . We do not know similar examples in the analytic setting. In Figure 15 one such domain is shown. This domain has a degenerate system of cuts and its count depends on the order of shortening of ovals. One specific order gives multiplicity 2 for this domain.

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INSTITUT FÜR MATHEMATIK, UNIVERSITÄT ZÜRICH, WINTERTHURERSTRASSE 190, CH-8057  
ZÜRICH, SWITZERLAND

*E-mail address:* `anna@math.unizh.ch`