Unified Quantum $SO(3)$ and $SU(2)$ Invariants for Rational Homology 3–Spheres

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In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics.

Hermann Weyl 1885–1955

To Manuel
who made a dream become real

and to Heidy Anna and Winfried
without whom I would not exist

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Abstract

Inspired by E. Witten's work, N. Reshetikhin and V. Turaev introduced in 1991 important invariants for 3–manifolds and links in 3–manifolds, the so–called quantum (WRT) $SU(2)$ invariants. Short after, R. Kirby and P. Melvin defined a modification of these invariants, called the quantum (WRT) $SO(3)$ invariants. Each of these invariants depends on a root of unity.

In this thesis, we give a unification of these invariants. Given a rational homology 3–sphere M and a link L inside, we define the unified invariants $I^{SU(2)}_{M,L}$ and $I^{SO(3)}_{M,L}$, such that the evaluation of these invariants at a root of unity equals the corresponding quantum (WRT) invariant. In the $SU(2)$ case, we assume the order of the first homology group of the manifold to be odd. Therefore, for rational homology 3–spheres, our invariants dominate the whole set of $SO(3)$ quantum (WRT) invariants and, for manifolds with the order of the first homology group odd, the whole set of $SU(2)$ quantum (WRT) invariants. We further show, that the unified invariants have a strong integrality property, i.e. that they lie in modifications of the Habiro ring, which is a cyclotomic completion of the polynomial ring $\mathbb{Z}[q]$.

We also give a complete computation of the quantum (WRT) $SO(3)$ and $SU(2)$ invariants of lens spaces with a colored unknot inside.

Zusammenfassung

Von E. Wittens Arbeit inspiriert definierten N. Reshetikhin und V. Turaev im Jahre 1991 wichtige Invarianten für 3–Mannigfaltikeiten und Verschlingungen in 3–Mannigfaltigkeiten, welche heute als (WRT) $SU(2)$ Quanteninvarianten bekannt sind. Wenig später führten R. Kirby und P. Melvin die (WRT) $SO(3)$ Quanteninvarianten ein, eine Modifikation der $SU(2)$ Invarianten. Alle diese Invarianten hängen von einer Einheitswurzel ab.

In dieser Dissertation geben wir eine Vereinigung dieser Invarianten an. Sei eine 3–dimensionale rationale Homologiesphäre M und eine Verschlingung L in M gegeben. Wir definieren die vereinigten Invarianten $I^{SU(2)}_{M,L}$ und $I^{SO(3)}_{M,L}$, so dass die Evaluierung dieser Invarianten an einer Einheitswurzel mit der entsprechenden (WRT) Quanteninvariante übereinstimmt. Im $SU(2)$ Fall verlangen wir, dass die Ordnung der ersten Homologiegruppe der 3–Mannigfaltigkeit M ungerade ist. Somit dominieren unsere Invarianten für rationale Homolgiesphären die Menge aller (WRT) $SO(3)$ Quanteninvarianten, respektive die Menge aller (WRT) $SU(2)$ Quanteninvarianten für Mannigfaltigkeiten mit erster Homolgiegruppe von ungerader Ordnung. Weiter zeigen wir, dass die vereinigten Invarianten eine starke Ganzzahligkeits–Eigenschaft besitzen: Sie liegen in Modifikationen des Habiro Rings, einem zyklotomischen Abschluss des Polynomrings $\mathbb{Z}[q]$.

Weiter geben wir eine vollständige Berechnung der (WRT) $SO(3)$ und $SU(2)$ Quanteninvarianten von Linsenräumen mit einem gefärbten trivialen Knoten darin an.

Contents

Introduction

In 1984, V. Jones [16] discovered the famous Jones polynomial, a strong link invariant which led to a rapid development of knot theory. Many new link invariants were defined short after, including the so–called colored Jones polynomial which uses representations of a ribbon Hopf algebra acting as colors attached to each link component. The whole collection of invariants of this spirit are called quantum link invariants.

In the 60' and 70' of the last century, Likorish [26], Wallace [35] and Kirby [18] showed, that there is a one–to–one correspondence via surgery between closed oriented 3–manifolds up to homeomorphisms and knots in the 3–dimensional sphere modulo Kirby–moves. This gives the possibility to study 3–manifolds using knot theory.

In 1989, E. Witten [36] considered quantum field theory defined by the noncommutative Chern–Simons action to define (on a physical level of rigor) certain invariants of closed oriented 3–manifolds and links in 3–manifolds. Inspired by this work, N. Reshetikhin and V. Turaev [33, 34] constructed in 1991 new topological invariants of 3–manifolds and of links in 3–manifolds. The construction goes as follows. Let M be a closed, oriented 3–manifold and L_M its corresponding surgery link. The quantum group $U_q(\mathfrak{sl}_2)$ is a deformation of the Lie algebra $\mathfrak{sl}_2$ and has the structure of a ribbon Hopf algebra. One now takes the sum of the colored Jones polynomial of L_M, normalized in an appropriate way, over all colors, i.e over all finite–dimensional irreducible representations of $U_q(\mathfrak{sl}_2)$. Evaluating at a root of unity ξ makes the sum finite and well–defined. These invariants are denoted by $\tau_M(\xi)$. Together they form a sequence of complex numbers parameterized by complex roots of unity and are known either as the Witten–Reshetikhin–Turaev invariants, short WRT invariants, or as the quantum invariants of 3–manifolds. Since the irreducible representations of the quantum group $U_q(\mathfrak{sl}_2)$ correspond to the irreducible representations of the Lie group $SU(2)$, they are sometimes also called the quantum (WRT) $SU(2)$ invariants.

R. Kirby and P. Melvin [20] defined the $SO(3)$ version of the quantum (WRT) invariants by summing only over representations of $U_q(\mathfrak{sl}_2)$ of *odd* dimension and evaluating at roots of unity of *odd* order. These invariants are known as the quantum (WRT) $SO(3)$ invariants. They have very nice properties. For example, A. Beliakova and T. Le [5] showed that they are algebraic integers, i.e. $\tau_M^{SO(3)}(\xi) \in \mathbb{Z}[\xi]$ for any closed oriented 3–manifold M and any root of unity ξ (of odd order). Similar results where also proven for the $SU(2)$ invariants with some restrictions on either the manifold M or the order of the root of unity ξ (see [12], [3], [11], [28]). The full integrality result is conjectured and work in progress.

The integrality results are based on a unification of the quantum (WRT) invariant. For any integral homology 3–sphere M, K. Habiro [12] constructed a unified invariant J_M whose evaluation at any root of unity coincides with the value of the quantum (WRT) $SU(2)$ invariant at that root. The unified invariant is an element of a certain cyclotomic completion of a polynomial ring, also known as the Habiro ring. This ring has beautiful properties. For example, we can think of its elements as analytic functions at roots of unity [12]. Therefore, the unified invariant belonging to the Habiro ring means that the collection of the quantum (WRT) invariants is far from a random collection of algebraic integers: together they form a nice function.

In this thesis, we give a similar unification result for rational homology 3–spheres which includes Habiro's result for integral homology 3–spheres. More precisely, for a rational homology 3–sphere M, we define unified invariants $I_M^{SO(3)}$ and $I_M^{SU(2)}$ such that the evaluation at a root of unity ξ gives the corresponding quantum (WRT) invariant (up to some renormalization). In the $SU(2)$ case, we assume the order of the first homology group of the manifold to be odd – the even case turns out to be quite different from the odd case and is part of ongoing research. Further, new rings, similar to the Habiro ring, are constructed which have the unified invariants as their elements. We show that these rings have similar properties to those of the Habiro ring. We also give a complete computation of the quantum (WRT) $SO(3)$ and $SU(2)$ invariants for lens spaces with a colored unknot inside at all roots of unity.

Additionally to the techniques developed by Habiro, we use deep results coming from number theory, commutative algebra, quantum group and knot theory. The new techniques developed in Chapters 3 and 5 about cyclotomic completions of polynomial rings could be of separate interest for analytic geometry (compare [27]), quantum topology, and representation theory. Further, even though integrality of the quantum (WRT) invariants does *not* in general follow directly from the unification of the quantum (WRT) invariants, it does help proving it and a conceptual solution of the integrality problem is of primary importance for any attempt of a categorification of the quantum (WRT) invariants (compare [17]). Our results are also a step towards the unification of quantum (WRT) $\mathfrak{g}$ invariants of any semi–simple Lie algebra $\mathfrak{g}$ (see [34] for a definition of quantum (WRT) $\mathfrak{g}$ invariants). K. Habiro and T. Le announced such unified $\mathfrak{g}$ invariants for integral homology 3–spheres. We expect that the techniques introduced here will help to generalize their results to rational homology 3–spheres.

Plan of the thesis

In Chapter 1, we give the definition of the colored Jones polynomial and state that it has a cyclotomic expansion with integral coefficients. The proof of this integrality result is postponed to the Appendix. This expansion is used for the definition of the unified invariant (Chapter 4). In Chapter 2, the quantum (WRT) invariants are defined and important facts about (generalized) Gauss sums are stated. Chapter 3 is devoted to the theory of cyclotomic completions of polynomial rings. For a given b, we define the rings $\mathcal{R}_b$ and $\mathcal{S}_b$ and discuss the evaluation at a root of unity in these rings. In Chapter 4, the unified invariants $I_M^{SO(3)}$ and $I_M^{SU(2)}$ of a rational homology 3–sphere M are defined and the main results of this thesis, i.e. the invariance of $I_M^{SO(3)}$ and $I_M^{SU(2)}$ and that their evaluation at a root of unity equals the corresponding quantum (WRT) invariant, are proven. Here we use (technical) results from Chapters 6 and

7. In Chapter 5, we prepare Chapters 6 and 7 by showing that certain roots appearing in the unified invariants exist in the rings $\mathcal{R}_b$ and $\mathcal{S}_b$. In Chapter 6, we compute the quantum (WRT) invariants of lens spaces with a colored unknot inside and define the unified invariants of lens spaces. In Chapter 7, we define a Laplace transform which we use to prove the main technical result of this thesis, namely that the unified invariant $I_M^{SO(3)}$ (respectively $I_M^{SU(2)}$) is indeed an element of $\mathcal{R}_b$ (respectively of $\mathcal{S}_b$), where b is the order of the first homology group of the rational homology 3–sphere M.

The material of Chapters 1 and 2 is partly taken from [12], [26], [20] and [4]. Chapter 3 includes results of Habiro [12, 14]. The $SO(3)$ case of the results from Chapters 3 to 7 as well as the Appendix appeared in our joint paper with A. Beliakova and T. Le [4]. The $SU(2)$ case has not yet been published anywhere else.

1 Colored Jones Polynomial

In this chapter, we first recall some basic concepts of knot theory and quantum groups. We then define the universal $\mathfrak{sl}_2$ invariant of knots and links which leads us to the definition of the colored Jones Polynomial. In the last section, we state a generalization of Habiro's Theorem 8.2 of [12] about a cyclotomic expansion of the colored Jones polynomial which we need for the definition of the unified invariant in Chapter 4. The proof of this theorem is postponed to the Appendix.

Throughout this thesis, we will use the following notation. The n–dimensional sphere will be denoted by S^n, the n–dimensional disc by D^n and the unit interval $[0,1] \subset \mathbb{R}$ by I. The boundary of a manifold M is denoted by ∂M. Except otherwise stated, a manifold M is always considered to be closed, oriented and 3–dimensional.

1.1 Links, tangles and bottom tangles

A *link* L with m *components* in a manifold M is an equivalence class by ambient isotopy of smooth embeddings of m disjoint circles S^1 into M. A one–component link is called a *knot*. The link is *oriented* when an orientation of the components is chosen.

A *(rational) framing* of a link is an assignment of a rational number to each component of the link. It is called *integral* when all numbers assigned are integral. A *link diagram* of a framed link is a generic projection onto the plane as depicted in Figure 1.1, where the framing is denoted by numbers next to each component.

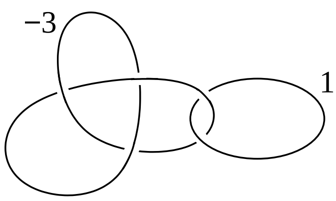


Figure 1.1: A link diagram of a framed link.

The *linking number* of two components L_1 and L_2 of an oriented link L is defined as follows. Each crossing in a link diagram of L between L_1 and L_2 counts as $+1$ or -1, see Figure 1.2 for the sign. The sum of all these numbers divided by 2 is called the linking number $\mathrm{lk}(L_1, L_2)$, which is independent of the diagram chosen for L. The *linking matrix* of a link L with components $L_1, L_2, \dots, L_n$ is a $n \times n$ matrix $(l_{ij})_{1 \le i,j \le n}$ with the framings of the L_i's on the diagonal and $l_{ij} = \mathrm{lk}(L_i, L_j)$ for $i \neq j$.

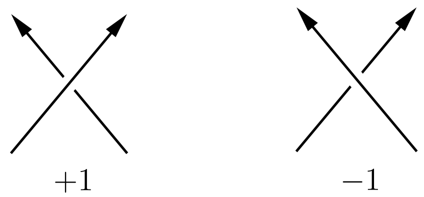


Figure 1.2: The assignment of $+1$ and -1 to the crossings.

A *tangle* T is an equivalence class by ambient isotopy (fixing $\partial I^3 \setminus \{\frac{1}{2}\} \times I \times \partial I$) of smooth embeddings of disjoint 1–manifolds into the unit cube I^3 in $\mathbb{R}^3 \subset S^3$ with $\partial T \subset \{\frac{1}{2}\} \times I \times \partial I$. We define $\partial_- T = T \cap (I^2 \times \{0\})$ and $\partial_+ T = T \cap (I^2 \times \{1\})$ and call T a (m,n)–tangle if $m = |\partial_+ T|$ and $n = |\partial_- T|$, where $|M|$ denotes the number of connected components of M. Thus a link is a $(0,0)$–tangle.

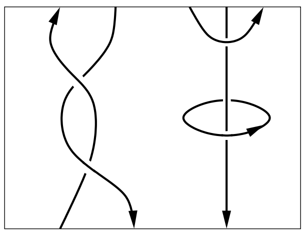


Figure 1.3: A diagram of an oriented $(5,3)$–tangle.

Framing, orientation and diagrams of tangles are defined analogously as for links. See Figure 1.3 for an example of a diagram of an oriented $(5,3)$–tangle. Every (oriented) tangle diagram can be factorized into the elementary diagrams shown in Figure 1.4 using composition $\circ$ (when defined) and tensor product $\otimes$ as defined in Figure 1.5. The oriented tangles can therefore be considered as the morphisms of a category $\mathcal{T}$ with objects $x_1 \otimes x_2 \otimes \cdots \otimes x_n$, $x_i \in \{\uparrow, \downarrow\}$.

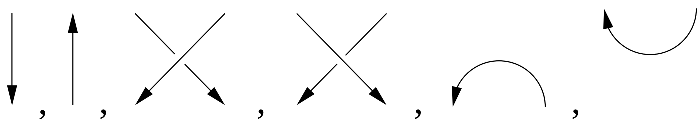


Figure 1.4: The fundamental tangles.

In the cube I^3, we define the points $p_i := \{\frac{1}{2}, \frac{i}{2n+1}, 0\}$ for $i = 1, \dots, 2n$, on the bottom of the cube. An *n–component bottom tangle* $T = T_1 \sqcup \cdots \sqcup T_n$ is an oriented $(0,n)$–tangle consisting

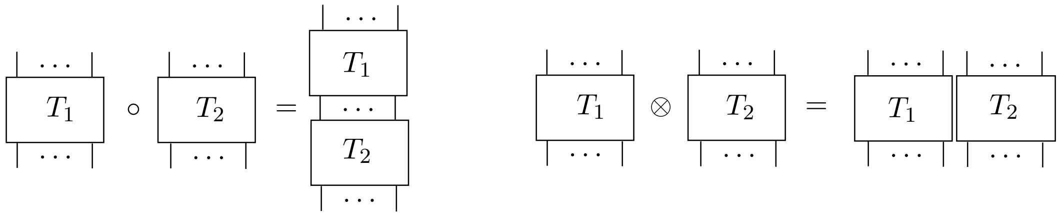


Figure 1.5: Composition and tensor product of tangle T_1 and tangle T_2.

of n arcs T_i homeomorphic to I and the i–th arc T_i starts at point p_{2i} and ends at p_{2i-1}. For an example, a diagram of the Borromean bottom tangle B is given in Figure 1.6.

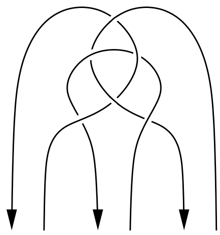


Figure 1.6: Borromean bottom tangle B.

The *closure* $\mathrm{cl}(T)$ of a bottom tangle T is the $(0,0)$–tangle obtained by taking the composition of T with the element $\cup\cup\ \cdots\ \cup$. See Figure 1.7 for an example.

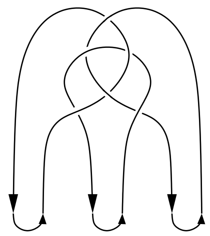


Figure 1.7: The closure $\mathrm{cl}(B)$ of B.

In [13], Habiro defined a subcategory B of the category of framed, oriented tangles $\mathcal{T}$. The objects of B are the symbols $\mathsf{b}^{\otimes n}$, $n \geq 0$, where $\mathsf{b} := \downarrow\uparrow$. A morphism X of B is a (m,n)–tangle mapping $\mathsf{b}^{\otimes m}$ to $\mathsf{b}^{\otimes n}$ for some $m, n \geq 0$. We can compose such a morphism with m–component bottom tangles to get n–component bottom tangles. Therefore, B acts on the bottom tangles by composition. The category B is braided: the monoidal structure is given by taking the tensor product of the tangles, the braiding for the generating object b with itself is given by $\psi_{\mathsf{b},\mathsf{b}} =$ .

1.2 The quantized enveloping algebra $U_h(\mathfrak{sl}_2)$

We follow the notation of [12]. We consider h as a free parameter and let

$$v = \exp\frac{h}{2} \in \mathbb{Q}[[h]], \qquad q = v^2 = \exp h,$$

$$\{n\} = v^n - v^{-n}, \quad \{n\}! = \prod_{i=1}^{n}\{i\}, \quad [n] = \frac{\{n\}}{\{1\}}, \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{\{n\}!}{\{k\}!\{n-k\}!}.$$

The quantized enveloping algebra $U_h := U_h(\mathfrak{sl}_2)$ is the quantum deformation of the universal enveloping algebra $U(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$. More precisely, it is the h–adically complete $\mathbb{Q}[[h]]$–algebra generated by the elements H, E and F satisfying the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = \frac{K - K^{-1}}{v - v^{-1}}$$

where $K := \exp\frac{hH}{2}$. It has a ribbon Hopf algebra structure with comultiplication $\Delta : U_h \to U_h\hat{\otimes}U_h$ (where $\hat{\otimes}$ denotes the h–adically complete tensor product), counit $\epsilon : U_h \to \mathbb{Q}[[h]]$ and antipode $S : U_h \to U_h$ defined by

$$\begin{array}{lll} \Delta(H) = H\otimes 1 + 1\otimes H & \epsilon(H) = 0 & S(H) = -H \\ \Delta(E) = E\otimes 1 + K\otimes E & \epsilon(E) = 0 & S(E) = -K^{-1}E \\ \Delta(F) = F\otimes K^{-1} + 1\otimes F & \epsilon(F) = 0 & S(H) = -FK. \end{array}$$

The universal R–matrix and its inverse are given by

$$\begin{aligned} R &= D\left(\sum_{n\geq 0} v^{n(n-1)/2}\frac{(v-v^{-1})^n}{[n]!}F^n\otimes E^n\right) \\ R^{-1} &= \left(\sum_{n\geq 0}(-1)^n v^{-n(n-1)/2}\frac{(v-v^{-1})^n}{[n]!}F^n\otimes E^n\right)D^{-1} \end{aligned}$$

where

$$D = \exp\left(\frac{h}{4}H\otimes H\right).$$

We will use the Sweedler notation $R = \sum \alpha\otimes\beta$ and $R^{-1} = \sum\overline{\alpha}\otimes\overline{\beta}$ when we refer to R. As always, the ribbon element and its inverse can be defined via the R–matrix and the associated grouplike element $\kappa \in U_h$ satisfies $\kappa = K^{-1}$.

By a *finite–dimensional representation* of U_h, we mean a left U_h–module which is free of finite rank as a $\mathbb{Q}[[h]]$–module. For each $n \geq 0$, there exists exactly one irreducible finite–dimensional representation V_n of rank $n+1$ up to isomorphism. It corresponds to the $(n+1)$–dimensional irreducible representation of the Lie algebra $\mathfrak{sl}_2$.

The structure of V_n is as follows. Let $\mathbf{v}_0^n \in V_n$ denote a highest weight vector of V_n which is characterized by $E\mathbf{v}_0^n = 0, H\mathbf{v}_0^n = n\mathbf{v}_0^n$ and $U_h\mathbf{v}_0^n = V_n$. Further we define the other basis elements of V_n by $\mathbf{v}_i^n := \frac{F^i}{[i]!}\mathbf{v}_0^n$ for $i = 1, \ldots, n$. Then the action ρ_{V_n} of U_h on V_n is given by

$$H\mathbf{v}_i^n = (n-2i)\mathbf{v}_i^n, \quad F\mathbf{v}_i^n = [i+1]\mathbf{v}_{i+1}^n, \quad E\mathbf{v}_i^n = [n+1-i]\mathbf{v}_{i-1}^n$$

where we understand $\mathbf{v}_i^n = 0$ unless $0 \leq i \leq n$. It follows that $K^{\pm 1}\mathbf{v}_i^n = v^{\pm(n-2i)}\mathbf{v}_i^n$.

If V is a finite–dimensional representation of U_h, then the *quantum trace* $\mathrm{tr}_q^V(x)$ in V of an element $x \in U_h$ is given by

$$\mathrm{tr}_q^V(x) = \mathrm{tr}^V(\rho_V(K^{-1}x)) \in \mathbb{Q}[[h]]$$

where $\mathrm{tr}^V : \mathrm{End}(V) \to \mathbb{Q}[[h]]$ denotes the trace in V.

1.3 Universal $\mathfrak{sl}_2$ invariant

For every ribbon Hopf algebra exists a *universal invariant* of links and tangles from which one can recover the operator invariants such as the colored Jones polynomial. Such universal invariants have been studied by Kauffman, Lawrence, Lee, Ohtsuki, Reshetikhin, Turaev and many others, see [12], [29], [34] and the references therein. Here we need only the case of bottom tangles.

Let $T = T_1 \sqcup T_2 \sqcup \ldots \sqcup T_n$ be an ordered oriented n–component framed bottom tangle. We define the universal $\mathfrak{sl}_2$ invariant $J_T \in U_h^{\hat{\otimes} n}$ as follows. We choose a diagram for T which is obtained by composition and tensor product of fundamental tangles (see Figure 1.4). On each fundamental tangle, we put elements of U_h as shown in Figure 1.8. Now we read off the elements on the i–th component following its orientation. Writing down these elements from right to left gives $J_{(T_i)}$. This is the i–th tensorand of the universal invariant $J_T = \sum J_{(T_1)} \otimes J_{(T_2)} \otimes \ldots \otimes J_{(T_n)}$. Here the sum is taken over all the summands of the R–matrices which appear. The result of this construction does not depend on the choice of diagram and defines an isotopy invariant of bottom tangles.

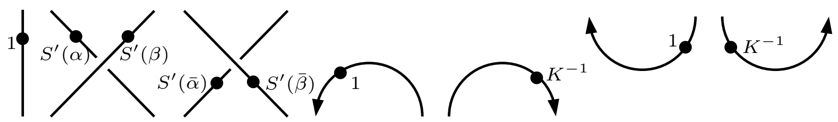


Figure 1.8: Assignment to the fundamental tangles. Here S' should be replaced with the identity map if the string is oriented downward and by S otherwise.

Example 1. For the Borromean tangle B, the assignment of elements of U_h to B are shown in Figure 1.9. The universal invariant is given by

$$J_B = \sum S(\alpha_6)S(\beta_5)S(\overline{\alpha_3})S(\beta_1) \otimes \alpha_1\beta_4 S(\alpha_5)S(\beta_2) \otimes \alpha_2\overline{\beta_3}\alpha_4\beta_6$$

where we use the Sweelder notation, i.e. we sum over all α_i, β_i for $i = 1, \ldots, 6$. Compare also with [13, Proof of Corollary 9.14] and [12, Proof of Theorem 4.1]. Habiro uses $(S \otimes S)R = R$ and $R^{-1} = (1 \otimes S^{-1})R$ therein.

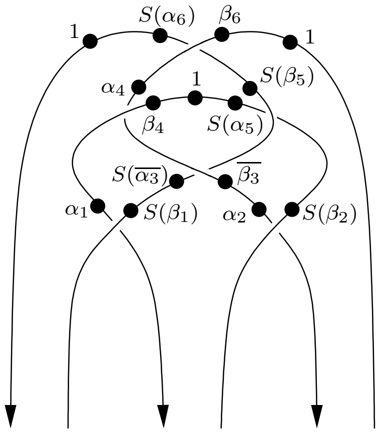


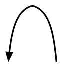
Figure 1.9: The assignments to the Borromean tangle.

1.4 Definition of the colored Jones Polynomial

Let $L = L_1 \sqcup L_2 \sqcup \ldots \sqcup L_m$ be an m–component framed oriented ordered link with associated positive integers $n_1, \ldots, n_m$ called the *colors associated with* L. Remember that the n–dimensional representation of U_h is denoted by V_{n-1}. Let further T be a bottom tangle with $\mathrm{cl}(T) = L$. The *colored Jones polynomial* of L with colors $n_1, \ldots, n_m$ is given by

$$J_L(n_1, \ldots, n_m) = (\mathrm{tr}_q^{V_{n_1-1}} \otimes \mathrm{tr}_q^{V_{n_2-1}} \otimes \ldots \otimes \mathrm{tr}_q^{V_{n_m-1}})(J_T).$$

For every choice of $n_1, \ldots, n_m$, this is an invariant of framed links (see e.g. [32] and [13, Section 1.2]).

Example 2. Let us calculate $J_U(n)$, where U denotes the unknot with zero framing. For $T =$ , we have $\mathrm{cl}(T) = U$. We choose for V_{n-1} the basis $\mathbf{v}_0^{n-1}, \mathbf{v}_1^{n-1}, \ldots, \mathbf{v}_{n-1}^{n-1}$ described in Section 1.2. Since $J_T = 1$, we have

$$\begin{aligned} J_U(n) &= \mathrm{tr}^{V_{n-1}}(\rho_{V_{n-1}}(K^{-1})) = \mathrm{tr}^{V_{n-1}}\left(\begin{pmatrix} v^{-n+2} & 0 & \ldots & \ldots & 0 \\ 0 & v^{-n+4} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & \ldots & 0 & v^{n-2} \end{pmatrix}\right) \\ &= v^{n-2} + v^{n-4} + \ldots + v^{-n+2} = [n]. \end{aligned}$$

We will need the following two important properties of the colored Jones polynomial.

Lemma 3. *[20, Lemma 3.27]*
If L_1 is obtained from L by increasing the framing of the ith component by 1, then

$$J_{L_1}(n_1, \ldots, n_m) = q^{(n_i^2-1)/4} J_L(n_1, \ldots, n_m).$$

Lemma 4. *[24, Strong integrality Theorem 2.2 and Corollary 2.4]*
There exists a number $p \in \mathbb{Z}$, depending only on the linking matrix of L, such that $J_L(n_1, \ldots, n_m) \in q^{\frac{p}{4}}\mathbb{Z}[q^{\pm 1}]$. Further, if all the colors n_i are odd, $J_L(n_1, \ldots, n_m) \in \mathbb{Z}[q^{\pm 1}]$.

1.5 Cyclotomic expansion of the colored Jones polynomial

Let L and L' have m and l components. Let us color L' by fixed $\mathbf{j} = (j_1, \dots, j_l)$ and vary the colors $\mathbf{n} = (n_1, \dots, n_m)$ of L.

For non–negative integers n, k we define

$$A(n,k) := \frac{\prod_{i=0}^{k} (q^n + q^{-n} - q^i - q^{-i})}{(1-q)\,(q^{k+1};q)_{k+1}}$$

where we use from q–calculus the definition

$$(x;q)_n := \prod_{j=1}^{n} (1 - xq^{j-1}).$$

For $\mathbf{k} = (k_1, \dots, k_m)$ let

$$A(\mathbf{n},\mathbf{k}) := \prod_{j=1}^{m} A(n_j, k_j).$$

Note that $A(\mathbf{n},\mathbf{k}) = 0$ if $k_j \geq n_j$ for some index j. Also

$$A(n,0) = q^{-1} J_U(n)^2.$$

The colored Jones polynomial $J_{L\sqcup L'}(\mathbf{n},\mathbf{j})$, when $\mathbf{j}$ is fixed, can be repackaged into the invariant $C_{L\sqcup L'}(\mathbf{k},\mathbf{j})$ as stated in the following theorem.

Theorem 5. *Suppose $L \sqcup L'$ is a link in S^3, with L having 0 linking matrix. Suppose the components of L' have fixed* odd *colors $\mathbf{j} = (j_1, \cdots j_l)$. Then there are invariants*

$$C_{L\sqcup L'}(\mathbf{k},\mathbf{j}) \in \frac{(q^{k+1};q)_{k+1}}{(1-q)}\ \mathbb{Z}[q^{\pm 1}] \quad \textit{where } k = \max\{k_1, \dots, k_m\} \tag{1.1}$$

such that for every $\mathbf{n} = (n_1, \dots, n_m)$

$$J_{L\sqcup L'}(\mathbf{n},\mathbf{j}) \prod_{i=1}^{m} [n_i] = \sum_{0 \leq k_i \leq n_i - 1} C_{L\sqcup L'}(\mathbf{k},\mathbf{j})\ A(\mathbf{n},\mathbf{k}). \tag{1.2}$$

When $L' = \emptyset$, this was proven by K. Habiro, see Theorem 8.2 in [12]. This generalization can be proved similarly as in [12]. For completeness, we give a proof in the Appendix. Note that the existence of $C_{L\sqcup L'}(\mathbf{k},\mathbf{j})$ as rational functions in q satisfying (1.2) is easy to establish. The difficulty here is to show the integrality of (1.1).

Remark 6. Since $A(\mathbf{n},\mathbf{k}) = 0$ unless $\mathbf{k} < \mathbf{n}$, in the sum on the right hand side of (1.2) one can assume that $\mathbf{k}$ runs over the set of all m–tuples $\mathbf{k}$ with non–negative integer components. We will use this fact later.

2 Quantum (WRT) invariant

In this chapter, we describe in Section 2.1 a one–to–one correspondence between 3–dimensional manifolds up to orientation preserving homeomorphisms and links up to Fenn–Rourke moves. We then state in Section 2.2 results about generalized Gauss sums and define a variation therefrom. We use this in Section 2.3 where we give the definition of the quantum (WRT) invariants and, for rational homology 3–spheres, a renormalization of these invariants. Finally, we describe the connection between the quantum (WRT) $SU(2)$ and $SO(3)$ invariants.

2.1 Surgery on links in S^3

Let K be a knot in S^3 and $N(K) = K \times D^2$ its tubular neighbourhood. The *knot exterior* E is defined as the closure of $S^3 \backslash N(K)$.

A 3–manifold M is obtained from S^3 by a *rational 1–surgery* along a framed knot $K \subset S^3$ with framing $\frac{p}{q}$, when $N(K)$ is removed from S^3 and a copy of $D^2 \times S^1$ is glued back in using a homeomorphism $h : \partial D^2 \times S^1 \to \partial E$. If $q = 1$, the 1–surgery is called *integral*. The homeomorphism h is completely determined by the image of any meridian $m := \partial D^2 \times \{*\}$ of $\partial D^2 \times S^1$. To describe this image it is enough to specify a canonical longitude l of ∂E and an orientation on m and l. The image will then be a simple closed curve on ∂E isotopic to a curve of the form $c = p \cdot m + q \cdot l$, where p and q are given by the framing of the knot. The canonical longitude l is, up to isotopy, uniquely defined as the curve homologically trivial in E and with $\mathrm{lk}(l, K) = 0$. For the orientation on m and l, we choose the standard orientation on S^3 which induces an orientation on E. The two curves m and l are then oriented such that the triple $\langle m, l, n\rangle$ is positively oriented. Here n is a normal vector to ∂E pointing inside E, see Figure 2.1.

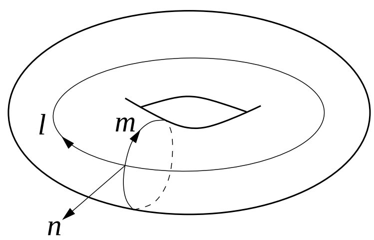


Figure 2.1: Orientation of meridian m and longitude l in ∂E.

Theorem 7 (Likorish, Wallace)**.** *Any closed connected orientable 3–manifold M can be obtained from S^3 by a collection of integral 1–surgeries.*

Proof. See for example [26] or [35]. □

Therefore any ordered framed link L gives a description of a collection of 1-surgeries and the manifold obtained in this way will be denoted by $M = S^3(L)$. Let L' be an other link in S^3. Surgery along L transforms (S^3, L') into (M, L'). We use the same notation L' to denote the link in S^3 and the corresponding one in M.

Example 8. The (b, a) lens space $L(b, a)$ is obtained by surgery along an unknot with rational framing $\frac{b}{a}$. Further we have $S^3 = S^3(U^1)$, where U^1 denotes the unknot with framing 1.

In [18], R. Kirby proved a one–to–one correspondence between 3–manifolds up to homeomorphisms and framed links up to the two so–called Kirby moves. In [9], R. Fenn and C. Rourke showed that these two moves are equivalent to the one Fenn–Rourke move (see Figure 2.2) and proved the following.

Theorem 9 (Fenn–Rourke)**.** *Two framed links in S^3 give, by surgery, the same oriented 3–manifold if and only if they are related by a sequence of Fenn Rourke moves. A Fenn Rourke move means replacing in the link locally T by T_+ or T_- as shown in Figure 2.2 where the non–negative integer m can be chosen arbitrary. The framings j and $j_\pm$ on corresponding components J and $J_\pm$ (before and after a move) are related by $j_\pm = j \pm \mathrm{lk}(K, J_\pm)^2$.*

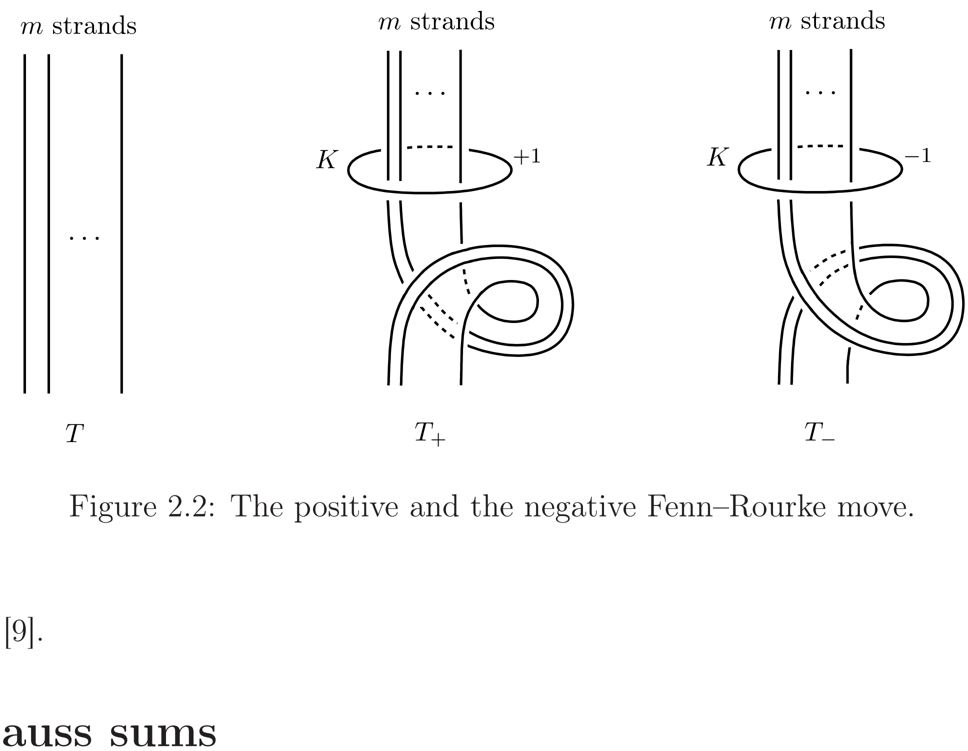


Figure 2.2: The positive and the negative Fenn–Rourke move.

Proof. See [9]. □

2.2 Gauss sums

We use the following notation. The greatest common divisor of two integers a and b is denoted by (a, b). If a does (respectively does not) divide b, we write $a \mid b$ (respectively $a \nmid b$).

Further, for b odd and positive, the Jacobi symbol, denoted by $\left(\frac{a}{b}\right)$, is defined as follows. First, $\left(\frac{a}{1}\right) = 1$. Then for p prime, $\left(\frac{a}{p}\right)$ represents the Legendre symbol, which is defined for all integers a and all odd primes p by

$$\left(\frac{a}{p}\right) = \begin{cases} \;\;0 \text{ if } a \equiv 0 \pmod{p} \\ +1 \text{ if } a \not\equiv 0 \pmod{p} \text{ and for some integer } x,\; a \equiv x^2 \pmod{p} \\ -1 \text{ else.} \end{cases}$$

Finally, if $b \neq 1$, we put

$$\left(\frac{a}{b}\right) = \prod_{i=1}^{m} \left(\frac{a}{p_i}\right)^{\alpha_i} \quad \text{where} \quad b = \prod_{i}^{m} p_i^{\alpha_i} \quad \text{for } p_i \text{ prime.}$$

Let $e_r := \exp(\frac{2\pi i}{r})$ where i denotes the positive primitive 4th root of 1. The generalized Gauss sum is defined as

$$G(r, x, y) := \sum_{j=0}^{r-1} e_r^{xj^2+yj}.$$

The values of $G(r, x, y)$ are well known:

Lemma 10. *For $r, x, y \in \mathbb{N}$ we have*

$$G(r, x, y) = \begin{cases} 0 & \text{if } (r,x) \nmid y \\ (r,x) \cdot G\left(\frac{r}{(r,x)}, \frac{x}{(r,x)}, \frac{y}{(r,x)}\right) & \text{else} \end{cases}$$

and for $(r, x) = 1$

$$G(r, x, y) = \begin{cases} \epsilon(r) \left(\frac{x}{r}\right) \sqrt{r} e_r^{-\frac{x_{*r} y^2}{4}(r+1)^2} & \text{if } r \text{ odd} \\ 0 & \text{if } r \equiv 2 \pmod{4} \text{ and } y \text{ even, or} \\ & \text{if } r \equiv 0 \pmod{4} \text{ and } y \text{ odd} \\ \epsilon\left(\frac{r}{2}\right) \left(\frac{2x}{r/2}\right) \sqrt{2r} e_r^{-\frac{x_{*r} y^2}{4}\left(\frac{r+2}{2}\right)^3} & \text{if } r \equiv 2 \pmod{4} \text{ and } y \text{ odd} \\ \overline{\epsilon(x)} \left(\frac{r}{x}\right) (1+i) \sqrt{r} e_r^{-\frac{x_{*r} y^2}{4}} & \text{if } r \equiv 0 \pmod{4} \text{ and } y \text{ even} \end{cases}$$

where x_{*r} is defined such that $xx_{*r} \equiv 1 \pmod{r}$ and $\epsilon(x) = 1$ if $x \equiv 1 \pmod{4}$ and $\epsilon(x) = i$ if $x \equiv 3 \pmod{4}$.

Proof. See e.g. [21] or any other text book in basic number theory. □

Let R be a unitary ring and $f(q; n_1, \ldots, n_m) \in R[q^{\pm 1}, n_1, n_2, \ldots, n_m]$. For each root of unity ξ of order r, in quantum topology, the following sum plays an important role:

$$\sum_{n_i}^{\xi, G} f := \sum_{n_i \in N_G} f(\xi; n_1, \ldots, n_m).$$

Here G stands for either the Lie group $SU(2)$ or the Lie group $SO(3)$ and $N_{SU(2)} := \{n \in \mathbb{Z} \mid 0 \leq n \leq 2r-1\}$ and $N_{SO(3)} := \{n \in \mathbb{Z} \mid 0 \leq n \leq 2r-1, n \text{ odd}\}$. If $G = SO(3)$, r is always assumed to be odd.

Let us explain the meaning of the N_G's. Roughly speaking, the set N_G corresponds to the set of irreducible representations of $U_q(\mathfrak{sl}_2)$ where q is a root of unity of order r.

In fact, the quantum invariants (see next Section 2.3) were originally defined by N. Reshetikhin and V. Turaev in [32, 33] by summing over all irreducible representations of the quantum group $U_q(\mathfrak{sl}_2)$ where q is chosen to be a root of unity of order r. The quantum group $U_q(\mathfrak{sl}_2)$ is defined similarly as $U_h(\mathfrak{sl}_2)$ (see Section 1.2) where q stands for $\exp(h)$. Similarly as for $U_h(\mathfrak{sl}_2)$, there exists exactly one free finite–dimensional irreducible representation of $U_q(\mathfrak{sl}_2)$ in each dimension. In the case when q is chosen to be a primitive rth root of unity, only the representations of dimension $\leq r$ are irreducible (see e.g. [20, Theorem 2.13]). Since for every irreducible representation of $U_q(\mathfrak{sl}_2)$ there is a corresponding irreducible representation of the Lie group $SU(2)$, the invariant is sometimes also called the *quantum (WRT) $SU(2)$ invariant*.

Kirby and Melvin showed in [20, Theorem 8.10], that summing over all irreducible representations of $U_q(\mathfrak{sl}_2)$ of *odd* dimension also gives an invariant. Since the Lie group $SO(3)$ is isomorphic to $SU(2)/\{\pm I\}$, where I stands for the identity matrix, a representation ρ_{n-1} of dimension $n-1$ of $SU(2)$ is a representation of $SO(3)$ if and only if $\rho_{n-1}(-I)$ is the identity map. This is true if and only if n is odd. Therefore it makes sense to call the invariant introduced by Kirby and Melvin the *quantum (WRT) $SO(3)$ invariant*.

As already mentioned above, if q is chosen to be a root of unity of order r, the irreducible representations of $U_q(\mathfrak{sl}_2)$ are actually of dimension $\leq r$, and not $\leq 2r-1$ which we use as upper bound in N_G. But summing up to $2r-1$ makes all calculations much simpler and due to the first symmetry principle of Le [24, Theorem 2.5], summing up to r or up to $2r-1$ does change the invariant only by some constant factor.

For ξ a root of unity, we define the following variation of the Gauss sum:

$$\gamma_b^G(\xi) := \sum_n^{\xi,G} q^{b\frac{n^2-1}{4}}.$$

Notice, that for $G = SO(3)$, $\gamma_b^{SO(3)}(\xi)$ is well–defined in $\mathbb{Z}[q]$ since, for odd n, $4 \mid n^2-1$. In the case $G = SU(2)$, the Gauss sum is dependent on a 4th root of ξ which we denote by $\xi^{\frac{1}{4}}$.

For an arbitrary primitive rth root of unity ξ, we define the Galois transformation

$$\begin{aligned} \varphi : \mathbb{Q}(e_r) &\rightarrow \mathbb{Q}(\xi) \\ e_r &\mapsto \xi \end{aligned}$$

which is a ring isomorphism.

Lemma 11. *Let ξ be a primitive rth root of unity and $b \in \mathbb{Z}$. The following holds.*

$$\begin{aligned} \gamma_b^{SO(3)}(\xi) &= \varphi(G(r,b,b)) \\ \gamma_b^{SU(2)}(\xi) &= \xi^{\frac{-b}{4}}\varphi(G(r,b,0)) + \varphi(G(r,b,b)). \end{aligned}$$

In particular, $\gamma_b^{SU(2)}(\xi)$ *is zero for* $\frac{r}{(r,b)}$ *odd,* $\frac{b}{(r,b)} \equiv 2 \pmod 4$ *and* $\gamma_b^G(\xi)$ *is nonzero in all other cases.*

Proof. It is enough to prove the claim at the root of unity e_r and then apply the Galois transformation to get the general result.

For $G = SO(3)$, we have r odd and

$$\gamma_b^{SO(3)}(e_r) = \sum_{\substack{n=0 \\ n \text{ odd}}}^{2r-1} e_r^{b\frac{n^2-1}{4}} = \sum_{n=0}^{r-1} e_r^{b(n^2+n)} = G(r,b,b)$$

which is nonzero for all b and odd r.

For $G = SU(2)$, we split the sum into the even and the odd part and get

$$\begin{aligned}\gamma_b^{SU(2)}(e_r) &= \sum_{n=0}^{2r-1} e_r^{b\frac{n^2-1}{4}} = \sum_{n=0}^{r-1} e_r^{b(n^2+n)} + e_r^{-\frac{b}{4}} \sum_{n=0}^{r-1} e_r^{bn^2} = G(r,b,b) + e_r^{-\frac{b}{4}} G(r,b,0) \\ &= c \cdot \begin{cases} \epsilon(r') \left(\frac{b'}{r'}\right) \left(e_{r'}^{-\frac{b'}{4}} + e_{r'}^{-\frac{b'}{4}(r'+1)^2}\right) & \text{if } r' \text{ odd} \\ G(r',b',b') & \text{if } r' \equiv 2 \pmod 4 \\ e_{r'}^{-\frac{b'}{4}} G(r',b',0) & \text{if } r' \equiv 0 \pmod 4 \end{cases}\end{aligned}$$

where $c = (r,b)$ and $r' = \frac{r}{c}$ and $b' = \frac{b}{c}$. Therefore, $\gamma_b^{SU(2)}(e_r)$ can only be zero if r' odd and $e_{r'}^{-\frac{b'}{4}} + e_{r'}^{-\frac{b'}{4}(r'+1)^2}$ equal zero, i.e. $e_{r'}^{-\frac{b'}{4}r'(r'+2)} = -1$. Since $e_{r'}^{\frac{r'}{4}}$ is a primitive 4th root of unity, this is true if and only if $b' \equiv 2 \pmod 4$. ☐

Example 12. For $b = 1$, we have

$$\gamma_1^{SO(3)}(e_r) = G(r,1,1) = \epsilon(r)\sqrt{r} e_r^{-4_{*r}} \tag{2.1}$$

where $4 \cdot 4_{*r} \equiv 1 \pmod r$. Further, for the $SU(2)$ case, fixing the 4th root of e_r as $e_r^{\frac{1}{4}} := e_{4r}$, we get

$$\gamma_1^{SU(2)}(e_r) = e_{4r}^{-1} G(r,1,0) + G(r,1,1) = (1+i)\sqrt{r} e_{4r}^{-1}. \tag{2.2}$$

2.3 Definition of the quantum (WRT) invariant

Suppose the components of L' are colored by fixed integers $j_1, \ldots, j_l$. Let

$$F_{L \sqcup L'}^G(\xi) := \sum_{n_i}^{\xi,G} \left\{ J_{L \sqcup L'}(n_1, \ldots, n_m, j_1, \ldots, j_l) \prod_{i=1}^m [n_i] \right\}. \tag{2.3}$$

Example 13. An important special case is when $L = U^b$, the unknot with framing $b \neq 0$, and $L' = \emptyset$. Then $J_{U^b}(n) = q^{b\frac{n^2-1}{4}}[n]$. Applying Lemma 10 we get

$$F_{U^b}^G(\xi) = \sum_n^{\xi,G} q^{b\frac{n^2-1}{4}} [n]^2 = 2\gamma_b^G(\xi) \operatorname{ev}_\xi \left(\frac{(1-q^{-b_{*r}})^{\chi(c)}}{(1-q)(1-q^{-1})} \right) \tag{2.4}$$

where $\chi(c) = 1$ if $c = 1$ and zero otherwise. Another (shorter) way to calculate $F^G_{U^b}(\xi)$ uses the Laplace transform method introduced in Section 7, see Equation (7.2). From Formula (2.4) and Lemma 11 we can see that $F^G_{U_b}(\xi)$ is non–zero except if $G = SU(2)$, $\frac{r}{(r,b)}$ odd, and $\frac{b}{(r,b)} \equiv 2 \pmod 4$.

For a better readability, we omit from now on the index G in all notations. If the dependency on G is not indicated, the notation should be understood in generality, in the sense that G can be $SU(2)$ or $SO(3)$. Otherwise the Lie group will be indicated as a superscript.

Let $M = S^3(L)$ and σ_+ (respectively σ_-) be the number of positive (respectively negative) eigenvalues of the linking matrix of L. Further, let L' be an ordered l–component framed link colored by fixed integers $j_1, j_2, \dots, j_l$. We define

$$\tau_{M,L'}(\xi) := \frac{F_{L\sqcup L'}(\xi)}{(F_{U^{+1}}(\xi))^{\sigma_+}\,(F_{U^{-1}}(\xi))^{\sigma_-}} \tag{2.5}$$

where $U^{\pm 1}$ is the unknot with framing ± 1.

Theorem 14 (Reshetikhin–Turaev)**.** *Assume the order of the root of unity ξ is odd if $G = SO(3)$ and arbitrary otherwise. Then $\tau_{M,L'}(\xi)$ is invariant under orientation preserving homeomorphisms of M and ambient isotopies of L' and is called the* quantum (WRT) invariant *of the pair (M, L').*

Proof. See [20] and [33, 34]. □

Example 15. The quantum invariant of the lens space $L(b, 1)$, obtained by surgery along U^b, is

$$\tau_{L(b,1)}(\xi) = \frac{F_{U^b}(\xi)}{F_{U^{\mathrm{sn}(b)}}(\xi)} = \frac{\gamma_b(\xi)}{\gamma_{\mathrm{sn}(b)}(\xi)} \cdot \frac{(1-\xi^{-b_* r})^{\chi(c)}}{1-\xi^{-\mathrm{sn}(b)}} \tag{2.6}$$

where $\mathrm{sn}(b)$ is the sign of the integer b. Since $S^3(U^1) = S^3$, we have $\tau_{S^3}(\xi) = 1$.

Theorem 16 (Reshetikhin–Turaev, Kirby–Melvin)**.** *The quantum invariant satisfies the following properties:*

(a) Multiplicativity: $\tau_{M\#N}(\xi) = \tau_M(\xi)\tau_N(\xi)$.

(b) Orientation: $\tau_{-M}(\xi) = \overline{\tau_M(\xi)}$.

(c) Normalization: $\tau_{S^3}(\xi) = 1$.

Here, $M\#N$ is the connected sum of the two manifolds M and N and $-M$ denotes M with orientation reversed. By $\overline{z}$ for $z \in \mathbb{C}$, we denote the complex conjugate of z.

Proof. We have proven the normalization property in Example 15. For the multiplicativity and orientation property see [20] and [33, 34]. □

2.3.1 Renormalization of quantum (WRT) invariant

Suppose that M is a rational homology 3–sphere. There is a unique decomposition $H_1(M;\mathbb{Z}) = \bigoplus_i \mathbb{Z}/b_i\mathbb{Z}$, where each b_i is a prime power. We put $b = \prod b_i = |H_1(M;\mathbb{Z})|$, the order of the first homology group $H_1(M;\mathbb{Z})$.

For the rest of this thesis, we allow b to be any number in the $SO(3)$ case but assume b to be *odd* in the $SU(2)$ case.

Let the absolute value of an integer x be denoted by $|x|$. We renormalize the quantum invariant of the pair (M, L') as follows:

$$\tau'_{M,L'}(\xi) := \frac{\tau_{M,L'}(\xi)}{\prod\limits_i \tau_{L(|b_i|,1)}(\xi)}. \tag{2.7}$$

Notice that due to Example 13, $\tau^{SO(3)}_{L(b,1)}(\xi)$ is always nonzero and $\tau^{SU(2)}_{L(b,1)}(\xi)$ is nonzero for b odd and therefore the renormalization is well–defined.

Let us focus on the special case when the linking matrix of L is diagonal, with $b_1, b_2, \ldots, b_m$ on the diagonal. Assume each b_i is a power of a prime up to sign. Then $H_1(M,\mathbb{Z}) = \oplus_{i=1}^m \mathbb{Z}/|b_i|\mathbb{Z}$, and

$$\sigma_+ = \text{card}\,\{i \mid b_i > 0\}, \quad \sigma_- = \text{card}\,\{i \mid b_i < 0\}.$$

Thus from Definitions (2.5) and (2.7) and Equation (2.6) we have

$$\tau'_{M,L'}(\xi) = \left(\prod_{i=1}^m \tau'_{L(b_i,1)}(\xi)\right) \frac{F_{L\sqcup L'}(\xi)}{\prod_{i=1}^m F_{U^{b_i}}(\xi)}, \tag{2.8}$$

with

$$\tau'_{L(b_i,1)}(\xi) = \frac{\tau_{L(b_i,1)}(\xi)}{\tau_{L(|b_i|,1)}(\xi)}.$$

For the renormalized quantum invariant, multiplicativity and normalization follows from Theorem 16, but reversing the orientation of the manifold does not induce complex conjugacy for the renormalized quantum invariant. For example $L(-b,1)$ is homeomorphic to $L(b,1)$ with orientation reversed. But for $b > 0$, $\tau'_{L(b,1)}(\xi) = 1$ but $\tau'_{L(-b,1)}(\xi) \neq 1$ (see Section 6 for the exact calculation). To achieve a better behavior under orientation reversing, we could instead renormalize the invariant as

$$\tilde{\tau}_{M,L'}(\xi) := \frac{\tau_{M,L'}(\xi)}{\prod\limits_{i=1}^m \sqrt{\tau_{L(b_i,1)\#L(-b_i,1)}(\xi)}}.$$

The disadvantage of this normalization is that to be allowed to take the square root of $\tau_{L(b_i,1)\#L(-b_i,1)}(\xi)$ we need to fix a 4th root of ξ. We have this in the $SU(2)$ case but not in the $SO(3)$ case.

2.3.2 Connection between $SU(2)$ and $SO(3)$ invariant

Theorem 17 (Kirby–Melvin)**.** *For* $\mathrm{ord}(\xi)$ *odd, we have*

$$\tau^{SU(2)}_{M,L'}(\xi) = \tau^{SO(3)}_{M,L'}(\xi) \cdot \tau^{SU(2)}_{M,L'}(e_3).$$

Therefore, for $\mathrm{ord}(\xi)$ *odd, the* $SO(3)$ *invariant is sometimes stronger than the* $SU(2)$ *invariant since* $\tau^{SU(2)}_{M,L'}(e_3)$ *is sometimes zero while the* $SO(3)$ *invariant is not.*

Proof. See [20, Corollary 8.9]. □

Remark 18. In [23], T. Le proved a similar result for arbitrary semi–simple Lie algebras $\mathfrak{g}$.

The result for the renormalized quantum invariant follows immediately:

Corollary 19. *For* $\mathrm{ord}(\xi)$ *odd, we have*

$$\tau'^{SU(2)}_{M,L'}(\xi) = \tau'^{SO(3)}_{M,L'}(\xi) \cdot \tau'^{SU(2)}_{M,L'}(e_3).$$

3 Cyclotomic completions of polynomial rings

In [14], Habiro develops a theory for cyclotomic completions of polynomial rings. In this chapter, we first recall some important results about cyclotomic polynomials and inverse limits before summarizing some of Habiro's results. We then define the rings $\mathcal{S}_b$ and $\mathcal{R}_b$ in which the unified invariants defined in Chapter 4 are going to lie. We also describe the evaluation in these rings. Most important, we prove that when the evaluation of two elements of the ring $\mathcal{S}_b$ coincide at all roots of unity, the two elements are actually identical in $\mathcal{S}_b$. A similar statement holds in the ring $\mathcal{R}_b$ for roots of unity of odd order.

3.1 On cyclotomic polynomial

Recall that $e_n := \exp(\frac{2\pi i}{n})$ and denote by $\Phi_n(q)$ the cyclotomic polynomial

$$\Phi_n(q) = \prod_{\substack{(j,n)=1 \\ 0<j\leq n}} (q - e_n^j).$$

Since $q^n - 1 = \prod_{j=0}^{n-1}(q - e_n^j)$, collecting together all terms belonging to roots of unity of the same order, we have

$$q^n - 1 = \prod_{d|n} \Phi_d(q).$$

The degree of $\Phi_n(q) \in \mathbb{Z}[q]$ is given by the Euler function $\varphi(n)$. Suppose p is a prime and n an integer. Then (see e.g. [21])

$$\Phi_n(q^p) = \begin{cases} \Phi_{np}(q) & \text{if } p \mid n \\ \Phi_{np}(q)\Phi_n(q) & \text{if } p \nmid n. \end{cases} \tag{3.1}$$

It follows that $\Phi_n(q^p)$ is always divisible by $\Phi_{np}(q)$. The ideal of $\mathbb{Z}[q]$ generated by $\Phi_n(q)$ and $\Phi_m(q)$ is well–known, see e.g. [22, Lemma 5.4]:

Lemma 20.

(a) If $\frac{m}{n} \neq p^e$ for any prime p and any integer $e \neq 0$, then $(\Phi_n(q)) + (\Phi_m(q)) = (1)$ in $\mathbb{Z}[q]$.

(b) If $\frac{m}{n} = p^e$ for a prime p and some integer $e \neq 0$, then $(\Phi_n(q)) + (\Phi_m(q)) = (1)$ in $\mathbb{Z}[1/p][q]$.

Remark 21. Note that in a commutative ring R, $(x) + (y) = (1)$ if and only if x is invertible in $R/(y)$. Therefore $(x) + (y) = (1)$ implies $(x^k) + (y^l) = (1)$ for any integers $k, l \geq 1$.

3.2 Inverse limit

Let $(I, \leq)$ be a partially ordered set, R_i, $i \in I$, unital commutative rings and for $i \leq j$ let $f_{ij} : R_j \to R_i$ be ring homomorphisms. We call (R_i, f_{ij}) an *inverse system* of rings and ring homomorphisms if f_{ii} is the identity map in R_i and $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \leq j \leq k$.

The *inverse limit* of an inverse system (R_i, f_{ij}) is defined to be the ring

$$\varprojlim_{i \in I} R_i = \left\{ (r_i) \in \prod_{i \in I} R_i \;\middle|\; r_i = f_{ij}(r_j) \text{ for all } i \leq j \right\}.$$

The following is well–known.

Lemma 22. *If the set $J \subset I$ is cofinal to I, i.e. for every element $i \in I$ exists an element $j \in J$ such that $j \geq i$, we have*

$$\varprojlim_{i \in I} R_i \simeq \varprojlim_{j \in J} R_j.$$

Proof. Let $\varphi : \varprojlim_{i \in I} R_i \to \varprojlim_{j \in J} R_j$ be the canonical projection. We have to show that φ is injective. Assume $\varphi((r_i)) = (0)$. Since J is cofinal to I, for every $i \in I$ we can choose $j_i \in J$ such that $j_i \geq i$. But $f_{ij_i}(r_{j_i}) = r_i$ and $r_{j_i} = 0$ for all j_i. Therefore we have $r_i = 0$ for all $i \in I$. ☐

For a ring R and $I \subset R$ an ideal, the inverse limit $\varprojlim_j R/I^j$ is called the *I-adic completion of* R. There is a map from this ring to the formal sums of elements of R. Namely, every element r in $\varprojlim_j R/I^j$ can be expressed in the form

$$r = \sum_{j \geq 0} s_j i_j$$

where $s_j \in R/I$ and $i_j \in I^j$. This decomposition is not unique.

Example 23. The (3)–adic completion of $\mathbb{Z}$ corresponds to the 3–adic expansion of $\mathbb{Z}$. For example, the number 124 corresponds to the element

$$(r_n) = (0, 1, 7, 16, 43, 124, 124, 124, \ldots)$$

in $\varprojlim_{n \in \mathbb{N}} \mathbb{Z}/(3^n)$ and as 3–adic number we write it as

$$124 = 1 + 2 \cdot 3 + 1 \cdot 9 + 1 \cdot 27 + 1 \cdot 81 = \sum_{n \geq 0} s_n 3^n$$

where $s_n = \frac{r_{n+1} - r_n}{3^n}$.

Example 24. In [14], Habiro defined the so–called *Habiro ring*

$$\widehat{\mathbb{Z}[q]} := \varprojlim_{n} \mathbb{Z}[q]/((q;q)_n).$$

Every element $f(q) \in \widehat{\mathbb{Z}[q]}$ can be written as an infinite sum

$$f(q) = \sum_{n\geq 0} f_n(q)\,(1-q)(1-q^2)...(1-q^n)$$

with $f_n(q) \in \mathbb{Z}[q]$. When evaluating $f(q)$ at a root of unity ξ, only a finite number of terms on the right hand side are not zero, hence the right hand side gives a well–defined value. Since $f_n(q) \in \mathbb{Z}[q]$, the evaluation of $f(q)$ is an algebraic integer, i.e. lies in $\mathbb{Z}[\xi]$.

3.3 Cyclotomic completions of polynomial rings

We now summarize further results of Habiro on cyclotomic completions of polynomial rings [14]. Let R be a commutative integral domain of characteristic zero and $R[q]$ the polynomial ring over R. We consider $\mathbb{N}$ as a directed set with respect to the divisibility relation. For any $S \subset \mathbb{N}$, the S–cyclotomic completion ring $R[q]^S$ is defined as

$$R[q]^S := \varprojlim_{f(q)\in\Phi^*_S} \frac{R[q]}{(f(q))} \tag{3.2}$$

where Φ^*_S denotes the multiplicative set in $\mathbb{Z}[q]$ generated by $\Phi_S = \{\Phi_n(q) \mid n \in S\}$ and directed with respect to the divisibility relation.

Example 25. Since the sequence $(q;q)_n$, $n \in \mathbb{N}$, is cofinal to $\Phi^*_{\mathbb{N}}$, Lemma 22 implies

$$\widehat{\mathbb{Z}[q]} \simeq \mathbb{Z}[q]^{\mathbb{N}}. \tag{3.3}$$

Note that if S is finite, $R[q]^S$ is identified with the $(\prod \Phi_S)$–adic completion of $R[q]$. In particular,

$$R[q]^{\{1\}} \simeq R[[q-1]], \quad R[q]^{\{2\}} \simeq R[[q+1]].$$

Two positive integers n, n' are called *adjacent* if $n'/n = p^e$ with a nonzero $e \in \mathbb{Z}$ and a prime p, such that the ring R is p–adically separated, i.e. $\bigcap_{n=1}^{\infty}(p^n) = 0$ in R. A set of positive integers is *R–connected* if for any two distinct elements n, n' there is a sequence $n = n_1, n_2, \dots, n_{k-1}, n_k = n'$ in the set, such that any two consecutive numbers of this sequence are adjacent.

Suppose $S' \subset S$, then $\Phi^*_{S'} \subset \Phi^*_S$, hence there is a natural map

$$\rho^R_{S,S'} : R[q]^S \to R[q]^{S'}.$$

Theorem 26 (Habiro)**.** *If S is R–connected, then for any subset $S' \subset S$ the natural map*

$$\rho^R_{S,S'} : R[q]^S \hookrightarrow R[q]^{S'}$$

is an embedding.

Proof. See [14, Theorem 4.1]. □

3.4 The rings $\mathcal{S}_b$ and $\mathcal{R}_b$

For any positive integer b, we define

$$\mathcal{R}_b := \lim_{\overleftarrow{k}} \frac{\mathbb{Z}[1/b][q]}{((q;q^2)_k)} \qquad \text{and} \qquad \mathcal{S}_b := \lim_{\overleftarrow{k}} \frac{\mathbb{Z}[1/b][q]}{((q;q)_k)} . \tag{3.4}$$

For every integer a, we put $\mathbb{N}_a := \{n \in \mathbb{N} \mid (a,n) = 1\}$. Since the sets $\Phi^*_{\mathbb{N}}$ and $\{(q;q)_n \mid n \in \mathbb{N}\}$, as well as $\Phi^*_{\mathbb{N}_2}$ and $\{(q;q^2)_n \mid n \in \mathbb{N}\}$, are cofinal we have due to Lemma 22

$$\mathcal{R}_b \simeq \mathbb{Z}[1/b][q]^{\mathbb{N}_2} \qquad \text{and} \qquad \mathcal{S}_b \simeq \mathbb{Z}[1/b][q]^{\mathbb{N}}.$$

Remark 27. For $b = 1$, we have $\mathcal{S}_1 = \widehat{\mathbb{Z}[q]}$. Further, if p is a prime divisor of b, we have $\mathcal{R}_p \subset \mathcal{R}_b$ and $\mathcal{S}_p \subset \mathcal{S}_b$.

3.4.1 Splitting of $\mathcal{S}_b$ and $\mathcal{R}_b$

Observe that $\mathbb{N}$ is not $\mathbb{Z}[1/b]$–connected for $b > 1$. In fact, for a prime p one has $\mathbb{N} = \amalg_{j=0}^{\infty} p^j \mathbb{N}_p$, where each $p^j\mathbb{N}_p$ is $\mathbb{Z}[1/p]$–connected.

Suppose p is a prime divisor of b. Let us define

$$\mathcal{S}^{p,0}_b := \mathbb{Z}[1/b][q]^{\mathbb{N}_p} , \qquad \mathcal{S}^{p,\bar{0}}_b := \prod_{j>0} \mathbb{Z}[1/b][q]^{p^j\mathbb{N}_p} , \quad \text{and} \quad \mathcal{S}_{p,j} := \mathbb{Z}[1/p][q]^{p^j\mathbb{N}_p}.$$

Notice that $\mathcal{S}^{p,0}_p = \mathcal{S}_{p,0}$ and $\mathcal{S}^{p,\bar{0}}_p = \prod_{j>0} \mathcal{S}_{p,j}$. Further is $\mathcal{S}^{p,\epsilon}_p \subset \mathcal{S}^{p,\epsilon}_b$ for ϵ either 0 or $\bar{0}$.

Proposition 28. *For p a prime divisor of b, we have*

$$\mathcal{S}_b \simeq \mathcal{S}^{p,0}_b \times \mathcal{S}^{p,\bar{0}}_b \tag{3.5}$$

and therefore there are canonical projections

$$\pi^p_0 : \mathcal{S}_b \to \mathcal{S}^{p,0}_b \quad \textit{and} \quad \pi^p_{\bar{0}} : \mathcal{S}_b \to \mathcal{S}^{p,\bar{0}}_b .$$

In particular, for every prime p one has

$$\mathcal{S}_p \simeq \prod_{j=0}^{\infty} \mathcal{S}_{p,j}$$

and canonical projections $\pi_j : \mathcal{S}_p \to \mathcal{S}_{p,j}$.

Proof. Suppose $n_i \in p^{j_i}\mathbb{N}_p$ for $i = 1, \dots, m$, with distinct j_i's. Then n_i/n_s, with $i \neq s$, is either not a power of a prime or a non–zero power of p. Hence by Lemma 20 (and Remark 21), for any positive integers $k_1, \dots, k_m$, we have

$$(\Phi_{n_i}^{k_i}(q)) + (\Phi_{n_s}^{k_s}(q)) = (1) \quad \text{in } \mathbb{Z}[1/b][q].$$

By the Chinese remainder theorem, we have

$$\frac{\mathbb{Z}[1/b][q]}{\left(\prod_{i=1}^m \Phi_{n_i}^{k_i}(q)\right)} \simeq \prod_{i=1}^m \frac{\mathbb{Z}[1/b][q]}{\left(\Phi_{n_i}^{k_i}(q)\right)}.$$

Taking the inverse limit, we get (3.5). □

We will also use the notation $\mathcal{S}_{b,0} := \mathbb{Z}[1/b][q]^{\mathbb{N}_b}$ and as above one can see that we have the projection $\pi_0 : \mathcal{S}_b \to \mathcal{S}_{b,0}$.

A completely similar splitting exists for $\mathcal{R}_b$, where $\mathcal{R}_b^{p,\epsilon}$, $\epsilon \in \{0, \bar{0}\}$, are defined analogously by replacing $\mathbb{N}_p$ by $\mathbb{N}_{2p}$ (only odd numbers coprime to p) as

$$\mathcal{R}_b^{p,0} := \mathbb{Z}[1/b][q]^{\mathbb{N}_{2p}} \quad \text{and} \quad \mathcal{R}_b^{p,\bar{0}} := \prod_{j>0} \mathbb{Z}[1/b][q]^{p^j\mathbb{N}_{2p}}$$

and the projections π_ϵ^p are defined analogously as above in the $\mathcal{S}_b$ case. If $2 \mid b$, then $\mathcal{R}_b^{2,0}$ coincides with $\mathcal{R}_b$.

We get the following.

Corollary 29. *For any odd divisor p of b, an element $x \in \mathcal{R}_b$ (or $\mathcal{S}_b$) determines and is totally determined by the pair $(\pi_0^p(x), \pi_{\bar{0}}^p(x))$. If $p = 2$ divides b, then for any $x \in \mathcal{R}_b$, $x = \pi_0^p(x)$.*

3.4.2 Further splitting of $\mathcal{S}_b$ and $\mathcal{R}_b$

Let $\{p_i \mid i = 1, \dots, m\}$ be the set of all distinct *odd* prime divisors of b. For $\mathbf{n} = (n_1, \dots, n_m)$, a tuple of numbers $n_i \in \mathbb{N}$, let $\mathbf{p}^\mathbf{n} = \prod_i p_i^{n_i}$. Let $A_\mathbf{n} = \mathbf{p}^\mathbf{n}\mathbb{N}_b$ and $O_\mathbf{n} := \mathbf{p}^\mathbf{n}\mathbb{N}_{2b}$. Then $\mathbb{N}_2 = \amalg_\mathbf{n} O_\mathbf{n}$ and, if b odd, $\mathbb{N} = \amalg_\mathbf{n} A_\mathbf{n}$. Moreover, for $a \in O_\mathbf{n}$, $a' \in O_{\mathbf{n}'}$, we have $(\Phi_a(q), \Phi_{a'}(q)) = (1)$ in $\mathbb{Z}[1/b]$ if $\mathbf{n} \neq \mathbf{n}'$. The same holds for $a \in A_\mathbf{n}$ and $a' \in A_{\mathbf{n}'}$. In addition, each $O_\mathbf{n}$ and $A_\mathbf{n}$ is $\mathbb{Z}[1/b]$–connected. An argument similar to that for Equation (3.5) gives

$$\mathcal{R}_b \simeq \prod_\mathbf{n} \mathbb{Z}[1/b][q]^{O_\mathbf{n}} \qquad \text{and} \quad \text{if } b \text{ odd} \quad \mathcal{S}_b \simeq \prod_\mathbf{n} \mathbb{Z}[1/b][q]^{A_\mathbf{n}}.$$

Proposition 30. *For odd b, the natural homomorphism $\rho_{\mathbb{N},\mathbb{N}_2} : \mathcal{S}_b \to \mathcal{R}_b$ is injective. If $2 \mid b$, then the natural homomorphism $\mathcal{S}_b^{2,0} \to \mathcal{R}_b$ is an isomorphism.*

Proof. By Theorem 26 of Habiro, the map

$$\mathbb{Z}[1/b][q]^{A_\mathbf{n}} \hookrightarrow \mathbb{Z}[1/b][q]^{O_\mathbf{n}}$$

is an embedding. Taking the inverse limit we get the result. If $2 \mid b$, then $\mathcal{S}_b^{2,0} := \mathbb{Z}[1/b][q]^{\mathbb{N}_2} \simeq \mathcal{R}_b$. □

3.5 Evaluation

Let ξ be a root of unity and R be a ring. We define the evaluation map

$$\begin{aligned} \mathrm{ev}_\xi : R[q] &\to R[\xi] \\ q &\mapsto \xi. \end{aligned}$$

Since for $r = \mathrm{ord}(\xi)$

$$R[\xi] \simeq \frac{R[q]}{(\Phi_r(q))},$$

the evaluation map ev_ξ can be defined analogously on $R[q]^S$ if ξ is a root of unity of order in S.

For a set Ξ of roots of unity whose orders form a subset $\mathcal{T} \subset S$, one defines the evaluation

$$\mathrm{ev}_\Xi : R[q]^S \to \prod_{\zeta\in\Xi} R[\zeta].$$

Theorem 31 (Habiro)**.** *If $R \subset \mathbb{Q}$, S is R–connected and there exists $n \in S$ that is adjacent to infinitely many elements in $\mathcal{T}$, then ev_Ξ is injective.*

Proof. See [14, Theorem 6.1]. □

For a prime p, while for every $f \in \mathcal{S}_p$ the evaluation $\mathrm{ev}_\xi(f)$ can be defined for every root of unity ξ, for $f \in \mathcal{S}_{p,j}$ the evaluation $\mathrm{ev}_\xi(f)$ can only be defined when ξ is a root of unity of order in $p^j\mathbb{N}_p$. Actually we have the following.

Lemma 32. *Suppose ξ is a root of unity of order $r = p^j r'$, with $(r', p) = 1$. Then for any $f \in \mathcal{S}_p$, one has*

$$\mathrm{ev}_\xi(f) = \mathrm{ev}_\xi(\pi_j(f)).$$

If $i \neq j$, then $\mathrm{ev}_\xi(\pi_i(f)) = 0$.

Proof. Note that $\mathrm{ev}_\xi(f)$ is the image of f under the projection $\mathcal{S}_p \to \mathcal{S}_p/(\Phi_r(q)) = \mathbb{Z}[1/p][\xi]$. It remains to notice that $\mathcal{S}_{p,i}/(\Phi_r(q)) = 0$ if $i \neq j$. □

Similarly, if $f \in \mathcal{S}_b$ and ξ is a root of unity of order coprime with p, then $\mathrm{ev}_\xi(f) = \mathrm{ev}_\xi(\pi_0^p(f))$. If the order of ξ is divisible by p, then $\mathrm{ev}_\xi(f) = \mathrm{ev}_\xi(\pi_0^p(f))$. The same holds when $f \in \mathcal{R}_b$.

Let T be an infinite set of powers of an odd prime not dividing b and let P be an infinite set of odd primes not dividing b. As above in Section 3.4.2, let $\{p_i \,|\, i = 1, \ldots, m\}$ be the set of all distinct *odd* prime divisors of b and for $\mathbf{n} = (n_1, \ldots, n_m)$, let $\mathbf{p}^\mathbf{n} = \prod_i p_i^{n_i}$.

Proposition 33. *For a given $\mathbf{n} = (n_1, \ldots, n_m)$, suppose $f, g \in \mathbb{Z}[1/b][q]^{O_\mathbf{n}}$ or $f, g \in \mathbb{Z}[1/b][q]^{A_\mathbf{n}}$ such that $\mathrm{ev}_\xi(f) = \mathrm{ev}_\xi(g)$ for any root of unity ξ with ord$(\xi) \in \mathbf{p}^\mathbf{n}T$, then $f = g$. The same holds true if $\mathbf{p}^\mathbf{n}T$ is replaced by $\mathbf{p}^\mathbf{n}P$.*

Proof. Since both sets T and P contain infinitely many numbers adjacent to $\mathbf{p}^\mathbf{n}$, the claims follows from Theorem 31. □

We can infer from this directly the following important result.

Corollary 34. *Let p be an odd prime not dividing b and T the set of all integers of the form $p^k b'$ with $k \in \mathbb{N}$ and b' any odd divisor of b^n for some n. Any element $f(q) \in \mathcal{R}_b$ or $f(q) \in \mathcal{S}_b$ is totally determined by the values at roots of unity with orders in T.*

4 Unified invariant

In [22, 3, 5], T. Le together with A. Beliakova and C. Blanchet defined unified invariants for rational homology 3–spheres with the restriction that one can only evaluate these invariants at roots of unity with order coprime to the order of the first homology group of the manifold.

In this chapter, we define unified invariants for the quantum (WRT) $SU(2)$ and $SO(3)$ invariants of rational homology 3–spheres with links inside, which can be evaluated at any root of unity. The only restriction which remains is that we assume for the $SU(2)$ case the order of the first homology group of the rational homology 3–sphere to be odd.

To be more precise, let M be a rational homology 3–sphere with $b := |H_1(M)|$ and let $L \subset M$ be a framed link with l components. Assume that L is colored by fixed $\mathbf{j} = (j_1, \ldots, j_l)$ with j_i odd for all i. The following theorem is the main result of this thesis.

Theorem 35. *There exist invariants* $I^{SO(3)}_{M,L} \in \mathcal{R}_b$ *and, for* b *odd,* $I^G_{M,L} \in \mathcal{S}_b$*, such that*

$$\mathrm{ev}_\xi(I^G_{M,L}) = \tau'^G_{M,L}(\xi) \tag{4.1}$$

for any root of unity ξ *(of odd order if* $G = SO(3)$*). Further, the invariants are multiplicative with respect to the connected sum, i.e. for* $L \subset M$ *and* $L' \subset M'$*,*

$$I_{M\#M',L\sqcup L'} = I_{M,L} \cdot I_{M',L'}.$$

The rest of this chapter is devoted to the proof of this theorem using technical results that will be proven later.

The following observation is important. By Corollary 34, there is *at most one* element $f^{SO(3)}(q) \in \mathcal{R}_b$ and, if b odd, *at most one* element $f^{SU(2)}(q) \in \mathcal{S}_b$, such that for every root ξ of odd order one has

$$\tau'^G_{M,L}(\xi) = \mathrm{ev}_\xi\left(f^G(q)\right).$$

That is, if we can find such an element, it is unique. Therefore, since τ'^G is an invariant of manifolds with links inside, so must be f^G and we put $I^G_{M,L'} := f^G(q)$. It follows directly from Theorem 16 that this unified invariant must be multiplicative with respect to the connected sum.

We say that M is *diagonal* if it can be obtained from S^3 by surgery along a framed link L_M with diagonal linking matrix where the diagonal entries are of the form $\pm p^k$ with $p = 0, 1$ or a prime.

To define the unified invariant for a general rational homology 3–sphere M, one first adds to M lens spaces to get a diagonal manifold M', for which the unified invariant $I_{M'}$ will be defined in Section 4.2. Then I_M is the quotient of $I_{M'}$ by the unified invariants of the lens spaces (see Section 4.1), which were added. But unlike the simpler case of [22] (where the orders of the roots of unity are always chosen to be coprime to $|H_1(M)|$), the unified invariant of lens spaces are *not* invertible in general. To overcome this difficulty we insert knots in lens spaces and split the unified invariant into different components.

4.1 Unified invariant of lens spaces

Suppose b, a, d are integers with $(b, a) = 1$ and $b \neq 0$. Let $M(b, a; d)$ be the pair of a lens space $L(b, a)$ and a knot $K \subset L(b, a)$, colored by d, as described in Figure 4.1. Among these pairs we want to single out some whose quantum invariants are invertible.

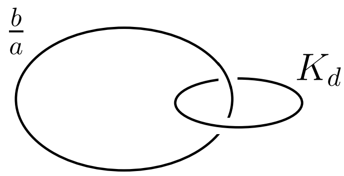


Figure 4.1: The lens space $(L(b, a), K_d)$ is obtained by b/a surgery on the first component of the link. The second component is the knot K colored by d.

Remark 36. It is known that if the color of a link component is 1, then the component can be removed from the link without affecting the value of quantum invariants (see [20, Lemma 4.14]). Hence $\tau_{M(b,a;1)} = \tau_{L(b,a)}$.

Let p be any prime divisor of b. Due to Corollary 29, to define I_M it is enough to fix $I^0_M = \pi^p_0(I_M)$ and $I^{\bar{0}}_M = \pi^p_{\bar{0}}(I_M)$.

For $\varepsilon \in \{0, \bar{0}\}$, let $M^\varepsilon(b, a) := M(b, a; d(\varepsilon))$, where $d(0) := 1$ and $d(\bar{0})$ is the smallest odd positive integer such that $|a|d(\bar{0}) \equiv 1 \pmod b$. First observe that such $d(\bar{0})$ always exists. Indeed, if b is odd, we can achieve this by adding b, otherwise the inverse of any odd number modulo an even number is again odd. Further observe that if $|a| = 1$, $d(0) = d(\bar{0}) = 1$.

Lemma 37. *Suppose $b = \pm p^l$ is a prime power. For $\varepsilon \in \{0, \bar{0}\}$, there exists an invertible invariant $I^\varepsilon_{M^\varepsilon(b,a)} \in \mathcal{R}^{p,\varepsilon}_p$ such that*

$$\tau'_{M^\varepsilon(b,a)}(\xi) \;=\; \mathrm{ev}_\xi\left(I^\varepsilon_{M^\varepsilon(b,a)}\right)$$

where $\varepsilon = 0$ if the order of ξ is not divisible by p (and odd if $G = SO(3)$), and $\varepsilon = \bar{0}$ otherwise. Moreover, if p is odd, then $I^\varepsilon_{M^\varepsilon(b,a)}$ belongs to and is invertible in $\mathcal{S}^{p,\varepsilon}_p$.

A proof of Lemma 37 will be given in Chapter 6.

4.2 Unified invariant of diagonal manifolds

Remember the definition given in Section 1.5

$$A(n,k)=\frac{\prod_{i=0}^{k}(q^n+q^{-n}-q^i-q^{-i})}{(1-q)\,(q^{k+1};q)_{k+1}}$$

for non–negative integers n,k.

We need the following main technical result of this thesis.

Theorem 38. *Suppose $b=\pm1$ or $b=\pm p^l$ where p is a prime and l is positive. For any non–negative integer k, there exists an element $Q^G_{b,k}\in\mathcal{R}_b$ such that for every root ξ of order r (odd, if $G=SO(3)$), one has*

$$\frac{\sum_n^{\xi,G} q^{b\frac{n^2-1}{4}}A(n,k)}{F^G_{U^b}(\xi)}=\mathrm{ev}_\xi(Q^G_{b,k}).$$

In addition, if b is odd, $Q^G_{b,k}\in\mathcal{S}_b$.

A proof will be given in Chapter 7.

Let $L_M\sqcup L$ be a framed link in S^3 with disjoint sublinks L_M and L, with m and l components, respectively. Assume that L is colored by a fixed $\mathbf{j}=(j_1,\dots,j_l)$ with j_i odd for all i. Surgery along the framed link L_M transforms (S^3,L) into (M,L). Assume now that L_M has diagonal linking matrix with nonzero entries $b_i=p_i^{l_i}$, p_i prime or 1, on the diagonal. Then $M=S^3(L_M)$ is a diagonal rational homology 3–sphere. Using (1.2) and Remark 6, taking into account the framings b_i, we have

$$J_{L_M\sqcup L}(\mathbf{n},\mathbf{j})\prod_{i=1}^{m}[n_i]=\sum_{\mathbf{k}\geq0}C_{L_{M,0}\sqcup L}(\mathbf{k},\mathbf{j})\prod_{i=1}^{m}q^{b_i\frac{n_i^2-1}{4}}A(n_i,k_i)$$

where $L_{M,0}$ denotes the link L_M with all framings switched to zero. By the Definition (2.3) of $F^G_{L_M\sqcup L}$, we get

$$F^G_{L_M\sqcup L}(\xi)=\sum_{\mathbf{k}\geq0}\mathrm{ev}_\xi(C_{L_{M,0}\sqcup L}(\mathbf{k},\mathbf{j}))\prod_{i=1}^{m}\sum_{n_i}^{\xi,G}q^{b_i\frac{n_i^2-1}{4}}A(n_i,k_i).$$

From (2.8) and Theorem 38, we get

$$\tau'_{M,L}(\xi)=\mathrm{ev}_\xi\left\{\prod_{i=1}^{m}I_{L(b_i,1)}\sum_{\mathbf{k}\geq0}C_{L_{M,0}\sqcup L}(\mathbf{k},\mathbf{j})\prod_{i=1}^{m}Q_{b_i,k_i}\right\}$$

where the unified invariant of the lens space $I_{L(b_i,1)}\in\mathcal{R}_b$, with $\mathrm{ev}_\xi(I_{L(b_i,1)})=\tau'_{L(b_i,1)}(\xi)$, exists by Lemma 37. Thus if we define

$$I_{(M,L)}:=\prod_{i=1}^{m}I_{L(b_i,1)}\sum_{\mathbf{k}\geq0}C_{L_{M,0}\sqcup L}(\mathbf{k},\mathbf{j})\prod_{i=1}^{m}Q_{b_i,k_i}\,,$$

then (4.1) is satisfied. By Theorem 5, $C_{L_{M,0}\sqcup L}(\mathbf{k},\mathbf{j})$ is divisible by $(q^{k+1};q)_{k+1}/(1-q)$ which is divisible by $(q;q)_k$ where $k=\max k_i$. It follows that $I^{SO(3)}_{(M,L)}\in\mathcal{R}_b$ and, if b is odd, $I^G_{(M,L)}\in\mathcal{S}_b$.

4.3 Definition of the unified invariant: general case

The general case reduces to the diagonal case by the well–known trick of diagonalization using lens spaces:

Lemma 39 (Le)**.** *For every rational homology sphere M, there are lens spaces $L(b_i, a_i)$ such that the connected sum of M and these lens spaces is diagonal. Moreover, each b_i is a prime power divisor of $|H_1(M, \mathbb{Z})|$.*

Proof. See [22, Proposition 3.2 (a)]. □

Suppose (M, L) is an arbitrary pair of a rational homology 3–sphere and a link L in it colored by odd numbers $j_1, \ldots, j_l$. Let $L(b_i, a_i)$ for $i = 1, \ldots, m$ be the lens spaces of Lemma 39. To construct the unified invariant of (M, L), we use induction on m. If $m = 0$, then M is diagonal and $I_{M,L}$ has been defined in Section 4.2.

Since $(M, L)\#M(b_1, a_1; d)$ becomes diagonal after adding $m - 1$ lens spaces, the unified invariant of $(M, L)\#M(b_1, a_1; d)$ can be defined by induction for any odd integer d. In particular, one can define I_{M^ε} where $M^\varepsilon := (M, L)\#M^\varepsilon(b_1, a_1)$. Here $\varepsilon = 0$ or $\varepsilon = \bar{0}$ and b_1 is a power of a prime p dividing b. It follows that the components $\pi^p_\varepsilon(I^{SO(3)}_{M^\varepsilon}) \in \mathcal{R}^{p,\varepsilon}_b$ and $\pi^p_\varepsilon(I^G_{M^\varepsilon}) \in \mathcal{S}^{p,\varepsilon}_b$, for b odd, are defined. By Lemma 37, $I^\varepsilon_{M^\varepsilon(b_1,a_1)}$ is defined and invertible. We put

$$I^\varepsilon_{M,L} := I^\varepsilon_{M^\varepsilon} \cdot (I^\varepsilon_{M^\varepsilon(b_1,a_1)})^{-1}$$

and due to our construction $I_{M,L} := (I^0_{M,L}, I^{\bar{0}}_{M,L})$ satisfies (4.1). This completes the construction of $I_{M,L}$.

Remark 40. The part $I^0_M = \pi^p_0(I_M)$, when $b = p$, was defined by T. Le [22] (up to normalization), where Le considered the case when the order of the roots of unity is coprime to b. More precisely, the invariant defined in [22] for M divided by the invariant of $\#_i L(b_i^{k_i}, 1)$ (which is invertible in $\mathcal{S}_{b,0}$, see [22, Subsection 4.1] and Remark 48 below) coincides with $\pi_0 I_M$ up to a factor $q^{\frac{1-b}{4}}$ by Theorem 35, [22, Theorem 3] and Proposition 33. Nevertheless, we give a self–contained definition of I^0_M here.

It remains to prove Lemma 37 (see Chapter 6) and Theorem 38 (see Chapter 7).

5 Roots in $\mathcal{S}_p$

The proof of the main theorem uses the Laplace transform method (see Chapter 7). The aim of this chapter is to show that the image of the Laplace transform belongs to $\mathcal{R}_b$ (respectively $\mathcal{S}_b$ if b odd), i.e. that certain roots of q exist in $\mathcal{R}_b$ (respectively $\mathcal{S}_b$). We achieve this by showing that a certain type of Frobenius endomorphism of $\mathcal{S}_{b,0}$ is in fact an isomorphism.

5.1 On the module $\mathbb{Z}[q]/(\Phi_n^k(q))$

Since cyclotomic completions of polynomial rings are built from modules like $\mathbb{Z}[q]/(\Phi_n^k(q))$, we first consider these modules. Fix $n, k \geq 1$. Let

$$E := \frac{\mathbb{Z}[q]}{(\Phi_n^k(q))} \quad \text{and} \quad G := \frac{\mathbb{Z}[e_n][x]}{(x^k)} \ .$$

The following is probably well–known.

Proposition 41.

(a) Both E and G are free $\mathbb{Z}$–modules of rank $k\varphi(n)$.

(b) The algebra map $h : \mathbb{Z}[q] \to \mathbb{Z}[e_n][x]$ defined by

$$h(q) = e_n + x$$

descends to a well–defined algebra homomorphism, also denoted by h, from E to G. Moreover, the algebra homomorphism $h : E \to G$ is injective.

Proof. (a) Since $\Phi_n^k(q)$ is a monic polynomial in q of degree $k\varphi(n)$, it is clear that

$$E = \mathbb{Z}[q]/(\Phi_n^k(q))$$

is a free $\mathbb{Z}$–module of rank $k\varphi(n)$. Since $G = \mathbb{Z}[e_n] \otimes_{\mathbb{Z}} \mathbb{Z}[x]/(x^k)$, we see that G is free over $\mathbb{Z}$ of rank $k\varphi(n)$.

(b) To prove that h descends to a map $E \to G$, one needs to verify that $h(\Phi_n^k(q)) = 0$. Note that

$$h(\Phi_n^k(q)) = \Phi_n^k(x + e_n) = \prod_{(j,n)=1} (x + e_n - e_n^j)^k.$$

When $j = 1$, the factor is x^k, which is 0 in $\mathbb{Z}[e_n][x]/(x^k)$. Hence $h(\Phi_n^k(q)) = 0$.

Now we prove that h is injective. Let $f(q) \in \mathbb{Z}[q]$. Suppose $h(f(q)) = 0$, or $f(x+e_n) = 0$ in $\mathbb{Z}[e_n][x]/(x^k)$. It follows that $f(x+e_n)$ is divisible by x^k; or that $f(x)$ is divisible by $(x-e_n)^k$. Since f is a polynomial with coefficients in $\mathbb{Z}$, it follows that $f(x)$ is divisible by all Galois conjugates $(x-e_n^j)^k$ with $(j,n)=1$. Then f is divisible by $\Phi_n^k(q)$. In other words, $f = 0$ in $E = \mathbb{Z}[q]/(\Phi_n^k(q))$.

□

5.2 Frobenius maps

5.2.1 A Frobenius homomorphism

We use E and G of the previous section. Suppose b is a positive integer coprime with n. If ξ is a primitive nth root of 1, i.e. $\Phi_n(\xi) = 0$, then ξ^b is also a primitive nth root of 1, i.e. $\Phi_n(\xi^b) = 0$. It follows that $\Phi_n(q^b)$ is divisible by $\Phi_n(q)$.

Therefore the algebra map $F_b : \mathbb{Z}[q] \to \mathbb{Z}[q]$, defined by $F_b(q) = q^b$, descends to a well–defined algebra map, also denoted by F_b, from E to E. We want to understand the image $F_b(E)$.

Proposition 42. *The image $F_b(E)$ is a free $\mathbb{Z}$–submodule of E of maximal rank, i.e.*

$$\mathrm{rk}(F_b(E)) = \mathrm{rk}(E).$$

Moreover, the index of $F_b(E)$ in E is $b^{k(k-1)\varphi(n)/2}$.

Proof. Using Proposition 41 we identify E with its image $h(E)$ in G.

Let $\tilde{F}_b : G \to G$ be the $\mathbb{Z}$–algebra homomorphism defined by $\tilde{F}_b(e_n) = e_n^b, \tilde{F}_b(x) = (x+e_n)^b - e_n^b$. Note that $\tilde{F}_b(x) = be_n^{b-1}x + O(x^2)$, hence $\tilde{F}_b(x^k) = 0$. Further, $\tilde{F}_b$ is a well–defined algebra homomorphism since $\tilde{F}_b(e_n + x) = e_n^b + (x+e_n)^b - e_n^b = (x+e_n)^b$, and $\tilde{F}_b$ restricted to E is exactly F_b. Since E is a lattice of maximal rank in $G \otimes \mathbb{Q}$, it follows that the index of F_b is exactly the determinant of $\tilde{F}_b$, acting on $G \otimes \mathbb{Q}$.

The elements $e_n^j x^l$ with $0 \le l < k$ and $(j,n) = 1$ for $0 < j < n$ or $j = 0$ form a basis of G. Note that

$$\tilde{F}_b(e_n^j x^l) = b^l e_n^{jb} e_n^{(b-1)l} x^l + O(x^{l+1}).$$

Since $(b,n) = 1$, the set e_n^{jb} with $(j,n)=1$ is the same as the set e_n^j with $(j,n)=1$. Let $f_1 : G \to G$ be the $\mathbb{Z}$–linear map defined by $f_1(e_n^{jb}x^l) = e_n^j x^l$. Since f_1 permutes the basis elements, its determinant is ± 1. Let $f_2 : G \to G$ be the $\mathbb{Z}$–linear map defined by $f_2(e_n^j x^l) = e_n^j (e_n^{1-b}x)^l$. The determinant of f_2 is again ± 1 because, for any fixed l, f_2 restricts to the automorphism of $\mathbb{Z}[e_n]$ sending a to $e_n^s a$, each of these maps has a well–defined inverse: $a \mapsto e_n^{-s}a$. Now

$$f_1 f_2 \tilde{F}_b(e_n^j x^l) = b^l e_n^j x^l + O(x^{l+1})$$

can be described by an upper triangular matrix with b^l's on the diagonal; its determinant is equal to $b^{k(k-1)\varphi(n)/2}$. □

From Proposition 42 we see that if b is invertible, then the index is equal to 1, and we have:

Proposition 43. *For $k \in \mathbb{N}$ and any n coprime with b, the Frobenius homomorphism F_b : $\mathbb{Z}[1/b][q]/\left(\Phi_n^k(q)\right) \to \mathbb{Z}[1/b][q]/\left(\Phi_n^k(q)\right)$, defined by $F_b(q) = q^b$, is an isomorphism.*

5.2.2 Frobenius endomorphism of $\mathcal{S}_{b,0}$

For finitely many $n_i \in \mathbb{N}_b$ and $k_i \in \mathbb{N}$, the Frobenius endomorphism

$$F_b : \frac{\mathbb{Z}[1/b][q]}{\left(\prod_i \Phi_{n_i}^{k_i}(q)\right)} \to \frac{\mathbb{Z}[1/b][q]}{\left(\prod_i \Phi_{n_i}^{k_i}(q)\right)}$$

sending q to q^b, is again well–defined. Taking the inverse limit, we get an algebra endomorphism

$$F_b : \mathbb{Z}[1/b][q]^{\mathbb{N}_b} \to \mathbb{Z}[1/b][q]^{\mathbb{N}_b}.$$

Theorem 44. *The Frobenius endomorphism $F_b : \mathbb{Z}[1/b][q]^{\mathbb{N}_b} \to \mathbb{Z}[1/b][q]^{\mathbb{N}_b}$, sending q to q^b, is an isomorphism.*

Proof. For finitely many $n_i \in \mathbb{N}_b$ and $k_i \in \mathbb{N}$, consider the natural algebra homomorphism

$$J : \frac{\mathbb{Z}[1/b][q]}{\left(\prod_i \Phi_{n_i}^{k_i}(q)\right)} \to \prod_i \frac{\mathbb{Z}[1/b][q]}{\left(\Phi_{n_i}^{k_i}(q)\right)}.$$

This map is injective, because in the unique factorization domain $\mathbb{Z}[1/b][q]$ one has

$$(\Phi_{n_1}(q)^{k_1} \dots \Phi_{n_s}(q)^{k_s}) = \bigcap_{j=1}^{s} \Phi_{n_j}(q)^{k_j} .$$

Since the Frobenius homomorphism commutes with J and is an isomorphism on the target of J by Proposition 43, it is an isomorphism on the domain of J. Taking the inverse limit, we get the claim. $\square$

5.3 Existence of bth root of q in $\mathcal{S}_{b,0}$

We want to show that there exists a bth root of q in $\mathcal{S}_{b,0}$. First we need the two following Lemmas.

Lemma 45. *Suppose that n and b are coprime positive integers and $y \in \mathbb{Q}[e_n]$ with $y^b = 1$. Then $y = \pm 1$. If b is odd then $y = 1$.*

Proof. Let $d \mid b$ be the order of y, i.e. y is a primitive dth root of 1. Then $\mathbb{Q}[e_n]$ contains y and hence e_d. Since $(n, d) = 1$, one has $\mathbb{Q}[e_n] \cap \mathbb{Q}[e_d] = \mathbb{Q}$ (see e.g. [21, Corollary of IV.3.2]). Hence if $e_d \in \mathbb{Q}[e_n]$, then $e_d \in \mathbb{Q}$ and it follows that $d = 1$ or 2. Thus $y = 1$ or $y = -1$. If b is odd, then y cannot be -1. $\square$

Lemma 46. *Let b be a positive integer, $T \subset \mathbb{N}_b$, and $y \in \mathbb{Q}[q]^T$ satisfying $y^b = 1$. Then $y = \pm 1$. If b is odd then $y = 1$.*

Proof. It suffices to show that for any $n_1, n_2 \dots n_m \in T$, the ring $\mathbb{Q}[q]/(\Phi_{n_1}^{k_1} \dots \Phi_{n_m}^{k_m})$ does not contains a bth root of 1 except possibly for ± 1. Using the Chinese remainder theorem, it is enough to consider the case where $m = 1$.

The ring $\mathbb{Q}[q]/(\Phi_n^k(q))$ is isomorphic to $\mathbb{Q}[e_n][x]/(x^k)$, by Proposition 41. If

$$y = \sum_{j=0}^{k-1} a_j x^j, \quad a_j \in \mathbb{Q}[e_n]$$

satisfies $y^b = 1$, then it follows that $a_0^b = 1$. By Lemma 45, we have $a_0 = \pm 1$. One can easily see that $a_1 = \dots = a_{k-1} = 0$. Thus $y = \pm 1$. □

In contrast with Lemma 46, we have the following.

Proposition 47. *For any odd positive b and any subset $T \subset \mathbb{N}_b$, the ring $\mathbb{Z}[1/b][q]^T$ contains a unique bth root of q, which is invertible in $\mathbb{Z}[1/b][q]^T$.*

For any even positive b and any subset $T \subset \mathbb{N}_b$, the ring $\mathbb{Z}[1/b][q]^T$ contains two bth roots of q which are invertible in $\mathbb{Z}[1/b][q]^T$; one is the negative of the other.

Proof. Let us first consider the case $T = \mathbb{N}_b$. Since F_b is an isomorphism by Theorem 44, we can define a bth root of q by

$$q^{1/b} := F_b^{-1}(q) \in \mathcal{S}_{b,0}\,.$$

If y_1 and y_2 are two bth root of the same element, then their ratio y_1/y_2 is a bth root of 1. From Lemma 46 it follows that if b is odd, there is only one bth root of q in $\mathbb{Z}[1/b][q]^{\mathbb{N}_b}$, and if b is even, there are 2 such roots, one is the minus of the other. We will denote them $\pm q^{1/b}$.

Further it is known that q is invertible in $\mathbb{Z}[q]^{\mathbb{N}}$ (see [14]). Actually, there is an explicit expression $q^{-1} = \sum_n q^n (q;q)_n$. Hence $q^{-1} \in \mathbb{Z}[1/b][q]^{\mathbb{N}_b}$, since the natural homomorphism from $\mathbb{Z}[q]^{\mathbb{N}}$ to $\mathbb{Z}[1/b][q]^{\mathbb{N}_b}$ maps q to q. In a commutative ring, if $x \mid y$ and y is invertible, then so is x. Hence any root of q is invertible.

In the general case of $T \subset \mathbb{N}_b$, we use the natural map $\mathbb{Z}[1/b][q]^{\mathbb{N}_b} \hookrightarrow \mathbb{Z}[1/b][q]^T$. □

Remark 48. By Proposition 47, $\mathcal{S}_{b,0}$ is isomorphic to the ring $\Lambda_b^{\mathbb{N}_b} := \mathbb{Z}[1/b][q^{1/b}]^{\mathbb{N}_b}$ used in [22].

5.4 Realization of $q^{a^2/b}$ in $\mathcal{S}_p$

We define another Frobenius type algebra homomorphism. The difference of the two types of Frobenius homomorphisms is in the target spaces of these homomorphisms.

Suppose m is a positive integer. Define the algebra homomorphism

$$G_m : R[q]^T \to R[q]^{mT} \quad \text{by} \qquad G_m(q) = q^m.$$

Since $\Phi_{mr}(q)$ always divides $\Phi_r(q^m)$, G_m is well–defined.

Throughout this section, let p be a prime or 1. Suppose $b = \pm p^l$ for an $l \in \mathbb{N}$ and let a be an integer. Let $B_{p,j} = G_{p^j}(\mathcal{S}_{p,0})$. Note that $B_{p,j} \subset \mathcal{S}_{p,j}$. If b is odd, by Proposition 47 there is a unique bth root of q in $\mathcal{S}_{p,0}$; we denote it by $x_{b;0}$. If b is even, by Proposition 47 there are exactly two bth root of q, namely $\pm q^{1/b}$. We put $x_{b;0} = q^{1/b}$.

We define the element $z_{b,a} \in \mathcal{S}_p$ as follows:

- If $b \mid a$, let $z_{b,a} := q^{a^2/b} \in \mathcal{S}_p$.
- If $b = \pm p^l \nmid a$, then $z_{b,a} \in \mathcal{S}_p$ is defined by specifying its projections $\pi_j(z_{b,a}) := z_{b,a;j} \in \mathcal{S}_{p,j}$ as follows. Suppose $a = p^s e$, with $(e,p) = 1$. Then $s < l$.

  For $j > s$ let $z_{b,a;j} := 0$.

  For $0 \le j \le s$ let $z_{b,a;j} := [G_{p^j}(x_{b;0})]^{a^2/p^j} = [G_{p^j}(x_{b;0})]^{e^2\, p^{2s-j}} \in B_{p,j} \subset \mathcal{S}_{p,j}$.

Similarly, for $b = \pm p^l$ we define the element $x_b \in \mathcal{S}_p$ as follows:

- We put $\pi_0(x_b) := x_{b;0}$.
- For $j < l$, $\pi_j(x_b) := [G_{p^j}(x_{b;0})]^{p^j}$.
- If $j \ge l$, $\pi_j(x_b) := q^b$.

Notice that for $c = (b, p^j)$ we have

$$\pi_j(x_b) = z_{b,c;j}. \tag{5.1}$$

Proposition 49. *Suppose ξ is a root of unity of order $r = cr'$, where $c = (r,b)$. Then*

$$\mathrm{ev}_\xi(z_{b,a}) = \begin{cases} 0 & \text{if } c \nmid a \\ (\xi^c)^{a_1^2 b'_{*r'}} & \text{if } a = ca_1, \end{cases}$$

where $b'_{*r'}$ is the unique element in $\mathbb{Z}/r'\mathbb{Z}$ such that $b'_{*r'}(b/c) \equiv 1 \pmod{r'}$. Moreover,

$$\mathrm{ev}_\xi(x_b) = (\xi^c)^{b'_{*r'}}\ .$$

Proof. Let us compute $\mathrm{ev}_\xi(z_{b,a})$. The case of $\mathrm{ev}_\xi(x_b)$ follows then from (5.1).

If $b \mid a$, then $c \mid a$, and the proof is obvious.

Suppose $b \nmid a$. Let $a = p^s e$ and $c = p^i$. Then $s < l$. Recall that $z_{b,a} = \prod_{j=0}^{\infty} z_{b,a;j}$. By Lemma 32,

$$\mathrm{ev}_\xi(z_{b,a}) = \mathrm{ev}_\xi(z_{b,a;i}).$$

If $c \nmid a$, then $i > s$. By definition, $z_{b,a;i} = 0$, hence the statement holds true. The case $c \mid a$, i.e. $i \le s$, remains. Note that $\zeta = \xi^c$ is a primitive root of order r' and $(p, r') = 1$. Since $z_{b,a;i} \in B_{p,i}$,

$$\mathrm{ev}_\xi(z_{b,a;i}) \in \mathbb{Z}[1/p][\zeta].$$

From the definition of $z_{b,a;i}$ it follows that $(z_{b,a;i})^{b/c} = (q^c)^{a^2/c^2}$, hence after evaluation we have

$$[\mathrm{ev}_\xi(z_{b,a;i})]^{b/c} = (\zeta)^{a_1^2}.$$

Note also that
$$[(\xi^c)^{a_1^2 b'_{*r'}}]^{b/c} = (\zeta)^{a_1^2}.$$

Using Lemma 45 we conclude $\mathrm{ev}_\xi(z_{b,a;i}) = (\xi^c)^{a_1^2 b'_{*r'}}$ if b is odd, and $\mathrm{ev}_\xi(z_{b,a;i}) = (\xi^c)^{a_1^2 b'_{*r'}}$ or $\mathrm{ev}_\xi(z_{b,a;i}) = -(\xi^c)^{a_1^2 b'_{*r'}}$ if b is even. Since $\mathrm{ev}_1(q^{1/b}) = 1$ and therefore $\mathrm{ev}_\xi(q^{1/b}) = \xi^{b_{*r}}$ (and not $-\xi^{b_{*r}}$), we get the claim. $\square$

6 Unified invariant of lens spaces

The purpose of this chapter is to prove Lemma 37. Recall that $M(b,a;d)$ is the lens space $L(b,a)$ together with an unknot K colored by d inside (see Figure 4.1). In Section 6.1, we compute the renormalized quantum invariant of $M(b,a;d)$ for arbitrary d. We then define in Section 6.2 the unified invariant of $M(b,a;d(\epsilon))$ (see Section 4.1 for the definition of $d(\epsilon)$).

Let us introduce the following notation. For $a,b \in \mathbb{Z}$, the Dedekind sum (see e.g. [19]) is defined by

$$s(a,b) = \sum_{n=0}^{b-1} \left(\left(\frac{n}{b}\right)\right) \left(\left(\frac{an}{b}\right)\right)$$

where

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

and $\lfloor x \rfloor$ denotes the largest integer not greater than x.

For $n,m \in \mathbb{Z}$ coprime and $0 < |n| < |m|$, we define n_{*m} and m_{*n} such that

$$nn_{*m} + mm_{*n} = 1, \text{ with } 0 < \operatorname{sn}(n) n_{*m} < |m|.$$

Notice that for $n = 1$, $m > 1$ we have $1_{*m} = 1$ and $m_{*1} = 0$.

Let r be a fixed integer denoting the order of ξ, a primitive root of unity. If $G = SO(3)$, r is always assumed to be odd.

When we write $\pm$ respectively $\mp$ in a formula, one can either choose everywhere the upper or everywhere the lower signs and the formula holds in both cases.

6.1 The quantum invariant of lens spaces with colored unknot inside

Proposition 50. *Suppose $c = (b,r)$ divides $|a|d \pm 1$. Then*

$$\tau'^{SO(3)}_{M(b,a;d)}(\xi) = (-1)^{\frac{c+1}{2}\,\frac{\operatorname{sn}(ab)-1}{2}} \left(\frac{|a|}{c}\right) \left(\frac{1-\xi^{\pm \operatorname{sn}(a) d b_{*r}}}{1-\xi^{\pm \operatorname{sn}(b) b_{*r}}}\right)^{\chi(c)} \xi^{4_{*r} u^{SO(3)} - 4_{*r} b'_{*r'} \frac{a(\pm a_{*b} - \operatorname{sn}(a) d)^2}{c}} \tag{6.1}$$

where

$$\begin{aligned} u^{SO(3)} \;&:=\; 12 s(1,b) - 12 \operatorname{sn}(b) s(a,b) \\ &\quad + \frac{1}{b}\left(a(1-d^2) + 2(\mp \operatorname{sn}(a) d - \operatorname{sn}(b)) + a(a_{*b} \pm \operatorname{sn}(a) d)^2\right) \in \mathbb{Z} \end{aligned} \tag{6.2}$$

and

$$\tau'^{SU(2)}_{M(b,a;d)}(\xi) = (-1)^{\frac{b'+1}{2}\frac{\mathrm{sn}(ab)-1}{2}} \left(\frac{|a|}{|b'|}\right) \left(\frac{1-\xi^{\pm\,\mathrm{sn}(a)db_{*r}}}{1-\xi^{\pm\,\mathrm{sn}(b)b_{*r}}}\right)^{\chi(c)} \xi^{\frac{u^{SU(2)}}{4} - \frac{b'_{*r'}a_{*b}(\mathrm{sn}(a)ad\pm1)^2(\mathrm{sn}(b)b'-1)^2}{4c}} \tag{6.3}$$

where

$$\begin{aligned} u^{SU(2)} \;:=\; & 12s(1,b) - 12\,\mathrm{sn}(b)s(a,b) + \frac{1}{b}(a(1-d^2)) \\ & + \frac{1}{b}(2(\mp\,\mathrm{sn}(a)d - \mathrm{sn}(b)) + a_{*b}(\mathrm{sn}(a)ad \pm 1)^2(\mathrm{sn}(b)b'-1)^2) \in \mathbb{Z} \end{aligned} \tag{6.4}$$

and $\chi(c) = 1$ *if* $c = 1$ *and is zero otherwise. If* $c \nmid (\mathrm{sn}(a)ad \pm 1)$, $\tau'^{G}_{M(b,a;d)}(\xi) = 0$.

In particular, it follows that $\tau'^{G}_{L(b,a)}(\xi) = 0$ if $c \nmid |a| \pm 1$.

Remark 51. For $G = SU(2)$, the quantum $SU(2)$ invariant of $M(b,a;d)$ is in general dependent on a 4th root of ξ (denoted by $\xi^{\frac{1}{4}}$). Here, we have only calculated the (renormalized) quantum invariant for a certain 4th root of ξ, namely $\xi^{\frac{1}{4}} = e^{l}_{4r}$ for $\xi = e^{l}_{r}$ where l and r are coprime. For the definition of the unified invariant of lens spaces in Section 6.2, we will choose d such that the quantum invariant is *independent* of the 4th root of ξ.

The rest of this section is devoted to the proof of Proposition 50.

6.1.1 The positive case

To start with, we consider the case when $b, a > 0$. Since two lens spaces $L(b,a_1)$ and $L(b,a_2)$ are homeomorphic if $a_1 \equiv a_2 \pmod b$, we can assume $a < b$. Let b/a be given by a continued fraction

$$\frac{b}{a} = m_n - \cfrac{1}{m_{n-1} - \cfrac{1}{m_{n-2} - \dots \cfrac{1}{m_2 - \cfrac{1}{m_1}}}}.$$

We can assume $m_i \geq 2$ for all i (see [15, Lemma 3.1]).

Representing the b/a–framed unknot in Figure 4.1 by a Hopf chain (as e.g. in Lemma 3.1 of [5]), $M(b,a;d)$ is obtained by integral surgery along the link $L_{M(b,a;d)}$ in Figure 6.1, where the m_i are the framing coefficients and K_d denotes the unknot with fixed color d and zero framing.

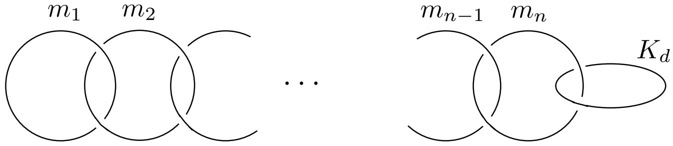


Figure 6.1: Surgery link $L_{M(b,a;d)}$ of $M(b,a;d)$ with integral framing.

The colored Jones polynomial of $L_{M(b,a;d)}$ is given by:

$$J_{L_{M(b,a;d)}}(j_1,\dots,j_n,d) = [j_1]\cdot\prod_{i=1}^{n-1}\frac{[j_ij_{i+1}]}{[j_i]}\cdot\frac{[j_nd]}{[j_n]}$$

(see e.g. [20, Lemma 3.2]). Applying (2.3) and taking the framing into account (let $L_{M(b,a;d),0}$ denote the link $L_{M(b,a;d)}$ with framing zero everywhere), we get

$$F^G_{L_{M(b,a;d),0}}(\xi) = \sum_{j_i}^{\xi,G}\prod_{i=1}^{n} q^{m_i\frac{j_i^2-1}{4}}\prod_{i=1}^{n-1}[j_ij_{i+1}]\cdot[j_nd][j_1] = \mathrm{ev}_\xi\left(\frac{q^{-\frac14\sum_{i=1}^n m_i}}{(q^{\frac12}-q^{-\frac12})^{n+1}}\right)\cdot S_n(a,d,\xi)$$

where

$$S_n(a,d,\xi) = \sum_{j_i}^{\xi,G} q^{\frac14\sum m_ij_i^2}(q^{\frac12 j_1}-q^{-\frac12 j_1})(q^{\frac12 j_1j_2}-q^{-\frac12 j_1j_2})\dots(q^{\frac12 j_{n-1}j_n}-q^{-\frac12 j_{n-1}j_n})(q^{\frac12 j_nd}-q^{-\frac12 j_nd})\,.$$

Inserting these formulas into the Definition (2.5) using $\sigma_+ = n$ and $\sigma_- = 0$ (compare [19, p. 243]) as well as (2.4), we get

$$\tau_{M(b,a;d)}(\xi) = \frac{\mathrm{ev}_\xi\left(q^{-\frac14\sum_{i=1}^n m_i+\frac{n}{2}}\right)\cdot S_n(a,d,\xi)}{(-2)^n(\gamma_1^G(\xi))^n\,\mathrm{ev}_\xi(q^{\frac12}-q^{-\frac12})}.$$

We restrict us now to the root of unity $e_r = \exp(\frac{2\pi i}{r})$. We look at arbitrary primitive roots of unity in Subsection 6.1.3.

Applying (2.1), (2.2) and the following formula for the Dedekind sum (compare [19, Theorem 1.12])

$$3n - \sum_i m_i = -12s(a,b) + \frac{a+a_{*b}}{b}, \tag{6.5}$$

we get

$$\tau_{M(b,a;d)}(e_r) = \mathrm{ev}_{e_r}\left(\frac{q^{\frac14(-12s(a,b)+\frac{a+a_{*b}}{b})}}{q^{\frac12}-q^{-\frac12}}\right)\cdot\frac{S_n(a,d,e_r)}{(-2)^n\sqrt{r}^n\epsilon^G(r)^n}, \tag{6.6}$$

where $\epsilon^{SO(3)}(r) = 1$ if $r\equiv 1 \pmod 4$, $\epsilon^{SO(3)}(r) = i$ if $r\equiv 3 \pmod 4$ and $\epsilon^{SU(2)}(r) = 1+i$.

Finally, for the renormalized version, we have to divide by

$$\tau^G_{L(b,1)}(e_r) = \mathrm{ev}_{e_r}\left(\frac{q^{\frac14(-12s(1,b)+\frac2b)}}{q^{\frac12}-q^{-\frac12}}\right)\cdot\frac{S_1(1,1,e_r)}{(-2)\sqrt{r}\epsilon^G(r)}.$$

This gives

$$\tau'_{M(b,a;d)}(e_r) = (-2\sqrt{r}\,\epsilon^G(r))^{1-n}\cdot\frac{S_n(a,d,e_r)}{S_1(1,1,e_r)}\cdot\mathrm{ev}_{e_r}\left(q^{\frac14(-12s(a,b)+\frac{a+a_{*b}}{b}+12s(1,b)-\frac2b)}\right). \tag{6.7}$$

We put $S_n(d) := S_n(a,d,e_r)$. To calculate $S_n(d)$, we need to look separately at the $SO(3)$ and the $SU(2)$ case.

The $SO(3)$ case

We follow the arguments of [25]. The $\tau_{M(b,a;d)}(e_r)$ can be computed in the same way as the invariant $\xi_r(L(b,a),A)$ in [25], after replacing A^2 (respectively A) by $e_r^{2_{*r}}$ (respectively $e_r^{4_{*r}}$).

Using Lemmas 4.11, 4.12 and 4.20 of [25][1] (and replacing e_r by $e_r^{4_{*r}}$, c_n by c, $N_{n,1}=p$ by b, $N_{n-1,1}=q$ by a, $N_{n,2}=q^*$ by a_{*b} and $-N_{n-1,2}=p^*$ by b_{*a}), we get

$$S_n(d) = (-2)^n(\sqrt{r}\epsilon(r))^n\sqrt{c}\epsilon(c)\left(\frac{\frac{b}{c}}{\frac{r}{c}}\right)\left(\frac{a}{c}\right)(-1)^{\frac{r-1}{2}\frac{c-1}{2}} \cdot \sum_{\pm}\chi^{\pm}(d)e_r^{-ca4_{*r}b'_{*r'}\left(\frac{d\mp a_{*b}}{c}\right)^2 \pm 2_{*r}b_{*a}(d\mp a_{*b})+4_{*r}a_{*b}b_{*a}} \tag{6.8}$$

where $\epsilon(x)=1$ if $x\equiv 1 \pmod 4$ and $\epsilon(x)=i$ if $x\equiv 3 \pmod 4$. Further, $\chi^{\pm}(d)=\pm 1$ if $c \mid d\mp a_{*b}$ and is zero otherwise. Since $(a,c)=(a,b)=1$ and $a(d\mp a_{*b})=ad\mp aa_{*b}=ad\mp(1-bb_{*a})$ and $c\mid b$, we have $c \mid d\mp a_{*b}$ if and only if $c\mid ad\mp 1$. This implies the last claim of Proposition 50 for $G=SO(3)$.

Note that when $c=1$, both $\chi^{\pm}(d)$ are nonzero. If $c>1$ and $c\mid(d-a_{*b})$, $\chi^{+}(d)=1$, but $\chi^{-}(d)=0$. Indeed, for c dividing $d-a_{*b}$, $c\mid(d+a_{*b})$ if and only if $c\mid a_{*b}$, which is impossible, because $c\mid b$ but $(b,a_{*b})=1$. For the same reason, if $c\mid d+a_{*b}$, then $\chi^{+}(d)=0$ and $\chi^{-}(d)=-1$.

Inserting (6.8) into (6.7) we get

$$\tau'^{SO(3)}_{M(b,a;d)}(e_r) = \left(\frac{a}{c}\right)\left(\frac{1-e_r^{\pm db'_{*r'}}}{1-e_r^{\pm b'_{*r'}}}\right)^{\chi(c)} e_r^{4_{*r}u-4_{*r}b'_{*r'}\frac{a(d\pm a_{*b})^2}{c}}$$

where

$$u=-12s(a,b)+12s(1,b)+\frac{1}{b}\left(a+a_{*b}-2-b_{*a}b(a_{*b}\pm 2d)\right).$$

Notice that $u\in\mathbb{Z}$. Further observe that by using $aa_{*b}+bb_{*a}=1$, we get

$$a+a_{*b}-2-b_{*a}b(a_{*b}\pm 2d)=2(\mp d-1)+a(1-d^2)+a(a_{*b}\pm d)^2.$$

Since $\tau_{M(b,a;d)}(\xi)=\tau_{M(-b,-a;d)}(\xi)$, this implies the result (6.1) for $0<a<b$ and $0>a>b$ for the root of unity e_r.

The SU(2) case

The way we calculate $S_n(d)$ in the $SO(3)$ case can not be adapted to the $SU(2)$ case. We use instead a Gauss sum reciprocity formula following the arguments of [15].

We use a well–known result by Cauchy and Kronecker.

Proposition 52 (Gauss sum reciprocity formula in one dimension). *For $m,n,\psi,\varphi\in\mathbb{Z}$ such that nm is even and $\varphi\mid n\psi$ we have*

$$\sum_{\lambda=0}^{n-1} e_{2n}^{m\lambda^2}e_{\varphi}^{\psi\lambda} = (1+i)\sqrt{\frac{n}{2m}}\sum_{\lambda=0}^{m-1} e_{2m\varphi^2}^{-n(\lambda\varphi+\psi)^2}.$$

[1]There are misprints in Lemma 4.21 of [25]: $q^*\pm n$ should be replaced by $q^*\mp n$ for $n=1,2$.

In particular for $n=\varphi$ even, we have

$$\sum_{\lambda=0}^{n-1} e_{2n}^{m\lambda^2+2\psi\lambda} = (1+i)\sqrt{\frac{n}{2m}}\sum_{\lambda=0}^{m-1} e_{2mn}^{-(n\lambda+\psi)^2}.$$

Proof. A proof can be found in e.g. [7, Chapter IX, Theorem 1]. For a generalization of this result see also [15, Propositions 2.3 and 4.3] and [8]. □

Lemma 53. *For $a,b>0$, we have*

$$S_n(d) = C_n \sum_{\substack{\gamma=0 \\ \gamma\equiv d \pmod{2r}}}^{2rb-1} \left(e_{4abr}^{-(a\gamma-1)^2} - e_{4abr}^{-(a\gamma+1)^2}\right)$$

where

$$C_n := (-2(1+i)\sqrt{r})^n \frac{1}{\sqrt{b}} e_{4ar}^{b_{*a}}.$$

Proof. We prove the claim by induction on n. Notice first, that for $x,y,z\in\mathbb{Z}$ we have

$$\sum_{j=0}^{2r-1} e_r^{zj^2}(e_r^{xj}-e_r^{-xj})(e_r^{yj}-e_r^{-yj}) = 2\sum_{j=0}^{2r-1} e_r^{zj^2}(e_r^{(y+x)j}-e_r^{(y-x)j}) \tag{6.9}$$

by replacing j by $-j$ in $-\sum_{j=0}^{2r-1} e_r^{zj^2}(e_r^{xj}-e_r^{-xj})e_r^{-yj}$. Using this and Proposition 52, we have for $n=1$

$$\begin{aligned} S_1(d) &= \sum_{j=0}^{2r-1} e_{4r}^{bj^2}(e_{2r}^{j}-e_{2r}^{-j})(e_{2r}^{jd}-e_{2r}^{-jd}) \\ &= 2\sum_{j=0}^{2r-1} e_{4r}^{bj^2}(e_{2r}^{j(d+1)}-e_{2r}^{j(d-1)}) \\ &= 2(1+i)\sqrt{\frac{r}{b}}\sum_{j=0}^{b-1}(e_{4br}^{-(2rj+d+1)^2}-e_{4br}^{-(2rj+d-1)^2}) \\ &= 2(1+i)\sqrt{\frac{r}{b}}\sum_{\substack{\gamma=0 \\ \gamma\equiv d \pmod{2r}}}^{2rb-1}(e_{4br}^{-(\gamma+1)^2}-e_{4br}^{-(\gamma-1)^2}). \end{aligned}$$

Now, we set

$$\frac{\tilde{b}}{\tilde{a}} = m_{n-1} - \frac{1}{m_{n-2} - \cdots \frac{1}{m_2 - \frac{1}{m_1}}}.$$

Notice that $\tilde{b}>\tilde{a}$. Assume the result of the lemma inductively. We have

$$\begin{aligned} S_n(d) &= \sum_{j_n=0}^{2r-1} e_{4r}^{m_n j_n^2}(e_{2r}^{j_n d}-e_{2r}^{-j_n d})S_{n-1}(j_n) \\ &= \sum_{j_n=0}^{2r-1} e_{4r}^{m_n j_n^2}(e_{2r}^{j_n d}-e_{2r}^{-j_n d})C_{n-1} \sum_{\substack{\gamma=0 \\ \gamma\equiv j_n \pmod{2r}}}^{2\tilde{b}r-1}(e_{4\tilde{a}\tilde{b}r}^{-(\tilde{a}\gamma-1)^2}-e_{4\tilde{a}\tilde{b}r}^{-(\tilde{a}\gamma+1)^2}) \end{aligned}$$

We replace j_n by γ everywhere and get

$$\begin{aligned} S_n(d) &= C_{n-1}\sum_{\gamma=0}^{2\tilde b r-1} e_{4r}^{m_n\gamma^2}(e_{2r}^{\gamma d}-e_{2r}^{-\gamma d})(e_{4\tilde a\tilde b r}^{-\tilde a^2\gamma^2+2\tilde a\gamma-1}-e_{4\tilde a\tilde b r}^{-\tilde a^2\gamma^2-2\tilde a\gamma-1}) \\ &= C_{n-1}e_{4\tilde a\tilde b r}^{-1}\sum_{\gamma=0}^{2\tilde b r-1} e_{4\tilde b r}^{(\tilde b m_n-\tilde a)\gamma^2}(e_{2r}^{\gamma d}-e_{2r}^{-\gamma d})(e_{2\tilde b r}^{\gamma}-e_{2\tilde b r}^{-\gamma}) \\ &= 2C_{n-1}e_{4\tilde a\tilde b r}^{-1}\sum_{\gamma=0}^{2\tilde b r-1} e_{4\tilde b r}^{(\tilde b m_n-\tilde a)\gamma^2}(e_{2\tilde b r}^{(\tilde b d+1)\gamma}-e_{2\tilde b r}^{(\tilde b d-1)\gamma}) \\ &= 2C_{n-1}e_{4\tilde a\tilde b r}^{-1}\sum_{\gamma=0}^{2ar-1} e_{4ar}^{b\gamma^2}(e_{2ar}^{(ad+1)\gamma}-e_{2ar}^{(ad-1)\gamma}) \end{aligned}$$

using first (6.9) and then $\tilde b = a$ and $\tilde b m_n - \tilde a = b$. Applying again Proposition 52 gives

$$\begin{aligned} S_n(d) &= 2(1+i)C_{n-1}e_{4\tilde a\tilde b r}^{-1}\sqrt{\frac{ar}{b}}\sum_{\lambda=0}^{b-1}(e_{4abr}^{-(2ar\lambda+ad+1)}-e_{4abr}^{-(2ar\lambda+ad-1)}) \\ &= -2(1+i)C_{n-1}e_{4\tilde a\tilde b r}^{-1}\sqrt{\frac{ar}{b}}\sum_{\substack{\gamma=0\\ \gamma\equiv d \ (\mathrm{mod}\ 2r-1)}}^{2rb-1}(e_{4abr}^{-(a\gamma-1)}-e_{4abr}^{-(a\gamma+1)}) \end{aligned}$$

where we put $\gamma = 2r\lambda + d$. Since

$$1 = \tilde a\tilde a_{*\tilde b} + \tilde b\tilde b_{*\tilde a} = \tilde a_{*\tilde b}(am_n - b) + \tilde b_{*\tilde a}a = a(\tilde b_{*\tilde a} + \tilde a_{*\tilde b}m_n) - b\tilde a_{*\tilde b} = aa_{*b} + bb_{*a}$$

and

$$0 < a(\tilde b_{*\tilde a} + \tilde a_{*\tilde b}m_n) = \tilde b\tilde b_{*\tilde a} + \tilde b\tilde a_{*\tilde b}m_n = 1 + \tilde a_{*\tilde b}(\tilde b m_n - \tilde a) = 1 + \tilde a_{*\tilde b}b < 1 + \tilde b b = 1 + ab$$

we have $0 < \tilde b_{*\tilde a} + \tilde a_{*\tilde b}m_n < b$ and therefore $b_{*a} = -\tilde a_{*\tilde b}$. Therefore

$$e_{4\tilde a r}^{\tilde b_{*\tilde a}}e_{4\tilde a\tilde b r}^{-1} = e_{4\tilde a\tilde b r}^{-\tilde a_{*\tilde b}\tilde a+1}e_{4\tilde a\tilde b r}^{-1} = e_{4\tilde b r}^{-\tilde a_{*\tilde b}} = e_{4ar}^{b_{*a}}$$

and we get

$$-2(1+i)C_{n-1}e_{4\tilde a\tilde b r}^{-1}\sqrt{\frac{ar}{b}} = C_n.$$

□

For further calculations on $S_n(d)$ we need the following result.

Lemma 54. *For $x, y \in \mathbb{N}$ and b odd with $(b, x) = 1$ we have*

$$\sum_{j=0}^{b-1} e_b^{-xj^2-yj} = \epsilon(b)\left(\frac{-x}{b}\right)\sqrt{b}e_b^{(b-1)^2\frac{x_{*b}y^2}{4}}.$$

Proof. This follows from Lemma 10 using $e_b^{-1} = e_b^{(b-1)}$ with $(b,(b-1)x) = (b,x) = 1$ and $(b-1)_{*b} \equiv -1 \pmod b$. □

Lemma 55. *For $c = (b,r)$ and $b' = \frac{b}{c}$, $r' = \frac{r}{c}$ we have*

$$S_n(d) = (-2(1+i)\sqrt{r})^n \sqrt{c}\epsilon(b') \left(\frac{-ar'}{b'}\right) \sum_{\pm} \chi(d) e_{4rb}^{-a_{*b}-ad^2\mp 2d} e_{r'b'}^{-\frac{a_{*b}(ad\pm1)^2(b'-1)^2(b'_{*r'}b'-1)}{4c^2}}$$

where $\chi(d) = \begin{cases} 0 & \text{if } c \nmid ad\pm 1 \\ \mp 1 & \text{if } c \mid ad \pm 1 \end{cases}$.

Proof. We put $\gamma = d + 2r\lambda$ and get

$$S_n(d) = C_n \sum_{\pm} \sum_{\lambda=0}^{b-1} \mp e_{4abr}^{-(a\gamma\pm1)^2}.$$

Notice that $(a\gamma \pm 1)^2 = a^2d^2 + 1 \pm 2ad + 4ar(ar\lambda^2 + ad\lambda \pm \lambda)$ and therefore

$$S_n(d) = C_n e_{4abr}^{-a^2d^2-1} \sum_{\pm}\sum_{\lambda=0}^{b-1} \mp e_b^{-ar\lambda^2-(ad\pm1)\lambda} e_{2br}^{\mp d}$$

and

$$C_n e_{4rab}^{-a^2d^2-1} = (-2(1+i)\sqrt{r})^n \frac{1}{\sqrt{b}} e_{4rb}^{-a_{*b}-ad^2}.$$

We use [25, Theorem 2.1] as well as Lemma 54 to get

$$\begin{aligned} \mp\sum_{\lambda=0}^{b-1} e_b^{-ar\lambda^2-(ad\pm1)\lambda} e_{2br}^{\mp d} &= \mp e_{2br}^{\mp d} \cdot c \sum_{\substack{\lambda=0 \\ c|ad\pm1}}^{b'-1} e_{b'}^{-ar'\lambda^2 - \frac{(ad\pm1)}{c}\lambda} \\ &= \chi(d) c e_{2br}^{\mp d} \epsilon(b') \left(\frac{-ar'}{b'}\right) \sqrt{b'} e_{b'r'}^{a_{*b}(1-b'b'_{*r})\left(\frac{ad\pm1}{c}\right)^2 \frac{(b'-1)^2}{4}} \end{aligned}$$

where we used that $a_{*b} \equiv a_{*b'} \pmod{b'}$, $(ar')_{*b'} \equiv a_{*b'} r'_{*b'} \pmod{b'}$ and $e_{b'}^{r'_{*b'}} = e_{b'r'}^{r'_{*b'}r'} = e_{b'r'}^{1-b'b'_{*r'}}$. □

Lemma 55 implies the last claim of Proposition 50 for $G = SU(2)$.

Since

$$\left(\frac{1 - e_{4rb}^{\pm 4d} e_{rb}^{\pm a_{*b}ad(b-1)^2(b_{*r}b-1)}}{1 - e_{rb}^{\pm1} e_{rb}^{\pm(b-1)^2(b_{*r}b-1)}}\right)^{\chi(c)} = \left(\frac{1 - e_{rb}^{\pm a_{*b}ad(b_{*r}b-1)-d}}{1-e_r^{\pm b_{*r}}}\right)^{\chi(c)} = \left(\frac{1-e_r^{\pm db_{*r}}}{1-e_r^{\pm b_{*r}}}\right)^{\chi(c)},$$

inserting the formula of $S_n^{SU(2)}(d)$ of Lemma 55 into (6.7), we get

$$\tau'^{SU(2)}_{M(b,a;d)}(e_r) = \left(\frac{a}{b'}\right)\left(\frac{1-e_r^{\pm db_{*r}}}{1-e_r^{\pm b_{*r}}}\right)^{\chi(c)} e_{4r}^{-12s(a,b)+12s(1,b)+\frac{1}{b}(a+a_{*b}-2)} e_{4br}^{-a_{*b}-ad^2\mp2d} e_{br}^{\frac{-a_{*b}(ad\pm1)^2(b'-1)^2(b'_{*r'}b'-1)}{4}}.$$

Therefore

$$\tau'^{SU(2)}_{M(b,a;d)}(e_r) = \left(\frac{a}{b'}\right)\left(\frac{1-e_r^{\pm db_{*r}}}{1-e_r^{\pm b_{*r}}}\right)^{\chi(c)} e_r^{\frac{u}{4}-\frac{b'_{*r'}a_{*b}(ad\pm 1)^2(b'-1)^2}{4c}}$$

where $u := -12s(a,b) + 12s(1,b) + \frac{1}{b}(a(1-d^2) + 2(\mp d - 1) + a_{*b}(ad \pm 1)^2(b'-1)^2)$. This implies (6.3) for $0 < a < b$ or $0 > a > b$ at e_r.

6.1.2 The negative case

To compute $\tau'^G_{M(-b,a;d)}(e_r)$, observe that, since $L(b,a)$ and $L(-b,a)$ are homeomorphic with opposite orientation, $\tau^G_{M(-b,a;d)}(\xi) = \tau^G_{M(b,-a;d)}(\xi)$ is equal to the complex conjugate of $\tau^G_{M(b,a;d)}(\xi)$ due to Theorem 16. The ratio

$$\tau'^G_{M(-b,a;d)}(\xi) = \frac{\overline{\tau^G_{M(b,a;d)}(\xi)}}{\tau^G_{L(b,1)}(\xi)}$$

can be computed analogously to the positive case. Using $\overline{\epsilon(c)} = (-1)^{\frac{c-1}{2}}\epsilon(c)$, we have for $a, b > 0$

$$\tau'^{SO(3)}_{M(-b,a,d)}(e_r) = (-1)^{\frac{c+1}{2}}\left(\frac{a}{c}\right)\left(\frac{1-e_r^{\mp db_{*r}}}{1-e_r^{\pm b_{*r}}}\right)^{\chi(c)} e_r^{4_{*r}\tilde{u}^{SO(3)}+4_{*r}b'_{*r'}\frac{a(d\pm a_{*b})^2}{c}}$$

where

$$\begin{aligned}\tilde{u}^{SO(3)} &= 12s(a,b) + 12s(1,b) + \frac{1}{b}\left(-a - a_{*b} - 2 + b_{*a}b(a_{*b} \pm 2d)\right)\\ &= 12s(a,b) + 12s(1,b) + \frac{1}{b}\left(2(\pm d - 1) - a(1-d^2) - a(a_{*b} \pm d)^2\right)\end{aligned}$$

and

$$\tau'^{SU(2)}_{M(-b,a:d)}(e_r) = (-1)^{\frac{b'+1}{2}}\left(\frac{a}{b'}\right)\left(\frac{1-e_r^{\mp db_{*r}}}{1-e_r^{\pm b_{*r}}}\right)^{\chi(c)} e_r^{\frac{\tilde{u}^{SU(2)}}{4}+\frac{b'_{*r'}a_{*b}(ad\pm 1)^2(b'-1)^2}{4c}}$$

where

$$\tilde{u}^{SU(2)} = 12s(a,b) + 12s(1,b) + \frac{1}{b}(-a(1-d^2) + 2(\pm d - 1) - a_{*b}(ad \pm 1)^2(b'-1)^2).$$

Using $s(a,b) = s(a,-b) = -s(-a,b)$, we get the claim of Proposition 50 at e_r if either a or b is negative.

6.1.3 Arbitrary primitive roots of unity

To get the result of Proposition 50 for an arbitrary primitive root of unity ξ of order r, notice that we can regard τ'^G as a map

$$\tau'^G : \{\xi^{\frac{1}{4}} \in \mathbb{C} \mid \xi = e_r^l, (r,l) = 1\} \to \mathbb{Q}(e_{4r})$$

and notice that for $\xi = e_r^l$ for some l coprime to r, we have the Galois transformation

$$\varphi : \mathbb{Q}(e_{4r}) \to \mathbb{Q}(\xi^{\frac{1}{4}}), \qquad e_{4r} \mapsto e_{4r}^l$$

which is a ring isomorphism and maps e_r to ξ. Therefore $\tau'^G_{M(b,a;d)}(\xi) = \tau'^G_{M(b,a;d)}(\varphi(e_r)) = \varphi(\tau'^G_{M(b,a;d)}(e_r))$ and we get (6.1) and (6.3) in general.

□

Example. For $b > 0$, we have

$$\tau'^{SO(3)}_{L(-b,1)}(\xi) = (-1)^{\frac{c+1}{2} - \chi(c)}\, \xi^{2_{*r}(b-3) + b_{*r}\chi(c)} \quad \text{and} \quad \tau'^{SU(2)}_{L(-b,1)}(\xi) = (-1)^{\frac{b'-1}{2} + \chi(c)} \xi^{\frac{b-3}{2} + b_{*r}\chi(c)}.$$

6.2 Proof of Lemma 37

Since $L(-b, a)$ and $L(b, -a)$ are homeomorphic, we can assume b to be positive, i.e. $b = p^l$ for a prime p. We have to define the unified invariant of $M^\varepsilon(b, a) := M(b, a; d(\varepsilon))$ where $d(0) = 1$ and $d(\bar{0})$ is the smallest odd positive integer such that $\operatorname{sn}(a)ad(\bar{0}) \equiv 1 \pmod b$.

Recall from Section 5.3 that we denote the unique positive bth root of q in $S_{p,0}$ by $q^{\frac{1}{b}}$. For $p \neq 2$, we define the unified invariant $I^G_{M^\varepsilon(b,a)} \in \mathcal{S}_p$ by specifying its projections

$$\pi_j I^{SO(3)}_{M^\varepsilon(b,a)} := \begin{cases} q^{3s(1,b) - 3\operatorname{sn}(b)\, s(a,b)} & \text{if } j = 0,\ \varepsilon = 0 \\ (-1)^{\frac{p^j+1}{2}\frac{\operatorname{sn}(a)-1}{2}} \left(\frac{|a|}{p}\right)^j q^{\frac{u'^{SO(3)}}{4}} & \text{if } 0 < j < l,\ \varepsilon = \bar{0} \\ (-1)^{\frac{p^l+1}{2}\frac{\operatorname{sn}(a)-1}{2}} \left(\frac{|a|}{p}\right)^l q^{\frac{u'^{SO(3)}}{4}} & \text{if } j \geq l,\ \varepsilon = \bar{0} \end{cases}$$

where $u'^{SO(3)} := u^{SO(3)} - \frac{a(a_{*b} - \operatorname{sn}(a)d(\bar{0}))^2}{b}$ and $u^{SO(3)}$ is defined in (6.2) and

$$\pi_j I^{SU(2)}_{M^\epsilon(b,a)} := \begin{cases} (-1)^{\frac{b+3}{2}\frac{\operatorname{sn}(a)-1}{2}} \left(\frac{|a|}{p}\right)^l q^{3s(1,b) - 3\operatorname{sn}(b)s(a,b)} & \text{if } j = 0, \epsilon = 0 \\ (-1)^{\frac{p^{l-j}+1}{2}\frac{\operatorname{sn}(a)-1}{2}} \left(\frac{|a|}{p}\right)^{l-j} q^{\frac{u'^{SU(2)}}{4}} & \text{if } 0 < j < l, \varepsilon = \bar{0} \\ (-1)^{\frac{\operatorname{sn}(a)-1}{2}} q^{\frac{u'^{SU(2)}}{4}} & \text{if } j > l, \varepsilon = \bar{0} \end{cases}$$

where $u'^{SU(2)} := u^{SU(2)} - \frac{a_{*b}(\operatorname{sn}(a)ad - 1)^2(\operatorname{sn}(b)b' - 1)^2}{b}$ and $u^{SU(2)}$ is defined in (6.4).

For $G = SO(3)$ and $p = 2$, only $\pi_0 I^{SO(3)}_{M(b,a)} \in \mathcal{S}_{2,0} = \mathcal{R}_2$ is non–zero and it is defined to be $q^{3s(1,b) - 3s(a,b)}$.

The $I^G_{M^\varepsilon(b,a)}$ is well–defined due to Lemma 56 below, i.e. all powers of q in $I^G_{M^\varepsilon(b,a)}$ are integers for $j > 0$ or lie in $\frac{1}{b}\mathbb{Z}$ for $j = 0$. Unlike the invariant for arbitrary d, there is no dependency on the 4th root of q. Further, for b odd (respectively even) $I^G_{M^\varepsilon(b,a)}$ is invertible in $\mathcal{S}_p^{p,\varepsilon}$ (respectively $\mathcal{R}_p^{p,\varepsilon}$) since q and $q^{\frac{1}{b}}$ are invertible in these rings.

In particular, for odd $b=p^l$, we have $I^G_{L(b,1)}=1$ and

$$\pi_j I^{SO(3)}_{L(-b,1)} = \begin{cases} q^{\frac{b-3}{2}+\frac{1}{b}} & \text{if } j=0 \\ (-1)^{\frac{p^j+1}{2}} q^{\frac{b-3}{2}} & \text{if } 0<j<l,\ p \text{ odd} \\ (-1)^{\frac{p^l+1}{2}} q^{\frac{b-3}{2}} & \text{if } j\geq l,\ p \text{ odd} \end{cases}$$

and

$$\pi_j I^{SU(2)}_{L(-b,1)} = \begin{cases} (-1)^{\frac{-b+3}{2}} q^{\frac{b-3}{2}+\frac{1}{b}} & \text{if } j=0 \\ (-1)^{\frac{p^{l-j}+1}{2}} q^{\frac{b-3}{2}} & \text{if } 0<j<l,\ p \text{ odd} \\ q^{\frac{b-3}{2}} & \text{if } j\geq l,\ p \text{ odd}\,. \end{cases}$$

It is left to show that for any ξ of order r coprime with p, we have

$$\mathrm{ev}_\xi(I^G_{M^0(b,a)}) = \tau'^G_{M^0(b,a)}(\xi)$$

and, if $r=p^jk$ with $j>0$, then

$$\mathrm{ev}_\xi(I^G_{M^{\bar{0}}(b,a)}) = \tau'^G_{M^{\bar{0}}(b,a)}(\xi)\,.$$

For $\varepsilon=0$, this follows directly from Propositions 49 and 50 with $c=d=1$ using

$$\frac{1-\xi^{-\operatorname{sn}(a)b_{*r}}}{1-\xi^{-b_{*r}}} = \begin{cases} 1 & \text{if } \operatorname{sn}(a)=1 \\ -\xi^{-\operatorname{sn}(a)b_{*r}} & \text{if } \operatorname{sn}(a)=-1 \end{cases}.$$

For $\varepsilon=\bar{0}$, we have $c=(p^j,b)>1$ and we get the claim by using Proposition 50 and for the $SO(3)$ case

$$\xi^{\frac{a(a_{*b}-\operatorname{sn}(a)d(\bar{0}))^2}{b}} = \xi^{c\,\frac{a(a_{*b}-\operatorname{sn}(a)d(\bar{0}))^2}{bc}} = \xi^{bb'_{*r'}\,\frac{a(a_{*b}-\operatorname{sn}(a)d(\bar{0}))^2}{bc}} = \xi^{b'_{*r'}\,\frac{a(a_{*b}-\operatorname{sn}(a)d(\bar{0}))^2}{c}}, \tag{6.10}$$

respectively for the $SU(2)$ case

$$\xi^{\frac{a_{*b}(\operatorname{sn}(a)ad-1)^2}{b}\cdot\frac{(b'-1)^2}{4}} = \xi^{c\,\frac{a_{*b}(\operatorname{sn}(a)ad-1)^2}{bc}\cdot\frac{(b'-1)^2}{4}} = \xi^{bb'_{*r'}\,\frac{a_{*b}(\operatorname{sn}(a)ad-1)^2}{bc}\cdot\frac{(b'-1)^2}{4}} = \xi^{b'_{*r'}\,\frac{a_{*b}(\operatorname{sn}(a)ad-1)^2}{c}\cdot\frac{(b'-1)^2}{4}}, \tag{6.11}$$

where for the second equalities in (6.10) and (6.11) we use $c\equiv bb'_{*r'} \pmod r$. For $G=SO(3)$, notice that due to part *(b)* of Lemma 56 below, b and c divide $a_{*b}-\operatorname{sn}(a)d(\bar{0})$ and therefore all powers of ξ in (6.10) are integers. For $G=SU(2)$, we can see that all powers of ξ in (6.11) are integers using part *(c)* of Lemma 56 and the fact that b' is odd and therefore $4\mid(b'-1)^2$. □

The following Lemma is used in the proof of Lemma 37.

Lemma 56. *We have*

(a) $3s(1,b) - 3\,\mathrm{sn}(b)\,s(a,b) \in \frac{1}{b}\mathbb{Z}$,

(b) $b \mid a_{*b} - \mathrm{sn}(a)d(\bar{0})$ *and therefore* $u' \in \mathbb{Z}$,

(c) $b \mid \mathrm{sn}(a)ad - 1$ *and therefore* $u' \in \mathbb{Z}$, *and*

(d) $4 \mid u'$ *for* $d = d(\bar{0})$.

Proof. For claim *(a)*, using the identity

$$12bs(a,b) \equiv (b-1)(b+2) - 4a(b-1) + 4\sum_{j<\frac{b}{2}} \left\lfloor \frac{2aj}{b} \right\rfloor \pmod 8$$

(e.g. see [2, Theorem 3.9]), we get

$$12bs(a,b) - 12bs(1,b) \equiv 4(1-a)(b-1) + 4\sum_{j<\frac{b}{2}} \left\lfloor \frac{2aj}{b} \right\rfloor - \left\lfloor \frac{2j}{b} \right\rfloor \pmod 8$$

which is divisible by 4 in $\mathbb{Z}$.

Claim *(b)* follows from the fact that $(a,b) = 1$ and

$$a(a_{*b} - \mathrm{sn}(a)d) = 1 - \mathrm{sn}(a)ad - bb_{*a} \equiv 0 \pmod b,$$

since d is chosen such that $\mathrm{sn}(a)ad \equiv 1 \pmod b$, which also proves the Claim *(c)*. For Claim *(d)*, notice that for odd d we have

$$4 \mid (1-d^2) \quad \text{and} \quad 4 \mid 2(\mathrm{sn}(a)d - \mathrm{sn}(b)).$$

∎

7 Laplace transform

This chapter is devoted to the proof of Theorem 38 by using Andrew's identity. Throughout this chapter, let p be a prime or $p = 1$ and $b = \pm p^l$ for some $l \in \mathbb{N}$.

7.1 Definition of Laplace transform

The Laplace transform is a $\mathbb{Z}[q^{\pm 1}]$–linear map defined by

$$\begin{aligned} \mathcal{L}_b : \mathbb{Z}[z^{\pm 1}, q^{\pm 1}] &\rightarrow \mathcal{S}_p \\ z^a &\mapsto z_{b,a}. \end{aligned}$$

In particular, we put $\mathcal{L}_{b;j} := \pi_j \circ \mathcal{L}_b$ and have $\mathcal{L}_{b;j}(z^a) = z_{b,a;j} \in \mathcal{S}_{p,j}$.

Further, for any $f \in \mathbb{Z}[z^{\pm 1}, q^{\pm 1}]$ and $n \in \mathbb{Z}$, we define

$$\hat{f} := f|_{z=q^n} \in \mathbb{Z}[q^{\pm n}, q^{\pm 1}] .$$

Lemma 57. *Suppose $f \in \mathbb{Z}[z^{\pm 1}, q^{\pm 1}]$. Then for any root of unity ξ of order r (odd for $G = SO(3)$),*

$$\sum_n^{\xi,G} q^{b\frac{n^2-1}{4}} \hat{f} = \gamma_b^G(\xi)\ \mathrm{ev}_\xi(\mathcal{L}_{-b}(f)).$$

Proof. It is sufficient to consider the case $f = z^a$. Then, by the same arguments as in the proof of [5, Lemma 1.3], we have

$$\sum_n^{\xi,G} q^{b\frac{n^2-1}{4}} q^{na} = \begin{cases} 0 & \text{if } c \nmid a \\ (\xi^c)^{-a_1^2 b'_*}\, \gamma_b^G(\xi) & \text{if } a = ca_1. \end{cases} \tag{7.1}$$

The result follows now from Proposition 49.

□

7.2 Proof of Theorem 38

Recall that

$$A(n,k) = \frac{\prod_{i=0}^{k} (q^n + q^{-n} - q^i - q^{-i})}{(1-q)\,(q^{k+1};q)_{k+1}}.$$

We have to show that there exists an element $Q_{b,k} \in \mathcal{R}_b$ (respectively $Q_{b,k} \in \mathcal{S}_b$ if b odd), such that for every root of unity ξ of order r (odd if $G = SO(3)$), one has

$$\frac{\sum_n^{\xi,G} q^{b\frac{n^2-1}{4}} A(n,k)}{F_{U^b}(\xi)} = \mathrm{ev}_\xi(Q_{b,k}).$$

Applying Lemma 57 to $F_{U^b}(\xi) = \sum_n^{\xi,G} q^{b\frac{n^2-1}{4}}[n]^2$, we get for $c = (b,r)$

$$F_{U^b}(\xi) = 2\gamma_b^G(\xi)\ \mathrm{ev}_\xi\left(\frac{(1-x_{-b})^{\chi(c)}}{(1-q^{-1})(1-q)}\right), \tag{7.2}$$

where, as usual, $\chi(c) = 1$ if $c = 1$ and zero otherwise. We will prove that for an odd prime p and any number $j \geq 0$, there exists an element $Q_k(q, x_b, j) \in \mathcal{S}_{p,j}$ such that

$$\frac{1}{(q^{k+1};q)_{k+1}} \mathcal{L}_{b;j}\left(\prod_{i=0}^{k}(z+z^{-1}-q^i-q^{-i})\right) = 2\, Q_k(q^{\mathrm{sn}(b)}, x_b, j). \tag{7.3}$$

If $p = 2$ we will prove the claim for $j = 0$ only, since $\mathcal{S}_{2,0} \simeq \mathcal{R}_2$.

Remark 58. The case $p = \pm 1$ was already done e.g. in [3].

Theorem 38 follows then from Lemma 57 and (7.2) where $Q_{b,k}$ is defined by its projections

$$\pi_j Q_{b,k} := \frac{1-q^{-1}}{(1-x_{-b})^{\chi(p^j)}}\ Q_k(q^{-\mathrm{sn}(b)}, x_{-b}, j).$$

We split the proof of (7.3) into two parts. In the first part we will show that there exists an element $Q_k(q, x_b, j)$ such that Equality (7.3) holds. In the second part we show that $Q_k(q, x_b, j)$ lies in $\mathcal{S}_{p,j}$.

7.2.1 Part 1: Existence of $Q_k(q, x_b, j)$, b odd case

Assume $b = \pm p^l$ with $p \neq 2$. We split the proof into several lemmas.

Lemma 59. *For $x_{b;j} := \pi_j(x_b)$ and $c = (b, p^j)$,*

$$\mathcal{L}_{b;j}\left(\prod_{i=0}^{k}(z+z^{-1}-q^i-q^{-i})\right) = 2\,(-1)^{k+1}\begin{bmatrix} 2k+1 \\ k \end{bmatrix} S_{b;j}(k,q)$$

where

$$S_{b;j}(k,q) := 1 + \sum_{n=1}^{\infty} \frac{q^{(k+1)cn}(q^{-k-1};q)_{cn}}{(q^{k+2};q)_{cn}}(1+q^{cn})x_{b;j}^{n^2}. \tag{7.4}$$

Observe that for $n > \frac{k+1}{c}$, the term $(q^{-k-1};q)_{cn}$ is zero and therefore the sum in (7.4) is finite.

Proof. Since $\mathcal{L}_b$ is invariant under $z \to z^{-1}$, one has

$$\mathcal{L}_b\left(\prod_{i=0}^{k}(z+z^{-1}-q^i-q^{-i})\right) = -2\mathcal{L}_b(z^{-k}(zq^{-k};q)_{2k+1}),$$

and the q–binomial theorem (e.g. see [10], II.3) gives

$$z^{-k}(zq^{-k};q)_{2k+1} = (-1)^k \sum_{i=-k}^{k+1}(-1)^i \begin{bmatrix} 2k+1 \\ k+i \end{bmatrix} z^i. \tag{7.5}$$

Notice that $\mathcal{L}_{b;j}(z^a) \neq 0$ if and only if $c \mid a$. Applying $\mathcal{L}_{b;j}$ to the RHS of (7.5), only the terms with $c \mid i$ survive and therefore

$$\mathcal{L}_{b;j}\left(z^{-k}(zq^{-k};q)_{2k+1}\right) = (-1)^k \sum_{n=-\lfloor k/c\rfloor}^{\lfloor (k+1)/c\rfloor}(-1)^{cn} \begin{bmatrix} 2k+1 \\ k+cn \end{bmatrix} z_{b,cn;j}.$$

Separating the case $n=0$ and combining positive and negative n, this is equal to

$$(-1)^k \begin{bmatrix} 2k+1 \\ k \end{bmatrix} + (-1)^k \sum_{n=1}^{\lfloor (k+1)/c\rfloor}(-1)^{cn}\left(\begin{bmatrix} 2k+1 \\ k+cn \end{bmatrix} + \begin{bmatrix} 2k+1 \\ k-cn \end{bmatrix}\right) z_{b,cn;j}$$

where we use the convention that $\begin{bmatrix} x \\ -1 \end{bmatrix}$ is zero for positive x. Further,

$$\begin{bmatrix} 2k+1 \\ k+cn \end{bmatrix} + \begin{bmatrix} 2k+1 \\ k-cn \end{bmatrix} = \frac{\{k+1\}}{\{2k+2\}} \begin{bmatrix} 2k+2 \\ k+cn+1 \end{bmatrix} (q^{cn/2}+q^{-cn/2})$$

and

$$\frac{\{k+1\}}{\{2k+2\}} \begin{bmatrix} 2k+2 \\ k+cn+1 \end{bmatrix} \begin{bmatrix} 2k+1 \\ k \end{bmatrix}^{-1} = (-1)^{cn} q^{(k+1)cn+\frac{cn}{2}} \frac{(q^{-k-1};q)_{cn}}{(q^{k+2};q)_{cn}}.$$

Using $z_{b,cn;j} = (z_{b,c;j})^{n^2} = x_{b;j}^{n^2}$, we get the result. ☐

To define $Q_k(q, x_b, j)$, we will need Andrew's identity (3.43) of [1]:

$$\sum_{n\geq 0}(-1)^n \alpha_n t^{-\frac{n(n-1)}{2}+sn+Nn} \frac{(t^{-N})_n}{(t^{N+1})_n} \prod_{i=1}^{s} \frac{(b_i)_n (c_i)_n}{b_i^n c_i^n (\frac{t}{b_i})_n (\frac{t}{c_i})_n} =$$

$$\frac{(t)_N (\frac{q}{b_s c_s})_N}{(\frac{t}{b_s})_N (\frac{t}{c_s})_N} \sum_{n_s \geq \cdots \geq n_2 \geq n_1 \geq 0} \beta_{n_1} \frac{t^{n_s}(t^{-N})_{n_s}(b_s)_{n_s}(c_s)_{n_s}}{(t^{-N}b_s c_s)_{n_s}} \prod_{i=1}^{s-1} \frac{t^{n_i}}{b_i^{n_i} c_i^{n_i}} \frac{(b_i)_{n_i}(c_i)_{n_i}}{(\frac{t}{b_i})_{n_{i+1}}(\frac{t}{c_i})_{n_{i+1}}} \frac{(\frac{t}{b_i c_i})_{n_{i+1}-n_i}}{(t)_{n_{i+1}-n_i}}.$$

Here and in what follows we use the notation $(a)_n := (a;t)_n$. The special Bailey pair (α_n, β_n) is chosen as follows

$$\begin{aligned} \alpha_0 &= 1, & \alpha_n &= (-1)^n t^{\frac{n(n-1)}{2}}(1+t^n) \\ \beta_0 &= 1, & \beta_n &= 0 \quad \text{for } n \geq 1. \end{aligned}$$

Lemma 60. *$S_{b;j}(k,q)$ is equal to the LHS of Andrew's identity with the parameters fixed below.*

Proof. Since

$$S_{b;j}(k,q) = S_{-b;j}(k,q^{-1}),$$

it is enough to look at the case $b>0$. Define $b' := \frac{b}{c}$ and let ω be a b'th primitive root of unity. For simplicity, put $N := k+1$ and $t := x_{b;j}$. Using the following identities

$$\begin{aligned}(q^y;q)_{cn} &= \prod_{l=0}^{c-1}(q^{y+l};q^c)_n\ ,\\ (q^{yc};q^c)_n &= \prod_{i=0}^{b'-1}(\omega^i t^y;t)_n\ ,\end{aligned}$$

where the later is true due to $t^{b'} = x_{b;j}^{b'} = q^c$ for all j, and choosing a cth root of t denoted by $t^{\frac{1}{c}}$, we can see that

$$S_{b;j}(k,q) = 1 + \sum_{n=1}^{\infty}\prod_{i=0}^{b'-1}\prod_{l=0}^{c-1}\frac{(\omega^i t^{\frac{-N+l}{c}})_n}{(\omega^i t^{\frac{N+1+l}{c}})_n}(1+t^{b'n})t^{n^2+b'Nn}.$$

Now we choose the parameters for Andrew's identity as follows. We put $a := \frac{c-1}{2}$, $d := \frac{b'-1}{2}$ and $m := \lfloor \frac{N}{c} \rfloor$. For $l \in \{1,\dots,c-1\}$, there exist unique $u_l, v_l \in \{0,\dots,c-1\}$, such that $u_l \equiv N+l \pmod{c}$ and $v_l \equiv N-l \pmod{c}$. Note that $v_l = u_{c-l}$. We define $U_l := \frac{-N+u_l}{c}$ and $V_l := \frac{-N+v_l}{c}$. Then $U_l, V_l \in \frac{1}{c}\mathbb{Z}$ but $U_l+V_l \in \mathbb{Z}$. We define

$$\begin{array}{rclrcll} b_l &:=& t^{U_l}, & c_l &:=& t^{V_l} & \text{for } l=1,\dots,a,\\ b_{a+i} &:=& \omega^i t^{-m}, & c_{a+i} &:=& \omega^{-i}t^{-m} & \text{for } i=1,\dots,d,\\ b_{a+ld+i} &:=& \omega^i t^{U_l}, & c_{a+ld+i} &:=& \omega^{-i}t^{V_l} & \text{for } i=1,\dots,d \text{ and } l=1,\dots,c-1,\\ b_{g+i} &:=& -\omega^i t, & c_{g+i} &:=& -\omega^{-i}t & \text{for } i=1,\dots,d,\\ b_{s-1} &:=& t^{-m}, & c_{s-1} &:=& t^{N+1}, &\\ b_s &\to& \infty, & c_s &\to& \infty, & \end{array}$$

where $g = a+cd$ and $s = (c+1)\frac{b'}{2}+1$.

We now calculate the LHS of Andrew's identity. Using the notation

$$(\omega^{\pm 1}t^x)_n = (\omega t^x)_n(\omega^{-1}t^x)_n$$

and the identities

$$\lim_{c\to\infty}\frac{(c)_n}{c^n} = (-1)^n t^{\frac{n(n-1)}{2}} \quad \text{and} \quad \lim_{c\to\infty}\left(\frac{t}{c}\right)_n = 1,$$

we get

$$\begin{aligned} LHS \;=\; 1 + \sum_{n\geq 1} t^{n(n-1+s+N-y)}\,(1+t^n)\frac{(t^{-N})_n}{(t^{N+1})_n}\cdot\prod_{l=1}^{a}\frac{(t^{U_l})_n(t^{V_l})_n}{(t^{1-U_l})_n(t^{1-V_l})_n}\cdot\prod_{i=1}^{d}\frac{(\omega^{\pm i}t^{-m})_n}{(\omega^{\pm i}t^{1+m})_n}\\ \cdot\prod_{i=1}^{d}\prod_{l=1}^{c-1}\frac{(\omega^i t^{U_l})_n(\omega^{-i}t^{V_l})_n}{(\omega^{-i}t^{1-U_l})_n(\omega^i t^{1-V_l})_n}\cdot\prod_{i=1}^{d}\frac{(-\omega^{\pm i}t)_n}{(-\omega^{\pm i})_n}\cdot\frac{(t^{-m})_n(t^{N+1})_n}{(t^{1+m})_n(t^{-N})_n}\end{aligned}$$

where

$$y := \sum_{l=1}^{a}(U_l+V_l) + \sum_{i=1}^{d}\sum_{l=1}^{c-1}(U_l+V_l) - m(2d+1) + 2d+1+N.$$

Since $\sum_{l=1}^{c-1}(U_l+V_l) = 2\sum_{l=1}^{a}(U_l+V_l) = 2(-N+m+\frac{c-1}{2})$ and $2d+1=b'$, we have

$$n-1+s+N-y = n+Nb'.$$

Further,

$$\prod_{i=1}^{d}\frac{(-\omega^{\pm i}t)_n}{(-\omega^{\pm i})_n} = \prod_{i=1}^{b'-1}\frac{1+\omega^i t^n}{1+\omega^i} = \frac{1+t^{b'n}}{1+t^n}$$

and

$$\prod_{l=1}^{a}\frac{(t^{U_l})_n(t^{V_l})_n}{(t^{1-U_l})_n(t^{1-V_l})_n}\cdot\prod_{i=1}^{d}\frac{(\omega^{\pm i}t^{-m})_n}{(\omega^{\pm i}t^{1+m})_n}\cdot\prod_{i=1}^{d}\prod_{l=1}^{c-1}\frac{(\omega^i t^{U_l})_n(\omega^{-i}t^{V_l})_n}{(\omega^{-i}t^{1-U_l})_n(\omega^i t^{1-V_l})_n}\cdot\frac{(t^{-m})_n}{(t^{1+m})_n}$$
$$= \prod_{i=0}^{b'-1}\prod_{l=0}^{c-1}\frac{(\omega^i t^{\frac{-N+l}{c}})_n}{(\omega^i t^{\frac{N+1+l}{c}})_n}.$$

Taking all the results together, we see that the LHS is equal to $S_{b;j}(k,q)$.

□

Let us now calculate the RHS of Andrew's identity with parameters chosen as above. For simplicity, we put $\delta_j := n_{j+1}-n_j$. Then the RHS is given by

$$\begin{aligned} RHS &= (t)_N \sum_{n_s\geq\cdots\geq n_2\geq n_1=0}\frac{t^x\cdot(t^{-N})_{n_s}(b_s)_{n_s}(c_s)_{n_s}}{\prod_{i=1}^{s-1}(t)_{\delta_i}(t^{-N}b_sc_s)_{n_s}}\cdot\frac{(t^{-m})_{n_{s-1}}(t^{N+1})_{n_{s-1}}(t^{m-N})_{\delta_{s-1}}}{(t^{m+1})_{n_s}(t^{-N})_{n_s}} \\ &\cdot\prod_{l=1}^{a}\frac{(t^{U_l})_{n_l}(t^{V_l})_{n_l}(t^{1-U_l-V_l})_{\delta_l}}{(t^{1-U_l})_{n_{l+1}}(t^{1-V_l})_{n_{l+1}}}\cdot\prod_{i=1}^{d}\frac{(\omega^{\pm i}t^{-m})_{n_{a+i}}(t^{2m+1})_{\delta_{a+i}}}{(\omega^{\pm i}t^{m+1})_{n_{a+i+1}}}\frac{(-\omega^{\pm i}t)_{n_{g+i}}(t^{-1})_{\delta_{g+i}}}{(-\omega^{\pm i})_{n_{g+i+1}}} \\ &\cdot\prod_{i=1}^{d}\prod_{l=1}^{c-1}\frac{(\omega^i t^{U_l})_{n_{a+ld+i}}(\omega^{-i}t^{V_l})_{n_{a+ld+i}}(t^{1-U_l-V_l})_{\delta_{a+ld+i}}}{(\omega^{-i}t^{1-U_l})_{n_{a+ld+i+1}}(\omega^i t^{1-V_l})_{n_{a+ld+i+1}}}\end{aligned}$$

where

$$\begin{aligned} x &= \sum_{l=1}^{a}(1-U_l-V_l)\,n_l + \sum_{i=1}^{d}(2m+1)\,n_{a+i} \\ &\quad + \sum_{i=1}^{d}\sum_{l=1}^{c-1}(1-U_l-V_l)\,n_{a+ld+i} - \sum_{i=1}^{d}n_{g+i} + (m-N)\,n_{s-1} + n_s.\end{aligned}$$

For $c=1$ or $d=0$, we use the convention that empty products are equal to 1 and empty sums are equal to zero.

Let us now have a closer look at the RHS. Notice that

$$\lim_{b_s,c_s\to\infty}\frac{(b_s)_{n_s}(c_s)_{n_s}}{(t^{-N}b_sc_s)_{n_s}} = (-1)^{n_s}t^{\frac{n_s(n_s-1)}{2}}t^{Nn_s}.$$

The term $(t^{-1})_{\delta_{g+i}}$ is zero unless $\delta_{g+i} \in \{0,1\}$. Therefore, we get

$$\prod_{i=1}^{d} \frac{(-\omega^{\pm i}t)_{n_{g+i}}}{(-\omega^{\pm i})_{n_{g+i+1}}} = \prod_{i=1}^{d} (1+\omega^{\pm i} t^{n_{g+i}})^{1-\delta_{g+i}}.$$

Due to the term $(t^{-m})_{n_s}$, we have $n_s \le m$ and therefore $n_i \le m$ for all i. Multiplying the numerator and denominator of each term of the RHS by

$$\prod_{l=1}^{a} (t^{1-U_l+n_{l+1}})_{m-n_{l+1}} (t^{1-V_l+n_{l+1}})_{m-n_{l+1}} \prod_{i=1}^{d} (\omega^{\pm i} t^{m+1+n_{a+i+1}})_{m-n_{a+i+1}}$$
$$\cdot \prod_{i=1}^{d} \prod_{l=1}^{c-1} (\omega^{-i} t^{1-U_l+n_{a+ld+i+1}})_{m-n_{a+ld+i+1}} (\omega^{i} t^{1-V_l+n_{a+ld+i+1}})_{m-n_{a+ld+i+1}}$$

gives in the denominator $\prod_{i=0}^{b'-1} \prod_{l=1}^{c-1} (\omega^i t^{1-U_l})_m \cdot \prod_{i=1}^{b'-1} (\omega^i t^{m+1})_m$. This is equal to

$$\prod_{l=1}^{c-1} (t^{b'(1-U_l)}; t^{b'})_m \cdot \frac{(t^{b'(m+1)}; t^{b'})_m}{(t^{m+1}; t)_m} = \frac{(q^{N+1}; q)_{cm}}{(t^{m+1}; t)_m}.$$

Further,

$$(t)_N (t^{N+1})_{n_{s-1}} = (t)_{N+n_{s-1}} = (t)_m (t^{m+1})_{N-m+n_{s-1}}.$$

The term $(t^{-N+m})_{\delta_{s-1}}$ is zero unless $\delta_{s-1} \le N-m$ and therefore

$$\frac{(t^{m+1})_{N-m+n_{s-1}}}{(t^{m+1})_{n_s}} = (t^{m+1+n_s})_{N-m-\delta_{s-1}}.$$

Using the above calculations, we get

$$RHS = \frac{(t;t)_{2m}}{(q^{N+1}; q)_{cm}} \cdot T_k(q,t) \tag{7.6}$$

where

$$\begin{aligned} T_k(q,t) \quad := \quad & \sum_{n_s \ge \cdots \ge n_2 \ge n_1 = 0} (-1)^{n_s} t^{x'} \cdot (t^{-m})_{n_{s-1}} \cdot (t^{m+1+n_s})_{N-m-\delta_{s-1}} \cdot \frac{(t^{-N+m})_{\delta_{s-1}}}{\prod_{i=1}^{s-1} (t)_{\delta_i}} \\ & \cdot \prod_{l=1}^{a} (t^{1-U_l-V_l})_{\delta_l} \cdot \prod_{i=1}^{d} (t^{2m+1})_{\delta_{a+i}} (t^{-1})_{\delta_{g+i}} \cdot \prod_{i=1}^{d} \prod_{l=1}^{c-1} (t^{1-U_l-V_l})_{\delta_{a+ld+i}} \\ & \cdot \prod_{l=1}^{a} (t^{U_l})_{n_l} (t^{V_l})_{n_l} (t^{1-U_l+n_{l+1}})_{m-n_{l+1}} (t^{1-V_l+n_{l+1}})_{m-n_{l+1}} \cdot \prod_{i=1}^{d} (1+\omega^{\pm i} t^{n_{g+i}})^{1-\delta_{g+i}} \\ & \cdot \prod_{i=1}^{d} (\omega^{\pm i} t^{-m})_{n_{a+i}} (\omega^{\pm i} t^{m+1+n_{a+i+1}})_{m-n_{a+i+1}} \cdot \prod_{i=1}^{d} \prod_{l=1}^{c-1} (\omega^i t^{U_l})_{n_{a+ld+i}} (\omega^{-i} t^{V_l})_{n_{a+ld+i}} \\ & \cdot \prod_{i=1}^{d} \prod_{l=1}^{c-1} (\omega^{-i} t^{1-U_l+n_{a+ld+i+1}})_{m-n_{a+ld+i+1}} (\omega^{i} t^{1-V_l+n_{a+ld+i+1}})_{m-n_{a+ld+i+1}} \end{aligned}$$

and $x' := x + \frac{n_s(n_s-1)}{2} + Nn_s$.

We define the element $Q_k(q, x_b, j)$ by

$$Q_k(q, x_b, j) := \left((-1)^{k+1} q^{-\frac{k(k+1)}{2}}\right)^{\frac{1+\mathrm{sn}(b)}{2}} \left(q^{(k+1)^2}\right)^{\frac{1-\mathrm{sn}(b)}{2}} \frac{(x_{b;j}; x_{b;j})_{2m}}{(q;q)_{N+cm}} T_k(q, x_{b;j}).$$

By Lemmas 59 and 60, Equation (7.6) and the following Lemma 61, we see that this element satisfies Equation (7.3).

Lemma 61. *The following formula holds.*

$$(-1)^{k+1} \left[\begin{array}{c} 2k+1 \\ k \end{array} \right] (q^{k+1}; q)_{k+1}^{-1} = (-1)^{k+1} \frac{q^{-k(k+1)/2}}{(q;q)_{k+1}} = \frac{q^{-(k+1)^2}}{(q^{-1}; q^{-1})_{k+1}}$$

Proof. This is an easy calculation using

$$(q^{k+1}; q)_{k+1} = (-1)^{k+1} q^{(3k^2+5k+2)/4} \frac{\{2k+1\}!}{\{k\}!}.$$

□

7.2.2 Part 1: Existence of $Q_k(q, x_b, j)$, b even case.

Let $b = \pm 2^l$. We have to prove Equality (7.3) only for $j = 0$, i.e. we have to show

$$\frac{1}{(q^{k+1}; q)_{k+1}} \mathcal{L}_{b;0} \left(\prod_{i=0}^{k} (z + z^{-1} - q^i - q^{-i}) \right) = 2\, Q_k(q^{\mathrm{sn}(b)}, x_b, 0).$$

The calculation works similarly to the odd case. Note that we have $c = 1$ here. This case was already done in [5] and [22]. Since their approaches are slightly different and for the sake of completeness, we will give the parameters for Andrew's identity and the formula for $Q_k(q, x_b, 0)$ nevertheless.

We put $t := x_{b;0}$, $d := \frac{b}{2} - 1$, ω a bth root of unity and choose a primitive square root ν of ω. Define the parameters of Andrew's identity by

$$\begin{array}{rclrcll} b_i & := & \omega^i t^{-N}, & c_i & := & \omega^{-i} t^{-N} & \text{for } i = 1, \dots, d, \\ b_{d+i} & := & -\nu^{2i-1} t, & c_{d+i} & := & -\nu^{-(2i-1)} t & \text{for } i = 1, \dots, d+1, \\ b_b & := & -t^{-N}, & c_b & := & -t^0 = -1, & \\ b_{s-1} & := & t^{-N}, & c_{s-1} & := & t^{N+1}, & \\ b_s & \to & \infty, & c_s & \to & \infty, & \end{array}$$

where $s = b + 2$. Now we can define the element

$$Q_k(q, x_b, 0) := \left((-1)^{k+1} q^{-\frac{k(k+1)}{2}}\right)^{\frac{1+\mathrm{sn}(b)}{2}} \left(q^{(k+1)^2}\right)^{\frac{1-\mathrm{sn}(b)}{2}} \frac{(x_{b;0}; x_{b;0})_{2N}}{(q;q)_{2N}} \frac{1}{(-x_{b;0}; x_{b;0})_N} T_k(q, x_{b;0})$$

where

$$
\begin{aligned}
T_k(q,t) \quad := \quad & \sum_{n_{s-1}\geq\cdots\geq n_1=0} (-1)^{n_{s-1}} t^{x''} \cdot \frac{\prod_{i=1}^{d}(t^{2N+1})_{\delta_i} \cdot \prod_{i=1}^{d+1}(t^{-1})_{\delta_{d+i}} \cdot (t^{N+1})_{\delta_b}}{\prod_{i=1}^{s-2}(t)_{\delta_i}} \\
& \cdot (t^{-N})_{n_{s-1}} \cdot (-t^{N+1+n_{s-1}})_{N-n_{s-1}} \cdot (-t^{-N})_{n_b} \cdot (-t)_{n_b-1} \cdot (-t^{n_{s-1}+1})_{N-n_{s-1}} \\
& \cdot \prod_{i=1}^{d} (\omega^{\pm i} t^{-N})_{n_i} (\omega^{\pm i} t^{N+1+n_{i+1}})_{N-n_{i+1}} \cdot \prod_{i=1}^{d+1} (1+\nu^{\pm(2i-1)} t^{n_{d+i}})^{1-\delta_{d+i}}
\end{aligned}
$$

and $x'' := \sum_{i=1}^{d}(2N+1)n_i - \sum_{i=1}^{d+1} n_{d+i} + \frac{n_{s-1}(n_{s-1}-1)}{2} + (N+1)(n_b+n_{s-1})$. We use the notation $(a;b)_{-1} := \frac{1}{1-ab^{-1}}$.

7.2.3 Part 2: $Q_k(q, x_b, j) \in \mathcal{S}_{p,j}$.

We have to show that $Q_k(q, x_b, j) \in \mathcal{S}_{p,j}$, where $j \in \mathbb{N} \cup \{0\}$ if p is odd, and $j = 0$ for $p = 2$. This follows from the next two lemmas.

Lemma 62. *For $t = x_{b;j}$,*

$$T_k(q,t) \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}].$$

Proof. Let us first look at the case b odd and positive. Since for $a \neq 0$, $(t^a)_n$ is always divisible by $(t)_n$, it is easy to see that the denominator of each term of $T_k(q,t)$ divides its numerator. Therefore we proved that $T_k(q,t) \in \mathbb{Z}[t^{\pm 1/c}, \omega]$. Since

$$S_{b;j}(k,q) = \frac{(t;t)_{2m}}{(q^{N+1};q)_{cm}} \cdot T_k(q,t), \tag{7.7}$$

there are $f_0, g_0 \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ such that $T_k(q,t) = \frac{f_0}{g_0}$. This implies that $T_k(q,t) \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ since f_0 and g_0 do not depend on ω and the cth root of t.

The proofs for the even and the negative case work analogously. □

Lemma 63. *For $t = x_{b;j}$,*

$$\frac{(t;t)_{2m}}{(q;q)_{N+cm}} \frac{1}{((-t;t)_N)^\lambda} \in \mathcal{S}_{p,j}$$

where $\lambda = 1$ and $j = 0$ if $p = 2$, and $\lambda = 0$ and $j \in \mathbb{N} \cup \{0\}$ otherwise.

Proof. Notice that

$$(q;q)_{N+cm} = \widetilde{(q;q)}_{N+cm} (q^c;q^c)_{2m}$$

where we use the notation

$$\widetilde{(q^a;q)}_n := \prod_{\substack{j=0 \\ c\nmid(a+j)}}^{n-1} (1-q^{a+j}).$$

We have to show that

$$\frac{(q^c;q^c)_{2m}}{(t;t)_{2m}} \cdot \widetilde{(q;q)}_{N+cm} \cdot ((-t;t)_N)^\lambda$$

is invertible in $\mathbb{Z}[1/p][q]$ modulo any ideal $(f) = (\prod_n \Phi_n^{k_n}(q))$ where n runs through a subset of $p^j\mathbb{N}_p$. Recall that in a commutative ring A, an element a is invertible in $A/(d)$ if and only if $(a)+(d)=(1)$. If $(a)+(d)=(1)$ and $(a)+(e)=(1)$, we get $(a)+(de)=(1)$. Hence, it is enough to consider $f=\Phi_{p^jn}(q)$ with $(n,p)=1$. For any $X \in \mathbb{N}$, we have

$$\widetilde{(q;q)_X} = \prod_{\substack{i=1 \\ c\nmid i}}^{X} \prod_{d|i} \Phi_d(q), \tag{7.8}$$

$$(-t;t)_X = \frac{(t^2;t^2)_X}{(t;t)_X} = \prod_{i=1}^{X}\prod_{d|i} \Phi_{2d}(t), \tag{7.9}$$

$$\frac{(q^c;q^c)_X}{(t;t)_X} = \frac{(t^{b'};t^{b'})_X}{(t;t)_X} = \frac{\prod_{i=1}^X \prod_{d|ib'} \Phi_d(t)}{\prod_{i=1}^X \prod_{d|i} \Phi_d(t),} \tag{7.10}$$

for $b'=b/c$. Recall that $(\Phi_r(q),\Phi_a(q))=(1)$ in $\mathbb{Z}[1/p][q]$ if either r/a is not a power of a prime or a power of p. For $r=p^jn$ odd and a such that $c\nmid a$, one of the conditions is always satisfied. Hence (7.8) is invertible in $\mathcal{S}_{p,j}$. If $b=c$ or $b'=1$, (7.9) and (7.10) do not contribute. For $c<b$, notice that q is a cnth primitive root of unity in $\mathbb{Z}[1/p][q]/(\Phi_{cn}(q)) = \mathbb{Z}[1/p][e_{cn}]$. Therefore $t^{b'}=q^c$ is an nth primitive root of unity. Since $(n,b')=1$, t must be a primitive nth root of unity in $\mathbb{Z}[1/p][e_{cn}]$ too, and hence $\Phi_n(t)=0$ in that ring. Since for j with $(j,p)>1$, $(\Phi_j(t),\Phi_n(t))=(1)$ in $\mathbb{Z}[1/p][t]$, $\Phi_j(t)$ is invertible in $\mathbb{Z}[1/p][e_{cn}]$, and therefore (7.9) and (7.10) are invertible too. □

A Proof of Theorem 5

The appendix is devoted to the proof of Theorem 5, a generalization of the deep integrality result of Habiro, namely Theorem 8.2 of [12]. The existence of this generalization and some ideas of the proof were kindly communicated to us by Habiro.

A.1 Reduction to a result on values of the colored Jones polynomial

We will use the notation of Section 1.2.

Let V_n be the unique $(n+1)$–dimensional irreducible U_h–module. In [12], Habiro defined a new basis $\tilde{P}'_k$, $k = 0, 1, 2, \ldots$, for the Grothendieck ring of finite–dimensional $U_h(sl_2)$–modules with

$$\tilde{P}'_k := \frac{v^{\frac{1}{2}k(1-k)}}{\{k\}!} \prod_{i=0}^{k-1} (V_1 - v^{2i+1} - v^{-2i-1}).$$

Put $\tilde{P}'_{\mathbf{k}} = \{\tilde{P}'_{k_1}, \ldots, \tilde{P}'_{k_m}\}$. It follows from Lemma 6.1 of [12] that we will have identity (1.2) of Theorem 5 if we substitute

$$C_{L\sqcup L'}(\mathbf{k}, \mathbf{j}) = J_{L\sqcup L'}\,(\tilde{P}'_{\mathbf{k}}, \mathbf{j}) \prod_i (-1)^{k_i} q^{k_i^2+k_i+1}\,.$$

Hence, to prove Theorem 5, it is enough to show the following.

Theorem A.1.1. *Suppose $L \sqcup L'$ is a colored framed link in S^3 such that L has 0 linking matrix and L' has odd colors. Then for $k = \max\{k_1, \ldots, k_m\}$, we have*

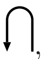
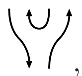
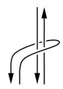
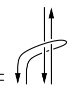
$$J_{L\sqcup L'}(\tilde{P}'_{\mathbf{k}}, \mathbf{j}) \in \frac{(q^{k+1}; q)_{k+1}}{1-q}\, \mathbb{Z}[q^{\pm 1}].$$

In the case $L' = \emptyset$, this statement was proven in [12, Theorem 8.2]. Since our proof is a modification of the original one, we first sketch Habiro's original proof for the reader's convenience.

A.2 Sketch of the proof of Habiro's integrality theorem

Corollary 9.13 in [13] states the following.

Proposition A.2.1. (Habiro) *If the linking matrix of a bottom tangle T is zero then T can be presented as $T = WB^{\otimes k}$, where $k \geq 0$ and $W \in \mathsf{B}(3k, n)$ is obtained by horizontal and vertical pasting of finitely many copies of 1_{b}, $\psi_{\mathsf{b},\mathsf{b}}$, $\psi^{-1}_{\mathsf{b},\mathsf{b}}$, and*

$\eta_{\mathsf{b}} =$ , $\mu_{\mathsf{b}} =$ , $\gamma_{+} =$ , $\gamma_{-} =$ .

Let $K = v^H = e^{\frac{hH}{2}}$. Habiro introduced the integral version $\mathcal{U}_q$, which is the $\mathbb{Z}[q, q^{-1}]$–subalgebra of U_h freely spanned by $\tilde{F}^{(i)} K^j e^k$ for $i, k \geq 0, j \in \mathbb{Z}$, where

$$\tilde{F}^{(n)} = \frac{F^n K^n}{v^{\frac{n(n-1)}{2}} [n]!} \qquad \text{and} \qquad e = (v - v^{-1})E.$$

There is a $\mathbb{Z}/2\mathbb{Z}$–grading, $\mathcal{U}_q = \mathcal{U}_q^0 \oplus \mathcal{U}_q^1$, where $\mathcal{U}_q^0$ (respectively $\mathcal{U}_q^1$) is spanned by $\tilde{F}^{(i)} K^{2j} e^k$ (respectively $\tilde{F}^{(i)} K^{2j+1} e^k$). We call this the ε–grading and $\mathcal{U}_q^0$ (respectively $\mathcal{U}_q^1$) the even (respectively odd) part.

The two–sided ideal $\mathcal{F}_p$ in $\mathcal{U}_q$ generated by e^p induces a filtration on $(\mathcal{U}_q)^{\otimes n}$, $n \geq 1$, by

$$\mathcal{F}_p((\mathcal{U}_q)^{\otimes n}) = \sum_{i=1}^{n} (\mathcal{U}_q)^{\otimes i-1} \otimes \mathcal{F}_p(\mathcal{U}_q) \otimes (\mathcal{U}_q)^{\otimes n-i} \subset (\mathcal{U}_q)^{\otimes n}.$$

Let $(\tilde{\mathcal{U}}_q)^{\tilde{\otimes} n}$ be the image of the homomorphism

$$\varprojlim_{p \geq 0} \frac{(\mathcal{U}_q)^{\otimes n}}{\mathcal{F}_p((\mathcal{U}_q)^{\otimes n})} \to U_h^{\hat{\otimes} n}$$

where $\hat{\otimes}$ is the h–adically completed tensor product. By using $\mathcal{F}_p^{\varepsilon}(\mathcal{U}_q^{\varepsilon}) := \mathcal{F}_p(\mathcal{U}_q) \cap \mathcal{U}_q^{\varepsilon}$ one defines $(\tilde{\mathcal{U}}_q^{\varepsilon})^{\tilde{\otimes} n}$ for $\varepsilon \in \{0, 1\}$ in a similar fashion.

By definition (Section 4.2 of [12]), the universal sl_2 invariant J_T of an n–component bottom tangle T is an element of $U_h^{\hat{\otimes} n}$. Theorem 4.1 in [12] states that, in fact, for any bottom tangle T with zero linking matrix, J_T is even, i.e.

$$J_T \in (\tilde{\mathcal{U}}_q^0)^{\tilde{\otimes} n}. \tag{A.2.1}$$

Further, using the fact that J_K of a 0–framed bottom knot K (i.e. a 1–component bottom tangle) belongs to the center of $\tilde{\mathcal{U}}_q^0$, Habiro showed that

$$J_K = \sum_{n \geq 0} (-1)^n q^{n(n+1)} \frac{(1-q)}{(q^{n+1}; q)_{n+1}} J_K(\tilde{P}'_n) \, \sigma_n$$

where

$$\sigma_n = \prod_{i=0}^{n} (C^2 - (q^i + 2 + q^{-i})) \quad \text{with} \quad C = (v - v^{-1}) \tilde{F}^{(1)} K^{-1} e + vK + v^{-1} K^{-1}$$

is the quantum Casimir operator. The σ_n provide a basis for the even part of the center. From this, Habiro deduced that $J_K(\tilde{P}'_n) \in \frac{(q^{n+1};q)_{n+1}}{(1-q)}\mathbb{Z}[q, q^{-1}]$.

The case of n–component bottom tangles reduces to the 1–component case by partial trace, using certain integrality of traces of even element (Lemma 8.5 of [12]) and the fact that J_T is invariant under the adjoint action.

The proof of (A.2.1) uses Proposition A.2.1, which allows us to build any bottom tangle T with zero linking matrix from simple parts, i.e. $T = W(B^{\otimes k})$.

On the other hand, the construction of the universal invariant J_T extends to the braided functor $J : \mathsf{B} \to \mathrm{Mod}_{U_h}$ from B to the category of U_h–modules. This means that $J_{W(B^{\otimes k})} = J_W(J_{B^{\otimes k}})$. Therefore, in order to show (A.2.1), we need to check that $J_B \in (\tilde{\mathcal{U}}_q^0)^{\tilde{\otimes}3}$ and then verify that J_W maps the even part to itself. The first check can be done by a direct computation [12, Section 4.3]. The last verification is the content of Corollary 3.2 in [12].

A.3 Strategy of the Proof of Theorem A.1.1

A.3.1 Generalization of Equation (A.2.1)

To prove Theorem A.1.1, we need a generalization of Equation (A.2.1) or Theorem 4.1 in [12] to tangles with closed components. To state the result, we first introduce two gradings.

Suppose T is an n–component bottom tangle in a cube, homeomorphic to the 3–ball D^3. Let $\tilde{\mathcal{S}}(D^3 \setminus T)$ be the $\mathbb{Z}[q^{\pm 1/4}]$–module freely generated by the isotopy classes of framed unoriented colored links in $D^3 \setminus T$, including the empty link. For such a link $L \subset D^3 \setminus T$ with m–components colored by $n_1, \dots, n_m$, we define our new gradings as follows. First provide the components of L with arbitrary orientations. Let l_{ij} be the linking number between the ith component of T and the jth component of L, and p_{ij} be the linking number between the ith and the jth components of L. For $X = T \sqcup L$, we put

$$\mathrm{gr}_\varepsilon(X) := (\varepsilon_1, \dots, \varepsilon_n) \in (\mathbb{Z}/2\mathbb{Z})^n \quad \text{where} \quad \varepsilon_i := \sum_j l_{ij} n'_j \pmod 2, \quad \text{and} \tag{A.3.1}$$

$$\mathrm{gr}_q(L) := \sum_{1 \le i,j \le m} p_{ij} n'_i n'_j + 2 \sum_{1 \le j \le m} (p_{jj} + 1) n'_j \pmod 4 \quad \text{where} \quad n'_i := n_i - 1.$$

It is easy to see that the definitions do not depend on the orientation of L.

The meaning of $\mathrm{gr}_q(L)$ is the following: The colored Jones polynomial of L, a priori a Laurent polynomial of $q^{1/4}$, is actually a Laurent polynomial of q after dividing by $q^{\mathrm{gr}_q(L)/4}$; see [24] for this result and its generalization to other Lie algebras.

We further extend both gradings to $\tilde{\mathcal{S}}(D^3 \setminus T)$ by

$$\mathrm{gr}_\varepsilon(q^{1/4}) = 0, \quad \mathrm{gr}_q(q^{1/4}) = 1 \pmod 4.$$

Recall that the universal invariant J_X can also be defined when X is the union of a bottom tangle and a colored link (see [13, Section 7.3]). In [13], it is proved that J_X is adjoint invariant. The generalization of Theorem 4.1 of [12] is the following.

Theorem A.3.1. *Suppose $X = T \sqcup L$ where T is a n–component bottom tangle with zero linking matrix and L is a framed unoriented colored link with $\mathrm{gr}_\varepsilon(X) = (\varepsilon_1, \dots, \varepsilon_n)$. Then*

$$J_X \in q^{\mathrm{gr}_q(L)/4} \left(\tilde{\mathcal{U}}_q^{\varepsilon_1} \tilde{\otimes} \dots \tilde{\otimes} \tilde{\mathcal{U}}_q^{\varepsilon_n} \right).$$

Corollary A.3.2. *Suppose L is colored by a tuple of odd numbers, then*

$$J_X \in (\tilde{\mathcal{U}}_q^0)^{\tilde{\otimes} n}.$$

Since J_X is invariant under the adjoint action, Theorem A.1.1 follows from Corollary A.3.2 by repeating Habiro's arguments. Hence it remains to prove Theorem A.3.1. In the proof we will need the notion of a *good morphism*.

Good morphisms

Let $I_m \in \mathsf{B}(m,m)$ be the identity morphism of $\mathsf{b}^{\otimes m}$ in the cube C. A framed link L in the complement $C \setminus I_m$ is *good* if L is geometrically disjoint from all the up arrows of $\mathsf{b}^{\otimes m}$, i.e. there is a plane dividing the cube into two halves, such that all the up arrows are in one half, and all the down arrows and L are in the other. Equivalently, there is a diagram in which all the up arrows are above all components of L. The union W of I_m and a colored framed good link L is called a *good morphism*. If Y is any bottom tangle so that we can compose $X = WY$, then it is easy to see that $\mathrm{gr}_\varepsilon(X)$ does not depend on Y, and we define $\mathrm{gr}_\varepsilon(W) := \mathrm{gr}_\varepsilon(X)$. Also define $\mathrm{gr}_q(W) := \mathrm{gr}_q(L)$.

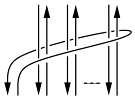
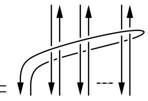
As in the case with $L = \emptyset$, the universal invariant extends to a map $J_W : \mathcal{U}_h^{\otimes m} \to \mathcal{U}_h^{\otimes m}$.

A.3.2 Proof of Theorem A.3.1

The strategy here is again analogous to the Habiro case. In Proposition A.3.3 we will decompose X into simple parts: the top is a bottom tangle with zero linking matrix, the next is a good morphism, and the bottom is a morphism obtained by pasting copies of μ_{b}. Since any bottom tangle with zero linking matrix satisfies Theorem A.3.1 and μ_{b} is the product in $\mathcal{U}_q$, which preserves gr_ε and gr_q, it remains to show that any good morphism preserves the gradings. This is done in Proposition A.3.4 below. □

Proposition A.3.3. *Assume $X = T \sqcup L$ where T is a n–component bottom tangle with zero linking matrix and L is a link. Then there is a presentation $X = W_2 W_1 W_0$ where W_0 is a bottom tangle with zero linking matrix, W_1 is a good morphism, and W_2 is obtained by pasting copies of μ_{b}.*

Proof. Let us first define $\tilde\gamma_\pm \in \mathsf{B}(i, i+1)$ for any $i \in \mathbb{N}$ as follows.

$\tilde\gamma_+ :=$  $\tilde\gamma_- :=$  .

If a copy of μ_{b} is directly above $\psi_{\mathsf{b},\mathsf{b}}^{\pm 1}$ or $\gamma_\pm$, one can move μ_{b} down by isotopy and represent the result by pasting copies of $\psi_{\mathsf{b},\mathsf{b}}^{\pm 1}$ and $\tilde\gamma_\pm$. It is easy to see that after the isotopy, $\gamma_\pm$ gets replaced by $\tilde\gamma_\pm$ and $\psi_{\mathsf{b},\mathsf{b}}^{\pm 1}$ by two copies of $\psi_{\mathsf{b},\mathsf{b}}^{\pm 1}$.

Using Proposition A.2.1 and reordering the basic morphisms so that the μ's are at the bottom, one can see that T admits the following presentation:

$$T = W_2 \tilde{W}_1(B^{\otimes k})$$

where B is the Borromean tangle, W_2 is obtained by pasting copies of μ_{b} and $\tilde{W}_1$ is obtained by pasting copies of $\psi^{\pm 1}_{\mathsf{b},\mathsf{b}}$, $\tilde{\gamma}_\pm$ and η_{b}.

Let P be the horizontal plane separating $\tilde{W}_1$ from W_2. Let P_+ (P_-) be the upper (respectively lower) half–space. Note that $W_0 = \tilde{W}_1(B^{\otimes k})$ is a bottom tangle with zero linking matrix lying in P_+ and does not have any minimum points. Therefore the pair (P_+, W_0) is homeomorphic to the pair (P_+, l trivial arcs). Similarly, W_2 does not have any maximum points; hence L can be isotoped off P_- into P_+. Since the pair (P_+, W_0) is homeomorphic to the pair (P_+, l trivial arcs) one can isotope L in P_+ to the bottom end points of down arrows. We obtain the desired presentation. □

Proposition A.3.4. *For every good morphism W, the operator J_W preserves* gr_ε *and* gr_q *in the following sense. If $x \in \mathcal{U}_q^{\varepsilon_1} \otimes \cdots \otimes \mathcal{U}_q^{\varepsilon_m}$, then*

$$J_W(x) \in q^{\mathrm{gr}_q(W)/4} \left(\mathcal{U}_q^{\varepsilon'_1} \otimes \cdots \otimes \mathcal{U}_q^{\varepsilon'_m} \right) \quad \textit{where} \quad (\varepsilon'_1, \dots, \varepsilon'_m) = (\varepsilon_1, \dots, \varepsilon_m) + \mathrm{gr}_\varepsilon(W).$$

The rest of the appendix is devoted to the proof of Proposition A.3.4.

A.3.3 Proof of Proposition A.3.4

We proceed as follows. Since J_X is invariant under cabling and skein relations, and by Lemma A.3.6 below, both relations preserve our gradings, we consider the quotient of $\tilde{S}(D^3 \backslash T)$ by these relations. It is known as a skein module of $D^3 \backslash T$. For $T = I_n$, this module has a natural algebra structure with good morphisms forming a subalgebra. By Lemma A.3.5 (see also Figure A.3.2), the basis elements W_γ of this subalgebra are labeled by n–tuples $\gamma = (\gamma_1, \dots, \gamma_n) \in (\mathbb{Z}/2\mathbb{Z})^n$. It is clear that if the proposition holds for W_{γ_1} and W_{γ_2}, then it holds for $W_{\gamma_1} W_{\gamma_2}$. Hence it remains to check the claim for W_γ's. This is done in Corollary A.3.8 for basic good morphisms corresponding to the γ whose non–zero γ_j's are consecutive. Finally, any W_γ can be obtained by pasting a basic good morphism with a few copies of $\psi^{\pm}_{\mathsf{b},\mathsf{b}}$. Since $J_{\psi^\pm}$ preserves gradings (compare (3.15), (3.16) in [12]), the claim follows from Lemmas A.3.5, A.3.6 and Proposition A.3.7 below. □

Cabling and skein relations

Let us introduce the following relations in $\tilde{\mathcal{S}}(D^3 \setminus T)$.

Cabling relations:

(a) Suppose $n_i = 1$ for some i. The first cabling relation is $L = \tilde{L}$ where $\tilde{L}$ is obtained from L by removing the ith component.

(b) Suppose $n_i \geq 3$ for some i. The second cabling relation is $L = L'' - L'$ where L' is the link L with the color of the ith component switched to $n_i - 2$, and L'' is obtained from L by replacing the ith component with two of its parallels, which are colored with $n_i - 1$ and 2.

Skein relations:

(a) The first skein relation is $U = q^{\frac{1}{2}} + q^{-\frac{1}{2}}$ where U denotes the unknot with framing zero and color 2.

(b) Let L_R, L_V and L_H be unoriented framed links with color 2 everywhere which are identical except in a disc where they are as shown in Figure A.3.1. Then the second skein relation is $L_R = q^{\frac{1}{4}} L_V + q^{-\frac{1}{4}} L_H$ if the two strands in the crossing come from different components of L_R, and $L_R = \epsilon(q^{\frac{1}{4}} L_V - q^{-\frac{1}{4}} L_H)$ if the two strands come from the same component of L_R, producing a crossing of sign $\epsilon = \pm 1$ (i.e. appearing as in L_ϵ of Figure A.3.1 if L_R is oriented).

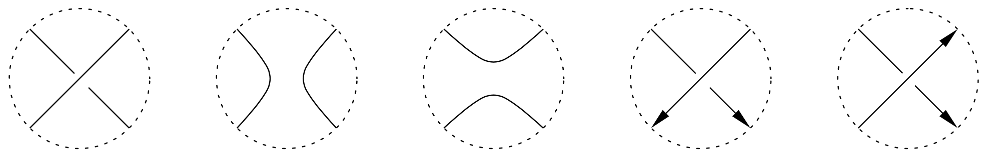


Figure A.3.1: L_R L_V L_H L_+ L_-

We denote by $S(D^3 \setminus T)$ the quotient of $\tilde{\mathcal{S}}(D^3 \setminus T)$ by these relations. It is known as the *skein module* of $D^3 \setminus T$ (compare [30], [31] and [6]). Recall that the ground ring is $\mathbb{Z}[q^{\pm 1/4}]$.

Using the cabling relations, we can reduce all colors of L in $S(D^3 \setminus T)$ to be 2. Note that the skein module $S(C \setminus I_n)$ has a natural algebra structure, given by putting one cube on top of the other. Let us denote by A_n the subalgebra of this skein algebra generated by good morphisms.

For a set $\gamma = (\gamma_1, \ldots, \gamma_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ let W_γ be a simple closed curve encircling the end points of those downward arrows with $\gamma_i = 1$. See Figure A.3.2 for an example.

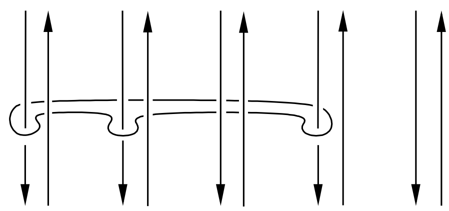


Figure A.3.2: The element $W_{(1,1,0,1,0)}$.

Similarly to the case of Kauffman bracket skein module [6], one can easily prove the following.

Lemma A.3.5. *The algebra $\mathcal{A}_n$ is generated by 2^n curves W_γ.*

Using linearity, we can extend the definition of J_X to $X = T \sqcup L$ where L is *any element* of $\tilde{\mathcal{S}}(D^3 \setminus T)$. It is known that J_X is invariant under the cablings and skein relations (Theorem 4.3 of [20]), hence J_X is defined for $L \in S(D^3 \setminus T)$. Moreover, we have the following.

Lemma A.3.6. *Both gradings gr_ε and gr_q are preserved under the cabling and skein relations.*

Proof. The statement is obvious for the ε–grading. For the q–grading, notice that

$$\mathrm{gr}_q(L) = 2 \sum_{1\le i<j\le m} p_{ij} n'_i n'_j + \sum_{1\le j\le m} p_{jj} n'^2_j + 2 \sum_{1\le j\le m} (p_{jj}+1) n'_j,$$

and therefore $\mathrm{gr}_q(L'') \equiv \mathrm{gr}_q(L') \equiv \mathrm{gr}_q(L) \pmod 4$. This takes care of the cabling relations.

Let us now assume that all colors of L are equal to 2 and therefore

$$\mathrm{gr}_q(L) = 2 \sum_{1\le i<j\le m} p_{ij} + 3 \sum_{i=1}^{m} p_{ii} + 2m.$$

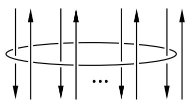
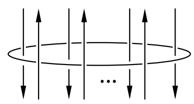
The statement is obvious for the first skein relation. For the second skein relation, choose an arbitrary orientation on L. Let us first assume that the two strands in the crossing depicted in Figure A.3.1 come from the same component of L_R and that the crossing is positive. Then, L_V and L_H have one positive self–crossing less, and L_V has one link component more than L_R. Therefore

$$\begin{aligned} \mathrm{gr}_q(q^{\frac{1}{4}} L_V) &= \mathrm{gr}_q(L_R) - 3 + 2 + 1 \equiv \mathrm{gr}_q L_R \pmod 4 \quad \text{and} \\ \mathrm{gr}_q(q^{-\frac{1}{4}} L_H) &= \mathrm{gr}_q(L_R) - 3 - 1 \equiv \mathrm{gr}_q L_R \pmod 4. \end{aligned}$$

It is obvious that this does not depend on the orientation of L_R. If the crossing of L_R is negative or the two strands do not belong to the same component of L_R, the proof works in a similar way. □

Basic good morphisms

Let $\hat{Z}_n$ be W_γ for $\gamma = (1, 1, \ldots, 1) \in (\mathbb{Z}/2\mathbb{Z})^n$. We will also need the tangle Z_n obtained from $\hat{Z}_n$ by removing the last up arrow.

$\hat{Z}_n =$  $Z_n =$ 

Let J_{Z_n} be the universal quantum invariant of Z_n, see [12].

Proposition A.3.7. *One has a presentation*

$$J_{\hat{Z}_n} = \sum z^{(n)}_{i_1} \otimes \sum z^{(n)}_{i_2} \otimes \cdots \otimes \sum z^{(n)}_{i_{2n}},$$

such that $z^{(n)}_{i_{2j-1}} z^{(n)}_{i_{2j}} \in v\, \mathcal{U}^1_q$ for every $j = 1, \ldots, n$.

Corollary A.3.8. *$J_{\hat{Z}_n}$ satisfies Proposition A.3.4.*

Proof. Assume $x \in \mathcal{U}_q^{\varepsilon_1} \otimes \cdots \otimes \mathcal{U}_q^{\varepsilon_n}$, we have

$$J_{\hat{Z}_n}(x) = \sum z_{i_1}^{(n)} x_1 z_{i_2}^{(n)} \otimes \cdots \otimes \sum z_{i_{2n-1}}^{(n)} x_n z_{i_{2n}}^{(n)}.$$

Hence, by Proposition A.3.7 we get

$$J_{\hat{Z}_n}(x) \in q^{1/2} \left(\mathcal{U}_q^{\varepsilon'_1} \otimes \cdots \otimes \mathcal{U}_q^{\varepsilon'_n} \right), \quad \text{where} \quad (\varepsilon'_1, \ldots, \varepsilon'_m) = (\varepsilon_1, \ldots, \varepsilon_n) + (1, 1, \ldots, 1).$$

The claim follows from the fact that $\mathrm{gr}_\varepsilon(\hat{Z}_n) = (1, 1, \ldots, 1)$ and $\mathrm{gr}_q(L) = 2$. $\square$

A.4 Proof of Proposition A.3.7

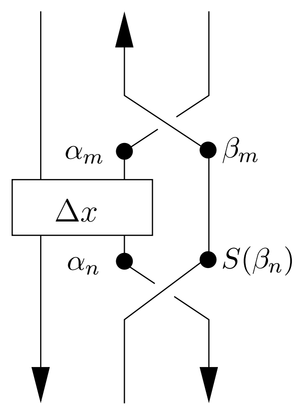
The statement holds true for $J_{\hat{Z}_1} = C \otimes \mathrm{id}_\uparrow$. Now Lemma 7.4 in [13] states that applying Δ to the ith component of the universal quantum invariant of a tangle is the same as duplicating the ith component. Using this fact, we represent

$$J_{Z_{n+1}} = (\mathrm{id}^{\otimes 2(n-1)} \otimes \Phi)\,(J_{Z_n})\,,$$

where Φ is defined as follows. For $x \in \mathcal{U}_q$ with $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$, we put

$$\Phi(x) := \sum_{(x),m,n} x_{(1)} \otimes \beta_m S(\beta_n) \otimes \alpha_n\, x_{(2)} \alpha_m$$

where the R–matrix is given by $R = \sum_l \alpha_l \otimes \beta_l$. See Figure below for a picture.



We are left with the computation of the ε–grading of each component of $\Phi(x)$.

In $\mathcal{U}_q$, in addition to the ε–grading, there is also the K–grading, defined by $|K| = |K^{-1}| = 0, |e| = 1, |F| = -1$. In general, the co–product Δ does not preserve the ε–grading. However, we have the following.

Lemma A.4.1. *Suppose $x \in \mathcal{U}_q$ is homogeneous in both ε–grading and K–grading. We have a presentation*

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$$

where each $x_{(1)}$, $x_{(2)}$ is homogeneous with respect to the ε–grading and K–grading. In addition,

$$\mathrm{gr}_\varepsilon(x_{(2)}) = \mathrm{gr}_\varepsilon(x) = \mathrm{gr}_\varepsilon(x_{(1)}\, K^{-|x_{(2)}|}).$$

Proof. If the statements hold true for $x, y \in \mathcal{U}_q$, then they hold true for xy. Therefore, it is enough to check the statements for the generators e, $\tilde{F}^{(1)}$, and K, for which they follow from explicit formulas of the co–product. □

Lemma A.4.2. *Suppose $x \in \mathcal{U}_q$ is homogeneous in both ε–grading and K–grading. There is a presentation*

$$\Phi(x) = \sum x_{i_0} \otimes x_{i_1} \otimes x_{i_2}$$

such that each x_{i_0} is homogeneous in both ε–grading and K–grading, and $\mathrm{gr}_\varepsilon(x_{i_2}) = \mathrm{gr}_\varepsilon(x_{i_0}\, x_{i_1}) = \mathrm{gr}_\varepsilon(x)$.

Proof. We put $D = \sum D' \otimes D'' := v^{\frac{1}{2}H \otimes H}$. Using (see e.g. [12])

$$R = D\left(\sum_n q^{\frac{1}{2}n(n-1)} \tilde{F}^{(n)} K^{-n} \otimes e^n\right),$$

we get

$$\begin{aligned} \Phi(x) &= \sum_{(x),n,m} q^{\frac{1}{2}(m(m-1)+n(n-1))} x_{(1)} \otimes D''_2 e^m S(D''_1 e^n) \otimes D'_1 \tilde{F}^{(n)} K^{-n} x_{(2)} D'_2 \tilde{F}^{(m)} K^{-m} \\ &= \sum_{(x),n,m} (-1)^n q^{-\frac{1}{2}m(m+1)-n(|x_{(2)}|+1)} x_{(1)} \otimes e^m e^n K^{-|x_{(2)}|} \otimes \tilde{F}^{(n)} x_{(2)} \tilde{F}^{(m)} \end{aligned}$$

where we used $(\mathrm{id} \otimes S)D = D^{-1}$ and $D^{\pm 1}(1 \otimes x) = (K^{\pm|x|} \otimes x)D^{\pm 1}$ for homogeneous $x \in \mathcal{U}_q$ with respect to the K–grading. Now, the claim follows from Lemma A.4.1. □

By induction, using the fact that $C \in v\,\mathcal{U}_q^1$, Lemma A.4.2 implies Proposition A.3.7.

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