Two Limit Laws in Random Matrix Theory and Statistical Mechanics

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Abstract

This thesis looks at two different problems in probability theory.
The first part of the thesis treats the problem of characterizing the law of the largest eigenvalue in the generalized Cauchy random matrix ensemble. The generalized Cauchy random matrix ensemble is an ensemble of Hermitian matrices with a weight that can be viewed as a generalization of the standard Cauchy probability distribution. Forrester and Witte describe the law of the largest eigenvalue of a matrix in such an ensemble of finite size $N \times N$ in earlier work (Nonlinearity, 13:1965–1986, 2000 and Nagoya Math. J., 174:29–114, 2004). They obtain a characterization of this law in terms of a Painlevé-VI equation using the theory of τ-functions. We show that under a restriction on the involved parameters, the same result can be obtained via the famous formalism of Tracy and Widom (Comm. Math. Phys., 163:33–72, 1994). Then, we show that when the largest eigenvalue is appropriately scaled, this law converges pointwise to a limiting law when the size of the ensemble tends to infinity. The limit law can be interpreted as the law of the largest point in a determinantal point process on the real line described by Borodin and Olshanski (Comm. Math. Phys., 223:87–123, 2001). We also characterize the limit law in terms of a Painlevé-V equation and give a sense to the convergence of the corresponding Painlevé-VI equation for the finite case to the former equation when $N \to \infty$. Finally, we also show that the pointwise convergence of the law is of order N^{-1}. The techniques we use to obtain the convergence results are completely elementary. They essentially involve checking pointwise convergence and domination of all quantities involved in the corresponding Fredholm determinants in order to apply dominated convergence.
In the second part of the thesis we deal with the asymptotic behavior of the perturbed weakly self-avoiding walk. The weakly self-avoiding walk is a random walk on $\mathbb{Z}^d$ where self-intersections are penalized by a factor $1 - \lambda$, $\lambda > 0$ a small parameter and the dimension $d \geq 9$ (respectively $d \geq 5$ in the symmetric case). We use the lace expansion to show that when starting the walk with a distribution which is a small perturbation of the standard nearest neighbor distribution $\frac{1}{2d}\mathbb{1}_{\{x:\|x\|=1\}}$, a local central limit Theorem holds with exponential error decay and a correction of order $n^{-d/2}$ near the mean of the walk. Our main Theorem in this part is in fact a more general central limit Theorem for convolution equations similar to the one given by the weakly self-avoiding walk. The lace expansion has been introduced by Brydges and Spencer (Comm. Math. Phys., 97:125–148, 1985). Most approaches to the lace expansion use Fourier methods. We however use the Banach fixed point Theorem for an appropriately chosen space and operator to show that the limiting density of the weakly self-avoiding walk is stable under small perturbations and close to a normal density. Our method is based on earlier work for the symmetric (standard) weakly self-avoiding walk by Ritzmann (PhD thesis, Universität Zürich, 2001). With this method we can work directly in $\mathbb{Z}^d$ and obtain the central limit Theorem in a more transparent way than with Fourier methods. Moreover, we can directly estimate the lace expansion diagrams via the connectivities of the walk.

Zusammenfassung

Diese Dissertation befasst sich mit zwei verschiedene Problemen aus der Wahrscheinlichkeitstheorie.
Im ersten Teil wird die Verteilung des grössten Eigenwertes im verallgemeinerten Cauchy Zufallsmatrizenensemble beschrieben. Das verallgemeinerte Cauchy Ensemble ist die Menge der Hermiteschen Matrizen mit einer verallgemeinerten Cauchy-Verteilung. Forrester und Witte beschreiben die Verteilung des grössten Eigenwertes einer solchen Matrix endlicher Grösse $N \times N$ (Nonlinearity, 13:1965–1986, 2000 und Nagoya Math. J., 174:29–114, 2004). Sie charakterisieren diese mit Hilfe von τ-Funktionen als Funktion der Lösung einer Painlevé-VI Gleichung. Wir zeigen, dass man unter einer kleinen Einschränkung für die involvierten Parameter dasselbe Resultat über die berühmte Methode von Tracy und Widom (Comm. Math. Phys., 163:33–72, 1994) herleiten kann. Weiter zeigen wir, dass diese Verteilung punktweise zu einer Grenzverteilung konvergiert, wenn die Grösse des Ensembles nach Unendlich strebt und der grösste Eigenwert richtig skaliert wird. Die Grenzverteilung interpretieren wir als Verteilung des grössten Punktes in einem von Borodin und Olshanski (Comm. Math. Phys., 223:87–123, 2001) eingeführten determinanten Punktprozess auf $\mathbb{R}$. Wir charakterisieren diese Grenzverteilung mit Hilfe der Lösung einer Painlevé-V Gleichung und geben der Konvergenz der entsprechenden Painlevé-VI Gleichung für den endlichen Fall zu der letztgenannten Gleichung für $N \to \infty$ einen mathematischen Sinn. Schliesslich zeigen wir auch, dass die punkweise Konvergenz der Verteilung von der Ordnung N^{-1} ist. Um die Konvergenzresultate zu zeigen benützen wir nur elementare Techniken. Im Wesentlichen prüfen wir die punkweise Konvergenz und geben obere Schranken für alle in den entsprechenden Fredholm-Determinanten involvierten Grössen. Dann benützen wir den Satz der majorisiteren Konvergenz.
Im zweiten Teil betrachten wir das asymptotische Verhalten der gestörten schwach selbst-abstossenden Irrfahrt. Die schwach selbst-abstossende Irrfahrt ist eine Irrfahrt auf $\mathbb{Z}^d$ bei der Selbstüberschneidungen durch einen Faktor $1 - \lambda$ bestraft werden, wobei $\lambda > 0$ ein kleiner Parameter ist und die Dimension d mindestens 9 ist (beziehungsweise 5 im symmetrischen Fall). Wir benützen die "Lace-Expansion" um zu zeigen, dass wir einen lokalen zentralen Grenzwertsatz mit exponentiellem Fehlerabfall und einer Korrektur von der Ordnung $n^{-d/2}$ nahe des Mittelwerts der Irrfahrt erhalten, falls wir mit einer leichten Störung der üblichen symmetrischen nächsten Nachbarn-Verteilung $\frac{1}{2d}\mathbb{1}_{\{x:\|x\|=1\}}$ starten. Unser zentrales Theorem in diesem Teil ist eigentlich allgemeiner und gilt für alle Faltungsgleichungen die in gewisser Weise ähnlich sind zu der Gleichung der schwach selbst-abstossenden Irrfahrt. Die Lace-Expansion wurde von Brydges und Spencer (Comm. Math. Phys., 97:125–148, 1985) eingeführt. Meistens wird die Lace-Expansion zusammen mit Fourier Methoden verwendet. Wir benützen jedoch den Banachschen Fixpunktsatz auf einem geeigneten Raum mit einem geeigneten Operator, um zu zeigen, dass die Grenzdichte der schwach selbst-abstossenden Irrfahrt nahe bei der Dichte der Normalveteilung liegt und zudem stabil unter kleinen Störungen ist. Unsere Technik basiert auf einer Arbeit von Ritzmann (PhD thesis, Universität Zürich, 2001). Dank dieser Technik können wir direkt in $\mathbb{Z}^d$ arbeiten und erhalten so den lokalen zentralen Grenzwertsatz in einer tranparenteren Art und Weise als mit Fourier Methoden. Ein weiterer Vorteil ist, dass die Diagramme der Lace-Expansion direkt duch die zwei-Punkte Funktion der Irrfahrt abgeschätzt werden können.

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Contents

Preface

In probability theory, the study of limiting distributions in various occurrences has a long history. In particular, one of the most universal and best studied limiting distributions is the *Normal* or *Gauss* distribution. It arises in the *Central Limit Theorem.* Roughly speaking this Theorem says that under certain (very mild) conditions, the appropriately normalized sum of independent random variables converges weakly to a standard normal random variable. This can be interpreted as follows: Modeling the outcome of an experiment with uncertainty by a random variable satisfying those mild conditions, the repeated and independent execution of this experiment under the same initial conditions implies that the normalized result over all experiments follows a Gauss distribution. The central limit Theorem was first proved by De Moivre around 1733 for independent and symmetric Bernoulli variables. Later on it was generalized by Laplace to the case of non-symmetric Bernoulli variables. A completely rigorous proof of the central limit Theorem for independent and identically distributed random variables with finite second moments was given in 1901–1902 by Lyapunov. This Theorem has wide applications ranging from game theory over financial mathematics to bio-statistics and physics. A variant of the central limit Theorem is the *Local Central Limit Theorem.* It states that the density of the normalized sum of the variables converges pointwise to the density of a normally distributed variable.

A completely different kind of limit Theorem arises in *Random Matrix Theory.* Random matrix theory was first encountered in statistics by Hsu, Wishart and others in the 1930's. However, it was only really intensively studied from the 1950's, starting with Wigner who used random matrices in nuclear physics. Since then random matrices have been used in various fields of physics such as chaotic systems and conductivity in disordered metals. They are even used to model the zeros of the Riemann-ζ-function (starting in the 1970's with a still open conjecture by Montgomery). One field of interest is the distribution of the (real) eigenvalues of a randomly distributed Hermitian matrix of size $N \times N$ whose probability law is independent under change of basis (ie. under conjugation by unitary matrices). From a physical point of view, these eigenvalues may model the energy levels of a random operator, or can give the distribution of electrical unit charges confined to be on the real line under a certain external potential and with a logarithmic interaction term. One can try to characterize the law of the largest eigenvalue in such a regime and try to understand the convergence of this law when the size N of the matrix ensemble tends to infinity.

In the first part of this thesis we deal with the latter problem. We consider the Hermitian matrix ensemble with a *Generalized Cauchy Weight.* This can be seen as a two-parameters extension of the well studied *Circular Unitary Ensemble (CUE)* (also called *Dyson Ensemble*). The CUE is an ensemble of unitary matrices distributed according to the normalized Haar measure on the unitary group of size $N \times N$. The Hermitian and the unitary matrices are linked via the Cayley transform. The generalized Cauchy weight is a weight that is invariant under unitary conjugation of the matrices and it generalizes the standard Cauchy distribution on the real line (Hermitian matrix of size 1×1). In this regime, we study the law of the largest eigenvalue and give a limiting law for this eigenvalue distribution under appropriate scaling and when the size N of the ensemble tends to infinity. In particular, we characterize the limiting law in terms of the solution of a Painlevé-V differential equation. *Painlevé* equations often enter the description of the law of the largest eigenvalue of a random matrix. These equations are second order ordinary differential equations in $\mathbb{C}$ with the property that their only movable singularities are poles and which are not solvable using elementary functions. They originate in the study of special functions and isomonodromic deformations of linear differential equations. In fact, we

also characterize the law of the largest eigenvalue in the finite N-case in terms of a Painlevé-VI equation. This is done via a very general method introduced by Tracy and Widom [37]. Using this method we unfortunately only get this characterization under a restriction on the set of parameters. However, Forrester and Witte [15] extend the Painlevé-VI characterization to the full set of parameters using a different method (τ-function theory). We show that this Painlevé-VI equation converges in some sense to the limiting Painlevé-V equation if $N \to \infty$. Finally, we also give the convergence speed for the law when $N \to \infty$.
In the second part of the thesis, we are interested in the local central limit Theorem for perturbed weakly self-avoiding random walks. In fact, a classical regime of the central limit Theorem is the standard random walk on a lattice ($\mathbb{Z}^d$, d being the dimension). The location of a random walker after n steps is then simply the sum of n independent identically distributed random variables. Therefore, one can give a central limit Theorem for this case. Here, we will not consider the standard random walk, but we will look at a random walk which is penalized whenever it intersects itself. This model has been introduced by Physicists and Chemists to study the growth of large polymer chains. We will assume that the initial distribution of the random walk need not to be symmetric and may be spread out. Note that the position of the weakly self-avoiding walk after n steps cannot be modeled by the sum of n independent and identically distributed random variables since the walk has to remember its complete past at any time. Nevertheless, we show that for high dimensions ($d \geq 9$, respectively $d \geq 5$ if we restrict to symmetric initial distributions), this random walk has diffusive behavior if its initial distribution is contained in a certain closed set around the standard symmetric initial distribution and the penalty for each self-intersection is not too large. That is, its probability density converges locally for each $x \in \mathbb{Z}^d$ to the density of a normal random variable. In other words, the perturbed weakly self-avoiding random walk satisfies a local central limit Theorem.

Chapter 1

Introduction to the Generalized Cauchy Random Matrix Ensemble

1.1 Introduction

This part of the thesis deals with the characterization of the law of the largest eigenvalue of a matrix in the Generalized Cauchy Random Matrix Ensemble (denoted by GCyE). In case of finite sized ensembles and under a restriction on the involved parameters we give a characterization of this law via a Painlevé-VI equation. We are also interested in the convergence of the law, when the size of the matrix tends to infinity and in the characterization of the limiting law in terms of a Painlevé-V equation. A result on the rate of convergence for the law is also given. All the results on the convergence and the limiting law are taken from the article [29] which is joint work with Joseph Najnudel and Ashkan Nikeghbali.

1.1.1 General Remarks on Random Matrix Theory

The theory of random matrices is essentially the theory of matrix valued random variables. A *Random Matrix Ensemble* is a set of matrices with an associated probability measure. One can imagine any kind of ensemble, but in general there are two groups of ensembles which are widely studied.

The first group are ensembles of matrices that contain entries which are chosen independently according to some given distribution. The most classical such example is the *Gaussian Unitary Ensemble (GUE)*:

Definition 1.1. *A random $N \times N$ Hermitian matrix belongs to the* GUE*, if the diagonal elements x_{jj} and the upper triangular elements $x_{jk} = u_{jk} + iv_{jk}$ ($j < k$) are chosen independently according to normal densities of the form:*

$$\begin{aligned} \frac{1}{\sqrt{\pi}} e^{-x_{jj}^2} &\sim \mathcal{N}(0, \frac{1}{2}) \text{ (diagonal elements)}, \\ \frac{2}{\pi} e^{-2(u_{jk}^2 + v_{jk}^2)} &\sim \mathcal{N}(0, \frac{1}{4}) + i\mathcal{N}(0, \frac{1}{4}) \text{ (upper triangular elements)}. \end{aligned}$$

Note that other conventions on the normalization of the variances exist.

That is, the GUE is an ensemble of Hermitian matrices with independent Gaussian entries. It is such that the law of a matrix is independent under conjugation by unitary matrices. Similar such examples are the *Gaussian Orthogonal Ensemble (GOE)* and the *Gaussian Symplectic Ensemble (GSE)*. The former is the ensemble of all real symmetric $N \times N$ matrices with Gaussian entries and the latter is the ensemble of all $N \times N$ symplectic matrices with Gaussian entries. These ensembles have been widely studied and applied in various fields. Already in the 50's Wigner (see [42]) used the GUE to model the statistical behavior of slow neutron resonances, and later in the 70's Montgomery [27] conjectured that the appropriately scaled zeros of the Riemann-Zeta

function on the critical line $\Re z = 1/2$ appear to have the same pair correlation as the eigenvalues of the GUE. There is still no proof of this numerical fact.
The second group of ensembles are obtained as follows: Consider a compact Lie group $\mathcal{G}$. Then, there exists a $\mathcal{G}$-invariant measure μ on $\mathcal{G}$ (unique up to scaling). Ie. $\mu(gA) = \mu(A)$, for all $g \in \mathcal{G}$ and A an open subset of $\mathcal{G}$ (see [8]). This measure is called the *Haar measure*. When normalized, it gives a probability measure on $\mathcal{G}$. The most classical such ensemble is the *Circular Unitary Ensemble (CUE)* (also called *Dyson Ensemble*). It is the unitary group $U(N)$ endowed with its normalized Haar measure. In the following, we will only be interested in the Generalized Cauchy Ensemble. It is an ensemble of this second kind and in fact, it is in some sense a generalization of the CUE.
A very detailed study of many ensembles and a good overview on random matrix theory can be found in the classical book by Mehta [26] and also in Forrester's new book [14].

1.1.2 The Eigenvalues of a Random Matrix

In random matrix theory one is often interested in the distribution of the eigenvalues in a certain ensemble. The measure on the eigenvalues is obtained by projecting the measure on the ensemble onto the space of eigenvalues. Given a random matrix of size $N \times N$ with real eigenvalues, the *eigenvalue probability distribution function (PDF)* on $\mathbb{R}^N/S(N)$ ($S(N)$ being the symmetric group of order N) often has the form

$$\text{const} \cdot \prod_{1 \le j < k \le N} (x_j - x_k)^2 \prod_{j=1}^{N} w(x_j) dx_j, \tag{1.1.1}$$

where $w(x)$ is a weight function on $\mathbb{R}$, and where $x_1, \ldots, x_n \in \mathbb{R}$ are the eigenvalues (considerer to be unordered here!). The term $\prod_{1\le j<k\le N}(x_j - x_k)$ is called *van der Monde Determinant* since it is equal to $-\det(x_k^{j-1})_{1\le j,k\le N}$. For example in the GUE case, the eigenvalue distribution has this form with the weight function $w_2(x) = e^{-x^2}$. On the other hand, the choices $w_L(x) = x^a e^{-x}$ on $\mathbb{R}_+$ with $a > -1$, or $w_J(x) = (1-x)^\alpha (1+x)^\beta$ for $-1 \le x \le 1$ with $\alpha, \beta > -1$, lead to the so called *Laguerre* or *Jacobi* ensembles respectively. The three weight functions w_2, w_L and w_J occur in the eigenvalue PDF for certain ensembles of Hermitian matrices based on matrices with independent Gaussian entries (see for example Forrester [14]) and are called *classical* weight functions. In Adler, Forrester, Nagao and van Moerbeke [1], the defining property of a classical weight function in this context was identified as the following fact: If one writes the weight function $w(x)$ of an ensemble as $w(x) = e^{-2V(x)}$, with $2V'(x) = g(x)/f(x)$, f and g being polynomials in x, then the operator $n := f(d/dx) + (f' - g)/2$ increases the degree of the polynomials by one, and thus, $\deg f \le 2$, and $\deg g \le 1$. For a long time, these three examples have been the only classical weight functions known.
In case the eigenvalue PDF has the form (1.1.1), there is a well known methodology for treating the distribution of the eigenvalues (see Mehta [26]). In fact we can rewrite formula (1.1.1) using elementary row and column operations from the second to the third line to obtain

$$\begin{aligned} &\text{const} \cdot \prod_{1 \le j < k \le N} (x_j - x_k)^2 \prod_{j=1}^{N} w(x_j) dx_j \\ =&\text{const} \cdot \left(\det(x_i^{j-1}\sqrt{w(x_i)})_{1\le i,j\le N}\right)^2 \\ =&\text{const} \cdot \left(\det(p_{j-1}(x_i)\sqrt{w(x_i)})_{1\le i,j\le N}\right)^2, \end{aligned}$$

where p_{j-1} is a monic polynomial of degree $j-1$. If now it is possible to define a set of monic orthogonal polynomials p_i with respect to the weight function $w(x)$ on $\mathbb{R}$, then one defines the integral operator K_N on $L_2(\mathbb{R})$, associated with the kernel $K_N(x,y) := \sum_{i=0}^{N-1} \frac{p_i(x)p_i(y)}{\|p_i\|^2}\sqrt{w(x)w(y)}$. Using a generalization of the Cauchy-Binet formula (see Johansson [21]), one can show that

$$\text{const} \cdot \left(\det(p_{j-1}(x_i)\sqrt{w(x_i)})_{1\le i,j\le N}\right)^2 = \text{const} \cdot \det(K_N(x_i,x_j))_{i,j=1}^N.$$

Note here that in the GUE, the monic orthogonal polynomial ensemble consists of the monic Hermite polynomials. Using the kernel K_N, it is possible to describe probabilities of the form:

$$E(k,J) := P[\text{there are exactly } k \text{ eigenvalues inside the interval } J],$$

where $J \subset \mathbb{R}$ and $k \in \mathbb{N}_0$, by the formula (see again Mehta [26]):

$$E(k,J) = \frac{(-1)^k}{k!}\frac{d^k}{dx^k}\det(I - xK_N)|_{x=1},$$

where the determinant is a *Fredholm Determinant* and the operator K_N is restricted to J. A definition of the Fredholm determinant is given in (1.2.15).
The distribution of the largest eigenvalue as well as the problem of the convergence of the scaled largest eigenvalue have received much attention (see e.g. [31], [34], [35], [38]). Also the problem on the rate of convergence has been studied, especially in [17] and [10] for GUE and LUE matrices, and in [22] as well as in [12] for Wishart matrices. To deal with the largest eigenvalue, one takes $J = (t,\infty)$ for some $t \in \mathbb{R}$. Then $E(0,(t,\infty))$ is simply the probability distribution of the largest eigenvalue, denoted from now on by $\lambda_1(N)$, of a $N \times N$ matrix in the respective ensemble. In their pioneering work [37], Tracy and Widom give a system of completely integrable differential equations to show how the probability $E(0,J)$ can be linked to solutions of certain Painlevé differential equations. Tracy and Widom apply their method to the finite Hermite, Laguerre and Jacobi ensembles. Moreover, one can also apply the method to scaling limits of random matrix ensembles when the dimension N goes to infinity. The famous sine kernel and its Painlevé-V representation for instance, as obtained by Jimbo, Miwa, Môri and Sato [20], arise if one takes the scaling limit in the bulk of the spectrum of the Gaussian Unitary Ensemble and of many other Hermitian matrix ensembles (see e.g. [23], [25], [28] and [30]). On the other hand, if one scales appropriately at the edge of the Gaussian Unitary Ensemble, one obtains an Airy kernel in the scaling limit with a Painlevé-II representation for the distribution of the largest eigenvalue (see Tracy and Widom [38]). Similar results have been obtained for the edge scalings of the Laguerre and Jacobi ensembles, where the Airy kernel has to be replaced by the Bessel kernel and the Painlevé-II equation by a Painlevé-V equation (see Tracy and Widom [39]). Soshnikov [35] gives an overview on scaling limit results for large random matrix ensembles.

1.1.3 The Generalized Cauchy Ensemble (GCyE)

Let $H(N)$ be the set of Hermitian matrices endowed with the measure

$$\text{const} \cdot \det(1+X^2)^{-N} \prod_{1\le j<k\le N} dX_{jk} \prod_{i=1}^N dX_{ii}, \qquad X \in H(N), \tag{1.1.2}$$

where const is a normalizing constant (depending on N), such that the total mass of $H(N)$ is equal to one. This measure is the analogue of the normalized Haar measure μ_N on the unitary group $U(N)$, if one relates $U(N)$ and $H(N)$ via the Cayley transform: $H(N) \ni X \mapsto U = \frac{X+i}{X-i} \in U(N)$. The measure (1.1.2) can be deformed to obtain the following two parameters probability measure:

$$\text{const} \cdot \det((1+iX)^{-s-N})\det((1-iX)^{-\overline{s}-N}) \prod_{1\le j<k\le N} dX_{jk} \prod_{i=1}^N dX_{ii}, \tag{1.1.3}$$

where s is a complex parameter such that $\Re s > -1/2$ (otherwise the quantity involved in (1.1.3) does not integrate as is proved in Borodin and Olshanski [5]). Following Forrester and Witte [15] and [43], we call this measure the *Generalized Cauchy Measure* on $H(N)$. $H(N)$ endowed with the generalized Cauchy measure shall be called the *Generalized Cauchy Random Matrix Ensemble*, noted *GCyE*. The name is chosen because if $s = 0$ and $N = 1$, (1.1.3) is nothing else than the density of a standard Cauchy random variable. We project this measure onto the space $\mathbb{R}^N/S(N)$ of all (unordered) sets of eigenvalues of matrices in $H(N)$, to obtain the eigenvalue density

$$\text{const} \cdot \prod_{1 \le j < k \le N} (x_j - x_k)^2 \prod_{j=1}^{N} w_H(x_j) dx_j. \tag{1.1.4}$$

Here $w_H(x_j) = (1 + ix_j)^{-s-N}(1 - ix_j)^{-\bar{s}-N}$, and the x_j's denote the eigenvalues. As usual, the constant is chosen so that the total mass of $\mathbb{R}^N/S(N)$ is equal to one.

As mentioned in the last Subsection 1.1.2, there have for a long time been three classical weight functions only (w_2, w_L and w_J). But for $s \in (-1/2, \infty)$, the property of being classical also holds for the weight function w_H of the *GCyE*. We thus have four classical weight functions (see also Witte and Forrester [43]). However, the construction of the matrix model for the *GCyE* is different from the construction of the other three classical ensembles: A matrix model for the *GCyE* will not have independent entries, but one can construct the ensemble via the Cayley transform. Indeed, following Borodin and Olshanski [5] (see also [14], [15] and [43]) the measure (1.1.3) is, via the Cayley transform, equivalent to the deformed normalized Haar measure const$\cdot\det((1-U)^{\bar{s}})\det((1-U^*)^s)\mu_N(dU)$, $U \in U(N)$. If we denote by $e^{i\theta_j}$, $j = 1, \dots, N$, the eigenvalues of a unitary matrix with $\theta_j \in [-\pi, \pi]$, the deformed Haar measure can, as in the Hermitian case, be projected to the eigenvalue space to give the PDF

$$\text{const} \cdot \prod_{1 \le j < k \le N} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{N} w_U(\theta_j) d\theta_j, \tag{1.1.5}$$

where $w_U(\theta_j) = (1 - e^{i\theta_j})^{\bar{s}}(1 - e^{-i\theta_j})^s$, and $\theta_j \in [-\pi, \pi]$. This measure is defined on $\mathbb{S}^N/S(N)$, where $\mathbb{S}$ is the complex unit circle. Note, that this eigenvalue measure has a singularity at $\theta = 0$, if $s \neq 0$. Borodin and Olshanski [5] studied the measures (1.1.3), (1.1.4) and (1.1.5) in great detail due to their connections with representation theory of the infinite dimensional unitary group $U(\infty)$.

When $s \in (-1/2, \infty)$, (1.1.5) is nothing else than the eigenvalue distribution of the circular Jacobi unitary ensemble. This is a generalization of the Circular Unitary ensemble corresponding to the case $s = 0$. In fact, if $s = 1$, this corresponds to the CUE case with one eigenvalue fixed at one. More generally, for $s \in (-1/2, \infty)$ the singularity at one corresponds, in the log-gas picture, to a impurity with variable charge fixed at one, and mobile unit charges represented by the eigenvalues (see Witte and Forrester [43], and also [16]). It is the singularity at one that makes the study of this ensemble more difficult than the CUE. In the special case when $s = 0$, one can obtain the eigenvalues with PDF (1.1.4) from the eigenvalues of the circular unitary ensemble using a stereographic projection (see the book of Forrester [14], Chapter 2, Section 5 on the Cauchy ensemble). In fact, in this case, we get that (1.1.4) represents the Boltzmann factor for a one-component log-gas on the real line subject to the potential $2V(x) = N \log(1 + x^2)$. This corresponds to an external charge of strength $-N$ placed at the point $(0, 1)$ in the plane (this can also be generalized to arbitrary inverse temperature β as shown in the previous reference). Moreover, note that when $s \neq 0$, a construction of a random matrix ensemble with eigenvalue PDF (1.1.5) is given in Bourgade, Nikeghbali and Rouault [7].

As already mentioned, we are interested in the law of the largest eigenvalue in the GCyE case (convergence, asymptotic distribution, rate of convergence and characterization in terms

of Painlevé equations) for all admissible values of the parameter s, namely $\Re s > -1/2$. Due to the form of the eigenvalue PDF (1.1.4), we will use the methodology with the Fredholm determinant briefly discussed in Subsection 1.1.2 above.
For the eigenvalue measure (1.1.4), Borodin and Olshanski [5] give the kernel in the finite N case, denoted by K_N in the following (see Theorem 1.2), as well as a scaling limit of this kernel, when $N \to \infty$, denoted by K_∞ (see (1.2.12)). Using the kernel K_N, one can set up the system of differential equations in the way of Tracy and Widom (see Subsection 1.1.2) for the law $E(0,(t,\infty))$ of the largest eigenvalue $\lambda_1(N)$, for any $t \in \mathbb{R}$. In the case of a real parameter s, this has been done by Witte and Forrester in [43]. They obtain a characterization of the law of the largest eigenvalue in terms of a Painlevé-VI equation. More precisely, $(1+t^2)$ times the logarithmic derivative of $E(0,(t,\infty))$ satisfies a Painlevé-VI equation. The same method suitably modified leads to a generalization of this result for complex s as we show in Chapter 2. However, the method of Tracy and Widom has the drawback that it only works for s with $\Re s > 1/2$. Forrester and Witte propose in [15] an alternative method which makes use of the τ-function theory to derive the Painlevé-VI characterization for $E(0,(t,\infty))$ for any s such that $\Re s > -1/2$.
To sum up, for the generalized Cauchy ensemble, it is known that for finite N, $(1+t^2)$ times the logarithmic derivative of $E(0,(t,\infty))$ satisfies a Painlevé-VI equation for $t \in \mathbb{R}$. The orthogonal polynomials associated with the measure w_H are known as well as the scaling limit of the associated kernel K_N, which we note K_∞. One naturally expects $\lambda_1(N)$, appropriately scaled, to converge in law to the probability distribution $F_\infty(t) := \det(I - K_\infty)|_{L_2(t,\infty)}$, for $t > 0$ ($t \leq 0$ is not permissible in this particular case, as we will see in Remark 1.10). We shall see below that this is indeed the case for all values of s such that $\Re s > -1/2$. A natural question is: does $(1+t^2)$ times the logarithmic derivative of $F_\infty(t)$ also satisfy some non-linear differential equation? And as previously mentioned, what is the rate of convergence to $F_\infty(t)$?

1.2 Results

In this Section, we state our main Theorems. These results are based on earlier work by Borodin and Olshanski [5] who obtained an explicit form for the orthogonal polynomials associated with the weight w_H as well as the scaling limit for the associated kernel, and Forrester and Witte [15] who express, for fixed N and for any complex number s with $\Re s > -1/2$, the probability distribution of the largest eigenvalue $\lambda_1(N)$ of a matrix in the Generalized Cauchy Ensemble in terms of some non-linear differential equation. For clarity and to fix the notations, we first state a Theorem of Borodin and Olshanski [5]. We refer the reader to the paper [5] for more information on the determinantal aspects. The discussion on the methods we use is postponed to the next Section 1.3.
Borodin and Olshanski [5] give the correlation kernel for the determinantal point process defined by the measure (1.1.4). In fact, the monic orthogonal polynomial ensemble $\{p_m;\ m < \Re s + N - \frac{1}{2}\}$ on $\mathbb{R}$ associated with the weight $w_H(x)$, is defined by $p_0 \equiv 1$, and

$$p_m(x) = (x-i)^m {}_2F_1\left[-m,\ s+N-m,\ 2\Re s + 2N - 2m;\ \frac{2}{1+ix}\right], \tag{1.2.1}$$

where ${}_2F_1[a,\ b,\ c;\ x] = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n n!} x^n$ is the *Gauss Hypergeometric Function*, and $(x)_n = x(x+1)\dots(x+n-1)$. Using the Christoffel-Darboux formula and the theory of orthogonal polynomials, the following was proven by Borodin and Olshanski [5]:

Theorem 1.2. *The n-point correlation function ($n \leq N$) for the eigenvalue distribution (1.1.4) is given by*

$$\rho_n^{s,N}(x_1, ..., x_n) = \det\left(K_{s,N}(x_i, x_j)\right)_{i,j=1}^n ,$$

where the kernel $K_{s,N}(x,y)$ *defined on* $\mathbb{R}^2$ *is given by:*

$$K_N(x,y) := K_{s,N}(x,y) = \sum_{m=0}^{N-1} \frac{p_m(x)p_m(y)}{\|p_m\|^2} \sqrt{w_H(x)w_H(y)} = \frac{\phi(x)\psi(y) - \phi(y)\psi(x)}{x-y}, \quad (1.2.2)$$

with

$$\phi(x) = \sqrt{Cw_H(x)}p_N(x), \quad (1.2.3)$$

and

$$\psi(x) = \sqrt{Cw_H(x)}p_{N-1}(x), \quad (1.2.4)$$

where $w_H(x) = (1+ix)^{-s-N}(1-ix)^{-\overline{s}-N} = (1+x^2)^{-\Re s - N} e^{2\Im s Arg(1+ix)}$ *and*

$$C := C_{N,s} = \frac{2^{2\Re s}}{\pi} \Gamma \left[\begin{array}{ccc} 2\Re s + N + 1, & s+1, & \overline{s}+1 \\ N, & 2\Re s + 1, & 2\Re s + 2 \end{array} \right]. \quad (1.2.5)$$

Here, we use the notation:

$$\Gamma \left[\begin{array}{cccc} a, & b, & c, & \ldots \\ d, & e, & f, & \ldots \end{array} \right] = \frac{\Gamma(a)\Gamma(b)\Gamma(c)\cdots}{\Gamma(d)\Gamma(e)\Gamma(f)\cdots}. \quad (1.2.6)$$

Moreover, if $x = y$*, the kernel is given by:*

$$K_N(x,x) = \phi'(x)\psi(x) - \phi(x)\psi'(x), \quad (1.2.7)$$

using the Bernoulli-Hôpital rule.

Note that p_N is well-defined (and in $L_2(w_H)$) only for $\Re s > 1/2$. However, it can be analytically continued to $\Re s > -1/2$ using the hypergeometric expression $p_N(x) = (x-i)^N {}_2F_1[-N,\ s,\ 2\Re s;\ 2/(1+ix)]$, except if $\Re s = 0$. Moreover, Borodin and Olshanski [5] give a way to get rid of the singularity at $\Re s = 0$. They introduce the polynomial

$$\begin{aligned} \tilde{p}_N(x) &= p_N(x) - \frac{iNs}{\Re s(2\Re s + 1)} p_{N-1}(x) \\ &= (x-i)^N {}_2F_1 \left[-N, s, 2\Re s + 1; \frac{2}{1+ix} \right]. \end{aligned} \quad (1.2.8)$$

This polynomial makes sense for any $s \in \mathbb{C}$ with $\Re s > -1/2$ and one can define the kernel in Theorem 1.2 equivalently by:

$$K_N(x,y) = C \frac{\tilde{p}_N(x)p_{N-1}(y) - p_{N-1}(x)\tilde{p}_N(y)}{x-y} \sqrt{w_H(x)w_H(y)}. \quad (1.2.9)$$

We are interested in the distribution of the largest eigenvalue $\lambda_1(N)$ of a matrix in the *GCyE*. We have already seen that the probability that $\lambda_1(N)$ is smaller than t, is

$$E(0,(t,\infty)) = \det(I - K_N)|_{L_2(t,\infty)}, \quad (1.2.10)$$

for any $t \in \mathbb{R}$. Hence, we need to consider the operator K_N with kernel $K_N(x,y)$ restricted to the interval (t,∞) to calculate the probability that no eigenvalue is in the interval (t,∞). This restriction is symmetric, with eigenvalues between 0 and 1. It is easy to see that K_N, restricted to any subinterval J (or finite union of subintervals) of $\mathbb{R}$, has no eigenvalue equal to 1, since $E(0,(t,\infty)) > 0$ for any $t \in \mathbb{R}$. This is true because

$$P(\lambda_1(N) \le t) = \text{cst} \cdot \int_{(-\infty,t)^N} \prod (x_j - x_k)^2 \prod w_H(x_j) dx_1 \ldots dx_N,$$

and the integrand is strictly positive. Moreover, restricting the correlation function $\rho_n^{s,N}$ of Theorem 1.2 to J gives

$$\rho_n^{s,N}(x_1,...,x_n)|_J = \prod_{j=1}^{n} \chi_J(x_j)\rho_n^{s,N}(x_1,...,x_n) \tag{1.2.11}$$

$$= \prod_{j=1}^{n} \chi_J(x_j)\det(K_N(x_i,x_j))_{i,j=1}^n = \det(\chi_J(x_i)K_N(x_i,x_j)\chi_J(x_j))_{i,j=1}^n,$$

where χ_J denotes the indicator function of the set J. Therefore, the restriction of K_N to J, denoted by $K_{N,J}$, defines a determinantal process on J with kernel $\chi_J(x)K_N(x,y)\chi_J(y) =: K_{N,J}(x,y)$.
Borodin and Olshanski [5] give a scaling limit for the kernel $K_N(x,y)$ given in Theorem 1.2. Namely, $\lim_{N\to\infty} NK_N(Nx,Ny) = K_\infty(x,y)$, for any $x,y \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}$, where the kernel K_∞ is defined by

$$K_\infty(x,y) = \frac{1}{2\pi}\frac{\Gamma(s+1)\Gamma(\bar{s}+1)}{\Gamma(2\Re s+1)\Gamma(2\Re s+2)}\frac{\tilde{P}(x)Q(y)-Q(x)\tilde{P}(y)}{x-y}, \tag{1.2.12}$$

if $x \neq y$, and,

$$K_\infty(x,x) = \frac{1}{2\pi}\frac{\Gamma(s+1)\Gamma(\bar{s}+1)}{\Gamma(2\Re s+1)\Gamma(2\Re s+2)}(\tilde{P}'(x)Q(x)-Q'(x)\tilde{P}(x)), \tag{1.2.13}$$

where

$$\tilde{P}(x) = |2/x|^{\Re s}e^{-i/x+\pi\Im s\mathrm{Sgn}(x)/2}{}_1F_1\left[s, 2\Re s+1; \frac{2i}{x}\right],$$
$$Q(x) = (2/x)|2/x|^{\Re s}e^{-i/x+\pi\Im s\mathrm{Sgn}(x)/2}{}_1F_1\left[s+1, 2\Re s+2; \frac{2i}{x}\right],$$

with

$${}_1F_1\left[r,q;x\right] = \sum_{n\geq 0}\frac{(r)_n}{(q)_n n!}x^n,$$

for any $r,q,x \in \mathbb{C}$.

Remark 1.3. The kernel K_∞ defines a determinantal point process (see [5] , Theorems IV and 6.1).

Remark 1.4. If $s=0$, the limiting kernel K_∞ writes as

$$K_\infty(x_1,x_2) = \frac{1}{\pi}\frac{\sin(1/x_2-1/x_1)}{x_1-x_2}.$$

Under the change of variable $y = \frac{1}{\pi x}$ and taking into account the corresponding change of the differential dx, K_∞ translates to the famous sine kernel with correlation function

$$\rho_n(y_1,\ldots,y_n) = \det\left(\frac{\sin(\pi(y_i-y_j))}{\pi(y_i-y_j)}\right)_{i,j=1}^n,$$

for any $n\in\mathbb{N}$ and $y_1,\ldots,y_n\in\mathbb{R}$ (see Borodin and Olshanski [5]).

Before stating our main results, we need to introduce one more notation: we note $K_{[N]}(x,y)$ the kernel

$$K_{[N]}(x,y) := NK_N(Nx,Ny), \tag{1.2.14}$$

and $K_{[N]}$ the associated integral operator. We also recall the definition of the Fredholm determinant: if K is an integral operator with kernel given by $K(x,y)$, then the k-correlation function ρ_k is defined by:

$$\rho_k(x_1,\ldots,x_k) := \det(K(x_i,x_j)_{1\le i,j\le k}).$$

The Fredholm determinant F, from $\mathbb{R}_+^*$ to $\mathbb{R}$, is then defined by

$$\begin{aligned} F(t) :=& 1+\sum_{k\ge1}\frac{(-1)^k}{k!}\int_{(t,\infty)^k}\rho_k(x_1,\ldots,x_k)dx_1\ldots dx_k \\ =& \det(I-K)|_{L_2(t,\infty)}. \end{aligned} \tag{1.2.15}$$

Our first Theorem states that $E(0,(t,\infty))$ from (1.2.10) can be interpreted in terms of the solution to an equation equivalent to a Painlevé-VI equation.

Theorem 1.5. *For $\Re s > 1/2$, define*

$$\begin{aligned} \sigma(t) =& (1+t^2)\frac{d}{dt}\log\det(I-K_N)|_{L_2(t,\infty)} \\ =& (1+t^2)\frac{d}{dt}\log P[\textit{there is no eigenvalue inside } (t,\infty)]. \end{aligned}$$

Then, for $t\in\mathbb{R}$, $\sigma(t)$ satisfies the equation:

$$\begin{aligned} &(1+t^2)(\sigma'')^2+4(1+t^2)(\sigma')^3-8t(\sigma')^2\sigma+4\sigma^2(\sigma'-(\Re s)^2)+8(t(\Re s^2)-\Re s\Im s \\ &-N\Im s)\sigma\sigma'+4(2t\Im s(N+\Re s)-(\Im s)^2-t^2(\Re s)^2+N(2\Re s+N))(\sigma')^2=0. \end{aligned} \tag{1.2.16}$$

Forrester and Witte [15] extend this Theorem using τ-function theory to the full set of admissible parameters s (ie. $\Re s > -1/2$). We will use this extension in the proof of the results below concerning the case $N\to\infty$. Note that this Theorem also generalizes the same result for s real with $\Re s > 1/2$ given by Witte and Forrester ([43], Proposition 4).

Remark 1.6. The ODE (1.2.16) is equivalent to the master Painlevé equation (SD-I) of Cosgrove and Scoufis [11]. Cosgrove and Scoufis, show that the solution of this equation can be expressed in terms of the solution of a Painlevé-VI equation using a Bäcklund transform. In the real case, this transformation is described in Witte and Forrester [43].

Remark 1.7. To prove the above Theorem, we use the method of Tracy and Widom [37]. Let us briefly explain the reason why we get the restriction $\Re s > 1/2$ here (more details follow in Chapter 2). The method of Tracy and Widom establishes a system of PDE's, the so called Jimbo-Miwa-Môri-Sato equations, which can be reduced to a Painlevé-type equation as for example the one in the above Theorem. These PDE's consist of a set of universal equations and a set of equations depending on the specific form of some recurrence differential equations for ϕ and ψ. The problem is that the method of Tracy and Widom has originally been developed for finite intervals (or unions of finite intervals). If one applies the method to the case of a semi-infinite interval (t,∞), one has to consider an interval (t,a), where $a>t$. Then, one writes down the PDE's of Tracy and Widom for that interval and takes the limit in all these equations as $a\to\infty$. Note that the variables in these PDE's are the end-points t and a of the interval. It is clear that one has to be careful about the convergence of the quantities involved in these equations, when $a\to\infty$. In particular, one needs in our case that the term $(1+a^2)Q(a)R(t,a)$, where $R(x,y)$ is the kernel of the resolvent operator $K_{N,J}(1-K_{N,J})^{-1}$, and $Q(x)=(I-K_{N,J})^{-1}\phi(x)$, which is of order $a^{1-2\Re s}$, tends to zero, when $a\to\infty$. This implies the restriction $\Re s > 1/2$. One might encounter the same type of obstacle in an attempt to prove Theorem 1.11 below with this method (we will give the corresponding recurrence equations for ϕ and ψ in the case of K_∞ in Remark 3.14).

The remaining results concern the limiting law and the convergence and can also be found in [29]:

Theorem 1.8. *For s such that $\Re s > -1/2$ and $t > 0$, let F_N be the Fredholm determinant associated with $K_{[N]}$, and let F_∞ be the Fredholm determinant associated with K_∞. Then, F_N and F_∞ are in $\mathcal{C}^3(\mathbb{R}_+^*, \mathbb{R})$, and for $p \in \{0,1,2,3\}$, the p-th derivative of F_N (with respect to t) converges pointwise to the p-th derivative of F_∞.*

As an immediate consequence, one obtains the following convergence in law for the scaled largest eigenvalue:

Corollary 1.9. *Given the set of $N \times N$ random Hermitian matrices $H(N)$ with the generalized Cauchy probability distribution (1.1.3), denote by $\lambda_1(N)$ the largest eigenvalue of such a randomly chosen matrix. Then, the law of $\lambda_1(N)/N$ converges to the distribution of the largest point of the determinantal process on $\mathbb{R}^*$ described by the limiting kernel $K_\infty(x,y)$ in the following sense:*

$$P\left[\frac{\lambda_1(N)}{N} \leq x_0\right] = \det(I - K_N)|_{L_2(Nx_0,\infty)} \longrightarrow \det(I - K_\infty)|_{L_2(x_0,\infty)}, \quad \text{as } N \to \infty,$$

for any $x_0 > 0$.

Remark 1.10. Note that in the case of finite N, the range of the largest eigenvalue is the whole real line, whereas in the limit case when $N \to \infty$, the range of the largest eigenvalue is $\mathbb{R}_+^*$. This is because an infinite number of points accumulate near 0 (0 itself being excluded however). The accumulation of the points can be seen from the fact that due to the form of $K_\infty(x,x)$ (see (1.2.13)), $\lim_{\epsilon \to 0} \int_\epsilon^\infty K_\infty(x,x)dx$ diverges.

Now, define

$$\theta_\infty(\tau) = \tau \frac{d \log \det(I - K_\infty)|_{L_2(\tau^{-1},\infty)}}{d\tau}, \qquad \tau > 0. \tag{1.2.17}$$

Using the result of Forrester and Witte [15] for the distribution of the largest eigenvalue for fixed N and Theorem 1.8, we are able to show:

Theorem 1.11. *Let s be such that $\Re s > -1/2$. Then the function θ_∞ given by (1.2.17) is well defined and is a solution to the Painlevé-V equation on $\mathbb{R}_+^*$:*

$$\begin{aligned} -\tau^2(\theta''(\tau))^2 = {} & \left[2(\tau\theta'(\tau) - \theta(\tau)) + (\theta'(\tau))^2 + i(\bar{s} - s)\theta'(\tau)\right]^2 \\ & - (\theta'(\tau))^2(\theta'(\tau) - 2is)(\theta'(\tau) + 2i\bar{s}). \end{aligned} \tag{1.2.18}$$

Remark 1.12. This implies in particular the result of Jimbo, Miwa, Môri and Sato [20] that the sine kernel, which is the special case of the K_∞ kernel with parameter $s = 0$ (see Remark 1.4), satisfies the Painlevé-V equation (1.2.18) with $s = 0$.

Eventually, following our initial motivation, we have the following result about the rate of convergence:

Theorem 1.13. *For all $x_0 > 0$, and for $x > x_0$,*

$$\left| P\left[\frac{\lambda_1(N)}{N} \leq x\right] - \det(I - K_\infty)|_{L_2(x,\infty)} \right| \leq \frac{1}{N} C(x_0, s),$$

where $C(x_0,s)$ is a constant depending only on x_0 and s.

1.3 Strategy of the Proof

We say a few words about the way we prove the above Theorems. The first Theorem 1.5 on the Painlevé-VI characterization in the finite N case is proven by simply checking that all the equations involved in the method of Tracy and Widom are fulfilled in our case.
For the remaining results on the limiting law we split our proofs into several technical Lemmas and only use elementary methods. Namely, our proofs only involve checking pointwise convergence and domination in all the quantities involved in the Fredholm determinants of $K_{[N]}$ and K_∞. We can then apply dominated convergence to show that the logarithmic derivative of the Fredholm determinant of $K_{[N]}$, as well as its derivatives, converge pointwise to the respective derivatives of the Fredholm determinant of K_∞. This suffices to show that the Fredholm determinant of K_∞ satisfies a Painlevé-V equation because we can write the rescaled finite N Painlevé-VI equation of Forrester and Witte ([15], [16] and Theorem 1.5 with the extension to $\Re s > -1/2$) as the sum of polynomial functions of the Fredholm determinant of $K_{[N]}$ and its first, second and third derivatives. Moreover, the various estimates and bounds we obtain for the different determinants and functions involved in our problem help us to obtain directly an estimate for the rate of convergence in Corollary 1.9 (that is Theorem 1.13).
Given Theorem 1.2 and the Painlevé-VI characterization of Forrester and Witte [15], the results contained in Theorem 1.8 and Corollary 1.9 are very natural; but yet they have to be rigorously checked. As far as Theorem 1.11 is concerned, Borodin and Deift [4] obtain the same equation as (1.2.18) from the scaling limit of a Painlevé-VI equation characterizing a general 2F1-kernel similar to our kernel K_N (Section 8 in [4]). They claim that it is natural to expect that the appropriately scaled logarithmic derivative of the Fredholm determinant of their 2F1-kernel solves this Painlevé-V equation. In fact, according to our Theorem 1.11, (1.2.17) corresponds to their limit, when $N \to \infty$, of the scaled solution of the Painlevé-VI equation and solves the Painlevé-V equation (1.2.18). Borodin and Deift's method is based on the combination of Riemann-Hilbert theory with the method of isomonodromic deformation of certain linear differential equations. This method is very powerful and general. However, we were not able to apply it in our situation; moreover, it seems that we would have to restrict ourselves to the values of s such that $0 \le \Re s \le 1$. Our method to prove Theorem 1.11 heavily relies on the result of Forrester and Witte [15] for fixed N: hence we do not provide a general method to obtain such Painlevé equations. Nevertheless it is an efficient approach to obtain some information about the rate of convergence in Corollary 1.9.

Chapter 2

The Painlevé Formulation via the Method of Tracy and Widom

Here, we derive the Painlevé-VI equation in Theorem 1.5 via the method of Tracy and Widom [37].
Note that this method has a major drawback. Namely, as already mentioned, it only works for $\Re s > 1/2$ (see Section 2.3), whereas using τ-function theory, Forrester and Witte [15] were able to prove the result for the full range of parameters. However, this Chapter provides an extension to the article [43] of Witte and Forrester, where they prove Theorem 1.5 for s real and $s > 1/2$ via the method of Tracy and Widom.

2.1 The Recurrence Equations

Let us denote the normalized polynomials $\frac{p_m(x)}{\|p_m\|}$ (see (1.2.1) for the definition of p_m) by $\hat{p}_m(x)$ for $m = 0, \ldots, N$. Then we have the following result:

Lemma 2.1. *We have*

$$\begin{aligned}\hat{p}_m(x) =& i^{-m} 2^{N+\Re s} \left[\frac{m!(N+\Re s - m - 1/2)\Gamma(N+s-m)\Gamma(N+\overline{s}-m)}{2\pi\Gamma(2\Re s + 2N - m)}\right]^{1/2} \cdot \\ & P_m^{-N-s,-N-\overline{s}}(-ix) \\ =:& Y(m) P_m^{-N-s,-N-\overline{s}}(-ix),\end{aligned} \tag{2.1.1}$$

where $P_m^{\alpha,\beta}(x)$ denotes the usual Jacobi polynomial (see for example Szegö [36] for a definition of those polynomials).

We use the formula

$$\begin{aligned}P_n^{\alpha,\beta}(x) =& \frac{1}{n!}\sum_{\nu=0}^{n}\binom{n}{\nu}(n+\alpha+\beta+1)\cdots(n+\alpha+\beta+\nu) \cdot \\ & (\alpha+\nu+1)\cdots(\alpha+n)\left(\frac{x-1}{2}\right)^{\nu}\end{aligned} \tag{2.1.2}$$

of Szegö [36] (p.62) to generalize the Jacobi polynomials to arbitrary complex parameters α, β and complex values of x.

Proof. From Borodin and Olshanski [5] we know,

$$\begin{aligned}\|p_m\|^2 =& \frac{\pi 2^{-2\Re s}}{2^{2(N-m-1)}} \cdot \\ & \Gamma\left[\begin{matrix} 2\Re s + 2(N-m) - 1, & 2\Re s + 2(N-m), & m+1 \\ s+N-m, & \overline{s}+N-m, & 2\Re s + 2N - m\end{matrix}\right].\end{aligned} \tag{2.1.3}$$

Thus, using (1.2.1) and setting $y = 1 + ix$,

$$\hat{p}_m(x) = \|p_m\|^{-1}(-2i)^m \sum_{k=0}^{m} \frac{(-m)_k(s+N-m)_k}{(2\Re s+2N-2m)_k} \frac{(\frac{y}{2})^{m-k}}{k!}.$$

Now use $(a)_k = \frac{(a)_m}{(-a-m+1)_{m-k}}(-1)^{m-k}$ to get

$$\begin{aligned}\hat{p}_m(x) =&\|p_m\|^{-1}(-2i)^m \frac{(-m)_m(s+N-m)_m}{(2\Re s+2N-2m)_m m!}\cdot \\ &{}_2F_1\left[1+m-2N-2\Re s,\ -m,\ 1-s-N;\ \frac{y}{2}\right] \\ =&\|p_m\|^{-1}(-2i)^m \frac{m!\Gamma(2\Re s+2N-2m)}{\Gamma(2\Re s+2N-m)} P_m^{-N-s,-N-\overline{s}}(-ix).\end{aligned}$$

The term in front of $P_m^{-N-s,-N-\overline{s}}(-ix)$ is equal to

$$(-1)^m i^m 2^{N+\Re s}\left(\frac{m!(\Re s+N-m-\frac{1}{2})\Gamma(s+N-m)\Gamma(\overline{s}+N-m)}{2\pi\Gamma(2\Re s+2N-m)}\right)^{\frac{1}{2}},$$

and the Lemma follows. □

Note that the definition of $\hat{p}_m(x)$ is equivalent to the one in Witte and Forrester [43] if $s \in \mathbb{R}$ and $s > -1/2$. This can easily be seen using the following symmetry property given by Borodin and Olshanski [5]:

$$p_m(-x) = (-1)^m p_m(x)|_{s\leftrightarrow\overline{s}}.$$

In order to find an ordinary differential equation in t for the probability $E(0,(t,\infty))$ to have no eigenvalue larger than t in (1.2.10), we will set up some general partial differential equations in Section 2.2 in accordance with the general method established by Tracy and Widom in [37]. There will be a set of universal equations and a set of equations depending on the specific form of the following recurrence differential equation for ϕ and ψ:

$$\begin{aligned}m(x)\phi'(x) &= A(x)\phi(x) + B(x)\psi(x) \\ m(x)\psi'(x) &= -C(x)\phi(x) - A(x)\psi(x),\end{aligned} \tag{2.1.4}$$

where A, B, C and m are polynomials in x. For that, the next Lemma will be useful.

Lemma 2.2. *Let $\alpha, \beta \in \mathbb{C}$ and $n \in \mathbb{N}$. Then, the following differential equation is satisfied by the Jacobi polynomials $P_n^{\alpha,\beta}(x)$, where $\alpha, \beta \in \mathbb{C}$ and $x \in [-1,1]$:*

$$\begin{aligned}(2n+\alpha+\beta)(1-x^2)\frac{d}{dx}P_n^{\alpha,\beta}(x) =&n(\alpha-\beta-(2n+\alpha+\beta)x)P_n^{\alpha,\beta}(x) \\ &+2(n+\alpha)(n+\beta)P_{n-1}^{\alpha,\beta}(x).\end{aligned}$$

This formula is also stated in Witte and Forrester [43], but they do not give a proof and only work with α and β real.

Proof. Suppose at first that α, β are real and strictly bigger than -1. Then, the equation in the Lemma is equivalent to

$$\frac{d}{dx}\{(1-x)^{\alpha+1}(1+x)^{\beta+1}y_n\} = (1-x)^{\alpha}(1+x)^{\beta}\left((ax+b)y_n + cy_{n-1}\right),$$

with $y_n = P_n^{\alpha,\beta}$, $y_{n-1} = P_{n-1}^{\alpha,\beta}$ and the constants a, b, c chosen accordingly. Note that

$$\frac{d}{dx}\{(1-x)^{\alpha+1}(1+x)^{\beta+1}y_n\} = \text{const} \cdot (1-x)^{\alpha}(1+x)^{\beta}z, \tag{2.1.5}$$

z being a polynomial of degree $\leq n+1$. Now, let us remark that

$$\int_{-1}^{1} \frac{d}{dx}\{(1-x)^{\alpha+1}(1+x)^{\beta+1}y_n\} \cdot \rho(x)dx = 0, \ \forall \rho \text{ of degree } n-2. \tag{2.1.6}$$

Indeed, integration by parts gives that the left hand side is equal to

$$-\int_{-1}^{1}(1-x)^{\alpha+1}(1+x)^{\beta+1}y_n\rho'(x)dx = \int_{-1}^{1} y_n \cdot r(x)(1-x)^{\alpha}(1+x)^{\beta}dx,$$

with r a polynomial of degree $\leq n-1$, and this last integral is equal to zero by the orthogonality property of the Jacobi polynomial y_n having degree n. The relation (2.1.6) implies that in (2.1.5), $z = (ax+b)y_n + cy_{n-1}$ for some a, b, c.
To finish the proof for $\alpha, \beta > -1$, all that is left to do is to check the values of a, b, c, or equivalently, to compare the coefficients of the three highest terms on both sides of the equation in the Lemma. First, note that both sides are polynomials of degree $n+1$ in $(x-1)$ since the original equation is equivalent to

$$\begin{aligned}
&-2(2n+\alpha+\beta)(x-1)\frac{d}{dx}P_n^{\alpha,\beta}(x) - (2n+\alpha+\beta)(x-1)^2\frac{d}{dx}P_n^{\alpha,\beta}(x)\\
=&(n(\alpha-\beta) - n(2n+\alpha+\beta))P_n^{\alpha,\beta}(x) - n(2n+\alpha+\beta)(x-1)P_n^{\alpha,\beta}(x)\\
&+2(n+\alpha)(n+\beta)P_{n-1}^{\alpha,\beta}(x),
\end{aligned}$$

and we can use formula (2.1.2) for the Jacobi polynomials. The coefficient of $(x-1)^n$ in that formula is

$$\frac{1}{n!}(n+\alpha+\beta+1)\cdots(2n+\alpha+\beta)\frac{1}{2^n}.$$

Thus, the coefficient of the highest term in the derivative $\frac{d}{dx}P_n^{\alpha,\beta}(x)$ is

$$\frac{1}{(n-1)!}(n+\alpha+\beta+1)\cdots(2n+\alpha+\beta)\frac{1}{2^n}.$$

Therefore, the coefficient of $(x-1)^{n+1}$, on the left-hand-side is

$$-(2n+\alpha+\beta)\frac{1}{2^n(n-1)!}(n+\alpha+\beta+1)\cdots(2n+\alpha+\beta).$$

This is clearly equal to the corresponding coefficient on the right-hand side. The verification for the terms $(x-1)^n$ and $(x-1)^{n-1}$ is left to the reader.
Finally, to extend the formula to $\alpha, \beta \in \mathbb{C}$, note that both sides of the equation are polynomials in α and β and equality holds by analytic continuation. □

We need this Lemma with $\alpha = -N-s$ and $\beta = -N-\bar{s}$. Moreover, we substitute the variable x by $-ix$. This is permissible since the equation can be continued analytically in x. The equation in the Lemma thus turns into:

$$\begin{aligned}
&(2n-2N-2\Re s)(1+x^2)\frac{d}{dx}P_n^{-N-s,-N-\bar{s}}(-ix) =\\
&n\left[-2\Im s + (2n-2N-2\Re s)x\right]P_n^{-N-s,-N-\bar{s}}(-ix)\\
&-2i(n-N-s)(n-N-\bar{s})P_{n-1}^{-N-s,-N-\bar{s}}(-ix).
\end{aligned} \tag{2.1.7}$$

2.1.1 The Equation for ϕ

Recall the definitions of ϕ, C and w_H from Theorem 1.2. A direct computation yields:

$$(1+x^2)\phi'(x) = \sqrt{Cw_H(x)}\left[(1+x^2)p_N'(x) + p_N(x)(\Im s - x(N+\Re s))\right]. \tag{2.1.8}$$

Since by (2.1.1), $p_N(x) = \|p_N\|\hat{p}_N(x) = \|p_N\|Y(N)P_N^{-N-s,-N-\bar{s}}(-ix)$, we have, using equation (2.1.7) with $n = N$,

$$(1+x^2)p_N'(x) = \left[N\frac{\Im s}{\Re s} + Nx\right]p_N(x) + i\frac{|s|^2}{\Re s}\frac{Y(N)}{Y(N-1)}\frac{\|p_N\|}{\|p_{N-1}\|}p_{N-1}(x). \tag{2.1.9}$$

Recall (2.1.3). Using this and the definition of $Y(m)$ in (2.1.1),

$$\frac{\|p_N\|}{\|p_{N-1}\|}\frac{Y(N)}{Y(N-1)} = -i\frac{N}{\Re s}\frac{2\Re s + N}{2\Re s + 1}.$$

Equations (2.1.8), (2.1.9) and the above finally give the desired equation for ϕ:

$$(1+x^2)\phi'(x) = \phi(x)\left[-x\Re s + \Im s\left(1 + \frac{N}{\Re s}\right)\right] + \frac{|s|^2}{\Re s^2}N\frac{2\Re s + N}{2\Re s + 1}\psi(x). \tag{2.1.10}$$

2.1.2 The Equation for ψ

As for ϕ', we have:

$$(1+x^2)\psi'(x) = \sqrt{Cw_H(x)}\left[(1+x^2)p_{N-1}'(x) + p_{N-1}(x)(\Im s - x(N+\Re s))\right]. \tag{2.1.11}$$

But $p_{N-1}(x) = \|p_{N-1}\|Y(N-1)P_{N-1}^{-N-s,-N-\bar{s}}(-ix)$, and we can, as for p_N above again put this into equation (2.1.7). However, the Jacobi polynomial $P_{N-2}^{-N-s,-N-\bar{s}}(-ix)$ will then appear. This calls for the following recurrence relation:

Lemma 2.3. *For general complex α, β and x, one has:*

$$\begin{aligned}
&2n(n+\alpha+\beta)(2n+\alpha+\beta-2)P_n^{\alpha,\beta}(x) \\
=&(2n+\alpha+\beta-1)\left[(2n+\alpha+\beta)(2n+\alpha+\beta-2)x + \alpha^2+\beta^2\right]P_{n-1}^{\alpha,\beta}(x) \\
&-2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)P_{n-2}^{\alpha,\beta}(x) \text{ for } n = 2,3,4,\ldots; \\
&P_0^{\alpha,\beta}(x) = 1, \qquad P_1^{\alpha,\beta}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta).
\end{aligned}$$

Proof. This formula is given in Szegö [36] (p.71) for $\alpha, \beta > -1$ and $x \in [-1,1]$. The extension to general α, β and x is done via analytic continuation, since both sides of the equation are polynomials in α, β and x if one uses the explicit expression (2.1.2) for the Jacobi polynomials. □

Using this Lemma, we express $P_{N-2}^{-N-s,-N-\bar{s}}(-ix)$ as:

$$\begin{aligned}
P_{N-2}^{-N-s,-N-\bar{s}}(-ix) &= \frac{N(N+2\Re s)(\Re s+1)}{|s+1|^2\Re s}P_N^{-N-s,-N-\bar{s}}(-ix) \\
&+ \frac{1+2\Re s}{|s+1|^2\Re s}\left[-i\Re s(\Re s+1)x + i\Im s(N+\Re s)\right]P_{N-1}^{-N-s,-N-\bar{s}}(-ix).
\end{aligned}$$

Combining this with equation (2.1.7) for $n = N-1$, one obtains

$$(1+x^2)\frac{d}{dx}P_{N-1}^{-N-s,-N-\bar{s}}(-ix) = i\frac{N(N+2\Re s)}{\Re s}P_N^{-N-s,-N-\bar{s}}(-ix)$$
$$+\left[x(N+2\Re s) - 2\Im s - N\frac{\Im s}{\Re s}\right]P_{N-1}^{-N-s,-N-\bar{s}}(-ix).$$

Therefore,

$$(1+x^2)p'_{N-1}(x) = i\frac{N(N+2\Re s)}{\Re s}\frac{\|p_{N-1}\|}{\|p_N\|}\frac{Y(N-1)}{Y(N)}p_N(x)$$
$$+\left[x(N+2\Re s) - 2\Im s - N\frac{\Im s}{\Re s}\right]p_{N-1}(x).$$

Inserting this into equation (2.1.11) finally gives the desired equation for ψ:

$$(1+x^2)\psi'(x) = \phi(x)(-(2\Re s+1)) + \psi(x)\left(x\Re s - \Im s\left(1+\frac{N}{\Re s}\right)\right). \qquad (2.1.12)$$

We can sum up equations (2.1.10) and (2.1.12) in the following Theorem:

Theorem 2.4. *For ϕ and ψ given in Theorem 1.2, the recurrence equations (2.1.4) hold with:*

$$m(x) = 1+x^2 =: \mu_0 + \mu_2 x^2,$$
$$A(x) = -x\Re s + \Im s\left(1+\frac{N}{\Re s}\right) =: -x\alpha_1 + \alpha_0,$$
$$B(x) = \frac{|s|^2}{\Re s^2}N\frac{2\Re s+N}{2\Re s+1} =: \beta_0,$$
$$C(x) = 2\Re s + 1 =: \gamma_0.$$

Note that this Theorem only makes sense if $\Re s \neq 0$, but we will have to restrict ourselves to $\Re s > 1/2$ anyway in this Chapter.

2.2 Some General PDE's

To obtain the desired differential equation in Theorem 1.5 for $E(0,(t,\infty))$ we need to establish some general partial differential equations (pde's). The following is a restriction to our particular case of the very general setting given by Tracy and Widom in [37].

Write $J = \bigcup_{i=0}^m (a_{2i+1}, a_{2i+2}) \subset \mathbb{R}$, for some $-\infty < a_1 < \ldots < a_{2m+2} < \infty$ and set $K := K_{N,J}$ for the operator with kernel $\chi_J(y)K_N(x,y)\chi_J(y)$ restricted to J. If A is an integral operator with kernel $A(x,y)$, we use the notation $A \doteq A(x,y)$ to relate an operator with its kernel. We introduce the following:

$$K(1-K)^{-1} \doteq R(x,y),$$
$$(1-K)^{-1} \doteq \rho(x,y) = \delta(x-y) + R(x,y),$$

where $(1-K)^{-1} := \sum_{i=0}^\infty K^i$ exists since all eigenvalues of K are strictly smaller than one (see Section 1.2). R is the resolvent kernel of K. Moreover, for $k \in \mathbb{N}_0$, let

$$Q_k(x) := \int_J \rho(x,y)y^k\phi(y)dy,$$
$$P_k(x) := \int_J \rho(x,y)y^k\psi(y)dy,$$

and set $q_{kj} := Q_k(a_j) = \lim_{x\to a_j} Q_k(x)$, and $p_{kj} := P_k(a_j) = \lim_{x\to a_j} P_k(x)$. Note that in the following, any quantity at some a_j is interpreted to be the limit as $x \to a_j$, with $x \in J$. We need the following scalar products:

$$
\begin{aligned}
u :=& \langle \phi, Q\rangle_J = \int_J Q(x)\phi(x)dx,\\
v :=& \langle \psi, Q\rangle_J = \int_J Q(x)\psi(x)dx = \int_J P(x)\phi(x)dx = \langle \phi, P\rangle_J,\\
w :=& \langle \psi, P\rangle_J = \int_J P(x)\psi(x)dx,
\end{aligned}
$$

where $P := P_0$ and $Q := Q_0$.
The following equations hold:

$$
\begin{aligned}
&\frac{\partial}{\partial a_j}\log\det(I-K) = (-1)^{j-1}R(a_j,a_j),\\
&R(a_j,a_k) = \frac{q_jp_k - q_kp_j}{a_j - a_k},\\
&\frac{\partial}{\partial a_k}R(a_j,a_j) = (-1)^kR(a_j,a_k)R(a_k,a_j),\\
&\frac{\partial q_j}{\partial a_k} = (-1)^kR(a_j,a_k)q_k,\\
&\frac{\partial p_j}{\partial a_k} = (-1)^kR(a_j,a_k)p_k,\\
&\frac{\partial u}{\partial a_k} = (-1)^kq_k^2, \qquad \frac{\partial v}{\partial a_k} = (-1)^kq_kp_k, \qquad \frac{\partial w}{\partial a_k} = (-1)^kp_k^2,
\end{aligned}
\tag{2.2.1}
$$

where $j \neq k$ and $q_j := q_{0j}$, $p_j := p_{0j}$. For the first equation, see for example Forrester [14] (p.325, ex.7.2). The others can be found in Tracy and Widom [37]. These equations are independent of the recurrence equations (2.1.4), whereas the following set of pde's does depend on these equations: Set

$$
\begin{aligned}
m(x) &=: \mu_0 + \mu_1x + \mu_2x^2,\\
A(x) &=: \alpha_0 + \alpha_1x,\\
B(x) &=: \beta_0 + \beta_1x,\\
C(x) &=: \gamma_0 + \gamma_1x.
\end{aligned}
$$

Then,

$$
\begin{aligned}
m(a_i)\frac{\partial q_i}{\partial a_i} =& (\alpha_0 + \alpha_1a_i + \gamma_1u - \beta_1w - \mu_2v)q_i\\
&+ (\beta_0 + \beta_1a_i + 2\alpha_1u + 2\beta_1v + \mu_2u)p_i\\
&- \sum_{k\neq i}(-1)^kR(a_i,a_k)q_km(a_k),
\end{aligned}
\tag{2.2.2}
$$

$$
\begin{aligned}
m(a_i)\frac{\partial p_i}{\partial a_i} =& (-\gamma_0 - \gamma_1a_i + 2\gamma_1v + 2\alpha_1w - \mu_2w)q_i\\
&+ (-\alpha_0 - \alpha_1a_i + \beta_1w - \gamma_1u + \mu_2v)p_i\\
&- \sum_{k\neq i}(-1)^kR(a_i,a_k)p_km(a_k),
\end{aligned}
\tag{2.2.3}
$$

$$\begin{aligned} m(a_i)R(a_i,a_i) =&(\gamma_0+\gamma_1 a_i - 2\gamma_1 v - 2\alpha_1 w + \mu_2 w)q_i^2 \\ &+ (\beta_0 + \beta_1 a_i + 2\alpha_1 u + 2\beta_1 v + \mu_2 u)p_i^2 \\ &+ (\alpha_0 + \alpha_1 a_i + \gamma_1 u - \beta_1 w - \mu_2 v)2p_i q_i \\ &+ \sum_{k\neq i}(-1)^k m(a_k)\frac{(q_i p_k - p_k q_i)^2}{a_i - a_k}, \end{aligned} \tag{2.2.4}$$

and

$$\begin{aligned} \frac{\partial}{\partial a_i}\left[m(a_i)R(a_i,a_i)\right] =& 2\alpha_1 q_i p_i + \beta_1 p_i^2 + \gamma_1 q_i^2 \\ &- \sum_{k\neq i}(-1)^k m(a_k) R^2(a_i,a_k). \end{aligned} \tag{2.2.5}$$

These are special cases of the more general equations in Tracy and Widom [37]. They are also stated in Witte and Forrester [43].
We restrict our attention to the single interval $J=(t,\infty)$, $t>0$. That is $a_1=t$ and $a_2\to\infty$. In order to make sense of the above equations in the limit, we need to make sure that q_2, p_2, as well as the last terms in the equations (2.2.2) to (2.2.5) tend to zero, when $a_2\to\infty$. Moreover, the integrals defining u, v and w have to be well defined as $a_2\to\infty$. In Section 2.3, we prove that these conditions are fulfilled in the case $\Re s > 1/2$ only. For now, we set

$$q_2 = p_2 = 0.$$

We also introduce the notations $q_1 =: q$ and $p_1 =: p$. Equations (2.2.1) now specialize to:

$$\frac{d}{dt}\log\det(I-K) = R(t,t), \tag{2.2.6}$$

$$\frac{du}{dt} = -q^2, \tag{2.2.7}$$

$$\frac{dv}{dt} = -qp, \tag{2.2.8}$$

$$\frac{dw}{dt} = -p^2. \tag{2.2.9}$$

The equations with $j\neq k$ vanish. The equations depending on the special form of (2.1.4) (Theorem 2.4 respectively) are:

$$\begin{aligned} (1+t^2)\frac{dq}{dt} =& \left(\Im s\left(1+\frac{N}{\Re s}\right) - t\Re s - v\right)q \\ &+ \left(\frac{|s|^2}{(\Re s)^2}N\frac{2\Re s + N}{2\Re s + 1} - u(2\Re s - 1)\right)p, \end{aligned} \tag{2.2.10}$$

$$\begin{aligned} (1+t^2)\frac{dp}{dt} =& \left(-2\Re s - 1 - w(2\Re s + 1)\right)q \\ &+ \left(-\Im s\left(1+\frac{N}{\Re s}\right) + t\Re s + v\right)p, \end{aligned} \tag{2.2.11}$$

$$\begin{aligned} (1+t^2)R(t,t) =& (2\Re s + 1 + w(2\Re s + 1))q^2 \\ &+ \left(\frac{|s^2|}{(\Re s)^2}N\frac{2\Re s + N}{2\Re s + 1} - u(2\Re s - 1)\right)p^2 \\ &+ \left(\Im s\left(1+\frac{N}{\Re s}\right) - t\Re s - v\right)2pq, \end{aligned} \tag{2.2.12}$$

$$\frac{d}{dt}\left[(1+t^2)R(t,t)\right] = -(2\Re s)pq. \tag{2.2.13}$$

2.3 Asymptotics for the PDE's in Section 2.2

In Section 2.2, we need to estimate some quantities which are related to the restriction $K_{N,J}$ of the operator K_N, where $J = (t, \infty)$ for some $t \in \mathbb{R}$. In particular, we need to prove that for $t > 0$ fixed, the quantities:

$$\begin{aligned} &R(t,x)Q(x)(1+x^2), \\ &R(t,x)P(x)(1+x^2), \\ &(1+x^2)\frac{P^2(x)+Q^2(x)}{x}, \quad \text{and} \\ &(1+x^2)R^2(t,x) \end{aligned}$$

tend to zero when x goes to infinity. In order to obtain the Painlevé formulation in the next subsection, we will need the following: To deduce (2.4.1) from (2.2.8) and (2.2.13), we have to prove that v and $(1+t^2)R(t,t)$ tend to zero when t goes to infinity. Finally, to deduce (2.4.5) from (2.4.4), u, w, $t(1+t^2)R(t,t)$ and $(1+t^2)((1+t^2)R(t,t))'$ also have to tend to zero.
In the following, $C(N,s)$ denotes a strictly positive real number, depending only on N and s. This constant may change from line to line. We also note that the following calculations require $\Re s > 1/2$. Now, for all $x \in \mathbb{R}$, we have from Theorem 1.2:

$$K_N(x,x) = C(N,s)\left(\tilde{p}_N'(x)p_{N-1}(x) - \tilde{p}_N(x)p_{N-1}'(x)\right)w_H(x).$$

Since $\tilde{p}_N' p_{N-1} - \tilde{p}_N p_{N-1}'$ is a polynomial of degree $2N-2$, the explicit form of w_H in Theorem 1.2 gives for all $x \in \mathbb{R}$:

$$K_N(x,x) \leq C(N,s)(1+|x|)^{-2-2\Re s},$$

and because K_N defines a positive self-adjoint operator on $L_2(\mathbb{R})$,

$$K_N(x,y) \leq C(N,s)[(1+|x|)(1+|y|)]^{-1-\Re s}, \quad \forall x,y \in \mathbb{R}.$$

Now, for $p \geq 2$, the kernel of the operator $K_{N,J}^p$, defined by

$$K_{N,J}^p(x,y) = \int_{J^{p-1}} K_N(x,z_1)\cdots K_N(z_{p-1},y)dz_1\cdots dz_{p-1},$$

satisfies for $x \geq t > 0$:

$$\begin{aligned} K_{N,J}^p(x,x) &\leq \int_{J^{p-1}} |K_N(x,z_1)||K_N(z_1,z_2)|\cdots|K_N(z_{p-1},x)|dz_1\cdots dz_{p-1} \\ &\leq C(N,s)(1+x)^{-2-2\Re s}\left(C(N,s)\int_t^\infty (1+z)^{-2-2\Re s}\,dz\right)^{p-1} \\ &\leq C(N,s)\,D_1(N,s,t)^{p-1}\,(1+x)^{-2-2\Re s}, \end{aligned}$$

where $D_1(N,s,t)$ depends only on N, s, t and tends to zero when t goes to infinity (recall that $\Re s > 1/2$). Since $R = \sum_{p\geq 1} K_{N,J}^p$, one obtains, for t large enough and $x \geq t$:

$$R(x,x) \leq D_2(N,s,t)(1+|x|)^{-2-2\Re s}, \tag{2.3.1}$$

if for N, s fixed, $D_2(N,s,t) > 0$ converges when t goes to infinity. Moreover, it is easy to check that:

$$\frac{K_{N,J}}{1-\lambda} - R$$

is a positive operator, if $\lambda < 1$ denotes the largest eigenvalue of $K_{N,J}$. Therefore:

$$R(x,x) \leq \frac{K_{N,J}(x,x)}{1-\lambda}$$

and (2.3.1) holds for any $x \in \mathbb{R}$, since λ depends only on N, s and t. By positivity of R:

$$R(x,y) \leq D_2(N,s,t)\left((1+|x|)(1+|y|)\right)^{-1-\Re s}.$$

Now,

$$\begin{aligned}|\phi(x)| &\leq C(N,s)(1+|x|)^{-\Re s},\\ |\psi(x)| &\leq C(N,s)(1+|x|)^{-1-\Re s},\end{aligned}$$

by (1.2.3) and (1.2.4). Hence,

$$\begin{aligned}|Q(x)| \leq & |\phi(x)| + \int_t^\infty |R(x,y)||\phi(y)|dy\\ \leq & C(N,s)(1+|x|)^{-\Re s}\\ & + D_2(N,s,t)C(N,s)(1+|x|)^{-1-\Re s}\int_t^\infty (1+|y|)^{-1-2\Re s}dy\\ \leq & D_3(N,s,t)(1+|x|)^{-\Re s},\end{aligned}$$

where $D_3(N,s,t)$ converges when t goes to infinity. By the same argument,

$$|P(x)| \leq D_4(N,s,t)(1+|x|)^{-1-\Re s},$$

where $D_4(N,s,t)$ converges when t goes to infinity. We deduce from (2.3.1) and the last two bounds that for t fixed and x going to infinity:

$$\begin{aligned}R(t,x)Q(x)(1+x^2) &= O(x^{1-2\Re s}),\\ R(t,x)P(x)(1+x^2) &= O(x^{-2\Re s}),\\ (1+x^2)\frac{P^2(x)+Q^2(x)}{x} &= O(x^{1-2\Re s}),\\ (1+x^2)R^2(t,x) &= O(x^{-2\Re s}).\end{aligned}$$

All these quantities tend to zero when x goes to infinity, as long as $\Re s > 1/2$. Moreover,

$$\begin{aligned}|u| &\leq \int_t^\infty |\phi(x)||Q(x)|dx\\ &\leq \int_t^\infty C(N,s)(1+|x|)^{-\Re s} D_3(N,s,t)(1+|x|)^{-\Re s}dx\\ &= O(t^{1-2\Re s}),\end{aligned}$$

when t goes to infinity (recall that $D_3(N,s,t)$ converges). By the same argument:

$$\begin{aligned}v &= O(t^{-2\Re s}),\\ w &= O(t^{-1-2\Re s}),\end{aligned}$$

for $t \to \infty$. Finally,

$$\begin{aligned}|t(1+t^2)R(t,t)| &\leq D_2(N,s,t)|t|(1+t^2)(1+|t|)^{-2-2\Re s}\\ &= O(t^{1-2\Re s}),\end{aligned}$$

and, using (2.2.13):

$$\begin{aligned}|(1+t^2)((1+t^2)R(t,t))'| &= 2\Re s|P(t)||Q(t)|(1+t^2)\\ &\leq 2\Re(s)D_3(N,s,t)(1+|t|)^{-\Re s}\\ &D_4(N,s,t)(1+|t|)^{-1-\Re s}(1+t^2)\\ &= O(t^{1-2\Re s}),\end{aligned}$$

as $t \to \infty$ goes to infinity.

2.4 Painlevé Formulation

We use equations (2.2.6)–(2.2.13) to state an ordinary differential equation (ODE) for $\sigma(t) := (1+t^2)R(t,t) = \frac{d}{dt}\log\det(I-K)$. Recall that $\Re s > 1/2$. With (2.2.8) and (2.2.13), one gets:

$$(1+t^2)R(t,t) = 2\Re s\, v. \tag{2.4.1}$$

The integration constant is equal to zero since both v and $(1+t^2)R(t,t)$ tend to zero (see Section 2.3) if $t \to \infty$. Now, using (2.4.1) and (2.2.12) and the notations of Theorem 2.4,

$$\begin{aligned}&[\gamma_0 + w(2\Re s+1)]\, q^2 + [\beta_0 - u(2\Re s-1)]\, p^2\\ &-[-\alpha_0 + t\Re s + v]\, 2pq - 2\Re s\, v = 0.\end{aligned} \tag{2.4.2}$$

Adding p times (2.2.10) and q times (2.2.11) gives:

$$(1+t^2)(pq)' + [\gamma_0 + w(2\Re s+1)]\, q^2 - [\beta_0 - u(2\Re s-1)]\, p^2 = 0. \tag{2.4.3}$$

Subtract $2\Re s$ times (2.4.3) from (2.4.2) to get:

$$\begin{aligned}&\{[\beta_0 - u(2\Re s-1)]\,[\gamma_0 + w(2\Re s+1)]\}' - 2\Re s(1+t^2)v'' - 4\Re s\, tv'\\ &+ 2\Re s\, tv' - 2vv' + 2\Re s\, v + 2\alpha_0 v' = 0\end{aligned} \tag{2.4.4}$$

Again using the fact that u, v, w all tend to zero if $t \to \infty$ (see Section 2.3) together with the equations (2.2.8) and (2.2.13), one integrates the above equation to:

$$\begin{aligned}&[\beta_0 - u(2\Re s-1)]\,[\gamma_0 + w(2\Re s+1)] = \beta_0\gamma_0 + (1+t^2)\left[(1+t^2)R(t,t)\right]'\\ &- t(1+t^2)R(t,t) + \frac{1}{4(\Re s)^2}((1+t^2)R(t,t))^2 - \frac{\alpha_0}{\Re s}[(1+t^2)R(t,t)].\end{aligned}$$

Setting $\sigma := \sigma(t) := (1+t^2)R(t,t)$, this turns into:

$$[\beta_0 - u(2\Re s-1)][\gamma_0 + w(2\Re s+1)] = \beta_0\gamma_0 + (1+t^2)\sigma' - t\sigma + \frac{1}{4(\Re s)^2}\sigma^2 - \frac{\alpha_0}{\Re s}\sigma. \tag{2.4.5}$$

With all these equations in hand, we can get the desired ODE as follows: Combining (2.4.1) and (2.4.3) gives:

$$-\frac{1}{2\Re s}(1+t^2)\sigma'' = [\beta_0 - u(2\Re s-1)]p^2 - [\gamma_0 + w(2\Re s+1)]q^2. \tag{2.4.6}$$

Combining (2.4.1) and (2.4.2) gives:

$$\sigma - \frac{1}{2(\Re s)^2}(\sigma + 2(\Re s)^2 t)\sigma' + \frac{\alpha_0}{\Re s}\sigma' = [\beta_0 - u(2\Re s-1)]p^2 + [\gamma_0 + w(2\Re s+1)]q^2. \tag{2.4.7}$$

Subtracting the square of equation (2.4.7) from the square of equation (2.4.6) leads to:

$$\begin{aligned}&\frac{1}{4(\Re s)^2}(1+t^2)^2(\sigma'')^2-\sigma^2+\frac{1}{(\Re s)^2}\sigma\sigma'(\sigma+2(\Re s)^2t)-2\frac{\alpha_0}{\Re s}\sigma\sigma'\\&+\frac{\alpha_0}{(\Re s)^3}(\sigma')^2(\sigma+2(\Re s)^2t)-\frac{1}{4(\Re s)^4}(\sigma+2(\Re s)^2t)^2(\sigma')^2-\left(\frac{\alpha_0}{\Re s}\right)^2(\sigma')^2\\&+4p^2q^2[\beta_0-u(2\Re s-1)][\gamma_0+w(2\Re s+1)]=0.\end{aligned} \tag{2.4.8}$$

In the last line of this equation, we can write $4p^2q^2=\frac{1}{(\Re s)^2}(\sigma')^2$ and for the product of the two brackets, one uses equation (2.4.5). Equation (2.4.8) is now equivalent to the ODE

$$\begin{aligned}&(1+t^2)(\sigma'')^2+4(1+t^2)(\sigma')^3-8t(\sigma')^2\sigma+4\sigma^2(\sigma'-(\Re s)^2)+8(t(\Re s^2)-\Re s\Im s\\&-N\Im s)\sigma\sigma'+4(2t\Im s(N+\Re s)-(\Im s)^2-t^2(\Re s)^2+N(2\Re s+N))(\sigma')^2=0.\end{aligned}$$

But this is precisely the ODE in Theorem 1.5 (note in particular that the equation itself is meaningful not only for $\Re s>1/2$ but for all s with $\Re s>-1/2$) with

$$\begin{aligned}\sigma(t)=&(1+t^2)R(t,t)=(1+t^2)\frac{d}{dt}\log\det(I-K_N)|_{L_2(t,\infty)}\\=&(1+t^2)\frac{d}{dt}\log P[\text{there is no eigenvalue inside }(t,\infty)].\end{aligned}$$

Thus, Theorem 1.5 is proved.

CHAPTER 3

Scaling Limit and Painlevé Characterization

In this Chapter we split the proofs of Theorems 1.8, and 1.11 into several technical Lemmas. The notations are those introduced in Chapter 1. Throughout the remainder of this part of the thesis, $C(a_0, a_1, \dots, a_n)$ stands for a positive constant which only depends on the parameters $a_0, a_1, \dots, a_n$, and whose value may change from line to line (we shall not be interested in explicit values for the different constants). We first bring in an ODE that θ_∞ should satisfy; then we prove several technical Lemmas about the convergence of the correlation functions and the derivatives of the kernel $K_{[N]}$. We shall use these Lemmas to show that $\theta_\infty(t)$ is indeed well defined (i.e. $F_\infty(t)$ is non-zero for any $t > 0$) and to prove Theorems 1.8 and 1.11.

3.1 Scaling Limits

We show that when $N \to \infty$, the ODE (1.2.16) converges to a σ-version of the Painlevé-V equation (for the full set of parameters s with $\Re s > -1/2$). This limiting equation is also given in Borodin and Deift [4] (Proposition 8.14). Borodin and Deift obtain this equation as a scaling limit of a Painlevé-VI equation characterizing their 2F1-kernel. However, their 2F1-kernel is different from our kernel K_N.

Set for $\tau > 0$,

$$\theta(\tau) := \theta_N(\tau) := \tau \frac{d \log \det(1 - K_N)|_{L_2(N\tau^{-1},\infty)}}{d\tau}. \tag{3.1.1}$$

Then,

$$\theta(\tau) = \tau \left(-\frac{N}{\tau^2}\right) R\left(\frac{N}{\tau}, \frac{N}{\tau}\right) = -\frac{N}{\tau}\left[\frac{\sigma\left(\frac{N}{\tau}\right)}{1 + \frac{N^2}{\tau^2}}\right].$$

where $R(x, y)$ is the kernel of the resolvent operator $K_{N,J}(1 - K_{N,J})^{-1}$ and we use (2.2.6). It follows that

$$\begin{aligned}
\sigma\left(\frac{N}{\tau}\right) &= -\theta(\tau)\left(\frac{\tau}{N} + \frac{N}{\tau}\right),\\
\sigma'\left(\frac{N}{\tau}\right) &= \frac{\tau^2}{N^2}(\tau\theta'(\tau) + \theta(\tau)) + (\tau\theta'(\tau) - \theta(\tau)),\\
\sigma''\left(\frac{N}{\tau}\right) &= -\frac{\tau^3}{N^3}[4\tau\theta'(\tau) + 2\theta(\tau) + \tau^2\theta''(\tau)] - \frac{\tau^3}{N}\theta''(\tau).
\end{aligned}$$

Now, put this into the ODE (1.2.16) with $t = \frac{N}{\tau}$. After dividing by N^2, we obtain:

$$\left(\frac{1}{\tau^2}\right)^2(\tau^3\theta''(\tau))^2 + 4\left(\frac{1}{\tau^2}\right)(\tau\theta'(\tau)-\theta(\tau))^3 + \frac{8}{\tau}(\tau\theta'(\tau)-\theta(\tau))^2\frac{\theta(\tau)}{\tau}$$
$$+4\left(\frac{\theta(\tau)}{\tau}\right)^2(\tau\theta'(\tau)-\theta(\tau)-(\Re s)^2) - 8\left(\frac{(\Re s)^2}{\tau}-\Im s\right)\frac{\theta(\tau)}{\tau}(\tau\theta'(\tau)-\theta(\tau))$$
$$+4\left[2\frac{\Im s}{\tau}-\frac{(\Re s)^2}{\tau^2}+1\right](\tau\theta'(\tau)-\theta(\tau))^2 = O(N^{-1}).$$

This gives

$$-\tau^2(\theta''(\tau))^2 = 4\left\{(\theta'(\tau))^2(\tau\theta'(\tau)-\theta(\tau)-(\Re s)^2) + 2\Im s\,\theta'(\tau)(\tau\theta'(\tau)-\theta(\tau))\right.$$
$$\left.+(\tau\theta'(\tau)-\theta(\tau))^2\right\} + O(N^{-1}).$$

Now if one neglects the terms of order $O(N^{-1})$, it is easy to see that this is precisely equation (1.2.18). But this is also exactly the σ-form of the Painlevé-V equation in Borodin and Deift [4], Proposition 8.14.
Hence, $\theta_N(\tau)(=\theta(\tau))$ satisfies a differential equation which tends to the σ-Painlevé-V equation and we have the following Proposition:

Proposition 3.1. *The ODE (1.2.16) with the change of variable $t = N/\tau$, $\tau > 0$, is solved by $\theta_N(\tau)$, and is of the form*

$$\sum_{k=0}^{m} N^{-k}\frac{P_k(\tau,\theta_N(\tau),\theta_N'(\tau),\theta_N''(\tau))}{\tau^q} = 0,$$

where m and q are universal integers and the P_k's are polynomials which are independent of N. Moreover, $P_0(\tau,\theta_N(\tau),\theta_N'(\tau),\theta_N''(\tau))\tau^{-q}$ corresponds to the σ-form of the Painlevé-V equation (1.2.18).

Remark 3.2. We note that $\theta_N(\tau)$, given by (3.1.1), is a solution of the ODE (1.2.16), with $t = N/\tau$. Moreover, we know that $\lim_{N\to\infty} NK_N(x,y) = K_\infty(x,y)$, for any $x, y \in \mathbb{R}^*$. Hence it is natural to guess that $\theta_\infty(\tau)$ should satisfy the ODE (1.2.18).

3.2 Some technical Lemmas

For clarity, we decompose the proof of our Theorems into several Lemmas about the convergence of correlation functions and the derivatives of the kernel $K_{[N]}$.

Lemma 3.3. *Let K be a function in $C^2((\mathbb{R}_+^*)^2,\mathbb{R})$, such that for all $k \in \mathbb{N}$, and $x_1, x_2, \dots, x_k > 0$, the matrix $K(x_i,x_j)_{1\le i,j\le k}$ is symmetric and positive. Define the k-correlation function ρ_k by:*

$$\rho_k(x_1,\dots,x_k) = \det(K(x_i,x_j)_{1\le i,j\le k}),$$

and suppose that for $(p,q) \in \{(i,j);\ i,j \in \mathbb{N}_0,\ i+j \le 2\}$, for some $\alpha > 1/2$, and for all $x_0 > 0$, one has the upper bound

$$\left|\frac{\partial^{p+q}}{\partial x^p \partial y^q}K(x,y)\right| \le \frac{C(x_0)}{(xy)^\alpha}, \tag{3.2.1}$$

if $x, y \ge x_0$. Then, ρ_k is in $C^2((\mathbb{R}_+^*)^k,\mathbb{R})$ for all k, and for all $x_0 > 0$, $x_1,\dots,x_k \ge x_0$, one has:

$$\left|\frac{\partial^p}{\partial x_j^p}\rho_k(x_1,\dots,x_k)\right| \le \frac{(C(x_0))^k}{(x_1\dots x_k)^{2\alpha}}, \tag{3.2.2}$$

if $p \in \{0,1,2\}$ and $j \in \{1,\dots,k\}$. Moreover,

$$\frac{\partial^p}{\partial x_j^p}\rho_k(x_1,\dots,x_k) = 0 \tag{3.2.3}$$

if $p \in \{0,1\}$, $j \in \{1,\dots,k\}$ and if there exists $j' \neq j$ such that $x_j = x_{j'}$.

Proof. Fix $k \in \mathbb{N}$. The fact that ρ_k is in C^2 is an immediate consequence of the fact that K is in C^2. For $x_1,...,x_{j-1},x_{j+1},...,x_k$ fixed, the function:

$$t \mapsto \rho_k(x_1,...,x_{j-1},t,x_{j+1},...,x_k)$$

is positive by the positivity of K, and equal to zero if $t = x_{j'}$ for some $j' \in \{1,\dots,j-1,j+1,\dots,k\}$. Therefore, $t = x_{j'}$ is a local minimum of this function and one deduces the equality (3.2.3). We now turn to the proof of (3.2.2). By symmetry of ρ_k, we only need to show the case $j=1$. We isolate the terms containing x_1 in the determinant defining ρ_k to obtain:

$$\begin{aligned}&\rho_k(x_1,\dots,x_k) = K(x_1,x_1)\det(K(x_{l+1},x_{m+1})_{1\le l,m\le k-1})\\&+\sum_{2\le i,j\le k}(-1)^{i+j-1}K(x_i,x_1)K(x_1,x_j)\det(K(x_{l+1+\mathbb{1}_{l\ge i-1}},x_{m+1+\mathbb{1}_{m\ge j-1}})_{1\le l,m\le k-2}),\end{aligned}$$

where we take the convention that an empty sum is equal to 0 and an empty determinant is equal to 1. One deduces:

$$\begin{aligned}&\frac{\partial}{\partial x_1}\rho_k(x_1,\dots,x_k) = (K_1'+K_2')(x_1,x_1)\det(K(x_{l+1},x_{m+1})_{1\le l,m\le k-1})\\&+\sum_{2\le i,j\le k}(-1)^{i+j-1}(K_2'(x_i,x_1)K(x_1,x_j)+K(x_i,x_1)K_1'(x_1,x_j))\\&\det(K(x_{l+1+\mathbb{1}_{l\ge i-1}},x_{m+1+\mathbb{1}_{m\ge j-1}})_{1\le l,m\le k-2}),\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial^2}{\partial x_1^2}\rho_k(x_1,...,x_k) = (K_{1,1}''+2K_{1,2}''+K_{2,2}'')(x_1,x_1)\det(K(x_{l+1},x_{m+1})_{1\le l,m\le k-1})\\&+\sum_{2\le i,j\le k}(-1)^{i+j-1}(K_2''(x_i,x_1)K(x_1,x_j)+2K_2'(x_i,x_1)K_1'(x_1,x_j)\\&+K(x_i,x_1)K_1''(x_1,x_j))\det(K(x_{l+1+\mathbb{1}_{l\ge i-1}},x_{m+1+\mathbb{1}_{m\ge j-1}})_{1\le l,m\le k-2}),\end{aligned}$$

where for $p,q \in \{1,2\}$, K_p' denotes the derivative of K with respect to the p-th variable, and $K_{p,q}''$ denotes the second derivative of K with respect to the p-th and the q-th variable. By the positivity of K, there exists, for all $r \in \mathbb{N}$ and $y_1,...,y_r,z_1,...,z_r > 0$, vectors $e_1,...,e_r,f_1,...,f_r$ of an Euclidian space E equipped with its usual scalar product $(.|.)$, such that $(e_i|f_j) = K(y_i,z_j)$ for all $i,j \in \{1,\dots,r\}$. Now, we can define a scalar product on the r-th exterior power of E by setting

$$(u_1\wedge...\wedge u_r|v_1\wedge...\wedge v_r) = \det((u_i|v_j)_{1\le i,j\le r}),$$

for all $u_1,...,u_r,v_1,...,v_r \in E$. Note that this scalar product is nothing else than a Gram determinant and we have the upper bound

$$|\det((e_i|f_j)_{1\le i,j\le r})| \le \prod_{i=1}^r \|e_i\|_E \prod_{i=1}^r \|f_i\|_E,$$

$\|.\|_E$ being the norm associated to $(.|.)$. This last bound is equivalent to

$$|\det(K(y_i, z_j)_{1\le i,j\le r})| \le \sqrt{\prod_{i=1}^{r} K(y_i, y_i) \prod_{i=1}^{r} K(z_i, z_i)}. \tag{3.2.4}$$

Now, let $x_0 > 0$ and $x_1, \dots, x_k \ge x_0$. The bound (3.2.1) given in the statement of the Lemma and the inequality (3.2.4) imply

$$|\det(K(x_{l+1}, x_{m+1})_{1\le l,m\le k-1})| \le \frac{(C(x_0))^{k-1}}{(x_2 x_3 \cdots x_k)^{2\alpha}}$$

and

$$\begin{aligned}&|\det(K(x_{l+1+\mathbb{1}_{l\ge i-1}}, x_{m+1+\mathbb{1}_{m\ge j-1}})_{1\le l,m\le k-2})|\\ \le&\frac{(C(x_0))^{k-2}}{(x_2x_3\cdots x_{i-1}x_{i+1}\cdots x_k)^{\alpha}(x_2x_3\cdots x_{j-1}x_{j+1}\cdots x_k)^{\alpha}}\\ =&\frac{(C(x_0))^{k-2}(x_ix_j)^{\alpha}}{(x_2...x_k)^{2\alpha}}.\end{aligned}$$

Hence, each term involved in the expressions of ρ_k and its two first derivatives with respect to x_1 is smaller than $4(C(x_0))^k/(x_1\cdots x_k)^{2\alpha}$ and therefore, the absolute values of ρ_k an its derivatives are bounded by $4((k-1)^2+1)(C(x_0))^k/(x_1\cdots x_k)^{2\alpha} \le 4^k(C(x_0))^k/(x_1\cdots x_k)^{2\alpha}$, implying the bound (3.2.2). □

Remark 3.4. In the above proof, the value of $C(x_0)$ does not change. It is thus possible to take $C(x_0)$ in the inequality (3.2.2) to be equal to 4 times the value of $C(x_0)$ in (3.2.1).

We now have to prove that the re-scaled kernel $K_{[N]}$ satisfies the hypothesis of Lemma 3.3, and that its partial derivatives converge pointwise to the partial derivatives of K_∞. In the following, we introduce the notation

$$F_{n,h,a}(x) = {}_2F_1\left[-n, h, a; 2/(1+ix)\right],$$

for $(n, h, a) \in \mathbb{N} \times \mathbb{C} \times \mathbb{R}_+^*$.

Lemma 3.5. *Let $\epsilon \in \{0, 1\}$, $h \in \mathbb{C}$, $a \in \mathbb{R}_+^*$. For $N \in \mathbb{N}$, we set $n := N - \epsilon$. Then, $x \mapsto F_{n,h,a}(Nx)$ and $x \mapsto {}_1F_1[h, a; 2i/x]$ are in $C^\infty(\mathbb{R}^*)$, and for all $p \in \mathbb{N}$ and $x \in \mathbb{R}^*$:*

$$\frac{d^p}{dx^p}(F_{n,h,a}(Nx)) \underset{N\to\infty}{\longrightarrow} \frac{d^p}{dx^p}({}_1F_1[h, a; 2i/x]).$$

Moreover, for all $x_0 > 0$ and for all $x \in \mathbb{R}$ such that $|x| \ge x_0$, one has the bound

$$\left|\frac{d^p}{dx^p}(F_{n,h,a}(Nx))\right| \le \frac{C(x_0, h, a, p)}{|x|^{p+\mathbb{1}_{p>0}}}.$$

Proof. One has

$$F_{n,h,a}(Nx) = \sum_{k=0}^{\infty} \frac{(-n)_k (h)_k}{(a)_k k!} \left(\frac{2}{1+Nix}\right)^k,$$

where only a finite number of the summands are different from zero. This implies that the function is C^∞ on $\mathbb{R}^*$, and

$$\frac{d^p}{dx^p}(F_{n,h,a}(Nx)) = \sum_{k=0}^{\infty} \frac{(-n)_k (h)_k}{(a)_k k!} (k)_p \left(\frac{2}{1+Nix}\right)^{k+p} \left(-\frac{iN}{2}\right)^p.$$

The term of order k in this sum is dominated by (note that $a > 0$)

$$\frac{(|h|)_k}{(a)_k k!}(k)_p \frac{2^k}{|x|^{k+p}},$$

and for fixed x, tends to

$$\frac{(h)_k}{(a)_k k!}(k)_p \frac{(2i)^k(-1)^p}{x^{k+p}},$$

when $N \to \infty$. One deduces, that for $|x| \geq x_0 > 0$:

$$\begin{aligned}\left|\frac{d^p}{dx^p}(F_{n,h,a}(Nx))\right| &\leq \sum_{k=0}^{\infty} \frac{(|h|)_k}{(a)_k k!}(k)_p \frac{2^k}{|x|^{k+p}} \\ &\leq \mathbb{1}_{p=0} + \frac{1}{|x|^{p+1}} \sum_{k=1}^{\infty} \frac{(|h|)_k}{(a)_k k!}(k)_p \frac{2^k}{x_0^{k-1}} \\ &\leq \frac{C(x_0,h,a,p)}{|x|^{p+\mathbb{1}_{p>0}}}\end{aligned}$$

which is the desired bound. Now, by dominated convergence, one has

$$\frac{d^p}{dx^p}(F_{n,h,a}(Nx)) \underset{N\to\infty}{\longrightarrow} \sum_{k=0}^{\infty} \frac{(h)_k}{(a)_k k!}(k)_p \frac{(2i)^k(-1)^p}{x^{k+p}}.$$

Hence, Lemma 3.5 is proved if we show that $x \mapsto_1 F_1[h,a;2i/x]$ is C^∞ on $\mathbb{R}^*$, and that

$$\frac{d^p}{dx^p}({}_1F_1[h,a;2i/x]) = \sum_{k=0}^{\infty} \frac{(h)_k}{(a)_k k!}(k)_p \frac{(2i)^k(-1)^p}{x^{k+p}}. \tag{3.2.5}$$

But the sum in (3.2.5) is obtained by taking the derivative of order p of each term of the sum defining ${}_1F_1$. Therefore, we are done, since this term by term derivation is justified by the domination of the right hand side of (3.2.5) by $C(x_0,h,a,p)/|x|^{p+\mathbb{1}_{p>0}}$ on $\mathbb{R}\backslash(-x_0,x_0)$. □

Lemma 3.6. *Fix s such that $\Re s > -\frac{1}{2}$. Define the functions $\tilde{P}_N$ and Q_N by*

$$\tilde{P}_N(x) = 2^{\Re s}\left(\frac{\Gamma(2\Re s+N+1)}{N\Gamma(N)}\right)^{1/2} \tilde{p}_N(Nx)\sqrt{w_H(Nx)},$$

$$Q_N(x) = 2^{\Re s+1}\left(\frac{N\Gamma(2\Re s+N+1)}{\Gamma(N)}\right)^{1/2} p_{N-1}(Nx)\sqrt{w_H(Nx)},$$

where $\tilde{p}_N$, p_{N-1} and w_H are given in Theorem 1.2 and the remark below that Theorem. Then, $\tilde{P}_N$ and Q_N are C^∞ on $\mathbb{R}$, $\tilde{P}$ and Q, defined below (1.2.12), are C^∞ on $\mathbb{R}^*$, and for all $x \in \mathbb{R}^*$, $p \in \mathbb{N}_0$,

$$(Sgn(x))^N \tilde{P}_N^{(p)}(x) \underset{N\to\infty}{\longrightarrow} \tilde{P}^{(p)}(x),$$

$$(Sgn(x))^N Q_N^{(p)}(x) \underset{N\to\infty}{\longrightarrow} Q^{(p)}(x).$$

Moreover, for all $p \in \mathbb{N}_0$, $x_0 > 0$, one has the following bounds:

$$\left|\tilde{P}_N^{(p)}(x)\right| \leq \frac{C(x_0,s,p)}{|x|^{p+\Re s}},$$

and

$$\left|Q_N^{(p)}(x)\right| \leq \frac{C(x_0,s,p)}{|x|^{p+1+\Re s}},$$

for all $|x| \geq x_0$.

Proof. We define

$$\Phi_N(x) = D(N,n,s)(Nx-i)^n F_{n,h,a}(Nx)(1+iNx)^{(-s-N)/2}(1-iNx)^{(-\overline{s}-N)/2},$$

where

$$D(N,n,s) = 2^{\Re s+(N-n)}\left(\frac{\Gamma(2\Re s+N+1)}{N\Gamma(N)}\right)^{1/2} N^{N-n},$$

and $N-n \in \{0,1\}$ (see Lemma 3.5). Then, if $(n,h,a) = (N,s,2\Re s+1)$, $\Phi_N(x) = \tilde{P}_N(x)$ and if $(n,h,a) = (N-1,s+1,2\Re s+2)$, $\Phi_N(x) = Q_N(x)$. Moreover, note that Φ_N is a product of C^∞ functions on $\mathbb{R}$.
Now, for $\delta \in \{-1,1\}$:

$$\log(1+\delta iNx) = \log(1-\delta i/Nx) + \log(N|x|) + i\frac{\pi}{2}\delta \mathrm{Sgn}(x),$$

because both sides of the equality have an imaginary part in $(-\pi,\pi)$ and their exponentials are equal. Hence,

$$\begin{aligned}
&\left(\frac{-s+N}{2}-(N-n)\right)\log(1+iNx) + \frac{-\overline{s}-N}{2}\log(1-iNx)\\
=&\left(\frac{-s+N}{2}-(N-n)\right)\log(1-i/Nx) + \frac{-\overline{s}-N}{2}\log(1+i/Nx)\\
&-(\Re s+(N-n))\log(N|x|) + ni\pi \mathrm{Sgn}(x)/2 + \pi\Im s \mathrm{Sgn}(x)/2.
\end{aligned}$$

This implies:

$$\begin{aligned}
\Phi_N(x) =& D(N,n,s)(-i)^n(1+iNx)^{(-s+N)/2-(N-n)}(1-iNx)^{(-\overline{s}-N)/2}F_{n,h,a}(Nx)\\
=& D(N,n,s)(-i)^n(N|x|)^{-\Re s-(N-n)}e^{ni\pi \mathrm{Sgn}(x)/2}e^{\pi\Im s \mathrm{Sgn}(x)/2}\\
&(1-i/Nx)^{(N-s)/2-(N-n)}(1+i/Nx)^{(-\overline{s}-N)/2}F_{n,h,a}(Nx)\\
=& D(N,n,s)(\mathrm{Sgn}(x))^n(2N)^{-\Re s-(N-n)}(2/|x|)^{\Re s+N-n}e^{\pi\Im s \mathrm{Sgn}(x)/2}\\
&(1-i/Nx)^{(N-s)/2-(N-n)}(1+i/Nx)^{(-\overline{s}-N)/2}F_{n,h,a}(Nx)\\
=& D'(N,s)(\mathrm{Sgn}(x))^N e^{\pi\Im s \mathrm{Sgn}(x)/2}(2/x)^{N-n}(2/|x|)^{\Re s}\\
&(1-i/Nx)^{(N-s)/2-(N-n)}(1+i/Nx)^{(-\overline{s}-N)/2}F_{n,h,a}(Nx),
\end{aligned} \tag{3.2.6}$$

where for s fixed,

$$D'(N,s) = D(N,n,s)(2N)^{-\Re s-(N-n)} = \left(\frac{\Gamma(2\Re s+N+1)}{N^{2\Re s+1}\Gamma(N)}\right)^{1/2}. \tag{3.2.7}$$

This tends to 1 when N goes to infinity. In particular $D'(N,s)$ can be bounded by some $C(s)$, not depending on N. We investigate all the terms in (3.2.6) separately in the following.
Let G be the function defined by:

$$G(y) := (1-iy/N)^{(N-s)/2-(N-n)}(1+iy/N)^{(-\overline{s}-N)/2}.$$

This function is C^∞ on $\mathbb{R}$ and one has:

$$\begin{aligned}
G^{(p)}(y) =& G(y)\sum_{q=0}^{p} C(p,q)(i/N)^q(-i/N)^{p-q}(-(N-s)/2+N-n)_q\\
&((N+\overline{s})/2)_{p-q}(1-iy/N)^{-q}(1+iy/N)^{-(p-q)}.
\end{aligned}$$

For s, y, p and $N-n \in \{0,1\}$ fixed, the last sum is dominated by some constant $C(s,p)$ only depending on s and p and tends to $(-i)^p$, as $N \to \infty$. Moreover, $G(y)$ tends to e^{-iy}, and

$$G(y) = \left(\frac{1-iy/N}{1+iy/N}\right)^{(N-i\Im s)/2} (1-iy/N)^{-(N-n)}(1+y^2/N^2)^{-\Re s/2}.$$

A simple computation, yields the following:

$$|G(y)| \leq C(s)\left(1+\frac{y^2}{N^2}\right)^{-\Re s/2} \leq C(s)(1+y^2)^{1/4}.$$

This implies that $G^{(p)}(y)$ tends to $(-i)^p e^{-iy}$ when N goes to infinity, and that

$$\left|G^{(p)}(y)\right| \leq C(s,p)(1+y^2)^{1/4}.$$

Now, for all f in $C^\infty(\mathbb{R})$, the function g defined by $x \mapsto f(1/x)$ is in $C^\infty(\mathbb{R}^*)$, and there exist universal integers $(\mu_{p,k})_{p\in\mathbb{N}_0, 0\leq k\leq p}$, such that $\mu_{p,0}=0$ for all $p \geq 1$, and for $p \in \mathbb{N}_0$,

$$g^{(p)}(x) = \sum_{k=0}^{p} \frac{\mu_{p,k}}{x^{p+k}} f^{(k)}(1/x).$$

Applying this formula to the functions G and $y \to e^{-iy}$, one obtains the following pointwise convergence (for $x \neq 0$):

$$\frac{d^p}{dx^p}\left[(1-i/Nx)^{(N-s)/2-(N-n)}(1+i/Nx)^{(-\bar{s}-N)/2}\right] \underset{N\to\infty}{\longrightarrow} \frac{d^p}{dx^p}(e^{-i/x}) \tag{3.2.8}$$

with, for $|x| \geq x_0 > 0$,

$$\left|\frac{d^p}{dx^p}\left[(1-i/Nx)^{(N-s)/2-(N-n)}(1+i/Nx)^{(-\bar{s}-N)/2}\right]\right| \leq \frac{C(x_0,s,p)}{|x|^{p+\mathbb{1}_{p>0}}}. \tag{3.2.9}$$

Recall that by Lemma 3.5, one has the convergence

$$\frac{d^p}{dx^p}(F_{n,h,a}(Nx)) \underset{N\to\infty}{\longrightarrow} \frac{d^p}{dx^p}({}_1F_1[h,a;2i/x]), \tag{3.2.10}$$

and the bound

$$\left|\frac{d^p}{dx^p}(F_{n,h,a}(Nx))\right| \leq \frac{C(x_0,h,a,p)}{|x|^{p+\mathbb{1}_{p>0}}} \leq \frac{C(x_0,s,p)}{|x|^{p+\mathbb{1}_{p>0}}}, \tag{3.2.11}$$

since (h,a) only depends on s in the relevant cases (see the beginning of the proof). Moreover,

$$\left|\frac{d^p}{dx^p}\left[(2/x)^{N-n}(2/|x|)^{\Re s}\right]\right| \leq \frac{C(s,p)}{|x|^{\Re s+(N-n)+p}}. \tag{3.2.12}$$

We can now give the derivatives of Φ_N, using (3.2.6). One has for $p \geq 0$:

$$\begin{aligned}(\mathrm{Sgn}(x))^N \frac{d^p}{dx^p}(\Phi_N(x)) =& D'(N,s)e^{\pi\Im s \mathrm{Sgn}(x)/2} \\ & \sum_{q_1+q_2+q_3=p} \frac{p!}{q_1!q_2!q_3!}\frac{d^{q_1}}{dx^{q_1}}\left[(2/x)^{N-n}(2/|x|)^{\Re s}\right] \\ & \frac{d^{q_2}}{dx^{q_2}}\left[(1-i/Nx)^{(N-s)/2-(N-n)}(1+i/Nx)^{(-N-\bar{s})/2}\right] \\ & \frac{d^{q_3}}{dx^{q_3}}\left[F_{n,h,a}(Nx)\right].\end{aligned}$$

By (3.2.7), (3.2.8) and (3.2.10), whenever s, x and $N-n \in \{0,1\}$ are fixed, this expression tends to

$$e^{\pi \Im s \mathrm{Sgn}(x)/2} \sum_{q_1+q_2+q_3=p} \frac{p!}{q_1!q_2!q_3!} \frac{d^{q_1}}{dx^{q_1}} \left[(2/x)^{N-n}(2/|x|)^{\Re s}\right] \frac{d^{q_2}}{dx^{q_2}} \left[e^{-i/x}\right] \frac{d^{q_3}}{dx^{q_3}} \left({}_1F_1[h,a;2i/x]\right),$$

for $N \to \infty$. But this is precisely the p-th derivative of $\tilde{P}$ at x if $\Phi_N = \tilde{P}_N$, and the p-th derivative of Q at x if $\Phi_N = Q_N$. Moreover, for $|x| \geq x_0 > 0$, one easily obtains the bound

$$\left|\frac{d^p}{dx^p}(\Phi_N(x))\right| \leq \frac{C(x_0,s,p)}{|x|^{\Re s+(N-n)+p}},$$

using (3.2.7), (3.2.9), (3.2.11) and (3.2.12). This completes the proof of the Lemma. □

Lemma 3.7. *Let f and g be two functions which are C^∞ from $\mathbb{R}^*$ to $\mathbb{R}$. We define the function ϕ from $(\mathbb{R}^*)^2$ to $\mathbb{R}$ by*

$$\phi(x,y) := \frac{f(x)g(y)-g(x)f(y)}{x-y},$$

for $x \neq y$, and

$$\phi(x,x) := f'(x)g(x) - g'(x)f(x).$$

Then, ϕ is C^∞ on $(\mathbb{R}^*)^2$ and for all $p,q \in \mathbb{N}_0$:
(a) If $x \neq y$:

$$\frac{\partial^{p+q}\phi}{\partial x^p \partial y^q} = \sum_{k=0}^{p}\sum_{l=0}^{q} C_p^k C_q^l \frac{f^{(k)}(x)g^{(l)}(y) - g^{(k)}(x)f^{(l)}(y)}{(x-y)^{p+q-k-l+1}}(-1)^{p-k}(p+q-k-l)!.$$

(b) If x and y have same sign:

$$\frac{\partial^{p+q}\phi}{\partial x^p \partial y^q} = \sum_{k=0}^{q} C_q^k \left[g^{(q-k)}(y)\int_0^1 f^{(k+p+1)}(y+\theta(x-y))\theta^p(1-\theta)^k d\theta \right. \left. - f^{(q-k)}(y)\int_0^1 g^{(k+p+1)}(y+\theta(x-y))\theta^p(1-\theta)^k d\theta\right].$$

Proof. (a) By induction, one proves that for all $p,q \in \mathbb{N}_0$, and for $x,y \in \mathbb{R}$ distincts and different from zero, it is possible to take, in a neighborhood of (x,y), p derivatives of ϕ with respect to x and q derivatives of ϕ with respect to y, in any order, with a result equal to the expression given in the statement of the Lemma. This implies the existence and the continuity of all partial derivatives of ϕ in $(\mathbb{R}^*)^2 \backslash \{(x,x), x \in \mathbb{R}^*\}$. Therefore, ϕ is C^∞ in this open subset of $(\mathbb{R}^*)^2$.
(b) With the same method as in (a), we obtain that ϕ is C^∞ on $(\mathbb{R}^*_-)^2 \cup (\mathbb{R}^*_+)^2$. The only technical issues are the continuity and the derivation under the integral. These can easily be justified by the boundedness of the derivatives of f and g in any compact set of $\mathbb{R}^*$. □

Proposition 3.8. *Let $x,y \in \mathbb{R}^*$ and let $\Re s > -1/2$. Then $K_{[N]}$ and K_∞ are C^∞ in $(\mathbb{R}^*)^2$ and for all $p,q \in \mathbb{N}_0$,*

$$(Sgn(xy))^N \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_{[N]}(x,y) \underset{N\to\infty}{\longrightarrow} \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_\infty(x,y).$$

Moreover, for any $x_0 > 0$, and $|x|,|y| \geq x_0 > 0$:

$$\left|\frac{\partial^{p+q}}{\partial x^p \partial y^q} K_{[N]}(x,y)\right| \leq \frac{C(x_0,s,p,q)}{|x|^{\Re s+p+1}|y|^{\Re s+q+1}}.$$

Note that the pointwise convergence in the case $p = q = 0$ corresponds to the convergence result for the kernels given by Borodin and Olshanski [5].

Proof. One has

$$(\mathrm{Sgn}(xy))^N K_{[N]}(x,y) = \frac{1}{2\pi}\frac{\Gamma(s+1)\Gamma(\overline{s}+1)}{\Gamma(2\Re s+1)\Gamma(2\Re s+2)}$$
$$\frac{(\mathrm{Sgn}(x))^N \tilde{P}_N(x)(\mathrm{Sgn}(y))^N Q_N(y) - (\mathrm{Sgn}(y))^N \tilde{P}_N(y)(\mathrm{Sgn}(x))^N Q_N(x)}{x-y}$$

for $x \neq y$, and

$$K_{[N]}(x,x) = \frac{1}{2\pi}\frac{\Gamma(s+1)\Gamma(\overline{s}+1)}{\Gamma(2\Re s+1)\Gamma(2\Re s+2)}(\tilde{P}'_N(x)Q_N(x) - Q'_N(x)\tilde{P}_N(x)),$$

with $\tilde{P}_N$ and Q_N defined in Lemma 3.6. Recall the definition of K_∞ in (1.2.12) and (1.2.13). Now, $\tilde{P}_N$, Q_N, $\tilde{P}$ and Q are in $C^\infty(\mathbb{R}^*)$ (see Lemma 3.6) and hence, by Lemma 3.7, $K_{[N]}$ and K_∞ are in $C^\infty((\mathbb{R}^*)^2)$.
Moreover, by Lemma 3.6, the derivatives of $x \mapsto \mathrm{Sgn}^N(x)\tilde{P}_N(x)$ and $x \mapsto \mathrm{Sgn}^N(x)Q_N(x)$ converge pointwise to the corresponding derivatives of $\tilde{P}$ and Q. Considering, for $x \neq y$, the expression (a) of Lemma 3.7, and for $x = y$, the expression (b), one easily deduces the pointwise convergence of the derivatives of $(x,y) \mapsto (\mathrm{Sgn}(xy))^N K_{[N]}(x,y)$ towards the corresponding derivatives of K_∞.
Finally, the bounds given in the statement of the Lemma can be obtained from the bounds of the derivatives of $\tilde{P}_N$ and Q_N, given in Lemma 3.6, and by applying the formula (a) of Lemma 3.7 if $xy < 0$ or $\max(|x|,|y|) > 2\min(|x|,|y|)$ (which implies $|x-y| \geq \max(|x|,|y|)/2$), or the formula (b) if $xy > 0$ and $\max(|x|,|y|) \leq 2\min(|x|,|y|)$. □

Summarizing, we have:

Proposition 3.9. *Let s be such that $\Re s > -\frac{1}{2}$. Then, the restriction to $\mathbb{R}_+^*$ of the scaled kernel $K_{[N]}$ and the kernel K_∞ satisfy the conditions of Lemma 3.3. Moreover, for all $p, q \in \mathbb{N}_0$, the partial derivatives*

$$Sgn(xy)^N \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_{[N]}(x,y)$$

converge pointwise to the corresponding partial derivatives of $K_\infty(x,y)$.

Proof. This follows immediately from Proposition 3.8 and the fact that these kernels are real symmetric and positive because they are kernels of determinantal processes on the real line (see remark 1.3 for the kernel K_∞). □

The next step is to analyze the convergence of the Fredholm determinant of $K_{N,J}$ and its derivatives to the corresponding derivatives of the Fredholm determinant of $K_{\infty,J}$, for $J = (t,\infty)$, $t > 0$.

Lemma 3.10. *Let F be a function defined from $(\mathbb{R}_+^*)^{k+1}$ to $\mathbb{R}$, for some $k \in \mathbb{N}$. We suppose that F is in C^1, and that there exists, for some $\alpha > 1$ and for all $x_0 > 0$, a bound of the form*

$$|F(t,x_1,x_2,...,x_k)| + \left|\frac{\partial}{\partial t}F(t,x_1,x_2,...,x_k)\right| \leq \frac{C(x_0)}{(x_1...x_k)^\alpha},$$

for all $t, x_1, ..., x_k \geq x_0$. Then, the integrals involved in the definitions of the following two functions from $\mathbb{R}_+^*$ to $\mathbb{R}$ are absolutely convergent:

$$H_0 : t \mapsto \int_{(t,\infty)^k} F(t,x_1,\ldots,x_k)dx_1 \ldots dx_k,$$

and

$$H_1 : t \mapsto \int_{(t,\infty)^k} \frac{\partial}{\partial t} F(t, x_1, \dots, x_k) dx_1 \dots dx_k$$
$$- \sum_{l=1}^{k} \int_{(t,\infty)^{k-1}} F(t, x_1, \dots, x_{l-1}, t, x_{l+1}, \dots, x_k) dx_1 \dots dx_{l-1} dx_{l+1} \dots dx_k.$$

Moreover, the first derivative of H_0 is continuous and equal to H_1.

Proof. Due to the bound given in the Lemma, it is clear that all the integrals in the definition of H_0 and H_1 are absolutely convergent. Therefore, for $0 < t < t'$, we can use Fubini's Theorem in order to compute the integral

$$\int_t^{t'} H_1(u) du.$$

Straightforward computations show that this integral is equal to $H_0(t') - H_0(t)$. Hence, if we prove that H_1 is continuous, we are done. Now, let $t > x_0 > 0$. For $t' > x_0$, one has

$$|H_1(t') - H_1(t)| \leq \int_{(x_0,\infty)^k} \Big| \frac{\partial}{\partial t'} F(t', x_1, \dots, x_k) \mathbb{1}_{\{x_1,\dots,x_k > t'\}}$$
$$- \frac{\partial}{\partial t} F(t, x_1, \dots, x_k) \mathbb{1}_{\{x_1,\dots,x_k > t\}} \Big| dx_1 \dots dx_k$$
$$+ \sum_{l=1}^{k} \int_{(x_0,\infty)^{k-1}} \big| F(t', x_1, \dots, x_{l-1}, t', x_{l+1}, \dots, x_k) \mathbb{1}_{\{x_1,\dots,x_{l-1},x_{l+1},\dots x_k > t'\}}$$
$$- F(t, x_1, \dots, x_{l-1}, t, x_{l+1}, \dots, x_k) \mathbb{1}_{\{x_1,\dots,x_{l-1},x_{l+1},\dots x_k > t\}} \big| dx_1 \dots dx_{l-1} dx_{l+1} \dots dx_k.$$

All the terms inside the integrals converge to zero almost everywhere when $t' \to t$ (more precisely, whenever the minimum of the x_j's is different from t). Hence, by dominated convergence, $|H_1(t') - H_1(t)|$ tends to zero when $t' \to t$. □

Lemma 3.11. *Let K be a function satisfying the conditions of Lemma 3.3. Then, using the notation of that Lemma,*

$$\sum_{k \geq 1} \frac{1}{k!} \int_{(t,\infty)^k} \rho_k(x_1, \dots, x_k) dx_1 \dots dx_k < \infty$$

for all $t > 0$. Moreover, the Fredholm determinant F, from $\mathbb{R}_+^*$ to $\mathbb{R}$, defined in (1.2.15) is in C^3, and its derivatives are given by

$$F'(t) = \sum_{k \geq 0} \frac{(-1)^k}{k!} \int_{(t,\infty)^k} \rho_{k+1}(t, x_1, \dots, x_k) dx_1 \dots dx_k,$$
$$F''(t) = \sum_{k \geq 0} \frac{(-1)^k}{k!} \int_{(t,\infty)^k} \frac{\partial}{\partial t} \rho_{k+1}(t, x_1, \dots, x_k) dx_1 \dots dx_k,$$
$$F'''(t) = \sum_{k \geq 0} \frac{(-1)^k}{k!} \int_{(t,\infty)^k} \frac{\partial^2}{\partial t^2} \rho_{k+1}(t, x_1, \dots, x_k) dx_1 \dots dx_k,$$

where all the sums and the integrals above are absolutely convergent.

Proof. For $k \geq 1$, we define F_k by

$$F_k(t) = \frac{(-1)^k}{k!} \int_{(t,\infty)^k} \rho_k(x_1, \dots, x_k) dx_1 \dots dx_k.$$

The integral is finite because of the bounds given in Lemma 3.3. By the same bounds, one can apply Lemma 3.10 three times, to obtain that F_k is in C^3, with the derivatives given by

$$F_k'(t) = \frac{(-1)^{k-1}}{(k-1)!}\int_{(t,\infty)^{k-1}} \rho_k(t, x_1, \ldots, x_{k-1}) dx_1 \ldots dx_{k-1},$$
$$F_k''(t) = \frac{(-1)^{k-1}}{(k-1)!}\int_{(t,\infty)^{k-1}} \frac{\partial}{\partial t}\rho_k(t, x_1, \ldots, x_{k-1}) dx_1 \ldots dx_{k-1},$$
$$F_k'''(t) = \frac{(-1)^{k-1}}{(k-1)!}\int_{(t,\infty)^{k-1}} \frac{\partial^2}{\partial t^2}\rho_k(t, x_1, \ldots, x_{k-1}) dx_1 \ldots dx_{k-1},$$

where again all the integrals are absolutely convergent by Lemma 3.3. Note that we use (3.2.3) to calculate the derivatives above. Moreover, for $p \in \{0, 1, 2, 3\}$, (3.2.2) gives the following bound for any $x_0 > 0$:

$$\sup_{t \geq x_0} |F_k^{(p)}(t)| \leq \frac{(C(x_0))^k}{(k-1)!}.$$

Using dominated convergence, we have that the sum

$$\sum_{k \geq 1} F_k(t)$$

is absolutely convergent, and that its p-th derivative, $p \in \{0, 1, 2, 3\}$ with respect to t is continuous and given by the absolutely convergent sum

$$\sum_{k \geq 1} F_k^{(p)}(t).$$

□

3.3 θ_∞ is well defined

In order to prove that θ_∞ is well defined, we need to prove that $F_\infty(t)$ never vanishes for $t > 0$ (recall from Remark 1.10 that the range of the largest eigenvalue is $\mathbb{R}_+^*$). We note that $F_\infty(t)$ is the Fredholm determinant of the restriction of the operator K_∞ to the space $L^2((t, \infty))$, which can also be seen as the operator on $L^2((t_0, \infty))$ with kernel $(x, y) \to K_\infty(x, y)\, \mathbb{1}_{x,y>t}$, for some t_0 such that $t > t_0 > 0$. This operator is positive, and we note that it is a trace class operator, since:

$$\int_{(t,\infty)} K_\infty(x, x)\, dx < \infty.$$

Therefore, the Fredholm determinant of this operator is given by the convergent product of $1 - \lambda_j$, where $(\lambda_j)_{j \in \mathbb{N}}$ is the decreasing sequence of its (positive) eigenvalues, with multiplicity. This implies that the determinant is zero if and only if 1 is an eigenvalue of the operator: hence, we only need to prove that this is not the case. Indeed, if 1 is an eigenvalue, there exists $f \neq 0$ in $L^2((t_0, \infty))$ such that for almost all $x \in (t_0, \infty)$:

$$f(x) = \mathbb{1}_{x>t} \int_t^\infty K_\infty(x, y)\, f(y)\, dy.$$

Therefore $f(x) = 0$ for almost every $x \leq t$, and

$$f = p_{(t,\infty)} K_{\infty,(t_0,\infty)} f$$

in $L^2((t_0,\infty))$, where $K_{\infty,(t_0,\infty)}$ is the operator on this space, with kernel K_∞, and $p_{(t,\infty)}$ is the projection on the space of functions supported by (t,∞). Now, if we denote $g := K_{\infty,(t_0,\infty)} f$,

$$||g||^2_{L^2((t_0,\infty))} = \int_{t_0}^{\infty}\int_{t_0}^{\infty}\int_{t_0}^{\infty} K_\infty(x,y)\, K_\infty(x,z)\, f(y)\, f(z)\, dx\, dy\, dz.$$

By dominated convergence, one can check that $||g||^2_{L^2((t_0,\infty))}$ is the limit of

$$\int_{t_0}^{\infty}\int_{t_0}^{\infty}\int_{t_0}^{\infty} K_{[N]}(x,y)\, K_{[N]}(x,z)\, f(y)\, f(z)\, dx\, dy\, dz$$

when N goes to infinity. This expression is equal to $||p_{(t_0,\infty)} K_{[N]} \tilde{f}||^2_{L^2(\mathbb{R})}$, and hence, smaller than or equal to $||K_{[N]}\tilde{f}||^2_{L^2(\mathbb{R})}$, where the operators $p_{(t_0,\infty)}$ and $K_{[N]}$ act on $L^2(\mathbb{R})$, and where $\tilde{f}$ is equal to f on (t_0,∞) and equal to zero on $(-\infty, t_0]$. Now, $K_{[N]}$ (as K_N) is an orthogonal projector on $L^2(\mathbb{R})$ (with an N-dimensional image), hence, $||K_{[N]}\tilde{f}||_{L^2(\mathbb{R})} \leq ||\tilde{f}||_{L^2(\mathbb{R})}$. This implies:

$$||g||_{L^2((t_0,\infty))} \leq ||f||_{L^2((t_0,\infty))}.$$

Now, with obvious notation:

$$\begin{aligned} ||g||^2_{L^2((t_0,\infty))} &= ||p_{(t,\infty)} g||^2_{L^2((t_0,\infty))} + ||p_{(t_0,t]} g||^2_{L^2((t_0,\infty))} \\ &= ||f||^2_{L^2((t_0,\infty))} + ||p_{(t_0,t]} g||^2_{L^2((t_0,\infty))} \end{aligned}$$

since $f = p_{(t,\infty)} g$. By comparing the last two equations, one deduces that

$$||p_{(t_0,t]} g||^2_{L^2((t_0,\infty))} = 0,$$

which implies that g is supported by (t,∞), and

$$f = p_{(t,\infty)} g = g = K_{\infty,(t_0,\infty)} f.$$

It follows that $K_{\infty,(t_0,\infty)} f$ (equal to f) takes the value zero a.e. on the interval (t_0,t). Since f is different from zero, one easily deduces a contradiction from the following Lemma:

Lemma 3.12. *Let f be a function in $L^2((t,\infty))$ for some $t > 0$. Then the function g from $\mathbb{R}_+^*$ to $\mathbb{R}$, defined by:*

$$g(x) = \int_t^\infty K_\infty(x,y)\, f(y)\, dy$$

is analytic on $\{z \in \mathbb{C};\ \Re(z) > 0\}$.

Proof. It is sufficient to prove that for all x_0 such that $0 < x_0 < t/2$, g can be extended to a holomorphic function on the set $H_{x_0} := \{x \in \mathbb{C}; \Re(x) > x_0\}$. Let (ϵ, h, a) be equal to $(0, s, 2\Re(s)+1)$ or $(1, s+1, 2\Re(s)+2)$, and Φ equal to $\tilde{P}$ in the first case, Q in the second case. One has for $x \in \mathbb{R}_+^*$:

$$\Phi(x) = \left(\frac{2}{x}\right)^{\Re(s)+\epsilon} e^{-i/x} e^{\pi \Im(s)/2}\, {}_1F_1[h, a; 2i/x].$$

Φ can easily be extended to H_{x_0}: for the first factor, one can use the standard extention of the logarithm (defined on $\mathbb{C}\backslash\mathbb{R}_-$), and the last factor is a hypergeometric series which is uniformly convergent on H_{x_0}. Moreover, it is easy to check (by using dominated convergence for the hypergeometric factor), that this extension of Φ is holomorphic with derivative:

$$\Phi'(x) =$$
$$e^{\pi\Im s/2}\left(\frac{2}{x}\right)^{\Re s+\epsilon} e^{-i/x}$$
$$\cdot\left[\frac{-(\Re s+\epsilon)}{x}{}_1F_1[h,a;2i/x] + \frac{i}{x^2}{}_1F_1[h,a;2i/x]\right.$$
$$\left.-\sum_{k=0}^{\infty}\frac{(h)_k(2i)^k k}{(a)_k k!}\left(\frac{1}{x}\right)^{k+1}\right].$$

With these formulae, one deduces the following bounds, available on the whole set H_{x_0}:

$$|\Phi(x)| \leq \frac{C(x_0,s)}{|x|^{\Re(s)+\epsilon}},$$

$$|\Phi'(x)| \leq \frac{C(x_0,s)}{|x|^{\Re(s)+\epsilon+1}}.$$

Now, let us fix $y \in (t,\infty)$. Recall that for $x \in \mathbb{R}_+^* \backslash \{y\}$:

$$K_\infty(x,y) = \frac{1}{2\pi}\frac{\Gamma(s+1)\Gamma(\overline{s}+1)}{\Gamma(2\Re s+1)\Gamma(2\Re s+2)}\frac{\tilde{P}(x)Q(y)-Q(x)\tilde{P}(y)}{x-y}. \tag{3.3.1}$$

This formula is meaningful for all $x \in H_{x_0}\backslash\{y\}$ and gives an analytic continuation of $x \mapsto K_\infty(x,y)$ to this set. Now, for $x > x_0$, one also has the formula:

$$K_\infty(x,y) = \frac{1}{2\pi}\frac{\Gamma(s+1)\Gamma(\overline{s}+1)}{\Gamma(2\Re s+1)\Gamma(2\Re s+2)}\mathbb{E}\left[\tilde{P}'(Z)Q(y)-Q'(Z)\tilde{P}(y)\right],$$

where Z is a uniform random variable on the segment $[x,y]$. By the bounds obtained for Φ and Φ', one deduces that the continuation of $x \mapsto K_\infty(x,y)$ to the set $H_{x_0}\backslash\{y\}$ is bounded in the neighborhood of y, and hence can be extended to H_{x_0}. By construction, this extension coincides with $K_\infty(x,y)$ for $x \in (x_0,\infty)\backslash\{y\}$, and in fact it coincides on the whole interval (x_0,∞), since $K_\infty(x,y)$ tends to $K_\infty(y,y)$ when x is real and tends to y. In other words, we have constructed an extension of $x \mapsto K_\infty(x,y)$ which is holomorphic on H_{x_0}. Now, let us take $x \in H_{x_0}$ such that $|x-y| \geq y/2$, which implies that $|x-y| \geq C(|x|+y)$ for a universal constant C. By using this inequality and the bounds on $\tilde{P}$ and Q, one obtains:

$$|K_\infty(x,y)| \leq \frac{C(s,x_0)}{|xy|^{\Re(s)+1}}.$$

By taking the derivative of the equation (3.3.1), one obtains the bound (again for $x \in H_{x_0}$ and $|x-y| \geq y/2$):

$$\left|\frac{\partial}{\partial x}K_\infty(x,y)\right| \leq \frac{C(s,x_0)}{|x|^{\Re(s)+2}y^{\Re(s)+1}}.$$

Now, the maximum principle implies that the condition $|x-y| \geq y/2$ can be removed in the last two bounds. By using these bounds, Cauchy-Schwarz inequality and dominated convergence, one deduces that the function:

$$x \mapsto \int_t^\infty K_\infty(x,y)\,f(y)\,dy$$

is well defined on the set H_{x_0}, and admits a derivative, given by the formula:

$$x \mapsto \int_t^\infty \left(\frac{\partial}{\partial x}K_\infty(x,y)\right) f(y)\,dy.$$

□

3.4 Proof of the Scaling Limit Theorem 1.8

Note that by Proposition 3.9, $K_{[N]}$ and K_∞ satisfy the conditions of Lemma 3.3. For $k, N \in \mathbb{N}$, let $\rho_{k,N}$ be the k-correlation function associated with $K_{[N]}$ and $\rho_{k,\infty}$ the k-correlation function associated with K_∞. By Lemma 3.11, F_N is well defined for $N \in \mathbb{N} \cup \{\infty\}$, and C^3. The explicit expressions of F_N and F_∞ and their derivatives are given in Lemma 3.11 by replacing ρ_k by $\rho_{k,N}$ and $\rho_{k,\infty}$ respectively. Now, for $k \geq 1$, all the partial derivatives of any order of $\rho_{k,N}$ converge pointwise to the corresponding derivatives of $\rho_{k,\infty}$ when N goes to infinity. This is due to the explicit expression of $\rho_{k,N}$ as a determinant and the convergence given by Proposition 3.9. Moreover, by that same Proposition, there exists $\alpha > 1/2$ only depending on s such that

$$\left| \frac{\partial^p}{\partial x_1^p} \rho_{k,N}(x_1, \ldots, x_k) \right| \leq \frac{C(x_0, s)^k}{(x_1 \ldots x_k)^{2\alpha}},$$

for $p \in \{0, 1, 2\}$, and for all $x_1, ..., x_k \geq x_0 > 0$. In particular, this bound is uniform with respect to N, and it is now easy to deduce the pointwise convergence of the derivatives of F_N (up to order 3), by dominated convergence.

3.5 Proof of the Painlevé Theorem 1.11

Theorem 1.11 follows immediately from Proposition 3.1 and the following Proposition:

Proposition 3.13. *Let s be such that $\Re s > -1/2$, and F_N, $N \in \mathbb{N}$, and F_∞ be as in Theorem 1.8. Then, for $N \in \mathbb{N} \cup \{\infty\}$, the function θ_N from $\mathbb{R}_+^*$ to $\mathbb{R}$, defined by*

$$\theta_N(\tau) = \tau \frac{d}{d\tau} \log(F_N(\tau^{-1})),$$

is well defined and C^2. Moreover, for $p \in \{0, 1, 2\}$, the derivatives $\theta_N^{(p)}$ converge pointwise to $\theta_\infty^{(p)}$ (defined by (1.2.17)).

Proof. Recall that for $t > 0$, $F_N(t)$ is the probability that a random matrix of dimension N, following the generalized Cauchy weight (1.1.3), has no eigenvalue in (Nt, ∞). Therefore, $F_N(t) > 0$, for any $t > 0$. Similarly, $F_\infty(t)$ is the probability that the limiting determinantal process has no point in (t, ∞), which is also different from zero for any $t > 0$, as we proved in section 3.3. Therefore, for all $N \in \mathbb{N} \cup \{\infty\}$, θ_N is well-defined and

$$\theta_N(\tau) = -\frac{F_N'(\tau^{-1})}{\tau F_N(\tau^{-1})}.$$

Since F_N is in C^3, θ_N is in C^2, for all $N \in \mathbb{N} \cup \{\infty\}$, and one can give explicit expressions for θ_N and for its first two derivatives (see Lemma 3.11). It is now easy to deduce from these explicit expressions and the pointwise convergence of the first three derivatives of F_N assured by Theorem 1.8, the pointwise convergence for the first two derivatives of θ_N, when $N \in \mathbb{N}$ goes to infinity. □

Remark 3.14. Note that most probably, it is also possible to derive the fact that the kernel K_∞ gives rise to a solution of the Painlevé-V equation (1.2.18) directly by the methods of Tracy and Widom [37] in an analogous way to the one used to obtain the Painlevé-VI equation (1.2.16) in the finite N case. In fact, the recurrence equations (2.1.4) in the infinite case are:

$$x^2 P'(x) = \left(-x\Re s + \frac{\Im s}{\Re s} \right) P(x) + \frac{|s|^2}{\Re s^2} \frac{1}{2\Re s + 1} Q(x),$$

$$x^2 Q'(x) = -\left(2\Re s + 1\right) P(x) - \left(-x\Re s + \frac{\Im s}{\Re s} \right) Q(x),$$

where P and Q are as in the definition of K_∞ in (1.2.12) and (1.2.13). However, this method has several drawbacks, as already mentioned in the Introduction (Chapter 1) and in Chapter 2.

Chapter 4

The Convergence Rate

We first need the rate of convergence for the scaled kernel $K_{[N]}(x,y) = NK_N(Nx,Ny)$:

Lemma 4.1. *Let $x, y > x_0 > 0$. Then there exists a constant $C(x_0, s) > 0$ only depending on x_0 and $s \in \mathbb{C}$ ($\Re s > -1/2$), such that*

$$\left|K_{[N]}(x,y) - K_\infty(x,y)\right| \leq \frac{1}{N}\frac{C(x_0,s)}{(xy)^{\Re s+1}}.$$

In the following proof, $C(a,b,\ldots)$ denotes a strictly positive constant only depending on $a,b,\ldots$ which may change from line to line.

Proof. Let $x, y > x_0$, $x \neq y$. Then, setting $C(s) = \left|\frac{1}{2\pi}\frac{\Gamma(s+1)\Gamma(\bar{s}+1)}{\Gamma(2\Re s+1)\Gamma(2\Re s+2)}\right|$, and using the notations from Lemma 3.6, we have

$$\begin{aligned}
&\left|K_{[N]}(x,y) - K_\infty(x,y)\right| = \qquad (4.0.1)\\
&C(s)\left|\frac{1}{x-y}\right|\left|\tilde{P}_N(x)Q_N(y) - \tilde{P}_N(y)Q_N(x) - (\tilde{P}(x)Q(y) - \tilde{P}(y)Q(x))\right|\\
\leq& C(s)\left|\frac{1}{x-y}\right|\left\{\left|\tilde{P}_N(x)Q_N(y) - \tilde{P}(x)Q(y)\right| + \left|\tilde{P}_N(y)Q_N(x) - \tilde{P}(y)Q(x)\right|\right\}\\
\leq& C(s)\left|\frac{1}{x-y}\right|\left\{\left|\tilde{P}_N(x) - \tilde{P}(x)\right||Q_N(y)| + |Q_N(y) - Q(y)|\left|\tilde{P}(x)\right|\right.\\
&\left. + \left|\tilde{P}_N(y) - \tilde{P}(y)\right||Q_N(x)| + |Q_N(x) - Q(x)|\left|\tilde{P}(y)\right|\right\}.
\end{aligned}$$

Similarly, if $x, y > x_0$, it is easy to check (by using the fundamental Theorem of calculus) that

$$\begin{aligned}
&\left|K_{[N]}(x,y) - K_\infty(x,y)\right| \leq C(s)\mathbb{E}\left[\left|\tilde{P}_N'(Z) - \tilde{P}'(Z)\right||Q_N(x)|\right. \qquad (4.0.2)\\
&\left. + |Q_N(x) - Q(x)|\left|\tilde{P}'(Z)\right| + \left|\tilde{P}_N(x) - \tilde{P}(x)\right||Q_N'(Z)| + |Q_N'(Z) - Q'(Z)|\left|\tilde{P}(x)\right|\right].
\end{aligned}$$

where Z is a uniform random variable in the interval $[x,y]$.
By using (4.0.1) if $\max(x,y) \geq 2\min(x,y)$ and (4.0.2) if $\max(x,y) < 2\min(x,y)$, one deduces that the Lemma is proved, if we show that for $p \in \{0,1\}$,

$$\left|\tilde{P}_N^{(p)}(x) - \tilde{P}^{(p)}(x)\right| \leq \frac{1}{N}\frac{C(x_0,s,p)}{x^{p+\Re s}}, \qquad (4.0.3)$$

and

$$\left|Q_N^{(p)}(x) - Q^{(p)}(x)\right| \leq \frac{1}{N}\frac{C(x_0,s,p)}{x^{p+1+\Re s}}, \qquad (4.0.4)$$

Recall from (3.2.6), the following function (note that $x > x_0 > 0$):

$$\begin{aligned}
\Phi_N(x) = &D'(N,s)e^{\pi\Im s/2}\left(\frac{2}{x}\right)^{N-n}\left(\frac{2}{x}\right)^{\Re s}\\
&\cdot\left(1 - \frac{i}{Nx}\right)^{(N-s)/2-(N-n)}\left(1 + \frac{i}{Nx}\right)^{-(\bar{s}+N)/2} F_{n,h,a}(Nx),
\end{aligned}$$

and let us define similarly:

$$\Phi(x) = e^{\pi \Im s/2} \left(\frac{2}{x}\right)^{N-n} \left(\frac{2}{x}\right)^{\Re s} e^{-i/x} {}_1F_1\left[h, a; 2i/x\right],$$

where $(n, h, a) = (N, s, 2\Re s + 1)$ and $\Phi_N(x) = \tilde{P}_N(x)$, or $(n, h, a) = (N-1, s+1, 2\Re s + 2)$ and $\Phi_N(x) = Q_N(x)$, for $N \in \mathbb{N}^*$ (recall that $N - n = 0$ in the first case and $N - n = 1$ in the second case). It suffices to show that for $p \in \{0, 1\}$, $|\Phi_N^{(p)}(x) - \Phi^{(p)}(x)| \leq \frac{C(x_0,s,p)}{N x^{\Re(s)+1+p}}$ to deduce (4.0.3) and (4.0.4). Let us first investigate the case $p = 0$:

$$\begin{aligned} |\Phi_N(x) - \Phi(x)| &\leq e^{\pi \Im s/2} \left(\frac{2}{x}\right)^{\Re s + (N-n)} \qquad (4.0.5) \\ &\cdot \Big\{ |D'(N,s) - 1| \left| (1 - i/(Nx))^{(N-s)/2-(N-n)} (1 + i/(Nx))^{-(N+\bar{s})/2} F_{n,h,a}(Nx) \right| \\ &+ \left| (1 - i/(Nx))^{(N-s)/2-(N-n)} (1 + i/(Nx))^{-(N+\bar{s})/2} - e^{-i/x} \right| |F_{n,h,a}(Nx)| \\ &+ \left| e^{-i/x} \right| |F_{n,h,a}(Nx) - {}_1F_1\left[h, a; 2i/x\right]| \Big\}. \end{aligned}$$

We show that the bracket $\{.\}$ is bounded uniformly by $\frac{1}{N} C(x_0, s)$. In the following, we look at the three summands in the bracket separately. For the first one, we have by (3.2.9) and (3.2.11) that

$$\left| (1 - i/(Nx))^{(N-s)/2-(N-n)} (1 + i/(Nx))^{-(N+\bar{s})/2} F_{n,h,a}(Nx) \right| \leq C(x_0, s).$$

Moreover, it is easy to check (for example, by using Stirling formula) that

$$\left| \frac{\Gamma(2\Re s + N + 1)}{N^{2\Re s + 1}\Gamma(N)} - 1 \right| \leq \frac{1}{N} C(s).$$

Now, if some sequence $a_N > 0$ converges to $a > 0$ in the order $1/N$ as $N \to \infty$, $\sqrt{a_N} \to \sqrt{a}$, in the order $1/N$ as well, for $N \to \infty$. Hence,

$$|D'(N,s) - 1| = \left| \left(\frac{\Gamma(2\Re s + N + 1)}{N^{2\Re s + 1}\Gamma(N)} \right)^{1/2} - 1 \right| \leq \frac{1}{N} C(s).$$

Thus, the first term in the bracket $\{.\}$ of (4.0.5) is bounded by $C(x_0, s)/N$. Let us look at the second term:

$$|F_{n,h,a}(Nx)| \leq C(x_0, s),$$

again according to (3.2.11). Moreover,

$$\begin{aligned} &\left| (1 - i/(Nx))^{(N-s)/2-(N-n)} (1 + i/Nx)^{-(N+\bar{s})/2} - e^{-i/x} \right| \qquad (4.0.6) \\ &\leq \left| (1 - i/(Nx))^{(N-s)/2} (1 + i/(Nx))^{-(N+\bar{s})/2} - e^{-i/x} \right| \left| (1 - i/(Nx))^{-(N-n)} \right| \\ &+ \left| e^{-i/x} \right| \left| (1 - i/(Nx))^{-(N-n)} - 1 \right|. \end{aligned}$$

It is clear, that the second term in the sum is bounded by $C(x_0)/N$. For the first term, the second factor is bounded by $C(x_0)$, whereas for the first factor, we have the following:

$$\begin{aligned}
&\left|\left(\frac{1-i/(Nx)}{1+i/(Nx)}\right)^{N/2}\left(\frac{1-i/(Nx)}{1+i/(Nx)}\right)^{-i\Im s/2}\left(1+1/(Nx)^2\right)^{-\Re s/2}-e^{-i/x}\right| \qquad (4.0.7)\\
\leq&\left|\left(\frac{1-i/(Nx)}{1+i/(Nx)}\right)^{N/2}-e^{-i/x}\right|\left|\left(\frac{1-i/(Nx)}{1+i/(Nx)}\right)^{-i\Im s/2}\right|\left|\left(1+1/(Nx)^2\right)^{-\Re s/2}\right|\\
+&\left|e^{-i/x}\right|\left|\left(\frac{1-i/(Nx)}{1+i/(Nx)}\right)^{-i\Im s/2}-1\right|\left|\left(1+1/(Nx)^2\right)^{-\Re s/2}\right|\\
+&\left|e^{-i/x}\right|\left|\left(1+1/(Nx)^2\right)^{-\Re s/2}-1\right|.
\end{aligned}$$

We investigate all terms in this sum separately: $|(1+1/(Nx)^2)^{-\Re s/2}-1|$ can be bounded by $C(x_0,s)/N$ using binomial series, and

$$\left|\left(\frac{1-i/(Nx)}{1+i/(Nx)}\right)^{-i\Im s/2}\right| = |\exp\{-\Im s\mathrm{Arg}(1+i/Nx)\}| \leq C(x_0,s).$$

Furthermore,

$$\begin{aligned}
&\left|\left(\frac{1-i/(Nx)}{1+i/(Nx)}\right)^{-i\Im s/2}-1\right| = |\exp\{-\Im s\mathrm{Arg}(1+i/(Nx))\}-1|\\
=&|\exp\{-\Im s\mathrm{Arctan}(1/(Nx))\}-1| \leq \left|\sum_{k=0}^{\infty}\frac{\left(-\Im s\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(1/(Nx)\right)^{2n+1}\right)^k}{k!}-1\right|\\
\leq&\frac{1}{N}C(x_0,s).
\end{aligned}$$

Here, we use the fact that the Taylor series for the arctangent is absolutely convergent if $0<1/(Nx)<1$, which is true for N large enough. Now, by considering the series of the complex logarithm of $1\pm i/(Nx)$ (absolutely convergent for N large enough), one can show that

$$\left|(1\pm i/(Nx))^{\mp N/2}-e^{-i/(2x)}\right| \leq \frac{1}{N}C(x_0).$$

The remaining terms in the sum (4.0.7) are clearly bounded by $C(x_0,s)$ and hence, the second term in the sum (4.0.5) converges to zero in the order $1/N$.

We investigate the third term in (4.0.5): Clearly, $\left|e^{-i/x}\right|=1$. The second factor in the third term requires somewhat more work:

$$\begin{aligned}
&|F_{n,h,a}(Nx)-{}_1F_1[h,a;2i/x]|\\
=&\left|\sum_{k=0}^{\infty}\frac{(-n)_k(h)_k2^k}{(a)_kk!}\left(\frac{1}{1+iNx}\right)^k-\sum_{k=0}^{\infty}\frac{(h)_k(2i)^k}{(a)_kk!}\left(\frac{1}{x}\right)^k\right|\\
\leq&\sum_{k=1}^{\infty}\frac{(|h|)_k2^k}{(a)_kk!}\left|(-n)_k\left(\frac{1}{i-Nx}\right)^k-\left(\frac{1}{x}\right)^k\right|,
\end{aligned}$$

where the last inequality is true because of the absolute convergence of both sums. Now,

$$\begin{aligned}
&\left|(-n)_k\left(\frac{1}{i-Nx}\right)^k-\left(\frac{1}{x}\right)^k\right| \\
\leq&\frac{1}{x_0^k}\left|1-\frac{(-n)_k}{((i/x)-N)^k}\right| \\
=&\frac{1}{x_0^k}\left|1-\prod_{l=N-n}^{N-n+k-1}\frac{l-N}{(i/x)-N}\right| \\
=&\frac{1}{x_0^k}\left|1-\prod_{l=N-n}^{N-n+k-1}\frac{(N-l)_+}{N-(i/x)}\right|.
\end{aligned}$$

Since all the factors in the last product have a module smaller than 1, it is possible to deduce:

$$\begin{aligned}
&\left|(-n)_k\left(\frac{1}{i-Nx}\right)^k-\left(\frac{1}{x}\right)^k\right| \\
\leq&\frac{1}{x_0^k}\sum_{l=N-n}^{N-n+k-1}\left|1-\frac{(N-l)_+}{N-(i/x)}\right| \\
\leq&\frac{1}{x_0^k}\sum_{l=N-n}^{N-n+k-1}\frac{l+1/x}{N} \\
\leq&\frac{1}{x_0^k}\frac{k^2+k/x_0}{N}.
\end{aligned}$$

This bound implies easily that:

$$|F_{n,h,a}(Nx)-{}_1F_1[h,a;2i/x]|\leq\frac{C(s,x_0)}{N},$$

and we can deduce:

$$|\Phi_N(x)-\Phi(x)|\leq\frac{1}{N}\frac{C(x_0,s)}{x^{\Re s+(N-n)}}.$$

Therefore, (4.0.3) and (4.0.4) are proved for $p=0$.
It remains to prove that

$$|\Phi_N'(x)-\Phi'(x)|\leq\frac{1}{N}\frac{C(x_0,s)}{x^{\Re s+(N-n)+1}},$$

to show (4.0.3) and (4.0.4) for $p=1$. But this is immediate using the same methods as above and the fact that we can write

$$\begin{aligned}
&\Phi_N'(x)= \\
&D'(N,s)e^{\pi\Im s/2}\left(\frac{2}{x}\right)^{\Re s+(N-n)}(1-i/(Nx))^{(N-s)/2-(N-n)}(1+i/(Nx))^{-(\bar{s}+N)/2} \\
&\cdot\left[\frac{-(\Re s+(N-n))}{x}F_{n,h,a}(Nx)\right. \\
&+\frac{i}{x^2}\left\{\left(\frac{1-s/N}{2}-\frac{N-n}{N}\right)\frac{1}{1-i/(Nx)}+\frac{1+\bar{s}/N}{2}\frac{1}{1+i/(Nx)}\right\}F_{n,h,a}(Nx) \\
&\left.+\sum_{k=0}^{\infty}\frac{(-n)_k(h)_k k2^{k+1}}{(a)_k k!}\left(-\frac{iN}{2}\right)\left(\frac{1}{1+iNx}\right)^{k+1}\right],
\end{aligned}$$

and

$$\Phi'(x) =$$
$$e^{\pi \Im s/2} \left(\frac{2}{x}\right)^{\Re s+(N-n)} e^{-i/x}$$
$$\cdot \left[\frac{-(\Re s + (N-n))}{x} {}_1F_1[h, a; 2i/x] + \frac{i}{x^2} {}_1F_1[h, a; 2i/x]\right.$$
$$\left. - \sum_{k=0}^{\infty} \frac{(h)_k (2i)^k k}{(a)_k k!} \left(\frac{1}{x}\right)^{k+1}\right].$$

This ends the proof. □

Now we prove Theorem 1.13. Let us first prove the following result: for all $n \in \mathbb{N}^*$, and for all symmetric and positive $n \times n$ matrices A and B such that $\sup_{1 \leq i,j \leq n} |A_{i,j}| \leq \alpha$, $\sup_{1 \leq i,j \leq n} |B_{i,j}| \leq \alpha$ and $\sup_{1 \leq i,j \leq n} |A_{i,j} - B_{i,j}| \leq \beta$ for some $\alpha, \beta > 0$, one has

$$|\det(B) - \det(A)| \leq \beta n^2 \alpha^{n-1}. \tag{4.0.8}$$

Indeed, the following formula holds:

$$\det(B) - \det(A) = \int_0^1 d\lambda \text{ Diff } \det[A + \lambda(B - A)].(B - A)$$

where for $C := A + \lambda(B - A)$, Diff $\det[C].(B - A)$ denotes the image of the matrix $B - A$ by the differential of the deteminant, taken at point C. Now, C is symmetric, positive, and $|C_{i,j}| \leq \alpha$ for all indices i, j, since C is a barycenter of A and B, with positive coefficients. Moreover, the derivative of C with respect to the coefficent of indices i, j is (up to a possible change of sign) the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing the line i and the column j of C. By using the same arguments as in the proof of inequality (3.2.4), one can easily deduce that this derivative is bounded by α^{n-1}. Hence:

$$|\det(B) - \det(A)| \leq \int_0^1 d\lambda \, \alpha^{n-1} \sum_{1 \leq i,j \leq n} |B_{i,j} - A_{i,j}|$$

which imples (4.0.8). Now, we can compare the determinants of $(K_{[N]}(x_i, x_j))_{i,j=1}^n$ and $(K_\infty(x_i, x_j))_{i,j=1}^n$ for $x_1, \dots, x_n > x_0$ by applying (4.0.8) to:

$$A_{i,j} = (x_i x_j)^{\Re(s)+1} K_{[N]}(x_i, x_j),$$

$$B_{i,j} = (x_i x_j)^{\Re(s)+1} K_\infty(x_i, x_j),$$

$$\alpha = C(x_0, s), \; \beta = C(x_0, s)/N.$$

Here, we use the bounds for $K_{[N]}$, K_∞ and their difference given in Proposition 3.8 and in Lemma 4.1. We obtain:

$$\left|\det(K_{[N]}(x_i, x_j)_{i,j=1}^n) - \det(K_\infty(x_i, x_j)_{i,j=1}^n)\right|$$
$$\leq \frac{1}{(x_1 \cdots x_n)^{2\Re(s)+2}} \frac{n^2}{N} (C(x_0, s))^n.$$

This implies

$$\begin{aligned}
&\left|P\left[\frac{\lambda_1(N)}{N}\leq x\right]-\det(I-K_\infty)|_{L_2(t,\infty)}\right|\\
\leq&\sum_{n=1}^\infty\frac{1}{n!}\int_{(x,\infty)^n}\left|\det(K_{[N]}(x_i,x_j)_{i,j=1}^n)-\det(K_\infty(x_i,x_j)_{i,j=1}^n)\right|dx_1\cdots dx_n\\
\leq&\sum_{n=1}^\infty\frac{1}{n!}\frac{n^2}{N}\left(\int_{(x,\infty)}\frac{C(x_0,s)}{y^{2\Re s+2}}dy\right)^n\\
\leq&\frac{1}{N}\sum_{n=1}^\infty\frac{n}{(n-1)!}\left(\int_{(x_0,\infty)}\frac{C(x_0,s)}{y^{2\Re s+2}}dy\right)^n\leq C(x_0,s)/N,
\end{aligned}$$

since the last sum is convergent and depends only on x_0 and s.

Chapter 5

Some Remarks about the Unitary Group $U(N)$

With Theorem 1.5 extended to the full range of parameters, we know that the distribution of $\lambda_1(N)$, the largest eigenvalue of a matrix in $H(N)$ can be written as

$$P[\lambda_1(N) \leq a] = \exp\left(-\int_a^\infty \frac{\sigma(t)}{1+t^2}dt\right) \tag{5.0.1}$$

under the distribution (1.1.3). Using the Cayley transform $H(N) \ni X \mapsto U = \frac{X+i}{X-i} \in U(N)$, we can map the generalized Cauchy measure from $H(N)$ to the measure (1.1.5) on $U(N)$. The inverse of the Cayley transform writes as

$$\theta \longmapsto i\frac{e^{i\theta}+1}{e^{i\theta}-1} = \cot\left(\frac{\theta}{2}\right),$$

for $\theta \in [-\pi, \pi]$. $\theta = 0$ is mapped to ∞ by definition. Using this application, equation (5.0.1) turns into:

$$P[\theta_1(N) \geq y] = \exp\left(-\frac{1}{2}\int_0^y d\phi\, \sigma\left(\cot\left(\frac{\phi}{2}\right)\right)\right), \tag{5.0.2}$$

for $y = 2\text{arccot}(a)$, $y \in [0, 2\pi]$, and $e^{i\theta_1(N)} = \frac{\lambda_1(N)+i}{\lambda_1(N)-i}$. $\theta_1(N)$ being here in $[0, 2\pi]$ (and not in $[-\pi, \pi]$!). In other words, the distribution of the largest eigenvalue on the real line of a random matrix $H \in H(N)$ with measure (1.1.3), maps to the distribution of the eigenvalue with smallest angle of a random matrix $U \in U(N)$ satisfying the law (1.1.5). Here, smallest angle has to be understood as the eigenvalue which is closest to 1 looking counterclockwise on the circle from the point 1.

According to Bourgade, Nikeghbali and Rouault [6], the eigenvalues $\{e^{i\theta_1}, \ldots, e^{i\theta_N}\}$, (recall that $\theta_i \in [-\pi, \pi]$) of a random unitary matrix U, satisfying the law (1.1.5), also determine a determinantal point process with correlation kernel

$$\begin{aligned} &K_N^U(e^{i\alpha}, e^{i\beta}) \\ &= d_N(s)\sqrt{w_U(\alpha)w_U(\beta)}\frac{e^{iN\frac{\alpha-\beta}{2}}Q_N^s(e^{-i\alpha})Q_N^{\bar{s}}(e^{i\beta}) - e^{-iN\frac{\alpha-\beta}{2}}Q_N^{\bar{s}}(e^{i\alpha})Q_N^s(e^{-i\beta})}{e^{i\frac{\alpha-\beta}{2}} - e^{-i\frac{\alpha-\beta}{2}}}, \end{aligned} \tag{5.0.3}$$

where $d_N(s) = \frac{1}{2\pi}\frac{(\bar{s}+1)_N(s+1)_N}{(2\Re s+1)_N N!}\frac{\Gamma(1+s)\Gamma(1+\bar{s})}{\Gamma(1+2\Re s)}$, $Q_N^s(x) = {}_2F_1[s, -n, -n-\bar{s}; x]$ and w_U is the weight defined after (1.1.5). If $N \to \infty$, the rescaled correlation kernel $\frac{1}{N}K_N^U(e^{i\alpha/N}, e^{i\beta/N})$ converges to

$$\begin{aligned} &K^U(\alpha, \beta) \\ &= e(s)|\alpha\beta|^{\Re s}e^{-\frac{\pi}{2}\Im s(\text{Sgn}(\alpha)+\text{Sgn}(\beta))}\frac{e^{i\frac{\alpha-\beta}{2}}Q^s(-i\alpha)Q^{\bar{s}}(i\beta) - e^{-i\frac{\alpha-\beta}{2}}Q^{\bar{s}}(i\alpha)Q^s(-i\beta)}{\alpha-\beta}, \end{aligned} \tag{5.0.4}$$

where $e(s) = \frac{1}{2\pi i}\frac{\Gamma(s+1)\Gamma(\bar{s}+1)}{\Gamma(2\Re s+1)^2}$, and $Q^s(x) = {}_1F_1[s, 2\Re s+1; x]$ (again according to Bourgade, Nikeghbali and Rouault [6]). In [6], it is also shown that the kernel K^U coincides up to multiplication by a constant with the limiting kernel K_∞ from (1.2.12) if one changes the variables in (5.0.4) to $\alpha = \frac{2}{x}$ and $\beta = \frac{2}{y}$, $x, y \in \mathbb{R}^*$. This not surprising because a scaling $x \mapsto Nx$ for the eigenvalues in the Hermitian case corresponds to a scaling $\alpha \mapsto \frac{\alpha}{N}$ for the eigenvalues in the unitary case as can be seen from the elementary fact that for $x \in \mathbb{R}^*$, and $N \in \mathbb{N}$, one has

$$\frac{Nx + i}{Nx - i} = e^{\frac{2i}{Nx} + O(N^{-2})}. \tag{5.0.5}$$

Remark 5.1. Note that because of the $O(N^{-2})$ term in the argument of (5.0.5), it is not possible to give an identity involving the kernel K_N of Theorem 1.2 and the kernel (5.0.3).

Chapter 6

Introduction to the Weakly Self-Avoiding Walk

In the second part of this thesis we look at the weakly self-avoiding random walk. We are interested in the diffusive behavior of this walk in high dimensions. Eventually, we prove a type of local central limit theorem for weakly self-avoiding walks in $\mathbb{Z}^d$, with dimension $d \geq 9$ ($d \geq 5$ in the restriction to the symmetric case), whose initial distributions are periodic and in a closed neighborhood of the standard symmetric nearest neighbor distribution $\frac{1}{2d}\mathbb{1}_{\{x:\|x\|=1\}}$. Due to the fact that we work on a lattice with discrete time, we need to take care of periodicity issues. In case of non-periodic initial distributions we get the same result for initial distributions in a closed neighborhood of the distribution giving uniform weight on the points $\pm e_i$ and $\pm 2e_i$, for $i = 1, \dots, d$, where e_i stands for the standard i-th unit vector in $\mathbb{R}^d$.

6.1 Introduction

Consider the following problem in two dimensions: You are standing at an intersection in a town where the streets are laid out in square-grid style. Now you start walking around. At each intersection you choose the next road that you take at random with the condition that you are never allowed to use a road leading you back to an intersection you have already visited. In other words you will walk along a random path which is self-avoiding. There are three basic questions you can ask for such a walk:

- How many paths of n steps starting from the origin are there?
- How many paths of n steps starting from the origin and ending at a given intersection are there?
- On average, how far from the starting point will you be after n steps?

This problem may be generalized to a self-avoiding random walk on the *Hypercubic Lattice* $\mathbb{Z}^d$, $d \geq 1$. Then, the transition from one *Vertex* along an edge to the next vertex is called a *Step*. The above questions are still very natural to ask in this setup. However, the answers are not known for any but very small values of $n \in \mathbb{N}$; except of course in the trivial case $d = 1$. A more simple question to ask might be to understand the asymptotic behavior of such a walk. This is still a hard question and no rigorous results are known for dimensions two and three. It is believed that the walk is not diffusive in these two dimensions though. Physicists and Chemists have introduced this type of model to study the growth of large polymer chains such as proteins. They have applied several methods and produced many results. However most of them are not proven in a mathematically rigorous way. Some of these results and details of some rigorous mathematical work on the self-avoiding walk can be found in the book of Madras and Slade [24], where the above 2-dimensional problem is taken from.
Most mathematically rigorous results have been obtained in high dimensions ($d \geq 5$). In the 1980's, Brydges and Spencer [9] introduced the *Lace Expansion* as a method to study the *Weakly*

Self-Avoiding Walk (or *Domb-Joyce Model*). This is a random walk which may intersect itself, but each self-intersection is penalized by a parameter $1-\lambda$, $\lambda \in [0,1]$. We explain the lace expansion in Appendix A. It is a renewal-type equation for the *Two-Point Function* (or *Connectivity*) of the walk. Other models for self-avoiding random walks have been proposed and studied, such as the true (or myopic) self-avoiding walk or the loop erased walk (see Madras and Slade [24]), but we will only be interested in the weakly self-avoiding walk model, often abbreviated as WSAW from now on. Using a perturbation technique Brydges and Spencer proved that this walk is diffusive for $d \geq 5$. For the self-avoiding walk ($\lambda = 1$, no intersection allowed), Hara and Slade ([19] and [18]) have been able to prove the diffusive behavior for $d \geq 5$ at the beginning of the 1990's also by using the lace expansion. However, their argument is still perturbative and relies on a number of computer-assisted estimates. Later on, van der Hofstad, den Hollander and Slade [40] presented an inductive approach to the lace expansion which they used to prove a *Local Central Limit Theorem* (noted local CLT) for a weakly self-avoiding walk in which the penalty for self-intersections decreases in time. Van der Hofstad and Slade [41] generalized and simplified this approach to prove a local CLT for the self-avoiding walk if $d \geq 5$. At this point it should be noted that the lace expansion was also applied to various other probabilistic problems, such as percolation theory and branched polymers.

The early approaches to the lace expansion usually rely on taking Laplace transforms in time and then inverting this transform. This is a rather difficult problem. The last two articles [40] and [41] mentioned above avoid this difficulty but the authors still work in Fourier space. A new approach has however been presented by Bolthausen and Ritzmann in the PhD-thesis [32] of the latter. They work directly in $\mathbb{Z}^d$, avoiding Fourier or Laplace transforms. Instead, they use Banach fixed point Theorem to show that the diffusive behavior of the weakly self-avoiding walk in dimensions $d \geq 5$ comes from the fact that the local CLT for a standard random walk remains stable under small perturbations. The perturbations are coming from the penalties for the self-intersections. The proof is done by showing that the fixed point of a certain convolution operator remains asymptotically close to the normal distribution. Note that due to the nature of the problem, a true local CLT cannot be obtained for the WSAW, since the decay of the error at the origin is of order $n^{-d/2}$ in time which is the same as the size of the approximating normal distribution at the origin. This is due to the fact that the walk will always remember that it started at zero. Nevertheless, in [32] Gaussian error decays in space are obtained.

In this work, we generalize the lace expansion to perturbed weakly self-avoiding walks. That is, we do allow not only nearest neighbor jumps and we weight jumps in different directions differently (we still keep spatial homogeneity though). The generalization of the lace expansion to these types of walks is straightforward and we present it in Section 6.2 and Appendix A. We are able to prove a local CLT for distributions which lie in a closed neighborhood of the standard nearest neighbor distribution (ie. for small perturbations of the standard nearest neighbor WSAW) for dimensions $d \geq 9$. We will make this more precise in the next Section 6.2 and in Chapter 8. We note here that due to the discrete nature of the problem we have to split the result into the periodic and the aperiodic case. To prove the local CLT, we use an operator which is slightly different from the one used in Ritzmann [32] switching from discrete to continuous time and then back. This has the side effect that we only obtain Exponential error decays instead of Gaussian error decays. For technical reasons we were unable to obtain the result for $d \geq 5$ because we lose the symmetry of the problem when allowing perturbations of the initial distribution of the walk. However, we show that if we restrict to WSAW's with symmetric and rotationally invariant initial distributions, we do obtain the result for $d \geq 5$ as in [32] (with Exponential error decays only though). The key issue is the change in the bound of the lace function (see Lemmas A.2 and A.4) where we obtain an extra polynomial decay in the symmetric case, allowing us to lower the dimension in the proof of the local CLT. We do however believe that it is possible to extend the result in the perturbed case to $d \geq 5$ too, since

the lace functions can be bounded down to $d \geq 5$ (see again Lemma A.2) and also, arguing heuristically, in the non-symmetric case the errors should really decay at least as fast as in the symmetric case. Possibly, this extension could be done by trying to find Gaussian decay for the errors and choosing a better norm for the Banach fixed point argument. An additional difficulty in the perturbed case is that one does not know the asymptotically correct drift of the walk initially. This is in particular problematic if this drift turns out to be zero. Then, one encounters continuity problems switching from a very small drift to drift zero if one attempts to lower the result to dimensions $d \geq 5$.
Finally, we remark that our method which uses a fixed point argument very similar to the one in Ritzmann [32] is rather general and can easily be extended to include the case where the initial distribution is not on $\mathbb{Z}^d$ but on $\mathbb{R}^d$. In fact, the main local CLT in Theorem 8.1 is then a lot simpler to prove. However, there is no lace expansion for random walks on $\mathbb{R}^d$. Therefore, this extension is only a toy result at the moment and we do not enter into details on this.

6.2 Notations and Result

Consider a random walk on $\mathbb{Z}^d$ with one step distribution $S(x) := s(x)/u$, having bounded support $\Omega \subset \mathbb{Z}^d$, where Ω cannot be embedded in some subspace of dimension strictly smaller than d. $u := |\Omega|$ is the total number of points in Ω and $s(x)$ is some positive function giving the proportion of weight assigned to each point in Ω. Of course s has to be chosen in such a way that S is normalized. Moreover, we assume that $0 \notin \Omega$, ie. $s(0) = 0$ and Ω is a set around 0. Also, we assume that $\mathbb{Z}^d$ is embedded in $\mathbb{R}^d$ in the canonical way. Now let $\lambda \in [0,1]$ be a given parameter and set for any $s, t \in \mathbb{N}_0$ and path $\omega = (\omega(0) = 0, \omega(1), \omega(2), \omega(3), ...) \in (\mathbb{Z}^d)^{\mathbb{N}}$ starting at the origin and with $\omega(i+1) - \omega(i) \in \Omega$ for all $i \geq 0$,

$$U_{st}(\omega) := \begin{cases} 1, & \text{if } \omega(s) = \omega(t), \\ 0, & \text{else.} \end{cases}$$

We define the *Connectivity* of the random walk to be the sequence $(C_n(x))_{n \geq 0}$, with $x \in \mathbb{Z}^d$, by $C_0(x) := \delta_0(x)$, and for the n-th step ($n \geq 1$) by

$$C_n(x) := \sum_{\substack{\omega: 0 \rightsquigarrow x \\ |\omega| = n}} \prod_{0 \leq l < t \leq n} (1 - \lambda U_{lt}(\omega)) \prod_{r=1}^{n} s(\omega(r) - \omega(r-1)). \tag{6.2.1}$$

$C_n(x)$ simply counts the weighted number of paths from 0 to x in n steps, penalizing each self-intersection of the path by $(1 - \lambda)$ for some $\lambda \in [0,1]$. The corresponding total mass sequence is defined by $c_0 := 0$, and for $n \geq 1$:

$$c_n := \sum_{x \in \mathbb{Z}^d} C_n(x). \tag{6.2.2}$$

(We will always denote measures by capital letters and the corresponding total mass by lower case letters). The quantity $C_n(x)/c_n$ gives a distribution for the end point of the random walk after n steps. Note that $C_n(x)/c_n$ is not the distribution of a Markov chain since the walk hast to remember its complete past at any time. In the case of the standard nearest neighbor initial distribution we have $\Omega = \{\pm e_i, i = 1, \ldots, d\}$, where $e_i = (0, \ldots, 1, \ldots, 0)$ is the standard unit vector in $\mathbb{R}^d$ in direction i, $s \equiv 1$ on Ω, and $u = 2d$. Then, $C_n(x) = \sum_{\substack{\omega: 0 \rightsquigarrow x \\ |\omega| = n}} \prod_{0 \leq l < t \leq n} (1 - \lambda U_{lt}(\omega))$. Ie. $C_n(x)$ is the total number of paths going from 0 to x in n steps where each self-intersection is penalized by a factor $(1 - \lambda)$. If $\lambda = 0$ we get the usual random walk, whereas if $\lambda = 1$, we get the fully self-avoiding random walk. In the following

we will be interested in the regime $0 < \lambda \ll 1$, and S a small perturbation of the usual nearest neighbor initial distribution.
Using the lace expansion, one may write the following renewal type equation for C_n:

$$C_n(x) = uS * C_{n-1}(x) + \sum_{m=2}^{n} \Pi_m * C_{n-m}(x), \tag{6.2.3}$$

where the $*$ refers to the convolution of two measures on $\mathbb{Z}^d$ and will from now on often be omitted. The sequence $(\Pi_m)_{m\geq 2}$ reflects the penalties for the self intersections. Of course if $\lambda = 0$, all Π_m's are equal to zero. We give the derivation of this equation and more details about the lace expansion (in particular on upper bounds for $\Pi_m(x)$) in Appendix A. Furthermore, we will write $\phi_{\kappa,\Delta}(x)$ for the d-dimensional normal density with mean $\kappa \in \mathbb{R}^d$ and covariance matrix Δ real, symmetric and positive semi-definite. That is,

$$\phi_{\kappa,\Delta}(x) := \frac{1}{(2\pi)^{d/2}|\Delta|^{1/2}} \exp\left(-\frac{1}{2}(x-\kappa)^t \Delta^{-1}(x-\kappa)\right).$$

Moreover, we write

$$\theta_{\kappa,\Delta}(x) := \frac{K(d)}{|\Delta|^{1/2}} \exp\left(-\sqrt{(x-\kappa)^t \Delta^{-1}(x-\kappa)}\right).$$

Ie. $\theta_{\kappa,\Delta}$ stands for a d-dimensional "doubly-exponential" distribution with mean κ and covariance matrix Δ. $K(d)$ is a norming constant. Finally, if X is a random variable with law S, we denote $s^{(i)} := E[X_i]$ the mean of X in direction i, $i = 1, \dots, d$, and $s^{(ij)} := E[X_i X_j]$, $i, j = 1, \dots, d$, the second moments of X. This notation is extended to arbitrary moments and to moments of (not necessarily positive) measures. Moreover, for a general measures G, $g := \sum_x G(x)$ is the zeroth moment.
Our main result is a local CLT for weakly self-avoiding walks in dimensions $d \geq 9$ with initial distributions S that can be viewed as a perturbation of the standard symmetric nearest neighbor distribution $S(x) = \frac{1}{2d}\mathbb{1}_{\{x:\|x\|=1\}}(x)$ ($\|.\|$ is the Euclidean norm on $\mathbb{R}^d$) and with a penalty parameter λ that is small enough (depending on the chosen initial distribution S). Before stating the Theorem, we have to introduce a few more notations and concepts. In the proof of our main result we will need a distribution p_t, $t \geq 0$ (see Section 8.2). This distribution depends on the following set of parameters: $a \leq \eta \leq b$ for some $0 < a \ll 1$ arbitrarily small and some $b \gg 1$ arbitrarily large, $\pi_i \in [0,1]$, for $i = 1, \dots, d$, $\pi_{ij} \in [0,1]$, for $1 \leq i < j \leq d$, $d_i \in [\epsilon', 1]$ for $i = 1, \dots, d$, and $d_{ij} \in [0,1]$, for $1 \leq i < j \leq d$, with $\sum_{i=1}^d d_i + \sum_{1\leq i<j\leq d} d_{ij} = 1$. $1 \gg \epsilon' > 0$ is some (arbitrarily chosen) parameter that will ensure that we only consider distributions having a covariance matrix of full rank d. Then, we define the subset $C = \{(x_1, \dots, x_d, y_1, \dots, y_{d(d+1)/2})\}$ of $\mathbb{R}^d \times \mathbb{R}^{d(d+1)/2}$ given by the following set of equations:

$$\begin{aligned} x_i &= \eta d_i(2\pi_i - 1), \quad \text{for } i = 1, \dots, d \\ y_i &= \eta(d_i + \sum_{j:j<i} d_{ji} + \sum_{j:j>i} d_{ij}), \quad \text{for } i = 1, \dots, d \\ y_i &= \eta d_{ij}(2\pi_{ij} - 1), \quad \text{for } i = d+1, \dots, d(d+1)/2. \end{aligned}$$

Now consider an arbitrary small parameter $1 \gg \epsilon > 0$ and set C^ϵ to be the closed set $C \backslash \{x \in C : \exists y \in \partial C \text{ with } \|y - x\| < \epsilon\}$. This set will determine the admissible mean (first d coordinates) and covariance structure (remaining coordinates) of the distribution S. As already mentioned, we have to deal with the case of periodic initial distributions and aperiodic initial distributions separately. In this thesis, we call S *two-periodic* if $S^n(x) = 0$ whenever n and $\|x\|_1$ do not have the same parity ($\|.\|_1$ is the L_1-norm and S^n the n-fold convolution of S with itself). Otherwise,

we call S *aperiodic*. Let now $R \in \mathbb{N}$ be a fixed number and write $\overline{B}(0,R)$ for the closed ball of radius R around 0 in the Euclidean norm on $\mathbb{R}^d$. Then, the set of all aperiodic distributions S with support in $(\overline{B}(0,R) \cap \mathbb{Z}^d)\backslash\{0\}$ such that the mean and covariance of S lie in C^ϵ is a closed subset of $L_+^\infty(\mathbb{R}^N)$, N being the number of points in $(\overline{B}(0,R) \cap \mathbb{Z}^d)\backslash\{0\}$. Moreover, this set is a closed neighborhood of the standard initial nearest neighbor distribution $\frac{1}{2d}\mathbb{1}_{\{x:\|x\|=1\}}$. Let us denote this set by $A_{N,\epsilon}$. Furthermore, the set of periodic distributions S with support in $(\overline{B}(0,R)\cap\mathbb{Z}^d)\backslash\{0\}$, for some $R \in \{2,3,\ldots\}$, such that mean and covariance of S lie in C^ϵ is again a closed subset of $L_+^\infty(\mathbb{R}^N)$. Also, this set is a closed neighborhood of the initial distribution giving weight $\frac{1}{4d}$ to the points $\pm e_i$ and $\pm 2e_i$, for $i = 1\ldots,d$, where e_i is the standard unit vector in direction i. Let us denote this set of periodic distributions by $P_{N,\epsilon}$. From now on, we may always assume that the upper bound b for η is equal to $2R$.
We are now able to state the main result. Note that if we take the standard symmetric nearest neighbor distribution $S(x) = \frac{1}{2d}\mathbb{1}_{\{x:\|x\|=1\}}(x)$ only, the result also follows by the corresponding Theorem in Ritzmann [32]. In this case, one can even prove the Theorem for $d \geq 5$ and with Gaussian decay of the error. In the first part the Theorem also recovers a result by Brydges and Spencer [9]:

Theorem 6.1. *Let $d \geq 9$ and let $R \in \mathbb{N}$ and $\epsilon > 0$ arbitrarily small. Then, there are closed neighborhoods in $L_+^\infty(\mathbb{R}^N) \times [0,1]$ of $(\frac{1}{2d}\mathbb{1}_{\{x:\|x\|=1\}}, 0)$ (periodic case) respectively of $(\frac{1}{4d}\mathbb{1}_{\{x:\|x\|\in\{1,2\}\}}, 0)$ (aperiodic case) containing $P_{N,\epsilon}$ and $A_{N,\epsilon}$ respectively, such that for any pair (S,λ) in the corresponding neighborhood,*

$$c_n = \alpha\mu^n(1 + O(n^{-3/2})),$$

for some $\alpha > 0$ and $\mu > 0$. For S fixed, the last coordinate λ takes values in some interval $[0, \lambda_0(S)]$, for some $\lambda_0(S) > 0$. The corresponding λ_0 will be determined more precisely in Chapter 8 (see in particular equations (8.3.25)–(8.3.27)).
Moreover, if S is aperiodic, we have for all $x \in \mathbb{Z}^d$ and all $n \in \mathbb{N}$,

$$\left|\frac{C_n(x)}{c_n} - \phi_{n\kappa,n\Delta}(x)\right| \leq K\left[n^{-1/2}\theta_{n\kappa,n\sigma}(x) + n^{-d/2}\sum_{j=1}^{n/2} j^2 \exp\left(-\sqrt{\frac{n-j}{\sigma}}\|\kappa\|\right)\theta_{j\kappa,j\sigma}(x)\right],$$

for some $\kappa \in \mathbb{R}^d$ and Δ a real symmetric and positive semi-definite matrix, as well as for some $K > 0$ and $\sigma > 0$ large enough. If S is two-periodic, and $n - \|x\|_1$ even,

$$\left|\frac{C_n(x)}{c_n} - 2\phi_{n\kappa,n\Delta}(x)\right| \leq K\left[n^{-1/2}\theta_{n\kappa,n\sigma}(x) + n^{-d/2}\sum_{j=1}^{n/2} j^2 \exp\left(-\sqrt{\frac{n-j}{\sigma}}\|\kappa\|\right)\theta_{j\kappa,j\sigma}(x)\right].$$

The constants α and μ and the mean κ and covariance matrix Δ depend on λ, d and S, whereas σ depends on d and S and K on d and R. Finally, λ_0 depends on d, S, R and ϵ.

We can also prove this Theorem for $d \geq 5$ in the case that S is symmetric and rotationally invariant (an thus $\kappa = 0$). This is stated in Chapter 10.

6.3 Strategy of the Proof

Consider the lace expansion formula for the C_n's:

$$C_n = uSC_{n-1} + \sum_{m=2}^{n} \Pi_m C_{n-m}.$$

Now suppose that C_n grows exponentially. Ie. $C_n = \mu^n A_n$ for some $\mu > 0$ (the so called *Connective Constant*), such that $a_n = \sum_{x\in\mathbb{Z}^d} A_n(x)$ tends to some $\alpha > 0$, for $n \to \infty$. Then we can re-write the lace expansion, setting $B_m := \frac{\Pi_m}{\lambda c_m}$ for all $m \geq 2$, as

$$A_n = u\mu^{-1}SA_{n-1} + \lambda \sum_{m=2}^{n} a_m B_m A_{n-m}, \tag{6.3.1}$$

and for the mass sequence:

$$a_n = u\mu^{-1}a_{n-1} + \lambda \sum_{m=2}^{n} a_m b_m a_{n-m}. \tag{6.3.2}$$

In this way, we cancel all the exponential growth out of the involved quantities $(C_n)_{n\geq 0}$ and $(\Pi_m)_{m\geq 2}$. Now, the proof is split into several parts. In a first step, we have to show the existence of the connective constant μ and the limit α. This is done by showing that if the b_m's decay fast enough, the sequence $(a_n)_{n\geq 0}$ given above is the unique fixed point of a certain operator in a normed Banach space of sequences. This has already been done by Ritzmann in [32] but will for completeness be included here in Chapter 7. In that Chapter, we also give explicit equations for α and μ as well as the convergence speed for $a_n \to \alpha$. In a second step (Chapter 8) we assume more specific pointwise decay rates for a general sequence $(B_m)_{m\geq 1}$. With these estimates in hand, we obtain a local CLT for A_n, again by showing that the sequence $(A_n)_{n\geq 0}$ is a fixed point of some operator. As in the case of the mass sequence, we use Banach fixed point theorem. This is our main result. It is more general than Theorem 6.1 but tailored to be applied to that Theorem. Finally, in order to show that Theorem 6.1 is true, we use the pointwise estimates of the Π_m's in terms of the C_n's from the Appendix A and apply an iterative procedure to show that the B_m's in question for the perturbed weakly self-avoiding walk indeed have the good decay needed for the local CLT in Chapter 8. This iterative procedure also yields the correct connective constant μ for the perturbed WSAW and a sequence of real-valued vectors κ_i converging to the correct asymptotic drift κ of the distribution C_n/c_n. This is Chapter 9. The method we use is an adaption from the one used by Ritzmann in [32]. Finally, in Chapter 10, we show that in the case where S is rotationally invariant and symmetric in each coordinate, we obtain the main Theorem of Chapter 8 and Theorem 6.1 (with a slightly different norm than in the general case) down to dimension $d \geq 5$.

CHAPTER 7

The Mass Constant

This Chapter is taken from Ritzmann's thesis [32]. We need Propositions 7.1 and Corollary 7.5 for later purposes. The Chapter is merely for completeness of the thesis.

7.1 Existence and Uniqueness

Proposition 7.1. *Consider a real-valued sequence $(b_m)_{m\geq 1}$ with $\beta := \sum_{m=1}^{\infty} m|b_m| < \infty$. Then, there is a $\lambda_0 = \lambda_0(\beta) > 0$ such that for all $\lambda \in [0, \lambda_0]$, there exists a unique real-valued sequence $(a_n)_{n\in\mathbb{N}_0}$ with $a_0 = 1$, and for $n \geq 1$*

$$a_n = (1 - \lambda \sum_{m=1}^{\infty} a_m b_m) a_{n-1} + \lambda \sum_{m=1}^{n} a_m b_m a_{n-m}, \tag{7.1.1}$$

such that $\sum_{n\geq 1} |a_n - a_{n-1}| < \infty$.

The proof of this Proposition is done via a fixed point argument. Thus we need to introduce a Banach space and an appropriate operator. Let $(l_\infty, \|.\|_\infty)$ be the Banach space of bounded real-valued sequences $g := (g_n)_{n\in\mathbb{N}_0}$ with the supremum norm. The difference operator $\triangle : \mathbb{R}^{\mathbb{N}_0} \to \mathbb{R}^{\mathbb{N}_0}$ is defined by $(\triangle g)_0 := g_0$ and

$$(\triangle g)_n := g_n - g_{n-1}, \quad \text{for } n \in \mathbb{N}.$$

For g a sequence with $\sum_{n\geq 0} |(\triangle g)_n| < \infty$, define the norm

$$\|g\|_{\mathcal{D}} := \sum_{n\geq 0} |(\triangle g)_n|.$$

Furthermore, define the operator $\tilde{.}$ on sequences by

$$\begin{aligned} \tilde{g}_0 :=& g_0, \quad \text{and} \\ \tilde{g}_n :=& \tilde{g}_{n-1} - \lambda \left[\sum_{m=1}^{n} g_m b_m (g_{n-1} - g_{n-m}) + g_{n-1} \sum_{j=n+1}^{\infty} g_j b_j \right]. \end{aligned} \tag{7.1.2}$$

This is the correct operator to use for the Banach fixed point Theorem. The following Lemmas prove that the necessary conditions for the fixed point Theorem are fulfilled on an appropriate subspace of l_∞.

Lemma 7.2. *Let $g \in l_\infty$ with $\|g\|_{\mathcal{D}} < \infty$. Then we also have $\|\tilde{g}\|_{\mathcal{D}} < \infty$.*

Proof. Let g with $\|g\|_{\mathcal{D}} < \infty$. We have to show that $\sum_{n\geq 0} |(\triangle \tilde{g})_n|$ is finite. First notice that

$$\|g\|_\infty = \sup_{n\in\mathbb{N}_0} |g_n| = \sum_{n\in\mathbb{N}_0} \left| \sum_{k=0}^{n} (\triangle g)_k \right| \leq \|g\|_{\mathcal{D}}. \tag{7.1.3}$$

From (7.1.2) we have

$$\begin{aligned}
\sum_{n\geq 0}|(\triangle\tilde{g})_n| =&|g_0| + \sum_{n\geq 1}|\tilde{g}_n - \tilde{g}_{n-1}| \\
=&|g_0| + \lambda\sum_{n\geq 1}\left|\sum_{m=1}^{n} g_m b_m(\underbrace{g_{n-1}-g_{n-m}}_{=\sum_{l=1}^{m-1}(\triangle g)_{n-l}}) + \sum_{j=n+1}^{\infty} g_j b_j g_{n-1}\right| \\
\leq&|g_0| + \lambda\left[\|g\|_\infty \sum_{n\geq 1}\sum_{l=1}^{n-1}|(\triangle g)_{n-l}| \underbrace{\sum_{m=l+1}^{n}|b_m|}_{\leq\sum_{m\geq l+1}|b_m|} + \|g\|_\infty^2 \underbrace{\sum_{n\geq 1}\sum_{j\geq n+1}|b_j|}_{\leq\sum_{j\geq 2} j|b_j|\leq\beta}\right] \\
\leq&|g_0| + \lambda\left[\|g\|_\infty \sum_{l\geq 1}\sum_{m\geq l+1}|b_m|\sum_{n\geq 1}|\triangle g_n| + \|g\|_\infty^2\beta\right] \\
\leq&|g_0| + 2\lambda\beta\|g\|_{\mathcal{D}}^2, \qquad (7.1.4)
\end{aligned}$$

where we used (7.1.3) in the last line. □

Lemma 7.3. *Let $\mathcal{D}_L := \{g \in l_\infty : g_0 = 1 \text{ and } \|g\|_{\mathcal{D}} \leq L\}$, where L is a constant greater than or equal to $3/2$. Then for all $\lambda \in [0, 1/(6\beta L)]$ the operator $\tilde{.}$ is a contraction with respect to $\|.\|_{\mathcal{D}}$ on $\mathcal{D}_L$.*

The value $3/2$ above is chosen to keep the constants simple. An analogous statement holds as long as L is bounded away from one.

Proof. We have to show:

1. $g \in \mathcal{D}_L \quad \Rightarrow \quad \tilde{g} \in \mathcal{D}_L$, and
2. there exists some $\epsilon < 1$ positive such that $\|\tilde{g} - \tilde{h}\|_{\mathcal{D}} \leq \epsilon\|g - h\|_{\mathcal{D}}$ for all $g, h \in \mathcal{D}_L$.

To see 1., let $g \in \mathcal{D}_L$ be given. We know $\tilde{g}_0 = g_0 = 1$. According to (7.1.4) we have

$$\|\tilde{g}\|_{\mathcal{D}} \leq 1 + 2\lambda\beta L^2 \leq \frac{2}{3}L + \frac{1}{3}L,$$

whenever $\lambda \leq 1/(6\beta L)$.
To see 2., take $g, h \in \mathcal{D}_L$. Since g_0 equals h_0, we have $\|\tilde{g} - \tilde{h}\|_{\mathcal{D}} = \sum_{n\geq 1}|(\triangle(\tilde{g} - \tilde{h}))_n|$. We have from the definition of the operator $\tilde{.}$ in (7.1.2):

$$\begin{aligned}
\sum_{n\geq 1}|(\triangle\tilde{g})_n - (\triangle\tilde{h})_n| = \lambda\sum_{n\geq 1}\Bigg|&\sum_{m=1}^{n}(g_m - h_m)b_m(g_{n-1} - g_{n-m}) \\
&+ \sum_{m=1}^{n} h_m b_m[g_{n-1} - h_{n-1} - (g_{n-m} - h_{n-m})] \\
&+ \sum_{j\geq n+1} b_j[g_j(g_{n-1} - h_{n-1}) + (g_j - h_j)h_{n-1}]\Bigg|
\end{aligned}$$

We estimate the absolute values of the three summands individually. The first one can be treated analogously to (7.1.4):

$$\lambda \sum_{n\geq 1} \sum_{m=1}^{n} |b_m||g_m - h_m||g_{n-1} - g_{n-m}| \leq \lambda\beta\|g-h\|_\infty \|g\|_{\mathcal{D}}.$$

Similarly, we obtain for the second one:

$$\lambda \sum_{n\geq 1} \sum_{m=1}^{n} |b_m||h_m||g_{n-1} - h_{n-1} - (g_{n-m} - h_{n-m})| \leq \lambda\beta\|h\|_\infty \|g-h\|_{\mathcal{D}},$$

and for the third one:

$$\lambda \sum_{n\geq 1} \sum_{j\geq n+1} |b_j||g_j(g_{n-1} - h_{n-1}) + (g_j - h_j)h_{n-1}| \leq \lambda\beta\|\|g-h\|_\infty(\|g\|_\infty + \|h\|_\infty).$$

Since both g and h are in $\mathcal{D}_L$ and $\lambda \leq 1/(6\beta L)$, this yields

$$\|\tilde{g} - \tilde{h}\|_{\mathcal{D}} \leq 4\lambda\beta L\|g-h\|_{\mathcal{D}} \leq \frac{2}{3}\|g-h\|_{\mathcal{D}}.$$

□

It remains to show the completeness of the space.

Lemma 7.4. *The elements of l_∞ with finite $\|.\|_{\mathcal{D}}$-norm form a Banach space with this norm, and $\mathcal{D}_L$ is a closed subset of this space.*

Proof. Clearly the set $\{g \in l_\infty : \|g\|_{\mathcal{D}} < \infty\}$ is a linear subspace of l_∞. Now let $(g^{(m)})_{m\in\mathbb{N}}$ be a Cauchy sequence in this space. Since $\|g\|_\infty \leq \|g\|_{\mathcal{D}}$ (see (7.1.3)), $(g^{(m)})_{m\in\mathbb{N}}$ is also a Cauchy sequence in $(l_\infty, \|.\|_\infty)$. Therefore it has a limit $g \in l_\infty$, and it suffices to show that $\|g\|_{\mathcal{D}} < \infty$. Since the difference operator $\triangle$ is continuous on l_∞, for each n, the term $\|(\triangle(g - g^{(m)}))_n\|_\infty$ will tend to zero as $m \to \infty$. Now choose a subsequence $(g^{(m_i)})_{i\in\mathbb{N}}$ with $\|g^{(m_i)} - g^{(m_{i-1})}\|_{\mathcal{D}} \leq 2^{-i}$ for all $i \in \mathbb{N}$. Then we have for each $i \in \mathbb{N}$:

$$\begin{aligned}
\sum_{n\geq 0} |(\triangle g)_n| &\leq \sum_{n\geq 0} |(\triangle g)_n^{(m_i)}| + \sum_{n\geq 0} |(\triangle(g - g^{(m_i)}))_n| \\
&\leq \|g^{(m_i)}\|_{\mathcal{D}} + \sum_{n\geq 0} \sum_{j\geq i+1} |(\triangle(g^{(m_j)} - g^{(m_{j-1})}))_n| \\
&\leq \|g^{(m_i)}\|_{\mathcal{D}} + \sum_{j\geq i+1} \underbrace{\|g^{(m_j)} - g^{(m_{j-1})}\|_{\mathcal{D}}}_{\leq 2^{-j}} \\
&< \infty.
\end{aligned}$$

The closedness of $\mathcal{D}_L$ follows from an analogous argument. □

Proof. (Proof of Proposition 7.1). Using Lemmas 7.2–7.4, the Banach fixed point theorem yields for small enough λ the existence and uniqueness of an element $a \in \mathcal{D}_L$ with $\tilde{a} = a$. Furthermore, the repeated iteration of $\tilde{\cdot}$ with starting point $(1, 1, \ldots)$ converges to a. As long as $L \geq 3/2$, the value of L has an influence only on the upper bound for λ. This proves the Proposition. □

7.2 The Connectivity μ, the Limit α and the Convergence Speed

We investigate the limit and the convergence behavior of the "fixed sequence" of Proposition 7.1 in a more particular setting. By choosing $L = 3/2$ we obtain for all $\lambda \leq 1/(9\beta)$ a sequence $(a_n)_{n\geq 0}$ with $a_0 = 1$, $\sum_{n\geq 1} |(\triangle a)_n| \leq 1/2$ and for all $n \in \mathbb{N}$

$$a_n = u\mu^{-1} a_{n-1} + \lambda \sum_{m=1}^{n} a_m b_m a_{n-m},$$

where

$$u\mu^{-1} = 1 - \lambda \sum_{m\geq 1} a_m b_m. \tag{7.2.1}$$

Hence, we have proved the existence and determined the value of the connectivity constant μ, because a is bounded and $\sum_{n\geq 1} |b_n| < \infty$. Thus $u\mu^{-1}$ remains finite. Note also that for all $n \in \mathbb{N}_0$ we have

$$1/2 \leq a_n \leq 3/2.$$

We now investigate the limiting value $\alpha = \lim_{n\to\infty} a_n$. Since the difference sequence of a is absolutely summable, α exists and we have

$$\alpha = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sum_{m=0}^{n} (\triangle a)_m.$$

Now recall (7.1.1) and consider for fixed $n \in \mathbb{N}$:

$$\begin{aligned}
a_n =& 1 + \sum_{k=1}^{n} (\triangle a)_k \\
=& 1 - \lambda \sum_{k=1}^{n} \left[\sum_{m\geq 1} a_m b_m a_{k-1} - \sum_{m=1}^{k} a_m b_m a_{k-m} \right] \\
=& 1 - \lambda \sum_{m\geq 1} a_m b_m \sum_{k=1}^{n} a_{k-1} + \lambda \sum_{m=1}^{n} a_m b_m \sum_{k=m}^{n} a_{k-m} \\
=& 1 - \lambda \underbrace{\sum_{m\geq n+1} a_m b_m \sum_{k=1}^{n} a_{k-1}}_{=:F_1} - \lambda \sum_{m=1}^{n} a_m b_m \sum_{l=1}^{m-1} \underbrace{a_{n-l}}_{=\alpha-(\alpha-a_{n-l})} \\
=& 1 - \lambda F_1 - \lambda \sum_{m=1}^{n} a_m b_m (m-1)\alpha + \lambda \underbrace{\sum_{m=1}^{n} a_m b_m \sum_{l=1}^{m-1} \sum_{k\geq n-l+1} (\triangle a)_k}_{=:F_2},
\end{aligned} \tag{7.2.2}$$

where

$$|F_1| = \left| \sum_{m\geq n+1} a_m b_m \sum_{k=1}^{n} a_{k-1} \right| \leq \sum_{m\geq n+1} L|b_m| nL \leq L^2 \sum_{m\geq n+1} m|b_m| \to 0,$$

as $n \to \infty$, and

$$|F_2| = \left| \sum_{m=1}^{n} a_m b_m \sum_{l=1}^{m-1} \sum_{k \geq n-l+1} (\triangle a)_k \right| \leq \sum_{l=1}^{n-1} \sum_{m \geq l+1} L|b_m| \sum_{k \geq n-l+1} |(\triangle a)_k|$$

$$\leq L \underbrace{\sum_{l=1}^{n/2} \sum_{m \geq l+1} |b_m|}_{\leq \beta} \sum_{k \geq n/2} |(\triangle a)_k| + L \sum_{l=n/2}^{n-1} \sum_{m \geq l+1} |b_m| \underbrace{\sum_{k \geq 1} |(\triangle a)_k|}_{\leq \|a\|_{\mathcal{D}}} \to 0,$$

as $n \to \infty$. Thus, letting n tend to infinity in (7.2.2), we obtain:

$$\alpha = 1 - \lambda \sum_{m \geq 1} (m-1) a_m b_m \alpha,$$

which yields

$$\alpha^{-1} = 1 + \lambda \sum_{m \geq 1} (m-1) a_m b_m = u\mu^{-1} + \lambda \sum_{m \geq 1} m a_m b_m.$$

In case we know the rate of decay of the b_m's, we can determine the speed of the convergence $a_n \to \alpha$ more precisely. This is the content of the following Corollary. It's proof is immediate from (7.2.2), as long as we keep λ small enough.

Corollary 7.5. *If there exist positive constants ϵ and β' such that*

$$|b_m| \leq \beta' m^{-2-\epsilon} \quad \text{for all } m \in \mathbb{N},$$

we get a decay of order $n^{-1-\epsilon}$ for the difference sequence $\triangle a$. More precisely we have

$$|(\triangle a)_n| \leq \beta' K n^{-1-\epsilon} \quad \text{for all } n \in \mathbb{N},$$

where K is a positive constant not depending on λ and β'. In particular we have another constant K such that

$$|\alpha - a_n| \leq \beta' K n^{-\epsilon} \quad \text{for all } n \in \mathbb{N}.$$

Chapter 8

A General Local CLT on $\mathbb{Z}^d$ – The Main Result

In this Chapter we consider measures of the type given by the lace expansion formula (6.3.1). The question we ask is: Considering such measures as the perturbation of the distribution of the sum of independent identically distributed random variables, how close are they to the normal density? We obtain a local CLT on $\mathbb{Z}^d$ in dimensions $d \geq 9$, with a correction term of order $n^{-d/2}$ near the mean of the walk, and with exponential error decay, improved by a factor $n^{-1/2}$ for x far away from the mean. As already mentioned, we use a method similar to the one introduced by Ritzmann in [32]. What we show here is more general than Theorem 6.1 but it is tailored to fit the WSAW case.

8.1 The Model

For technical reasons we only treat the case of aperiodic initial distributions S. The two-periodic case will be treated in a short Section at the end of this Chapter.
Let us start by introducing the ingredients we need. Consider at first some positive number $R \in \mathbb{N}$ and $\epsilon > 0$ arbitrarily small. Then choose a non-degenerate and aperiodic probability measure $S = s/u$ with bounded support $\Omega \subset \overline{B}(0,R)\backslash\{0\}$ in the set $A_{N,\epsilon}$ (see Section 6.2). Also, let $(B_m)_{m\geq 1}$ be a sequence of finite signed measures on $\mathbb{Z}^d$ such that $\sum_{m\geq 1} m|b_m| < \infty$ and $\sum_{m\geq 1} b_m^{(i)} < \infty$, for $i = 1,\dots,d$. We apply Proposition 7.1 to obtain the existence of a unique sequence $(a_n)_{n\geq 0}$, with $a_0 = 1$ and

$$a_n = u\mu^{-1}a_{n-1} + \lambda \sum_{m=1}^{n} a_m b_m a_{n-m},$$

where we write $u\mu^{-1} = 1 - \lambda \sum_{m\geq 1} a_m b_m$ for $\lambda > 0$ small enough. We also know that $a_n \in [1/2, 3/2]$, for all n and we set

$$\rho := \sum_{m\geq 1} a_m b_m \tag{8.1.1}$$

for later use. We may define the quantities

$$\kappa^{(i)} := \frac{u\mu^{-1}s^{(i)} + \lambda \sum_{m\geq 1} a_m b_m^{(i)}}{1 + \lambda \sum_{m\geq 1} a_m b_m (m-1)}, \quad \text{for } i = 1,\dots,d. \tag{8.1.2}$$

Now, we assume that the B_m's have the following pointwise decay (uniformly for all $m \geq 1$ and $x \in \mathbb{Z}^d$):

$$|B_m(x)| \leq K m^{-d/2} \sum_{k=1}^{m/2} \exp\left(-\frac{\sqrt{m-k}}{\sqrt{\sigma}}\|\kappa\|\right) \theta_{k\kappa,k\sigma}(x). \tag{8.1.3}$$

Here, K is a positive constant whose value may from now on change from line to line and $\sigma > 0$ is to be determined later. Note that since $d \geq 9$, Proposition 7.1 is satisfied even if $\kappa = 0$ and Corollary 7.5 holds with $\epsilon = 3/2$.

Using the above setting, we are able to define a sequence of measures $(A_n)_{n\geq 0}$ on $\mathbb{Z}^d$ as follows: Set $A_0 := \delta_0$, and for $n \geq 1$:

$$A_n := u\mu^{-1} S A_{n-1} + \lambda \sum_{m=1}^{n} a_m B_m A_{n-m}. \tag{8.1.4}$$

One might worry that the use of the sequence $(a_m)_m$ is ambiguous here. But it is readily checked by summing over all $x \in \mathbb{Z}^d$ that $(a_m)_m$ is indeed the weight sequence corresponding to $(A_n)_n$. We are now interested in the asymptotic behavior of the sequence $(A_n/a_n)_n$. We will see that the right asymptotic drift (for large n) is given by $n\kappa$, κ from (8.1.2), and the right asymptotic covariance matrix by $n\Delta = n(\delta_{ij})_{i,j=1}^d$, with

$$\begin{aligned}\delta_{ij} =& \frac{1}{1+\lambda\sum_{m\geq 1}(m-1)a_m b_m}\Bigg(u\mu^{-1}s^{(ij)} + \lambda\sum_{m\geq 1} a_m b_m^{(ij)} - \kappa^{(i)}\kappa^{(j)} \\ &+\lambda\kappa^{(i)}\kappa^{(j)}\sum_{m\geq 1}(m-1)^2 a_m b_m - \lambda\kappa^{(i)}\sum_{m\geq 1} a_m(m-1)b_m^{(j)} - \lambda\kappa^{(j)}\sum_{m\geq 1} a_m(m-1)b_m^{(i)}\Bigg),\end{aligned} \tag{8.1.5}$$

for $i,j = 1,\dots,d$ (of course $\delta_{ij} = \delta_{ji}$). From now on, we will always assume that λ is small enough to assure that $|\kappa^{(i)}| \leq 2R$, for $i = 1,\dots,d$, and $\delta_{ij} \in [(s^{(ij)} - s^{(i)}s^{(j)})/2, 2(s^{(ij)} - s^{(i)}s^{(j)})]$, for $i,j = 1,\dots,d$, where $s^{(ij)} - s^{(i)}s^{(j)}$ is the (i,j)-th entry in the covariance matrix of S. In fact it is not important how big these intervals are chosen precisely as long as the bounds are determined by R and S only. Note also that the definition of Δ requires that if $\kappa \neq 0$, $b_m \leq Km^{-7/2}$, $b_m^{(i)} \leq Km^{-5/2}$ and $b_m^{(ij)} \leq Km^{-3/2}$ for all $i,j = 1,\dots,d$. This is of course guaranteed by the exponential decay of the moments of the sequence $(B_m)_m$ in m (see (8.1.3)). However, we have to make sure that Δ does not explode, if κ tends to zero, or if $\kappa = 0$. In other words, we have to check that the constant K is independent of κ. If $\kappa = 0$, we only need the term with $b_m^{(ij)}$ in the definition of Δ, but this term then decays like $m^{2-d/2}$ which is fast enough if $d \geq 9$. If $\kappa \neq 0$, we have: $|b_m^{(ij)}| \leq \sum_x |x_i x_j||B_m(x)|$ for $i,j = 1,\dots,d$. Combining with the bound (8.1.3), we see that we have to first bound $\sum_x |x_i x_j|\theta_{k\kappa,k\sigma}(x)$. But

$$\begin{aligned}\sum_x |x_i x_j|\theta_{k\kappa,k\sigma}(x) \leq& \sum_x |x_i - k\kappa^{(i)} + k\kappa^{(i)}||x_j - k\kappa^{(j)} + k\kappa^{(j)}|\theta_{k\kappa,k\sigma}(x) \\ \leq& \sum_x \theta_{k\kappa,k\sigma}(x)(|x_i - k\kappa^{(i)}||x_j - k\kappa^{(j)}| + k|x_i - k\kappa^{(i)}|\|\kappa\| \dots \\ &\dots + k|x_j - k\kappa^{(j)}|\|\kappa\| + k^2\|\kappa\|^2) \\ \leq& K(k + k^{3/2}\|\kappa\| + k^2\|\kappa\|^2).\end{aligned}$$

Therefore, $|b_m^{(ij)}| \leq K\exp\left(-\frac{\sqrt{m}}{\sqrt{2\sigma}}\|\kappa\|\right) m^{-d/2}(m^2 + m^{5/2} + m^3) \leq Km^{-3/2}$, as long as $d \geq 9$ and where K is independent of κ. Similar considerations for the terms involving b_m and $b_m^{(i)}$ show that Δ remains bounded for all values of κ near zero.
The goal of this Chapter is to prove the following Theorem:

Theorem 8.1. *In the above setting, there exists $\lambda_0 > 0$ such that for any $\lambda \in [0,\lambda_0]$, we have*

$$\left|\frac{A_n(x)}{a_n} - \phi_{n\kappa,n\Delta}(x)\right| \leq K\left[n^{-1/2}\theta_{n\kappa,n\sigma}(x) + n^{-d/2}\sum_{j=1}^{n/2} j^2 \exp\left(-\frac{\sqrt{n-j}}{\sqrt{\sigma}}\|\kappa\|\right)\theta_{j\kappa,j\sigma}(x)\right],$$

where the parameters κ and Δ depend on λ, d, S and on the sequence $(B_m)_{m\geq 1}$ and are defined above in (8.1.2) and (8.1.5). $K = K(R,d)$ and $\sigma = \sigma(d,S)$ are positive constants independent of the sequence $(B_m)_m$ and will be determined in the proof of the Theorem.

8.2 The Distribution $p_t(x)$

Before turning to the proof of Theorem 8.1 we need to introduce a new distribution on $\mathbb{Z}^d$. Consider $(Y_i)_{i\geq 1}$ a sequence of iid random variables on $\mathbb{Z}^d$ with the following distribution: Let e_i be the standard i-th unit vector in $\mathbb{Z}^d$ (or $\mathbb{R}^d$), and choose π_i, $i = 1,\dots,d$, and π_{ij}, $1 \leq i < j \leq d$, in $[0,1]$. Moreover, let d_i, $i = 1,\dots,d$, and d_{ij}, $1 \leq i < j \leq d$, be in $[0,1]$ with $\sum_{i=1}^d d_i + \sum_{1\leq i<j\leq d} d_{ij} = 1$. The distribution of Y_1 is defined by

$$\begin{aligned}
P[Y_1 = e_i] &:= \pi_i d_i, \quad i = 1,\dots,d\\
P[Y_1 = -e_i] &:= (1-\pi_i)d_i, \quad i = 1,\dots,d\\
P[Y_1 = e_i + e_j] = P[Y_1 = -e_i - e_j] &:= \frac{1}{2}\pi_{ij}d_{ij}, \quad 1 \leq i < j \leq d\\
P[Y_1 = e_i - e_j] = P[Y_1 = -e_i + e_j] &:= \frac{1}{2}(1-\pi_{ij})d_{ij}, \quad 1 \leq i < j \leq d.
\end{aligned}$$

Now set $S_n := \sum_{i=1}^n Y_i$ for $n \in \mathbb{N}$, and consider another parameter $\eta > 0$. Then, for $t \geq 0$, the distribution $p_t(x)$ on $\mathbb{Z}^d$ is defined by $p_0(x) := \delta_0(x)$, and for $t > 0$,

$$p_t(x) := \sum_{n\geq 0} e^{-\eta t}\frac{(\eta t)^n}{n!}P[S_n = x]. \tag{8.2.1}$$

That is, p_t is the distribution of a random walk $(X_t)_{t\geq 0} = (X_t^{(1)},\dots,X_t^{(d)})_{t\geq 0}$ with jumps Y_i and a Poisson distributed number of jumps up to time t (with parameter $t\eta$). Note that $p_t * p_s(x) = p_{t+s}(x)$. Indeed, we use Fubini to get

$$\begin{aligned}
p_t * p_s(x) &= \sum_{y\in\mathbb{Z}^d} p_t(y)p_s(x-y) = \sum_y \sum_{n\geq 0} e^{-\eta t}\frac{(\eta t)^n}{n!}P[S_n = y]\sum_{m\geq 0} e^{-\eta s}\frac{(\eta s)^m}{m!}P[S_m = x-y]\\
&= e^{-\eta(t+s)}\sum_{n,m\geq 0}\eta^{n+m}\frac{t^n s^m}{n!m!}\sum_y P[S_n = y]P[S_m = x-y]\\
&= e^{-\eta(t+s)}\sum_{k\geq 0}\eta^k P[S_k = x]\sum_{l=0}^k \frac{t^l s^{k-l}}{l!(k-l)!}\frac{k!}{k!} = \sum_{k\geq 0} e^{-\eta(t+s)}\frac{(\eta(t+s))^k}{k!}P[S_k = x]\\
&= p_{t+s}(x).
\end{aligned}$$

It is also immediate to see that $E[X_t^{(i)}] = t\eta E[Y_1^{(i)}] = t\eta(2\pi_i - 1)d_i$, for $i = 1,\dots,d$. Moreover, using the formula of the total covariance, we have $\mathrm{cov}(X_t^{(i)}, X_t^{(j)}) = t\eta E[Y_1^{(i)}Y_1^{(j)}] = t\eta(2\pi_{ij} - 1)d_{ij}$, for $1 \leq i < j \leq d$, and for $i = 1,\dots,d$, $\mathrm{var}(X_t^{(i)}) = \mathrm{cov}(X_t^{(i)}, X_t^{(i)}) = t\eta E[Y_1^{(i)\,2}] = t\eta d_{ii}$, where $d_{ii} := d_i + \sum_{j:j<i} d_{ji} + \sum_{j:i<j} d_{ij}$. We can show the following Lemma:

Lemma 8.2. *Setting $\kappa := \eta(d_1(2\pi_1 - 1),\dots,d_d(2\pi_d - 1))$, and $\Delta = (\delta_{ij})_{i,j=1}^d$, with $\delta_{ii} := \eta d_{ii}$, for $i = 1,\dots,d$, $\delta_{ij} := \eta d_{ij}(2\pi_{ij} - 1)$, and $\delta_{ji} = \delta_{ij}$ for $1 \leq i < j \leq d$, we have for all $x \in \mathbb{Z}^d$,*

$$|p_n(x) - \phi_{n\kappa,n\Delta}(x)| \leq \frac{K}{\sqrt{n}}\theta_{n\kappa,n\sigma' Id_d}(x), \quad \text{for all } n > 0,$$

where $K > 0$ and $\sigma' > 0$ have to be chosen big enough and depend only on d and an upper bound for η.

In the remainder of this chapter, we will write $\phi_{t\kappa,t\Delta} =: \phi_t$ and $\theta_{t\kappa,t\sigma' Id_d} =: \theta_t$. Also, K and σ' might have to be adapted but will always only depend on d and the distribution of p_t (that is on an upper bound for η).

Proof. The proof of the Lemma is a combination of large deviation theory and tilting of the measure p_n.
Let $Z_1 \sim p_1(x)$. Then, $Z(t) := \sum_{x\in\mathbb{Z}^d} \exp(\langle t, x\rangle)\, p_1(x)$ exists for all $t \in \mathbb{R}^d$, and we may define the entropy function $I(\xi) := \sup_{t\in\mathbb{R}^d}\{\langle t, \xi\rangle - \log Z(t)\}$. By standard large deviation theory (see eg. Ellis [13]) the laws of $p_n(nx) = p_1^{*n}(nx)$, $n \in \mathbb{N}$, obey a large deviation principle with entropy function I and rate n. The following properties of I and Z are then easily obtained: I is strictly convex on $\mathbb{R}^d$ with $I(\kappa) = 0$ and $I \geq 0$ on $\mathbb{R}^d$. This implies that κ is a global minimum for I and therefore, the first partial derivatives of I vanish at κ.
The function $t \mapsto \nabla \log Z(t)$ is an analytic diffeomorphism in $\mathbb{R}^d$ (see Ellis [13] and note that Z is analytic). Thus, for any $\xi \in \mathbb{R}^d$, there exists a unique $t_\xi \in \mathbb{R}^d$ such that $\nabla \log Z(t_\xi) = \xi$. Finally, the following points are easily obtained by the aforementioned facts and simple calculations:

- $\nabla \log Z(0) = \kappa$,
- $\nabla^2 \log Z(0) = \Delta = \mathrm{Cov}(Z_1)$ (here, $\nabla^2 = \nabla^t \cdot \nabla$),
- $I(\xi) = \langle t_\xi, \xi\rangle - \log Z(t_\xi)$ for any $\xi \in \mathbb{R}^d$, and
- $\nabla^2 I(\kappa) = \Delta^{-1}$.

Moreover, for $k = 3, 4, 5$, $\nabla^k I(\kappa)$ depend only on the moments of Z_1 up to order 5.
Now, for $t \in \mathbb{R}^d$, set

$$P_t(x) := \frac{p_1(x)\exp(\langle t, x\rangle)}{Z(t)}.$$

Then, for $\xi = x/n$, we can write

$$p_n(x) = \exp(-nI(\xi))\, P_{t_\xi}^{*n}(x), \tag{8.2.2}$$

and we have $E[Z^{(P_{t_\xi})}] = \nabla \log Z(t_\xi) = \xi$, if $Z^{(P_{t_\xi})} \sim P_{t_\xi}$.
We now consider ξ such that $\|\xi - \kappa\| \leq n^{-5/12}$. Denoting by Δ_ξ the covariance matrix of P_{t_ξ}, we have $\Delta_\kappa = \Delta$ and Δ_ξ depends analytically on ξ. Setting δ_1 the smallest eigenvalue of Δ, we consider the set R_P of all ξ such that the smallest eigenvalue of Δ_ξ is greater or equal to $\delta_1/2$. This is a closed neighborhood of κ, and for $\|\xi - \kappa\| \leq n^{-5/12}$, ξ is in R_P for almost all n. More precisely, there is some $N \in \mathbb{N}$ such that $\xi \in R_P$, for all $n \geq N$. We only prove the estimate for such ξ since the remaining cases are contained in a compact and bounded set and may be dealt with by simply choosing K large enough.
Thus, let $\xi = x/n \in R_P$ with $\|\xi - \kappa\| = \|(x - n\kappa)/n\| \leq n^{-5/12}$. We estimate the two factors in (8.2.2) separately. For the first one we get, doing a Taylor expansion for I around κ and then a Taylor expansion of exp around 0:

$$\begin{aligned}\exp(-nI(\xi)) &= \exp\left(-\frac{1}{2n}(x-n\kappa)^t\Delta^{-1}(x-n\kappa) + n\sum_{i=3}^{5} T^{(i)}(\xi-\kappa) + nO((\xi-\kappa)^{(6)})\right)\\ &= \exp\left(-\frac{1}{2n}(x-n\kappa)^t\Delta^{-1}(x-n\kappa)\right)\left[1 + \frac{1}{\sqrt{n}}T^{(3)}\left(\frac{x-n\kappa}{\sqrt{n}}\right)\right.\\ &\quad \left. + \frac{1}{n}T^{(4)}\left(\frac{x-n\kappa}{\sqrt{n}}\right) + \frac{1}{n^{3/2}}T^{(5)}\left(\frac{x-n\kappa}{\sqrt{n}}\right) + O(n^{-3/2})\right],\end{aligned} \tag{8.2.3}$$

where $T^{(i)}$ denotes a polynomial containing i-th order terms only. The coefficients of the polynomials are rational functions of the moments of Z_1 up to order 5.
We still need to estimate the second factor in (8.2.2). For that, we use Corollary 8.3 stated below. Using the notations of that Corollary, we obtain

$$\left| P_{t_\xi}^{*n}(x) - n^{-d/2} \sum_{r=0}^{3} n^{-r/2} P_r(-\phi_{0,\Delta_\xi} : \{\chi_\nu\})((x-n\xi)/\sqrt{n}) \right| = o(n^{-(d+3)/2}). \tag{8.2.4}$$

Note that the constant on the right-hand side of (8.2.4) needs to be independent of ξ which is a priori not guaranteed by Corollary 8.3. However, calculating the constants in the proof of that Corollary given by Bhattacharya and Rao in [2], it can be shown that they only depend on the moments of Z_1 up to order 5.
Now, in the above so called Edgeworth polynomials $P_r(-\phi_{0,\Delta_\xi} : \{\chi_\nu\})((x-n\xi)/\sqrt{n})$, the coefficients of P_r depend on the moments of P_{t_ξ} up to order 5, and since $\xi = x/n$, only $P_r(.)(0)$ appear in (8.2.4). But $P_0(.)(0)$ is the centered normal density with covariance matrix Δ_ξ itself. Thus, Taylor expansion around κ yields $P_0(.)(0) = \frac{1}{\sqrt{2\pi}^d |\Delta|^{1/2}} + T^{(2)}(\xi-\kappa) + O((\xi-\kappa)^{(4)})$, and $P_2(.)(0) = \frac{K}{\sqrt{2\pi}^d |\Delta|^{1/2}} + O((\xi-\kappa)^{(2)})$. Moreover, P_1 and P_3 vanish at zero because the odd derivatives of a centered normal density do so. K and the error term depend only on the moments of Z_1 up to order 5. Thus, (8.2.4) simplifies to

$$\begin{aligned} P_{t_\xi}(x) =& \frac{1}{\sqrt{2\pi n}^d |\Delta|^{1/2}} \Big[1 + T^{(2)}(\xi-\kappa) + O((\xi-\kappa)^{(4)}) + n^{-1} T^{(0)}(\xi-\kappa) \\ & + O(n^{-1}(\xi-\kappa)^{(2)}) + o(n^{-3/2}) \Big] \\ =& \frac{1}{\sqrt{2\pi n}^d |\Delta|^{1/2}} \left[1 + n^{-1} T^{(2)}\left(\frac{x-n\kappa}{\sqrt{n}}\right) + n^{-1} T^{(0)}\left(\frac{x-n\kappa}{\sqrt{n}}\right) + O(n^{-3/2}) \right]. \end{aligned} \tag{8.2.5}$$

Inserting (8.2.3) and (8.2.5) into (8.2.2) yields the desired estimate whenever σ' is chosen large enough (depending on Δ and thus on an upper bound for η) and $\|\xi - \kappa\| \le n^{-5/12}$ with $\xi \in R_P$. Note that the constants only depend on the first five moments of Z_1 and therefore only on η since the π_i's and the π_{ij}'s as well as the d_i's and the d_{ij}'s which are also involved in the definition of the distribution of Z_1 are contained in a bounded compact set.
It remains to check the case $\|\xi - \kappa\| \ge n^{-5/12}$. We estimate $p_n(x)$ and ϕ_n separately. For ϕ_n, since $n^{1/2} \le ((x-n\kappa)/\sqrt{n})^6$, and $\|x\|^k \exp(-x^2) \le K \exp(-x^2/\sqrt{2})$ for any fixed $k \in \mathbb{N}$ and for all $x \in \mathbb{Z}^d$, we have

$$\phi_n(x) \le K n^{-1/2} \theta_n(x), \tag{8.2.6}$$

for $\sigma' > 0$ large enough depending on Δ and thus again on an upper bound for η. For $p_n(x)$, we note that I is strictly convex with $I(\kappa) = 0$ a global minimum. Thus, we may bound $I(\xi)$ away from zero by $I(\xi) \ge \frac{1}{c}\|\xi - \kappa\| = \frac{1}{c}\frac{\|x-n\kappa\|}{n}$, for some $c > 0$. c will again depend on an upper bound for η. Thus, with (8.2.2), we obtain

$$p_n(x) \le \exp\left(-\frac{n}{c}\frac{\|x-n\kappa\|}{n}\right) \le \exp\left(-\frac{1}{c}\frac{\|x-n\kappa\|}{\sqrt{n}}\right).$$

Choosing $\sigma' > 0$ large enough and using the same argument as for (8.2.6), we obtain the desired estimate in the latter case. This finishes the proof. □

The following Corollary is from Bhattacharya and Rao [2].

Corollary 8.3. *Let $(X_j)_{j\ge1}$ be a sequence of iid lattice random vectors with values in $\mathbb{R}^k$. Assume that $E[X_1] = \mu$ and that $Cov(X_1) = TT^t = V$, where T is a nonsingular $k \times k$-matrix,* *and $P[X_1 \in \mathbb{Z}^k] = 1$, $\mathbb{Z}^k$ being also the minimal lattice of X_1. If $\rho_s := E[\|X_1 - \mu\|^s] < \infty$ for some integer $s \geq 2$, then*

$$\sup_{\alpha\in\mathbb{Z}^k} (1 + \|y_{\alpha,n}\|^s) |p_n(y_{\alpha,n}) - q_{n,s}(y_{\alpha,n})| = o\left(n^{-(k+s-2)/2}\right),$$
$$\sum_{\alpha\in\mathbb{Z}^k} |p_n(y_{\alpha,n}) - q_{n,s}(y_{\alpha,n})| = o\left(n^{-(s-2)/2}\right) \quad (n \to \infty),$$

where

$$\begin{aligned} y_{\alpha,n} =& n^{-1/2}(\alpha - n\mu), \qquad p_n(y_{\alpha,n}) = P(X_1 + \ldots + X_n = \alpha) \quad (\alpha \in \mathbb{Z}^k), \\ =& P\left(n^{-1/2}\sum_{j=1}^{n}(X_j - \mu) = y_{\alpha,n}\right), \end{aligned}$$

$$q_{n,s} = n^{-k/2}\sum_{r=0}^{s-2} n^{-r/2} P_r(-\phi_{0,V} : \{\chi_\nu\}),$$

χ_ν being the ν-th cumulant of X_1 and $P_r(-\phi_{0,V} : \{\chi_\nu\})$ the r-th Edgeworth polynomial (see also [2] for a definition of these polynomials). In particular, $P_0(-\phi_{0,V} : \{\chi_\nu\}) = \phi_{0,V}$.

Note that in our case, $X_1 = Z_1$, $V = \Delta$, $\mu = \kappa$ and all cumulants χ_ν exist.
Now consider a function f on $\mathbb{Z}^d$. We define the forward difference of f in direction i, $i = 1, \ldots, d$, by $\triangle_i f(x) := f(x + e_i) - f(x)$, where e_i is the i-th unit vector in canonical coordinates. We also denote $\triangle_{ij} f(x) := \triangle_i(\triangle_j f)(x)$, for $i, j = 1, \ldots, d$, and similarly for higher order differences (note that $\triangle_{ij} = \triangle_{ji}$). We get the following Lemma for forward differences of $p_t(x)$:

Lemma 8.4. *For all $x \in \mathbb{Z}^d$ and all $t > 0$, we have the following estimates:*

$$|\triangle_i p_t(x)| \leq \frac{K}{\sqrt{t}}\theta_t(x) \quad \text{for } i = 1, \ldots, d \tag{8.2.7}$$
$$|\triangle_{ij} p_t(x)| \leq \frac{K}{t}\theta_t(x) \quad \text{for } i, j = 1, \ldots, d \tag{8.2.8}$$
$$|\triangle_{ijk} p_t(x)| \leq \frac{K}{t^{3/2}}\theta_t(x) \quad \text{for } i, j, k = 1, \ldots, d, \tag{8.2.9}$$

where $\theta_t(x) = \theta_{t\kappa,t\sigma'}(x)$. $\sigma' > 0$ and $K > 0$ have to be chosen large enough.

Proof. For (8.2.7), recall from Lemma 8.2 that $|p_t(x) - \phi_t(x)| \leq \frac{K}{\sqrt{t}}\theta_t(x)$. This immediately implies that for $i \in \{1, \ldots, d\}$,

$$|p_t(x + e_i) - p_t(x)| \leq \int_0^1 |\phi_t^{(i)}(x + y_i e_i)| dy_i + \frac{K}{\sqrt{t}}\theta_t(x + e_i) + \frac{K}{\sqrt{t}}\theta_t(x) \leq \frac{K}{\sqrt{t}}\theta_t(x).$$

Here, $\phi_t^{(i)}$ is the partial derivative of ϕ_t in the direction of e_i.
For (8.2.8) and (8.2.9), we first note the following relation valid for all $x \in \mathbb{Z}^d$, $t > 0$ and $i = 1, \ldots, d$:

$$\begin{aligned} \triangle_i p_t(x) =& \triangle_i\left(\sum_{y\in\mathbb{Z}^d} p_{t/2}(y)p_{t/2}(x - y)\right) = \sum_y p_{t/2}(y)\triangle_i p_{t/2}(x - y) \\ =& \sum_y \triangle_i p_{t/2}(y)p_{t/2}(x - y). \end{aligned}$$

Using this relation, we obtain for $i, j \in \{1, \ldots, d\}$,

$$|\triangle_{ij} p_t(x)| = |\sum_y \triangle_i p_{t/2}(y) \triangle_j p_{t/2}(x-y)| \le \frac{K}{t} \sum_y \theta_{t/2}(y)\theta_{t/2}(x-y) \le \frac{K}{t}\theta_t(x),$$

where we use Lemma B.1 for the last inequality and of course (8.2.7). This proves (8.2.8) and a similar procedure yields (8.2.9). □

We also need a "diffusion equation", ie. an expression for the time derivative of $p_t(x)$ in terms of the forward differences of $p_t(x)$:

Lemma 8.5. *We have*

$$\frac{1}{\eta}\frac{\partial}{\partial t} p_t(x) = \sum_{i=1}^{d} \left(d_i \pi_i + \sum_{j:j<i} \frac{d_{ji}}{2} + \sum_{j:j>i} \frac{d_{ij}}{2} \right) \triangle_{ii} p_t(x) - \sum_{i=1}^{d} d_i (2\pi_i - 1) \triangle_i p_t(x)$$
$$+ \sum_{1 \le i < j \le d} d_{ij}(2\pi_{ij} - 1)\triangle_{ij} p_t(x) + E(p,t,x),$$

$E(p,t,x)$ being the error term. For this error term the following estimate holds:

$$|E(p,t,x)| \le \frac{K}{t^{3/2}} \theta_{t\kappa, t\sigma'}(x),$$

where $K > 0$ and $\sigma' > 0$ have to be chosen large enough.

Proof. We have

$$\frac{\partial p_t}{\partial t}(x) = \sum_{n\ge 0} \frac{\partial}{\partial t}\left(e^{-\eta t}\frac{(\eta t)^n}{n!}\right) P[S_n = x] = \eta \sum_{n \ge 0}\left(-e^{-\eta t}\frac{(\eta t)^n}{n!} + e^{-\eta t} n \frac{(\eta t)^{n-1}}{n!}\right) P[S_n = x]$$
$$= \eta \sum_{n\ge 0} e^{-\eta t}\frac{(\eta t)^n}{n!} P[S_{n+1} = x] - \eta p_t(x)$$
$$= \eta \sum_{n\ge 0} e^{-\eta t}\frac{(\eta t)^n}{n!} \left\{ \sum_{i=1}^{d} d_i \left(P[S_n = x - e_i]\pi_i + P[S_n = x + e_i](1-\pi_i) - p_t(x) \right) \right.$$
$$+ \sum_{1\le i<j\le d} \frac{d_{ij}}{2} \left(\pi_{ij}(P[S_n = x - e_i - e_j] + P[S_n = x + e_i + e_j]) \right.$$
$$\left. + (1-\pi_{ij})(P[S_n = x - e_i + e_j] + P[S_n = x + e_i - e_j]) - 2p_t(x)) \right\}$$
$$= \eta \sum_{i=1}^{d} d_i (\pi_i p_t(x - e_i) + (1-\pi_i) p_t(x + e_i) - p_t(x)) \tag{8.2.10}$$
$$+ \eta \sum_{1\le i<j\le d} \frac{d_{ij}}{2} \left[\pi_{ij}(p_t(x - e_i - e_j) + p_t(x + e_i + e_j)) \right.$$
$$\left. + (1-\pi_{ij})(p_t(x - e_i + e_j) + p_t(x + e_i - e_j)) - 2p_t(x) \right]. \tag{8.2.11}$$

Hence, for the first sum over the i's we have:

$$\begin{aligned}\frac{1}{\eta}(8.2.10) =& \sum_{i=1}^{d}(1-\pi_i)d_i\triangle_{ii}p_t(x-e_i) + 2\sum_{i=1}^{d}(1-\pi_i)d_ip_t(x) - \sum_{i=1}^{d}(1-\pi_i)d_ip_t(x-e_i)\\ &+ \sum_{i=1}^{d}\pi_id_ip_t(x-e_i) - \sum_{i=1}^{d}d_ip_t(x)\\ =& \sum_{i=1}^{d}(1-\pi_i)d_i\triangle_{ii}p_t(x-e_i) - \sum_{i=1}^{d}(2\pi_i-1)d_i\triangle_ip_t(x-e_i)\\ =& \sum_{i=1}^{d}d_i(\pi_i\triangle_{ii}p_t(x) - (2\pi_i-1)\triangle_ip_t(x)) + \sum_{i=1}^{d}E_1(p,t,x,i),\end{aligned}$$

with $E_1(p,t,x,i) = -\pi_id_i\triangle_{iii}p_t(x-e_i)$. Using Lemma 8.4, we can bound this error by $\frac{K}{t^{3/2}}\theta_t(x)$. It remains to check the second summand (8.2.11): Note that for any $1 \le i < j \le d$, we have

$$\triangle_{ij}p_t(x-e_j) = p_t(x+e_i) - p_t(x) - p_t(x-e_j+e_i) + p_t(x-e_j),$$

and

$$\triangle_{ij}p_t(x-e_i) = p_t(x+e_j) - p_t(x-e_i+e_j) - p_t(x) + p_t(x-e_i).$$

Moreover,

$$\triangle_{ij}p_t(x) = p_t(x+e_i+e_j) - p_t(x+e_j) - p_t(x+e_i) + p_t(x),$$

and

$$\triangle_{ij}p_t(x-e_i-e_j) = p_t(x) - p_t(x-e_i) - p_t(x-e_j) + p_t(x-e_i-e_j).$$

Thus, in (8.2.11), we have for each summand:

$$\begin{aligned}&\pi_{ij}(p_t(x-e_i-e_j) + p_t(x+e_i+e_j))\\ &+ (1-\pi_{ij})(p_t(x-e_i+e_j) + p_t(x+e_i-e_j)) - 2p_t(x)\\ =&\pi_{ij}(\triangle_{ij}p_t(x) + p_t(x+e_j) + p_t(x+e_i) + \triangle_{ij}p_t(x-e_i-e_j) + p_t(x-e_i)\dots\\ &\dots + p_t(x-e_j) - 2p_t(x))\\ &+ (1-\pi_{ij})(-\triangle_{ij}p_t(x-e_i) + p_t(x+e_j) + p_t(x-e_i) - \triangle_{ij}p_t(x-e_j) + p_t(x+e_i)\dots\\ &\dots + p_t(x-e_j) - 2p_t(x)) - 2p_t(x)\\ =&\pi_{ij}(\triangle_{ij}p_t(x) + \triangle_{ij}p_t(x-e_i) + \triangle_{ij}p_t(x-e_j) + \triangle_{ij}p_t(x-e_i-e_j))\\ &- \triangle_{ij}p_t(x-e_i) - \triangle_{ij}p_t(x-e_j) + p_t(x-e_j) + p_t(x+e_j) + p_t(x-e_i) + p_t(x+e_i)\\ &- 4p_t(x).\end{aligned}$$

The last expression is equal to

$$\begin{aligned}=&4\pi_{ij}\triangle_{ij}p_t(x) - 2\triangle_{ij}p_t(x) + \triangle_{ii}p_t(x) + \triangle_{jj}p_t(x) + E_2(p,t,x,i,j)\\ =&2(2\pi_{ij}-1)\triangle_{ij}p_t(x) + \triangle_{ii}p_t(x) + \triangle_{jj}p_t(x) + E_2(p,t,x,i,j).\end{aligned}$$

As for the error E_1, we can write this error in terms of 3rd order forward differences and thus $|E_2(p,t,x,i,j)| \le \frac{K}{t^{3/2}}\theta_t(x)$ for all $1 \le i < j \le d$. Together with the calculation on the term (8.2.10), this finishes the proof of the Lemma by setting $E(p,t,x) := \sum_{i=1}^{d}E_1(p,t,x,i) + \sum_{i<j}\frac{d_{ij}}{2}E_2(p_t,t,x,i,j)$. □

Finally, we need to establish a "discrete Taylor Theorem" for $p_t(x+y)$, $x,y \in \mathbb{Z}^d$, $t>0$, yielding a development of $p_t(x+y)$ around x in a forward differences series. We also need to estimate the error terms in these series. Let us first consider the case $d=1$. In this case, $p_t(x+y)$ may be written in the following way:

$$p_t(x+y) = p_t(x) + y v_t(x,y) \tag{8.2.12}$$
$$= p_t(x) + y \triangle p_t(x) + y(y-1)\triangle v_t(x,y) \tag{8.2.13}$$
$$= p_t(x) + y \triangle p_t(x) + \frac{y(y-1)}{2}\triangle^2 p_t(x) + \frac{y(y-1)(y-2)}{2}\triangle^2 v_t(x,y), \tag{8.2.14}$$

where for the error terms we have:

$$v_t(x,y) = \frac{p_t(x+y) - p_t(x)}{y} \tag{8.2.15}$$

if $y \neq 0$, and $v_t(x,x) \equiv 0$. Moreover,

$$\begin{aligned}\triangle v_t(x,y) =& \frac{p_t(x+y) - p_t(x+1)}{y-1} - \frac{p_t(x+y) - p_t(x)}{y} \\ =& \frac{p_t(x+y) - p_t(x) - y(p_t(x+1) - p_t(x))}{y(y-1)},\end{aligned} \tag{8.2.16}$$

if $y \notin \{0,1\}$ ($v_t(x,y) \equiv 0$ in that case) and

$$\begin{aligned}\triangle^2 v_t(x,y) =& \frac{p_t(x+y) - p_t(x+2)}{y-2} - 2\frac{p_t(x+y) - p_t(x+1)}{y-1} + \frac{p_t(x+y) - p_t(x)}{y} \\ =& \frac{1}{y(y-1)(y-2)}\left(y(y-1)(p_t(x+y) - p_t(x+2)) \dots\right. \\ & \left.\dots - 2y(y-2)(p_t(x+y) - p_t(x+1)) + (y-2)(y-1)(p_t(x+y) - p_t(x))\right),\end{aligned} \tag{8.2.17}$$

if $y \notin \{0,1,2\}$ ($v_t(x,y) \equiv 0$ again in that case). These developments can be found in Boole [3]. Note that in the above, $\triangle$ acts on the first coordinate of $v_t(.,q) = \frac{p_t(q) - p_t(.)}{q - .}$.
We turn to the estimate of the error terms above. In the following, we freely use Lemma 8.4. For (8.2.15) we have (again with $\theta_{t\kappa, t\sigma'} =: \theta_t$):

$$\begin{aligned}|v_t(x,y)| =& \left|\frac{1}{y}\sum_{i=0}^{y-1}\triangle p_t(x+i)\right| \le \frac{1}{y}\frac{K}{\sqrt{t}}\sum_{i=0}^{y-1}\theta_t(x+i) \\ \le& \frac{K}{\sqrt{t}y}\int_0^y \theta_t(x+i)di \le \frac{K}{\sqrt{t}}\int_0^1 \theta_t(x+iy)di.\end{aligned} \tag{8.2.18}$$

For the error (8.2.16), we may write

$$\begin{aligned}& p_t(x+y) - p_t(x) - y(p_t(x+1) - p_t(x)) \\ =& \sum_{i=0}^{y-1}(p_t(x+i+1) - p_t(x+i) - (p_t(x+1) - p_t(x))) \\ =& \sum_{j=0}^{y-2}\sum_{i=0}^{j}\triangle^2 p_t(x+i).\end{aligned}$$

Using this, we obtain:

$$\begin{aligned}|\triangle v_t(x,y)| \le& \left|\frac{1}{y(y-1)}\sum_{j=0}^{y-2}\sum_{i=0}^{j}\triangle^2 p_t(x+i)\right| \\ \le& \frac{1}{y(y-1)}\int_0^{y-1}dj\int_0^j di\frac{K}{t}\theta_t(x+i) \le \frac{K}{t}\int_0^1(1-i)\theta_t(x+iy)di.\end{aligned} \tag{8.2.19}$$

Finally, for the last error (8.2.17), we obtain similarly:

$$
\begin{aligned}
|\triangle^2 v_t(x,y)| =& \left| \frac{2}{y(y-1)(y-2)} \sum_{j=0}^{y-3} \sum_{l=1}^{y-2-j} \sum_{i=0}^{j} \triangle^3 p_t(x+i) \right| \\
\leq & \frac{2}{y(y-1)(y-2)} \int_0^{y-3} dj \int_1^{y-2-j} dl \int_0^j di \frac{K}{t^{3/2}} \theta_t(x+i) \\
\leq & \frac{K}{t^{3/2}} \int_0^1 dl (1-l)^2 \theta_t(x+ly). \qquad (8.2.20)
\end{aligned}
$$

For higher dimensions $d > 1$, one can apply the development in (8.2.12)–(8.2.14) iteratively to each coordinate of $p_t(x)$ to obtain:

$$
\begin{aligned}
p_t(x+y) =& p_t(x) + \sum_{i=1}^{d} y_i v_t^{(i)}(x,y) \qquad (8.2.21)\\
=& p_t(x) + \sum_{i=1}^{d} y_i \triangle_i p_t(x) + \sum_{i=1}^{d} y_i(y_i-1)\triangle_i v_t^{(i)}(x,y) + \sum_{i<j} y_i y_j v_t^{(\triangle_i j)}(x,y) \qquad (8.2.22)\\
=& p_t(x) + \sum_{i=1}^{d} y_i \triangle_i p_t(x) + \sum_{i=1}^{d} \frac{y_i(y_i-1)}{2} \triangle_{ii} p_t(x) + \sum_{i<j} y_i y_j \triangle_{ij} p_t(x) \\
&+ \sum_{i=1}^{d} \frac{y_i(y_i-1)(y_i-2)}{2} \triangle_{ii} v_t^{(i)}(x,y) + \sum_{i<j} y_i y_j (y_j-1) \triangle_j v_t^{(\triangle_i j)}(x,y) \\
&+ \sum_{i<j} \frac{y_i(y_i-1)}{2} y_j v_t^{(\triangle_{ii} j)}(x,y) + \sum_{i<j<k} y_i y_j y_k v_t^{(\triangle_{ij} k)}(x,y). \qquad (8.2.23)
\end{aligned}
$$

Here, we have for the error terms:

$$
\begin{aligned}
|v_t^{(i)}(x,y)| =& \left| \frac{1}{y_i} \left(p_t(x_1,\dots,x_{i-1},x_i+y_i,\dots,x_d+y_d) - p_t(x_1,\dots,x_{i-1},x_i,\dots,x_d+y_d) \right) \right| \\
\leq & \frac{K}{\sqrt{t}} \sum_{i=1}^{d} \int_0^1 dl \theta_t(x_1,\dots,x_{i-1},x_i+ly_i,x_{i+1}+y_{i+1},\dots,x_d+y_d) \qquad (8.2.24)
\end{aligned}
$$

for the error terms in (8.2.21). For the error terms in (8.2.22), we have

$$
\begin{aligned}
\triangle_i v_t^{(i)}(x,y) = & \frac{1}{y_i(y_i-1)} \left(p_t(x_1,\dots,x_{i-1},x_i+y_i,\dots,x_d+y_d) \dots \right. \\
& \dots - p_t(x_1,\dots,x_i,x_{i+1}+y_{i+1},\dots,x_d+y_d) \dots \\
& \left. \dots - y_i(p_t(x_1,\dots,x_{i-1},x_i+1,x_{i+1}+y_{i+1},\dots,x_d+y_d) - p_t(x_1,\dots,x_i,\dots,x_d+y_d)) \right),
\end{aligned}
$$

and

$$
\begin{aligned}
v_t^{(\triangle_i j)}(x,y) = & \frac{1}{y_j} \left(\triangle_i p_t(x_1,\dots,x_{j-1},x_j+y_j,\dots,d_d+y_d) \dots \right. \\
& \left. \dots - \triangle_i p_t(x_1,\dots,x_j,x_{j+1}+y_{j+1},\dots,x_d+y_d) \right).
\end{aligned}
$$

Hence,

$$
|\triangle_i v_t^{(i)}(x,y)| \leq \frac{K}{t} \int_0^1 dl (1-l) \theta_t(x_1,\dots,x_{i-1},x_i+ly_i,x_{i+1}+y_{i+1},\dots,x_d+y_d), \qquad (8.2.25)
$$

and

$$|v_t^{(\triangle_i j)}(x,y)| \le \frac{K}{t}\int_0^1 dl\theta_t(x_1,\dots,x_{j-1},x_j+ly_j,x_{j+1}+y_{j+1},x_d+y_d). \tag{8.2.26}$$

Finally, reasoning as before and extending the notation for the errors from (8.2.21) and (8.2.22) in a natural way, we obtain for the error terms in (8.2.23):

$$|\triangle_{ii}v_t^{(i)}(x,y)| \le \frac{K}{t^{3/2}}\int_0^1 dl(1-l)^2\theta_t(x_1,\dots,x_{i-1},x_i+ly_i,x_{i+1}+y_{i+1},\dots,x_d+y_d), \tag{8.2.27}$$

$$|\triangle_j v_t^{(\triangle_i j)}(x,y)| \le \frac{K}{t^{3/2}}\int_0^1 dl(1-l)\theta_t(x_1,\dots,x_{j-1},x_j+ly_j,x_{j+1}+y_{j+1},\dots,x_d+y_d), \tag{8.2.28}$$

$$|v_t^{(\triangle_{ii} j)}(x,y)| \le \frac{K}{t^{3/2}}\int_0^1 dl\theta_t(x_1,\dots,x_{j-1},x_j+ly_j,x_{j+1}+y_{j+1},\dots,x_d+y_d), \tag{8.2.29}$$

and

$$|v_t^{(\triangle_{ij} k)}(x,y)| \le \frac{K}{t^{3/2}}\int_0^1 dl\theta_t(x_1,\dots,x_{k-1},x_k+ly_k,x_{k+1}+y_{k+1},\dots,x_d+y_d). \tag{8.2.30}$$

8.3 Proof of the Main Theorem 8.1

We prove Theorem 8.1 by establishing a norm on the space of sequences $A=(A_n)_{n\ge 0}$ which turns this space into a Banach space. Then, we define a contraction operator on a subspace of this Banach space of sequences and use Banach fixed point Theorem. Finally, we show that the sequence defined in (8.1.4) is the limit point of a sequence of sequences given by the iterated application of the contraction operator applied to a certain initial point.
Let

$$\mathcal{W} := \left\{ G=(G_n)_{n\in\mathbb{N}_0} \,\middle|\, \sup_{n\ge 1,\, x\in\mathbb{Z}^d}\frac{|G_n(x)|}{\chi_n(x)} + \sup_{x\in\mathbb{Z}^d}|G_0(x)|<\infty,\ G_n \text{ a signed measure on } \mathbb{Z}^d \right\}, \tag{8.3.1}$$

where

$$\chi_n(x) := n^{-1/2}\theta_{n\kappa,n\sigma}(x) + n^{-d/2}\sum_{j=1}^{n/2} j^2\exp\left(-\frac{\sqrt{n-k}}{\sqrt{\sigma}}\|\kappa\|\right)\theta_{j\kappa,j\sigma}(x).$$

$\sigma>0$ is to be determined later (this is the same σ as in Theorem 8.1). On this space, we define the obvious norm $\|G\|_{1/2} := \sup_{n\ge 1,\,x\in\mathbb{Z}^d}|G_n(x)|/\chi_n(x) + \sup_{x\in\mathbb{Z}^d}|G_0(x)|$. Then, $(\mathcal{W},\|.\|_{1/2})$ is a Banach space. Now consider $F:=(F_n)_{n\in\mathbb{N}_0}$ a sequence of probability distributions on $\mathbb{Z}^d$ with $F_0=\delta_0$, and define the operator ψ_F acting on elements of $\mathcal{W}$ by $\psi_F(\xi)_0 := \xi_0$, and for $n\ge 1$:

$$\psi_F(\xi)_n := F_n\xi_0 + \sum_{l=1}^{n}\xi_{n-l}\left[(1-\lambda\rho)SF_{l-1} - F_l + \lambda\sum_{m=1}^{l}a_m B_m F_{l-m}\right], \tag{8.3.2}$$

for any $\xi\in\mathcal{W}$. The sequence F we will use from now on is:

$$F_k := \left(1-\frac{k}{N}\right)S^k + \frac{k}{N}p_k, \quad \text{for } k\le N,$$
$$F_k := p_k, \quad \text{for } k>N,$$

where $N\in\mathbb{N}$ has to be chosen large enough and will be determined later. S^k stands for the k-fold convolution of S with itself. The distribution of p_t, $t\ge 0$, will be chosen such that its mean is exactly $t\kappa$ and its covariance $t\Delta$ at time t, where κ and Δ are given in (8.1.2) and (8.1.5). This is why we have to choose $S \in A_{N,\epsilon}$. Moreover, this also poses restrictions on the size of λ as we will see in (8.3.25)–(8.3.27). To simplify notations, we set $\psi := \psi_F$. We now show that ψ is a contraction on the subspace $\mathcal{W}_0 := \{\xi \in \mathcal{W} | \, \xi_0 = 0\} \subset \mathcal{W}$.

Lemma 8.6. *Let $\xi \in \mathcal{W}_0$. Then, for N big enough and λ small enough (depending on N), there exists $\epsilon \in (0, 1)$ with*

$$\|\psi(\xi)\|_{1/2} \leq \epsilon \|\xi\|_{1/2}.$$

Recall that K denotes a positive constant, possibly changing from line to line, and depending only on d, on S and on an upper bound for η. But due to the choice of S and because $\delta_{ij} \in [(s^{(ij)} - s^{(i)}s^{(j)})/2, 2(s^{(ij)} - s^{(i)}s^{(j)})]$, we obtain that η is bounded by $2R$ and K depends on d and R only.

Proof. Let us do some preliminary calculations around p_l: Using (8.2.23) we have

$$\begin{aligned} Sp_{l-1}(x) = \sum_{y \in \mathbb{Z}^d} S(y) p_{l-1}(x - y) = p_{l-1}(x) - \sum_{i=1}^{d} s^{(i)} \triangle_i p_{l-1}(x) \\ + \frac{1}{2} \sum_{i,j=1}^{d} s^{(ij)} \triangle_{ij} p_{l-1}(x) + \frac{1}{2} \sum_{i=1}^{d} s^{(i)} \triangle_{ii} p_{l-1}(x) \\ + \sum_{y} S(y) E^p(x, y, l-1), \end{aligned} \tag{8.3.3}$$

where

$$\begin{aligned} E^p(x, y, l-1) = \sum_{i=1}^{d} \frac{-y_i(y_i+1)(y_i+2)}{2} \triangle_{ii} v_{l-1}^{(i)}(x, -y) + \sum_{i<j} y_i y_j (-y_j - 1) \triangle_j v_{l-1}^{(\triangle_i j)}(x, -y) \\ + \sum_{i<j} \frac{y_i(-y_i - 1)}{2} y_j v_{l-1}^{(\triangle_{ii} j)}(x, -y) + \sum_{i<j<k} (-y_i y_j y_k) v_{l-1}^{(\triangle_{ij} k)}(x, -y). \end{aligned} \tag{8.3.4}$$

But the support of S is a bounded set around 0. Therefore, we can bound the error in (8.3.3) simply by

$$|\sum_{y} S(y) E^p(x, y, l-1)| \leq \frac{K}{t^{3/2}} \theta_{t\kappa, t\sigma'}(x), \tag{8.3.5}$$

using (8.2.27)–(8.2.30) and choosing K and σ' large enough. From Lemma 8.5, we also have:

$$\begin{aligned} \dot{p}_{l-1}(x) =& \eta \sum_{i=1}^{d} \left(d_i \pi_i + \sum_{j:j<i} \frac{d_{ji}}{2} + \sum_{j:j>i} \frac{d_{ij}}{2} \right) \triangle_{ii} p_{l-1}(x) - \eta \sum_{i=1}^{d} d_i (2\pi_i - 1) \triangle_i p_{l-1}(x) \\ &+ \eta \sum_{1 \leq i < j \leq d} d_{ij} (2\pi_{ij} - 1) \triangle_{ij} p_{l-1}(x) + E(p, l-1, x), \end{aligned}$$

with $|E(p, l-1, x)| \leq K l^{-3/2} \theta_{l\kappa, l\sigma'}(x)$, and one more application of that Lemma together with Lemma 8.4 yield:

$$\ddot{p}_{l-1}(x) = \eta^2 \sum_{i=1}^{d} d_i^2 (2\pi_i - 1)^2 \triangle_{ii} p_{l-1}(x) + \eta^2 \sum_{i \neq j} d_i d_j (2\pi_i - 1)(2\pi_j - 1) \triangle_{ij} p_{l-1}(x) + E^2(p, l-1, x), \tag{8.3.6}$$

where for the error E^2 we again have $|E^2(p,l-1,x)| \le \frac{K}{l^{3/2}}\theta_{l\kappa,l\sigma'}(x)$. Hence, we get from Taylor expansion of p_l in time:

$$\begin{aligned} p_l(x) =& p_{l-1}(x) - \eta\sum_{i=1}^{d} d_i(2\pi_i-1)\triangle_i p_{l-1}(x) + \eta\sum_{i=1}^{d}\left(d_i\pi_i + \sum_{j:j<i}\frac{d_{ji}}{2} + \sum_{j:j>i}\frac{d_{ij}}{2}\right)\triangle_{ii}p_{l-1}(x) \\ &+ \eta\sum_{1\le i<j\le d} d_{ij}(2\pi_{ij}-1)\triangle_{ij}p_t(x) + \frac{\eta^2}{2}\sum_{i=1}^{d} d_i^2(2\pi_i-1)^2\triangle_{ii}p_{l-1}(x) \\ &+ \eta^2\sum_{1\le i<j\le d} d_i d_j(2\pi_i-1)(2\pi_j-1)\triangle_{ij}p_{l-1}(x) + E^{time}(p,l-1,x), \end{aligned} \tag{8.3.7}$$

with $|E^{time}(p,l-1,x)| = |E(p,l-1,x) + E^2(p,l-1,x) + \dddot{p}_{l-\xi}(x)| \le \frac{K}{l^{3/2}}\theta_{l\kappa,l\sigma'}(x)$, since ξ is in $[0,1]$ and $\dddot{p}_{l-\xi}(x)$ is bounded using Lemmas 8.4 and 8.5.
Similarly, for $l/2 \le l-m < l-1$,

$$\begin{aligned} p_{l-m}(x) =& p_{l-1}(x) + (m-1)\eta\sum_{i=1}^{d} d_i(2\pi_i-1)\triangle_i p_{l-1}(x) \\ &- (m-1)\eta\sum_{i=1}^{d}\left(d_i\pi_i + \sum_{j:j<i}\frac{d_{ji}}{2} + \sum_{j:j>i}\frac{d_{ij}}{2}\right)\triangle_{ii}p_{l-1}(x) \\ &- (m-1)\eta\sum_{1\le i<j\le d} d_{ij}(2\pi_{ij}-1)\triangle_{ij}p_t(x) + \frac{(m-1)^2}{2}\eta^2\sum_{i=1}^{d} d_i^2(2\pi_i-1)^2\triangle_{ii}p_{l-1}(x) \\ &+ (m-1)^2\eta^2\sum_{1\le i<j\le d} d_i d_j(2\pi_i-1)(2\pi_j-1)\triangle_{ij}p_{l-1}(x) \\ &- (m-1)E(p,l-1,x) + (m-1)^2E^2(p,l-1,x) + (m-1)^3\dddot{p}_{l-\xi}(x), \end{aligned} \tag{8.3.8}$$

where $l-\xi \in [l/2, l-1]$.
Finally, we have for $q, r = 1,\dots,d$,

$$\begin{aligned} B_m p_{l-1}(x) =& b_m p_{l-1}(x) - \sum_{i=1}^{d} b_m^{(i)}\triangle_i p_{l-1}(x) + \frac{1}{2}\sum_{i,j=1}^{d} b_m^{(ij)}\triangle_{ij}p_{l-1}(x) \\ &+ \sum_{i=1}^{d}\frac{b_m^{(i)}}{2}\triangle_{ii}p_{l-1}(x) + \sum_y B_m(y)E^p(x,y,l-1), \end{aligned} \tag{8.3.9}$$

$$B_m\triangle_q p_{l-1}(x) = b_m\triangle_q p_{l-1}(x) - \sum_{i=1}^{d} b_m^{(i)}\triangle_{qi}p_{l-1}(x) + \sum_y B_m(y)E^{\triangle_q p}(x,y,l-1), \tag{8.3.10}$$

$$B_m\triangle_{qr}p_{l-1}(x) = b_m\triangle_{qr}p_{l-1}(x) + \sum_y B_m(y)E^{\triangle_{qr}p}(x,y,l-1), \tag{8.3.11}$$

where $E^p(x,y,l-1)$ is given in (8.3.4),

$$E^{\triangle_q p}(x,y,l-1) = \sum_i y_i(y_i+1)\triangle_i v_{l-1}^{(\triangle_q i)}(x,-y) + \sum_{i<j} y_i y_j v_{l-1}^{(\triangle_q ij)} v_{l-1}(x,-y),$$

and

$$E^{\triangle_{qr}p}(x,y,l-1) = -\sum_i y_i v_{l-1}^{(\triangle_{qr} i)}(x,-y).$$

Using these equations together with (8.3.8), we get

$$\begin{aligned}
B_m p_{l-m}(x) =& b_m p_{l-1}(x) + \sum_{i=1}^{d} \triangle_i p_{l-1}(x) \left(-b_m^{(i)} + b_m(m-1)\eta d_i(2\pi_i - 1)\right) \\
&+ \sum_{i=1}^{d} \triangle_{ii} p_{l-1}(x) \left(\frac{b_m^{(ii)}}{2} + \frac{b_m^{(i)}}{2} - (m-1)\eta d_i b_m^{(i)}(2\pi_i - 1)\right. \\
&\left. -(m-1)b_m\eta \left(d_i\pi_i + \sum_{j:j<i}\frac{d_{ji}}{2} + \sum_{j:j>i}\frac{d_{ij}}{2}\right) + \frac{(m-1)^2}{2}\eta^2 d_i^2 b_m(2\pi_i - 1)^2\right) \\
&+ \sum_{1\le i<j\le d} \triangle_{ij} p_{l-1}(x) \left(b_m^{(ij)} - (m-1)\eta b_m^{(i)} d_j(2\pi_j - 1) - (m-1)\eta b_m^{(j)} d_i(2\pi_i - 1)\right. \\
&\left. + b_m(m-1)^2\eta^2 d_i d_j(2\pi_i - 1)(2\pi_j - 1) - b_m(m-1)\eta d_{ij}(2\pi_{ij} - 1)\right) \\
&+ \sum_y B_m(y) E(m,p,x,y,l-1). \qquad (8.3.12)
\end{aligned}$$

Here,

$$\begin{aligned}
E(m,p,x,y,l-1) =& E^p(x,y,l-1) + (m-1)\eta \sum_{i=1}^{d} d_i(2\pi_i - 1) E^{\triangle_i p}(x,y,l-1) \qquad (8.3.13) \\
&+ \sum_{i=1}^{d}\left(-(m-1)\eta\left(d_i\pi_i + \sum_{j:j<i}\frac{d_{ji}}{2} + \sum_{j:j>i}\frac{d_{ij}}{2}\right) + \frac{(m-1)^2}{2}\eta^2 d_i^2(2\pi_i - 1)^2\right) E^{\triangle_{ii} p}(x,y,l-1) \\
&- \sum_{i<j}\left((m-1)\eta d_{ij}(2\pi_{ij} - 1) - (m-1)^2\eta^2 d_i d_j(2\pi_i - 1)(2\pi_j - 1)\right) E^{\triangle_{ij} p}(x,y,l-1) \\
&- (m-1)E(p,l-1,x-y) + (m-1)^2 E^2(p,l-1,x-y) + (m-1)^3 \dddot{p}_{l-\xi}(x-y),
\end{aligned}$$

where $l-\xi \in [l/2, l-1]$, and the error terms are collected from the above calculations (8.3.8)–(8.3.11). Note that there are two types of errors in the above formula. $E^p(x,y,l-1)$, $E^{\triangle_i p}(x,y,l-1)$, $E^{\triangle_{ii} p}(x,y,l-1)$ and $E^{\triangle_{ij} p}(x,y,l-1)$ are coming from the discrete Taylor development of $p_{l-1}(x-y)$ and its discrete derivatives, whereas the errors on the last line come from the Taylor development of $p_{l-m}(x-y)$ in time.
We want to show that

$$\sum_y |B_m(y)E(m,p,x,y,l-1)| \le \frac{K}{m^{3/2}l^{3/2}}\theta_{l\kappa,l\sigma}(x) + \frac{K}{m^{1/2}l^2}\theta_{l\kappa,l\sigma}(x) \qquad (8.3.14)$$

for some $\sigma > 0$ large enough in order that $\sum_{m=1}^{l/2}\sum_y |B_m(y)E(m,p,x,y,l-1)| \le \frac{K}{l^{3/2}}\theta_{l\kappa,l\sigma}(x)$. We check all terms in (8.3.13) separately, freely making reference to (8.2.21)–(8.2.30). The first term, $E^p(x,y,l-1)$, can be bounded by:

$$\frac{K}{l^{3/2}}\sum_{i\le j\le q}|y_i y_j y_q|\int_0^1 ds\,\theta_{l\kappa,l\sigma'}(x_1,\dots,x_{q-1},x_q - sy_q, x_{q+1} - y_{q+1},\dots,x_d - y_d). \qquad (8.3.15)$$

This has to be folded with B_m. But now recall the bound (8.1.3) for B_m and note that $|y_i y_j y_q|\theta_{k\kappa,k\sigma}(y) = |(y_i - k\kappa^{(i)} + k\kappa^{(i)})(y_j - k\kappa^{(j)} + k\kappa^{(j)})(y_q - k\kappa^{(q)} + k\kappa^{(q)})|\theta_{k\kappa,k\sigma}(y) \le$

$K(k^3\|\kappa\|^3 + k^{5/2}\|\kappa\|^2 + k^2\|\kappa\| + k^{3/2})\theta_{k\kappa,k\sqrt{2}\sigma}(y)$. Thus, folding B_m with (8.3.15) we get:

$$\begin{aligned}
&\sum_y |B_m(y)E^p(x,y,l-1)| \\
\le&\frac{K}{l^{3/2}}\sum_{i\le j\le q}|y_iy_jy_q|\int_0^1 ds\sum_y B_m(y)\theta_{l\kappa,l\sigma'}(x_1,\dots,x_{q-1},x_q-sy_q,x_{q+1}-y_{q+1},\dots,x_d-y_d)| \\
\le&\frac{K}{l^{3/2}}\sum_q\int_0^1 dsm^{-d/2}\sum_{k=1}^{m/2}(k^3\|\kappa\|^3+k^{5/2}\|\kappa\|^2+k^2\|\kappa\|+k^{3/2})\exp\left(-\frac{\sqrt{m-k}}{\sqrt{\sigma}}\|\kappa\|\right) \\
&\cdot\sum_y\theta_{k\kappa,\sqrt{2}k\sigma}(y)\theta_{l\kappa,l\sigma'}(x_1,\dots,x_{q-1},x_q-sy_q,x_{q+1}-y_{q+1},\dots,x_d-y_d).
\end{aligned}$$

We want to fold each of the d coordinates separately in the last line above (using Lemma B.1). This can be done by multiplying the variances by 2 and bounding the above by a multiplication of corresponding one-dimensional independent doubly-exponential distributions, since $\frac{1}{\sqrt{\sqrt{2}k\sigma}}\|y\| \ge \frac{1}{\sqrt{2\sqrt{2}k\sigma}}(|y_1|+\dots+|y_d|)$ and similarly for $\theta_{l\kappa,l\sigma'}(.)$. Special care has to be taken when folding the q-th coordinate with the integral over s. For that coordinate we have (now assuming that the θ's are one-dimensional):

$$\begin{aligned}
&\int_0^1 ds\sum_{y_q\in\mathbb{Z}}\theta_{k\kappa_q,2\sqrt{2}k\sigma}(y_q)\theta_{l\kappa_q,2l\sigma'}(x_q-sy_q) \\
\le&K\int_0^1 dss^{-1}\theta_{k\kappa_q+l\kappa_q/s,2\sqrt{2}k\sigma+2l\sigma'/s^2}(x_q/s) \\
\le&K\int_0^1 ds\theta_{sk\kappa_q+l\kappa_q,2\sqrt{2}s^2k\sigma+2l\sigma'}(x_q) \\
\le&K\int_0^1 ds\theta_{sk\kappa_q+l\kappa_q,2\sqrt{2}k\sigma+2l\sigma'}(x_q) \\
\le&K\int_0^1 ds\theta_{sk\kappa_q+l\kappa_q,l\sigma}(x_q),
\end{aligned}$$

where we use that $k\le m/2\le l/4$ and set $\sigma\ge\sigma'\frac{2}{1-\sqrt{2}/2}$. Now note that $\frac{sk|\kappa_q|}{\sqrt{l\sigma}}\le\frac{\sqrt{k}|\kappa_q|}{\sqrt{\sigma}}\frac{\sqrt{k}}{\sqrt{l}}\le\frac{\sqrt{m}|\kappa_q|}{2\sqrt{2\sigma}}$. Thus, using the subadditivity of any norm,

$$\begin{aligned}
&\exp\left(-\frac{\sqrt{m-k}}{\sqrt{2\sigma}}|\kappa_q|\right)\int_0^1 ds\theta_{sk\kappa_q+l\kappa_q,l\sigma}(x_q) \\
\le&\exp\left(-\frac{\sqrt{m}|\kappa_q|}{2\sqrt{\sigma}}(1-1/\sqrt{2})\right)\theta_{l\kappa_q,l\sigma}(x_q).
\end{aligned}$$

Together with much simpler calculations for the remaining coordinates (we do not have to integrate over $[0,1]$!) and again with the subadditivity of the norm we end up with:

$$\begin{aligned}
&\sum_y|B_m(y)E^p(x,y,l-1)| \\
\le&\frac{K}{l^{3/2}}m^{-d/2}\exp\left(-\frac{\sqrt{m}}{2\sqrt{\sigma}}\|\kappa\|(1-1/\sqrt{2})\right)(m^4\|\kappa\|^3+m^{7/2}\|\kappa\|^2+m^3\|\kappa\|+m^{5/2})\theta_{l\kappa,l\sigma}(x) \\
\le&\frac{K}{m^{3/2}l^{3/2}}\theta_{l\kappa,l\sigma}(x),
\end{aligned}$$

since $d \geq 9$. Thus (8.3.14) is satisfied. This finishes the calculation for the first term in the error (8.3.13). We turn to the second term in that error. Recall from the beginning of this Section that by choice of the distribution of p_t, $\eta d_i(2\pi_i - 1) = \kappa^{(i)}$ for all $i = 1, \ldots, d$. Therefore, the second term in the error can be bounded by terms of the form

$$m\|\kappa\|\frac{K}{l^{3/2}}\sum_{i\leq j}|y_iy_j|\int_0^1 ds\sum_y|B_m(y)|\theta_{l\kappa,l\sigma'}(x_1,\ldots,x_{j-1},x_j-sy_j,x_{j+1}-y_{j+1},\ldots,x_d-y_d).$$

Similar considerations as for the first term again lead to the desired bound (8.3.14). Alike calculations lead to the desired bounds for all errors on the first three lines of (8.3.13).
We turn to the last line of (8.3.13). For the first summand on that line, we have $(m-1)E(p,l-1,x-y) \leq m\frac{K}{l^{3/2}}\theta_{l\kappa,l\sigma'}(x-y)$ by Lemma 8.5. Thus, folding with B_m, we have (again using Lemma B.1):

$$\begin{aligned}
&\left|\sum_y B_m(y)m\frac{K}{l^{3/2}}\theta_{l\kappa,l\sigma'}(x-y)\right|\\
\leq&\frac{K}{l^{3/2}}m^{1-d/2}\sum_{k=1}^{m/2}\exp\left(-\frac{\sqrt{m-k}}{\sqrt{\sigma}}\|\kappa\|\right)\sum_y\theta_{k\kappa,k\sigma}(y)\theta_{l\kappa,l\sigma'}(x-y)\\
\leq&\frac{K}{l^{3/2}}m^{1-d/2}\sum_{k=1}^{m/2}\exp\left(-\frac{\sqrt{m}}{2\sqrt{2\sigma}}\|\kappa\|\right)\theta_{l\kappa,l\sigma}(x)\\
\leq&\frac{K}{l^{3/2}}m^{2-d/2}\exp\left(-\frac{\sqrt{m}}{2\sqrt{2\sigma}}\|\kappa\|\right)\theta_{l\kappa,l\sigma}(x)\\
\leq&\frac{K}{l^{3/2}m^{3/2}}\theta_{l\kappa,l\sigma}(x),
\end{aligned}$$

since $d \geq 9$ and $\sigma \geq \sigma'\frac{2}{1-\sqrt{2}/2}$, and using again that $k \leq m/2 \leq l/4$ and the subadditivity of the norm. To handle the second term on the last line of (8.3.13), we need to look at the error term in (8.3.6). Analyzing this error term shows:

$$E^2(p,l-1,x-y) \leq K\left(\frac{1}{l^2}+\|\kappa\|\frac{1}{l^{3/2}}\right)\theta_{l\kappa,l\sigma'}(x-y).$$

Folding this bound with B_m, we obtain by similar arguments to the ones use for the first error on the last line of (8.3.13) that the second error on the last line is also bounded by (8.3.14). Finally, for the last error, we obtain by a still more careful analysis of the iterated time derivatives in Lemma 8.5:

$$|\dddot{p}_{l-\xi}(x-y)| \leq K\left(\frac{1}{l^3}+\|\kappa\|\frac{1}{l^{5/2}}+\|\kappa\|^2\frac{1}{l^2}+\|\kappa\|^3\frac{1}{l^{3/2}}\right)\theta_{l\kappa,l\sigma'}(x-y),$$

and folding this with B_m again leads to the good bound (8.3.14).
Therefore, we obtain for the error in (8.3.12),

$$|\sum_y B_m(y)E(m,p,x,y,l-1)| \leq \frac{K}{m^{3/2}l^{3/2}}\theta_{l\kappa,l\sigma}(x)+\frac{K}{m^{1/2}l^2}\theta_{l\kappa,l\sigma}(x), \qquad (8.3.16)$$

for all values of κ (in particular also for $\kappa = 0$), as long as $d \geq 9$. Moreover we set from now on $\sigma = \sigma'\frac{2}{1-\sqrt{2}/2}$.
With these preliminary calculations in hand, we turn to the main part of the proof.

What we need to show is $|\psi(\xi)_n| \leq \epsilon \|\xi\|_{1/2}\chi_n$, for all $n \in \mathbb{N}$ and some $\epsilon \in (0,1)$. For this we split (8.3.2) as follows:

$$\begin{aligned} |\psi(\xi)_n| \leq & \sum_{l=1}^{n} |\xi_{n-l}| * \left\| \left[(1-\lambda\rho)SF_{l-1} - F_l + \lambda \sum_{m=1}^{l/2} a_m B_m F_{l-m} \right] \right\| \\ & + \sum_{l=1}^{n} |\xi_{n-l}| * \left| \lambda \sum_{m=l/2}^{l} a_m B_m F_{l-m} \right| . \end{aligned} \tag{8.3.17}$$

We start with the second term. Note that $|\xi_{n-l}| \leq \|\xi\|_{1/2}\chi_{n-l}$ and split χ_{n-l} into $(n-l)^{-1/2}\theta_{n-l}$ and $(n-l)^{-d/2}\sum_{j=1}^{(n-l)/2} j^2 \exp\left(-\frac{\sqrt{n-l-j}}{\sqrt{\sigma}}\|\kappa\|\right)\theta_j$, where for simplicity, $\theta_k := \theta_{k\kappa,k\sigma}$ and $\theta'_k := \theta_{k\kappa,k\sigma'}$ in the following. For the first part this leads to

$$\begin{aligned} & \sum_{l=1}^{n-1}(n-l)^{-1/2}\theta_{n-l} * \sum_{m=l/2}^{l} a_m|B_m| * F_{l-m} \\ \leq & K\sum_{l=1}^{n-1}(n-l)^{-1/2} \sum_{m=l/2}^{l} m^{-d/2}\sum_{k=1}^{m/2} \exp\left(-\frac{\sqrt{m-k}}{\sqrt{\sigma}}\|\kappa\|\right) \underbrace{\theta_{n-l} * \theta_k * \theta'_{l-m}}_{\leq K\theta_{n-m+k}} \\ \leq & Kn^{-1/2}\theta_n \sum_{l=1}^{n/2}\sum_{m=l/2}^{l} m^{-d/2}\sum_{k=1}^{m/2} 1 \\ & + Kn^{-d/2}\sum_{l=n/2}^{n-1}(n-l)^{-1/2}\sum_{m=l/2}^{l}\sum_{k=1}^{m/2}\exp\left(-\sqrt{\frac{m-k}{\sigma}}\|\kappa\|\right)\theta_{n-m+k} \\ \leq & K\left[n^{-1/2}\theta_n + n^{-d/2}\sum_{m=n/4}^{n-1}\sum_{k=1}^{m/2}\exp\left(-\sqrt{\frac{m-k}{\sigma}}\|\kappa\|\right)\theta_{n-m+k}\underbrace{\sum_{l=m}^{n-1}(n-l)^{-1/2}}_{\leq n-m}\right] \\ \leq & K\left[n^{-1/2}\theta_n + n^{-d/2}\sum_{m=n/4}^{n-1}\sum_{k=n-m+1}^{n-m/2} k\exp\left(-\sqrt{\frac{n-k}{\sigma}}\|\kappa\|\right)\theta_k\right] \\ \leq & K\left[n^{-1/2}\theta_n + n^{-d/2}\sum_{k=1}^{7n/8} k\theta_k\exp\left(-\sqrt{\frac{n-k}{\sigma}}\|\kappa\|\right)\underbrace{\sum_{m=n-k+1}^{n-1}}_{\leq k}\right] \\ \leq & K\chi_n, \end{aligned}$$

where we use $\sigma > \sigma'$ and $d \geq 9$. For the second part, we split the sum and find

$$\begin{aligned}
&\sum_{l=1}^{n/2}(n-l)^{-d/2}\sum_{j=1}^{(n-l)/2} j^2 \exp\left(-\sqrt{\frac{n-l-j}{\sigma}}\|\kappa\|\right)\theta_j * \sum_{m=l/2}^{l} a_m|B_m| * F_{l-m}\\
\leq &Kn^{-d/2}\sum_{l=1}^{n/2}\sum_{j=1}^{(n-l)/2} j^2 \exp\left(-\sqrt{\frac{n-l-j}{\sigma}}\|\kappa\|\right)\\
&\cdot \sum_{m=l/2}^{l} m^{-d/2}\sum_{k=1}^{m/2}\exp\left(-\sqrt{\frac{m-k}{\sigma}}\|\kappa\|\right)\theta_{j+k+l-m}\\
\leq &Kn^{-d/2}\sum_{m=1}^{n/2} m^{-d/2}\\
&\cdot \sum_{l=m}^{(n/2)\wedge(2m)}\sum_{k=1}^{m/2}\sum_{j=1+k}^{(n-l)/2+k} j^2 \underbrace{\exp\left(-\sqrt{\frac{n-l-j+k}{\sigma}}\|\kappa\| - \sqrt{\frac{m-k}{\sigma}}\|\kappa\|\right)}_{\leq \exp\left(-\sqrt{\frac{n-l-j+m}{\sigma}}\|\kappa\|\right)}\theta_{j+l-m}\\
\leq &Kn^{-d/2}\sum_{m=1}^{n/2} m^{1-d/2}\sum_{l=m}^{(n/2)\wedge(2m)}\sum_{j=1+l-m}^{(n+l-m)/2} j^2 \exp\left(-\sqrt{\frac{n-j}{\sigma}}\|\kappa\|\right)\theta_j\\
\leq &Kn^{-d/2}\sum_{j=1}^{3n/4} j^2 \exp\left(-\sqrt{\frac{n-j}{\sigma}}\|\kappa\|\right)\theta_j\\
\leq &K\chi_n,
\end{aligned}$$

again using $d \geq 9$, and finally

$$\begin{aligned}
&\sum_{l=n/2}^{n-1}(n-l)^{-d/2}\sum_{j=1}^{(n-l)/2} \underbrace{j^2}_{\leq (n-l)^2} \exp\left(-\sqrt{\frac{n-l-j}{\sigma}}\|\kappa\|\right)\\
&\cdot \sum_{m=l/2}^{l} m^{-d/2}\sum_{k=1}^{m/2}\exp\left(-\sqrt{\frac{m-k}{\sigma}}\|\kappa\|\right)\theta_{j+k+l-m}\\
\leq &K\sum_{l=n/2}^{n-1}(n-l)^{2-d/2}\sum_{m=l/2}^{l} m^{-d/2}\sum_{k=1}^{m/2}\exp\left(-\sqrt{\frac{m-k}{\sigma}}\|\kappa\|\right)\\
&\cdot \sum_{j=1}^{(n-l)/2}\exp\left(-\sqrt{\frac{n-l-j}{\sigma}}\|\kappa\|\right)\theta_{j+k+l-m}\\
\leq &Kn^{-d/2}\sum_{l=n/2}^{n-1}(n-l)^{2-d/2}\\
&\cdot \sum_{m=0}^{l/2}\sum_{k=1}^{(l-m)/2}\sum_{j=1}^{(n-l)/2}\underbrace{\exp\left(-\sqrt{\frac{l-m-k}{\sigma}}\|\kappa\| - \sqrt{\frac{n-l-j}{\sigma}}\|\kappa\|\right)}_{\leq \exp\left(-\sqrt{\frac{n-m-k-j}{\sigma}}\|\kappa\|\right)}\theta_{j+k+m}
\end{aligned}$$

$$
\begin{aligned}
&\leq K n^{1-d/2} \sum_{l=n/2}^{n-1} (n-l)^{2-d/2} \sum_{m=0}^{l/2} \sum_{j=1+m}^{(n+m)/2} \exp\left(-\sqrt{\frac{n-j}{\sigma}} \|\kappa\|\right) \theta_j \\
&\leq K n^{1-d/2} \sum_{l=n/2}^{n-1} (n-l)^{2-d/2} \sum_{j=1}^{(2n+l)/4} \exp\left(-\sqrt{\frac{n-j}{\sigma}} \|\kappa\|\right) \theta_j \sum_{m=0}^{j-1} 1 \\
&\leq K n^{-d/2} \sum_{j=1}^{3n/4} j^2 \exp\left(-\sqrt{\frac{n-j}{\sigma}} \|\kappa\|\right) \theta_j \\
&\leq K \chi_n,
\end{aligned}
$$

where in the ninth line we use $d \geq 9$. Thus we get

$$
\text{second summand of (8.3.17)} \leq K\lambda \|\xi\|_{1/2} \chi_n,
$$

and it suffices to choose λ small enough.

It remains to check the first summand of (8.3.17). For this summand we have that it is equal to

$$
\begin{aligned}
&\sum_{l=1}^{(n-1)\wedge N} |\xi_{n-l}| * \Bigg| (1-\lambda\rho)(1-\frac{l-1}{N})S^l + (1-\lambda\rho)\frac{l-1}{N} S p_{l-1} - (1-\frac{l}{N})S^l - \frac{l}{N} p_l \dots \\
&\qquad \dots + \lambda \sum_{m=1}^{l/2} a_m B_m (1-\frac{l-m}{N}) S^{l-m} + \lambda \sum_{m=1}^{l/2} a_m B_m \frac{l-m}{N} p_{l-m} \Bigg| \qquad (8.3.18) \\
&+ \sum_{l=N+1}^{n-1} |\xi_{n-l}| * \Bigg| (1-\lambda\rho) S p_{l-1} - p_l + \lambda \sum_{m=1}^{(l/2)\wedge(l-N)} a_m B_m p_{l-m} \dots \\
&\qquad \dots + \lambda \sum_{m=l-N+1}^{l/2} a_m B_m \left((1-\frac{l-m}{N}) S^{l-m} + \frac{l-m}{N} p_{l-m} \right) \Bigg|. \qquad (8.3.19)
\end{aligned}
$$

For the moment, we are interested in the second sum which is present only if $n > N+1$. We have:

$$
\begin{aligned}
(8.3.19) \leq \sum_{l=N+1}^{n-1} \|\xi\|_{1/2} \chi_{n-l} * &\left(\left| (1-\lambda\rho) S p_{l-1} - p_l + \lambda \sum_{m=1}^{l/2} a_m B_m p_{l-m} \right| \right. \\
&\left. + \left| \lambda \sum_{m=l-N+1}^{l/2} a_m B_m \left((1-\frac{l-m}{N})(S^{l-m} - p_{l-m}) \right) \right| \right). \qquad (8.3.20)
\end{aligned}
$$

Now we use the calculations from (8.3.3) to (8.3.16) and collect terms coming with the same discrete derivatives to write:

$$
\begin{aligned}
&(1-\lambda\rho)Sp_{l-1} - p_l + \lambda\sum_{m=1}^{l/2} a_m B_m p_{l-m} \\
=&\left((1-\lambda\rho) - 1 + \lambda\sum_{m=1}^{l/2} a_m b_m\right) p_{l-1} && (8.3.21)\\
&+\sum_{i=1}^{d}\left(\eta d_i(2\pi_i-1) - (1-\lambda\rho)s^{(i)} - \lambda\sum_{m=1}^{l/2} a_m b_m^{(i)} + \lambda\eta d_i(2\pi_i-1)\sum_{m=1}^{l/2} a_m b_m(m-1)\right)\triangle_i p_{l-1} \\
& && (8.3.22)\\
&+\sum_{i=1}^{d}\left(\frac{(1-\lambda\rho)}{2}s^{(ii)} - \frac{\eta^2 d_i^2(2\pi_i-1)^2}{2} + \frac{\lambda}{2}\sum_{m=1}^{l/2} a_m b_m^{(ii)} - \lambda\eta d_i(2\pi_i-1)\sum_{m=1}^{l/2} a_m b_m^{(i)}(m-1)\right. \\
&+\frac{\lambda}{2}\eta^2 d_i^2(2\pi_i-1)^2\sum_{m=1}^{l/2} a_m b_m(m-1)^2 - \eta\left(d_i\pi_i + \sum_{j:j<i}\frac{d_{ji}}{2} + \sum_{j:j>i}\frac{d_{ij}}{2}\right) \\
&\left.-\lambda\eta\left(d_i\pi_i + \sum_{j:j<i}\frac{d_{ji}}{2} + \sum_{j:j>i}\frac{d_{ij}}{2}\right)\sum_{m=1}^{l/2} a_m b_m(m-1) + \frac{s^{(i)}}{2}(1-\lambda\rho) + \frac{\lambda}{2}\sum_{m=1}^{l/2} a_m b_m^{(i)}\right)\triangle_{ii} p_{l-1} \\
& && (8.3.23)\\
&+\sum_{1\le i<j\le d}\left((1-\lambda\rho)s^{(ij)} - \eta^2 d_i d_j(2\pi_i-1)(2\pi_j-1) - \eta d_{ij}(2\pi_{ij}-1) + \lambda\sum_{m=1}^{l/2} a_m b_m^{(ij)}\right. \\
&-\lambda\eta d_j(2\pi_j-1)\sum_{m=1}^{l/2} a_m b_m^{(i)}(m-1) - \lambda\eta d_i(2\pi_i-1)\sum_{m=1}^{l/2} a_m b_m^{(j)}(m-1) \\
&\left.+\lambda\eta^2 d_i d_j(2\pi_i-1)(2\pi_j-1)\sum_{m=1}^{l/2} a_m b_m(m-1)^2 - \lambda\eta d_{ij}(2\pi_{ij}-1)\sum_{m=1}^{l/2} a_m b_m(m-1)\right)\triangle_{ij} p_{l-1} \\
& && (8.3.24)\\
&+E(l)(.),
\end{aligned}
$$

where $|E(l)(.)| \le \frac{K}{l^{3/2}}\theta_{l\kappa,l\sigma}(.)$ due to (8.3.16) and the bounds on all other errors at the beginning of the proof. We analyze the terms above separately: For the term in front of p_{l-1} in (8.3.21), we have:

$$
\left|(1-\lambda\rho) - 1 + \lambda\sum_{m=1}^{l/2} a_m b_m\right| \le K\sum_{m=l/2}^{\infty}|b_m| = O(l^{-3/2}),
$$

using the definition of ρ in (8.1.1) and the decay rate of the b_m's. For the remaining terms, we ask $\eta \in [a, 2R]$, $\pi_i \in [0,1]$, $i = 1,\dots,d$, $\pi_{ij} \in [0,1]$, $1 \le i < j \le d$, $d_i \in [0,1]$, $i = 1,\dots,d$, and $d_{ij} \in [0,1]$, $1 \le i < j \le d$, to satisfy the following system of equations (recall that

$d_{ii} = d_i + \sum_{j:j<i} d_{ji} + \sum_{j:j>i} d_{ij}$):

$$\sum_{i=1}^{d} d_i + \sum_{1\le i<j\le d} d_{ij} = 1,$$

$$\kappa^{(i)} := \eta d_i(2\pi_i - 1) = \frac{u\mu^{-1}s^{(i)} + \lambda\sum_{m\ge1} a_m b_m^{(i)}}{1+\lambda\sum_{m\ge1} a_m b_m(m-1)}, \tag{8.3.25}$$

$$\delta_{ii} := \eta d_{ii} = \frac{1}{1+\lambda\sum_{m\ge1} a_m b_m(m-1)} \left(u\mu^{-1}s^{(ii)} + \lambda\sum_{m\ge1} a_m b_m^{(ii)} - \kappa^{(i)\,2} + \lambda\kappa^{(i)\,2}\sum_{m\ge1}(m-1)^2 a_m b_m - 2\lambda\kappa^{(i)}\sum_{m\ge1} a_m(m-1)b_m^{(i)} \right), \tag{8.3.26}$$

$$\delta_{ij} := \eta d_{ij}(2\pi_{ij} - 1) = \frac{1}{1+\lambda\sum_{m\ge1} a_m b_m(m-1)} \left(u\mu^{-1}s^{(ij)} + \lambda\sum_{m\ge1} a_m b_m^{(ij)} - \kappa^{(i)}\kappa^{(j)} + \lambda\kappa^{(i)}\kappa^{(j)}\sum_{m\ge1}(m-1)^2 a_m b_m - \lambda\kappa^{(i)}\sum_{m\ge1} a_m b_m^{(j)}(m-1) - \lambda\kappa^{(j)}\sum_{m\ge1} a_m b_m^{(i)}(m-1) \right). \tag{8.3.27}$$

Thus as already mentioned, the mean and the covariance of p_t should be exactly the (scaled) asymptotic mean and covariance of A_n/a_n. We now need $\lambda \ll 1$ in order that these equations can be satisfied. In fact, the above system of equations should be viewed as a perturbation of the same system with $\lambda = 0$. One has to first make sure that the system with $\lambda = 0$ has a solution. This will work only, if S is in $A_{N,\epsilon}$. Then, after fixing such an S, one may increase λ slightly in order to perturb that initial system of equations. In order that this is always possible, we took away an ϵ boundary from the set C (see Section 6.2).

Plugging the $\kappa^{(i)}$'s into (8.3.22), we get that the terms in front of the $\triangle_i p_{l-1}$'s are of order $l^{-3/2}$. Plugging the δ_i's and $\kappa^{(i)}$'s into the terms in front of (8.3.23), and using that for $i = 1, \dots, d$, the last four terms in that summand converge to $-\eta d_{ii}/2$ at rate $l^{-3/2}$, we have that for each summand, the term inside the bracket converges to 0 at rate $l^{-1/2}$. The same is true for (8.3.24). Thus, collecting the above and combining with the bounds on p_{l-1} and its discrete derivatives from Lemma 8.4, we have:

$$\left| (1-\lambda\rho)Sp_{l-1}(x) - p_l(x) + \lambda\sum_{m=1}^{l/2} a_m B_m p_{l-m}(x) \right| \le \frac{K}{l^{3/2}}\theta_{l\kappa,l\sigma}(x),$$

and hence the first summand in (8.3.20) can be bounded by

$$\begin{aligned}
&K\|\xi\|_{1/2}\sum_{l=N+1}^{n-1}\frac{1}{l^{3/2}}\chi_{n-l}*\theta_{l\kappa,l\sigma}\\
\leq &K\|\xi\|_{1/2}\sum_{l=N+1}^{n-1} l^{-3/2}\\
&\cdot\left[(n-l)^{-1/2}\theta_{n\kappa,n\sigma}+(n-l)^{-d/2}\sum_{j=1}^{(n-l)/2}\exp\left(-\sqrt{\frac{n-l-j}{\sigma}}\|\kappa\|\right)j^2\theta_{(l+j)\kappa,(l+j)\sigma}\right]\\
\leq &\|\xi\|_{1/2}\\
&\cdot\left[N^{-1/2}Kn^{-1/2}\theta_{n\kappa,n\sigma}+Kn^{-d/2}\sum_{l=N+1}^{n/2} l^{-3/2}\sum_{j=1}^{(n-l)/2}\exp\left(-\sqrt{\frac{n-l-j}{\sigma}}\|\kappa\|\right)j^2\theta_{(l+j)\kappa,(l+j)\sigma}\right.\\
&\left.+N^{-1/2}Kn^{-1}\sum_{l=(n/2)\vee(N+1)}^{n-1}(n-l)^{-d/2}\sum_{j=1}^{(n-l)/2}\exp\left(-\sqrt{\frac{n-l-j}{\sigma}}\|\kappa\|\right)j^2\theta_{(l+j)\kappa,(l+j)\sigma}\right]\\
\leq &C(N)K\|\xi\|_{1/2}\chi_n,
\end{aligned}$$

where $C(N)$ goes to zero when $N\to\infty$, and hence $C(N)K\leq\epsilon$, if N is large enough.

For the second summand in (8.3.20) we have:

$$\begin{aligned}
&\sum_{l=N+1}^{n-1}\|\xi\|_{1/2}\chi_{n-l}*\left|\lambda\sum_{m=l-N+1}^{l/2}a_mB_m\left((1-\frac{l-m}{N})(S^{l-m}-p_{l-m})\right)\right|\\
\leq&\lambda K\|\xi\|_{1/2}\sum_{l=N+1}^{n-1}(n-l)^{-1/2}\sum_{m=l-N+1}^{l/2}m^{-d/2}\sum_{k=1}^{m/2}\exp\left(-\sqrt{\frac{m-k}{\sigma}}\|\kappa\|\right)\theta_{k+n-m}\\
&+\lambda K\|\xi\|_{1/2}\sum_{l=N+1}^{n-1}(n-l)^{-d/2}\sum_{j=1}^{(n-l)/2}j^2\sum_{m=l-N+1}^{l/2}m^{-d/2}\\
&\cdot\sum_{k=1}^{m/2}\exp\left(-\sqrt{\frac{n-l+m-j-k}{\sigma}}\|\kappa\|\right)\theta_{k+j+l-m}.
\end{aligned}$$

We need to make a distinction again. Suppose at first that $n\geq 2N$ and note that the sum over m is empty as soon as $l>2(N-1)$. Then, it follows that $(n-l)^{-\alpha}\approx n^{-\alpha}$. Therefore, it is clear that the above can be bounded by $\epsilon\|\xi\|_{1/2}\chi_n$ if λ is chosen small enough (reasoning similarly as for the second part of (8.3.17)). On the other hand, if $n\in[N+2,3N]$, it suffices to choose $\lambda=N^{-k}e^{-sN}$ for some k and s large enough to get the desired bound. Thus, we end up with (8.3.19) $\leq\epsilon\|\xi\|_{1/2}\chi_n$.

We turn to the term (8.3.18). It can be rewritten in three sums as follows:

$$
\begin{aligned}
(8.3.18) \leq & \frac{1}{N}\|\xi\|_{1/2} \sum_{l=1}^{(n-1)\wedge N} \chi_{n-l} * \Bigg| (1-\lambda\rho)S^l - (1-\lambda\rho)Sp_{l-1} + \lambda \sum_{m=1}^{l/2} a_m B_m m S^{l-m} \\
& -\lambda \sum_{m=1}^{l/2} a_m B_m m p_{l-m} \Bigg| && (8.3.28)\\
& + \|\xi\|_{1/2} \sum_{l=1}^{(n-1)\wedge N} \chi_{n-l} * \Bigg| -\lambda\rho S^l + \lambda \sum_{m=1}^{l/2} a_m B_m S^{l-m} \Bigg| && (8.3.29)\\
& + \|\xi\|_{1/2} \sum_{l=1}^{(n-1)\wedge N} \chi_{n-l} * \frac{l}{N} \Bigg| \lambda\rho S^l + (1-\lambda\rho)Sp_{l-1} - p_l - \lambda \sum_{m=1}^{l/2} a_m B_m S^{l-m} \\
& + \lambda \sum_{m=1}^{l/2} a_m B_m p_{l-m} \Bigg| . && (8.3.30)
\end{aligned}
$$

The three sums have to be treated separately. Let us first consider the last one (8.3.30). We have:

$$
\begin{aligned}
(8.3.30) \leq & \|\xi\|_{1/2} \sum_{l=1}^{(n-1)\wedge N} \chi_{n-l} * \frac{l}{N} \lambda \Bigg| \rho S^l - \sum_{m=1}^{l/2} a_m B_m S^{l-m} \Bigg| \\
& + \|\xi\|_{1/2} \sum_{l=1}^{(n-1)\wedge N} \chi_{n-l} * \frac{l}{N} \Bigg| (1-\lambda\rho)Sp_{l-1} - p_l + \lambda \sum_{m=1}^{l/2} a_m B_m p_{l-m} \Bigg| .
\end{aligned}
$$

Using again the calculations from (8.3.3) to (8.3.16) we can regroup the terms on the second line as in (8.3.21)–(8.3.24) to get

$$
\begin{aligned}
(8.3.30) \leq & \|\xi\|_{1/2} \sum_{l=1}^{(n-1)\wedge N} \chi_{n-l} * \frac{l}{N} \left(\lambda \Bigg| \rho S^l - \sum_{m=1}^{l/2} a_m B_m S^{l-m} \Bigg| + K l^{-3/2} \theta_{l\kappa, l\sigma} \right) && (8.3.31)\\
\leq & \frac{K}{N}\|\xi\|_{1/2} \sum_{l=1}^{(n-1)\wedge N} l^{-1/2} \chi_{n-l} * \theta_l + \frac{\lambda}{N}\|\xi\|_{1/2} \sum_{l=1}^{(n-1)\wedge N} l \chi_{n-l} * \Bigg| \rho S^l - \sum_{m=1}^{l/2} a_m B_m S^{l-m} \Bigg| .
\end{aligned}
$$

The first term above is easily bounded by $\epsilon\|\xi\|_{1/2}\chi_n$ by considering the two cases $n \leq N+1$ and $n > N+1$ separately and by noting that in the first case, $\sum_{l=1}^{n-1} l^{-1/2}(n-l)^{-1/2}$ is of order one, whereas in the latter case the same sum is of order $n^{-1/2}\sqrt{N}$ when folding with the first part of χ_{n-l} (ie. with $(n-l)^{-1/2}\theta_{n-l}$), and by splitting the sum in two cases with $n \geq 2N$ and $n < 2N$ for the folding with the second part of χ_{n-l} (ie. with $(n-l)^{-d/2}\sum_{j=1}^{(n-l)/2} j^2 \exp(-\frac{\sqrt{n-l-j}}{\sqrt{\sigma}}\|\kappa\|)\theta_j$).

For the second term, we get

$$\frac{\lambda}{N}\|\xi\|_{1/2}\sum_{l=1}^{(n-1)\wedge N} l\chi_{n-l} * \left|\rho S^l - \sum_{m=1}^{l/2} a_m B_m S^{l-m}\right| \tag{8.3.32}$$

$$\leq K\frac{\lambda}{N}\|\xi\|_{1/2}\sum_{l=1}^{(n-1)\wedge N} l\chi_{n-l} * \theta_l \tag{8.3.33}$$

$$+ K\frac{\lambda}{N}\|\xi\|_{1/2}\sum_{l=1}^{(n-1)\wedge N} l\sum_{m=1}^{l/2} \chi_{n-l} * |B_m| * \theta_{l-m} \tag{8.3.34}$$

$$\leq \epsilon\|\xi\|_{1/2}\chi_n,$$

where (8.3.33) is bounded using Lemma 8.7 and choosing λ small enough for the folding with the first part of χ_{n-l} and again by splitting into the cases $n \geq 2N$ and $n < 2N$ for the folding with the second part of χ_{n-l}. (8.3.34) is similar to the second term in (8.3.17). Thus, we get that (8.3.30) $\leq \epsilon\|\xi\|_{1/2}n^{-1/2}$.
Let us look at the second term (8.3.29). This one is essentially equivalent to (8.3.32) and using again Lemma 8.7, we obtain (8.3.29) $\leq \epsilon\|\xi\|_{1/2}\chi_n$.
Finally, for the first term (8.3.28), we have

$$(8.3.28) \leq \frac{K}{N}\|\xi\|_{1/2}\sum_{l=1}^{(n-1)\wedge N} \chi_{n-l} * |S^{l-1} - p_{l-1}|$$

$$+ 2\|\xi\|_{1/2}\frac{\lambda}{N}\sum_{l=1}^{(n-1)\wedge N} \chi_{n-l} * \sum_{m=1}^{l/2} a_m m|B_m| * \theta_{l-m}$$

$$\leq K\frac{1}{N}\|\xi\|_{1/2}\sum_{l=1}^{(n-1)\wedge N} \chi_{n-l} * |S^{l-1} - p_{l-1}| + \epsilon\|\xi\|_{1/2}\chi_n, \tag{8.3.35}$$

where we have to choose N large enough and λ small enough and the second term is dealt with similarly as the term (8.3.34). For the first summand in (8.3.35), we are using the fact that $|S^l - p_l| \leq |S^l - \tilde{\phi}_l| + |\phi_l - p_l| + |\phi_l - \tilde{\phi}_l|$, where $\tilde{\phi}_l$ is a normal density with mean vector $l(s^{(1)}, \ldots, s^{(d)})$ and covariance matrix $l\mathrm{Cov}(S)$, and ϕ_l is a normal density with mean $l\kappa$ and covariance matrix $l\Delta = l(\delta_{ij})_{i,j=1}^d$. Using Lemma 8.2 and an argument similar to the one in the proof of that Lemma for the difference $|S^l - \tilde{\phi}_l|$, we get that $|S^l - \tilde{\phi}_l| + |\phi_l - p_l| \leq \frac{K}{l^{-1/2}}\theta_l$. Moreover, since $(\kappa, \Delta) \to ((s^{(1)}, \ldots, s^{(d)}), \mathrm{Cov}(S))$ as $\lambda \to 0$, it follows that $|\phi_l - \tilde{\phi}_l| \leq C(\lambda)\theta_l$, where $C(\lambda)$ is a quantity tending to zero as $\lambda \to 0$. Thus,

$$K\frac{1}{N}\|\xi\|_{1/2}\sum_{l=1}^{(n-1)\wedge N} \chi_{n-l} * |S^{l-1} - p_{l-1}|$$

$$\leq \frac{K}{N}\|\xi\|_{1/2}\sum_{l=1}^{(n-1)\wedge N} l^{-1/2}\chi_{n-l} * \theta_l + \frac{C(\lambda)}{N}\|\xi\|_{1/2}\sum_{l=1}^{(n-1)\wedge N} \chi_{n-l} * \theta_l \leq \epsilon\|\xi\|_{1/2}\chi_n,$$

where we argue for the two terms separately as for the second term in (8.3.31) and for the term in (8.3.33) (without l here!) respectively. This implies (8.3.28) $\leq \epsilon\|\xi\|_{1/2}\chi_n$, and the proof of Lemma 8.6 is finished. □

Lemma 8.7. *If N is large enough, then*

$$\frac{1}{N}\sum_{l=1}^{(n-1)\wedge N} (n-l)^{-1/2} \leq 4n^{-1/2}.$$

Proof. If $n \geq N+1$, then

$$\frac{1}{N}\sum_{l=1}^{N}(n-l)^{-1/2} \leq \frac{1}{N}\int_{n-N-1}^{n} x^{-1/2}dx = \frac{2}{N}[n^{1/2}-(n-N-1)^{1/2}]$$
$$= 2\frac{N+1}{N}n^{-1/2}\left[\frac{n}{N+1}-\left(\frac{n}{N+1}\right)^{1/2}\left(\frac{n}{N+1}-1\right)^{1/2}\right].$$

If N is large enough, we have $2\frac{N+1}{N} \leq 4$, and since for $t \geq 1$,

$$t\left(1-\left(1-\frac{1}{t}\right)^{1/2}\right) \leq 1,$$

we get the desired bound in this case.
If $n \leq N$, we simply get

$$\frac{1}{N}\sum_{l=1}^{n-1}(n-l)^{-1/2} \leq \frac{2}{N}n^{1/2} \leq 2n^{-1/2}.$$

□

We also need the following Lemma to prove Theorem 8.1:

Lemma 8.8. *Let $\xi_n = a_n F_n$. Then,*

$$\|\psi(\xi)-\xi\|_{1/2} \leq K,$$

where K is a positive constant depending on R and on the constant N.

Proof. First of all, we note that we can re-write $\psi(\xi)_n$ given in (8.3.2) in the following way:

$$\psi(\xi)_n = \xi_n - \sum_{l=1}^{n} F_{n-l}\left[\xi_l - (1-\lambda\rho)S\xi_{l-1} - \lambda\sum_{m=1}^{l} a_m B_m \xi_{l-m}\right]. \tag{8.3.36}$$

Thus, it suffices to show that

$$\sum_{l=1}^{n}\left|F_{n-l}\left[a_lF_l - (1-\lambda\rho)Sa_{l-1}F_{l-1} - \lambda\sum_{m=1}^{l} a_m a_{l-m}B_mF_{l-m}\right]\right| \leq K\chi_n. \tag{8.3.37}$$

Now it is obvious that it suffices to consider the case $F_{n-l} = p_{n-l}$ and $F_l = p_l$ since for the finite number of remaining cases it follows immediately that if $l \leq N$, $|F_{n-l}F_l| \leq Kp_n$, using the fact that S^l has compact support for $l \leq N$ and vice versa if $n-l \leq N$. Then the left hand side in (8.3.37) is bounded by

$$\sum_{l=1}^{n}\left|a_lF_n - (1-\lambda\rho)Sa_{l-1}F_{n-1} - \lambda\sum_{m=1}^{l\wedge(n/2)} a_m a_{l-m}B_mF_{n-m}\right|$$
$$+\lambda\sum_{l=n/2}^{n}\sum_{m=n/2}^{l}|a_m a_{l-m}B_mF_{n-m}|. \tag{8.3.38}$$

For the second term in (8.3.38) we get:

$$
\begin{aligned}
&\lambda \sum_{l=n/2}^{n} \sum_{m=n/2}^{l} |a_m a_{l-m} B_m F_{n-m}| \\
\leq &\lambda K \sum_{l=n/2}^{n} \sum_{m=n/2}^{l} m^{-d/2} \sum_{k=1}^{m/2} \exp\left(-\sqrt{\frac{m-k}{\sigma}}\|\kappa\|\right) \theta_{k+n-m} \\
\leq &\lambda K n^{-d/2} \sum_{m=n/2}^{n} (n-m) \sum_{k=1+n-m}^{n-m/2} \exp\left(-\sqrt{\frac{n-k}{\sigma}}\|\kappa\|\right) \theta_k \\
\leq &\lambda K n^{-d/2} \sum_{k=1}^{3n/4} \theta_k \exp\left(-\sqrt{\frac{n-k}{\sigma}}\|\kappa\|\right) \underbrace{\sum_{m=(n/2)\vee(n-k+1)}^{n} k}_{\leq k^2} \\
\leq &\lambda K \chi_n.
\end{aligned}
$$

It remains to check the first term in (8.3.38). Writing $\tilde{d}_i = \pi_i d_i + \sum_{j:j<i} \frac{d_{ji}}{2} + \sum_{j:j>i} \frac{d_{ij}}{2}$, that term is given by

$$
\begin{aligned}
&\sum_{l=1}^{n} \left| a_l p_n - (1-\lambda\rho) a_{l-1} S p_{n-1} - \lambda \sum_{m=1}^{l\wedge(n/2)} a_m a_{l-m} B_m p_{n-m} \right| \qquad (8.3.39)\\
\leq &\sum_{l=1}^{n} \left| a_l - (1-\lambda\rho) a_{l-1} - \lambda \sum_{m=1}^{l\wedge(n/2)} a_{l-m} a_m b_m \right| |p_{n-1}| \\
&+ \sum_{l=1}^{n} \sum_{i=1}^{d} \Bigg| (1-\lambda\rho) s^{(i)} a_{l-1} + \lambda \sum_{m=1}^{l\wedge(n/2)} a_{l-m} a_m b_m^{(i)} - \eta d_i (2\pi_i - 1) a_l \\
&-\lambda \eta d_i (2\pi_i - 1) \sum_{m=1}^{l\wedge(n/2)} (m-1) a_{l-m} a_m b_m \Bigg| |\triangle_i p_{n-1}| \\
&+ \sum_{l=1}^{n} \sum_{i=1}^{d} \Bigg| \frac{\eta^2 d_i^2 (2\pi_i - 1)^2}{2} a_l - \frac{1-\lambda\rho}{2} s^{(ii)} a_{l-1} - \frac{\lambda}{2} \eta^2 d_i^2 (2\pi_i - 1)^2 \sum_{m=1}^{l\wedge(n/2)} (m-1)^2 a_m a_{l-m} b_m \\
&+ \lambda \eta d_i (2\pi_i - 1) \sum_{m=1}^{l\wedge(n/2)} a_m a_{l-m} (m-1) b_m^{(i)} - \frac{\lambda}{2} \sum_{m=1}^{l\wedge(n/2)} a_{l-m} a_m b_m^{(ii)} \\
&+ \eta \tilde{d}_i a_l + \lambda \eta \tilde{d}_i \sum_{m=1}^{l\wedge(n/2)} a_m a_{l-m} b_m (m-1) - \frac{s^{(i)}}{2} (1-\lambda\rho) a_{l-1} - \frac{\lambda}{2} \sum_{m=1}^{l\wedge(n/2)} a_m a_{l-m} b_m^{(i)} \Bigg| |\triangle_{ii} p_{n-1}|
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{l=1}^{n}\sum_{1\le i<j\le d}\Big|\eta^2 d_i d_j(2\pi_i-1)(2\pi_j-1)a_l-(1-\lambda\rho)s^{(ij)}a_{l-1}+\eta d_{ij}(2\pi_{ij}-1)a_l\\
&-\lambda\sum_{m=1}^{l\wedge(n/2)}a_m a_{l-m}b_m^{(ij)}+\lambda\eta d_j(2\pi_j-1)\sum_{m=1}^{l\wedge(n/2)}a_m a_{l-m}b_m^{(i)}(m-1)\\
&+\lambda\eta d_i(2\pi_i-1)\sum_{m=1}^{l\wedge(n/2)}a_m a_{l-m}b_m^{(j)}(m-1)-\lambda\eta^2 d_i d_j(2\pi_i-1)(2\pi_j-1)\sum_{m=1}^{l\wedge(n/2)}a_m a_{l-m}b_m(m-1)^2\\
&+\lambda\eta d_{ij}(2\pi_{ij}-1)\sum_{m=1}^{l\wedge(n/2)}a_m a_{l-m}b_m(m-1)\Big|\,|\triangle_{ij}p_{l-1}|+E(n)(.),
\end{aligned}
$$

where we use the calculations from (8.3.3) to (8.3.16) in the proof of Lemma 8.6 and regroup terms according to their discrete derivative in the place variable as in that Lemma. Similarly to those calculations we have $E(n)(.)\le \frac{K}{n^{1/2}}\theta_n(.)$. We check the remaining terms above separately. For the term in front of $|p_{n-1}|$, we get

$$
\left|a_l-(1-\lambda\rho)a_{l-1}-\lambda\sum_{m=1}^{l\wedge(n/2)}a_{l-m}a_m b_m\right|\le\lambda\sum_{m=n/2}^{l}a_m a_{l-m}b_m\le Kn^{-3/2},
$$

using the decay of the b_m's, (6.3.2) and Proposition 7.1. Thus, $\sum_{l=1}^{n}|.||p_{n-1}(x)|\le Kn^{-1/2}\theta_n(x)$. For the terms in front of the $|\triangle_i p_{n-1}|$'s, $i=1,\dots,d$, we use the fact that $a_l\to\alpha$ when $l\to\infty$, for some $\alpha>0$, and $|a_l-\alpha|\le Kl^{-3/2}$, which follows directly from Corollary 7.5 with $d\ge 9$, to get

$$
\begin{aligned}
&\left|(1-\lambda\rho)s^{(i)}a_{l-1}+\lambda\sum_{m=1}^{l\wedge(n/2)}a_{l-m}a_m b_m^{(i)}-\eta d_i(2\pi_i-1)a_l-\lambda\eta d_i(2\pi_i-1)\sum_{m=1}^{l\wedge(n/2)}(m-1)a_{l-m}a_m b_m\right|\\
&\le\alpha\left|(1-\lambda\rho)s^{(i)}+\lambda\sum_{m=1}^{l\wedge(n/2)}a_m b_m^{(i)}-\eta d_i(2\pi_i-1)-\lambda\eta d_i(2\pi_i-1)\sum_{m=1}^{l\wedge(n/2)}(m-1)a_m b_m\right|+Kl^{-3/2}\\
&\le Kl^{-3/2},
\end{aligned}
$$

where the last line follows from the definition of η, d_i and π_i in (8.3.25), and the decay rates for the moments of the B_m's. Thus, $\sum_{l=1}^{n}\sum_{i=1}^{d}|.||\triangle_i p_{n-1}(x)|\le Kn^{-1/2}\theta_n$, because $|\triangle_i p_{n-1}(x)|\le \frac{K}{\sqrt{n}}\theta_n(x)$, for $i=1,\dots,d$ (see Lemma 8.4). By the same Lemma, $|\triangle_{ij}p_{n-1}(x)|\le\frac{K}{n}\theta_n(x)$ for $i,j=1,\dots,d$, and using the same considerations for the terms in front of the $|\triangle_{ii}p_{n-1}(x)|$'s as for the terms in front of the $|\triangle_i p_{n-1}(x)|$'s we obtain for any $i=1,\dots,d$:

$$
\begin{aligned}
&\Big|\frac{\eta^2 d_i^2(2\pi_i-1)^2}{2}a_l-\frac{1-\lambda\rho}{2}s^{(ii)}a_{l-1}-\frac{\lambda}{2}\eta^2 d_i^2(2\pi_i-1)^2\sum_{m=1}^{l\wedge(n/2)}(m-1)^2 a_m a_{l-m}b_m\\
&+\lambda\eta d_i(2\pi_i-1)\sum_{m=1}^{l\wedge(n/2)}a_m a_{l-m}(m-1)b_m^{(i)}-\frac{\lambda}{2}\sum_{m=1}^{l\wedge(n/2)}a_{l-m}a_m b_m^{(ii)}\\
&+\eta\tilde{d}_i a_l+\lambda\eta\tilde{d}_i\sum_{m=1}^{l\wedge(n/2)}a_m a_{l-m}b_m(m-1)-\frac{s^{(i)}}{2}(1-\lambda\rho)a_{l-1}-\frac{\lambda}{2}\sum_{m=1}^{l\wedge(n/2)}a_m a_{l-m}b_m^{(i)}\Big|\,|\triangle_{ii}p_{n-1}(x)|\\
&\le Kl^{-1/2}n^{-1}\theta_n(x).
\end{aligned}
$$

Thus $\sum_{l=1}^n \sum_{i=1}^d |.||\triangle_{ii} p_{n-1}(x)| \le Kn^{-1}\theta_n(x) \sum_{l=1}^n l^{-1/2} \le Kn^{-1/2}\theta_n(x)$. With the same arguments, the terms in front of the $|\triangle_{ij} p_{n-1}(x)|$'s $(i<j)$ are also of order $l^{-1/2}$, implying that also $\sum_{l=1}^n \sum_{1\le i<j\le d} |.||\triangle_{ij} p_{n-1}(x)| \le Kn^{-1/2}\theta_n(x)$, and hence (8.3.39) is bounded by $Kn^{-1/2}\theta_n(x)$ and finally by $K\chi_n$ as desired. This finishes the proof of Lemma 8.8. □

Using Lemmas 8.6 and 8.8, we are now able to prove Theorem 8.1:

Proof. (Proof of Theorem 8.1).
Let $\lambda_0 > 0$ be such that for $\lambda \in [0, \lambda_0]$, Lemma 8.6 and Proposition 7.1 hold. Now, taking $(a_n)_{n\ge 0}$ from that Proposition and $(F_n)_{n\ge 0}$ from below (8.3.2) with κ and Δ from (8.1.2) and (8.1.5), we set $F := (a_n F_n)_{n\ge 0}$. Then, by Lemma 8.8, $\psi(F) - F \in \mathcal{W}_0$. Hence we may apply Banach fixed point Theorem to the sequence $(\psi^{(k)}(\psi(F) - F))_{k\ge 0}$ in the Banach space $(\mathcal{W}_0, \|.\|_{1/2})$ to find that it converges to the unique fixed point $(G_n)_{n\ge 0} \in \mathcal{W}_0$. But that fixed point is G with $G_n \equiv 0$, for all $n \ge 0$. Since ψ is linear this implies that the sequence $(\psi^{(k)}(F))_{k\ge 0}$ converges to a (unique) limit, say $(A_n)_{n\ge 0}$, satisfying

1. $A_0 = \delta_0$,
2. $\psi(A)_n(x) = A_n(x)$, for all $n \ge 0$ and $x \in \mathbb{Z}^d$, and
3. $\|F - A\|_{1/2} \le K$.

The last point follows because

$$\|F - A\|_{1/2} = \|\sum_{l\ge 1}(\psi^{(l)}(F) - \psi^{(l-1)}(F))\|_{1/2} \le \|\psi(F) - F\|_{1/2} \sum_{l\ge 1} \epsilon^l \le K,$$

using Lemma 8.6 in the first inequality and Lemma 8.8 in the last inequality. But clearly, $(A_n)_{n\ge 0}$ is the sequence defined in (8.1.4). Thus, $|A_n(x) - a_n F_n(x)| \le K\chi_n(x)$, for all $n \ge 1$, and using Lemma 8.2 we get $|a_n F_n(x) - a_n \phi_n(x)| \le Kn^{-1/2}\theta_n(x)$, for all $n \ge 1$. This implies Theorem 8.1 with the parameters λ, κ and Δ given above. □

8.4 The Two-Periodic Case

In case we start with an initial distribution $S \in P_{N,\epsilon}$ which is two-periodic, and also with a two-periodic sequence $(B_m)_{m\ge 1}$, we obtain the following variant of our main Theorem 8.1:

Theorem 8.9. *In the above setting, there exists $\lambda_0 > 0$, such that for any $\lambda \in [0, \lambda_0]$, we have that if n and $\|x\|_1$ have same parity:*

$$\left|\frac{A_n(x)}{a_n} - 2\phi_{n\kappa, n\Delta}(x)\right| \le K\left[n^{-1/2}\theta_{n\kappa, n\sigma}(x) + n^{-d/2}\sum_{j=1}^{n/2} j^2 \exp\left(-\frac{\sqrt{n-j}}{\sqrt{\sigma}}\|\kappa\|\right)\theta_{j\kappa, j\sigma}(x)\right],$$

where the parameters are as in Theorem 8.1.

The only change is that we have a factor 2 in front of the approximating normal density due to the parity issue. The proof is exactly the same as for the Theorem in the non-periodic case, except that we plug in $2p_t$ instead of p_t. This is in particular necessary at the end of the proof of Lemma 8.6 where instead of $|S^l - \tilde{\phi}_l|$, one plugs in $|S^l - 2\tilde{\phi}_l|$ in order to make the argument in Lemma 8.2 work. The reason for this is that the minimal lattice changes if one starts with a two-periodic distribution S (see Bhattacharya and Rao [2] for details).

Chapter 9

Application to Perturbed Weakly Self-Avoiding Walks

We come back to the specific context of the perturbed weakly self-avoiding random walks, where the main objects of study are the two-point functions C_n with total mass c_n. Recall that they satisfy the lace expansion formula (6.2.3). That is,

$$C_n = uSC_{n-1} + \sum_{m=2}^{n} \Pi_m C_{n-m}.$$

In order to prove Theorem 6.1, we have to show that $B_m := \Pi_m/(\lambda c_m)$ actually has the decay behavior assumed in (8.1.3) for any $S \in A_{N,\epsilon}$, respectively $S \in P_{N,\epsilon}$ and λ small enough.

Lemma 9.1. *There are positive constants λ_0, σ and L, such that for all $\lambda \in [0, \lambda_0]$ and for all $m \geq 2$ we have, setting $B_m(x) := \frac{\Pi_m(x)}{\lambda c_m}$,*

$$|B_m(x)| \leq Lm^{-d/2} \sum_{k=1}^{m/2} \exp\left(-\frac{\sqrt{m-k}}{\sqrt{\sigma}} \|\kappa\|\right) \theta_{k\kappa,k\sigma}(x),$$

with $\kappa = (\kappa^{(1)}, \ldots, \kappa^{(d)})$ and where

$$\kappa^{(i)} = \frac{u\mu^{-1}s^{(i)} + \lambda \sum_{m\geq 1} a_m b_m^{(i)}}{1 + \lambda \sum_{m\geq 1} a_m b_m (m-1)} \quad \textit{for } i = 1, \ldots, d. \tag{9.0.1}$$

Proof. We construct a sequence $\kappa_i \in \mathbb{R}^d$ converging to κ as $i \to \infty$ and show that the B_m's have the right decay both at the same time via a double iteration technique. For notational convenience, we set

$$\psi_m^{(i)}(x) := m^{-d/2} \sum_{k=1}^{m/2} \exp\left(-\frac{\sqrt{m-k}}{\sqrt{\sigma}} \|\kappa_i\|\right) \theta_{k\kappa_i,k\sigma}(x).$$

Note first that to be consistent with the definition of the sequence $(B_m)_{m\geq 1}$ in the last Chapter 8, we may set $B_1(x) \equiv 0$. Now consider $\kappa_1 := s^{(1)}$, $s^{(1)}$ being the expectation of S. We easily have

$$|B_2(x)| \leq L^{(1)} \psi_2^{(1)}(x),$$

for $L^{(1)} = L^{(1)}(S, d, \sigma)$ large enough, since $\psi_2^{(1)}(0) \leq 2^{-d/2} \frac{K(d)}{\sqrt{\sigma}^d} e^{-2\|\kappa_1\|/\sqrt{\sigma}}$, and $B_2(x) = 0$ if $x \neq 0$. Now, choose $B_3^{(1)}(x)$ in such a way that κ_1 can be defined via the sequence $(B_2, B_3^{(1)}, 0, 0, \ldots)$. Ie., for $i = 1, \ldots, d$,

$$\kappa_1^{(i)} = \frac{u\overline{\mu}^{-1}s^{(i)} + \lambda \sum_{m\geq 2} \overline{a}_m b_m^{(i)\,(1)}}{1 + \lambda \sum_{m\geq 2} \overline{a}_m b_m^{(1)}(m-1)},$$

where the quantities with a bar $(\bar{\cdot})$ are the usual quantities defined via the auxiliary sequence $(B_2^{(1)}, B_3^{(1)}, 0, , 0 \ldots)$, with $B_2^{(1)} = B_2$ and $B_k^{(1)} \equiv 0$ for $k > 3$. Note that $B_3^{(1)}$ can be chosen with support in the whole of $\mathbb{Z}^d$ and such that $|B_3^{(1)}(x)| \leq 2L^{(1)}\psi_3^{(1)}(x)$, if λ is small enough. An application of the main Theorem 8.1 and then of Lemma A.2 yields

$$|B_3(x)| \leq KL_1 \frac{\overline{\mu}^3}{c_3} 3^{-d/2} \sum_{k=1}^{3/2} e^{-\frac{\sqrt{3-k}}{\sqrt{\sigma}}\|\kappa_1\|} \theta_{k\kappa_1, k\sigma}(x) \leq L\frac{\overline{\mu}^3}{c_3}\psi_3^{(1)}(x), \tag{9.0.2}$$

where K does not depend on $L^{(1)}$, if λ is chosen small enough (see Lemma 10.5 for some more details on this step). The second inequality follows by choosing L large enough. We fix this L for the rest of the proof.
The next step is to define κ_2 in the same way as κ in (8.1.2), but using the sequence $(B_2, B_3, 0, 0, \ldots)$ and to set $L^{(2)} = L^{(2)}(S, d, \sigma)$ such that $|B_2(x)| \leq L^{(2)}\psi_2^{(2)}(x)$. We now choose $B_3^{(2)}(x)$ with support in $\mathbb{Z}^d$ such that κ_2 is defined via the sequence $(B_2, B_3^{(2)}, 0, \ldots)$, where for λ small enough, we may assume that $|B_3^{(2)}(x)| \leq 2L^{(2)}\psi_3^{(2)}(x)$. Applying again Theorem 8.1 and Lemma A.2, we obtain $|B_3(x)| \leq L\frac{\overline{\mu}^3}{c_3}\psi_3^{(2)}(x)$ for λ small enough. We thus have that $|B_k(x)| \leq L\frac{\overline{\mu}^k}{c_k}\psi_k^{(2)}(x)$, $k = 2, 3$ and one more application of Theorem 8.1 and Lemma A.2 yields $|B_4(x)| \leq L\frac{\overline{\mu}^4}{c_4}\psi_4^{(2)}(x)$. We now define κ_3 via the sequence $(B_2, B_3, B_4, 0, 0, \ldots)$.
Continuing this scheme, we assume that for some general $k \geq 3$, we are given κ_k via the sequence $(B_2, \ldots, B_{k+1}, 0, 0, \ldots)$, and we have $L^{(k)} = L^{(k)}(S, d, \sigma)$ such that $|B_2(x)| \leq L^{(k)}\psi_2^{(k)}$. Now suppose that for some $m \leq k+2$, $|B_l(x)| \leq L\psi_l^{(k)}(x)$, for $2 \leq l \leq m-1$ and choose $B_m^{(k)}$ with support in $\mathbb{Z}^d$ in such a way that κ_k is defined via the sequence $(B_2, \ldots, B_{m-1}, B_m^{(k)}, 0, \ldots)$ where we assume λ small enough such that $|B_m^{(k)}(x)| \leq 2L^{(k)}\psi_m^{(k)}(x)$. Use once more Theorem 8.1 and Lemma A.2 to obtain $|B_m(x)| \leq L\psi_m^{(k)}(x)$ for λ small enough. The procedure is repeated up to $m = k+2$. Then, we define κ_{k+1} via the sequence $(B_2, \ldots, B_{k+2}, 0, 0, \ldots)$ and restart the procedure.
By repeatedly applying Theorem 8.1 and Lemma A.2, we thus obtain

$$|B_m(x)| \leq L\frac{\overline{\mu}^m}{c_m} m^{-d/2} \sum_{i=1}^{m/2} e^{-\sqrt{\frac{m-i}{\sigma}}\|\kappa^{(k)}\|} \theta_{i\kappa^{(k)}, i\sigma}(x),$$

where we still have to show that $\frac{\overline{\mu}^m}{c_m}$ remains bounded in order to prove the Lemma. However, defining $\overline{C}_n$ via the sequence $(B_2, \ldots, B_{m-1}, B_m^{(k)}, 0, 0, \ldots)$ in the m-th step of the k-th iteration by setting $\overline{C}_n = \overline{\mu}^n \overline{A}_n$, we have $\overline{C}_n = C_n$ for $n < m$, by an application of the lace expansion formula (6.2.3). Now apply the lace expansion formula to the total weight sequence to obtain:

$$\begin{aligned} c_m =& u\overline{c}_{m-1} + \sum_{k=2}^{m} \pi_k \overline{c}_{m-k} \\ =& u\overline{\mu}^{m-1}\overline{a}_{m-1} + \sum_{k=2}^{m} \pi_k \overline{\mu}^{m-k}\overline{a}_{m-k}, \end{aligned} \tag{9.0.3}$$

where π_k denotes the total mass of Π_k. An application of Lemma A.2 to π_k yields $|\pi_k| \leq \lambda K L_1 \overline{\mu}^k k^{1-d/2}$. Inserting this into equation (9.0.3) leads to

$$\frac{c_m}{\overline{\mu}^m} \geq \frac{1}{2}u\overline{\mu}^{-1} - \lambda K L_1 \geq K,$$

if $\lambda = \lambda(d, S, L_1)$ is small enough. Hence, $\frac{\overline{\mu}^m}{c_m}$ does remain bounded.

With the above procedure we obtain $\kappa^{(k)} \to \kappa$ and the bound in the Lemma for L large enough and $\lambda \in [0, \lambda_0]$, for some $\lambda_0 > 0$, if the sequence $(L^{(k)})_k$ remains bounded. However, this sequence does remain bounded because of the decay of the moments of the B_m's. This proves the Lemma. □

Proof. (Proof of Theorem 6.1). Using Lemma 9.1, the second part of the Theorem follows directly from either Theorem 8.1 (non-periodic case) or from Theorem 8.9 (two-periodic case). Using Corollary A.3, the first part follows from Corollary 7.5. □

Chapter 10

Restriction to the Symmetric Case

As already mentioned, we can extend both, the local CLT 6.1 for perturbed weakly self-avoiding walks and the main Theorem 8.1 (8.9 respectively) to dimensions $d \geq 5$ in case we start the walk with an initial distribution S which is *symmetric in each coordinate and rotationally invariant.* In case of the weakly self-avoiding walk, this implies that the B_m's are also rotationally invariant and symmetric. Also, there is considerable simplification of the main proof in this case. In fact, this extension is true not only in the above case, but whenever we know a priori that the asymptotic drift is equal to zero (ie. $\kappa = 0$). However, this is essentially impossible to know from the initial setting unless we are in the symmetric and rotationally invariant case.
The main reason why the results can be extended to dimensions $d = 5, 6, 7$ and 8 is because we can apply Lemma A.4 instead of Lemma A.2 where we gain an extra $k^{1-d/2}$ in the bound for the Π_m's. Moreover, we never have to account for corrections in the mean and in Lemma 8.8, it suffices that $|\alpha - a_n| \leq \frac{K}{n^{-1/2}}$ and it need not to be summable since in (8.3.39), the term in front of $\triangle_i p_{n-1}$ vanishes for all $i = 1, \ldots, d$, and $n \geq 1$. We give the proof of the local CLT in this special case in the following.

10.1 The Main Theorem in the Symmetric Case

For the entire Chapter, $d \geq 5$. Consider a positive number $R \in \mathbb{N}$. Then choose an aperiodic symmetric and rotationally invariant probability measure S with bounded support $\Omega \subset \overline{B}(0, R)\backslash\{0\}$ (in the following, we abbreviate rotationally invariant and symmetric by simply writing symmetric). As in the non-symmetric case we only treat aperiodic measures, but the extension to the two-periodic case is again a triviality. Now let $(B_m)_{m\geq 1}$ be a sequence of symmetric and rotationally invariant measures with

$$|B_m(x)| \leq Km^{-d/2} \sum_{k=1}^{m/2} k^{1-d/2} \theta_{k\sigma}(x), \tag{10.1.1}$$

where we write $\theta_{k\sigma} := \theta_{0,k\sigma}$ and K is a positive constant whose value may change from line to line. Again, $\sigma > 0$ will be determined later. This definition of the B_m's immediately implies that $|b_m| \leq Km^{-d/2}$ and $b_m^{(ii)} \leq Km^{-(d-1)/2}$, for all $i = 1 \ldots, d$, and of course the first moments vanish. Thus, Proposition 7.1 and Corollary 7.5 are satisfied and we may define the mass sequence $(a_n)_{n\geq 0}$ with $a_0 := 1$ and for $n \geq 1$:

$$a_n := u\mu^{-1} a_{n-1} + \lambda \sum_{m=1}^{n} a_m b_m a_{n-m}, \tag{10.1.2}$$

where $u\mu^{-1} = 1 - \lambda\rho$ and $a_n \in [1/2, 3/2]$. Using this mass sequence, we define the sequence of measures $(A_n)_{n\geq 0}$ by $A_0 := \delta_0$ and for $n \geq 1$:

$$A_n := u\mu^{-1} S A_{n-1} + \lambda \sum_{m=1}^{n} a_m B_m A_{n-m}. \tag{10.1.3}$$

In this case, the right covariance matrix to approximate the asymptotic behavior of the sequence $(A_n/a_n)_n$ is given by $\Delta = \delta Id_d$, where

$$\delta := \frac{u\mu^{-1}s^{(11)} + \lambda \sum_{m\geq 1} a_m b_m^{(11)}}{1+\lambda\sum_{m\geq 1}(m-1)a_m b_m}. \tag{10.1.4}$$

Of course we have $s^{(11)} = s^{(ii)}$ for any $i = 2, \ldots, d$, and $s^{(ij)} = 0$ if $i \neq j$. The same applies to all B_m's. We assume from now on that λ is small enough such that $s^{(11)}/2 \leq d\delta \leq 2s^{(11)}$.
The main Theorem then states as:

Theorem 10.1. *In the above setting there exists $\lambda_0 > 0$, such that for any $\lambda \in [0, \lambda_0]$, we have that*

$$\left|\frac{A_n(x)}{a_n} - \phi_{n\delta}(x)\right| \leq K\left[n^{-1/2}\theta_{n\sigma}(x) + n^{-d/2}\sum_{j=1}^{n/2} j\theta_{j\sigma}(x)\right],$$

where the parameter δ depends on λ, S and the sequence $(B_m)_{m\geq1}$ and is defined above in (10.1.4). $K = K(R, d)$ and $\sigma = \sigma(d, S)$ are positive constants independent of the sequence $(B_m)_m$ and will be determined in the proof of the Theorem.

10.2 The Symmetric Distribution $p_t(x)$

We again need the distribution $p_t(x)$ of Chapter 8. However, we may fix $d_1 = \ldots = d_d = 1/d$ and $\pi_1 = \ldots = \pi_d = 1/2$. Also, we set $d_{ij} = 0$, for all $i, j = 1, \ldots, d$. Then, $E[X_t^{(i)}] = 0$, for any $i = 1, \ldots, d$, and $\mathrm{var}(X_t^{(1)}) = \ldots = \mathrm{var}(X_t^{(1)}) = t\eta/d$. Of course, all off-diagonal entries in the covariance matrix of p_t now vanish. The relation between the time derivative and discrete space derivatives of p_t given in Lemma 8.5 changes to:

Lemma 10.2. *We have*

$$\frac{1}{\eta}\frac{\partial}{\partial t}p_t(x) = \sum_{i=1}^{d}\frac{1}{2d}\triangle_{ii}p_t(x) + E(p,t,x),$$

where $|E(p,t,x)| \leq \frac{K}{t^{3/2}}\theta_{t\kappa,t\sigma'}(x)$, for $K > 0$ and $\sigma' > 0$ large enough.

Note that in the case of symmetric and rotationally invariant initial distributions, we automatically have that $S \in A_{N,\epsilon}$ for any small $\epsilon > 0$, and also, we may even choose $\epsilon = 0$ (see equation (10.3.19)).

10.3 Proof of Theorem 10.1

This time we set

$$\mathcal{W} := \left\{ G = (G_n)_{n\geq 0} \,\middle|\, \sup_{n\geq 1, x\in\mathbb{Z}^d}\frac{|G_n(x)|}{\chi_n(x)} + \sup_{x\in\mathbb{Z}^d}|G_0(x)| < \infty, \right.$$
$$\left. G_n \text{ a signed symmetric and rotationally invariant measure on } \mathbb{Z}^d \right\},$$

with

$$\chi_n(x) := n^{-1/2}\theta_{n\sigma}(x) + n^{-d/2}\sum_{j=1}^{n/2} j\theta_{j\sigma}(x).$$

σ is to be determined. The operator ψ from Chapter 8 remains unchanged.
The contraction Lemma still holds:

Lemma 10.3. *Let $\xi \in \mathcal{W}_0$. Then, for N big enough and λ small enough (depending on N), there exists $\epsilon \in (0,1)$ with*

$$\|\psi(\xi)\|_{1/2} \leq \epsilon \|\xi\|_{1/2}.$$

The proof of this Lemma is almost a copy of the proof of the corresponding Lemma 8.6:

Proof. Let us do the same preliminary calculations around p_l as in the proof of Lemma 8.6: Using (8.2.23) we have

$$\begin{aligned} Sp_{l-1}(x) = &\sum_{y\in\mathbb{Z}^d} S(y)p_{l-1}(x-y) = p_{l-1}(x) \\ &+ \frac{1}{2}\sum_{i=1}^{d} s^{(11)} \triangle_{ii} p_{l-1}(x) + \sum_y S(y)E^p(x,y,l-1), \end{aligned} \tag{10.3.1}$$

where again

$$\begin{aligned} E^p(x,y,l-1) = &\sum_{i=1}^{d} \frac{-y_i(y_i+1)(y_i+2)}{2} \triangle_{ii} v_{l-1}^{(i)}(x,-y) + \sum_{i<j} y_i y_j(-y_j-1)\triangle_j v_{l-1}^{(\triangle_i j)}(x,-y) \\ &+ \sum_{i<j} \frac{y_i(-y_i-1)}{2} y_j v_{l-1}^{(\triangle_i i j)}(x,-y) + \sum_{i<j<k} (-y_i y_j y_k) v_{l-1}^{(\triangle_i j k)}(x,-y), \end{aligned} \tag{10.3.2}$$

and we get

$$|\sum_y S(y)E^p(x,y,l-1)| \leq \frac{K}{t^{3/2}}\theta_{t\sigma'}(x), \tag{10.3.3}$$

using (8.2.27)–(8.2.30) and choosing K and σ' large enough. From Lemma 10.2, we have:

$$\dot{p}_{l-1}(x) = \eta \sum_{i=1}^{d} \frac{1}{2d} \triangle_{ii} p_{l-1}(x) + E(p,l-1,x),$$

with $|E(p,l-1,x)| \leq K l^{-3/2}\theta_{l\sigma'}(x)$. Hence, we obtain, doing a Taylor expansion in time,

$$\begin{aligned} p_l(x) = &p_{l-1}(x) + \eta \sum_{i=1}^{d} \frac{1}{2d} \triangle_{ii} p_{l-1}(x) \\ &+ E^{time}(p,l-1,x), \end{aligned} \tag{10.3.4}$$

with $|E^{time}(p,l-1,x)| = |E(p,l-1,x) + \ddot{p}_{l-\xi}(x)| \leq \frac{K}{l^{3/2}}\theta_{l\kappa,l\sigma'}(x)$ since ξ is in $[0,1]$ and $\ddot{p}_{l-\xi}(x)$ is bounded using Lemmas 10.2 and 8.4. Note here that we only have to expand p_l to the first time derivative instead of the second one as in Lemma 8.6. Similarly, for $l/2 \leq l-m < l-1$,

$$\begin{aligned} p_{l-m}(x) = &p_{l-1}(x) - (m-1)\eta \sum_{i=1}^{d} \frac{1}{2d} \triangle_{ii} p_{l-1}(x) \\ &- (m-1)E(p,l-1,x) + (m-1)^2 \ddot{p}_{l-\xi}(x), \end{aligned} \tag{10.3.5}$$

where $l-\xi \in [l/2, l-1]$.
Finally, we have for $q, r = 1, \ldots, d$,

$$\begin{aligned} B_m p_{l-1}(x) = &b_m p_{l-1}(x) + \frac{1}{2}\sum_{i=1}^{d} b_m^{(11)} \triangle_{ii} p_{l-1}(x) \\ &+ \sum_y B_m(y) E^p(x,y,l-1), \end{aligned} \tag{10.3.6}$$

$$B_m \triangle_{qq} p_{l-1}(x) = b_m \triangle_{qq} p_{l-1}(x) + \sum_y B_m(y) E^{\triangle_{qq} p}(x,y,l-1), \tag{10.3.7}$$

where $E^p(x,y,l-1)$ is given in (10.3.2), and

$$E^{\triangle_{qq}p}(x,y,l-1) = -\sum_i y_i v_{l-1}^{(\triangle_{qq}i)}(x,-y).$$

Using these equations together with (10.3.5) we get

$$\begin{aligned} B_m p_{l-m}(x) =& b_m p_{l-1}(x) \\ &+ \sum_{i=1}^d \triangle_{ii} p_{l-1}(x) \left(\frac{b_m^{(11)}}{2} - (m-1) b_m \frac{\eta}{2d} \right) \\ &+ \sum_y B_m(y) E(m,p,x,y,l-1). \end{aligned} \tag{10.3.8}$$

Here,

$$\begin{aligned} E(m,p,x,y,l-1) = E^p(x,y,l-1) - \sum_{i=1}^d (m-1)\eta \frac{1}{2d} E^{\triangle_{ii}p}(x,y,l-1) \\ - (m-1)E(p,l-1,x-y) + (m-1)^2 \ddot{p}_{l-\xi}(x-y), \end{aligned} \tag{10.3.9}$$

where $l-\xi \in [l/2, l-1]$, and the error terms are collected from the above calculations (10.3.1)–(10.3.7). We again need to show that $\sum_y |B_m(y)E(m,p,x,y,l-1)| \leq \frac{K}{m^{3/2}l^{3/2}}\theta_{l\sigma}(x) + \frac{K}{m^{1/2}l^2}\theta_{l\sigma}(x)$ for some $\sigma > 0$ large enough. Thus we again check all terms in (10.3.9) separately. The first one, $E^p(x,y,l-1)$, can be bounded by:

$$\frac{K}{l^{3/2}} \sum_{i\leq j\leq q} |y_i y_j y_q| \int_0^1 ds \theta_{l\sigma'}(x_1,\ldots,x_{q-1},x_q - sy_q, x_{q+1}-y_{q+1},\ldots,x_d-y_d). \tag{10.3.10}$$

This has to be folded with B_m. But recalling the bound on B_m from (10.1.1) and noting that $|y_i y_j y_q|\theta_{k\sigma}(y) \leq K k^{3/2} \theta_{k\sqrt{2}\sigma}(y)$ we get:

$$\begin{aligned} &\sum_y |B_m(y)E^p(x,y,l-1)| \\ \leq& \frac{K}{l^{3/2}} \sum_{i\leq j\leq q} |y_i y_j y_q| \int_0^1 ds \sum_y B_m(y) \theta_{l\sigma'}(x_1,\ldots,x_{q-1},x_q-sy_q,x_{q+1}-y_{q+1},\ldots,x_d-y_d)| \\ \leq& \frac{K}{l^{3/2}} \sum_q \int_0^1 ds m^{-d/2} \sum_{k=1}^{m/2} k^{3/2} k^{1-d/2} \\ &\cdot \sum_y \theta_{\sqrt{2}k\sigma}(y) \theta_{l\sigma'}(x_1,\ldots,x_{q-1},x_q-sy_q,x_{q+1}-y_{q+1},\ldots,x_d-y_d). \end{aligned}$$

We want to fold each coordinate separately above (as in the corresponding Lemma 8.6). As in the proof of that Lemma, this is done by multiplying the variances by 2 and bounding the above by a multiplication of corresponding one-dimensional independent doubly-exponential distributions. Again, special care has to be taken when folding the q-th coordinate with the integral over s.

Here, we have (now assuming the θ's are one-dimensional):

$$\begin{aligned}
&\int_0^1 ds \sum_{y_q\in\mathbb{Z}} \theta_{2\sqrt{2}k\sigma}(y_q)\theta_{2l\sigma'}(x_q - sy_q)\\
\leq &K\int_0^1 dss^{-1}\theta_{2\sqrt{2}k\sigma+2l\sigma'/s^2}(x_q/s)\\
\leq &K\int_0^1 ds\theta_{2\sqrt{2}s^2k\sigma+2l\sigma'}(x_q)\\
\leq &K\int_0^1 ds\theta_{2\sqrt{2}k\sigma+2l\sigma'}(x_q)\\
\leq &K\int_0^1 ds\theta_{l\sigma}(x_q)\\
\leq &K\theta_{l\sigma}(x_q),
\end{aligned}$$

where we use that $k \leq m/2 \leq l/4$ and set $\sigma \geq \sigma' \frac{2}{1-\sqrt{2}/2}$. As in Lemma 8.6 we end up with:

$$\begin{aligned}
&\sum_y |B_m(y)E^p(x,y,l-1)|\\
\leq &\frac{K}{l^{3/2}} m^{-d/2}\theta_{l\sigma}\sum_{k=1}^{m/2} k^{5/2-d/2}\\
\leq &\frac{K}{m^{3/2}l^{3/2}}\theta_{l\sigma}(x),
\end{aligned} \tag{10.3.11}$$

since $d \geq 5$. This finishes the calculation for the first term in the error (10.3.9). For the second error term, similar considerations as for the first term again lead to the desired bound (10.3.11). We turn to the second line of (10.3.9). For the first error on that line, we have $(m-1)E(p, l-1, x-y) \leq m\frac{K}{l^{3/2}}\theta_{l\sigma'}(x-y)$ by Lemma 10.2. Thus, folding with B_m, we have:

$$\begin{aligned}
&\left|\sum_y B_m(y) m\frac{K}{l^{3/2}}\theta_{l\sigma'}(x-y)\right|\\
\leq &\frac{K}{l^{3/2}} m^{1-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\sum_y \theta_{k\sigma}(y)\theta_{l\sigma'}(x-y)\\
\leq &\frac{K}{l^{3/2}} m^{1-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\theta_{l\sigma}(x)\\
\leq &\frac{K}{l^{3/2}} m^{1-d/2}\theta_{l\sigma}(x)\\
\leq &\frac{K}{l^{3/2}m^{3/2}}\theta_{l\sigma}(x),
\end{aligned}$$

since $d \geq 5$ and $\sigma \geq \sigma' \frac{2}{1-\sqrt{2}/2}$. To handle the second term on the last line of (10.3.9), we note that $|\frac{1}{\eta^2}\frac{\partial^2}{\partial(l-1)^2}p_{l-1}(x)| \leq \frac{K}{l^2}\theta_{l\sigma'}$. Folding this bound with B_m, we obtain by similar arguments to the ones used for the former error that the second error on the last line is bounded by $\frac{K}{l^2m^{1/2}}\theta_{l\sigma}$. Therefore, we obtain in (10.3.8),

$$|\sum_y B_m(y)E(m,p,x,y,l-1)| \leq \frac{K}{l^{3/2}m^{3/2}}\theta_{l\sigma}(x) + \frac{K}{l^2m^{1/2}}\theta_{l\sigma}(x), \tag{10.3.12}$$

as long as $d \geq 5$, and we set from now on $\sigma = \sigma' \frac{2}{1-\sqrt{2}/2}$.

We now turn to the main part of the proof.

Again, we need to show that $|\psi(\xi)_n| \leq \epsilon \|\xi\|_{1/2}\chi_n$, for all $n \in \mathbb{N}$ and some $\epsilon \in (0,1)$. Thus we split (8.3.2):

$$|\psi(\xi)_n| \leq \sum_{l=1}^{n} |\xi_{n-l}| * \left\| \left[(1-\lambda\rho)SF_{l-1} - F_l + \lambda \sum_{m=1}^{l/2} a_m B_m F_{l-m} \right] \right\| + \sum_{l=1}^{n} |\xi_{n-l}| * \left| \lambda \sum_{m=l/2}^{l} a_m B_m F_{l-m} \right|. \tag{10.3.13}$$

As in the non-symmetric case, we start with the second term. Note that $|\xi_{n-l}| \leq \|\xi\|_{1/2}\chi_{n-l}$ and we split χ_{n-l} into $(n-l)^{-1/2}\theta_{n-l}$ and $(n-l)^{-d/2}\sum_{j=1}^{(n-l)/2} j\theta_j$, where for simplicity, $\theta_k := \theta_{k\sigma}$ and $\theta'_k := \theta_{k\sigma'}$ in the following. For the first part this leads to

$$\begin{aligned}
&\sum_{l=1}^{n-1}(n-l)^{-1/2}\theta_{n-l} * \sum_{m=l/2}^{l} a_m|B_m| * F_{l-m} \\
\leq &K\sum_{l=1}^{n-1}(n-l)^{-1/2}\sum_{m=l/2}^{l} m^{-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\underbrace{\theta_{n-l}*\theta_k*\theta'_{l-m}}_{\leq K\theta_{n-m+k}} \\
\leq &Kn^{-1/2}\theta_n\sum_{l=1}^{n/2}\sum_{m=l/2}^{l} m^{-d/2}\sum_{k=1}^{m/2} k^{1-d/2} \\
&+ Kn^{-d/2}\sum_{l=n/2}^{n-1}(n-l)^{-1/2}\sum_{m=l/2}^{l}\sum_{k=1}^{m/2} k^{1-d/2}\theta_{n-m+k} \\
\leq &K\left[n^{-1/2}\theta_n + n^{-d/2}\sum_{m=n/4}^{n-1}\sum_{k=1}^{m/2} k^{1-d/2}\theta_{n-m+k}\underbrace{\sum_{l=m}^{n-1}(n-l)^{-1/2}}_{\leq n-m}\right] \\
\leq &K\left[n^{-1/2}\theta_n + n^{-d/2}\sum_{m=n/4}^{n-1}\sum_{k=n-m+1}^{n-m/2} k(k-(n-m))^{1-d/2}\theta_k\right] \\
\leq &K\left[n^{-1/2}\theta_n + n^{-d/2}\sum_{k=1}^{7n/8} k\theta_k\underbrace{\sum_{m=n-k+1}^{n-1}(k-(n-m))^{1-d/2}}_{\leq K}\right] \\
\leq &K\chi_n,
\end{aligned}$$

where we use $\sigma > \sigma'$ and $d \geq 5$. For the second part, we split the sum and find

$$\begin{aligned}
&\sum_{l=1}^{n/2}(n-l)^{-d/2}\sum_{j=1}^{(n-l)/2} j\theta_j * \sum_{m=l/2}^{l} a_m|B_m| * F_{l-m}\\
\leq& Kn^{-d/2}\sum_{l=1}^{n/2}\sum_{j=1}^{(n-l)/2} j \sum_{m=l/2}^{l} m^{-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\theta_{j+k+l-m}\\
\leq& Kn^{-d/2}\sum_{m=1}^{n/2} m^{-d/2}\sum_{l=m}^{(n/2)\wedge(2m)}\sum_{k=1}^{m/2} k^{1-d/2}\sum_{j=1+k}^{(n-l)/2+k} j\theta_{j+l-m}\\
\leq& Kn^{-d/2}\sum_{m=1}^{n/2} m^{-d/2}\sum_{l=m}^{(n/2)\wedge(2m)}\sum_{j=1+l-m}^{(n+l-m)/2} j\theta_j\\
\leq& Kn^{-d/2}\sum_{j=1}^{3n/4} j\theta_j\\
\leq& K\chi_n,
\end{aligned}$$

again using $d \geq 5$ and finally

$$\begin{aligned}
&\sum_{l=n/2}^{n-1}(n-l)^{-d/2}\sum_{j=1}^{(n-l)/2}\underbrace{j}_{\leq n-l}\sum_{m=l/2}^{l} m^{-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\theta_{j+k+l-m}\\
\leq& K\sum_{l=n/2}^{n-1}(n-l)^{1-d/2}\sum_{m=l/2}^{l} m^{-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\sum_{j=1}^{(n-l)/2}\theta_{j+k+l-m}\\
\leq& Kn^{-d/2}\sum_{l=n/2}^{n-1}(n-l)^{1-d/2}\sum_{m=0}^{l/2}\sum_{k=1}^{(l-m)/2} k^{1-d/2}\sum_{j=1}^{(n-l)/2}\theta_{j+k+m}\\
\leq& Kn^{-d/2}\sum_{l=n/2}^{n-1}(n-l)^{1-d/2}\sum_{m=0}^{l/2}\sum_{j=1+m}^{(n+m)/2}\theta_j\\
\leq& Kn^{-d/2}\sum_{l=n/2}^{n-1}(n-l)^{1-d/2}\sum_{j=1}^{(2n+l)/4}\theta_j\sum_{m=0}^{j-1} 1\\
\leq& Kn^{-d/2}\sum_{j=1}^{3n/4} j\theta_j\\
\leq& K\chi_n.
\end{aligned}$$

Thus we get

$$\text{second summand of (10.3.13)} \leq K\lambda\|\xi\|_{1/2}\chi_n,$$

and it suffices to choose λ small enough.

It remains to check the first summand of (10.3.13). For this summand we again have that it is equal to

$$\sum_{l=1}^{(n-1)\wedge N} |\xi_{n-l}| * \Bigg|(1-\lambda\rho)(1-\frac{l-1}{N})S^l + (1-\lambda\rho)\frac{l-1}{N}Sp_{l-1} - (1-\frac{l}{N})S^l - \frac{l}{N}p_l$$

$$+\lambda\sum_{m=1}^{l/2} a_m B_m(1-\frac{l-m}{N})S^{l-m} + \lambda\sum_{m=1}^{l/2} a_m B_m \frac{l-m}{N}p_{l-m}\Bigg| \tag{10.3.14}$$

$$+\sum_{l=N+1}^{n-1} |\xi_{n-l}| * \Bigg|(1-\lambda\rho)Sp_{l-1} - p_l + \lambda\sum_{m=1}^{(l/2)\wedge(l-N)} a_m B_m p_{l-m}$$

$$+\lambda\sum_{m=l-N+1}^{l/2} a_m B_m\left((1-\frac{l-m}{N})S^{l-m} + \frac{l-m}{N}p_{l-m}\right)\Bigg|. \tag{10.3.15}$$

For the moment, we are interested in the second sum which is present only if $n > N+1$. We have:

$$(10.3.15) \le \sum_{l=N+1}^{n-1} \|\xi\|_{1/2}\chi_{n-l} * \left(\left|(1-\lambda\rho)Sp_{l-1} - p_l + \lambda\sum_{m=1}^{l/2} a_m B_m p_{l-m}\right|\right.$$

$$\left.+\left|\lambda\sum_{m=l-N+1}^{l/2} a_m B_m\left((1-\frac{l-m}{N})(S^{l-m}-p_{l-m})\right)\right|\right). \tag{10.3.16}$$

Now we use the calculations from (10.3.1) to (10.3.12) and again collect terms coming with the same discrete derivatives to write:

$$(1-\lambda\rho)Sp_{l-1} - p_l + \lambda\sum_{m=1}^{l/2} a_m B_m p_{l-m}$$

$$= \left((1-\lambda\rho) - 1 + \lambda\sum_{m=1}^{l/2} a_m b_m\right)p_{l-1} \tag{10.3.17}$$

$$+\sum_{i=1}^{d}\left(\frac{(1-\lambda\rho)}{2}s^{(11)} + \frac{\lambda}{2}\sum_{m=1}^{l/2} a_m b_m^{(11)}\right.$$

$$\left.-\eta\frac{1}{2d} - \lambda\eta\frac{1}{2d}\sum_{m=1}^{l/2} a_m b_m(m-1)\right)\triangle_{ii}p_{l-1} \tag{10.3.18}$$

$$+E(l)(.),$$

where $|E(l)(.)| \le \frac{K}{l^{3/2}}\theta_{l\sigma}(.)$ due to the error bound in (10.3.12) and the bounds on the other errors at the beginning of the proof. We analyze the terms above separately: For the term in front of p_{l-1} in (10.3.17) we have:

$$\left|(1-\lambda\rho) - 1 + \lambda\sum_{m=1}^{l/2} a_m b_m\right| \le K\sum_{m=l/2}^{\infty} |b_m| = O(l^{-3/2}),$$

using the definition of ρ in (8.1.1) and the decay of the b_m's. For the remaining terms, we choose $\eta > 0$ such that

$$\delta := \frac{\eta}{2d} = \frac{u\mu^{-1}s^{(11)} + \lambda\sum_{m\ge1} a_m b_m^{(11)}}{1+\lambda\sum_{m\ge1} a_m b_m(m-1)}. \tag{10.3.19}$$

Plugging δ into the terms in (10.3.18), we get that for each summand, the term inside the bracket converges to 0 at rate $l^{-1/2}$. Thus, collecting the above and combining with the bounds on p_{l-1} and it's discrete derivatives from Lemma 8.4 we have:

$$\left|(1-\lambda\rho)Sp_{l-1}(x) - p_l(x) + \lambda \sum_{m=1}^{l/2} a_m B_m p_{l-m}(x)\right| \leq \frac{K}{l^{3/2}}\theta_{l\sigma}(x),$$

and hence the first summand in (10.3.16) can be bounded by

$$\begin{aligned}
&K\|\xi\|_{1/2} \sum_{l=N+1}^{n-1} \frac{1}{l^{3/2}} \chi_{n-l} * \theta_{l\sigma} \\
\leq &K\|\xi\|_{1/2} \sum_{l=N+1}^{n-1} l^{-3/2} \left[(n-l)^{-1/2}\theta_{n\sigma} + (n-l)^{-d/2} \sum_{j=1}^{(n-l)/2} j\theta_{(l+j)\sigma}\right] \\
\leq &\|\xi\|_{1/2} \left[N^{-1/2}Kn^{-1/2}\theta_{n\sigma} + Kn^{-d/2} \sum_{l=N+1}^{n/2} l^{-3/2} \sum_{j=1}^{(n-l)/2} j\theta_{(l+j)\sigma}\right. \\
&\left. + N^{-1/2}Kn^{-1} \sum_{l=(n/2)\vee(N+1)}^{n-1} (n-l)^{-d/2} \sum_{j=1}^{(n-l)/2} j\theta_{(l+j)\sigma}\right] \\
\leq &C(N)K\|\xi\|_{1/2}\chi_n,
\end{aligned}$$

where $C(N)$ goes to zero when $N \to \infty$, and thus $C(N)K \leq \epsilon$, if N is large enough.
The remainder of the proof essentially carries over word by word from the proof of Lemma 8.6 (setting $\kappa = 0$ and changing the norm appropriately). □

We still also have the symmetric version of Lemma 8.8:

Lemma 10.4. *Let $\xi_n := a_n F_n$, for all $n \geq 0$. Then,*

$$\|\psi(\xi) - \xi\|_{1/2} \leq K,$$

where K is a positive constant depending on R and on the constant N.

Proof. Again, the proof carries over almost word by word from the proof of Lemma 8.8: First of all, we note that it again suffices to show that

$$\sum_{l=1}^{n} \left| F_{n-l} \left[a_l F_l - (1-\lambda\rho)Sa_{l-1}F_{l-1} - \lambda \sum_{m=1}^{l} a_m a_{l-m} B_m F_{l-m} \right]\right| \leq K\chi_n.$$

But the left hand side above is again bounded by

$$\begin{aligned}
&\sum_{l=1}^{n} \left| a_l F_n - (1-\lambda\rho)Sa_{l-1}F_{n-1} - \lambda \sum_{m=1}^{l\wedge(n/2)} a_m a_{l-m} B_m F_{n-m}\right| \\
&+ \lambda \sum_{l=n/2}^{n} \sum_{m=n/2}^{l} |a_m a_{l-m} B_m F_{n-m}|, \qquad (10.3.20)
\end{aligned}$$

and we may assume that $F_n = p_n$, for all n as in Lemma 8.8. For the second term in (10.3.20) we get:

$$\begin{aligned}
&\lambda \sum_{l=n/2}^{n} \sum_{m=n/2}^{l} |a_m a_{l-m} B_m F_{n-m}| \\
\leq & \lambda K \sum_{l=n/2}^{n} \sum_{m=n/2}^{l} m^{-d/2} \sum_{k=1}^{m/2} k^{1-d/2} \theta_{k+n-m} \\
\leq & \lambda K n^{-d/2} \sum_{m=n/2}^{n} (n-m) \sum_{k=1+n-m}^{n-m/2} (k-(n-m))^{1-d/2} \theta_k \\
\leq & \lambda K n^{-d/2} \sum_{k=1}^{3n/4} \theta_k \sum_{m=(n/2)\vee(n-k+1)}^{n} (k-(n-m))^{1-d/2} k \\
\leq & \lambda K \chi_n .
\end{aligned}$$

It remains to check the first term in (10.3.20):

$$\begin{aligned}
&\sum_{l=1}^{n} \left| a_l p_n - (1-\lambda\rho) a_{l-1} S p_{n-1} - \lambda \sum_{m=1}^{l\wedge(n/2)} a_m a_{l-m} B_m p_{n-m} \right| \qquad (10.3.21) \\
\leq & \sum_{l=1}^{n} \left| a_l - (1-\lambda\rho) a_{l-1} - \lambda \sum_{m=1}^{l\wedge(n/2)} a_{l-m} a_m b_m \right| |p_{n-1}| \\
&+ \sum_{l=1}^{n} \sum_{i=1}^{d} \left| -\frac{1-\lambda\rho}{2} s^{(11)} a_{l-1} - \frac{\lambda}{2} \sum_{m=1}^{l\wedge(n/2)} a_{l-m} a_m b_m^{(11)} \right. \\
&\left. + \eta \frac{1}{2d} a_l + \lambda\eta \frac{1}{2d} \sum_{m=1}^{l\wedge(n/2)} a_m a_{l-m} b_m (m-1) \right| |\triangle_{ii} p_{n-1}| \\
&+ E(n)(.),
\end{aligned}$$

where we use the calculations from (10.3.1) to (10.3.12) in the proof of Lemma 10.3 and regroup terms according to their discrete derivative in the place variable as in that Lemma. Similarly to those calculations, $E(n)(.) \leq \frac{K}{n^{1/2}} \theta_n(.)$. We check the remaining terms above separately. For the term in front of $|p_{n-1}|$, we get

$$\left| a_l - (1-\lambda\rho) a_{l-1} - \lambda \sum_{m=1}^{l\wedge(n/2)} a_{l-m} a_m b_m \right| \leq \lambda \sum_{m=n/2}^{l} a_m a_{l-m} b_m \leq K n^{-3/2},$$

using the decay of the b_m's, (10.1.2) and Corollary 7.5. Thus, $\sum_{l=1}^{n} |.| p_{n-1}(x) \leq K n^{-1/2} \theta_n(x)$. By Lemma 8.4, $|\triangle_{ii} p_{n-1}(x)| \leq \frac{K}{n} \theta_n(x)$ for $i = 1, \ldots, d$, and using again Corollary 7.5 we have for $i = 1 \ldots, d$,

$$\begin{aligned}
&\left| -\frac{1-\lambda\rho}{2} s^{(11)} a_{l-1} - \frac{\lambda}{2} \sum_{m=1}^{l\wedge(n/2)} a_{l-m} a_m b_m^{(11)} + \eta \frac{1}{2d} a_l + \lambda\eta \frac{1}{2d} \sum_{m=1}^{l\wedge(n/2)} a_m a_{l-m} b_m (m-1) \right| |\triangle_{ii} p_{n-1}(x)| \\
\leq & K l^{-1/2} n^{-1} \theta_n(x).
\end{aligned}$$

Thus $\sum_{l=1}^{n}\sum_{i=1}^{d}|.||\triangle_{ii}p_{n-1}(x)| \leq Kn^{-1}\theta_n(x)\sum_{l=1}^{n} l^{-1/2} \leq Kn^{-1/2}\theta_n(x)$. Therefore (10.3.21) is bounded by $Kn^{-1/2}\theta_n(x)$ and hence by $K\chi_n$ as desired. This finishes the proof of Lemma 10.4. □

The proof of Theorem 10.1 is now a copy the proof of Theorem 8.1.

10.4 Application to Symmetric Weakly Self-Avoiding Walks

It remains again to apply Theorem 10.1 to weakly self-avoiding walks with symmetric and rotationally invariant initial distributions S. The Lemma corresponding to Lemma 9.1 giving the good decay for $\Pi_m/(\lambda c_m)$ now states as:

Lemma 10.5. *There are positive constants λ_0, σ and L, such that for all $\lambda \in [0, \lambda_0]$ and for all $m \geq 2$ we have, setting $B_m(x) := \frac{\Pi_m(x)}{\lambda c_m}$,*

$$|B_m(x)| \leq Lm^{-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\theta_{k\sigma}(x).$$

The proof of this Lemma is a lot simpler than the proof of Lemma 9.1, since we already know the correct asymptotic mean (it is zero!). Therefore, we do not need to make a second iteration. The proof can also be found (with Gaussian decay and for S the symmetric nearest neighbor distribution only) in Ritzmann [32].

Proof. For convenience, we set

$$\psi_m(x) := m^{-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\theta_{k\sigma}(x).$$

It is easy to see that $|B_2(x)| \leq L\psi_2(x)$, for L large enough (depending on d, σ and S). The induction now goes as follows: For $m \geq 3$, assume that $|B_k(x)| \leq L\psi_k(x)$ for all $2 \leq k < m$. Define the truncated sequence $(\overline{B}_n)_{n\geq 2}$ by

$$\overline{B}_n(x) := \begin{cases} B_n(x), & \text{if } |B_n(x)| \leq L\psi_n(x), \\ L\psi_n(x), & \text{else.} \end{cases}$$

This sequence satisfies Theorem 10.1 and we thus obtain a sequence $(\overline{A}_n)_{n\geq 0}$ of measures with

$$\left|\frac{\overline{A}_n(x)}{\overline{a}_n} - \phi_{n\overline{\delta}}(x)\right| \leq K\left[n^{-1/2}\theta_{n\sigma}(x) + n^{-d/2}\sum_{k=1}^{n/2} k\theta_{k\sigma}(x)\right]. \tag{10.4.1}$$

(In the two-periodic case, put a factor 2 in front of $\phi_{n\overline{\delta}}$ and consider only n and $\|x\|_1$ of same parity). As long as λ is small enough, the positive constants K and σ do not depend on L. Defining $\overline{C}_n := \overline{\mu}\overline{A}_n$, and using (10.4.1) as well as the fact that $\overline{\delta} \leq \sigma$ and both are of comparable size, we have

$$\overline{C}_n(x) \leq K\overline{c}_n\theta_{n\sigma}(x) \leq K(\overline{\alpha} + Kn^{-1/2})\overline{\mu}^n\theta_{n\sigma}(x) \leq L\overline{\mu}^n\theta_{n\sigma}(x),$$

where L is a positive constant. But since $\overline{B}_n = B_n$, if $n < m$, we also have $\overline{C}_n = C_n$, for $n < m$, using the lace expansion formula (6.2.3). Therefore, applying Lemma A.4 we have

$$|B_m(x)| \leq KL_1\frac{\overline{\mu}^m}{c_m}m^{-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\theta_{k\sigma}(x).$$

It remains to show that $\overline{\mu}^m/c_m$ is bounded. But this is done exactly as in the proof of Lemma 9.1. This finishes the proof. □

With this Lemma, the second part of the local CLT-Theorem for symmetric distributions follows immediately:

Theorem 10.6. *Let $d \geq 5$ and let $R \in \mathbb{N}$. Then, for any symmetric and rotationally invariant distribution S with support in $\overline{B}(0,R)\backslash\{0\}$, there exists $\lambda_0(S) > 0$, such that for all $\lambda \in [0, \lambda_0]$, and for all $n \in \mathbb{N}$,*

$$c_n = \alpha\mu^n(1 + O(n^{-1/2})).$$

Moreover, for if S is aperiodic, we have for $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$,

$$\left|\frac{C_n(x)}{c_n} - \phi_{n\delta}(x)\right| \leq K\left[n^{-1/2}\theta_{n\sigma}(x) + n^{-d/2}\sum_{j=1}^{n/2} j\theta_{j\sigma}(x)\right],$$

and if S is periodic and $n - \|x\|_1$ even,

$$\left|\frac{C_n(x)}{c_n} - 2\phi_{n\delta}(x)\right| \leq K\left[n^{-1/2}\theta_{n\sigma}(x) + n^{-d/2}\sum_{j=1}^{n/2} j\theta_{j\sigma}(x)\right].$$

The constants $\alpha > 0$ and $\mu > 0$ and the variance δ depend on λ, d and S, whereas σ depends on d and S and K only depends on d and R.

The first part of the Theorem again follows from Corollary 7.5.

Appendix A

The Lace Expansion and Bounds for the Lace Expansion

A.1 The Lace Expansion

We give a short introduction to the *Lace Expansion* and show how equation (6.2.3) is obtained in the fist part below. For more details on this topic we refer to the book by Slade [33] or by Madras and Slade [24]. The lace expansion was first introduced by Brydges and Spencer in [9]. The following Subsection is an adaption of van der Hofstad, den Hollander and Slade [40] (they deal with the symmetric nearest neighbor initial distribution only). The second part is devoted to bounds for the lace expansion in terms of the connectivity.

A.1.1 Definition

First, we introduce some terminology. Given an interval of integers $I = [a, b] \subset \mathbb{Z}$ with $0 \le a < b$, we call the pair $\{s, t\} =: st$ $(s < t)$ in I an *Edge*. A set of edges is called a *Graph*. A graph Γ on $[a, b]$ is said to be *connected* if both a and b are endpoints of edges in Γ and if, in addition, for any $c \in (a, b)$ there is an edge $st \in \Gamma$ such that $s < c < t$. The set of all graphs on $[a, b]$ is denoted by $\mathcal{B}[a, b]$, and the subset consisting of all connected graphs is denoted by $\mathcal{G}[a, b]$. A *Lace* is a minimally connected graph. That is a connected graph for which the removal of any edge would result in a disconnected graph. The set of laces on $[a, b]$ is denoted by $\mathcal{L}[a, b]$, and the set of laces on $[a, b]$ consisting of exactly N edges is denoted by $\mathcal{L}^{(N)}[a, b]$. It is possible to associate a unique lace L_Γ to each connected graph Γ in the following way: L_Γ consists of the edges $s_1t_1, s_2t_2, \ldots$, with t_1, s_1, t_2, $s_2, \ldots$ determined in that order by

$$t_1 := \max\{t : \ at \in \Gamma\}, \qquad s_1 := a,$$
$$t_{i+1} := \max\{t : \ \exists s < t_i \text{ such that } st \in \Gamma\}, \quad s_{i+1} := \min\{s : \ st_{i+1} \in \Gamma\}.$$

Given a lace L, the set of all edges $st \notin L$ such that $L_{L\cup\{st\}} = L$ is denoted by $\mathcal{C}(L)$. Edges in $\mathcal{C}(L)$ are said to be *compatible* with L.
Recall the definition of C_n from (6.2.1). We can rewrite that definition as:

$$C_n(x) = \sum_{\substack{\omega:0\rightsquigarrow x\\ |\omega|=n}} K[0, n](\omega) \prod_{r=1}^{n} s(\omega(r) - \omega(r-1)), \tag{A.1.1}$$

where the sum is over all permissible paths from 0 to x of length n and for $a < b$,

$$K[a, b](\omega) := \prod_{a\le s<t\le b} (1 - \lambda U_{st}(\omega)) = \sum_{\Gamma\in\mathcal{B}[a,b]} \prod_{st\in\Gamma} (-\lambda U_{st}(\omega)). \tag{A.1.2}$$

We also define a similar quantity in which the sum is restricted to connected graphs:

$$J[a, b](\omega) := \sum_{\Gamma\in\mathcal{G}[a,b]} \prod_{st\in\Gamma} (-\lambda U_{st}(\omega)). \tag{A.1.3}$$

This last definition leads us to the definition of the *Lace Functions*:

$$\Pi_m(x) := \sum_{\substack{\omega:0\rightsquigarrow x\\|\omega|=m}} J[0,m](\omega)\prod_{r=1}^m s(\omega(r)-\omega(r-1)), \tag{A.1.4}$$

for any $m \geq 2$ and $x \in \mathbb{Z}^d$. Note here that $\Pi_1 \equiv 0$ since $\omega(i+1) \neq \omega(i)$, for any path ω and any $i \geq 0$. The lace expansion formula (6.2.3) is given in the following Lemma:

Lemma A.1. *For $n \geq 1$ and $x \in \mathbb{Z}^d$,*

$$C_n(x) = u\sum_{y:y\in\Omega} S(y)C_{n-1}(x-y) + \sum_{m=2}^n\sum_{z\in\mathbb{Z}^d}\Pi_m(z)C_{n-m}(x-z).$$

Proof. It suffices to show that for each path ω we have (suppressing ω in the formulas):

$$K[0,n] = K[1,n] + \sum_{m=2}^n J[0,m]K[m,n]. \tag{A.1.5}$$

Indeed, the Lemma is obtained by summing on both sides over all paths ω of length n, going from 0 to x, multiplying each summand with the product $\prod_{r=1}^n s(\omega(r)-\omega(r-1))$ and factorizing the sum. To prove (A.1.5), we note from (A.1.2) that the contribution to $K[0,n]$ from all graphs Γ for which 0 is not in an edge is exactly $K[1,n]$. For the contribution of the remaining graphs, we proceed as follows: If Γ does contain an edge starting at 0 we suppose that $m \leq n$ is the largest integer such that the set of edges in Γ with at least one end in the interval $[0,m]$ forms a connected graph on $[0,m]$. Then, resummation over graphs on $[m,n]$ gives

$$K[0,n] = K[1,n] + \sum_{m=2}^n\sum_{\Gamma\in\mathcal{G}[0,m]}\prod_{st\in\Gamma}(-\lambda U_{st})K[m,n].$$

Together with (A.1.3) this proves (A.1.5). □

Note that this Lemma is of course valid in the particular case where $S = \frac{1}{2d}\mathbb{1}_{\{x:\,\|x\|=1\}}$ (symmetric nearest neighbor initial distribution).
We need to bound the lace functions (A.1.4) in a good way. This in fact turns out to be rather tricky and we have to rewrite the definition of these functions: First, we rewrite the right-hand side of (A.1.3) to obtain

$$\begin{aligned} J[a,b] &= \sum_{L\in\mathcal{L}[a,b]}\sum_{\Gamma:\,L_\Gamma=L}\prod_{st\in L}(-\lambda U_{st})\prod_{s't'\in\Gamma\setminus L}(-\lambda U_{s't'}) \\ &= \sum_{L\in\mathcal{L}[a,b]}\prod_{st\in L}(-\lambda U_{st})\prod_{s't'\in\mathcal{C}(L)}(1-\lambda U_{s't'}). \end{aligned} \tag{A.1.6}$$

For $0 \leq a < b$, we define $J^{(N)}[a,b]$ to be, up to the factor $(-\lambda)^N$, the contribution to (A.1.6) coming from laces consisting of exactly N edges ($N \geq 1$):

$$J^{(N)}[a,b] := \sum_{L\in\mathcal{L}^{(N)}[a,b]}\prod_{st\in L}U_{st}\prod_{s't'\in\mathcal{C}(L)}(1-\lambda U_{s't'}).$$

Then,

$$J[a,b] = \sum_{N=1}^\infty(-\lambda)^N J^{(N)}[a,b],$$

and defining

$$\Pi_m^{(N)}(x) := \sum_{\substack{\omega:0\rightsquigarrow x\\|\omega|=m}} J^{(N)}[0,m](\omega)\prod_{r=1}^{m} s(\omega(r)-\omega(r-1))$$

$$= \sum_{\substack{\omega:0\rightsquigarrow x\\|\omega|=m}} \sum_{L\in\mathcal{L}^{(N)}[0,m]} \prod_{st\in L} U_{st} \prod_{s't'\in\mathcal{C}(L)} (1-\lambda U_{s't'}) \prod_{r=1}^{m} s(\omega(r)-\omega(r-1)), \tag{A.1.7}$$

we have together with (A.1.4)

$$\Pi_m(x) = \sum_{N=1}^{\infty} (-\lambda)^N \Pi_m^{(N)}(x). \tag{A.1.8}$$

A.1.2 Bounds on the Lace Expansion

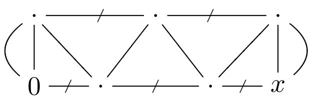
We bound the Π_m's in terms of C_n's. This allows us to give specific bounds for $\Pi_m(x)$ if we assume Exponential decay for the C_n's in x. We bound the terms in the sum (A.1.8) separately. For $N = 1$, we have:

$$\begin{aligned}
\Pi_m^{(1)}(x) =& \delta_{0x} \sum_{\substack{\omega:0\rightsquigarrow x\\|\omega|=m}} \prod_{\substack{0\le s'<t'\le m\\ s't'\neq 0m}} (1-\lambda U_{s't'}(\omega)) \prod_{r=1}^{m} s(\omega(r)-\omega(r-1))\\
\le& \delta_{0x} \sum_{y:y\in\Omega} s(y) \sum_{\substack{\omega:y\rightsquigarrow 0\\|\omega|=m-1}} \prod_{1\le s'<t'\le m} (1-\lambda U_{s't'}(\omega)) \prod_{r=2}^{m} s(\omega(r)-\omega(r-1))\\
=& \delta_{0x} uS * C_{m-1}(0),
\end{aligned} \tag{A.1.9}$$

where we use that $(1-\lambda U_{0l}(\omega)) \le 1$, for any $l \in \{1,\ldots,m-1\}$. Turning to the case $N \ge 2$, we remark that a walk giving a non-zero contribution to $\Pi_m^{(N)}$ must intersect itself at least N times in order that $U_{st} \neq 0$, for all $st \in L$. Then we can split the walk into $2N-1$ subwalks of lengths $m_1,\ldots,m_{2N-1}$, where only $m_3, m_5, \ldots, m_{2N-3}$ may be zero, and $\sum_i m_i = m$. Using again the fact that $1-\lambda U_{s't'} \le 1$, and replacing $1-\lambda U_{s't'}$ by 1 if $\omega(s')$ and $\omega(t')$ belong to different subwalks, we get the estimate

$$\begin{aligned}
\Pi_m^{(N)}(x) \le& \sum_{m_i} \sum_{x_1,\ldots,x_{N-2}\in\mathbb{Z}^d} C_{m_1}(x_1)C_{m_2}(-x_1)C_{m_3}(x_2)C_{m_4}(x_1-x_2)C_{m_5}(x_3-x_1)\cdots\\
&\cdots C_{m_{2N-3}}(x-x_{N-3})C_{m_{2N-2}}(x_{N-2}-x)C_{m_{2N-1}}(x-x_{N-2}),
\end{aligned} \tag{A.1.10}$$

where the sum over the m_i's is restricted to the set described above. Below, we give an example of a trajectory for $N = 7$ (slashed lines denote subwalks which may be zero, non-slashed ones must contain at least one step).



Lemma A.2. *Fix $m \ge 2$, and $d \ge 5$. Assume that for all $x \in \mathbb{Z}^d$, and $n < m$, $n \in \mathbb{N}$,*

$$|C_n(x)| \le L_1 \mu^n \theta_{n\nu,n\sigma}(x),$$

with constants $\mu > 0$, $L_1 \geq 1$ and $\theta_{n\nu,n\sigma}$ a "doubly-exponential" density with mean $n\nu$, $\nu \in \mathbb{R}^d$ and covariance matrix $n\sigma Id_d$, $\sigma > 0$ (see Section 6.2). Then, for $\lambda = \lambda(d,\sigma,\nu,L_1)$ small enough, we have

$$|\Pi_m(x)| \leq \lambda L_1 K \mu^m m^{-d/2} \sum_{k=1}^{m/2} \exp\left(-\frac{\sqrt{m-k}}{\sqrt{\sigma}}\|\nu\|\right)\theta_{k\nu,k\sigma}(x), \tag{A.1.11}$$

where $K = K(d,\nu,\sigma,S) > 0$.

Note that if $\nu = 0$, the rate of decay of the sequence $(\Pi_m/\mu^m)_m$ is completely different than if $\nu \neq 0$. In the former case, the decay is merely polynomially in m, whereas in the latter case, the decay is exponential in m (see also Corollary A.3).

Proof. For notational convenience we set

$$\psi_m(x) := m^{-d/2} \sum_{k=1}^{m/2} \exp\left(-\frac{\sqrt{m-k}}{\sqrt{\sigma}}\|\nu\|\right)\theta_{k\nu,k\sigma}. \tag{A.1.12}$$

The idea is to again use the sum (A.1.8) and bound each term $\Pi_m^{(N)}(x)$ separately by induction. For $N = 1$, we have, using inequality (A.1.9) and the assumptions in the Lemma:

$$\Pi_m^{(1)}(x) \leq \delta_{0x} u L_1 \mu^{m-1} \sum_{y\in\Omega} S(y)\theta_{(m-1)\nu,(m-1)\sigma}(-y).$$

But since the support Ω of S is bounded,

$$\begin{aligned}
&\sum_{y\in\Omega} S(y)\theta_{(m-1)\nu,(m-1)\sigma}(-y)\\
\leq &\frac{K(d,S)}{\sqrt{(m-1)\sigma}^d} \max_{y\in\Omega} \exp\left(-\frac{1}{\sqrt{(m-1)\sigma}}\|-y-(m-1)\nu\|\right)\\
\leq &\frac{K(d,S)}{(m\sigma)^{d/2}} \max_{y\in\Omega} \exp\left(-\frac{1}{\sqrt{m\sigma}}\|-y-(m-1)\nu\|\right)\\
\leq &\frac{K(d,S)}{(m\sigma)^{d/2}} \exp\left(-\sqrt{\frac{m-1}{\sigma}}\|\nu\|\right) \max_{y\in\Omega} \exp\left(\frac{1}{\sqrt{m\sigma}}\|y\|\right)\\
\leq &\frac{K(d,S,\sigma)}{(m\sigma)^{d/2}} \exp\left(-\sqrt{\frac{m-1}{\sigma}}\|\nu\|\right).
\end{aligned}$$

Hence,

$$\Pi_m^{(1)}(x) \leq K(d,S,\sigma,\nu) L_1 \mu^m \psi_m(x). \tag{A.1.13}$$

For $N \geq 2$, we set $\theta_{0,0}(x) := \delta_{0x}$ and define

$$\begin{aligned}
P_m^{(N)}(x) := \sum_{y_i,m_i} &\theta_{m_1\nu,m_1\sigma}(y_1)\theta_{m_2\omega,m_2\sigma}(-y_1)\theta_{m_3\omega,m_3\sigma}(y_2)\theta_{m_4\omega,m_4\sigma}(y_1-y_2)\cdots\\
&\cdots\theta_{m_{2N-3}\omega,m_{2N-3}\sigma}(x-y_{N-3})\theta_{m_{2N-2}\omega,m_{2N-2}\sigma}(y_{N-2}-x)\theta_{m_{2N-1}\omega,m_{2N-1}\sigma}(x-y_{N-2}),
\end{aligned}$$

where $y_1,\ldots,y_{N-2} \in \mathbb{Z}^d$, and $m_1,m_3,m_5,\ldots,m_{2N-3} \in \mathbb{N}_0$, $m_2,m_4,\ldots,m_{2N-2},m_{2N-1} \in \mathbb{N}$ such that $\sum_i m_i = m$ (note that in contrast to the lace expansion, we allow the first path to be zero in order to give the induction below). With (A.1.10) and the assumptions in the Lemma it follows that

$$\Pi_m^{(N)}(x) \leq L_1^{2N-1}\mu^m P_m^{(N)}(x). \tag{A.1.14}$$

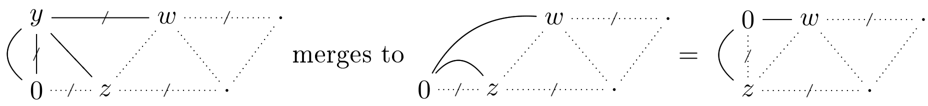
We use induction to show that there is a constant L_2 depending on d, σ and ν such that

$$P_m^{(N)}(x) \leq L_2^N \psi_m(x). \tag{A.1.15}$$

(A.1.13), (A.1.14) and (A.1.15) then imply

$$\Pi_m^{(N)}(x) \leq K(d,\nu,\sigma,S) L_2^N L_1^{2N+1} \mu^m \psi_m(x).$$

The Lemma follows by plugging the above inequality into (A.1.8) and choosing $\lambda = \lambda(d,\sigma,\nu,L_1)$ small enough.
The induction step for $N \geq 3$ reduces $P_m^{(N)}$ to $P_m^{(N-1)}$ by merging four subwalks in the lace into two as shown in the following figure:



We use Cauchy-Schwarz inequality and abbreviate $\theta_t := \theta_{t\nu,t\sigma}$ in what follows.

$$\begin{aligned}
&\sum_{\substack{u_1+u_2=u\\u_1,u_2\geq 0}} \sum_{\substack{t_1+t_2=t\\t_1,t_2\geq 1}} \sum_{y\in\mathbb{Z}^d} \theta_{u_1}(y)\theta_{u_2}(w-y)\theta_{t_1}(-y)\theta_{t_2}(y-z)\\
&\leq \sum_{\substack{u_1+u_2=u\\t_1+t_2=t\\t_1,t_2,\geq 1,u_1,u_2\geq 0}} \Bigg[\underbrace{\sum_{y\in\mathbb{Z}^d}\theta_{u_1}^2(y)\theta_{u_2}^2(w-y)}_{A}\Bigg]^{1/2} \Bigg[\underbrace{\sum_{y\in\mathbb{Z}^d}\theta_{t_1}^2(-y)\theta_{t_2}^2(y-z)}_{B}\Bigg]^{1/2}.
\end{aligned} \tag{A.1.16}$$

If $u=0$, $u_1=u_2=0$, and $A=\delta_0(w)=\theta_0(w)$, and hence, $A^{1/2}\leq K\theta_0(w)$. If $u>0$, and $u_1=0$, $u=u_2$, and

$$A^{1/2} = \Bigg[\sum_{y\in\mathbb{Z}^d}\delta_0(y)\theta_{u_2}^2(w-y)\Bigg]^{1/2} = \theta_u(w).$$

The case $u>0$, and $u_2=0$ is equivalent. Finally, if $u_1,u_2>0$, we use (B.0.1) and Lemma B.1 to get

$$\begin{aligned}
&\Bigg[\sum_{y\in\mathbb{Z}^d}\theta_{u_1}^2(y)\theta_{u_2}^2(w-y)\Bigg]^{1/2} \leq K(d,\sigma)\left[\theta_{u_1}^2 * \theta_{u_2}^2(w)\right]^{1/2}\\
\leq &K(d,\sigma)(u_1u_2)^{-d/4}\sigma^{-d/2}(\theta_{u\nu,(u/2)\sigma}(w))^{1/2} \leq K(d,\sigma)u^{d/4}(u_1u_2)^{-d/4}\theta_u(w),
\end{aligned}$$

and therefore,

$$\begin{aligned}
&\sum_{\substack{u_1+u_2=u\\u_1,u_2\geq 0}}\left(\sum_{y\in\mathbb{Z}^d}\theta_{u_1}^2(y)\theta_{u_2}^2(w-y)\right)^{1/2}\\
\leq &K(d,\sigma)\left(2+u^{d/4}\sum_{u_1=1}^{u-1}u_1^{-d/4}(u-u_1)^{-d/4}\right)\theta_u(w)\\
\leq &K(d,\sigma)\theta_{u\nu,u\sigma}(w),
\end{aligned}$$

where we use that $d \geq 5$ and hence, $u^{d/4} \sum_{u_1}^{u-1} u_1^{-d/4}(u-u_1)^{-d/4}$ can be bounded uniformly for all u. For B, we apply the same reasoning as for A. Inserting these bounds into (A.1.16) yields

$$\sum \theta_{u_1}(y)\theta_{u_2}(w-y)\theta_{t_1}(-y)\theta_{t_2}(y-z) \leq K(d,\sigma)\theta_u(w)\theta_t(-z). \tag{A.1.17}$$

Using (A.1.17) with y_1, y_3, y_2, m_1, m_5, m_2, and m_4 instead of y, w, z, u_1, u_2, t_1 and t_2 respectively, we get

$$\begin{aligned} P_m^{(N)}(x) &= \sum_{m_i, y_i} \theta_{m_3}(y_2)\theta_{m_4}(y_1-y_2)\theta_{m_2}(-y_1)\theta_{m_1}(y_1)\theta_{m_5}(y_3-y_1)\theta_{m_6}(y_2-y_3)\cdots \\ &\leq K(d,\sigma)\sum \theta_{m_3}(y_2)\theta_{m_2+m_4}(-y_2)\theta_{m_1+m_5}(y_3)\theta_{m_6}(y_2-y_3)\cdots \\ &\leq K(d,\sigma)P_m^{(N-1)}(x), \end{aligned} \tag{A.1.18}$$

finishing the induction step.
It remains to show the case $N=2$: In this case, the lace is three-legged and we have

$$P_m^{(2)}(x) = \sum_{\substack{k+l+j=m \\ l,j\geq 1, k\geq 0}} \theta_k(x)\theta_l(-x)\theta_j(x) = \delta_{0x}I + J,$$

where

$$\begin{aligned} I &= \sum_{l=1}^{m-1} \theta_l(0)\theta_{m-l}(0) \\ &= \sum_{l=1}^{m-1} \frac{K(d)}{\sigma^d}\frac{1}{(l(m-l))^{d/2}} \exp\left(-\frac{1}{\sqrt{l\sigma}}\|(-l\nu)\| - \frac{1}{\sqrt{(m-l)\sigma}}\|(-(m-l)\nu)\|\right) \\ &\leq \sum_{l=1}^{m-1} \frac{K(d)}{(l(m-l))^{d/2}\sigma^d} \exp\left(-\left[\frac{1}{l\sigma}\|(l\nu)\|^2 + \frac{1}{(m-l)\sigma}\|((m-l)\nu)\|^2\right]^{1/2}\right) \\ &\leq \frac{K(d,\sigma)}{m^{d/2}}\frac{1}{\sigma^{d/2}} \exp\left(-\sqrt{\frac{m}{\sigma}}\|\nu\|\right) \\ &\leq K(d,\sigma,\nu)\psi_m(0), \end{aligned}$$

and

$$\begin{aligned} J &= \sum_{\substack{k+j+l=m \\ k,l,j\geq 1}} \theta_k(x)\theta_l(-x)\theta_j(x) \\ &= \sum_{\substack{k+j+l=m \\ k,l,j\geq 1}} \frac{K(d)}{\sigma^{(3/2)d}(kjl)^{d/2}} \exp\Bigg(-\frac{1}{\sqrt{k\sigma}}\|x-k\nu\| - \frac{1}{\sqrt{j\sigma}}\|x-j\nu\| \\ &\quad -\frac{1}{\sqrt{l\sigma}}\|-x-l\nu\|\Bigg) \\ &\leq K(d,\sigma) \sum_{\substack{1\leq k\leq j \\ k+l+j=m}} l^{-d/2}m^{-d/2}\frac{1}{(k\sigma)^{d/2}} \exp\left(-\frac{\|x-k\nu\|}{\sqrt{k\sigma}}\right) \\ &\quad \cdot \exp\left(-\frac{1}{\sqrt{\sigma}}\left(\frac{1}{j}\|x\|^2 - \langle x,\nu\rangle + j\|\nu\|^2 + \frac{1}{l}\|x\|^2 + \langle x,\nu\rangle + l\|\nu\|^2\right)^{1/2}\right) \\ &\leq K(d,\sigma)m^{-d/2}\sum_{k=1}^{m/2} \exp\left(-\frac{\sqrt{m-k}}{\sqrt{\sigma}}\|\nu\|\right)\theta_{k\nu,k\sigma}(x) \\ &\leq K(d,\sigma)\psi_m(x), \end{aligned}$$

where we use the subadditivity of the square-root function. I and J together thus imply

$$P_m^{(2)}(x) \leq K(d,\sigma,\nu)\psi_m(x). \tag{A.1.19}$$

Setting L_2 to be the maximum of the constant in (A.1.18) and (A.1.19), we get (A.1.15). □

Under the assumptions of Lemma A.2, the next Corollary which gives explicit decay rates in m for the moments of the B_m's follows immediately.

Corollary A.3. *Let $\nu \neq 0$. Then, for B_m defined by $B_m(x) := \frac{\Pi_m(x)}{\lambda\mu^m}$ for all $m \geq 2$, with C_n, λ and μ satisfying the assumptions of Lemma A.2, we have that the first three moments of the B_m's decay exponentially in m. More precisely,*

$$\begin{aligned}
\sum_x |B_m(x)| &\leq K_0 \exp(-k_0\sqrt{m}),\\
\sum_x |x_i||B_m(x)| &\leq K_1 \exp(-k_1\sqrt{m}),\\
\sum_x |x_i x_j||B_m(x)| &\leq K_2 \exp(-k_2\sqrt{m}), \quad \textit{and}\\
\sum_x |x_i x_j x_k||B_m(x)| &\leq K_3 \exp(-k_3\sqrt{m}), \quad \textit{for } i,j,k = 1,\dots,d,
\end{aligned}$$

where $K_1, K_2, K_3, k_1, k_2, k_3$ are positive constants depending on d, σ, ν, S and L_1.
If $\nu = 0$, this changes completely. The first moments of course vanish. For the zeroth moments we get a decay of order $m^{-(d-2)/2}$, and for the second moments a decay of order $m^{-(d-4)/2}$. The decay of the third moments is then of order $m^{-(d-5)/2}$.

The above Lemma A.2 is valid for any drift ν. However, in case $\nu = 0$, we can give a slightly better bound for the Π_m's than the one stated above. This is the content of the next Lemma:

Lemma A.4. *Under the same assumptions as in Lemma A.2 with $\nu = 0$, and for $\lambda = \lambda(d, L_1, \sigma)$ small enough, we have*

$$|\Pi_m(x)| \leq \lambda L_1 K \mu^m m^{-d/2} \sum_{k=1}^{m/2} k^{1-d/2}\theta_{0,k\sigma}(x), \tag{A.1.20}$$

where $K = K(d, \sigma, S) > 0$.

Proof. The proof of this Lemma is essentially a copy of the proof of Lemma A.2. First, we replace (A.1.12) by

$$\psi_m(x) := m^{-d/2} \sum_{k=1}^{m/2} k^{1-d/2}\theta_{0,k\sigma}, \tag{A.1.21}$$

and put $\nu = 0$ everywhere in that proof. The only place where things change is the calculation of J:

$$\begin{aligned}
J =& \sum_{\substack{k+j+l=m \\ k,l,j\geq 1}} \theta_k(x)\theta_l(-x)\theta_j(x) \\
=& \sum_{\substack{k+j+l=m \\ k,l,j\geq 1}} \frac{K(d)}{\sigma^{(3/2)d}(kjl)^{d/2}} \exp\left(-\frac{1}{\sqrt{k\sigma}}\|x\| - \frac{1}{\sqrt{j\sigma}}\|x\| - \frac{1}{\sqrt{l\sigma}}\|x\|\right) \\
\leq& K(d,\sigma) \sum_{\substack{1\leq k\leq l\leq j \\ k+l+j=m}} l^{-d/2} m^{-d/2} \frac{1}{(k\sigma)^{d/2}} \exp\left(-\frac{\|x\|}{\sqrt{k\sigma}}\right) \\
\leq& K(d,\sigma) m^{-d/2} \sum_{k=1}^{m/2} k^{1-d/2} \theta_{0,k\sigma}(x) \\
\leq& K(d,\sigma)\psi_m(x),
\end{aligned}$$

where we use the symmetry of J in k, l and j. This amounts for the change in the Lemma. □

This of course also gives better bounds for the moments of the B_m's:

Corollary A.5. *Under the hypotheses of the last Lemma and setting* $B_m(x) := \frac{\Pi_m(x)}{\lambda\mu^m}$ *for all* $m \geq 2$, *we have that the first moments of the* B_m*'s vanish. The zeroth moments are of order* $m^{-d/2}$, *the second moments of order* $m^{-(d-3)}$ *and the third moments are of order* $m^{-(d-7/2)}$.

Appendix B

The Discrete and the Continuous Folding of "Doubly-Exponential" Distributions

We explain here how to fold (or better bound the folding of) two "doubly-exponential" distributions on $\mathbb{R}^d$. For that purpose, let $t_1, t_2 \in \mathbb{N}$, $\mu_1, \mu_2 \in \mathbb{R}^d$, and $\sigma_1, \sigma_2 > 0$. Then, for $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} dy \theta_{t_1\mu_1, t_1\sigma_1}(y)\theta_{t_2\mu_2, t_2\sigma_2}(x-y) \le K(d)\theta_{t_1\mu_1+t_2\mu_2, t_1\sigma_1+t_2\sigma_2}(x). \tag{B.0.1}$$

Indeed, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} dy \theta_{t_1\mu_1, t_1\sigma_1}(y)\theta_{t_2\mu_2, t_2\sigma_2}(x-y)\\
=&\int_{\mathbb{R}^d} dy \frac{K(d)}{\sqrt{t_1t_2\sigma_1\sigma_2}^d} \exp\left(-\frac{1}{\sqrt{t_1\sigma_1}}\|y-t_1\mu_1\| - \frac{1}{\sqrt{t_2\sigma_2}}\|x-t_2\mu_2-y\|\right)\\
=&\int_{\mathbb{R}^d} dy \frac{K(d)}{\sqrt{t_1t_2\sigma_1\sigma_2}^d} \exp\left(-\frac{1}{\sqrt{t_1\sigma_1}}\|y\| - \frac{1}{\sqrt{t_2\sigma_2}}\|x-t_1\mu_1-t_2\mu_2-y\|\right)\\
=&\frac{K(d)}{\sqrt{t_1t_2\sigma_1\sigma_2}^d}\int_{\mathbb{R}^d} dy \exp\left(-\frac{1}{\sqrt{t_1\sigma_1}}\|y\| - \frac{1}{\sqrt{t_2\sigma_2}}\left(\sum_{i=1}^d (x_i - t_1\mu_{1i} - t_2\mu_{2i} - y_i)^2\right)^{1/2}\right).
\end{aligned}$$

For simplicity, we set $p_i := x_i - t_1\mu_{1i} - t_2\mu_{2i}$ for $i = 1, \dots, d$. In the following we change variables from $y_1, \dots, y_d$ to hyperspheric coordinates $(r, \phi_1, \dots, \phi_{d-1})$, and denote by $J := r^{d-1}\sin^{d-2}(\phi_1)\cdots\sin(\phi_{d-2})$ the corresponding Jacobian. Moreover, $I := [0,\infty) \times [0,\pi]^{(d-2)} \times [0, 2\pi]$ is the area of integration under hyperspheric coordinates. The above then turns into:

$$\begin{aligned}
=&\frac{K(d)}{\sqrt{t_1t_2\sigma_1\sigma_2}^d}\int_{\mathbb{R}^d} dy \exp\left(-\frac{1}{\sqrt{t_1\sigma_1}}\|y\| - \frac{1}{\sqrt{t_2\sigma_2}}\left(\|y\|^2 + \|p\|^2 - 2\langle y, p\rangle\right)^{1/2}\right)\\
\le&\frac{K(d)}{\sqrt{t_1t_2\sigma_1\sigma_2}^d}\int_{\mathbb{R}^d} dy \exp\left(-\frac{1}{\sqrt{t_1\sigma_1}}\|y\| - \frac{1}{\sqrt{t_2\sigma_2}}\left(\|y\|^2 + \|p\|^2 - 2\|y\|\|p\|\right)^{1/2}\right)\\
=&\frac{K(d)}{\sqrt{t_1t_2\sigma_1\sigma_2}^d}\int_{\mathbb{R}^d} dy \exp\left(\frac{1}{\sqrt{t_1\sigma_1}}\|y\| - \frac{1}{\sqrt{t_2\sigma_2}}\left|\|y\| - \|p\|\right|\right)\\
=&\frac{K(d)}{\sqrt{t_1t_2\sigma_1\sigma_2}^d}\int_I dr d\phi_1 \cdots d\phi_{d-1} |J| \exp\left(-\frac{1}{\sqrt{t_1\sigma_1}} r - \frac{1}{\sqrt{t_2\sigma_2}}\left|r - \|p\|\right|\right)\\
\le&\frac{K(d)}{\sqrt{t_1t_2\sigma_1\sigma_2}^d}\int_{[0,\infty)} dr r^{d-1} \exp\left(-\frac{\sqrt{t_2\sigma_2}}{\sqrt{t_1t_2\sigma_1\sigma_2}} r - \frac{\sqrt{t_1\sigma_1}}{\sqrt{t_1t_2\sigma_1\sigma_2}}\left|r - \|p\|\right|\right).
\end{aligned}$$

It remains to integrate this. Assume that $t_2\sigma_2 < t_1\sigma_1$ and apply $d-1$ times partial integration to obtain:

$$=\frac{K(d)}{\sqrt{t_1t_2\sigma_1\sigma_2}^d}\left(\frac{\sqrt{t_1t_2\sigma_1\sigma_2}^d}{(\sqrt{t_1\sigma_1}+\sqrt{t_2\sigma_2})^d}e^{-\|p\|/\sqrt{t_1\sigma_1}}+\frac{\sqrt{t_1t_2\sigma_1\sigma_2}^d}{(\sqrt{t_1\sigma_1}-\sqrt{t_2\sigma_2})^d}\left(e^{-\|p\|/\sqrt{t_1\sigma_1}}-e^{-\|p\|/\sqrt{t_2\sigma_2}}\right)\right)$$
$$\leq\frac{K(d)}{\sqrt{t_1\sigma_1+t_2\sigma_2}^d}e^{-\|p\|/\sqrt{t_1\sigma_1+t_2\sigma_2}}=\frac{K(d)}{\sqrt{t_1\sigma_1+t_2\sigma_2}^d}\exp\left(-\frac{\|x-t_1\mu_1-t_2\mu_2\|}{\sqrt{t_1\sigma_1+t_2\sigma_2}}\right).$$

This is precisely the desired result in (B.0.1). The reasoning for $t_t\sigma_2 > t_1\sigma_1$ is analogous. Moreover, if $t_1\sigma_1 = t_2\sigma_2$, partial integration again yields the same result.
We still need to check that the discrete folding of two doubly-exponential distributions can be bounded from above by a continuous folding of the same two distributions.

Lemma B.1. *Let $\theta_{t_1\mu_1,t_1\sigma_1}$ and $\theta_{t_2\mu_2,t_2\sigma_2}$ be two doubly exponential densities with $t_1,t_2\in\mathbb{N}$, $\mu_1,\mu_2\in\mathbb{R}^d$ and $\sigma_1,\sigma_2>0$. Let $x\in\mathbb{Z}^d$. Then*

$$\sum_{y\in\mathbb{Z}^d}\theta_{t_1\mu_1,t_1\sigma_1}(y)\theta_{t_2\mu_2,t_2\sigma_2}(x-y)\leq K(d,\sigma_1,\sigma_2)\int_{\mathbb{R}^d}dy\,\theta_{t_1\mu_1,t_1\sigma_1}(y)\theta_{t_2\mu_2,t_2\sigma_2}(x-y)$$
$$\leq K(d,\sigma_1,\sigma_2)\theta_{t_1\mu_1+t_2\mu_2,t_1\sigma_1+t_2\sigma_2}(x).$$

Proof. Note that it is sufficient to prove the Lemma for $\mu_1=\mu_2=0$. Let $I^d := [-1/2,1/2]^d$ and $y+I^d$ be the shifted cube. Then, using (B.0.1) and Jensen's inequality on the third line,

$$\theta_{0,t_1\sigma_1+t_2\sigma_2}(x)\geq K(d)\sum_{y\in\mathbb{Z}^d}\int_{y+I^d}\theta_{0,t_1\sigma_1}(s)\theta_{0,t_2\sigma_2}(x-s)ds \tag{B.0.2}$$
$$=C\sum_{y\in\mathbb{Z}^d}\int_{I^d}\exp\left(-\frac{1}{\sqrt{t_1\sigma_1}}\|s+y\|-\frac{1}{\sqrt{t_2\sigma_2}}\|x-s-y\|\right)ds$$
$$\geq C\sum_{y\in\mathbb{Z}^d}\exp\left(-\int_{I^d}\frac{1}{\sqrt{t_1\sigma_1}}\|s+y\|+\frac{1}{\sqrt{t_2\sigma_2}}\|x-s-y\|ds\right),$$

where $C=\frac{K(d)}{\sqrt{t_1\sigma_1}^d\sqrt{t_2\sigma_2}^d}$. For the first term in the exponent, we use again Jensen to obtain

$$\int_{I^d}\frac{1}{\sqrt{t_1\sigma_1}}\left(\sum_{i=1}^d(s_i+y_i)^2\right)^{1/2}ds\leq\frac{1}{\sqrt{t_1\sigma_1}}\left(\sum_{i=1}^d\int_{I^d}(s_i+y_i)^2ds\right)^{1/2}$$
$$=\frac{1}{\sqrt{t_1\sigma_1}}\left(\sum_{i=1}^d\int_{I^d}s_i^2+2s_iy_i+y_i^2ds\right)^{1/2}\leq\frac{1}{\sqrt{t_1\sigma_1}}\left(\sum_{i=1}^d y_i^2\right)^{1/2}+K(d,\sigma_1).$$

The second term is treated analogously. Reinserting into (B.0.2) gives

$$\theta_{0,t_1\sigma_1+t_2\sigma_2}(x)\geq K(d)\exp\left(-K(d,\sigma_1)-K(d,\sigma_2)\right)\theta_{0,t_1\sigma_1}(y)\theta_{0,t_2\sigma_2}(x-y).$$

This finishes the proof of the Lemma. □

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