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# Correctors for Some Asymptotic Problems

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**Abstract**—In the theory of anisotropic singular perturbation boundary value problems, the solution  $u_\varepsilon$  does not converge, in the  $H^1$ -norm on the whole domain, towards some  $u_0$ . In this paper we construct correctors to have good approximations of  $u_\varepsilon$  in the  $H^1$ -norm on the whole domain. Since the anisotropic singular perturbation problems can be connected to the study of the asymptotic behaviour of problems defined in cylindrical domains becoming unbounded in some directions, we transpose our results for such problems.

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## 1. INTRODUCTION

Let  $\mathcal{O} = (-1, 1) \times \omega$  be a bounded open subset of  $\mathbb{R}^{p+1}$ ,  $p \geq 1$ ,  $\omega$  being a bounded open subset of  $\mathbb{R}^p$ . We denote by  $x = (X_1, X_2)$  the points of  $\mathcal{O}$  with

$$X_1 = x_1, \quad X_2 = (x'_1, \dots, x'_p).$$

With this notation we set

$$\nabla u = (\partial_{x_1} u, \partial_{x'_1} u, \dots, \partial_{x'_p} u)^T = \begin{pmatrix} \partial_{X_1} u \\ \nabla_{X_2} u \end{pmatrix},$$

where

$$\nabla_{X_2} u = (\partial_{x'_1} u, \dots, \partial_{x'_p} u)^T.$$

For  $f \in L^2(\mathcal{O})$  and  $\varepsilon > 0$ , there exists a unique solution  $u_\varepsilon$  (in a weak sense) of

$$\begin{cases} u_\varepsilon \in H_0^1(\mathcal{O}), \\ -\varepsilon^2 \partial_{X_1}^2 u_\varepsilon - \Delta_{X_2} u_\varepsilon = f \quad \text{in } \mathcal{O}. \end{cases} \quad (1.1)$$

We denote by  $\Delta_{X_2}$  the Laplace operator defined by

$$\Delta_{X_2} = \partial_{x'_1}^2 + \dots + \partial_{x'_p}^2.$$

For a.e.  $X_1 \in (-1, 1)$  one can define a solution  $u_0$  to

$$\begin{cases} u_0(X_1, \cdot) \in H_0^1(\omega), \\ -\Delta_{X_2} u_0(X_1, \cdot) = f(X_1, \cdot) \quad \text{in } \omega. \end{cases} \quad (1.2)$$

It is shown in [3, 4] that

$$u_\varepsilon \rightarrow u_0 \quad \text{in } L^2(\mathcal{O}) \quad \text{as } \varepsilon \rightarrow 0. \quad (1.3)$$

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Even if  $\nabla_{X_2} u_\varepsilon \rightarrow \nabla_{X_2} u_0$  in  $(L^2(\mathcal{O}))^p$  (see [3, 4]), one cannot expect in general that

$$u_\varepsilon \rightarrow u_0 \quad \text{in } H^1(\mathcal{O}). \tag{1.4}$$

Indeed, if, for instance,  $f$  is independent of  $X_1$ , then so is  $u_0$  and clearly, for  $f \neq 0$ ,  $u_0 \notin H_0^1(\mathcal{O})$  when  $u_\varepsilon$  does belong to  $H_0^1(\mathcal{O})$ , which makes (1.4) impossible. The goal of this paper is to “correct”  $u_\varepsilon - u_0$  by a simple function  $w_\varepsilon$  which gives the behaviour of  $u_\varepsilon - u_0$  near the end sections  $\{-1, 1\} \times \omega$  and is such that

$$u_\varepsilon - u_0 - w_\varepsilon \rightarrow 0 \quad \text{in } H_0^1(\mathcal{O}). \tag{1.5}$$

Due to the uniqueness of a solution of (1.1), one has (see Lemma 3.4)

$$u_\varepsilon(-X_1, X_2) = u_\varepsilon(X_1, X_2),$$

and this clearly implies that

$$\frac{\partial u_\varepsilon}{\partial X_1}(0, X_2) = 0. \tag{1.6}$$

Thus, to study and correct the behaviour of  $u_\varepsilon - u_0$ , one can consider  $u_\varepsilon$  as the solution to

$$\begin{cases} -\varepsilon^2 \partial_{X_1}^2 u_\varepsilon - \Delta_{X_2} u_\varepsilon = f & \text{in } \Omega = (0, 1) \times \omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \setminus \{0\} \times \omega, \\ \frac{\partial u_\varepsilon}{\partial X_1} = 0 & \text{on } \{0\} \times \omega. \end{cases} \tag{1.7}$$

This is what we will do in the next section. Note that this is inspired by [5], where a similar analysis was carried out for the Stokes problem. In Section 3 we will transpose our results—via a scaling argument—to the Dirichlet problem set in cylinders becoming infinite in various directions.

For more details about the anisotropic singular perturbation problems, as well as for details on the problems considered in Section 3, we refer the reader to [1–6, 8, 9]. The classic singular perturbation problems are dealt with in [10].

## 2. THE CASE OF ANISOTROPIC PROBLEMS IN ONE DIRECTION

Let  $\Omega$  be defined as

$$\Omega = (0, 1) \times \omega,$$

where  $\omega$  is a bounded domain of  $\mathbb{R}^p$ , and  $V$  be the space

$$V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega \setminus \{0\} \times \omega\}. \tag{2.1}$$

There exists a unique solution  $u_\varepsilon$  to

$$\begin{cases} u_\varepsilon \in V, \\ \int_\Omega (\varepsilon^2 \partial_{X_1} u_\varepsilon \partial_{X_1} v + \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v) dx = \int_\Omega f v dx \quad \forall v \in V. \end{cases} \tag{2.2}$$

Clearly (2.2) is the weak formulation of (1.7). We assume that  $f \in L^2(\Omega)$ , and the existence of a unique solution to (2.2) follows from the Lax–Milgram theorem. The weak formulation of (1.2) reads for a.e.  $X_1 \in (0, 1)$  as

$$\begin{cases} u_0(X_1, \cdot) \in H_0^1(\omega), \\ \int_\omega \nabla_{X_2} u_0(X_1, \cdot) \cdot \nabla_{X_2} v dX_2 = \int_\omega f(X_1, \cdot) v dX_2 \quad \forall v \in H_0^1(\omega). \end{cases} \tag{2.3}$$

In the case where

$$f \in L^2(\Omega), \quad \partial_{X_1} f \in L^2(\Omega), \quad (2.4)$$

one can show (see [4]) that

$$u_0 \in H^1(\Omega).$$

Now (see, for instance, [2]) if  $v \in V$ , then for a.e.  $X_1 \in (0, 1)$  one has

$$v(X_1, \cdot) \in H_0^1(\omega). \quad (2.5)$$

Using this test function in (2.3), one derives, after an integration in  $X_1$ , that

$$\int_{\Omega} \nabla_{X_2} u_0 \cdot \nabla_{X_2} v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in V. \quad (2.6)$$

In order to construct a corrector for  $u_\varepsilon$ , we denote by  $S_\ell$  the half-cylinder

$$S_\ell = (\ell, +\infty) \times \omega,$$

where  $\ell \in \mathbb{R}$ . Then if  $\rho: [0, +\infty) \rightarrow [0, 1]$  is the function defined by

$$\rho(x) = \begin{cases} 1 - x & \text{on } [0, 1], \\ 0 & \text{on } (1, +\infty), \end{cases}$$

we introduce  $u$  as the solution to

$$\begin{cases} u \in H_0^1(S_0), \\ \int_{S_0} \nabla u \cdot \nabla v \, dx = \int_{S_0} \nabla(\rho(X_1)u_0) \cdot \nabla v \, dx \quad \forall v \in H_0^1(S_0). \end{cases} \quad (2.7)$$

Since  $u_0 \in H^1(\Omega)$ , the existence and uniqueness of  $u$  follows from the Lax–Milgram theorem. Then we set

$$w(X_1, X_2) = u(X_1, X_2) - \rho(X_1)u_0(X_1, X_2) = u - \rho u_0 \quad (2.8)$$

and denote by  $w_\varepsilon$  the function defined as

$$w_\varepsilon(X_1, X_2) = w\left(\frac{1 - X_1}{\varepsilon}, X_2\right). \quad (2.9)$$

Note that  $w \in H^1(S_0)$  and satisfies in a weak sense

$$\begin{cases} \Delta w = 0 & \text{in } S_0, \\ w = -u_0 & \text{on } \{0\} \times \omega, \quad w = 0 & \text{on } (0, +\infty) \times \partial\omega. \end{cases} \quad (2.10)$$

**2.1. Some preliminary results.** We denote by  $\Omega_-$  the domain defined by

$$\Omega_- = (-1, 0) \times \omega.$$

For  $v \in V$  we define by  $\hat{v}$  the function given by

$$\hat{v}(X_1, X_2) = \begin{cases} v(X_1, X_2), & X_1 \geq 0, \\ v(-X_1, X_2), & X_1 < 0. \end{cases} \quad (2.11)$$

Then we have

**Lemma 2.1.** *For every  $v \in V$  the following equality holds:*

$$\int_{\Omega} (\varepsilon^2 \partial_{X_1} w_\varepsilon \partial_{X_1} v + \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2} v) dx = - \int_{\Omega_-} (\varepsilon^2 \partial_{X_1} w_\varepsilon \partial_{X_1} \widehat{v} + \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2} \widehat{v}) dx.$$

**Proof.** For  $\ell > 0$  we set  $\Omega_\ell = (0, \ell) \times \omega$ . Then first note that for  $v \in V$  we have  $\widehat{v}(1 - \varepsilon X_1, X_2) \in H_0^1(\Omega_{2/\varepsilon})$ . Thus we derive, from (2.10),

$$\int_{\Omega_{2/\varepsilon}} \nabla w \cdot \nabla \widehat{v}(1 - \varepsilon X_1, X_2) dx = 0,$$

whence

$$\int_{\Omega_{1/\varepsilon}} \nabla w \cdot \nabla \widehat{v}(1 - \varepsilon X_1, X_2) dx = - \int_{\Omega_{2/\varepsilon} \setminus \Omega_{1/\varepsilon}} \nabla w \cdot \nabla \widehat{v}(1 - \varepsilon X_1, X_2) dx. \tag{2.12}$$

Making the change of variable  $X'_1 = 1 - \varepsilon X_1$  in the integrals above, we obtain respectively

$$\int_{\Omega_{1/\varepsilon}} \nabla w \cdot \nabla \widehat{v}(1 - \varepsilon X_1, X_2) dx = \frac{1}{\varepsilon} \int_{\Omega} (\varepsilon^2 \partial_{X_1} w_\varepsilon \partial_{X_1} v + \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2} v) dx$$

and

$$\begin{aligned} \int_{\Omega_{2/\varepsilon} \setminus \Omega_{1/\varepsilon}} \nabla w \cdot \nabla \widehat{v}(1 - \varepsilon X_1, X_2) dx &= \int_{\Omega_{2/\varepsilon} \setminus \Omega_{1/\varepsilon}} (-\varepsilon \partial_{X_1} w \partial_{X_1} \widehat{v}(X'_1, X_2) + \nabla_{X_2} w \cdot \nabla_{X_2} \widehat{v}(X'_1, X_2)) dx \\ &= \frac{1}{\varepsilon} \int_{\Omega_-} (\varepsilon^2 \partial_{X_1} w_\varepsilon \partial_{X_1} \widehat{v} + \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2} \widehat{v}) dx. \end{aligned}$$

The lemma follows from (2.12).

We will also need the following lemma.

**Lemma 2.2.** *There exist positive constants  $C > 0$  and  $\alpha > 0$  independent of  $\varepsilon$  such that*

$$\int_{S_{1/\varepsilon}} |\nabla w|^2 dx \leq C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2 dx. \tag{2.13}$$

**Proof.** Without loss of generality, we assume that  $\varepsilon < 1$ . Let  $\gamma_\varepsilon: \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function such that  $\gamma_\varepsilon = 0$  in  $(-\infty, \frac{1}{\varepsilon} - 1)$ ,  $\gamma_\varepsilon = 1$  in  $(\frac{1}{\varepsilon}, +\infty)$  and  $\gamma_\varepsilon$  is linear in  $[\frac{1}{\varepsilon} - 1, \frac{1}{\varepsilon}]$ . Since  $\gamma_\varepsilon(X_1)w \in H_0^1(S_0)$ , we have, by (2.10),

$$\int_{S_0} \nabla w \cdot \nabla (\gamma_\varepsilon(X_1)w) dx = 0. \tag{2.14}$$

Thus

$$\begin{aligned} \int_{S_0} |\nabla w|^2 \gamma_\varepsilon(X_1) dx &= - \int_{S_{1/\varepsilon-1} \setminus S_{1/\varepsilon}} \partial_{X_1} w \partial_{X_1} \gamma_\varepsilon(X_1)w dx \leq \int_{S_{1/\varepsilon-1} \setminus S_{1/\varepsilon}} |\partial_{X_1} w| |w| dx \\ &\leq \frac{1}{2} \int_{S_{1/\varepsilon-1} \setminus S_{1/\varepsilon}} |\partial_{X_1} w|^2 dx + \frac{1}{2} \int_{S_{1/\varepsilon-1} \setminus S_{1/\varepsilon}} |w|^2 dx. \end{aligned}$$

Applying the Poincaré inequality in  $X_2$  to the last integral, we get for some constant  $C_\omega$

$$\int_{S_{1/\varepsilon-1} \setminus S_{1/\varepsilon}} |w|^2 dx = \int_{1/\varepsilon-1}^{1/\varepsilon} \int_{\omega} |w|^2 dx \leq C_\omega \int_{1/\varepsilon-1}^{1/\varepsilon} \int_{\omega} |\nabla_{X_2} w|^2 dx.$$

This leads to

$$\begin{aligned} \int_{S_{1/\varepsilon}} |\nabla w|^2 dx &\leq \frac{\max(1, C_\omega)}{2} \int_{S_{1/\varepsilon-1} \setminus S_{1/\varepsilon}} |\nabla w|^2 dx \\ &= \frac{\max(1, C_\omega)}{2} \int_{S_{1/\varepsilon-1}} |\nabla w|^2 dx - \frac{\max(1, C_\omega)}{2} \int_{S_{1/\varepsilon}} |\nabla w|^2 dx, \end{aligned}$$

and thus

$$\int_{S_{1/\varepsilon}} |\nabla w|^2 dx \leq r \int_{S_{1/\varepsilon-1}} |\nabla w|^2 dx,$$

where  $r = \frac{\max(1, C_\omega)}{2 + \max(1, C_\omega)}$ . Iterating  $[\frac{1}{\varepsilon}]$  times this formula ( $[\frac{1}{\varepsilon}]$  is the integer part of  $\frac{1}{\varepsilon}$ ), we obtain

$$\int_{S_{1/\varepsilon}} |\nabla w|^2 dx \leq r^{[\frac{1}{\varepsilon}]} \int_{S_{1/\varepsilon-[\frac{1}{\varepsilon}]}} |\nabla w|^2 dx.$$

Since  $\frac{1}{\varepsilon} - 1 < [\frac{1}{\varepsilon}] \leq \frac{1}{\varepsilon}$ , we deduce

$$\int_{S_{1/\varepsilon}} |\nabla w|^2 dx \leq \frac{1}{r} e^{\ln r \frac{1}{\varepsilon}} \int_{S_0} |\nabla w|^2 dx.$$

This completes the proof by setting  $C = \frac{1}{r}$  and  $\alpha = -\ln r$ .

The theorem below will play an important role in the following.

**Theorem 2.3.** *Let  $u_\varepsilon$  and  $u_0$  be the solutions to (1.7) and (1.2), respectively. Then under the assumption (2.4) there exist two constants  $C$  and  $\alpha > 0$  independent of  $\varepsilon$ , such that*

$$\begin{aligned} \frac{3}{4} \int_{\Omega} (\varepsilon^2 (\partial_{X_1} (u_\varepsilon - u_0 - w_\varepsilon))^2 + |\nabla_{X_2} (u_\varepsilon - u_0 - w_\varepsilon)|^2) dx \\ \leq C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2 dx - \varepsilon^2 \int_{\Omega} \partial_{X_1} u_0 \partial_{X_1} (u_\varepsilon - u_0 - w_\varepsilon) dx. \end{aligned} \quad (2.15)$$

**Proof.** Subtracting (2.6) from (2.2), we obtain

$$\int_{\Omega} (\varepsilon^2 \partial_{X_1} (u_\varepsilon - u_0) \partial_{X_1} v + \nabla_{X_2} (u_\varepsilon - u_0) \cdot \nabla_{X_2} v) dx = -\varepsilon^2 \int_{\Omega} \partial_{X_1} u_0 \partial_{X_1} v dx \quad \forall v \in V. \quad (2.16)$$

Since  $u_\varepsilon - u_0 - w_\varepsilon \in V$ , we get

$$\begin{aligned} \int_{\Omega} (\varepsilon^2 \partial_{X_1} (u_\varepsilon - u_0) \partial_{X_1} (u_\varepsilon - u_0 - w_\varepsilon) + \nabla_{X_2} (u_\varepsilon - u_0) \cdot \nabla_{X_2} (u_\varepsilon - u_0 - w_\varepsilon)) dx \\ = -\varepsilon^2 \int_{\Omega} \partial_{X_1} u_0 \partial_{X_1} (u_\varepsilon - u_0 - w_\varepsilon) dx. \end{aligned}$$

According to Lemma 2.1, the identity above can be written as

$$\begin{aligned}
& \int_{\Omega} (\varepsilon^2 (\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon))^2 + |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|^2) dx \\
&= - \int_{\Omega} (\varepsilon^2 \partial_{X_1} w_\varepsilon \partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon) + \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)) dx \\
&\quad - \varepsilon^2 \int_{\Omega} \partial_{X_1} u_0 \partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon) dx \\
&= \int_{\Omega_-} (\varepsilon^2 \partial_{X_1} w_\varepsilon \partial_{X_1}(u_\varepsilon - \widehat{u_0} - w_\varepsilon) + \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2}(u_\varepsilon - \widehat{u_0} - w_\varepsilon)) dx \\
&\quad - \varepsilon^2 \int_{\Omega} \partial_{X_1} u_0 \partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon) dx. \tag{2.17}
\end{aligned}$$

We separately estimate the integral over  $\Omega_-$  using the Cauchy–Schwarz and Young’s inequalities. We derive

$$\begin{aligned}
& \int_{\Omega_-} (\varepsilon^2 \partial_{X_1} w_\varepsilon \partial_{X_1}(u_\varepsilon - \widehat{u_0} - w_\varepsilon) + \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2}(u_\varepsilon - \widehat{u_0} - w_\varepsilon)) dx \\
&\leq \left( \int_{\Omega_-} (\varepsilon^2 (\partial_{X_1} w_\varepsilon)^2 + |\nabla_{X_2} w_\varepsilon|^2) dx \right)^{1/2} \\
&\quad \times \left( \int_{\Omega_-} (\varepsilon^2 (\partial_{X_1}(u_\varepsilon - \widehat{u_0} - w_\varepsilon))^2 + |\nabla_{X_2}(u_\varepsilon - \widehat{u_0} - w_\varepsilon)|^2) dx \right)^{1/2} \\
&\leq \int_{\Omega_-} (\varepsilon^2 (\partial_{X_1} w_\varepsilon)^2 + |\nabla_{X_2} w_\varepsilon|^2) dx + \frac{1}{4} \int_{\Omega} (\varepsilon^2 |\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)|^2 + |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|^2) dx.
\end{aligned}$$

Going back to (2.17), we find that

$$\begin{aligned}
& \frac{3}{4} \int_{\Omega} (\varepsilon^2 (\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon))^2 + |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|^2) dx \\
&\leq \int_{\Omega_-} (\varepsilon^2 (\partial_{X_1} w_\varepsilon)^2 + |\nabla_{X_2} w_\varepsilon|^2) dx - \varepsilon^2 \int_{\Omega} \partial_{X_1} u_0 \partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon) dx.
\end{aligned}$$

Making the change of variable  $X_1 \rightarrow \frac{1-X_1}{\varepsilon}$  in the first integral of the second line, we get

$$\begin{aligned}
& \frac{3}{4} \int_{\Omega} (\varepsilon^2 (\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon))^2 + |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|^2) dx \\
&\leq \varepsilon \int_{\Omega_{2/\varepsilon} \setminus \Omega_{1/\varepsilon}} |\nabla w|^2 dx - \varepsilon^2 \int_{\Omega} \partial_{X_1} u_0 \partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon) dx \\
&\leq \varepsilon \int_{S_{1/\varepsilon}} |\nabla w|^2 dx - \varepsilon^2 \int_{\Omega} \partial_{X_1} u_0 \partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon) dx. \tag{2.18}
\end{aligned}$$

Combining (2.13) and (2.18) leads to the basic inequality (2.15). This completes the proof of the theorem.

**2.2. Convergence results.** As a first application of Theorem 2.3 we have

**Theorem 2.4.** *The solution  $u_0$  is a strong limit of the sequence  $u_\varepsilon - w_\varepsilon$  in  $H^1(\Omega)$  and the following error estimate is valid:*

$$\begin{aligned} |u_\varepsilon - u_0 - w_\varepsilon|_{L^2(\Omega)}, |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)} &= o(\varepsilon), \\ |\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)} &= o(1). \end{aligned}$$

**Proof.** Applying the Cauchy–Schwarz inequality to the last term of (2.15), we derive

$$\begin{aligned} \frac{3}{4} \int_{\Omega} (\varepsilon^2 (\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon))^2 + |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|^2) dx \\ \leq C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2 dx + \varepsilon^2 |\partial_{X_1} u_0|_{L^2(\Omega)} |\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)}. \end{aligned}$$

Then by Young’s inequality we get for some constant  $C$

$$\varepsilon^2 |\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)}^2 + |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)}^2 \leq C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2 dx + C \varepsilon^2 |\partial_{X_1} u_0|_{L^2(\Omega)}^2.$$

This estimate shows in particular that

$$|\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)} = O(\varepsilon) \tag{2.19}$$

since  $e^{-\frac{\alpha}{\varepsilon}} = o(\varepsilon^2)$ . At the same time we have proved the boundedness of  $|\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)}$ . This allows us to extract a weakly convergent subsequence of  $\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)$  in  $L^2(\Omega)$  and according to (2.19) it follows that the whole sequence converges weakly to 0, i.e.

$$\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon) \rightharpoonup 0 \quad \text{in } L^2(\Omega).$$

Going back to (2.15), using the fact that  $e^{-\frac{\alpha}{\varepsilon}} = o(\varepsilon^2)$  and the weak convergences above, we obtain

$$|\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)} = o(\varepsilon), \quad |\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)} = o(1).$$

Finally, using the Poincaré inequality in the direction  $X_2$ , with the help of the estimates above we complete the proof of the theorem.

We can improve the rate of convergence above if we assume more smoothness of  $f$  as in the following theorem.

**Theorem 2.5.** *Under the assumptions of Theorem 2.3 and if*

$$\partial_{X_1}^2 u_0 \in L^2(\Omega) \quad \text{and} \quad \partial_{X_1} u_0 = 0 \quad \text{on } \{0\} \times \omega, \tag{2.20}$$

we have

$$\begin{aligned} |u_\varepsilon - u_0 - w_\varepsilon|_{L^2(\Omega)}, |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)} &= O(\varepsilon^2), \\ |\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)} &= O(\varepsilon) \end{aligned}$$

as  $\varepsilon \rightarrow 0$ .



**Remark 2.6.** (i) The second hypothesis in (2.20) means that for a.e.  $X_2 \in \omega$  we have

$$\partial_{X_1} u_0(0, X_2) = 0.$$

(ii) For instance, if  $f$  is smooth enough, we can show that the hypotheses

$$\partial_{X_1}^2 f \in L^2(\Omega) \quad \text{and} \quad \partial_{X_1} f = 0 \quad \text{on} \quad \{0\} \times \omega$$

imply (2.20) using the representation formula

$$u_0(x) = \int_{\omega} f(X_1, y) G(X_2, y) dy,$$

where  $G$  is the Green function (see [7]).

**Proof of Theorem 2.5.** Integrating by parts the last integral of (2.15), we get

$$\begin{aligned} \frac{3}{4} \int_{\Omega} (\varepsilon^2 (\partial_{X_1} (u_{\varepsilon} - u_0 - w_{\varepsilon}))^2 + |\nabla_{X_2} (u_{\varepsilon} - u_0 - w_{\varepsilon})|^2) dx \\ \leq C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2 dx + \varepsilon^2 \int_{\Omega} \partial_{X_1}^2 u_0 (u_{\varepsilon} - u_0 - w_{\varepsilon}) dx \\ + \varepsilon^2 \int_{\omega} \partial_{X_1} u_0(0, X_2) (u_{\varepsilon} - u_0 - w_{\varepsilon})(0, X_2) dX_2 \\ = C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2 dx + \varepsilon^2 \int_{\Omega} \partial_{X_1}^2 u_0 (u_{\varepsilon} - u_0 - w_{\varepsilon}) dx \end{aligned}$$

( $u_{\varepsilon} - u_0 - w_{\varepsilon} \in V$  and  $\partial_{X_1} u_0 = 0$  on  $\{0\} \times \omega$ ). By the Cauchy-Schwarz and Young's inequalities it follows that

$$\begin{aligned} \frac{3}{4} \int_{\Omega} (\varepsilon^2 (\partial_{X_1} (u_{\varepsilon} - u_0 - w_{\varepsilon}))^2 + |\nabla_{X_2} (u_{\varepsilon} - u_0 - w_{\varepsilon})|^2) dx \\ \leq C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2 dx + \varepsilon^2 |\partial_{X_1}^2 u_0|_{L^2(\Omega)} |u_{\varepsilon} - u_0 - w_{\varepsilon}|_{L^2(\Omega)} \\ \leq C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2 dx + \mu \varepsilon^4 |\partial_{X_1}^2 u_0|_{L^2(\Omega)}^2 + \frac{1}{4\mu} |u_{\varepsilon} - u_0 - w_{\varepsilon}|_{L^2(\Omega)}^2 \\ \leq C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2 dx + \mu \varepsilon^4 |\partial_{X_1}^2 u_0|_{L^2(\Omega)}^2 + \frac{C_{\omega}}{4\mu} |\nabla_{X_2} (u_{\varepsilon} - u_0 - w_{\varepsilon})|_{L^2(\Omega)}^2, \end{aligned}$$

where  $C_{\omega}$  is the Poincaré inequality constant. Choosing  $\mu = C_{\omega}$  and since  $e^{-\frac{\alpha}{\varepsilon}} = o(\varepsilon^4)$ , we are ending up with

$$\varepsilon^2 \int_{\Omega} ((\partial_{X_1} (u_{\varepsilon} - u_0 - w_{\varepsilon}))^2 + |\nabla_{X_2} (u_{\varepsilon} - u_0 - w_{\varepsilon})|^2) dx \leq C \varepsilon^4.$$

Applying the Poincaré inequality to  $u_{\varepsilon} - u_0 - w_{\varepsilon} \in V$ , we complete the proof of the theorem.

Thanks to Theorem 2.3, if we assume that  $f$  is independent of  $X_1$ , we get an exponential rate of convergence. This is the following theorem.

**Theorem 2.7.** *Under the assumptions above and if in addition  $f$  is independent of  $X_1$ , we have an exponential convergence of  $u_\varepsilon - w_\varepsilon$  to  $u_0$  in the whole domain  $\Omega$ , i.e. there exist positive constants  $C$  and  $\alpha$  independent of  $\varepsilon$  such that*

$$\int_{\Omega} |\nabla(u_\varepsilon - u_0 - w_\varepsilon)|^2 dx \leq C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2 dx.$$

**Proof.** This is an immediate consequence of (2.15).

### 3. PROBLEMS IN DOMAINS BECOMING UNBOUNDED

Let  $\Omega_\ell^m$  be a bounded open subset of  $\mathbb{R}^{m+p}$  defined by

$$\Omega_\ell^m = \begin{cases} (0, \ell)^m \times \omega & \text{if } \ell > 0, \\ (\ell, 0)^m \times \omega & \text{if } \ell < 0, \end{cases}$$

where  $m, p > 0$  are integers and  $\omega$  is a bounded open subset of  $\mathbb{R}^p$ . For simplicity we drop the index 1 in  $\Omega_\ell^1$  and  $\Omega_1$ ; i.e., to be consistent with our notation of Section 1, we set

$$\Omega_\ell^1 := \Omega_\ell, \quad \Omega_1 := \Omega.$$

The points in  $\mathbb{R}^{m+p}$  will be denoted by  $x = (X_1, X_2) = (x_1, \dots, x_m, x'_1, \dots, x'_p)$  with

$$X_1 = (x_1, \dots, x_m), \quad X_2 = (x'_1, \dots, x'_p).$$

With this notation we set

$$\nabla u = (\partial_{x_1} u, \dots, \partial_{x_m} u, \partial_{x'_1} u, \dots, \partial_{x'_p} u)^T = \begin{pmatrix} \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix},$$

where

$$\nabla_{X_1} u = (\partial_{x_1} u, \dots, \partial_{x_m} u)^T, \quad \nabla_{X_2} u = (\partial_{x'_1} u, \dots, \partial_{x'_p} u)^T.$$

We divide the boundary  $\Gamma_\ell^m$  of  $\Omega_\ell^m$  into two parts  $\mathcal{D}_\ell^m$  and  $\mathcal{N}_\ell^m$  such that

$$\mathcal{N}_\ell^m = \bigcup_{i=1, \dots, m} \{x_i = 0\} \cap \overline{\Omega}_\ell^m, \quad \mathcal{D}_\ell^m = \Gamma_\ell^m \setminus \mathcal{N}_\ell^m.$$

We also set

$$\mathcal{N}_\ell^1 := \mathcal{N}_\ell, \quad \mathcal{N}_1 := \mathcal{N}, \quad \mathcal{D}_\ell^1 := \mathcal{D}_\ell, \quad \mathcal{D}_1 := \mathcal{D}.$$

In this section we deal with the asymptotic behaviour, when  $\ell \rightarrow +\infty$ , of the solution  $u_\ell^m$  to the Laplace boundary value problem

$$\begin{cases} -\Delta u_\ell^m = f & \text{in } \Omega_\ell^m, \\ u_\ell^m = 0 & \text{on } \mathcal{D}_\ell^m, \\ \partial_\eta u_\ell^m = 0 & \text{on } \mathcal{N}_\ell^m, \end{cases} \tag{3.1}$$

where  $f$  is independent of  $X_1$ , i.e.

$$f(x) = f(X_2) \in L^2(\omega).$$

Here  $\partial_\eta$  denotes the derivative along the outward normal to the boundary  $\Gamma_\ell^m$ . The existence of a weak solution  $u_\ell^m$  of (3.1) is ensured by the Lax–Milgram theorem in the space

$$V_\ell^m = \{v \in H^1(\Omega_\ell^m) \mid v = 0 \text{ on } \mathcal{D}_\ell^m\}.$$

Thanks to Lemma 3.1 below and [6, Theorem 1.1] it follows that  $u_\ell^m$  converges towards the solution  $u_0$  to (1.2), as  $\ell \rightarrow +\infty$ , in  $H^1(\Omega_{\ell_0}^m)$  where  $\ell_0 < \ell$  is a constant. More precisely, we have in fact

$$\int_{\Omega_{\ell/2}^m} |\nabla(u_\ell^m - u_0)|^2 dx \leq C e^{-\alpha\ell}, \tag{3.2}$$

where  $C$  and  $\alpha$  are positive constants independent of  $\ell$ . In this section we are interested in the asymptotic behaviour of  $u_\ell^m$  in the neighbourhood of  $\mathcal{D}_\ell^m$ . We start with the case  $m = 1$  in the following subsection and next we consider the general case.

**3.1. Domains becoming unbounded in one direction.**

3.1.1. *Mixed boundary value problems.* We consider here the special case  $m = 1$ . By making the change of variable

$$X_1 \rightarrow \frac{1}{\varepsilon} X_1, \tag{3.3}$$

where  $\varepsilon = \frac{1}{\ell}$ , we deduce that  $u_\ell^1$  is a solution of (3.1) if and only if the function

$$\Omega \rightarrow \mathbb{R}, \quad x \rightarrow u_\ell^1 \left( \frac{1}{\varepsilon} X_1, X_2 \right)$$

is a solution of (1.7). Then we set

$$w_\ell(X_1, X_2) := w_\varepsilon \left( \frac{1}{\ell} X_1, X_2 \right) = w(\ell - X_1, X_2),$$

where  $w$  is a solution of (2.10). The following theorem is a direct consequence of Theorem 2.7 and (3.3).

**Theorem 3.1.** *We have the convergence  $u_\ell^1 - w_\ell \rightarrow u_0$  on the whole domain  $\Omega_\ell$ , i.e. in  $H_0^1(\Omega_\ell)$ , and the following estimate is true:*

$$\int_{\Omega_\ell} |\nabla(u_\ell^1 - u_0 - w_\ell)|^2 dx \leq C e^{-\alpha\ell} \int_{S_0} |\nabla w|^2 dx, \tag{3.4}$$

where  $C$  and  $\alpha$  are positive constants independent of  $\ell$ .

**Remark 3.2.** Estimate (3.2) is a corollary of Theorem 3.1. Indeed, we have

$$\begin{aligned} \int_{\Omega_{\ell/2}} |\nabla(u_\ell^1 - u_0)|^2 dx &\leq 2 \int_{\Omega_{\ell/2}} |\nabla(u_\ell^1 - u_0 - w_\ell)|^2 dx + 2 \int_{\Omega_{\ell/2}} |\nabla w_\ell|^2 dx \\ &\leq C e^{-\alpha\ell} \int_{S_0} |\nabla w|^2 dx + 2 \int_{S_{\ell/2}} |\nabla w|^2 dx, \end{aligned}$$

by a change of variable. The last integral converges towards 0 at an exponential rate by Lemma 2.2, which shows (3.2).

**Remark 3.3.** For any  $a > 0$ ,

$$\int_{\Omega_{\ell-a}} |\nabla(u_\ell^1 - u_0)|^2 dx \rightarrow 0.$$

To show this, one notices that

$$\begin{aligned} \int_{\Omega_{\ell-a}} |\nabla(u_\ell^1 - u_0)|^2 dx &= \int_{\Omega_{\ell-a}} |\nabla(u_\ell^1 - u_0 - w_\ell) + \nabla w_\ell|^2 dx \\ &\geq \frac{1}{2} \int_{\Omega_{\ell-a}} |\nabla w_\ell|^2 dx - \int_{\Omega_{\ell-a}} |\nabla(u_\ell^1 - u_0 - w_\ell)|^2 dx \\ &= \frac{1}{2} \int_{S_a} |\nabla w|^2 dx + o(1). \end{aligned}$$

Since  $w$  is a harmonic function, one has for every  $a$

$$\int_{S_a} |\nabla w|^2 dx > 0.$$

Then the convergence of  $u_\ell^1$  towards  $u_0$  may not occur in  $H^1(\Omega)$ .

3.1.2. *Dirichlet boundary value problems.* Let us consider in  $\mathcal{O}_\ell = (-\ell, \ell) \times \omega$  the Dirichlet boundary value problem

$$\begin{cases} -\Delta U_\ell = f & \text{in } \mathcal{O}_\ell, \\ U_\ell = 0 & \text{on } \partial\mathcal{O}_\ell. \end{cases}$$

It is clear that  $U_\ell$  is a unique function of  $H_0^1(\mathcal{O}_\ell)$  satisfying

$$\int_{\mathcal{O}_\ell} \nabla U_\ell \cdot \nabla v dx = \int_{\mathcal{O}_\ell} f v dx \quad \forall v \in H_0^1(\mathcal{O}_\ell). \quad (3.5)$$

The following lemma summarizes some useful properties of the solution  $U_\ell$ .

**Lemma 3.4.** *Under the previous assumptions, we have*

- $U_\ell(-X_1, X_2) = U_\ell(X_1, X_2)$  for a.e.  $x \in \mathcal{O}_\ell$ ;
- the restriction of  $U_\ell$  to  $\Omega_\ell$  is a unique solution to

$$\begin{cases} U_\ell \in V_\ell, \\ \int_{\Omega_\ell} \nabla U_\ell \cdot \nabla v dx = \int_{\Omega_\ell} f v dx \quad \forall v \in V_\ell. \end{cases}$$

**Proof.** For  $v \in H_0^1(\mathcal{O}_\ell)$  denote by  $\tilde{v}$  the function defined by  $\tilde{v}(X_1, X_2) = v(-X_1, X_2)$ . It is clear that  $\tilde{v} \in H_0^1(\mathcal{O}_\ell)$ , and if we make the change of variable  $\bar{X}_1 = -X_1$  in (3.5), we derive

$$\begin{aligned} \int_{\mathcal{O}_\ell} \nabla \tilde{U}_\ell \cdot \nabla v dx &= \int_{\mathcal{O}_\ell} \nabla \tilde{U}_\ell \cdot \nabla \tilde{v} dx \\ &= \int_{\mathcal{O}_\ell} \{-\partial_{\bar{X}_1} U_\ell(-\partial_{\bar{X}_1} \tilde{v}) + \nabla_{X_2} U_\ell \cdot \nabla_{X_2} \tilde{v}\}(\bar{X}_1, X_2) dx \\ &= \int_{\mathcal{O}_\ell} \nabla U_\ell \cdot \nabla \tilde{v} dx = \int_{\mathcal{O}_\ell} f \tilde{v} dx = \int_{\mathcal{O}_\ell} f v dx, \end{aligned}$$

since  $f$  is independent of  $X_1$ . This means that  $\tilde{U}_\ell$  is also a weak solution to (3.5) and by uniqueness of the solution we deduce the first point of the lemma. For  $v \in V$  we can easily check that  $\hat{v}$  defined by (2.11) belongs to  $H_0^1(\mathcal{O}_\ell)$ . Moreover, we have

$$\int_{\mathcal{O}_\ell} \nabla U_\ell \cdot \nabla \hat{v} \, dx = \int_{\Omega_\ell} \nabla U_\ell \cdot \nabla \hat{v} \, dx + \int_{\Omega_{-\ell}} \nabla U_\ell \cdot \nabla \hat{v} \, dx.$$

Thanks to the first point, the last integral can be written as ( $\bar{X}_1 = -X_1$ )

$$\int_{\Omega_{-\ell}} \nabla U_\ell \cdot \nabla \hat{v} \, dx = \int_{\Omega_{-\ell}} (-\partial_{X_1} U_\ell(-\bar{X}_1, X_2) \partial_{\bar{X}_1} v(\bar{X}_1, X_2) + \nabla_{X_2} U_\ell \cdot \nabla_{X_2} \hat{v}) \, dx = \int_{\Omega_\ell} \nabla U_\ell \cdot \nabla v \, dx.$$

Thus we have for every  $v \in V_\ell$

$$\int_{\mathcal{O}_\ell} \nabla U_\ell \cdot \nabla \hat{v} \, dx = 2 \int_{\Omega_\ell} \nabla U_\ell \cdot \nabla v \, dx. \tag{3.6}$$

Also by (3.5) we have

$$\int_{\mathcal{O}_\ell} \nabla U_\ell \cdot \nabla \hat{v} \, dx = \int_{\mathcal{O}_\ell} f \hat{v} \, dx = 2 \int_{\Omega_\ell} f v \, dx. \tag{3.7}$$

Combining (3.6) and (3.7) shows the second point.

As a consequence of the second point of Lemma 3.4, it follows that

$$U_\ell = u_\ell^1 \quad \text{on } \Omega_\ell.$$

Then, thanks to Theorem 3.1 and the first point of Lemma 3.4 we can state

**Theorem 3.5.** *There exist positive constants  $C$  and  $\alpha$  independent of  $\ell$  such that*

$$\int_{\mathcal{O}_\ell} |\nabla(U_\ell - u_0 - \hat{w}_\ell)|^2 \, dx \leq C e^{-\alpha \ell} \int_{S_0} |\nabla w|^2 \, dx.$$

**3.2. More general domains.** For  $m = 1$ , we defined in the previous subsection a corrector  $w_\ell^1 := w_\ell$  satisfying (3.4). In order to construct a corrector for  $m = 2$ , we introduce a function  $w^2 \in H_0^1((0, +\infty) \times \Omega_\ell^1)$  as a solution to

$$\begin{cases} \Delta w^2 = 0 & \text{in } (0, +\infty) \times \Omega_\ell^1, \\ w^2 = -u_0 - w_\ell^1 & \text{on } \{0\} \times \Omega_\ell^1, \quad w^2 = 0 & \text{on } (0, +\infty) \times \partial\Omega_\ell^1. \end{cases}$$

The existence of  $w^2$  is ensured by the Lax–Milgram theorem. The corrector candidate in this case is  $w_\ell^1 + w_\ell^2$  where  $w_\ell^2$  is given by

$$w_\ell^2(x_1, x_2, X_2) = w^2(\ell - x_1, x_2, X_2).$$

Instead of showing this only for the case  $m = 2$ , we construct by induction for  $i = 2, \dots, m$  functions  $w_\ell^i: S_0^i \rightarrow \mathbb{R}$  defined as follows. For a solution  $u$  to

$$\begin{cases} u \in H_0^1(S_0^i), \\ \int_{S_0^i} \nabla u \cdot \nabla v \, dx = \int_{S_0^i} \nabla \left[ \rho(x_1) \left( u_0 + \sum_{j=1}^{i-1} w_\ell^j \right) \right] \cdot \nabla v \, dx & \forall v \in H_0^1(S_0^i), \end{cases} \tag{3.8}$$

where  $S_a^i = (a, +\infty) \times \Omega_\ell^{i-1}$  ( $a \in \mathbb{R}$ ), we set

$$w^i(x_1, \dots, x_i, X_2) = u(x_1, \dots, x_i, X_2) - \rho(x_1) \left( u_0(X_2) + \sum_{j=1}^{i-1} w_\ell^j(x_{i-j+1}, \dots, x_i, X_2) \right) \tag{3.9}$$

and denote by  $w_\ell^i$  the function defined as

$$w_\ell^i(x) = w^i(\ell - x_1, x_2, \dots, x_i, X_2).$$

Then we have

**Theorem 3.6.** *Under the assumptions above, the difference  $u_\ell^m - \sum_{j=1}^m w_\ell^j$  converges towards  $u_0$  on the whole domain  $\Omega_\ell^m$ , i.e. in  $H_0^1(\Omega_\ell^m)$ , and there exist positive constants  $C$  and  $\alpha$  independent of  $\ell$  such that*

$$\int_{\Omega_\ell^m} \left| \nabla \left( u_\ell^m - u_0 - \sum_{j=1}^m w_\ell^j \right) \right|^2 dx \leq C e^{-\alpha \ell}. \tag{3.10}$$

**Proof.** In order to check that  $\sum_{j=1}^m w_\ell^j(x_{m-j+1}, \dots, x_m, X_2)$  is a corrector corresponding to problem (3.1) and satisfying (3.10), we will argue by induction. According to Theorem 2.4 the statement holds when  $m = 1$ ; then we assume that  $\sum_{j=1}^{m-1} w_\ell^j(x_{m-j+1}, \dots, x_m, X_2)$  is a corrector satisfying

$$\int_{\Omega_\ell^{m-1}} \left| \nabla \left( u_\ell^{m-1} - u_0 - \sum_{j=1}^{m-1} w_\ell^j \right) \right|^2 dx \leq C e^{-\alpha \ell}, \tag{3.11}$$

where  $C$  and  $\alpha$  are some positive constants independent of  $\ell$ . In the following we show the same estimate for  $m$ . Let us introduce a function  $\bar{w}_\ell^m$  defined as below. For a solution  $\bar{u}$  to

$$\begin{cases} \bar{u} \in H_0^1(S_0^m), \\ \int_{S_0^m} \nabla \bar{u} \cdot \nabla v \, dx = \int_{S_0^m} \nabla(\rho(x_1)u_\ell^{m-1}) \cdot \nabla v \, dx \quad \forall v \in H_0^1(S_0^m), \end{cases} \tag{3.12}$$

we set

$$\bar{w}(x) = \bar{u}(x) - \rho(x_1)u_\ell^{m-1}(x_2, \dots, x_m, X_2) \tag{3.13}$$

( $\bar{w}$  depends on  $\ell$ ) and denote by  $\bar{w}_\ell^m$  the function defined as

$$\bar{w}_\ell^m(x) = \bar{w}(\ell - x_1, x_2, \dots, x_m, X_2). \tag{3.14}$$

Then we have

**Lemma 3.7.** *For any  $\ell > 0$ , there exist constants  $C > 0$  and  $\alpha' > 0$  such that*

$$\int_{\Omega_\ell^m} |\nabla(u_\ell^m - u_\ell^{m-1} - \bar{w}_\ell^m)|^2 dx \leq C e^{-\alpha' \ell}. \tag{3.15}$$

**Proof.** Without loss of generality, we assume that  $\ell > 1$ . Arguing as in the previous section and replacing  $\omega$  by  $\Omega_\ell^{m-1}$ , we can show an estimate similar to (3.4), i.e.

$$\int_{\Omega_\ell^m} |\nabla(u_\ell^m - u_\ell^{m-1} - \bar{w}_\ell^m)|^2 dx \leq C e^{-\alpha \ell} \int_{S_0^m} |\nabla \bar{w}|^2 dx. \tag{3.16}$$

(We use the fact that  $\Omega_\ell^{m-1}$  is bounded in the direction  $X_2$  to get a Poincaré constant independent of  $\ell$ .) We have now to estimate the last integral in (3.16). By using, in (3.12),  $v = \bar{u}$  we obtain easily

$$|\nabla \bar{u}|_{L^2(S_0^m)}^2 = \int_{S_{\ell-1}^m \setminus S_\ell^m} \nabla(\rho(x_1)u_\ell^{m-1}) \cdot \nabla \bar{u} \, dx \leq C |\nabla u_\ell^{m-1}|_{L^2(\Omega_\ell^{m-1})} |\nabla \bar{u}|_{L^2(S_0^m)},$$

whence

$$|\nabla \bar{u}|_{L^2(S_0^m)} \leq C |\nabla u_\ell^{m-1}|_{L^2(\Omega_\ell^{m-1})}.$$

Then by (3.13) we derive

$$|\nabla \bar{w}|_{L^2(S_0^m)} \leq |\nabla \bar{u}|_{L^2(S_0^m)} + |\nabla(\rho(x_1)u_\ell^{m-1})|_{L^2(S_0^m)} \leq C |\nabla u_\ell^{m-1}|_{L^2(\Omega_\ell^{m-1})},$$

where  $C$  is independent of  $\ell$ . Next, taking in the weak formulation of (3.1), written for  $m - 1$ ,  $v = u_\ell^{m-1}$  yields

$$|\nabla u_\ell^{m-1}|_{L^2(\Omega_\ell^{m-1})} \leq C |f|_{L^2(\Omega_\ell^{m-1})}^2 = C \ell^{m-1} |f|_{L^2(\omega)}^2,$$

since  $f$  is independent of  $X_1$ . Thus, it follows that

$$|\nabla \bar{w}|_{L^2(S_0^m)} \leq C \ell^{m-1} |f|_{L^2(\omega)}^2. \tag{3.17}$$

Going back to (3.16), we have

$$\int_{\Omega_\ell^m} |\nabla(u_\ell^m - u_\ell^{m-1} - \bar{w}_\ell^m)|^2 \, dx \leq C \ell^{m-1} |f|_{L^2(\omega)}^2 e^{-\alpha \ell}.$$

Since  $\ell \rightarrow +\infty$ , there exist constants  $0 < \alpha' < \alpha$  and  $C > 0$  such that

$$\int_{\Omega_\ell^m} |\nabla(u_\ell^m - u_\ell^{m-1} - \bar{w}_\ell^m)|^2 \, dx \leq C e^{-\alpha' \ell}.$$

This completes the proof of the lemma.

We now return to the proof of the theorem. The integral in (3.10) can be estimated as

$$\begin{aligned} \int_{\Omega_\ell^m} \left| \nabla \left( u_\ell^m - u_0 - \sum_{j=1}^m w_\ell^j \right) \right|^2 \, dx &\leq 3 \int_{\Omega_\ell^m} |\nabla(u_\ell^m - u_\ell^{m-1} - \bar{w}_\ell^m)|^2 \, dx \\ &+ 3 \int_{\Omega_\ell^m} \left| \nabla \left( u_\ell^{m-1} - \sum_{j=1}^{m-1} w_\ell^j - u_0 \right) \right|^2 \, dx + 3 \int_{\Omega_\ell^m} |\nabla(\bar{w}_\ell^m - w_\ell^m)|^2 \, dx. \end{aligned}$$

The exponential convergences to 0 of the first and the second integral of the right-hand side are given by (3.15) and the induction hypothesis (3.11), respectively. Then it remains to show the same rate of convergence for the last integral to complete the proof. First, we estimate the difference between  $\bar{w}$  and  $w^m$ , defined in (3.13) and (3.9), respectively, as

$$|\nabla(\bar{w} - w^m)|_{L^2(S_0^m)} \leq |\nabla(\bar{u} - u)|_{L^2(S_0^m)} + \left| \nabla \left[ \rho(x_1) \left( u_\ell^{m-1} - u_0 - \sum_{j=1}^{m-1} w_\ell^j \right) \right] \right|_{L^2(S_0^m)}. \tag{3.18}$$

We estimate the last term in the inequality above using the Poincaré inequality and the induction hypothesis (3.11); then we have

$$\begin{aligned} \left| \nabla \left[ \rho(x_1) \left( u_\ell^{m-1} - u_0 - \sum_{j=1}^{m-1} w_\ell^j \right) \right] \right|_{L^2(S_0^m)} &\leq C \left| \nabla \left( u_\ell^{m-1} - u_0 - \sum_{j=1}^{m-1} w_\ell^j \right) \right|_{L^2(\Omega_\ell^{m-1})} \\ &\leq C e^{-\alpha \ell}. \end{aligned} \quad (3.19)$$

For the first term of the right-hand side of (3.18), we compare (3.12) and (3.8) for  $i = m - 1$  and, taking  $v = \bar{u} - u \in H_0^1(S_0^m)$  as a test function, obtain

$$|\nabla(\bar{u} - u)|_{L^2(S_0^m)}^2 \leq \left| \nabla \left[ \rho(x_1) \left( u_\ell^{m-1} - u_0 - \sum_{j=1}^{m-1} w_\ell^j \right) \right] \right|_{L^2(S_0^m)} |\nabla(\bar{u} - u)|_{L^2(S_0^m)}.$$

Applying (3.19) here and in (3.18), we get

$$|\nabla(\bar{w} - w^m)|_{L^2(S_0^m)} \leq C e^{-\alpha \ell}. \quad (3.20)$$

Finally, the change of variable  $x_1 \rightarrow \ell - x_1$  and (3.20) lead to

$$\begin{aligned} |\nabla(\bar{w}_\ell^m - w_\ell^m)|_{L^2(\Omega_\ell^m)} &= |\nabla(\bar{w} - w^m)|_{L^2((0,2\ell) \times \Omega_\ell^{m-1})} \\ &\leq |\nabla(\bar{w} - w^m)|_{L^2(S_0^m)} \\ &\leq C e^{-\alpha \ell}. \end{aligned}$$

This completes the proof.

**Remark 3.8.** As in Theorem 3.5, using symmetries, we can construct correctors for the Laplace equation defined in  $(-\ell, \ell)^m \times \omega$  coupled with the homogeneous Dirichlet boundary conditions.

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