Year: 2012

Regulating collateral-Requirements when markets are incomplete

Kübler, Felix ; Araujo, Aloisio ; Schommer, Susan

Abstract: In this paper we examine the effects of default and scarcity of collateralizable durable goods on risk-sharing. We assume that there is a large set of assets which all promise a risk-less payo but which distinguish themselves by the collateral requirement. In equilibrium agents default and the assets have dierent payoffs. Assets with very low collateral requirements can be interpreted as sub-prime loans and these assets are often traded actively in the competitive equilibrium. If there is an abundance of commodities that can be used as collateral and if each agent owns a large fraction of these commodities, markets are complete and competitive equilibrium allocations Pareto optimal. If, on the other hand, the collateralizable durable good is scarce or if some agents do not own enough of the collateralizable durable good in the first period, markets can be endogenously incomplete, not all of the available assets are traded in the competitive equilibrium and allocations are not Pareto optimal. We give examples that show that welfare losses can be quantitatively large. We also examine the scope for government intervention. In particular we ask who in the economy gains and who loses if collateral requirements are regulated exogenously. In our examples, regulation never leads to a Pareto-improvement. Often, the rich and the poor agents gain if trade is restricted to subprime contracts and lose if trade in these contracts is not allowed.

REGULATING COLLATERAL-REQUIREMENTS WHEN MARKETS ARE INCOMPLETE

ALOÍSIO ARAÚJO, FELIX KUBLER, AND SUSAN SCHOMMER

ABSTRACT. In this paper we examine the effects of default and scarcity of collateralizable durable goods on risk-sharing. We assume that there is a large set of assets which all promise a risk-less payoff but which distinguish themselves by the collateral requirement. In equilibrium agents default and the assets have different payoffs. Assets with very low collateral requirements can be interpreted as sub-prime loans and these assets are often traded actively in the competitive equilibrium.

If there is an abundance of commodities that can be used as collateral and if each agent owns a large fraction of these commodities, markets are complete and competitive equilibrium allocations Pareto optimal. If, on the other hand, the collateralizable durable good is scarce or if some agents do not own enough of the collateralizable durable good in the first period, markets can be endogenously incomplete, not all of the available assets are traded in the competitive equilibrium and allocations are not Pareto optimal. We give examples that show that welfare losses can be quantitatively large.

We also examine the scope for government intervention. In particular we ask who in the economy gains and who loses if collateral requirements are regulated exogenously. In our examples, regulation never leads to a Pareto-improvement. Often, the rich and the poor agents gain if trade is restricted to subprime contracts and lose if trade in these contracts is not allowed.

1. Introduction

We examine how scarcity and an unequal distribution of collateralizable goods affects risk-sharing in an economy with default. Individuals have to put up durable goods as collateral when they want to take short positions in financial markets. They are allowed to default on their promises without any punishment, but in the case of default, the collateral associated with the promise is seized and distributed among the creditors. We show that if agents do not own enough durable goods that can be used as collateral, markets can be endogenously incomplete and competitive equilibrium allocations suboptimal. The amount of collateralizable goods needed to achieve an efficient allocation is often unrealistically large.

The vast majority of debt, especially if it extends over a long period of time, is guaranteed by tangible assets called collateral. For example, residential homes serve as collateral for short-term and long-term loans to households, equipment and plants are often used as collateral for corporate bonds, and investors can borrow money to establish a position in stocks, using these as collateral. [DGS] and [GZ] incorporate default and collateral into the standard two-period general equilibrium with incomplete markets (GEI) model. In a two period model, default can be prevented by collateral and by utility-penalties that can be thought of a reduced form representation of reputation effects present in more realistic dynamic models. While it is true that in developed economies it is also possible to take substantial positions in unsecured debt and default on collateralized obligations has consequences beyond the loss of the collateral, to simplify our analysis we follow [GZ] and abstract from these considerations by assuming that collateral constitutes the only enforcement mechanism.

In our model, there are two periods, with uncertainty over the states of the world in the second period. There are two commodities, one perishable and the second durable. The durable good serves as collateral. An asset in this model is characterized by its collateral- (or margin-) requirement which specifies how much of the durable good needs to be used (i.e. held by the borrower) to back a short-position in this asset and by its state-contingent promises in the second period. The actual payoff of the asset will be the minimum of this

promise and the value of the associated collateral. The margin requirement that dictates how much collateral one has to hold in order to borrow one dollar and via the possibility of default also determines the payoff of an asset.

In the simplest version of the model, the only assets available for trade are promising one unit of the perishable good across all states. Throughout the work, we consider the generic case where the price of collateral will differ across states. If there are S states, S assets with safe promises but with appropriately chosen collateral requirements will span the marketed sub-space, i.e. it appears as if they will 'complete markets'. However, scarcity of collateralizable durable goods will generally imply that agents cannot hold arbitrary portfolios but will instead lead to allocations which can be very far away from the complete markets competitive equilibrium.

While it appears somewhat arbitrary to limit the set of available securities to assets that promise a safe payout, this assumption can be motivated by the observation that few individuals hold short-position in assets other than debt. If everybody owns enough of the collateralizable durable good, the resulting equilibrium allocation does not depend on the exact promises of the assets. However, with scarce collateralizable durable good, this assumption turns out to be crucial for our analysis – although the payoffs of the S assets are linearly independent, large positions are necessary to derive the desired payoff of a portfolio in the second period.

The first observation of the model is that even if there are many more securities than states of the world available for trade, if they all promise a safe payoff, it is without loss of generality to consider S different securities. While in the standard model this is a trivial observation, in our model with collateral, it is crucial that these margin requirements to these securities are chosen to ensure that for each security there is exactly one state where the price of the underlying collateral is exactly equal to the promise of the asset.

If there is enough of the collateralizable durable good in the sense that each agent is endowed with so much of the durable good that he can hold a portfolio of all assets without

the collateral constraint being binding, the model is equivalent to a standard Arrow-Debreu model and competitive equilibrium allocations are Pareto optimal. A similar result was obtained by [Ki] who considers a model with production and arbitrary promises and who shows that with a sufficiently large but finite amount of collateral full risk sharing and complete markets can be obtained in that model.

If, on the other hand, the collateralizable durable good is scarce, most assets are not traded in equilibrium and markets 'appear' incomplete. As [GZ] and [G] point out, 'scarce collateral' rations the volume of trade since there will always be a gap between utility of buying and dis-utility of selling an asset. The rationing does not reduce volume of trade proportionally but chokes off all trade in many contracts. We show that with several states and several agents there are generally still more than one contract being actively traded in equilibrium. However, it is also true that usually not all contracts are traded and that many agents trade only in one of the available assets.

It is easy to see that the resulting equilibrium allocation is not Pareto optimal. This simply follows from the fact that not every agent trades every security, but would also be true in a model with certainty, if collateral requirements are binding. This in itself is not very surprising. Given the simple 'contracting' technology that does not allow for agents to deliver on promises without collateral, it is clear that with too little collateralizable goods in the economy, there cannot be complex trade in securities and full risk-sharing is impossible.

A more interesting question is whether the allocation is constrained sub-optimal in the sense that a government intervention can lead to a Pareto-improvement. There are several interventions one can think of (see e.g. [GP]). The most interesting one in this model is clearly to consider the effects of a regulation of margin-requirements. In other words, can it be Pareto-improving to force agents to trade only a subset of the available assets. [GZ] are the first to address this question. They show that without price effects, competitive equilibrium allocations are constrained efficient and a regulation can never be Pareto-improving.

However, it is possible that the majority of agents benefits from a regulation, although this is not Pareto-improving.

It is a quantitative question how much collateralizable goods needs to be in the economy to achieve Pareto optimality and how the collateralizable goods needs to be distributed. In the absence of Pareto-optimality, it is also a quantitative question who in the economy gains and who loses through a regulation of collateral-requirements. We provide a series of examples, some of them illustrative and some realistically calibrated in order to address this quantitative question.

Our examples show how for realistic levels of collateralizable goods welfare losses due to endogenously incomplete markets can be large. Furthermore, the examples illustrate that regulation of margin requirements generally does not lead to Pareto-improvements. However, often a majority of agents would favor a regulation since it is welfare improving for them.

In our model, we can interpret the asset with the lowest collateral-requirement as a 'subprime loan'. What is typically understood as subprime loans are mortgage loans for borrowers who do not qualify for prime loans, due to weak credit history, low incomes or missing collateral. To compensate for these credit risks subprime loans carry higher interest rates compared to prime loans. In our collateral model, assets with low margin-requirements do carry higher interest rates, and are bought by agents who lack collateralizable durable goods in the present. It is clear that understanding the effect of subprime loans on the economy is of fundamental importance for today's highly sophisticated financial markets. In this paper we ask whether the existence of subprime loans can lead to a Pareto-improvement and who in the economy gains and who loses if sub-prime loans are regulated.

In our examples, it is never optimal to regulate the market for sub-prime loans. In some cases, the rich and the poor agents gain if only subprime loans can be traded and markets for prime loans are shut down. However, the middle-class loses from such a regulation and it is therefore not Pareto-improving.

The examples also elucidate that only subprime loans are optimal (in the sense that no other assets are traded) when the borrower owns almost no collateralizable goods. In one example, where all borrowers own substantial amounts of the durable good agents endogenously avoid subprime loans, even when there is little certainty about future income.

While this work does not intend to explain the subprime mortgage crisis, we hope that the computation of equilibrium will eventually lead to a good understanding of the role subprime should play in a healthy market. In our model, default does not have any negative externalities and all agents are completely rational and agree on the probabilities of default. Our results can therefore be interpreted as benchmarks in which subprime loans play an important role for the economy.

In our model, collateral levels are exogenously given, but since all possible collateral levels are in principal available for trade, one can think of the market picking out the collateral levels. [AOP] and [AFP] develop a model where collateral levels are determined endogenously and set by the lender. It is subject to further research to compare the welfare consequences and assets traded in our model to their analysis.

Clearly our focus on a two period model has important implications for our welfare analysis. In a dynamic model (see e.g. [APT] or [KS]), where agents can accumulate the durable good over time, the distribution of collateral is endogenous. It is an important open question to evaluate the welfare consequences in such a model.

The paper is organized as follows. In Section 2 we present the GEI model with collateral (GEIC). In Section 3, we discuss some theoretical results. In Section 4 we provide stylized examples that illustrate our main points. In Section 5 we discuss a larger example that is realistically calibrated along several dimensions.

2. The model

We consider a pure exchange economy over two time periods $t = 0, 1$ with uncertainty over the state of nature in period 1 denoted by the subscript $s \in \mathcal{S} = \{1, \ldots, S\}$. For convenience, the first period will sometimes be called state 0 so that in total there are $\mathcal{S}^* = S + 1$ states.

The economy consists of a finite number of H agents denoted by the superscript $h \in \mathcal{H} = \{1, \ldots H\}$ and $L = 2$ goods or commodities, denoted by the subscript $l \in \mathcal{L} = \{1, 2\}$. Throughout the analysis we assume that good 1 is perishable and good 2 is durable, i.e. there is a possibly risky (and possibly productive) storage technologies, represented by $Y_s \in \mathbb{R}^L_+$. Using one unit of the durable good in the first period yields Y_{sl} units of good l in state s. The most natural assumption is of course that $Y_s = (0, 1)$ for each state s, i.e. it is possible to store the durable good without depreciation. Each agent has an initial endowment of the L goods in each state, $e^h \in \mathbb{R}^{S^*L}_+$. The preference ordering of agent h is represented by a utility function $u^h : \mathbb{R}^{S^*L}_+ \to \mathbb{R}$, defined over consumption $x^h = (x^h_0, x^h_1, \ldots, x^h_S) \in \mathbb{R}^{S^*L}_+$.

The characteristics of agent h are summarized by a utility function and endowment vector (u^h, e^h) satisfying the following standard assumptions:

A 1. $u^h : \mathbb{R}^{S^*L}_+ \to \mathbb{R}$ is continuous on $\mathbb{R}^{S^*L}_+$ and C^2 on $\mathbb{R}^{S^*L}_{++}$;

A 2. for each $x^h \in \mathbb{R}^{S^*L}_{++}$, $\nabla u^h(x) \in \mathbb{R}^{S^*L}_{++}$, and $f^T \nabla^2 u^h(x) f < 0$ for all $f \neq 0$ such that $\nabla u^h(x) f = 0$;

A 3. $\sum_h e^h_0 > 0$;

A 4. $\sum_h e^h_s + \sum_h Y_s e^h_0 > 0$; $\quad \forall s \in \mathcal{S}$;

In each state $s = 0, \ldots, S$ there are complete spot markets - the spot prices of the commodities across states are denoted by $p \in \mathbb{R}^{S^*L}_{++}$.

There are J real assets denoted by the subscript $j \in \mathcal{J} = \{1, \ldots, J\}$, and $A_j \in \mathbb{R}^{SL}_+$ denotes the promises per unit of asset j across states $s = 1, \ldots, S$. We will assume that:

A 5. Each asset only promises payments in commodity 1.

We will also assume that the promises are independent of the states of nature. Hence we can normalize so that each asset promises one unit of commodity 1 in each state of nature.

We associate with each asset $j \in \mathcal{J}$ a collateral requirement $C_j \geq 0$. We assume that the collateral always has to be held by the borrower, in order to simplify our analysis. Agents have to hold C_j units of good 2 in order to sell one unit of asset j. We assume that collateral being held to secure some asset j cannot be used for any other asset and long position of other assets cannot be used to secure a short-position. This will turn out to be a strong assumption with important welfare implications.

Agents default on their promises whenever the market value of the durable good they hold as collateral is lower than the face value of their promise. Given equilibrium prices p, the actual payoff of asset j in states s is therefore $\min(p_1(s), p_2(s)C_j)$. Let $q \in \mathbb{R}_+^J$ denote the prices of assets in period zero. It is useful to define the margin requirement on asset j as $\mu_j = \frac{C_j p_2(0) - q_j}{q_j}$.

Finally, let $\theta^h = (\theta_1^h, \ldots, \theta_J^h) \in \mathbb{R}_+^J$ denote the number of units of each of the J assets bought by agent h, and $\varphi^h = (\varphi_1^h, \ldots, \varphi_J^h) \in \mathbb{R}_+^J$ the short-positions in the assets.

The economy with collateral, E_{GEIC}, is characterized by the agents' utility functions $u = (u^h)_{h \in \mathcal{H}}$, the agents' endowment process $e = (e^h)_{h \in \mathcal{H}}$, the asset structure $(C_j)_{j \in \mathcal{J}}$.

Given $p \in \mathbb{R}_{++}^{S*L}$, and $q \in \mathbb{R}_+^J$ the agent h chooses consumption and portfolios $(x^h, \theta^h, \varphi^h)$, to maximize utility subject to the budget constraints.

$$\max_{x \geq 0, \theta \geq 0, \varphi \geq 0} \quad u^h(x^h)$$

$$\text{s.t.} \quad p(0) \cdot (x^h(0) - e^h(0)) + q \cdot (\theta - \varphi) \leq 0;$$

$$p(s) \cdot (x^h(s) - e^h(s) - Y_s x_2^h(0)) - \sum_{j \in \mathcal{J}} (\theta_j - \varphi_j) \min\{p_1(s), p_2(s)C_j\} \leq 0; \ \forall s \in \mathcal{S}$$

$$(2.1) \quad x_2^h(0) - \sum_{j \in \mathcal{J}} \varphi_j C_j \geq 0.$$

In state s, an asset j pays $\min\{p_1(s), p_2(s)C_j\}$, an agent has endowments and receives $x_2(0)$ units from his 'investment' in the first period. We refer to the last inequality constraint in the agent's problem as the 'collateral' constraint.

A competitive equilibrium is defined as usual by agents' optimality and market clearing.

Definition 1. A GEIC equilibrium for the economy E_{GEIC} is a vector $[(\overline{x}, \overline{\theta}, \overline{\varphi}); (\overline{p}, \overline{q})]$ with $(\overline{x}, \overline{\theta}, \overline{\varphi}) = (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h)_{h \in H}$ such that:

   (i) $(\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h)$ solves problem 2.1.

   (ii) $\sum_{h=1}^{H}(\overline{x}^h(0) - e^h(0)) = 0$

   (iii) $\sum_{h=1}^{H}(\overline{x}^h(s) - e^h(s) - Y_s\overline{x}_2^h(0)) = 0$

   (iv) $\sum_{h=1}^{H}(\overline{\theta}^h - \overline{\varphi}^h) = 0$

The following assumption is natural.

A 6. $C_j \geq 0$ for all $j \in J$;

The following theorem follows from Geanakoplos and Zame (2007).

Theorem 1. *For an economy E_{GEIC}, under assumptions A1-A4, and A6 there exists a GEIC equilibrium.*

As a benchmark for our welfare-analysis, we also consider the Arrow Debreu (AD) equilibrium in the examples below. An AD equilibrium consists of an allocation $(x^h)_{h \in \mathcal{H}}$ and prices p such that each agent maximizes his utility subject to the budget constraint $p \cdot (x^h - e^h) \leq 0$ and such that markets clear.

3. Some theoretical observations

Throughout the analysis we want to think of the set of assets $\mathcal{J}$ as a finite but very large set. It is easy to see that 'generically' (e.g. for an open and full-measure set of individual endowments) for each GEI equilibrium we have $p_2(s)/p_1(s) \neq p_2(s')/p_1(s')$ for each $s, s' \in \mathcal{S}$. For the remainder of this paper, we will only consider this generic case.

The assets in $\mathcal{J}$ could distinguish themselves both with respect to promises and collateral requirements. As explained in the introduction, we focus on the case where all assets promise a safe pay-off.

3.1. Complete set of collateral requirements.

We assume throughout that each asset j promises one unit of good 1 in each state $s = 1, ..., S$. The assets therefore distinguish themselves only by their collateral requirement C_j and not by their promises. Of course, given the assumptions on default, this will also imply that the assets have different payoffs. We write $(\mathbf{1}, C_j)_{j \in \mathcal{J}}$ to characterize all assets.

The first insight is that if the set $\mathcal{J}$ is very large many assets are collinear and not all assets are being traded in equilibrium, or differently put, there exists an equivalent equilibrium with trade in only a few assets (see also [GZ] for an explanation). In fact, it is clear that if for two assets j and j', $C_j < C_{j'} \leq \min_s p_1(s)/p_2(s)$, assets j and j' are collinear and it is without loss of generality to only consider equilibria with $\varphi_j = 0$.

More interestingly, if there are S states and there is a set of asset $\mathcal{J}^{CC}$ (CC standing for complete set of collateral requirements) that consists of S assets $j \in \mathcal{J}^{CC} \subset \mathcal{J}$ such that for each state $s = 1, ..., S$ there is a $j \in \mathcal{J}^{CC}$ with $C_j p_2(s) = p_1(s)$, then it is without loss of generality to assume that only these S assets are traded. The set $\mathcal{J}^{CC}$ denotes the set of 'active' assets and contains S securities such that for each one, there is a different state in which the holder is indifferent between defaulting and paying the full promise. If $\mathcal{J}^{CC} \subset \mathcal{J}$, we say there is a complete set of collateral requirements.

The following proposition formalizes this issue.

Proposition 1. *Given an economy $\left((u^h)_{h \in \mathcal{H}}, (e^h)_{h \in \mathcal{H}}, (\mathbf{1}, C_j)_{j \in \mathcal{J}}, (Y_s)_{s \in \mathcal{S}}\right)$ and a GEI equilibrium $[(\overline{x}, \overline{\theta}, \overline{\varphi}); (\overline{p}, \overline{q})]$, suppose that for each s there is a $j \in \mathcal{J}$ with $C_j p_2(s) = p_1(s)$, then $\overline{x}$ and $\overline{p}, \overline{q}$ are GEIC equilibrium consumptions and prices for any economy $\left((u^h)_{h \in \mathcal{H}}, (e^h)_{h \in \mathcal{H}}, (\mathbf{1}, C_j)_{j \in \tilde{\mathcal{J}}}, (Y_s)_{s \in \mathcal{S}}\right)$ if $\mathcal{J} \subset \tilde{\mathcal{J}}$.*

Proof. Let $\delta \in \mathbb{R}^{HS^*}$ denote the vector of multipliers associated with the $S+1$ spot budget constraints and let κ denote the multiplier associated with the collateral constraints. Agent h chooses positive θ_j^h if

$$(3.1) \qquad -\delta_0^h q_j + \sum_{s \in \mathcal{S}} \delta_s^h \min\{p_1(s), p_2(s)C_j\} = 0;$$

Agent h chooses positive φ_j^h if

$$(3.2) \qquad \delta_0^h q_j - \sum_{s \in \mathcal{S}} \delta_s^h \min\{p_1(s), p_2(s)C_j\} - \kappa C_j = 0;$$

Assume now that $i, j \in \mathcal{J}$, $C_i < C_j$ and that there is no $k \in \mathcal{J}$ with $C_i < C_k < C_j$. If an individual holds an asset $\tilde{j} \in \tilde{\mathcal{J}}$ with $C_i < C_{\tilde{j}} < C_j$, there obviously exists a $\eta > 0$ such that $\eta C_i + (1 - \eta)C_j = C_{\tilde{j}}$. The individual can then simply hold η units of asset i and $(1-\eta)$ units of asset j, obtaining the same payoff in the second period and holding the exact same collateral. By Equation (3.1), the first order condition of the lender, if the individual is indifferent between holding asset i at price q_i, asset j at price q_j or holding asset j' at price $\eta q_i + (1 - \eta)q_j$. Therefore the cost to the borrower of holding asset $\tilde{j}$ is the same as holding the portfolio of i and j that gives the same payoff. $\square$

Having established that it is without loss of generality[1] to limit ourselves to S assets $\mathcal{J}^{CC}$, the question is then if scarce durable good held as collateral implies that not all of these S assets are traded.

We denote a GEIC equilibrium in which the set of assets available for trade contains $\mathcal{J}^{CC}$ by GEICC, a GEI equilibrium with complete collateral. On the other, we might consider a situation where the set of assets available for trade is regulated exogenously and might not contain all potential assets would like to trade. We refer to a GEI equilibrium with an exogenously fixed set of collateral requirements as a GEIRC equilibrium – GEI with regulated collateral. For a given economy, there might a unique GEICC equilibrium,

---

[1] Of course, there is a subtle issue concerning the possibility of multiple equilibria. Throughout the paper, all statements are made about one of the equilibria.

there are obviously always infinitely many GEIRC equilibria depending on different available assets. The GEICC equilibrium can be viewed as that specific GEIRC equilibrium for which adding assets (differing only by the collateral requirement) does not change the equilibrium allocation.

3.2. **Complete versus incomplete markets.** We first consider the situation where there is an abundance of collateralizable goods in the economy and each individual owns enough of it to back his promises. Since the payoffs of the S securities in $\mathcal{J}$ are linearly independent, markets are complete. The first welfare theorem applies. It is a quantitative question, how much collateralizable goods is needed for this. In the numerical section below we address this question. For now it is useful to note that this is a joint conditions on endowments and preferences. What is important is that each agent owns collateral and that the value of this collateral is sufficiently large to secure all the short positions the agent has to take in order to span his desired consumption. Theorem 4 in [GZ] illustrates this point for the trivial case of no uncertainty and [Ki] in the Proposition 4 showed that complete markets can be obtained with a sufficiently large amount of collateral, under different hypothesis. As we will see in the next section, to assume that collateralizable durable good is 'sufficiently large' in a world where all assets promise a safe payoff and variations in payoffs are only due to variations in prices is generally unrealistic. Agents have to take on huge short-position in some assets and would need to own a lot of collateralizable goods.

If, on the other hand, the amount of collateralizable goods is small or some agent own no collateralizable goods at all, not all S contracts in $\mathcal{J}^{CC}$ will be traded in equilibrium. Instead, the examples below show that only very few assets are traded. Just like in the GEI model with incomplete markets, allocations will not be Pareto-efficient.

In the situation of scarce collateralizable goods the collateral constraints will be binding and $\kappa > 0$ in Equation (3.2). Whether a particular contract will now be traded depends on the multipliers δ^h across agents. Generally they will not be collinear and some assets

are more attractive to both agents than others. These will be the only asset traded in equilibrium.

In particular, it is easy to see that if an agent is poor in the first period and owns no durable good, he wants to finance his first period consumption in the durable good by selling an asset which promises to hand the durable good over to the lender in the second period. As [GZ] observe, this is essentially a rental contract. The agent buys the durable good and borrows as much money as possible on an asset that defaults for sure, i.e. hands the durable good to the lender in all states of the world tomorrow.

If the agent is very poor and his marginal utility for the durable good is large, this will be the only asset he will sell. He will not trade in any asset with a large collateral requirement since this cannot be used to finance extra consumption in the first period.

More interestingly, we give an example below, where two agents trade in a unique asset that does not default in all states. We will argue that this asset has the best risk-sharing characteristics for the two agents.

3.3. **Welfare when markets are incomplete.** Following [GZ] it is a natural question whether it is efficient that precisely the assets in $\mathcal{J}$ are traded or if it is possible to make everybody in the economy better off by restricting trade to take place in other (possible fewer) assets. As [GZ] put it, 'Given that the markets choose the asset structure, we are compelled to ask whether the market chooses the asset structure efficiently'.

Neither [GZ] nor we can provide a complete answer to the problem. If all agents have identical homothetic utility (an assumption often made in applied work), the answer is simple. The market chooses the asset structure efficiently. The following result is a (trivial) special case of Theorem 6 in [GZ], since identical homothetic utility is sufficient for prices to be independent of the wealth distribution.

Theorem 2. *If all agents have identical homothetic utility, given a GEICC equilibrium with actively traded assets $\mathcal{J}^{CC}$, there is no other set of assets $\mathcal{J}'$ such that in the resulting GEIRC equilibrium all agents are better off.*

Below, we give an example where agents do not have identical utility but the GEICC equilibrium allocation is still constrained efficient. The example suggests, that there is no generic sense in which GEICC allocations are always inefficient when preferences are heterogeneous. However, this is as much as we know.

4. Risk-sharing with scarce collateralizable goods

In this section, we describe three numerical examples. The first example illustrates the point that the Arrow-Debreu allocation can be achieved if there is sufficient collateralizable goods in the economy.

We then give two examples that illustrate how scarce collateralizable goods leads to a situation where only very few assets are traded and welfare losses due to imperfect risk-sharing are large. In these examples, allocations are always far from the Arrow-Debreu allocation. We use the algorithm described in [Sc] to approximate equilibrium numerically.

4.1. **Example 1: Plentiful durable good can lead to complete markets.** The first example illustrates how with a complete set of margin requirements, plentiful collateralizable goods can lead to the Arrow-Debreu allocation, while an unequal distribution of collateralizable goods might imply a situation where markets are endogenously incomplete and welfare losses compared to the Arrow-Debreu allocation are large.

We first consider the simplest two period model with two states in period 1 $\mathcal{S}^* = \{0, 1, 2\}$ and two agents, $\mathcal{H} = \{1, 2\}$. Each individual $h = 1, 2$ has a utility function of the form:

$$u^h(x) = 0.2 \log(x_1(0)) + 0.8 \log(x_2(0)) + \frac{1}{2} \sum_{s=1}^{2} (0.2 \log(x_1(s)) + 0.8 \log(x_2(s)))$$

Suppose first that collateralizable durable goods is plentiful and endowments are:

$$e^1 = (e_1^1(0), e_2^1(0), e_1^1(1), e_2^1(1), e_1^1(2), e_2^1(2)) = (4, 2, 4, 0, 4, 0);$$
$$e^2 = (e_1^2(0), e_2^2(0), e_1^2(1), e_2^2(1), e_1^2(2), e_2^2(2)) = (2, 2, 6, 0, 2, 0).$$

The GEICC equilibrium allocation is identical to the (unique because of Cobb-Douglas utility) Arrow-Debreu allocation for this economy. The collateral requirements for the two assets traded are $C_1 = 0.1$ and $C_2 = 0.16667$. With two states, these two assets are sufficient to complete the markets. The crucial point of this example is that each agent has so much collateralizable goods (and its price is so high) that collateral constraints are not binding and agents can trade to the complete markets allocation. Agent 1' portfolio $z_j^1 = \theta_j^1 - \phi_j^1$ is given by $z_1^1 = 5.2$ and $z_2^1 = -4$.

The situation is very different if instead of taking the durable good to be evenly distributed among agents, we assume that agent 1 initially owns the entire amount of the durable good, i.e.

$$e_2^1(0) = 4, \quad e_2^2(0) = 0.$$

Since we assumed identical homothetic utility, collateral requirements on the two assets are as above, $C_1 = 0.1$ and $C_2 = 0.16667$. However, in this case, agent 2 needs to borrow just to be able to consume the durable good in period 0. As this is relatively more important to him than any risk-sharing considerations, the agent will not establish a long position in asset 2 and the only asset traded in the GEICC equilibrium is asset 1. Agent 1's portfolio is now $z_1^1 = 2.66667$ and $z_2^1 = 0$ and second period risk is not shared at all. It is clear that now there should be substantial welfare losses compared to the Arrow-Debreu allocation.

Throughout the paper, we report welfare numbers in terms of wealth equivalence compared to the Arrow-Debreu allocation. That is, for log-utility and the case of no discounting (these are the preferences considered throughout the paper), if u^{hGEIC} denotes an agent's utility in the GEIC equilibrium and u^{hAD} denotes his utility in the Arrow-Debreu equilibrium, we compute Welfare Rate $WR^h = \exp(\frac{u^{hGEIC} - u^{hAD}}{2})$. If we multiply consumption in the Arrow-Debreu equilibrium by WR in all states, we obtain an allocation that gives the agent the same utility as in the GEIC equilibrium. That is a number of say 0.95 means that an agent would be willing to decrease his consumption in the Arrow-Debreu allocation by 5 percent in each state to avoid the incomplete markets consumption.

The welfare rate in this example are:

$$(WR^1, WR^2) = (0.9998, 0.9567).$$

Naturally, the collateral requirement hurts agent 2 much more than agent 1. But also agent 1's welfare is clearly below its Arrow-Debreu level. Both the borrower and the lender are hurt by the fact that the borrower faces a binding collateral constraint and cannot take simultaneous long and short-positions to share risk in the second period.

As mentioned in the previous section, 'plentiful' collateralizable durable goods can lead to complete markets, while in a situation where an agent owns very little or no collateralizable goods, often only one asset is traded and welfare losses can be substantial. The example illustrates this point, but leaves open the question of what happens when there is little collateralizable goods in the economy as a whole and of what assets are traded when $\mathcal{J}^{CC}$ consists of more than only two assets. The next example provides some insights to this question.

4.2. **Example 2: Scarce durable goods markets 'appear' incomplete.** We now consider an example with four states in period 1 $\mathcal{S}^* = \{0, 1, \ldots, 4\}$ and two agents, $\mathcal{H} = \{1, 2\}$, each with identical utility,

$$u^h(x) = \log(x_1(0)) + \log(x_2(0)) + \frac{1}{4} \sum_{s=1}^{4} (\log(x_1(s)) + \log(x_2(s)))$$

We consider a variety of profiles of endowments, differing by the distribution of the durable (collateralizable) good in the first period.

$$e^1(0) = (4, \eta), e^1(1) = e^1(2) = (1, 0), e^1(3) = e^1(4) = (2, 0);$$

$$e^2(0) = (1, (1 - \eta)), e^2(1) = e^2(3) = (1, 0), e^2(2) = e^2(4) = (2, 0.2).$$

In the first period agent 1 is rich (the natural lender in the example) and agent 2 is poor. In the second period both agents face identically distributed shocks to endowments of good 1 that are independent across agents. In addition, agent 2 has random endowments in

the durable good. The parameter η determines how the collateralizable durable good is distributed between the two agent. We consider $\eta \geq 1/2$.

Since we assume identical homothetic utility, spot-prices do not depend on η. The set $\mathcal{J}^{CC}$ consists of the four assets with collateral requirements $C_1 = 0.5$, $C_2 = 0.4$, $C_3 = 0.333$ and $C_4 = 0.3$. The margin requirements depend on the interest rate and therefore on the distribution of first period endowments, η. The assets' payment in the states defined by $\frac{\min\{p_1(s), p_2(s)C_j\}}{p_1(s)}$ follows in Table 1:

TABLE 1. Assets' payment in the states

Assets	state 1	state 2	state 3	state 4
j=1	1	1	1	1
j=2	0.8	1	1	1
j=3	0.667	0.833	1	1
j=4	0.6	0.75	0.9	1

Obviously, if agents would not face a collateral constraint in period 0, markets would be complete and the Arrow-Debreu allocation (which is unique since we assume Cobb-Douglas utility) would be achieved. However, since agents do face collateral constraints, the Arrow-Debreu allocation is not achieved for any value of η.

4.2.1. *Assets traded.* We first examine, how portfolios depend on the distribution of the durable good and how with an unequal distribution, the fact that collateralizable goods is scarce implies that only one of the four assets is traded and the collateral requirement is uniquely determined endogenously.

The following Table 2 denotes the portfolio-holding $\theta - \varphi$ of agent 1 for different values of η.

TABLE 2. Portfolio agent 1

η	asset 1	asset 2	asset 3	asset 4
0.5	-0.48	1.08	-0.81	0
0.75	0	0.77	-0.15	0
0.8	0	0.70	-0.01	0
0.81	0	0.68	0	0
0.85	0	0.15	0	0.62
0.9	0	0	0	0.67
0.95	0	0	0	0.51

If the borrower owns almost nothing of the durable good in period zero and has to buy it to consume it, the only asset traded in equilibrium is the one with the lowest possible margin requirement. The borrower buys the durable good and borrows as much as possible for it while he is for sure going to default in all states in the second period. Any asset with a higher margin requirement is not optimal since the additional price the lender would be willing to pay to get a payoff above one in states 1,2 or 3 is not sufficiently high for the borrower to forgo extra consumption in period zero. The little collateralizable goods he owns he needs to use to finance the margin on the loan he takes out just to buy more of the durable good.

For $\eta = 0.85$, two assets are traded, the full default asset, but also asset 2, that pays back in full in states 2, 3 and 4 (see Table 1). This is traded for risk-sharing in the second period. In state 2, the borrower is rich (has endowments of 2) while the lender is poor (has endowments of 1). So the lender values this asset relatively more than the borrower, making its price high enough so that the borrower is willing to take on the extra collateral (compared to assets 3 and 4). Around $\eta = 0.81$ there is a robust region of endowments in good 2, for which in fact asset 2 is the only asset traded. For these distributions of endowments, remarkably there is a unique asset determined collateral-requirement that is not equal to the lowest collateral available. Since the two agents want to share the risk in the second period, if agent 2 has

sufficient collateralizable goods to sell only asset 2 to finance first period consumption, he does so because this asset fetches the relatively best price among the 4 available assets.

For $\eta = 0.8$, the borrower, agent 2, himself starts buying assets in the first period. He holds a long-position in asset 3 and borrows money in asset 2. Asset 2 pays in full in states 2, 3 and 4 but defaults in state 1. In state 3, the borrower is poor (endowments of one) while the lender is rich (endowments of 2), so buying asset 3 is a way for the borrower to insure himself against the second period risk. As the Table 2 shows this is true for all η between 0.75 and 0.8.

Finally, for $\eta = 0.5$ both agents have sufficient collateralizable goods to establish large short positions in some assets, however, the collateral requirement is still binding for both agents. Asset 4 (which obviously would be traded in a situation with collateral and default) is still not traded in equilibrium. Instead, agent 1 has relatively large short positions in assets 1 and 3. Agent 2 only uses asset 3 to borrow and finance first period consumption. He takes long-positions in assets 1 and 3 to insure against the second period risk. Even with both agents owning substantial amounts of the durable good in the first period, trade only takes place in 3 of the four available assets.

This obviously raises the question how large the welfare losses due to this trade in a restricted set of assets are and if welfare can be improved by 'forcing' agents to trade in other assets through a regulation of the margin requirements.

4.2.2. *Welfare.* We first consider the case of a complete set of collateral requirements and ask how large the welfare losses are that are implied by the fact that agents cannot commit to pay their promises, i.e. we compare the GEICC equilibrium welfares to the welfares agents would obtain in an Arrow Debreu equilibrium. The following Table 3 shows this welfare loss that due to default and collateral for different values of η.

As one would expect, the welfare losses due to default and collateral are large when the borrower has little collateralizable goods. In particular for $\eta > 0.9$, the possible welfare

TABLE 3. Welfare rate for distribution of durable good: agent 1 and 2

η	Lender (agent 1)	Borrower (agent 2)
0.5	0.993	0.996
0.75	0.992	0.989
0.8	0.992	0.982
0.85	0.991	0.978
0.9	0.989	0.969
0.95	0.988	0.933

gains from better enforcement of intertemporal contracts are very large. Note that for each value of η, we compare the GEICC welfare to the Arrow-Debreu equilibrium welfare for that given economy. So for $\eta = 0.95$, agent 2 would gain more than 7 percent if he could commit to pay back all promises and trade in all assets without holding any collateral. This would allow the agent to buy more of the durable good in the first period and to ensure against his endowment risk in the second period. Since with log-utility risk aversion is relatively low, the endowment risk is actually not the main source of the welfare losses. They are mostly due to the fact that the agent cannot afford to consume very much of the durable good. In fact, agent 2's GEICC equilibrium consumption in period zero is $x_1^2(0) = 0.76$ and $x_2^2(0) = 0.15$ while in the Arrow-Debreu equilibrium it is $x_1^2(0) = 1.10$ and $x_2^2(0) = 0.22$. Of course, his second period consumption is higher in all states, but this shows how collateral skews consumption away from the efficient Arrow-Debreu allocation.

Even for the case of an equal distribution of collateralizable durable goods, the welfare losses are still substantial. As we argued in the previous section, not all assets are traded and collateralizable goods is still scarce. Moreover, it is surprising that he fact that the natural borrower (agent 2) has more collateralizable goods available to finance large short-positions, does not necessarily bring the other agent closer to the complete markets welfare. Between $\eta = 0.8$ and $\eta = 0.5$ the welfare losses remain more or less constant for agent 1, while there are still substantial improvements for agent 2.

The fact that even for $\eta = 0.5$, the welfare losses due to default and collateral are still substantial (certainly significantly positive) makes highlight that it is not clear ex ante what 'plentiful collateralizable goods' means. In this example, agents spend 50 percent of their income on the consumption of the durable good. That seems to large fraction but not suffices to create so much collateralizable durable goods that markets are complete.

4.2.3. *Regulating the margin requirement.* What happens if, instead of allowing the agents to trade in assets with arbitrary margin requirements, we consider a situation where there is a fixed set of S assets with 'optimally' set margin requirements?

As explained above, under the assumption made in this example that all agents have identical homothetic utility, equilibrium allocations must be constrained efficient, and it is impossible to make both agents better off by exogenously selecting margin requirements. This obviously does not imply, however, that all possible margin-requirements are Pareto-ranked.

To illustrate these points, we consider two cases. First we assume $\eta = 0.95$. In this case, it seems likely that GEIC equilibrium allocations are in fact Pareto-ranked, with the GEICC allocation yielding the highest utility for both agents: Forcing both agents to trade in an asset that does not default in all states will reduce trade in the single asset traded and most likely make most agents worse off. This intuition turns out to be correct. In Figure 1, we show a few different points in utility (measured in terms of welfare rate) space for a sample 895 values of collateral between 0 and 1.

The figure clearly shows that all equilibria are Pareto-ranked with the largest utility arising from the GEICC allocation

We also consider the case $\eta = 0.85$. In this case, the GEIC equilibria are not Pareto-ranked as Figure 2 shows for a sample 350 values of collateral between 0 and 1.

While it is true that there is no GEIC equilibrium that is Pareto-better than the complete collateral equilibrium, there are GEIC equilibria that make agent 1 better and there are other GEIC equilibria that make agent 2 better off. In particular, somewhat counter-intuitively,
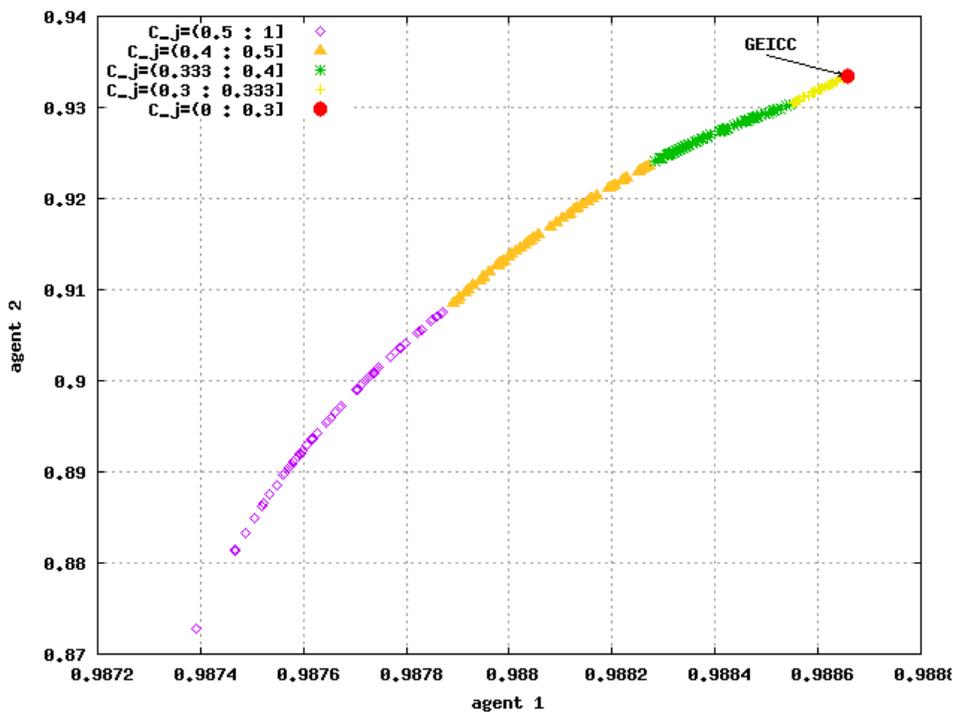
FIGURE 1. Welfare Rate for regulated collateral ($\eta = 0.95$)

the lender, agent 1, would be better off if is trade only takes place in the asset that defaults in all states. The point in the graph that gives him the highest utility corresponds to a situation where only the full default asset is available for trade. In this case, all agents of type 2 borrow heavily, since the collateral requirement is not an issue. The equilibrium interest rate is so high that agent 1 is compensated by the high interest rate for the fact that the only available asset has bad risk-sharing properties.

On the other hand, the point that gives agent 2 the highest utility corresponds to the case where only one asset with collateral requirement 0.4 is traded. This asset only defaults in one state. If it is the only asset traded in equilibrium, its interest rate is so favorable that agent 2 is well off. Agent 1 naturally is hurt by the low interest rate.
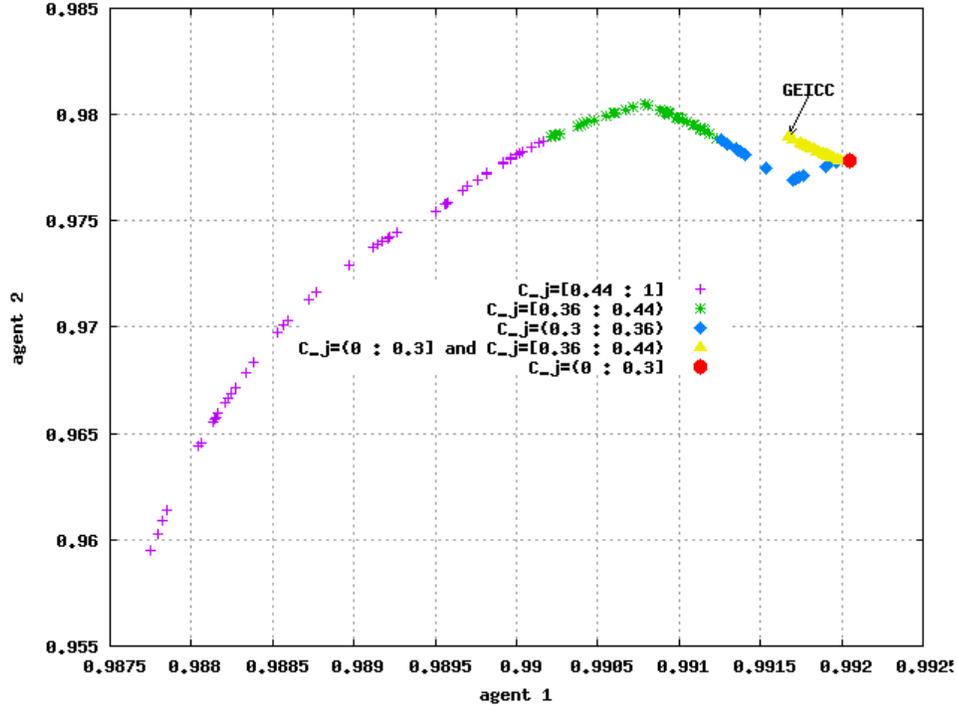
FIGURE 2. Welfare Rate for regulated collateral ($\eta = 0.85$)

4.2.4. *Subprime loans.* As explained above, if the borrower owns almost no durable goods in the first period (case when $(1 - \eta) = 0.05$), the GEIC equilibrium allocations are Pareto-ranked and the only asset traded is the one with lower collateral requirement. Figure 1 illustrates this. The highest utility[2] for both agents is with collateral between 0 and 0.3, this is true for all $\eta \geq 0.88$. In these cases, subprime is good for agent 1 (he benefits with higher interest rate) and agent 2 (the poor borrower who buys the durable good), because the only asset traded in equilibrium is the one with the lowest collateral. When the borrower has more of the durable good (case when $(1 - \eta) = 0.15$), the GEIC equilibria are not Pareto-ranked and now subprime is good only for agent 1 (rich). Figure 2 shows the largest utility for agent

---

[2]Measure by Welfare Rate.

1 with collateral between 0 and 0.3 and the largest utility for agent 2 with collateral between 0.36 and 0.44, this behavior holding for all $0.82 \leq \eta \leq 0.87$.

4.3. **Example 3: Regulating collateral with heterogeneous utility.** Finally, we consider an example with 3 agents. It seems clear that enough heterogeneity among agents should lead to trade in several assets even when collateralizable goods is scarce and unequally distributed among agents. As in Example 2, we first discuss portfolios, then report welfare rate due to collateral and finally show how the GEICC equilibrium welfare compares to welfare achieved in regulated economies.

To keep the example relatively simple, we assume that there are $S = 3$ states in the second period. The three agents' endowments are given by:

$$e^1(0) = (4, \eta), e^1(1) = (1, 0), e^1(2) = (4, 0), e^1(3) = (2, 0);$$

$$e^2(0) = (1, \gamma), e^2(1) = (1, 0), e^2(2) = (2, 0), e^2(3) = (4, 0);$$

$$e^3(0) = (2, 1 - \eta - \gamma), e^3(1) = (2, 0.2), e^3(2) = (2, 0), e^3(3) = (2, 0.2).$$

We assume that agents have heterogeneous utility. Under this assumption, Theorem 2 from above does not imply and there could be GEIRC allocations that are Pareto-better than the GEICC allocation. However, in this example this turns out not to be the case. Utility functions are:

$$u^h(x) = \alpha^h \log(x_1(0)) + (1 - \alpha^h) \log(x_2(0)) + \frac{1}{3} \sum_{s=1}^{3} (\alpha^h \log(x_1(s)) + (1 - \alpha^h) \log(x_2(s))),$$

with $\alpha^1 = 0.7, \quad \alpha^2 = 0.77, \quad \alpha^3 = 0.625$.

With these preferences, the collateral requirements of the traded assets (i.e. of the assets in $\mathcal{J}^{CC}$) obviously vary with η and γ since spot prices will vary. However, since preferences are quite similar, the variation is relatively small. In most cases, the collateral requirements for the three assets traded are around $C_1 = 0.644, C_2 = 0.288$ and $C_3 = 0.362$. In all the cases we consider, asset 1 pays back in full in all states, asset 2 pays one unit in state 2,

but defaults in states 1 and 3 and asset 3 defaults only in state 1, but pays one unit in both states 2 and 3.

4.3.1. *Portfolios.* As before, we first examine which assets are actively traded in the GEICC equilibrium, depending on the distribution of collateralizable goods which we parametrize by η, γ. We report portfolios of agent 1 and of agent 2 in Table 4.

TABLE 4. Portfolio agent 1 and 2

η	γ	Portfolio agent 1	Portfolio agent 2
0.9	0.05	(0.00,0.34,0.76)	(0,-0.34,0)
0.8	0.1	(0.06,0.42,0.74)	(0,-0.42,0)
0.8	0.2	(0.00,0.00,1.15)	(0,0,-0.48)
0.6	0.3	(0.18,-0.15,0.98)	(0.05,0.15,-0.54)
0.6	0.2	(0.55,0.00,0.46)	(0.00,0.00,-0.46)
0.6	0.1	(0.64,0.04,0.24)	(0.00,-0.1,-0.24)
0.4	0.3	(0.49,-0.32,0.52)	(0.13,0.13,-0.52)

Although now, there are almost always more than just one asset being traded in equilibrium, the same logic as in Example 2 above can be applied to understand which assets are traded. In terms of risk-sharing, agent 1 would ideally want to hold an asset that pays a lot in state 1, little in state 2 and substantial in state 3. Among the available assets, asset 1 comes the closest to this pattern in the sense that it is the only asset that pays in full in state 1. Asset 2 seems the worst since it only pays in full in state 2. Agent 2, on the other hand, would like to borrow since he is poor today and pay back more in state 3 than in state 2. He is also poor in state 1, so asset 1 is not a good asset to borrow in. Agent 3 is relatively rich in states 1 and 3, but wants to pay less in state 2. So asset 1 seems to be a good asset to be traded between agent 1 and 3, while asset 3 seems to be a good asset for trade between agent 1 and 2. But asset 3 is also attractive for agent 3 since he pays in full in state 3, where he is relatively rich. Asset 2 is the worst for agent 3.

Unfortunately, in the first case ($\eta = 0.9$, $\gamma = 0.05$) agent 2 is so poor that he only borrows to finance consumption in the durable good, therefore he trades in asset 2 which defaults in all states and, as we pointed out, can be interpreted as a rental contract. In this case, agent 3, although poor, still trades in asset 3 which gives him a relatively better price although it requires him to hold more collateralizable good.

In the second case ($\eta = 0.8$, $\gamma = 0.1$), both agents become more wealthy, but still agent 2 is stuck with asset 2 (he is still too poor to trade in any other asset). Instead agent 3 starts borrowing both in asset 1 and in asset 3, which are both good assets for him to share risk with agent 1.

In the third case ($\eta = 0.8$, $\gamma = 0.2$), agent 3 has no durable goods. Agent 2 is not yet rich enough to trade in anyting but asset 3. There is a unique asset being traded in equilibrium by all three agents.

Let us now consider $\eta = 0.6$ – there is a lot of collateralizable goods both for agents 2 and 3. In the first case ($\gamma = 0.3$), agent 2 owns a lot of collateralizable good, now instead of using asset 2 to borrow, he actually goes long in asset 2 and borrows only in asset 3. Agent 3 borrows in assets 1 and 3. Agent 1 goes short in asset 2, to share risk with agent 2. For $\gamma = 0.2$ agent 2 has not enough collateralizable good anymore to borrow enough so that he can take long-positions in some assets. He exclusively borrows in asset 3, which with some collateralizable good is the best way to at the same time borrow and share risk with agent 1. For $\gamma = 0.1$, agent 2 is quite poor again and has to do some of the borrowing in the full default asset.

Finally, for the case $\eta = 0.4, \gamma = 0.3$, we have a situation where the three available assets go a long way to share second period risk for the three agents. We now ask, how much of their Arrow-Debreu welfare the agents can achieve for the different distributions of collateralizable durable goods.

4.3.2. *Welfare.* As in Examples 1 and 2, we now want to examine how scarce collateralizable goods leads to welfare loss. In this example, we in addition want to point out how one agent's

welfare-losses can depend on the distribution of collateralizable durable good between the to other agents. The following Table 5 shows this welfare rate that due to default and collateral for different values of η and γ.

TABLE 5. Welfare rate for the distribution of durable good: agent 1, 2 and 3

η	γ	Agent 1	Agent 2	Agent 3
0.9	0.05	0.9756	0.9470	0.9773
0.8	0.1	0.9775	0.9718	0.9845
0.8	0.2	0.9783	0.9973	0.9612
0.6	0.3	0.9815	0.9926	0.9810
0.6	0.2	0.9831	0.9856	0.9906
0.6	0.1	0.9804	0.9662	0.9966
0.4	0.3	0.9836	0.9886	0.9928

For all distributions of collateralizable goods, welfare losses are substantial for all three agents. In the first case, despite the fact that agent 1 is the lender and his collateral constraint is not binding, his welfare losses due to collateral are substantial. The fact that with scarce collateralizable durable good agents 2 and 3 are not willing to trade in asset 1 leads to little risk sharing in the second period. The table also shows that even for $\eta = 0.4, \gamma = 0.3$, i.e. in a situation where every agent owns substantial collateralizable good, welfare losses compared to incomplete markets are fairly large, in particular for agents 1 and 2.

Agent 2 is actually the one who seems to be hurt relatively least by default and collateral requirements. This is surprising since he is relatively poor in the first period, having the greatest need to borrow. However, the payoffs of the available assets are best for him - in states where he is poor he has to pay back relatively less than in states where he is rich.

4.3.3. *Regulating the collateral requirements.* As before, we ask how regulating the collateral requirements and not allowing agents to trade in arbitrary assets can influence welfare. Note that since preferences are heterogeneous, Theorem 2 no longer applies and it could be possible

that the GEICC allocation is constrained optimal. To investigate this question, we search for GEIRC equilibria that could be Pareto better.

We examine the case $\eta = 0.8, \gamma = 0.1$. The Figure 3 shows a three dimensional scatter plot of different utility (measured in terms of welfare rate) levels corresponding to different GEIRC allocations for a sample 813 values of collateral between 0 and 1.
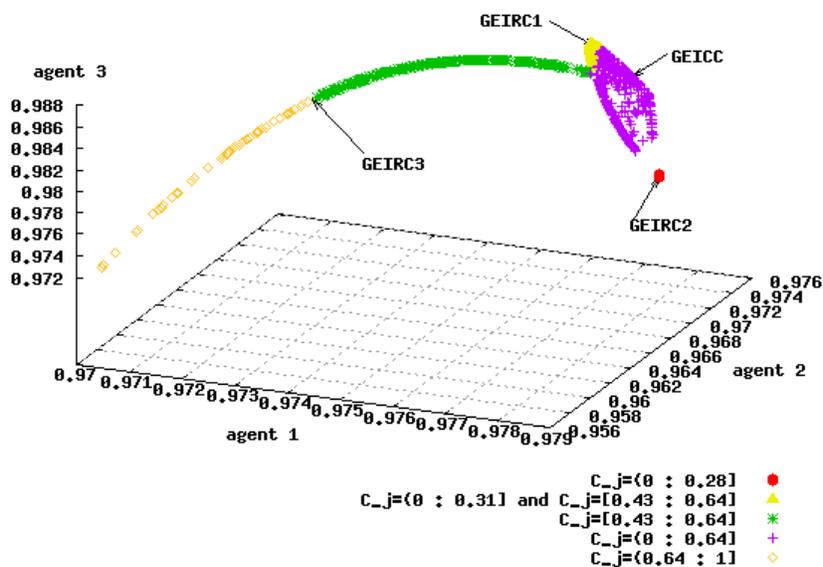

FIGURE 3. Welfare Rate for regulated collateral ($\eta = 0.8$ and $\gamma = 0.1$)

While it is still impossible to Pareto-improve on the GEICC allocation, both agents 2 and 3 can obtain relatively gains through a regulation. It is clear that the allocations are not Pareto-ranked, but the Figure 3 also shows that the GEICC allocation cannot be Pareto-dominated. It is a bit difficult to see in a 3D scatter plot, but it turns out that both agent 2 and 3 prefer a regulated equilibrium. Figure 4 illustrates this.

While the GEICC allocation is not Pareto-dominated, the figure shows that agents 2 and 3 can do much better. At the GEICC equilibrium the collateral requirements are $C_1 =$
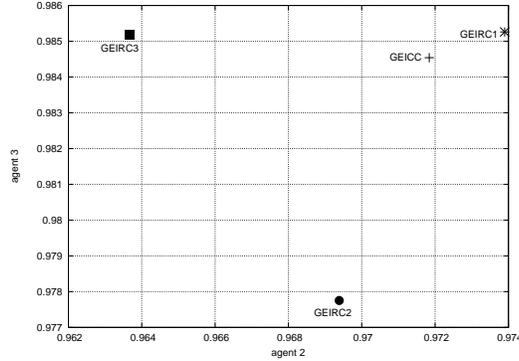
FIGURE 4. Welfare Rate for regulated collateral ($\eta = 0.8$ and $\gamma = 0.1$) - Agents 2 and 3

$0.644, C_2 = 0.288$ and $C_3 = 0.362$. The point denoted by GEIRC1 in the figure corresponds to the equilibrium where trade is restricted to take place only in two assets and collateral levels are set exogenously to $C_2 = 0.286$ and $C_3 = 0.45$. Since trade takes place in two assets, there is still significant risk sharing possible, however, the equilibrium interest rate decreases making it easier for the borrowers to finance first period consumption.

The first asset has been eliminated and the collateral requirement on the third asset is higher than its endogenous level. While agents 2 and 3 benefit from this, the next figure shows that agent 1 loses.

Figure 5 shows that agent 1 is best off in a GEIRC where all agents are only allowed to trade in a asset that fully defaults. The point GEIRC2 in the figure corresponds to this point.

The situation here is similar to the previous example. The lender is better off in a situation where all trade takes place in the full default asset (i.e. essentially only rental contracts are traded). While trade in this asset does not allow for any risk-sharing, if all borrowers are forced to borrow only in this asset, the equilibrium interest rate rises so dramatically that in fact the lender is compensated for the lack of risk sharing by a high interest rate.
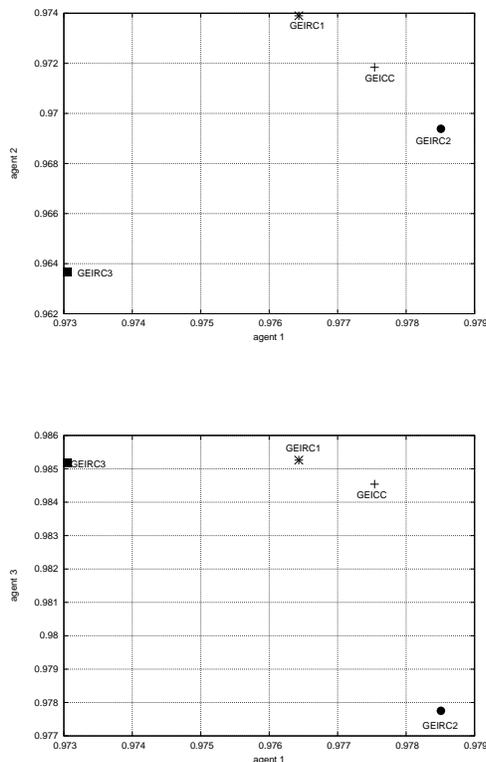
FIGURE 5. Welfare Rate for regulated collateral ($\eta = 0.8$ and $\gamma = 0.1$) - Agents 1 and 2 (top) and 1 and 3 (below)

Finally, restricting all agents to trade in an asset that never defaults makes all agents worse off. The point GEIRC3 shows the equilibrium that arises if there is only one asset available for trade and its collateral requirement ensures (exactly) full delivery in all three states.

4.3.4. *Subprime loans.* As in the Example 2 we examine in which situations subprime is optimal for agents. In this example for all η and γ, GEIRC equilibria are not Pareto-ranked, then there not exist an economy where subprime is optimal for all agents. The case examined above with $\eta = 0.8, \gamma = 0.1$, subprime is good for agent 1 (rich) and subprime is good for agent 2 (poor) in one asset. The Figure 3 shows the largest utility for agent 1 with collateral

between 0 and 0.28, the largest utility for agent 2 with collateral between 0 and 0.31 in one asset traded and another with collateral between 0.43 and 0.64. The largest utility for agent 3 with collateral between 0.43 and 0.64.

5. A CALIBRATED EXAMPLE

As a last example, we assume that there are 4 types of agents whose endowments we calibrated to (roughly) match the income and wealth distribution in US data. We want to investigate the role of sub-prime loans for risk-sharing as well as who in the economy gains and who loses through regulation.

We assume that there are 4 states and as before preferences are Cobb-Douglas with:

$$u^h(x) = \alpha^h \log(x_1(0)) + (1 - \alpha^h) \log(x_2(0)) + \sum_{s=1}^{4} \pi_s(\alpha^h \log(x_1(s)) + (1 - \alpha^h) \log(x_2(s)))$$

We interpret endowments in the non-durable as income while endowments in the durable good are interpreted as wealth. The estimates on wealth and income distribution in the US for 1995 and 2004 follow ([Di], Table 1). We choose α^h to rougly match a relative price of durable to non-durable good of $1/2$ and take endowments to be:

$$e^1(0) = (0.61, 0.84), \; e^1(1) = e^1(3) = (0.63, 0), \; e^1(2) = e^1(4) = (0.21, 0);$$

$$e^2(0) = (0.22, 0.12), \; e^2(1) = e^2(3) = (0.21, 0), \; e^2(2) = e^2(4) = (0.63, 0);$$

$$e^3(0) = (0.12, 0.04), \; e^3(1) = e^3(2) = (0.11, 0), \; e^3(3) = e^3(4) = (0.05, 0);$$

$$e^4(0) = (0.05, 0.00), \; e^4(1) = e^4(2) = (0.05, 0), \; e^4(3) = e^4(4) = (0.11, 0).$$

Probabilities are given by: $\pi = (0.6, 0.18, 0.18, 0.04)$.

While this example does a good job to match income and wealth inequality at $t = 0$ it is very simplistic in the stochastic dynamics of income over time. To keep things simple we merely assume that agents 1 and 2 and agents 3 and 4 can interchange. A more realistic calibration where there is a large number of ex ante identical agents in each income class is computationally not feasible.

5.1. **The role of subprime loans.** To fix ideas, suppose first that:

$$\alpha^h = (\alpha^1, \alpha^2, \alpha^3, \alpha^4) = (0.5, 0.4, 0.3, 0.6).$$

This results in prices that are slightly off in the first period but match well the data for the second period (first period prices are $p(0) = (0.30, 0.70)$ but second period prices all lie between 0.45 and 0.55). In the GEICC equilibrium only the sub-prime loan that defaults in all states and the safe asset that defaults in no state are traded. The rich agent 1 lends in the sub-prime asset (0.37 units) and borrows (0.16 units) in the safe asset. Agents 2-4 borrow exclusively sub-prime, while agents 2 and 3 (the middle-class) actually saves some money in the safe bond.

From this situation, one can ask which possible regulation of collateral requirements are improving. As Figures 6-8[3] show, it turns out that agents 2 and 3 cannot be made better off through any regulation, while agents 1 and 4 gain simultaneously if only subprime borrowing is allowed.

The point denoted by GEIRC1 in Figures 6-8 corresponds to the equilibrium where there is full default for all traded assets. Figure 8 shows that agents 1 and 4 are best off in a GEIRC. The rich agent 1 benefits from lending more units in the subprime asset (0.43 units), due to higher interest rate, in relation to GEICC equilibrium, while agents 2 and 3 are now allowed to save some money in safe bonds. Agent 4 benefits from the increased borrowing in the sub-prime asset.

The point denoted by GEIRC2 corresponds to the equilibrium with default occurring only for asset 4 in the states 1 and 3 (i. e. in this economy the sub-prime loans are not available). Agents 1 and 4 lose, since they can't lend and borrow, respectively, in the sub-prime asset (see Figure 8). Agents 2 and 3 are better off in GEIRC2, through lending in the safe asset.

---

[3]In these Figures, we show different points in utility (measured in terms of welfare rate) space for a sample 3,798 values of collateral between 0 and 1.5. We use the algorithm described in Schommer (2008) to approximate equilibrium numerically

Finally, restricting all agents to trade in an asset that never defaults makes all agents worse off (point GEIRC3) as in Example 3.
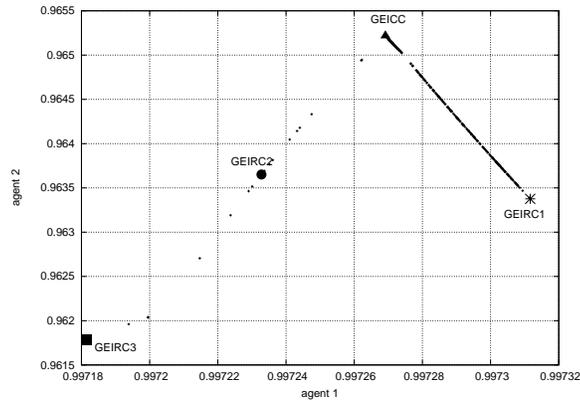

FIGURE 6. Welfare Rate for regulated collateral $(\alpha^h = (0.5, 0.4, 0.3, 0.6))$ - Agents 1 and 2
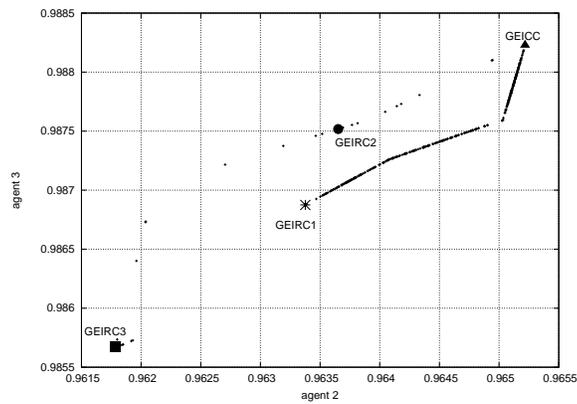

FIGURE 7. Welfare Rate for regulated collateral $(\alpha^h = (0.5, 0.4, 0.3, 0.6))$ - Agents 2 and 3
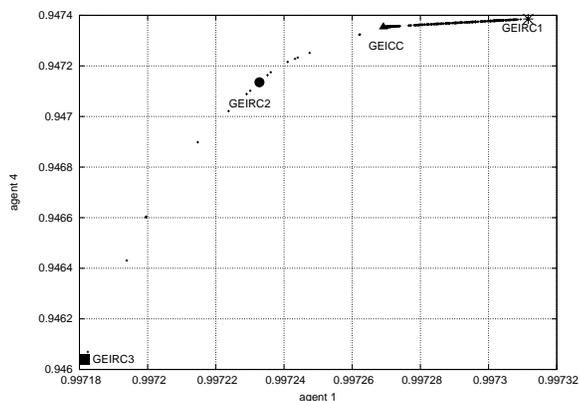
FIGURE 8. Welfare Rate for regulated collateral ($\alpha^h = (0.5, 0.4, 0.3, 0.6)$) -
Agents 1 and 4

5.2. **Robustness analysis.** In order to verify if the previous specification are robust we
consider eight more specifications for preferences.

If we consider the opposite case, when the durable good is very expensive in the first period
(i.e. if the rich agents propensity to consume the durable good is high) then regulation of
the collateral requirement does not make anyone better off. For example we considered:

$$\alpha^h = (0.3, 0.4, 0.5, 0.8), \ \alpha^h = (0.3, 0.4, 0.5, 0.6) \text{ or } \alpha^h = (0.2, 0.4, 0.6, 0.7).$$

In the GEICC equilibria two assets are traded one that defaults in all states and other
that ensures full delivery in four states. The rich agent 1 lends in the safe asset and borrows
in the sub-prime asset. Agents 2 and 3 lend in the sub-prime asset, while agents 2 and 3
(the middle-class) actually save some money in the safe bond.

In [YZh] preferences over housing and other goods consumption are represented by the
Cobb-Douglas function. They estimate the housing preference $(1 - \alpha^h)$ as 0.2 for US in 2001
based on the average proportion of household housing expenditure according to the Bureu of
Labor Statistics (BLS) of the US Department of Labor. Here we estimate the durable good
preference $(1 - \alpha^h)$ for each type of agent, based on the proportions of housing, furniture

and vehicle purchases in the Consumer Expenditure Survey of BLS for 2004[4]. In this case the preference are:

$$\alpha^h = (0.74, 0.73, 0.72, 0.71).$$

As in the above example, both agents 1 and 4 can be made better off when trade is restricted to be in the sub-prime asset. The following values for α also give this result:

$$\alpha^h = (0.7, 0.6, 0.5, 0.4),\ \alpha^h = (0.4, 0.3, 0.5, 0.7),\ \alpha^h = (0.5, 0.4, 0.3, 0.2)\ \text{or}\ \alpha^h = (0.5, 0.3, 0.4, 0.6).$$

indicating that what seems to be the most robust case is a case as in the above example.

Note that in all cases the subprime asset is traded actively and it always hurts all agents if trade in this asset is restricted. As mentioned in the introduction, this is not meant to say that markets in sub-prime loans are always Pareto-improving as our model is clearly very specialized, default is rationally anticipated by all market participants and there are no externalities of default. However, this result should still provide a benchmark since it shows that in this economy (where there are no rental markets) subprime loans provide the only possibility for the poor to purchase the durable good.

6. Conclusion

In this paper we consider a model with default and collateral and demonstrate how scarcity of collateralizable goods can lead to large welfare losses in the absence of other mechanisms to enforce intertemporal contracts.

Following [GZ] we allow for a complete set of collateral requirements and show that in the GEIC equilibrium generally only few of the possible contracts are traded.

In our examples, equilibrium is constrained efficient in the sense that a regulation of collateral requirements never leads to a Pareto-improvement for all agents. However, we show that equilibria corresponding to different regulated collateral requirements are often not Pareto-ranked, and some agents can be benefited with regulation.

---

[4]In Table 46 of U.S. Department of Labor (2004) consumers are split into nine groups, of uneven size. In order to reach four classes, we regroup and perform an weighted average.

We show, through examples, that when the borrower owns almost nothing of the durable good a subprime loan is optimal for both agents (lender and borrower). The lender always benefits from subprime loans, his risk being completely absorbed by the higher interest rate carried by these loans.

The welfare losses due to collateral can be large if some agents own small amounts of the collateralizable durable goods.

References

[AFP]   A. Araújo, J. Fajardo and M. Páscoa. Endogenous collateral. *Journal of Mathematical Economics*, 41 (2005), 439–462.

[AOP]   A. Araújo, J. Orrillo and M. Páscoa. Equilibrium with Default and Endogenous collateral. *Mathematical Finance*, 10 (2000), 1–21.

[APT]   A. Araújo, M. Páscoa and J.P. Torres-Martinez. Collateral Avoids Ponzi Schemes in Incomplete Markets. *Econometrica*, 70 (2002), 1613–1638.

[Di]   Z. Di. Growing Wealth, Inequality, and Housing in the United States. Harvard University's Joint Center for Housing Studies W07-1. (2007).

[DGS]   P. Dubey, J. Geanakoplos and M. Shubik. Default and punishment in general equilibrium. *Econometrica*, 73 (2005), 1–37.

[G]   J. Geanakoplos. Liquidity, Default, and Crashes: Endogenous Contracts in General Equilibrium. in: *Advances in Economics and Econometrics: Theory and Applications, Eighth World Conference, Volume II*, ed. Mathias Dewatripont, Lars Peter Hansen, and Stephen J. Turnovsky, Econometric Society Monographs, (2003), 170–205.

[GP]   J. Geanakoplos, and H.P. Polemarchakis, 1986, Existence, Regularity, and Constrained Suboptimality of Competitive Allocations when the Asset Market is Incomplete. in *Essays in Honor of Kenneth Arrow* Cambridge University Press, edited by W. Heller, R. Starr, and D. Starrett, 3 (1986), 65–95.

[GZ]   J. Geanakoplos and W. Zame. Collateralized Asset Markets. Unpublished (2007).

[Ki]   W.T. Kilentong. Collateral premia and risk sharing under limited commiment. http://econ.ucsb.edu/∼tee/ (2008).

[KS]   F. Kubler and K. Schmedders. Stationary equilibria in asset-pricing models with incomplete markets and collateral. *Econometrica*, 71 (2003), 1767–1793.

[Sc]   S. Schommer. Computing general equilibrium with incomplete markets and default. http://sites.google.com/site/susanschommersite/working-papers (2008).

[US]   U.S. Departament of Labor, Bureau of Labor Statistics, Consumer Expenditures Survey (2004).

[YZh] R. Yao and H. Zhang. Optimal Consumption and Portfolio Choices with Risky Housing and Borrowing Constraints. *The Review of Financial Studies*, 18 (2005), 197–239.

ALOÍSIO ARAÚJO, IMPA

E-mail address: `aloisio@impa.br`

FELIX KUBLER, UNIVERSITY OF ZURICH

E-mail address: `fkubler@gmail.com`

SUSAN SCHOMMER, IMPA

E-mail address: `schommer@impa.br`