Spatial Asymptotic Behavior of Elliptic Equations and Variational Inequalities

Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde

(Dr.sc.nat.)

vorgelegt der

Mathematisch-naturwissenschaftlichen Fakultät

der Universität Zürich

von

Karen Yeressian Negarchi

aus dem

Iran

Promotionskomitee

Prof. Dr. Michel Chipot (Vorsitz)

Prof. Dr. Stephan Sauter

Zürich 2011

Contents

Zusammenfassung

In dieser Arbeit untersuchen wir das räumliche asymptotische Verhalten von Lösungen elliptischer partieller Differentialgleichungen und Variationsungleichungen.

Wir betrachten zwei Probleme. Erstens, a priori Fehlerabschätzungen der Finite-Elemente-Methode für elliptische partielle Differentialgleichungen zweiter Ordnung. Zweitens, das asymptotische Verhalten von Lösungen elliptischer Variationsungleichungen mit punktweiser Beschränkung des Wertes der Lösung und deren Gradient in zylindrischen Bereichen.

Das erste Kapitel ist eine allgemeine Einführung in räumlich asymptotisches Verhalten und in die historische Erforschung dieses Problems.

Das zweite Kapitel enthält eine Publikation [**CY08**] mit Michel Chipot über das asymptotische Verhalten von elliptischen Gleichungen zweiter Ordnung in Zylindern mittels einer Iterationsmethode.

Im dritten Kapitel verallgemeinern wir die Iterationsmethode für eine grosse Klasse von Gleichungen auf Bereiche mit fast beliebiger Geometrie. Das ist eine Verallgemeinerung der Arbeiten [**CY08**] und [**OY77**]. Dann passen wir die Iterationsmethode an, um lokale a priori Fehlerabschätzungen der Finite Elemente Approximation von elliptischen Neumann und Dirichlet Randwertprobleme zweiter Ordnung zu erhalten. Dieses Resultat zeigt, wie man ein Gitter um einen Bereich verfeinert, in dem man an der Lösung interessiert ist. Diese Art von Abschätzungen wurde in [**W91**] studiert.

Im vierten Kapitel führen wir elliptische Variationsungleichungen in Zylindern mit punktweiser Beschränkung des Wertes der Lösung und deren Gradient ein. Spezielle Formen dieses Problems werden "ellastic-plastic torsion" Probleme genannt. Im selben Kapitel betrachten wir das asymptotische Verhalten einer einzelnen Lösung. Wir zeigen, dass das asymptotische Verhalten hauptsächlich von den Randwerten an den Enden des Zylinders abhängt.

Im fünften Kapitel studieren wir das asymptotische Verhalten der Differenz zweier Lösungen in dem Fall, wenn die angewandte Kraft entweder periodisch oder konstant in Richtung des Zylinders ist.

Abstract

In this thesis the spatial asymptotic behavior of the solutions to elliptic partial differential equations and variational inequalities is studied.

We consider two problems. First, a priori error estimates of the finite element method for the elliptic second order partial differential equations and second, the asymptotic behavior of solutions to elliptic variational inequalities with pointwise constraint on the value of the solution and its gradient in cylindrical domains, are studied.

The first chapter is a general introduction to spatial asymptotic behavior estimates and the history of the problem.

The second chapter contains a published note [**CY08**] which was a joint work with Michel Chipot and is about the asymptotic behavior of second order elliptic equations in cylinders by an iteration technique.

In the third chapter first we generalize the iteration technique by which we may obtain the asymptotic estimate for a large class of equations to domains of rather general geometry, this is a generalization of the works [**CY08**] and [**OY77**], then we adapt the iteration technique to obtain a priori local error estimates of finite element approximation to elliptic second order Neumann and Dirichlet boundary value problems. These results show how one may a priori grade the mesh around a region of interest. This kinds of estimates have been studied in [**W91**].

In the forth chapter we introduce the elliptic variational inequalities in cylinders with a pointwise constraint on the value of the function and its gradient. Special forms of these problems are called elastic-plastic torsion problems. In the same chapter we consider the asymptotic behavior of a single solution. We show that the asymptotic behavior is depending mainly on the boundary data at the ends of the cylinder.

In the fifth chapter we study the asymptotic behavior of the difference of two solutions in the case when the applied force is either periodic in the lateral direction of the cylinder or is defined in the cross section of the cylinder.

Acknowledgment

This work would not be possible without the guidance of Prof. Michel Chipot, I am very grateful to him for giving me this interesting topic and for his supervision.

CHAPTER 1

Introduction

1. Spatial Asymptotic Behavior

The spatial asymptotic behavior of solutions to partial differential equations have been studied for many problems in many contexts and in these different contexts these results have been given different names. In complex analysis and sometimes in partial differential equations these are known as Phragmén-Lindelöf principle, in Mechanics similar results are called Saint-Venant's principle and in partial differential equations depending on the type of the equations we have different names, for example we will see in the introductory examples that the notion of spatial asymptotic behavior for linear wave equation coincides with the fact of finite speed of propagation.

1.1. Phragmén-Lindelöf Principle in Complex Analysis. In complex analysis the Phragmén-Lindelöf principle extends the maximum modulus principle to unbounded domains. Let us bring its statement for a horizontal half cylinder in the complex plane. Let $\alpha > 0$ and

$$\Omega = \left\{ z \in \mathbb{C} \;\middle|\; \mathrm{Re}z \geq 0, -\frac{1}{2}\frac{\pi}{\alpha} \leq \mathrm{Im}z \leq \frac{1}{2}\frac{\pi}{\alpha} \right\}$$

be the set depicted in the figure 1. Let us consider the function f such that it is holomorphic in Ω and for some positive constant $M > 0$, $|f(z)| \leq M$ for all $z \in \partial\Omega$ and $f(z) = O(e^{e^{\rho \mathrm{Re} z}})$ for some constant $0 < \rho < \alpha$, then $|f(z)| \leq M$ for all $z \in \Omega$. Except for the condition on the growth of the module of $f(z)$ at infinity

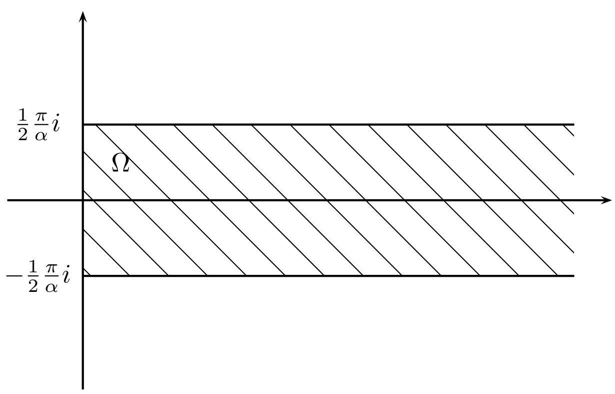


FIGURE 1. Ω

the Phragmén-Lindelöf principle is the same as the maximum modulus principle which applies to bounded domains. This growth condition is necessary, for example the function $\phi(z) = e^{e^{\alpha z}}$ is holomorphic in Ω and $|\phi(z)| = \exp(e^{\alpha \mathrm{Re} z} \cos(\alpha \mathrm{Im} z))$, when $\mathrm{Re} z = 0$ then $|\phi(z)| = \exp(\cos(\alpha \mathrm{Im} z)) \leq e$ and when $\mathrm{Im} z = \pm \frac{1}{2} \frac{\pi}{\alpha}$ then $\cos(\alpha \mathrm{Im} z) = 0$ so $|\phi(z)| = 1$, hence we have $|\phi(z)| \leq e$ on $\partial\Omega$, but on the real axis we have $|\phi(z)| = \exp(e^{\alpha \mathrm{Re} z})$ which converges to $+\infty$ as $\mathrm{Re} z$ approaches $+\infty$.

In the Phragmén-Lindelöf principle we see that when we look at wider cylinders, that is when α is smaller, then the condition on the growth of the module of the function states that the module of the function should grow slower at the infinity. We will see later that similar situations hold for solutions to PDE in cylinders and domains of general geometry.

1.2. Saint-Venant's Principle in Mechanics. In mechanics the spatial asymptotic analysis arises mainly in the elasticity theory, where it is called Saint-Venant's principle. The simplest problem in which to illustrate this principle would be the following. Consider the cylindrical shaped elastic body Ω_ℓ with length ℓ and arbitrary cross section depicted in figure 2. Let this body be loaded only on the end C_0 with an arbitrary system of equilibrated forces. Then the stored elastic energy $E(\tau)$ in Ω_τ compared to the total stored energy $E(\ell)$ satisfies the inequality

$$\frac{E(\tau)}{E(\ell)} \leq e^{-c\omega_0(\ell-\tau)} \tag{1}$$

where $c > 0$ depends on mechanical properties of the linear elastic material and ω_0 is the smallest characteristic frequency of free vibration of the cross section which depends on the shape and the size of the cross section. The dependence of the inverse of this frequency on the size of the cross section is linear, that is if the cross section is proportionally doubled then ω_0 is halved, this is similar to the situation that we have seen in the Phragmén-Lindelöf principle concerning the dependence of the growth condition and the width of the cylinder.

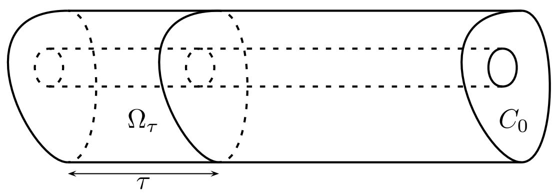


FIGURE 2. Cylindrical linear elastic body Ω_ℓ.

1.3. Partial Differential Equations. Here we bring the well known results about the spatial asymptotic estimates of the elliptic equations with Laplace operator, the heat equation and the wave equation.

In the following we will consider equations defined on the family of domains $\Omega_\ell \subset \mathbb{R}^d$ parametrized by the positive real parameter ℓ such that if $\ell_1 \leq \ell_2$ then $\Omega_{\ell_1} \subset \Omega_{\ell_2}$, and the boundary of Ω_ℓ is piecewise smooth and composed of two parts $\partial\Omega_\ell = \Gamma_\ell \cup \Delta_\ell$. The boundary Δ_ℓ moves smoothly as the parameter ℓ changes such that there exists a positive continuous function $h(x)$ defined on Ω_ℓ and for $0 < \tau < \ell$ we have

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{\Omega_{\tau+\delta}\setminus\Omega_\tau} f(x)d\mu_d(x) = \int_{\Delta_\tau} f(x)h(x)d\mu_{d-1}(x)$$

for any smooth function f, here μ_{d-1} is the surface measure. Figure 3 shows an example of this kind of domain.

1.3.1. *Linear Elliptic Equations.* In [**OY77**] Oleinik and Yosifian presented a general result about the asymptotic behavior of second order linear elliptic equations. To introduce the asymptotic analysis in the case of linear elliptic equations we bring parts of their results. Let us consider the following simple example of Dirichlet problem in domains Ω_ℓ defined above

$$\begin{cases} -\triangle u_\ell = 0 \text{ in } \Omega_\ell, \\ u_\ell = g_\ell \text{ on } \Delta_\ell, \\ u_\ell = 0 \text{ on } \Gamma_\ell. \end{cases} \tag{2}$$

Let us denote

$$\lambda_\ell = \inf_{v\in C_c^1(\Delta_\ell)} \frac{\int_{\Delta_\ell} |D_T v|^2 h d\mu_{d-1}(x)}{\int_{\Delta_\ell} v^2 \frac{1}{h} d\mu_{d-1}(x)} \tag{3}$$

here $D_T v$ denotes the gradient vector of v which is in the tangent plane to Δ_ℓ. Taking $0 < \tau < \ell$ and multiplying the equation by u_ℓ and integrating on Ω_τ we obtain

$$\begin{aligned} \int_{\Omega_\tau} |Du_\ell|^2 = \int_{\Delta_\tau} u_\ell \frac{\partial u_\ell}{\partial \nu} &\le \frac{\sqrt{\lambda_\tau}}{2}\int_{\Delta_\tau} u_\ell^2 \frac{1}{h} + \frac{1}{2\sqrt{\lambda_\tau}}\int_{\Delta_\tau} |Du_\ell\cdot\nu|^2 h \\ &\le \frac{1}{2\sqrt{\lambda_\tau}}\int_{\Delta_\tau} |D_T u_\ell|^2 h + \frac{1}{2\sqrt{\lambda_\tau}}\int_{\Delta_\tau} |Du_\ell\cdot\nu|^2 h \\ &= \frac{1}{2\sqrt{\lambda_\tau}}\int_{\Delta_\tau}\big(|D_T u_\ell|^2 + |Du_\ell\cdot\nu|^2\big)h = \frac{1}{2\sqrt{\lambda_\tau}}\int_{\Delta_\tau} |Du_\ell|^2 h. \end{aligned} \tag{4}$$

Let us denote

$$E(\tau) = \int_{\Omega_\tau} |Du_\ell|^2$$

then we have

$$E'(\tau) = \int_{\Delta_\tau} |Du_\ell|^2 h$$

then by (4) we have

$$E(\tau) \le \frac{1}{2\sqrt{\lambda_\tau}} E'(\tau).$$

From here we obtain that the following expression

$$e^{-\int_0^\tau 2\sqrt{\lambda_y}dy} E(\tau)$$

as a function of τ, is increasing in τ, and in particular this means that

$$\frac{E(\tau)}{E(\ell)} \le e^{-\int_\tau^\ell 2\sqrt{\lambda_y}dy}. \tag{5}$$

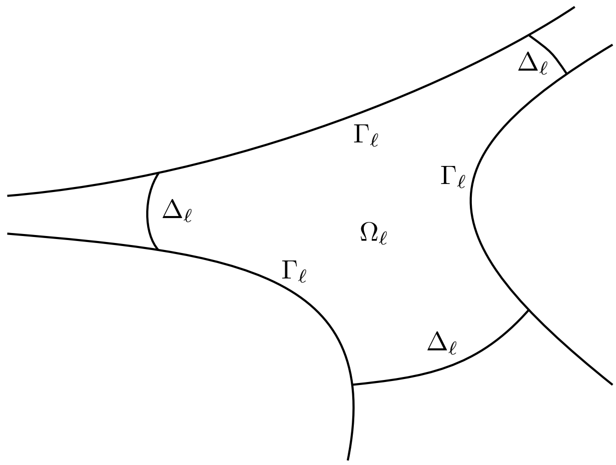


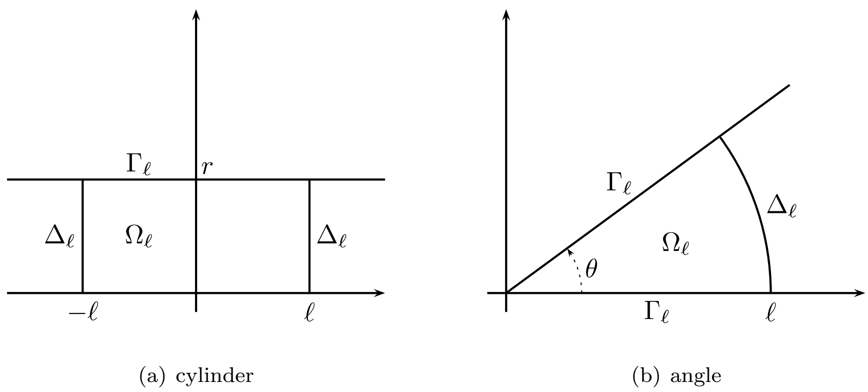
FIGURE 3. Ω_ℓ

In the case of a cylinder $\Omega_\ell = (-\ell, \ell) \times (0, r)$ as depicted in figure 4(a) we have that $h = 1$ and λ_y is constant equal to $(\frac{\pi}{r})^2$, so we have

$$\frac{E(\tau)}{E(\ell)} \leq e^{-\frac{2\pi}{r}(\ell-\tau)}.$$

In the case of an angle of degree θ as depicted in figure 4(b) again $h = 1$. Then by doing a polar coordinate transformation one may see that $\lambda_y = (\frac{\pi}{\theta})^2 \frac{1}{y^2}$. So we have $\int_\tau^\ell 2\sqrt{\lambda_y} dy = \frac{2\pi}{\theta} \ln(\frac{\ell}{\tau})$, hence

$$\frac{E(\tau)}{E(\ell)} \leq \left(\frac{\tau}{\ell}\right)^{\frac{2\pi}{\theta}}.$$



(a) cylinder (b) angle

FIGURE 4. Ω_ℓ

We observe that in the case of a cylinder the estimate for $\frac{E(\tau)}{E(\ell)}$ from above is of the same form as the one we have seen for the Saint-Venant principle.

In the case of a cylinder the estimate grows as the cylinder widens, that is as r grows, and in the case of an angle the estimate grows when θ grows. In both cases the phenomena is that when the boundary Γ_ℓ, where the solution is equal to zero, is far from the interior of the domain then the estimate grows.

To show that these estimates for the case of a cylindrical domain are asymptotically sharp, let us bring an example of an explicit solution. Consider the case of the cylinder $\Omega_\ell = (-\ell,\ell)\times\omega$ where ω is a bounded domain in $\mathbb{R}^{d-1}$, $\Gamma_\ell = (-\ell,\ell)\times\partial\omega$ and $\Delta_\ell = \{-\ell,\ell\}\times\omega$. Let us denote a point $X\in\Omega_\ell$ by $X=(x_1,X_2)$ such that $x_1\in(-\ell,\ell)$ and $X_2\in\omega$. Let us consider the normalized eigenfunction $w(X_2)$ corresponding to the first eigenvalue of the Dirichlet homogeneous boundary value problem in ω, that is

$$\begin{cases} -\triangle w = \lambda w \text{ in } \omega, \\ w = 0 \text{ on } \partial\omega, \\ \displaystyle\int_\omega w^2 = 1. \end{cases}$$

By multiplying the equation by w and integrating we obtain

$$\lambda = \int_\omega |D_{X_2} w|^2.$$

Now let us consider the function

$$u_\ell(X) = \cosh(\sqrt{\lambda}x_1)w(X_2),$$

then it is clear that $-\triangle u_\ell = 0$ in Ω_ℓ and $u = 0$ on Γ_ℓ, and for $0<\tau<\ell$ we may compute

$$\begin{aligned} E(\tau) &= \int_{\Omega_\tau} |Du_\ell|^2 dX = \int_{\Omega_\tau} \Big\{\lambda\sinh^2(\sqrt{\lambda}x_1)w^2(X_2) + \cosh^2(\sqrt{\lambda}x_1)|D_{X_2}w|^2\Big\} dX \\ &= \int_{-\tau}^{\tau} \lambda\sinh^2(\sqrt{\lambda}x_1)dx_1 + \int_{-\tau}^{\tau}\cosh^2(\sqrt{\lambda}x_1)dx_1 \int_\omega |D_{X_2}w|^2 dX_2 \\ &= \lambda\int_{-\tau}^{\tau}\big(\sinh^2(\sqrt{\lambda}x_1)+\cosh^2(\sqrt{\lambda}x_1)\big)dx_1 = \lambda\int_{-\tau}^{\tau}\cosh(2\sqrt{\lambda}x_1)dx_1 \\ &= \sqrt{\lambda}\sinh(2\sqrt{\lambda}\tau). \end{aligned}$$

Thus

$$\frac{E(\tau)}{E(\ell)} = \frac{\sinh(2\sqrt{\lambda}\tau)}{\sinh(2\sqrt{\lambda}\ell)} = h_{\tau,\ell}e^{-2\sqrt{\lambda}(\ell-\tau)} \tag{6}$$

where

$$h_{\tau,\ell} = \frac{1-e^{-4\sqrt{\lambda}\tau}}{1-e^{-4\sqrt{\lambda}\ell}}.$$

To compare this with the general estimate (5) let us note that for our choice of Ω_ℓ the constant λ_ℓ defined in (3) is the first eigenvalue of the Dirichlet problem in Δ_ℓ that is in ω, so λ_ℓ is constant and equal to λ, so $\int_\tau^\ell 2\sqrt{\lambda_y}dy = 2\sqrt{\lambda}(\ell-\tau)$ and from the general estimate (5) we have

$$\frac{E(\tau)}{E(\ell)} \le e^{-2\sqrt{\lambda}(\ell-\tau)}$$

but this differs from the equality (6) only by the factor $h_{\tau,\ell}$, and we have

$$h_{\tau,\ell} \longrightarrow 1 \quad \text{as} \quad \tau, \ell \longrightarrow +\infty.$$

So our estimate is asymptotically sharp for large τ and ℓ in the case of this cylindrical domain.

More generally we may consider u_ℓ^i for $i = 1, 2$ the solutions to the following equation

$$\begin{cases} -\triangle u_\ell^i = f_\ell \text{ in } \Omega_\ell, \\ u_\ell^i = g_\ell^i \text{ on } \Delta_\ell, \\ u_\ell^i = h_\ell \text{ on } \Gamma_\ell \end{cases}$$

as we see in both problems the boundary condition on Γ_ℓ and the right hand side of the equation, that is f_ℓ, are the same. So the solutions only differ by their boundary value on Δ_ℓ. In this case it is very natural to expect that as ℓ grows then because the part of the boundary where the solutions are different is getting away from the domain Ω_{ℓ_0} for fixed $\ell_0 > 0$, the solutions converge to each other on this fixed domain. Actually in this linear case we may consider the difference $w_\ell = u_\ell^2 - u_\ell^1$, then w_ℓ satisfies the equation (2) with $g_\ell = g_\ell^2 - g_\ell^1$. Hence from (5) we have the estimate

$$\big\||Du_\ell^2 - Du_\ell^1|\big\|_{L^2(\Omega_\tau)} \leq e^{-\int_\tau^\ell \sqrt{\lambda_y}dy} \big\||Du_\ell^2 - Du_\ell^1|\big\|_{L^2(\Omega_\ell)}$$

so we have

$$\big\||Du_\ell^2 - Du_\ell^1|\big\|_{L^2(\Omega_\tau)} \leq C_\ell e^{-\int_\tau^\ell \sqrt{\lambda_y}dy}$$

where

$$C_\ell = C\|g_\ell^2 - g_\ell^1\|_{H^{\frac{1}{2}}(\Delta_\ell)}.$$

So if the norm of the difference of the boundary values on Δ_ℓ does not grow too fast then in any fixed domain the two solutions u_ℓ^2 and u_ℓ^1 will converge to each other.

1.3.2. *Heat Equation.* The heat equation has also been studied by many people. The interesting general phenomena is that spatially this problem behave more locally than the elliptic case and in the next section we will see that in the case of wave equation this is even more. To describe this let us consider the case of initial-boundary value heat equation and bring briefly an estimate proved in [**HPW84**]. Consider the cylindrical domains Ω_ℓ for $\ell > 0$ and for $T > 0$ consider $u_\ell(t, x)$ the solution to

$$\begin{cases} \dot{u}_\ell - \triangle u_\ell = 0 \text{ in } (0, T) \times \Omega_\ell, \\ u_\ell = g_\ell \text{ on } (0, T) \times \Delta_\ell, \\ u_\ell = 0 \text{ on } (0, T) \times \Gamma_\ell, \\ u_\ell = 0 \text{ on } \{0\} \times \Omega_\ell \end{cases}$$

here $\dot{u}_\ell = \frac{\partial u_\ell}{\partial t}$.

Multiplying the equation by u_ℓ and integrating on Ω_τ for $0 < \tau < \ell$ we get

$$\int_{\Omega_\tau} \dot{u}_\ell u_\ell dx + \int_{\Omega_\tau} |Du_\ell|^2 dx = \int_{\Delta_\tau} u_\ell \frac{\partial u_\ell}{\partial \nu} ds(x)$$

from here we obtain

$$\int_{\Omega_\tau} \dot{u}_\ell u_\ell dx \leq \int_{\Delta_\tau} u_\ell \frac{\partial u_\ell}{\partial \nu} ds(x). \tag{7}$$

Defining

$$f\,(t,\tau)=\frac{1}{2}\int_{\Omega_\tau}u_\ell^2(t,x)dx$$

then we have

$$\frac{\partial f}{\partial t}=\int_{\Omega_\tau}\dot{u}_\ell u_\ell dx,\ \frac{\partial^2 f}{\partial\tau^2}=\int_{\Delta_\tau}u_\ell\frac{\partial u_\ell}{\partial\nu}ds(x).$$

Then from (7) we have

$$\frac{\partial f}{\partial t}\leq\frac{\partial^2 f}{\partial\tau^2}$$

so we see that f is a subsolution of the heat equation in the domain $(0,T)\times(0,\ell)$, and we have the boundary values $f(0,\tau)=0,\ f(t,0)=0$ and $f(t,\ell)=\frac{1}{2}\int_{\Omega_\ell}u_\ell^2(t,x)dx$.

Let us consider a solution to the following heat equation

$$\begin{cases}\dot{w}-\triangle w=0 \text{ in } (0,T)\times(0,\ell),\\ w=f(t,\ell) \text{ on } (0,T)\times\{\ell\},\\ w\geq 0 \text{ on } (0,T)\times\{0\},\\ w=0 \text{ on } \{0\}\times(0,\ell).\end{cases}$$

The following is a solution to this problem

$$w(t,\tau)=\frac{1}{2\sqrt{\pi}}\int_0^t(\ell-\tau)(t-\tilde{t})^{-\frac{3}{2}}e^{-\frac{(\ell-\tau)^2}{4(t-\tilde{t})}}f(\tilde{t},\ell)d\tilde{t}$$

and by the comparison principle for the heat equation because f is a subsolution and w a solution and $f\leq w$ on the parabolic boundary, so we have

$$f\leq w \text{ in } (0,T)\times(0,\ell). \tag{8}$$

We may estimate w as

$$w(t,\tau)\leq\frac{1}{2\sqrt{\pi}}\Big(\sup_{\tilde{t}\in(0,t)}f(\tilde{t},\ell)\Big)\int_0^t(\ell-\tau)(t-\tilde{t})^{-\frac{3}{2}}e^{-\frac{(\ell-\tau)^2}{4(t-\tilde{t})}}d\tilde{t}.$$

Now let us simplify the integral on the right hand side, by the change of variable $\zeta^2=\frac{(\ell-\tau)^2}{4(t-\tilde{t})}$ we obtain

$$\int_0^t(\ell-\tau)(t-\tilde{t})^{-\frac{3}{2}}e^{-\frac{(\ell-\tau)^2}{4(t-\tilde{t})}}d\tilde{t}=4\int_{\frac{\ell-\tau}{2\sqrt{t}}}^{+\infty}e^{-\zeta^2}d\zeta=8\sqrt{\pi}\,\mathrm{erfc}\big(\frac{\ell-\tau}{2\sqrt{t}}\big)$$

where

$$\mathrm{erfc}(x)=\frac{1}{2\sqrt{\pi}}\int_x^{+\infty}e^{-\zeta^2}d\zeta.$$

So we have

$$w(t,\tau)\leq 4\,\mathrm{erfc}\big(\frac{\ell-\tau}{2\sqrt{t}}\big)\sup_{\tilde{t}\in(0,t)}f(\tilde{t},\ell)$$

and from this by the inequality (8) we obtain

$$\|u_\ell(t,\cdot)\|^2_{L^2(\Omega_\tau)}\leq 4\,\mathrm{erfc}\big(\frac{\ell-\tau}{2\sqrt{t}}\big)\sup_{\tilde{t}\in(0,t)}\|u_\ell(\tilde{t},\cdot)\|^2_{L^2(\Omega_\ell)}$$

from here clearly we have

$$\frac{\|u_\ell\|^2_{L^\infty(0,T,L^2(\Omega_\tau))}}{\|u_\ell\|^2_{L^\infty(0,T,L^2(\Omega_\ell))}}\leq 4\,\mathrm{erfc}\big(\frac{\ell-\tau}{2\sqrt{T}}\big) \tag{9}$$

where $\|u_\ell\|_{L^\infty(0,T,L^2(\Omega_\tau))} = \sup_{t\in(0,T)} \|u_\ell(t,\cdot)\|_{L^2(\Omega_\tau)}$. This estimate resembles the estimate (5) which we have for the linear elliptic problem, but there are significant differences in this case. First we notice that we have nothing in the estimate similar to λ_ℓ which somehow measures the diameter of Δ_ℓ, even more in this case the part of the boundary where we have 0 boundary condition can be empty, that is $\Gamma_\ell = \emptyset$. A simple example of this kind of situation is the family of balls $\Omega_\ell = B_\ell(0) = \{x \in \mathbb{R}^d \mid |x| < \ell\}$, $\Delta_\ell = S_\ell(0) = \{x \in \mathbb{R}^d \mid |x| = \ell\}$. Second, in the case the domain is a cylinder we see that the right hand side of the estimate for the heat equation is very small compared to the one we have for the elliptic case for large $\ell - \tau$, because we have the estimate

$$\sqrt{\pi}\,\mathrm{erfc}(x) \le \frac{e^{-x^2}}{x}$$

for $x > 0$, so we have

$$4\,\mathrm{erfc}\Big(\frac{\ell-\tau}{2\sqrt{T}}\Big) \le \frac{8}{\sqrt{\pi}}\frac{\sqrt{T}}{\ell-\tau}e^{-\frac{(\ell-\tau)^2}{4T}}$$

this shows that in cylindrical domains the respective norm of the solutions is concentrated near to the boundary Δ_ℓ more than the similar thing is in the case of harmonic functions.

1.3.3. *Wave Equation.* In the case of wave equation actually the spatial asymptotic analysis comparable to the cases of elliptic and parabolic equations is the very well known finite speed of propagation. To clarify this let us bring a simple proof of this well known phenomena, similar to the proofs that we have seen in the elliptic and parabolic cases. Consider the cylindrical domain Ω_ℓ and u_ℓ the solution of

$$\begin{cases} \ddot{u}_\ell - \triangle u_\ell = 0 \text{ in } (0,T)\times\Omega_\ell, \\ u_\ell = g_\ell \text{ on } (0,T)\times\Delta_\ell, \\ u_\ell = 0 \text{ on } (0,T)\times\Gamma_\ell, \\ u_\ell = \dot{u}_\ell = 0 \text{ on } \{0\}\times\Omega_\ell \end{cases}$$

here $\ddot{u}_\ell = \frac{\partial^2 u_\ell}{\partial t^2}$. Then as usual multiplying the equation by $\dot{u}_\ell$ and integrating on the subdomain Ω_τ we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\Big(\int_{\Omega_\tau} \dot{u}_\ell^2 + |Du_\ell|^2\Big) = \int_{\Delta_\tau} \frac{\partial u_\ell}{\partial \nu}\dot{u}_\ell \le \frac{1}{2}\Big(\int_{\Delta_\tau} \dot{u}_\ell^2 + |Du_\ell|^2\Big)$$
$$= \frac{1}{2}\frac{\partial}{\partial \tau}\Big(\int_{\Omega_\tau} \dot{u}_\ell^2 + |Du_\ell|^2\Big). \quad (10)$$

Defining

$$f(t,\tau) = \int_{\Omega_\tau} \dot{u}_\ell^2 + |Du_\ell|^2$$

then from (10) we have

$$\frac{\partial f}{\partial t} \le \frac{\partial f}{\partial \tau}. \quad (11)$$

For $(t,\tau) \in [0,T]\times[0,\ell]$ such that $t+\tau \le \ell$ let us consider $g(s) = f(t-s,\tau+s)$. Then by (11) we have $0 \le g'(s)$ so $g(0) \le g(t)$ that is $f(t,\tau) \le f(0,t+\tau)$. Because $0 \le f(t,\tau)$ and $f(0,t+\tau) = 0$ we have $f(t,\tau) = 0$.

Now if $T + \tau < \ell$ then for all $t \in (0, T)$ we have $f(t, \tau) = 0$ so by the definition of f we have the inequality

$$\frac{\|\dot{u}_\ell\|^2_{L^2(0,T,L^2(\Omega_\tau))} + \|Du_\ell\|^2_{L^2(0,T,L^2(\Omega_\tau))}}{\|\dot{u}_\ell\|^2_{L^2(0,T,L^2(\Omega_\ell))} + \|Du_\ell\|^2_{L^2(0,T,L^2(\Omega_\ell))}} \leq \chi_{T \geq \ell - \tau} \tag{12}$$

where $\chi_{\{T \geq \ell-\tau\}}$ is equal to 1 if $T \geq \ell - \tau$ and 0 otherwise. This estimate is actually the well known finite speed of propagation and resembles the estimates (5) and (9) that we have for elliptic and parabolic cases. Let us notice that if $T \neq \ell - \tau$ then

$$\chi_{T \geq \ell - \tau} = \lim_{\beta \to +\infty} e^{-(\frac{\ell - \tau}{T})^\beta}.$$

1.3.4. *Comparison of Linear Cases.* To compare the asymptotic behavior estimates that we have for the linear equations, let us consider the cylindrical domains Ω_ℓ, then we have the estimates (5),(9) and (12), a unified way to write these estimates would be

$$\frac{\|u_\ell\|_{X(\Omega_\tau)}}{\|u_\ell\|_{X(\Omega_\ell)}} \leq h(\ell - \tau)$$

where $X(\Omega)$ is a function space defined on Ω and

$$h(z) = \begin{cases} e^{-2\sqrt{\lambda_1}z} & \text{,Laplace equation} \\ 4\,\mathrm{erfc}(\dfrac{z}{2\sqrt{T}}) \approx e^{-\frac{z^2}{4T}} & \text{,heat equation} \\ \chi_{T \geq z} \approx e^{-(\frac{z}{T})^{+\infty}} & \text{,wave equation.} \end{cases}$$

In the figure 5 we have the graph of these functions for $T = 0.5$ and $\lambda_1 = 0.5$. As we see corresponding norm of the solution to the wave equation is more concentrated near the boundary Δ_ℓ than the one for the heat equation and the corresponding norm of the solution to heat equation is more concentrated near Δ_ℓ than the one for the elliptic equation with Laplace operator.

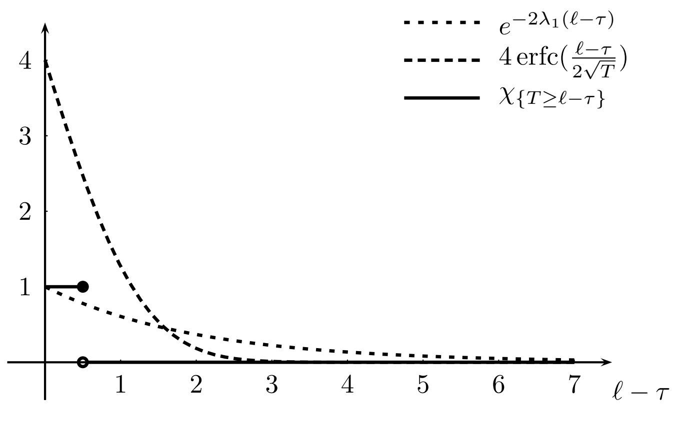


FIGURE 5. Graph of $h(\Delta\ell)$ in the three cases.

2. Brief Research History of Spatial Asymptotic Behavior

Toupin brought a rigorous proof of the Saint-Venant problem in mechanics in [**T65**] where he showed the inequality (1) for cylinders and outlined the way to obtain the result for more general geometries. Oleinik in many papers dealt with the asymptotic problems in unbounded domains. Together with Yosifian, she developed in [**OY77**] a rather general formulation of results for second order linear elliptic problems. Horgan together with Knowles wrote the review paper [**HK83**] about developments concerning the Saint-Venant principle and later Horgan had two updates to these reviews [**H89**], [**H96**]. Chipot studied the asymptotic behavior in cylinders for elliptic and parabolic problems when the applied force is periodic or defined in the cross section. In the case the force is periodic, he has shown that for many problems the solution will converge to a periodic solution in the middle of the cylinder as the length of the cylinder grows, and in the case the force is defined in the cross section the solution will converge to a solution defined in the cross section. In the book [**C02**] he collected some of his results.

3. Notation

Everywhere standard notation are used and in the beginning of the chapters 4 and 5, notation and definitions used in those chapters are brought.

Generally by the notation $|\cdot|$ the euclidean norm of vectors is denoted or the Lebesgue measure of sets. To emphasize the euclidean norm we occasionally also use the notation $|\cdot|_2$.

CHAPTER 2

Exponential Rates of Convergence by an Iteration Technique

1. Introduction

As it is described in the subsection 1.3 in the previous chapter, the asymptotic behavior of some PDE might be achieved by a differential inequality for some quantity depending on the solution. In general this might be generalized to a difference inequality, in this case we should iterate this inequality to obtain the desired estimates.

This chapter contains the published note [**CY08**]. The goal in [**CY08**] is to introduce the difference inequality technique leading to a convergence of exponential type for the solution of problems set in cylinders becoming unbounded in some directions.

Suppose that for $\ell > 0$, Ω_ℓ is the rectangle

$$\Omega_\ell = (-\ell, \ell) \times (-1, 1).$$

Let us denote by

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x_2) \\ a_{21}(x) & a_{22}(x_2) \end{pmatrix}$$

a positive definite matrix with bounded coefficients i.e. such that

$$a_{ij} \in L^\infty(\mathbb{R} \times (-1,1)), \ \ \lambda|\xi|^2 \le A(x)\xi.\xi \quad \forall \xi \in \mathbb{R}^2, \text{ a.e. } x \in \mathbb{R} \times (-1,1)$$

(The points in $\mathbb{R}^2$ are denoted by $x = (x_1, x_2)$, "." is the usual euclidean scalar product, $|\ |$ the associated norm, λ is some positive constant).

Let $f = f(x_2)$ be a function (or distribution) depending on x_2 only, for instance

$$f \in L^2((-1,1))$$

then there exists a unique u_ℓ solution to

$$\begin{cases} \displaystyle\int_{\Omega_\ell} A(x) Du_\ell \cdot Dv dx = \int_{\Omega_\ell} f v dx, \quad \forall v \in H_0^1(\Omega_\ell) \\ u_\ell \in H_0^1(\Omega_\ell). \end{cases}$$

We would like to show that when ℓ goes to plus infinity u_ℓ converges on any subdomain Ω_{ℓ_0}, $\ell_0 > 0$ towards u_∞, where u_∞ is the solution to

$$\begin{cases} \displaystyle\int_{-1}^{1} \partial_{x_2} u_\infty \partial_{x_2} v dx_2 = \int_{-1}^{1} f(x_2) v dx_2 \quad \forall v \in H_0^1((-1,1)) \\ u_\infty \in H_0^1((-1,1)) \end{cases}$$

this convergence being at a rate $e^{-\alpha\ell}$ for some positive constant α.

2. A General Result for Second Order Elliptic Equations

We denote the point $x \in \mathbb{R}^d$ also as $x = (X_1, X_2)$ where

$$X_1 = (x_1, \ldots, x_q), \quad X_2 = (x_{q+1}, \ldots, x_d)$$

i.e. we split the components of a point in $\mathbb{R}^d$ into q first components and the $d - q$ last ones.

Let ω_1 be an open subset of $\mathbb{R}^q$ that we suppose to satisfy

$$\omega_1 \text{ is bounded and star-shaped with respect to } 0. \tag{13}$$

Let ω_2 be a bounded open subset of $\mathbb{R}^{d-q}$, then we set

$$\Omega_\ell = \ell\omega_1 \times \omega_2.$$

Remark 1. *In our introduction we had $\omega_1 = \omega_2 = (-1, 1)$. ω_1 can be for instance a unit ball B_1 (for an arbitrary norm), then $\Omega_\ell = B_\ell \times \omega_2$.*

We denote by

$$A(x) = \begin{pmatrix} A_{11}(X_1, X_2) & A_{12}(X_2) \\ A_{21}(X_1, X_2) & A_{22}(X_2) \end{pmatrix} = (a_{ij}(x))$$

a $d \times d$-matrix divided into four blocks such that

$$A_{11} \text{ is a } q \times q \text{ matrix, } A_{22} \text{ is a } (d-q) \times (d-q) - matrix.$$

We assume that

$$a_{ij} \in L^\infty(\mathbb{R}^q \times \omega_2)$$

and that for some constants λ, Λ we have

$$\lambda |\xi|^2 \leq A(x)\xi.\xi, \quad \forall \xi \in \mathbb{R}^d, \text{ a.e. } x \in \mathbb{R}^q \times \omega_2, \tag{14}$$

$$|A(x)\xi| \leq \Lambda|\xi|, \quad \forall \xi \in \mathbb{R}^d, \text{ a.e. } x \in \mathbb{R}^q \times \omega_2. \tag{15}$$

Then by the Lax-Milgram theorem (see [**DL88, GT83**]) for $f \in H^{-1}(\omega_2)$ there exists a unique u_∞ solution to

$$\begin{cases} \displaystyle\int_{\omega_2} A_{22} D_{X_2} u_\infty \cdot D_{X_2} v dX_2 = \langle f, v \rangle, \quad \forall v \in H_0^1(\omega_2) \\ u_\infty \in H_0^1(\omega_2) \end{cases} \tag{16}$$

(in the above system D_{X_2} stands for the gradient in X_2, that is $(\partial_{x_{q+1}}, \ldots, \partial_{x_d})$, $dX_2 = dx_{q+1} \cdots dx_d$ and $\langle \, , \rangle$ denotes the duality between $H^{-1}(\omega_2)$ and $H_0^1(\omega_2)$).

Let us denote

$$H^1_{lat}(\Omega_\ell) = \left\{ v \in H^1(\Omega_\ell) \mid v = 0 \text{ on } \ell\omega_1 \times \partial\omega_2 \right\}$$

i.e. the set of functions in $H^1(\Omega_\ell)$ vanishing on the lateral boundary of Ω_ℓ. Then for $v \in H^1_{lat}(\Omega_\ell)$

$$v \longmapsto \int_{\ell\omega_1} \langle f, v(X_1, \cdot) \rangle \, dX_1 \tag{17}$$

defines a continuous linear form that we will yet denote by $\langle f, \cdot \rangle$.

Let V_ℓ be a closed subspace of $H^1_{lat}(\Omega_\ell)$, equipped with the $H^1(\Omega_\ell)$ topology such that

$$H_0^1(\Omega_\ell) \subset V_\ell \subset H^1_{lat}(\Omega_\ell). \tag{18}$$

By the Lax-Milgram theorem there exists a unique u_ℓ solution to

$$\begin{cases} \displaystyle\int_{\Omega_\ell} ADu_\ell \cdot Dv dx = \langle f, v \rangle \,, \quad \forall v \in V_\ell \\ u_\ell \in V_\ell. \end{cases} \tag{19}$$

Moreover we have

THEOREM 1. *There exists two constants $c, \alpha > 0$ independent of ℓ such that*

$$\int_{\Omega_{\frac{\ell}{2}}} |D(u_\ell - u_\infty)|^2 dx \le c e^{-\alpha \ell} |f|_\star^2 \tag{20}$$

($|\ |_\star$ denotes the strong dual norm in $H^{-1}(\omega_2)$).

Proof. The proof is divided into three steps.
Step 1. The equation satisfied by $u_\ell - u_\infty$.
If $v \in H^1_{lat}(\Omega_\ell)$ then for almost every X_1 in $\ell\omega_1$ we have

$$v(X_1, .) \in H^1_0(\omega_2)$$

and thus by (16)

$$\int_{\omega_2} A_{22} D_{X_2} u_\infty \cdot D_{X_2} v(X_1, \cdot) dX_2 = \langle f, v(X_1, \cdot) \rangle \,.$$

Integrating in X_1 we get

$$\int_{\Omega_\ell} A_{22} D_{X_2} u_\infty \cdot D_{X_2} v dx = \langle f, v \rangle \,, \quad \forall v \in H^1_{lat}(\Omega_\ell)$$

(see (17) for the definition of $\langle f, v \rangle$).
Now for $v \in H^1_0(\Omega_\ell)$ we have

$$\begin{aligned} \int_{\Omega_\ell} ADu_\infty \cdot Dv dx &= \int_{\Omega_\ell} A_{12} D_{X_2} u_\infty \cdot D_{X_1} v dx + \int_{\Omega_\ell} A_{22} D_{X_2} u_\infty \cdot D_{X_2} v dx \\ &= \int_{\Omega_\ell} A_{22} D_{X_2} u_\infty \cdot D_{X_2} v dx = \langle f, v \rangle \end{aligned} \tag{21}$$

(since A_{12}, u_∞ are depending on X_2 only).
Combining (18),(19) and (21) we get

$$\int_{\Omega_\ell} AD(u_\ell - u_\infty) \cdot Dv dx = 0, \quad \forall v \in H^1_0(\Omega_\ell). \tag{22}$$

Step 2. An iteration technique.
Set $0 < \ell_0 \le \ell - 1$. In addition to (13) let us assume that there exists ρ a function of X_1 only such that

$$0 \le \rho \le 1, \; \rho = 1 \text{ on } \ell_0\omega_1, \; \rho = 0 \text{ on } \mathbb{R}^q \backslash (\ell_0 + 1)\omega_1, \; |D_{X_1}\rho| \le c_0 \tag{23}$$

where c_0 is a universal constant. Such a function does exist in many instances. Then we have

$$(u_\ell - u_\infty)\rho^2 \in H^1_0(\Omega_\ell)$$

and from (22) we derive

$$\int_{\Omega_\ell} AD(u_\ell - u_\infty)\cdot D(u_\ell - u_\infty)\rho^2 dx = -2\int_{\Omega_\ell} AD(u_\ell - u_\infty)\cdot \begin{pmatrix} D_{X_1}\rho \\ 0 \end{pmatrix}(u_\ell - u_\infty)\rho dx \leq 2\int_{\Omega_{\ell_0+1}\setminus\Omega_{\ell_0}} |AD(u_\ell - u_\infty)||D_{X_1}\rho||u_\ell - u_\infty|\rho dx.$$

Using (14),(15),(23) and the Cauchy-Schwarz inequality we derive

$$\lambda\int_{\Omega_\ell} |D(u_\ell - u_\infty)|^2\rho^2 dx \leq 2c_0\Lambda\int_{\Omega_{\ell_0+1}\setminus\Omega_{\ell_0}} |D(u_\ell - u_\infty)|\rho|u_\ell - u_\infty|dx \leq 2c_0\Lambda\Big(\int_{\Omega_\ell} |D(u_\ell - u_\infty)|^2\rho^2 dx\Big)^{\frac{1}{2}}\Big(\int_{\Omega_{\ell_0+1}\setminus\Omega_{\ell_0}} (u_\ell - u_\infty)^2 dx\Big)^{\frac{1}{2}}.$$

It follows that (recall that $\rho = 1$ on Ω_{ℓ_0})

$$\int_{\Omega_{\ell_0}} |D(u_\ell - u_\infty)|^2 dx \leq \Big(2c_0\frac{\Lambda}{\lambda}\Big)^2\int_{\Omega_{\ell_0+1}\setminus\Omega_{\ell_0}} (u_\ell - u_\infty)^2 dx. \tag{24}$$

Since $u_\ell - u_\infty$ vanishes on the lateral boundary of Ω_ℓ there exists a constant c_{ω_2} independent of ℓ such that (see [**C02**])

$$\int_{\Omega_{\ell_0+1}\setminus\Omega_{\ell_0}} (u_\ell - u_\infty)^2 dx \leq c_{\omega_2}^2\int_{\Omega_{\ell_0+1}\setminus\Omega_{\ell_0}} |D_{X_2}(u_\ell - u_\infty)|^2 dx.$$

Combining this Poincaré inequality with (24) we get

$$\int_{\Omega_{\ell_0}} |D(u_\ell - u_\infty)|^2 dx \leq C\int_{\Omega_{\ell_0+1}\setminus\Omega_{\ell_0}} |D(u_\ell - u_\infty)|^2 dx$$

which is also

$$\int_{\Omega_{\ell_0}} |D(u_\ell - u_\infty)|^2 dx \leq \frac{C}{1+C}\int_{\Omega_{\ell_0+1}} |D(u_\ell - u_\infty)|^2 dx$$

where $C = (2c_0 c_{\omega_2}\frac{\Lambda}{\lambda})^2$. Iterating this formula starting from $\frac{\ell}{2}$ we obtain

$$\int_{\Omega_{\frac{\ell}{2}}} |D(u_\ell - u_\infty)|^2 dx \leq \big(\frac{C}{1+C}\big)^{[\frac{\ell}{2}]}\int_{\Omega_{\frac{\ell}{2}+[\frac{\ell}{2}]}} |D(u_\ell - u_\infty)|^2 dx$$

where $[\frac{\ell}{2}]$ denotes the integer part of $\frac{\ell}{2}$. Since $\frac{\ell}{2} - 1 < [\frac{\ell}{2}] \leq \frac{\ell}{2}$, it comes

$$\int_{\Omega_{\frac{\ell}{2}}} |D(u_\ell - u_\infty)|^2 dx \leq e^{(-\frac{\ell}{2}+1)\ln(\frac{1+C}{C})}\int_{\Omega_\ell} |D(u_\ell - u_\infty)|^2 dx = c_1 e^{-\alpha_0\ell}\int_{\Omega_\ell} |D(u_\ell - u_\infty)|^2 dx \tag{25}$$

where $c_1 = \frac{1+C}{C}$ and $\alpha_0 = \frac{1}{2}\ln(\frac{1+C}{C})$.

Step 3. Evaluation of the last integral.

Taking $v = u_\ell$ in (19) we get

$$\int_{\Omega_\ell} ADu_\ell \cdot Du_\ell dx = \langle f, u_\ell \rangle = \int_{\ell\omega_1} \langle f, u_\ell(X_1, .)\rangle \, dX_1$$

$$\leq \int_{\ell\omega_1} |f|_\star |D_{X_2} u_\ell(X_1, .)|_{L^2(\omega_2)} dX_1 \leq |\ell\omega_1|^{\frac{1}{2}} \Big(\int_{\Omega_\ell} |Du_\ell|^2 dx \Big)^{\frac{1}{2}} |f|_\star$$

($|\ |$ denotes the measure of sets), from which we derive

$$\int_{\Omega_\ell} |Du_\ell|^2 dx \leq \frac{|\ell\omega_1|}{\lambda^2} |f|_\star^2 = \frac{|\omega_1||f|_\star^2}{\lambda^2} \ell^q.$$

Similarly taking $v = u_\infty$ in (16) we get

$$\int_{\omega_2} |D_{X_2} u_\infty|^2 dX_2 \leq \frac{|f|_\star^2}{\lambda^2}$$

and thus

$$\int_{\Omega_\ell} |D(u_\ell - u_\infty)|^2 dx \leq 2 \int_{\Omega_\ell} (|Du_\ell|^2 + |Du_\infty|^2) dx \leq \frac{4|\omega_1||f|_\star^2}{\lambda^2} \ell^q.$$

The estimate (20) follows then from (25) where α can be chosen any constant smaller than α_0 and c large enough. □

REMARK 2. *In the case of a diagonal matrix A such result was already known (see for instance* [**C02, CR02**]*).*

REMARK 3. *A function f independent of X_1 is a periodic function-for any period. The independence of f from X_1 forces u_ℓ at the limit to depend only on X_2. We can adapt our method in the case of a periodic f , in the spirit of* [**CX04a, CX06, X06**]*, showing that f periodic forces u_ℓ to become periodic at the limit exponentially quickly.*

The method is not restricted to second order elliptic equations. It extends to many other situations, to more general domains, to nonlinear problems and systems (see [**CM08, C02, G06, X06**]*). Note also that our convergence technique applies to other norms and is not restricted to the H^1-norms.*

CHAPTER 3

Local Error Estimates in Finite Element Method

In this chapter we adjust the iteration method that we have to domains of general geometry and to the finite element method to obtain local error estimates. This estimates are particularly useful for a priori mesh adaptation to a region of interest. Local behavior of finite element approximations have been studied by Wahlbin as presented in [**W91**], here for a subclass of problems we bring a simplified and general proof of these results.

In section 1 we introduce the iteration for domains of general geometry, in section 2 we bring some preliminary results for the local error estimates in finite element methods. In section 3 we will consider an elliptic problem with Neumann boundary condition and in section 4 an elliptic problem with Dirichlet boundary condition and in both cases we prove the local error estimate. In section 5 using the local error estimates we obtain the optimal triangulation formulas. In section 6 we apply the optimal triangulation to the cylindrical domains.

1. Iteration Technique for Domains of General Geometry

1.1. Growing Subdomains. Let us consider the bounded domain $\Omega \subset \mathbb{R}^d$ and for $\ell \geq 2$ the class of growing domains

$$\Upsilon_1 \subset \Upsilon_2 \subset \Upsilon_3 \subset \cdots \subset \Upsilon_\ell$$

such that

$$\Upsilon_1 \cap \Omega \neq \emptyset, \ \Omega \subset \Upsilon_\ell, \ \bar{\Omega} \cap \partial\Upsilon_\ell \neq \emptyset$$

and for $i = 2, \dots, \ell$

$$d_i = \operatorname{dist}\big(\bar{\Omega} \cap \partial\Upsilon_{i-1}, \bar{\Omega} \cap \partial\Upsilon_i\big) > 0. \tag{26}$$

Let us for $i = 1, \dots, \ell$ define

$$\Omega_i = \Upsilon_i \cap \Omega$$

and

$$\triangle\Omega_1 = \Omega_1, \quad \triangle\Omega_i = \Omega_i \backslash \bar{\Omega}_{i-1} \ \text{ for } \ i = 2, \dots, \ell.$$

An example of this kind of domain is depicted in the figure 1.

For $i = 2, \dots, \ell$ let us define the cutoff function φ_i such that $\varphi_i = 1$ in $\bar{\Omega}_{i-1}$, $\varphi_i = 0$ in $\overline{\Omega \backslash \Omega_i}$ and on $\triangle\Omega_i$, φ_i is defined by

$$\varphi_i(x) = \frac{\operatorname{dist}\big(x, \bar{\Omega} \cap \partial\Upsilon_i\big)}{\operatorname{dist}\big(x, \bar{\Omega} \cap \partial\Upsilon_{i-1}\big) + \operatorname{dist}\big(x, \bar{\Omega} \cap \partial\Upsilon_i\big)}. \tag{27}$$

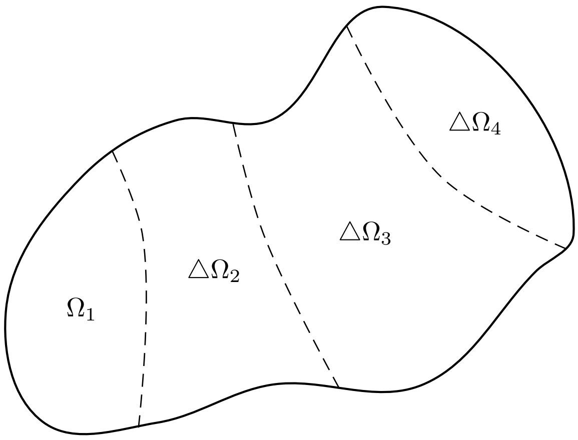


FIGURE 1. Ω

It is clear that $0 \le \varphi_i \le 1$. Let us estimate the gradient of φ_i on $\triangle\Omega_i$, for simplicity let us denote $A = \bar{\Omega} \cap \partial\Upsilon_{i-1}$ and $B = \bar{\Omega} \cap \partial\Upsilon_i$, then we may estimate

$$
\begin{aligned}
|D\varphi_i|_2 &= \frac{\big|\operatorname{dist}(x,A) D \operatorname{dist}(x,B) - \operatorname{dist}(x,B) D \operatorname{dist}(x,A)\big|_2}{\big(\operatorname{dist}(x,A) + \operatorname{dist}(x,B)\big)^2} \\
&\le \frac{\operatorname{dist}(x,A) |D \operatorname{dist}(x,B)|_2 + \operatorname{dist}(x,B) |D \operatorname{dist}(x,A)|_2}{\big(\operatorname{dist}(x,A) + \operatorname{dist}(x,B)\big)^2} \\
&\le \frac{1}{\operatorname{dist}(x,A) + \operatorname{dist}(x,B)} \le \frac{1}{d_i}.
\end{aligned}
$$

1.2. An Introductory Example. To introduce the iteration technique for general domains let us consider the Neumann problem

$$
\begin{cases} -\triangle u = f, & \text{in } \ \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \ \partial\Omega, \end{cases} \tag{28}
$$

where $f \in L^2(\Omega)$ and for $i = 1, \dots, \ell - 1$

$$
0 = \int_{\triangle\Omega_i} f. \tag{29}
$$

In the case of cylindrical domains in [**O96**] Oleinik has given the asymptotic behavior of this problem.

By testing the equation (28) with

$$
v(x) = \varphi_m(x)\big(u(x) - \frac{1}{|\triangle\Omega_m|} \int_{\triangle\Omega_m} u\big)
$$

we obtain

$$
\int_\Omega Du \cdot D\big(\varphi_m\big(u - \frac{1}{|\triangle\Omega_m|} \int_{\triangle\Omega_m} u\big)\big) = \int_\Omega f\varphi_m\big(u - \frac{1}{|\triangle\Omega_m|} \int_{\triangle\Omega_m} u\big)
$$

from here we have

$$\int_\Omega |Du|^2\varphi_m = -\int_{\triangle\Omega_m} Du\cdot D\varphi_m\big(u-\frac{1}{|\triangle\Omega_m|}\int_{\triangle\Omega_m} u\big) + \int_\Omega f\varphi_m\big(u-\frac{1}{|\triangle\Omega_m|}\int_{\triangle\Omega_m} u\big). \quad (30)$$

We may estimate

$$-\int_{\triangle\Omega_m} Du\cdot D\varphi_m\big(u-\frac{1}{|\triangle\Omega_m|}\int_{\triangle\Omega_m} u\big) \le \frac{c_m}{d_m}\int_{\triangle\Omega_m}|Du|^2 \quad (31)$$

here c_m is the Poincaré constant for the domain $\triangle\Omega_m$.

By (29) for $i=1,\dots,m-1$ we have

$$\int_{\triangle\Omega_i} f\big(u-\frac{1}{|\triangle\Omega_m|}\int_{\triangle\Omega_m} u\big) = \int_{\triangle\Omega_i} fu = \int_{\triangle\Omega_i} f\big(u-\frac{1}{|\triangle\Omega_i|}\int_{\triangle\Omega_i} u\big)$$

so we may estimate

$$\begin{aligned}
&\int_\Omega f\varphi_m\big(u-\frac{1}{|\triangle\Omega_m|}\int_{\triangle\Omega_m} u\big)\\
&\quad=\sum_{i=1}^{m-1}\int_{\triangle\Omega_i} f\big(u-\frac{1}{|\triangle\Omega_m|}\int_{\triangle\Omega_m} u\big)+\int_{\triangle\Omega_m} f\varphi_m\big(u-\frac{1}{|\triangle\Omega_m|}\int_{\triangle\Omega_m} u\big)\\
&\quad=\sum_{i=1}^{m-1}\int_{\triangle\Omega_i} f\big(u-\frac{1}{|\triangle\Omega_i|}\int_{\triangle\Omega_i} u\big)+\int_{\triangle\Omega_m} f\varphi_m\big(u-\frac{1}{|\triangle\Omega_m|}\int_{\triangle\Omega_m} u\big)\\
&\quad\le\sum_{i=1}^{m}\int_{\triangle\Omega_i} |f|\big|u-\frac{1}{|\triangle\Omega_i|}\int_{\triangle\Omega_i} u\big| \le \sum_{i=1}^{m}\frac{c_i^2}{2}\int_{\triangle\Omega_i} f^2+\frac{1}{2}\int_{\Omega_m}|Du|^2. \quad (32)
\end{aligned}$$

By (30),(31) and (32) we have the estimate

$$\int_\Omega |Du|^2\varphi_m \le \frac{c_m}{d_m}\int_{\triangle\Omega_m}|Du|^2+\sum_{i=1}^{m}\frac{c_i^2}{2}\int_{\triangle\Omega_i} f^2+\frac{1}{2}\int_{\Omega_m}|Du|^2$$

from here by observing that $\varphi_m\ge 0$ and $\varphi_m=1$ on Ω_{m-1} we obtain

$$\frac{1}{2}\int_{\Omega_{m-1}}|Du|^2 \le \Big(\frac{1}{2}+\frac{c_m}{d_m}\Big)\int_{\triangle\Omega_m}|Du|^2+\sum_{i=1}^{m}\frac{c_i^2}{2}\int_{\triangle\Omega_i} f^2.$$

So denoting

$$\gamma_m = 1+2\frac{c_m}{d_m}$$

we have

$$\int_{\Omega_{m-1}}|Du|^2 \le \gamma_m\int_{\triangle\Omega_m}|Du|^2+\sum_{i=1}^{m}c_i^2\int_{\triangle\Omega_i} f^2. \quad (33)$$

We may write this as

$$\int_{\Omega_{m-1}}|Du|^2 \le \frac{\gamma_m}{1+\gamma_m}\int_{\Omega_m}|Du|^2+\frac{1}{1+\gamma_m}\sum_{i=1}^{m}c_i^2\int_{\triangle\Omega_i} f^2,$$

iterating this inequality from $m = 2$ to $m = \ell$ we obtain the estimate

$$\int_{\Omega_1} |Du|^2 \le \Big\{\prod_{i=2}^{\ell} \frac{\gamma_i}{1+\gamma_i}\Big\} \int_{\Omega_\ell} |Du|^2 + \sum_{m=2}^{\ell} \Big\{\prod_{i=2}^{m} \frac{\gamma_i}{1+\gamma_i}\Big\} \frac{1}{\gamma_m} \sum_{j=1}^{m} c_j^2 \int_{\triangle\Omega_j} f^2. \quad (34)$$

To see that how this estimate gives some information on the asymptotic behavior let us consider the special case when $f = 0$ in $\Omega_{\ell-1}$ and for some $d, c > 0$, $d_i \ge d$ and $c_i \le c$, then we have the estimate

$$\gamma_m \le \gamma = 1 + 2\frac{c}{d}$$

and by the estimate (34) we obtain

$$\int_{\Omega_1} |Du|^2 \le \Big(\frac{\gamma}{1+\gamma}\Big)^{\ell-1} \int_{\Omega_\ell} |Du|^2 + c^2 \Big(\frac{\gamma}{1+\gamma}\Big)^{\ell-1} \int_{\triangle\Omega_\ell} f^2. \quad (35)$$

By testing the equation (28) by u we obtain the inequality

$$\int_{\Omega} |Du|^2 \le \int_{\Omega} f^2 = \int_{\triangle\Omega_\ell} f^2$$

and by this inequality and the inequality (35) we obtain

$$\int_{\Omega_1} |Du|^2 \le \big(1 + c^2\big) \Big(\frac{\gamma}{1+\gamma}\Big)^{\ell-1} \int_{\triangle\Omega_\ell} f^2.$$

Denoting $\alpha = \ln(1 + \frac{1}{\gamma})$ then for some constant $C > 0$ we have

$$\|Du\|^2_{L^2(\Omega_1)} \le C e^{-\alpha\ell} \|f\|^2_{L^2(\triangle\Omega_\ell)}$$

this inequality shows that for example if the norm of f grows polynomially as ℓ grows then the derivatives of u in Ω_1 converges to zero exponentially fast.

2. Local Error Estimates

Let us consider the bounded open polygonal domain $\Omega \subset \mathbb{R}^d$ and assume that one is interested in the solution only in a polygonal sub-domain $\Omega' \subset \Omega$. To compute the solution by finite element method, it is natural to have the finest refinement of the triangulation in the domain Ω' and gradually the triangulation to get coarse away from this region of interest. We will prove error estimates that suggest this coarsening to obtain less error with the same number of elements.

Let us consider the definitions in subsection 1.1 and in addition let us consider the condition that for $i = 1, \dots, \ell$, Ω_i is a polygonal domain and $\Omega' = \Omega_1$.

Now let us consider a triangulation $\mathcal{T}$ of the domain Ω consisting of simplices, which is compatible with the domains Ω_i in the sense that each Ω_i is a union of simplices in this triangulation.

Let us consider the affine linear P_1 finite element method and denote by V the corresponding finite element space.

Let the triangulation $\mathcal{T}$ be regular in the usual sense that for some fixed $\sigma > 0$

$$\frac{h_T}{\rho_T} \le \sigma \quad \text{for all } T \in \mathcal{T}$$

where h_T is the diameter of the simplex T and ρ_T is the diameter of the largest ball contained in T.

Let us denote

$$\overline{h}_i = \max_{T \in \triangle\Omega_i} h_T, \quad \underline{h}_i = \min_{T \in \triangle\Omega_i} h_T$$

and the triangulation $\mathcal{T}$ be also regular in the sense that for some fixed $\tilde{\sigma} > 0$

$$\overline{h}_i \leq \tilde{\sigma} \underline{h}_i, \quad i = 1, \dots, \ell.$$

Let us denote by $\hat{T}$ the reference simplex

$$\hat{T} = \Big\{ (x_1, \dots, x_d) \in \mathbb{R}^d \;\Big|\; 0 \leq x_i, \ \forall i = 1, \dots, d \ \text{ and } \ \sum_{i=1}^{d} x_i \leq 1 \Big\}$$

and by s_i for $i = 1, \dots, d+1$ the linear shape functions defined on the reference simplex $\hat{T}$.

For each $T \in \mathcal{T}$ let us denote by ϕ the affine function mapping $\hat{T}$ to T.

Let us denote by $\{x_1, \dots, x_M\}$ the set of all vertices in the triangulation $\mathcal{T}$. Let us denote by v_i the basis function which has the value 1 at x_i.

In the following lemma we prove two inequalities and one equality which will be used in lemma 2.

Lemma 1. *There exists constants $C_1, C_2 > 0$ which are independent of the triangulation and a constant C_3 which only depends on σ, such that*

$$|\gamma|_2^2 \int_T \sum_{x_i \in T} v_i^2 \leq C_1 \int_T \big\{ \sum_{x_i \in T} \gamma_i v_i \big\}^2, \ \forall \gamma \in \mathbb{R}^{d+1}, \forall T \in \mathcal{T} \tag{36}$$

$$|T| = C_2 \int_T \sum_{x_i \in T} v_i^2, \quad \forall T \in \mathcal{T} \tag{37}$$

and for all $T \in \mathcal{T}$ we have

$$|Dv_i(x)|_2 \leq \frac{C_3}{h_T}, \quad \forall x_i, x \in T. \tag{38}$$

Proof. First we show the inequality (36), we may compute

$$\begin{aligned} \int_T \big\{ \sum_{x_i \in T} \gamma_i v_i \big\}^2 &= \int_T \big\{ \sum_{n=1}^{d+1} \gamma_{i_n} s_n(\phi^{-1}(x)) \big\}^2 dx \\ &= \int_{\hat{T}} \big\{ \sum_{n=1}^{d+1} \gamma_{i_n} s_n(y) \big\}^2 d\phi(y) = |\det D\phi| \int_{\hat{T}} \big\{ \sum_{n=1}^{d+1} \gamma_{i_n} s_n(y) \big\}^2 dy \end{aligned} \tag{39}$$

here i_n is a surjective mapping from $\{1, \dots, d+1\}$ to $\{1, \dots, M\}$. We have that the functions $s_i \in L^2(\hat{T})$ are independent, hence there exists a constant $\mu > 0$ such that

$$\mu |\gamma|_2^2 \leq \int_{\hat{T}} \big\{ \sum_{n=1}^{d+1} \gamma_{i_n} s_n(y) \big\}^2 dy \tag{40}$$

and we have

$$\begin{aligned} \int_T \sum_{x_i \in T} v_i^2 &= \int_T \sum_{n=1}^{d+1} s_n^2(\phi^{-1}(x)) dx \\ &= \int_{\hat{T}} \sum_{n=1}^{d+1} s_n^2(y) d\phi(y) = |\det D\phi| \int_{\hat{T}} \sum_{n=1}^{d+1} s_n^2(y) dy = C |\det D\phi|. \end{aligned} \tag{41}$$

From the (39),(40) and (41) we obtain

$$|\gamma|_2^2 \int_T \sum_{x_i \in T} v_i^2 \leq \frac{C}{\mu} \int_T \{ \sum_{x_i \in T} \gamma_i v_i \}^2$$

so the inequality (36) holds with the constant $C_1 = \frac{C}{\mu}$.

Now let us show the equality (37). By (41) we may compute

$$\int_T \sum_{x_i \in T} v_i^2 = C |\det D\phi| = C |\det D\phi| \frac{|T|}{\int_T 1 dx} = C |\det D\phi| \frac{|T|}{\int_{\hat{T}} 1 d\phi(y)} = \frac{C}{|\hat{T}|} |T|$$

so the equality (37) holds with $C_2 = \frac{|\hat{T}|}{C}$.

Now let us show the inequality (38).

For $x_i, x \in T$ we may compute

$$\begin{aligned} |Dv_i(x)|_2 &= |D_x(s_{n_i}(\phi^{-1}(x)))|_2 = |D_y s_{n_i}(\phi^{-1}(x)) D_x \phi^{-1}(x)|_2 \\ &= |D_x \phi^{-1}(x)^T D_y s_{n_i}(\phi^{-1}(x))^T|_2 \leq |D_x \phi^{-1}(x)|_{2,2} |D_y s_{n_i}|_2 \quad (42) \end{aligned}$$

here n_i is in $\{1, \ldots, d+1\}$ and depends on i, and $| \cdot |_{2,2}$ is the matrix norm corresponding to euclidean norm of vectors.

To estimate the norm of $D\phi^{-1}$ let us denote by x_0 the center of the largest inscribed sphere in T. Then let us consider the linear function

$$\nu(x) = \phi^{-1}(x + x_0) - \phi^{-1}(x_0).$$

which maps $T - x_0$ onto $\hat{T} - \phi^{-1}(x_0)$.

Because ν is linear its derivative is constant and equal to $D\phi^{-1}$. Taking any $x \in \mathbb{R}^d$ we have $\frac{\rho_T}{2} \frac{x}{|x|_2} \in T - x_0$ so we have

$$\begin{aligned} |D\phi^{-1} x|_2 = \frac{2|x|_2}{\rho_T} \big|\nu\big(\frac{\rho_T}{2} \frac{x}{|x|_2}\big)\big|_2 &= \frac{2|x|_2}{\rho_T} \big|\phi^{-1}\big(\frac{\rho_T}{2} \frac{x}{|x|_2} + x_0\big) - \phi^{-1}(x_0)\big|_2 \\ &\leq \frac{2|x|_2}{\rho_T} \operatorname{diam}(\hat{T}) \end{aligned}$$

so we have the estimate

$$|D\phi^{-1}|_{2,2} \leq \frac{2 \operatorname{diam}(\hat{T})}{\rho_T}. \quad (43)$$

Let us show that $\operatorname{diam}(\hat{T}) \leq \sqrt{2}$. Let $x, y \in \hat{T}$ then

$$|x - y|_2^2 = \sum_{i=1}^{d} |x_i - y_i|^2$$

and because $0 \leq x_i, y_i \leq 1$ we have $|x_i - y_i| \leq 1$ hence

$$|x - y|_2^2 \leq \sum_{i=1}^{d} |x_i - y_i| \leq \sum_{i=1}^{d} (x_i + y_i) \leq 2$$

hence we have the estimate

$$\operatorname{diam}(\hat{T}) \leq \sqrt{2}. \quad (44)$$

By the estimates (43) and (44) we obtain

$$|D\phi^{-1}|_{2,2} \leq \frac{2\sqrt{2}}{\rho_T}. \quad (45)$$

One may also obtain the estimate

$$|D_y s_n|_2 \leq \sqrt{d}. \tag{46}$$

Hence by the inequalities (42), (45) and (46) we have the estimate

$$|Dv_i(x)|_2 \leq 2\sqrt{2d}\frac{1}{\rho_T} = 2\sqrt{2d}\frac{h_T}{\rho_T}\frac{1}{h_T} \leq 2\sqrt{2d}\frac{\sigma}{h_T}$$

by this the estimate (38) holds with $C_3 = 2\sqrt{2d}\sigma$. □

Let us consider the interpolation operator $I : C^0(\overline{\Omega}) \to V$ defined for each $w \in C^0(\overline{\Omega})$ as

$$(I(w))(x) = \sum_{x_i \in T} w(x_i)v_i(x), \quad \forall x \in T.$$

LEMMA 2. *For all $w \in V$ and $m = 2, \dots, \ell$ the following inequality holds*

$$\|I(\varphi_m w)\|_{H^1(\triangle\Omega_m)} \leq \frac{c}{\min(1, d_m)}\|w\|_{H^1(\triangle\Omega_m)} \tag{47}$$

where φ_m is defined by (27) and $c > 0$ depends on σ.

PROOF. Consider $w \in V$, so we have

$$w = \sum_{i=1}^{N} w(x_i)v_i, \quad I(\varphi_m w) = \sum_{i=1}^{N} w(x_i)\varphi_m(x_i)v_i.$$

Then we may compute

$$\begin{aligned}
&\|I(\varphi_m w)\|^2_{H^1(\triangle\Omega_m)} \\
&\quad = \sum_{T\subset\overline{\triangle\Omega_m}} \int_T \Big\{ \Big| \sum_{x_i\in T} w(x_i)\varphi_m(x_i)Dv_i \Big|^2 + \Big(\sum_{x_i\in T} w(x_i)\varphi_m(x_i)v_i \Big)^2 \Big\} \\
&\quad = \sum_{T\subset\overline{\triangle\Omega_m}} \int_T \Big\{ \Big| \sum_{x_i\in T} w(x_i)(\varphi_m(x_i) - c_T)Dv_i + c_T \sum_{x_i\in T} w(x_i)Dv_i \Big|^2 \\
&\qquad\qquad + \Big(\sum_{x_i\in T} w(x_i)\varphi_m(x_i)v_i \Big)^2 \Big\}
\end{aligned}$$

here $c_T > 0$ is a constant depending on the triangle T to be chosen later.

We may continue the computation

$$
\begin{aligned}
&\|I(\varphi_m w)\|^2_{H^1(\triangle\Omega_m)} \\
&\quad\le \sum_{T\subset\overline{\triangle\Omega_m}} \int_T \Big\{ 2\Big|\sum_{x_i\in T} w(x_i)(\varphi_m(x_i)-c_T)Dv_i\Big|^2 + 2|c_T|^2\Big|\sum_{x_i\in T} w(x_i)Dv_i\Big|^2 \\
&\qquad\qquad + \Big(\sum_{x_i\in T} w(x_i)\varphi_m(x_i)v_i\Big)^2\Big\} \\
&\qquad \le \sum_{T\subset\overline{\triangle\Omega_m}} \int_T \Big\{ 2\Big\{\sum_{x_i\in T} w^2(x_i)\Big\}\Big\{\sum_{x_i\in T}\big|(\varphi_m(x_i)-c_T)Dv_i\big|^2\Big\} \\
&\qquad\qquad + 2|c_T|^2\Big|\sum_{x_i\in T} w(x_i)Dv_i\Big|^2 \\
&\qquad\qquad\quad + \Big\{\max_{x_i\in T}\varphi_m^2(x_i)\Big\}\Big\{\sum_{x_i\in T} w^2(x_i)\Big\}\Big\{\sum_{x_i\in T} v_i^2\Big\}\Big\}.
\end{aligned}
$$

Considering that Dv_i is constant on T and $\varphi_m^2 \le 1$ using lemma 1 we obtain

$$
\begin{aligned}
&\|I(\varphi_m w)\|^2_{H^1(\triangle\Omega_m)} \\
&\qquad \le \sum_{T\subset\overline{\triangle\Omega_m}} \Big\{ 2|T|\Big\{\sum_{x_i\in T} w^2(x_i)\Big\}\Big\{\sum_{x_i\in T}\big|(\varphi_m(x_i)-c_T)Dv_i\big|^2\Big\} \\
&\qquad + \int_T \Big\{2|c_T|^2\Big|\sum_{x_i\in T} w(x_i)Dv_i\Big|^2 + \Big\{\sum_{x_i\in T} w^2(x_i)\Big\}\Big\{\sum_{x_i\in T} v_i^2\Big\}\Big\}\Big\} \\
&\quad\le \sum_{T\subset\overline{\triangle\Omega_m}} \int_T \Big\{ 2C_2\Big\{\sum_{x_i\in T} w^2(x_i)\Big\}\Big\{\sum_{x_i\in T}\big|(\varphi_m(x_i)-c_T)Dv_i\big|^2\Big\}\Big\{\sum_{x_i\in T} v_i^2\Big\} \\
&\qquad\qquad + 2|c_T|^2\Big|\sum_{x_i\in T} w(x_i)Dv_i\Big|^2 + C_1\Big(\sum_{x_i\in T} w(x_i)v_i\Big)^2\Big\} \\
&\quad\le \sum_{T\subset\overline{\triangle\Omega_m}} \int_T \Big\{ 2C_1C_2\Big\{\sum_{x_i\in T}\big|(\varphi_m(x_i)-c_T)Dv_i\big|^2\Big\}\Big(\sum_{x_i\in T} w(x_i)v_i\Big)^2 \\
&\qquad\qquad + 2|c_T|^2\Big|\sum_{x_i\in T} w(x_i)Dv_i\Big|^2 + C_1\Big(\sum_{x_i\in T} w(x_i)v_i\Big)^2\Big\} \\
&\quad= \sum_{T\subset\overline{\triangle\Omega_m}} \int_T \Big\{\Big\{ C_1 + 2C_1C_2\Big\{\sum_{x_i\in T}\big|(\varphi_m(x_i)-c_T)Dv_i\big|^2\Big\}\Big\}\Big(\sum_{x_i\in T} w(x_i)v_i\Big)^2 \\
&\qquad\qquad\qquad + 2|c_T|^2\Big|\sum_{x_i\in T} w(x_i)Dv_i\Big|^2\Big\}.
\end{aligned}
$$

Now we take $c_T = \varphi_m(x_j)$ for some $x_j \in T$ and we may estimate

$$
\begin{aligned}
&\sum_{x_i\in T}\big|(\varphi_m(x_i)-c_T)Dv_i\big|^2 \le \Big\{\max_{x_i\in T}|Dv_i|^2\Big\}\sum_{x_i\in T}(\varphi_m(x_i)-\varphi_m(x_j))^2 \\
&\qquad \le \|D\varphi_m\|^2_{L^\infty(T)}\Big\{\max_{x_i\in T}|Dv_i|^2\Big\}\sum_{x_i\in T}|x_i-x_j|^2 \le d\frac{1}{d_m^2}\frac{C_3^2}{h_T^2}h_T^2 = \frac{dC_3^2}{d_m^2}
\end{aligned}
$$

so we have

$$\begin{aligned}\|I(\varphi_m w)\|^2_{H^1(\triangle\Omega_m)} \leq \sum_{T\subset\overline{\triangle\Omega}_m} \int_T \Big\{ \Big\{ C_1 + 2dC_1C_2C_3^2 \frac{1}{d_m^2} \Big\} \Big(\sum_{x_i\in T} w(x_i)v_i \Big)^2 \\ + 2\Big| \sum_{x_i\in T} w(x_i) Dv_i \Big|^2 \Big\} \\ = \sum_{T\subset\overline{\triangle\Omega}_m} \int_T \Big\{ \Big\{ C_1 + 2dC_1C_2C_3^2 \frac{1}{d_m^2} \Big\} w^2 + 2|Dw|^2 \Big\} \\ \leq \max\Big\{ 2, C_1 + 2dC_1C_2C_3^2 \frac{1}{d_m^2} \Big\} \|w\|^2_{H^1(\triangle\Omega_m)}. \quad (48)\end{aligned}$$

We may estimate

$$\begin{aligned}\max\Big\{ 2, C_1 + 2dC_1C_2C_3^2 \frac{1}{d_m^2} \Big\} \leq \max\Big\{ 2, C_1 + 2dC_1C_2C_3^2 \Big\} \max\big(1, \frac{1}{d_m^2}\big) \\ = \Big(\frac{c}{\min(1, d_m)} \Big)^2 \quad (49)\end{aligned}$$

where

$$c = \sqrt{\max\{2, C_1 + 2dC_1C_2C_3^2\}}.$$

The estimate (48) together with (49) prove the lemma. □

3. Neumann Problem

Let us consider in the bounded open polygonal domain Ω the problem

$$\begin{cases} -\operatorname{div} A(x,u,Du) + b(x,u,Du) = f & \text{in } \Omega, \\ A(x,u,Du)\cdot\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\nu = \nu(x)$ is the outer normal to the domain Ω at $x \in \partial\Omega$. As usual

$$A : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d,$$
$$b : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$$

such that there exists positive numbers $0 < \lambda \leq \Lambda$ that for all $p_1, p_2 \in \mathbb{R}^d$, $z_1, z_2 \in \mathbb{R}$ and a.e. $x \in \Omega$

$$\begin{aligned}\lambda\big(|p_2 - p_1|_2^2 + |z_2 - z_1|^2\big) \leq \big(A(x,z_2,p_2) - A(x,z_1,p_1)\big)\cdot(p_2 - p_1) \\ + \big(b(x,z_2,p_2) - b(x,z_1,p_1)\big)(z_2 - z_1) \quad (50)\end{aligned}$$

and

$$\begin{aligned}|A(x,z_2,p_2) - A(x,z_1,p_1)|_2^2 + \big(b(x,z_2,p_2) - b(x,z_1,p_1)\big)^2 \\ \leq \Lambda^2\big(|p_2 - p_1|_2^2 + (z_2 - z_1)^2\big) \quad (51)\end{aligned}$$

and for fixed z and p, $A(x,z,p)$ and $b(x,z,p)$ are measurable in x.

The weak formulation of our problem is

$$\begin{cases} \displaystyle\int_\Omega A(x,u,Du)\cdot Dv + b(x,u,Du)v = \langle f, v\rangle, \quad \forall v \in H^1(\Omega) \\ u \in H^1(\Omega) \end{cases} \quad (52)$$

where $f \in (H^1(\Omega))^*$.

Let us now consider the finite element solution of the equation (52) in V

$$\begin{cases} \int_\Omega A(x,\hat{u},D\hat{u})\cdot Dv + b(x,\hat{u},D\hat{u})v = \langle f,v\rangle\,, \quad \forall v\in V \\ \hat{u}\in V. \end{cases} \tag{53}$$

In the following theorem we perform iterations to obtain a local error estimate. For ease of notation let us define

$$\beta_i = \frac{c}{\min(1,d_i)}, \tag{54}$$

$$\alpha_i = \frac{\sqrt{2}\beta_i}{\frac{\lambda}{2\Lambda}+\sqrt{2}\beta_i} \tag{55}$$

and

$$\nu_m = \prod_{i=1}^{m}\alpha_i. \tag{56}$$

THEOREM 2. *Let u be the solution of* (52) *and $\hat{u}$ the solution of* (53) *then for any $w\in V$ we have the estimate*

$$\|u-\hat{u}\|^2_{H^1(\Omega_1)} \le \frac{C}{\nu_1}\sum_{i=1}^{\ell}\Big\{\sum_{m=i}^{\ell}\nu_m\Big\}\|u-w\|^2_{H^1(\triangle\Omega_i)}. \tag{57}$$

PROOF. For any $v\in V$ subtracting the equation (53) from (52) we obtain

$$\int_\Omega \big(A(x,u,Du)-A(x,\hat{u},D\hat{u})\big)\cdot Dv + \big(b(x,u,Du)-b(x,\hat{u},D\hat{u})\big)v = 0. \tag{58}$$

Now for any $w\in V$ we may take

$$v = I(\varphi_{m+1}(w-\hat{u})).$$

We may decompose the gradient of v as

$$Dv = \Big\{\chi_{\Omega_m}Du - \chi_{\Omega_m}D\hat{u}\Big\} + \Big\{\chi_{\Omega_m}Dw - \chi_{\Omega_m}Du + \chi_{\triangle\Omega_{m+1}}DI(\varphi_{m+1}(w-\hat{u}))\Big\}$$

and similarly for the value of the function v

$$v = \Big\{\chi_{\Omega_m}u - \chi_{\Omega_m}\hat{u}\Big\} + \Big\{\chi_{\Omega_m}w - \chi_{\Omega_m}u + \chi_{\triangle\Omega_{m+1}}I(\varphi_{m+1}(w-\hat{u}))\Big\}.$$

Substituting this in the equation (58) we have

$$\begin{aligned} &\int_{\Omega_m}\big(A(x,u,Du)-A(x,\hat{u},D\hat{u})\big)\cdot D(u-\hat{u}) + \big(b(x,u,Du)-b(x,\hat{u},D\hat{u})\big)(u-\hat{u}) \\ &= -\int_{\Omega_m}\big(A(x,u,Du)-A(x,\hat{u},D\hat{u})\big)\cdot D(w-u) + \big(b(x,u,Du)-b(x,\hat{u},D\hat{u})\big)(w-u) \\ &\qquad - \int_{\triangle\Omega_{m+1}}\Big\{\big(A(x,u,Du)-A(x,\hat{u},D\hat{u})\big)\cdot DI(\varphi_{m+1}(w-\hat{u})) \\ &\qquad\qquad + \big(b(x,u,Du)-b(x,\hat{u},D\hat{u})\big)I(\varphi_{m+1}(w-\hat{u}))\Big\}. \end{aligned}$$

Then by the inequalities (50), (51) and Young's inequality, for some $\eta, \epsilon > 0$, we deduce

$$\lambda \int_{\Omega_m} |D(u-\hat{u})|_2^2 + (u-\hat{u})^2$$
$$\leq \Lambda \int_{\Omega_m} \frac{\eta}{2}\big(|D(u-\hat{u})|_2^2 + (u-\hat{u})^2\big) + \frac{1}{2\eta}\big(|D(w-u)|_2^2 + (w-u)^2\big)$$
$$+ \Lambda \int_{\triangle\Omega_{m+1}} \Big\{ \frac{\epsilon}{2}\big(|D(u-\hat{u})|_2^2 + (u-\hat{u})^2\big)$$
$$+ \frac{1}{2\epsilon}\big(|DI(\varphi_{m+1}(w-\hat{u}))|_2^2 + (I(\varphi_{m+1}(w-\hat{u})))^2\big)\Big\}$$

so we have

$$(\frac{\lambda}{\Lambda} - \frac{\eta}{2})\|u-\hat{u}\|^2_{H^1(\Omega_m)} \leq \frac{\epsilon}{2}\|u-\hat{u}\|^2_{H^1(\triangle\Omega_{m+1})}$$
$$+ \frac{1}{2\eta}\|w-u\|^2_{H^1(\Omega_m)} + \frac{1}{2\epsilon}\|I(\rho_{m+1}(w-\hat{u}))\|^2_{H^1(\triangle\Omega_{m+1})}$$
$$\leq \frac{\epsilon}{2}\|u-\hat{u}\|^2_{H^1(\triangle\Omega_{m+1})} + \frac{1}{2\eta}\|w-u\|^2_{H^1(\Omega_m)} + \frac{\beta_{m+1}^2}{2\epsilon}\|w-\hat{u}\|^2_{H^1(\triangle\Omega_{m+1})}$$
$$\leq \Big(\frac{\epsilon}{2} + \frac{\beta_{m+1}^2}{\epsilon}\Big)\|u-\hat{u}\|^2_{H^1(\triangle\Omega_{m+1})} + \frac{1}{2\eta}\|w-u\|^2_{H^1(\Omega_m)}$$
$$+ \frac{\beta_{m+1}^2}{\epsilon}\|w-u\|^2_{H^1(\triangle\Omega_{m+1})}.$$

Considering that

$$\|u-\hat{u}\|^2_{H^1(\triangle\Omega_{m+1})} = \|u-\hat{u}\|^2_{H^1(\Omega_{m+1})} - \|u-\hat{u}\|^2_{H^1(\Omega_m)}$$

and taking the value $\eta = \frac{\lambda}{\Lambda}$ we have

$$\Big(\frac{\lambda}{2\Lambda} + \big(\frac{\epsilon}{2} + \frac{\beta_{m+1}^2}{\epsilon}\big)\Big)\|u-\hat{u}\|^2_{H^1(\Omega_m)}$$
$$\leq \big(\frac{\epsilon}{2} + \frac{\beta_{m+1}^2}{\epsilon}\big)\|u-\hat{u}\|^2_{H^1(\Omega_{m+1})} + \frac{\Lambda}{2\lambda}\|w-u\|^2_{H^1(\Omega_m)}$$
$$+ \frac{\beta_{m+1}^2}{\epsilon}\|w-u\|^2_{H^1(\triangle\Omega_{m+1})}.$$

The value $\epsilon = \sqrt{2}\beta_{m+1}$ minimizes the quantity

$$\frac{\frac{\epsilon}{2} + \frac{\beta_{m+1}^2}{\epsilon}}{\frac{\lambda}{2\Lambda} + \big(\frac{\epsilon}{2} + \frac{\beta_{m+1}^2}{\epsilon}\big)} = \alpha_{m+1}$$

so we have

$$\|u-\hat{u}\|^2_{H^1(\Omega_m)} \leq \alpha_{m+1}\|u-\hat{u}\|^2_{H^1(\Omega_{m+1})}$$
$$+ \frac{1}{2\sqrt{2}}\frac{\Lambda}{\lambda}\frac{\alpha_{m+1}}{\beta_{m+1}}\|w-u\|^2_{H^1(\Omega_m)} + \frac{1}{2}\alpha_{m+1}\|w-u\|^2_{H^1(\triangle\Omega_{m+1})}.$$

Multiplying with ν_m we obtain

$$\nu_m \|u-\hat{u}\|^2_{H^1(\Omega_m)} \leq \nu_{m+1}\|u-\hat{u}\|^2_{H^1(\Omega_{m+1})} + \frac{1}{2\sqrt{2}}\frac{\Lambda}{\lambda}\frac{\nu_{m+1}}{\beta_{m+1}}\|w-u\|^2_{H^1(\Omega_m)} + \frac{1}{2}\nu_{m+1}\|w-u\|^2_{H^1(\triangle\Omega_{m+1})}$$

iterating this inequality for $m = 1, \ldots, \ell-1$ we obtain

$$\begin{aligned}
\nu_1\|u-\hat{u}\|^2_{H^1(\Omega_1)} &\leq \nu_\ell\|u-\hat{u}\|^2_{H^1(\Omega_\ell)} \\
&+ \frac{1}{2\sqrt{2}}\frac{\Lambda}{\lambda}\sum_{m=1}^{\ell-1}\frac{\nu_{m+1}}{\beta_{m+1}}\|w-u\|^2_{H^1(\Omega_m)} + \frac{1}{2}\sum_{m=1}^{\ell-1}\nu_{m+1}\|w-u\|^2_{H^1(\triangle\Omega_{m+1})} \\
&= \nu_\ell\|u-\hat{u}\|^2_{H^1(\Omega_\ell)} \\
&+ \frac{1}{2\sqrt{2}}\frac{\Lambda}{\lambda}\sum_{m=1}^{\ell-1}\frac{\nu_{m+1}}{\beta_{m+1}}\sum_{i=1}^{m}\|w-u\|^2_{H^1(\triangle\Omega_i)} + \frac{1}{2}\sum_{m=1}^{\ell-1}\nu_{m+1}\|w-u\|^2_{H^1(\triangle\Omega_{m+1})} \\
&= \nu_\ell\|u-\hat{u}\|^2_{H^1(\Omega_\ell)} \\
&+ \frac{1}{2\sqrt{2}}\frac{\Lambda}{\lambda}\sum_{i=1}^{\ell-1}\{\sum_{m=i+1}^{\ell}\frac{\nu_m}{\beta_m}\}\|w-u\|^2_{H^1(\triangle\Omega_i)} + \frac{1}{2}\sum_{i=2}^{\ell}\nu_i\|w-u\|^2_{H^1(\triangle\Omega_i)} \\
&\leq \nu_\ell\|u-\hat{u}\|^2_{H^1(\Omega_\ell)} + \sum_{i=1}^{\ell}\Big(\frac{1}{2}\nu_i + \frac{1}{2\sqrt{2}}\frac{\Lambda}{\lambda}\sum_{m=i+1}^{\ell}\frac{\nu_m}{\beta_m}\Big)\|w-u\|^2_{H^1(\triangle\Omega_i)}.
\end{aligned}$$

Because

$$\frac{1}{\beta_m} = \frac{1}{c}\min(1, d_m) \leq \frac{1}{c}$$

we may estimate

$$\frac{1}{2}\nu_i + \frac{1}{2\sqrt{2}}\frac{\Lambda}{\lambda}\sum_{m=i+1}^{\ell}\frac{\nu_m}{\beta_m} \leq \max(\frac{1}{2}, \frac{1}{2\sqrt{2}}\frac{\Lambda}{\lambda}\frac{1}{c})\sum_{m=i}^{\ell}\nu_m$$

so we have

$$\nu_1\|u-\hat{u}\|^2_{H^1(\Omega_1)} \leq \nu_\ell\|u-\hat{u}\|^2_{H^1(\Omega)} + C\sum_{i=1}^{\ell}\Big\{\sum_{m=i}^{\ell}\nu_m\Big\}\|u-w\|^2_{H^1(\triangle\Omega_i)}. \tag{59}$$

By Cea's lemma we have

$$\|u-\hat{u}\|^2_{H^1(\Omega)} \leq \Big(\frac{\Lambda}{\lambda}\Big)^2\|u-w\|^2_{H^1(\Omega)} = \Big(\frac{\Lambda}{\lambda}\Big)^2\sum_{i=1}^{\ell}\|u-w\|^2_{H^1(\triangle\Omega_i)}. \tag{60}$$

Now by (59) and (60) we obtain

$$\begin{aligned}
&\nu_1\|u-\hat{u}\|^2_{H^1(\Omega_1)}\\
&\quad\le \Big(\frac{\Lambda}{\lambda}\Big)^2\nu_\ell\sum_{i=1}^{\ell}\|u-w\|^2_{H^1(\triangle\Omega_i)}+C\sum_{i=1}^{\ell}\Big\{\sum_{m=i}^{\ell}\nu_m\Big\}\|u-w\|^2_{H^1(\triangle\Omega_i)}\\
&\qquad= C\sum_{i=1}^{\ell}\Big\{\Big(\frac{\Lambda}{\lambda}\Big)^2\nu_\ell+\sum_{m=i}^{\ell}\nu_m\Big\}\|u-w\|^2_{H^1(\triangle\Omega_i)}\\
&\qquad\qquad\le C\sum_{i=1}^{\ell}\Big\{\sum_{m=i}^{\ell}\nu_m\Big\}\|u-w\|^2_{H^1(\triangle\Omega_i)} \qquad (61)
\end{aligned}$$

which proves the desired inequality. □

4. Dirichlet Problem

Let us consider in the bounded open polygonal domain Ω the problem

$$\begin{cases} -\operatorname{div} A(x,Du)=f \quad \text{in } \Omega,\\ u=0 \quad \text{on } \partial\Omega,\end{cases}$$

where as usual

$$A:\Omega\times\mathbb{R}^d\to\mathbb{R}^d$$

is such that there exist positive numbers $0<\lambda\le\Lambda$ that for all $p_1,p_2\in\mathbb{R}^d$ and a.e. $x\in\Omega$

$$\lambda|p_2-p_1|_2^2\le\big(A(x,p_2)-A(x,p_1)\big)\cdot(p_2-p_1) \qquad (62)$$

and

$$|A(x,p_2)-A(x,p_1)|_2\le\Lambda|p_2-p_1|_2 \qquad (63)$$

and for fixed p, $A(x,p)$ is measurable in x. The weak formulation of our problem is

$$\begin{cases} \displaystyle\int_\Omega A(x,Du)\cdot Dv=\langle f,v\rangle\,, \quad \forall v\in H^1_0(\Omega)\\ u\in H^1_0(\Omega)\end{cases} \qquad (64)$$

where $f\in H^{-1}(\Omega)$.

Let us denote by V_0 the subspace of the finite dimensional space V where the coefficient of the basis functions whose central vertex is on the boundary of the domain are 0.

Let us consider the finite element solution in V_0

$$\begin{cases} \displaystyle\int_\Omega A(x,D\hat{u})\cdot Dv=\langle f,v\rangle\,, \quad \forall v\in V_0\\ \hat{u}\in V_0.\end{cases} \qquad (65)$$

In the following theorem as before by iterations we obtain a local error estimate.

Let us impose the condition that each component of the domain $\triangle\Omega_i$ contains at least one face on the boundary of the domain Ω. In this case we will have the Poincaré inequalities

$$\int_{\triangle\Omega_i}v^2\le\gamma_i^2\int_{\triangle\Omega_i}|Dv|_2^2.$$

Again we use the notation (54) and (56) and define

$$\alpha_i = \frac{\sqrt{2(1+\gamma_i^2)}\beta_i}{\frac{\lambda}{2\Lambda} + \sqrt{2(1+\gamma_i^2)}\beta_i}. \tag{66}$$

THEOREM 3. *Let u be the solution of (64) and $\hat{u}$ be the solution of (65) then for any $w \in V_0$ we have the estimate*

$$\||D(u-\hat{u})|_2\|^2_{L^2(\Omega_1)} \leq \frac{C}{\nu_1} \sum_{i=1}^{\ell} \Big\{ \sum_{m=i}^{\ell} \nu_m \Big\} \||D(u-w)|_2\|^2_{L^2(\triangle\Omega_i)}. \tag{67}$$

PROOF. For any $v \in V_0$ subtracting the equation (65) from (64) we obtain

$$\int_\Omega \big(A(x, Du) - A(x, D\hat{u})\big) \cdot Dv = 0. \tag{68}$$

Now for any $w \in V_0$ we may take

$$v = I(\varphi_{m+1}(w - \hat{u})).$$

We may decompose the gradient as follows

$$Dv = \Big\{\chi_{\Omega_m} Du - \chi_{\Omega_m} D\hat{u}\Big\} + \Big\{\chi_{\Omega_m} Dw - \chi_{\Omega_m} Du + \chi_{\triangle\Omega_{m+1}} DI(\varphi_{m+1}(w - \hat{u}))\Big\}.$$

Substituting this in the equation (68) we have

$$\begin{aligned}\int_{\Omega_m} \big(A(x, Du) - A(x, D\hat{u})\big) \cdot D(u - \hat{u}) = &- \int_{\Omega_m} \big(A(x, Du) - A(x, D\hat{u})\big) \cdot D(w - u) \\ &- \int_{\triangle\Omega_{m+1}} \big(A(x, Du) - A(x, D\hat{u})\big) \cdot DI(\varphi_{m+1}(w - \hat{u})).\end{aligned}$$

From here by the inequalities (62), (63) and Young's inequality for some $\eta, \epsilon > 0$ we have

$$\begin{aligned}\lambda \int_{\Omega_m} |D(u-\hat{u})|_2^2 \leq\ & \Lambda \int_{\Omega_m} \frac{\eta}{2} |D(u-\hat{u})|_2^2 + \frac{1}{2\eta} |D(w-u)|_2^2 \\ &+ \Lambda \int_{\triangle\Omega_{m+1}} \frac{\epsilon}{2} |D(u-\hat{u})|_2^2 + \frac{1}{2\epsilon} |DI(\varphi_{m+1}(w-\hat{u}))|_2^2\end{aligned}$$

so we have

$$
\begin{aligned}
(\frac{\lambda}{\Lambda} - \frac{\eta}{2})\| |D(u-\hat{u})|_2\|^2_{L^2(\Omega_m)} &\le \frac{\epsilon}{2}\| |D(u-\hat{u})|_2\|^2_{L^2(\triangle\Omega_{m+1})} \\
&+ \frac{1}{2\eta}\| |D(w-u)|_2\|^2_{L^2(\Omega_m)} + \frac{1}{2\epsilon}\| |DI(\varphi_{m+1}(w-\hat{u}))|_2\|^2_{L^2(\triangle\Omega_{m+1})} \\
&\le \frac{\epsilon}{2}\| |D(u-\hat{u})|_2\|^2_{L^2(\triangle\Omega_{m+1})} + \frac{1}{2\eta}\| |D(w-u)|_2\|^2_{L^2(\Omega_m)} \\
&+ \frac{\beta^2_{m+1}}{2\epsilon}\|w-\hat{u}\|^2_{H^1(\triangle\Omega_{m+1})} \\
&\le \frac{\epsilon}{2}\| |D(u-\hat{u})|_2\|^2_{L^2(\triangle\Omega_{m+1})} + \frac{1}{2\eta}\| |D(w-u)|_2\|^2_{L^2(\Omega_m)} \\
&+ \frac{\beta^2_{m+1}}{2\epsilon}(1+\gamma^2_{m+1})\| |D(w-\hat{u})|_2\|^2_{L^2(\triangle\Omega_{m+1})} \\
&\le \Big(\frac{\epsilon}{2} + \frac{\beta^2_{m+1}}{\epsilon}(1+\gamma^2_{m+1})\Big)\| |D(u-\hat{u})|_2\|^2_{L^2(\triangle\Omega_{m+1})} \\
&+ \frac{1}{2\eta}\| |D(w-u)|_2\|^2_{L^2(\Omega_m)} + \frac{\beta^2_{m+1}}{\epsilon}(1+\gamma^2_{m+1})\| |D(w-u)|_2\|^2_{L^2(\triangle\Omega_{m+1})}.
\end{aligned}
$$

Considering that

$$\| |D(u-\hat{u})|_2\|^2_{L^2(\triangle\Omega_{m+1})} = \| |D(u-\hat{u})|_2\|^2_{L^2(\Omega_{m+1})} - \| |D(u-\hat{u})|_2\|^2_{L^2(\Omega_m)}$$

and taking the value $\eta = \frac{\lambda}{\Lambda}$ we have

$$
\begin{aligned}
\Big(\frac{\lambda}{2\Lambda} + \big(\frac{\epsilon}{2} + \frac{\beta^2_{m+1}}{\epsilon}(1+\gamma^2_{m+1})\big)\Big)\| |D(u-\hat{u})|_2\|^2_{L^2(\Omega_m)} \\
\le \big(\frac{\epsilon}{2} + \frac{\beta^2_{m+1}}{\epsilon}(1+\gamma^2_{m+1})\big)\| |D(u-\hat{u})|_2\|^2_{L^2(\Omega_{m+1})} \\
+ \frac{\Lambda}{2\lambda}\| |D(w-u)|_2\|^2_{L^2(\Omega_m)} + \frac{\beta^2_{m+1}}{\epsilon}(1+\gamma^2_{m+1})\| |D(w-u)|_2\|^2_{L^2(\triangle\Omega_{m+1})}.
\end{aligned}
$$

The value

$$\epsilon = \sqrt{2(1+\gamma^2_{m+1})}\beta_{m+1}$$

minimizes the quantity

$$\frac{\frac{\epsilon}{2} + \frac{\beta^2_{m+1}}{\epsilon}(1+\gamma^2_{m+1})}{\frac{\lambda}{2\Lambda} + \big(\frac{\epsilon}{2} + \frac{\beta^2_{m+1}}{\epsilon}(1+\gamma^2_{m+1})\big)} = \alpha_{m+1}$$

so we have

$$
\begin{aligned}
\| |D(u-\hat{u})|_2\|^2_{L^2(\Omega_m)} \le \alpha_{m+1}\| |D(u-\hat{u})|_2\|^2_{L^2(\Omega_{m+1})} \\
+ \frac{1}{2\sqrt{2(1+\gamma^2_{m+1})}}\frac{\Lambda}{\lambda}\frac{\alpha_{m+1}}{\beta_{m+1}}\| |D(w-u)|_2\|^2_{L^2(\Omega_m)} \\
+ \frac{1}{2}\alpha_{m+1}\| |D(w-u)|_2\|^2_{L^2(\triangle\Omega_{m+1})}.
\end{aligned}
$$

Multiplying by ν_m and iterating, and then using Cea's lemma as in theorem 2 we prove the theorem. □

5. Optimal Triangulation

Let us denote by N the total number of simplices and by N_i the number of simplices in the domain $\triangle\Omega_i$.

Let us obtain an upper estimate of N_i in terms of $\overline{h}_i$ and $|\triangle\Omega_i|$. To do this for $T \subset \overline{\triangle\Omega_i}$ let us estimate

$$|T| \geq |B_{\frac{1}{2}\rho_T}| = |B_1|\big(\frac{1}{2}\rho_T\big)^d = \frac{|B_1|}{2^d}\big(\frac{\rho_T}{h_T}\big)^d h_T^d \geq \frac{|B_1|}{(2\sigma)^d} h_T^d \geq \frac{|B_1|}{(2\sigma)^d}\underline{h}_i^d$$
$$= \frac{|B_1|}{(2\sigma)^d}\big(\frac{\underline{h}_i}{\overline{h}_i}\big)^d \overline{h}_i^d \geq \frac{|B_1|}{(2\sigma\tilde{\sigma})^d}\overline{h}_i^d$$

and by summing the inequality above for all $T \subset \overline{\triangle\Omega_i}$ we get

$$|\triangle\Omega_i| = \sum_{T \subset \overline{\triangle\Omega_i}} |T| \geq \frac{|B_1|}{(2\sigma\tilde{\sigma})^d}\overline{h}_i^d N_i.$$

So we have the estimate

$$N_i \leq C\frac{|\triangle\Omega_i|}{\overline{h}_i^d}$$

where $C = \frac{(2\sigma\tilde{\sigma})^d}{|B_1|}$, and for N we have the estimate

$$N \leq C\sum_{i=1}^{\ell}\frac{|\triangle\Omega_i|}{\overline{h}_i^d}. \tag{69}$$

Let us define

$$\tilde{N} = \sum_{i=1}^{\ell}\frac{|\triangle\Omega_i|}{\overline{h}_i^d} \tag{70}$$

then the estimate (69) becomes

$$N \leq C\tilde{N}. \tag{71}$$

In the following our aim is to minimize the error while keeping the number $\tilde{N}$ constant. By doing this we will bound the number of simplices by the inequality (71) and choose $\overline{h}_i$ such that the error is minimized, and hence we obtain the optimal coarsening.

In the formulas in this section we use the convention that in the case of Neumann problem we have $\gamma_i = 0$.

In the following in the case of Neumann problem u is the solution of (52) and $\hat{u}$ the solution of (53) and in the case of Dirichlet problem u is the solution of (64) and $\hat{u}$ the solution of (65).

For ease of notation let us define

$$\kappa_i = \sum_{m=i}^{\ell}\nu_m. \tag{72}$$

Lemma 3. *If $d \leq 3$ and $u \in H^2(\Omega)$ then we have the estimate*

$$\|u-\hat{u}\|^2_{H^1(\Omega_1)} \leq C\frac{1+\gamma_1^2}{\nu_1}\sum_{i=1}^{\ell}\kappa_i\overline{h}_i^2\|u\|^2_{H^2(\triangle\Omega_i)}. \tag{73}$$

PROOF. Because $d \leq 3$ and $u \in H^2(\Omega)$ we have the interpolation error estimate (cf. [**BS94**])

$$\|u - I(u)\|^2_{H^1(T)} \leq C h_T^2 \|u\|^2_{H^2(T)}$$

from here we have the estimate

$$\|u - I(u)\|^2_{H^1(\triangle\Omega_i)} = \sum_{T \subset \overline{\triangle\Omega_i}} \|u - I(u)\|^2_{H^1(T)} \leq C \sum_{T \subset \overline{\triangle\Omega_i}} h_T^2 \|u\|^2_{H^2(T)}$$

$$\leq C\overline{h}_i^2 \sum_{T \subset \overline{\triangle\Omega_i}} \|u\|^2_{H^2(T)} = C\overline{h}_i^2 \|u\|^2_{H^2(\triangle\Omega_i)}. \quad (74)$$

In the case of the Neumann problem by theorem 2 by taking $w = I(u)$ in the inequality (57) and the estimate (74) we may estimate

$$\|u - \hat{u}\|^2_{H^1(\Omega_1)} \leq \frac{C}{\nu_1} \sum_{i=1}^{\ell} \kappa_i \|u - I(u)\|^2_{H^1(\triangle\Omega_i)} \leq \frac{C}{\nu_1} \sum_{i=1}^{\ell} \kappa_i \overline{h}_i^2 \|u\|^2_{H^2(\triangle\Omega_i)}$$

which is the desired inequality.

In the case of Dirichlet problem similarly by theorem 3 by taking $w = I(u)$ in the inequality (67) and the estimate (74) we obtain the inequality

$$\||D(u - \hat{u})|_2\|^2_{L^2(\Omega_1)} \leq \frac{C}{\nu_1} \sum_{i=1}^{\ell} \kappa_i \overline{h}_i^2 \|u\|^2_{H^2(\triangle\Omega_i)}.$$

By the Poincaré inequality we have the estimate

$$\|u - \hat{u}\|^2_{H^1(\Omega_1)} \leq (1 + \gamma_1^2) \||D(u - \hat{u})|_2\|^2_{L^2(\Omega_1)}$$

so we have

$$\|u - \hat{u}\|^2_{H^1(\Omega_1)} \leq C \frac{1 + \gamma_1^2}{\nu_1} \sum_{i=1}^{\ell} \kappa_i \overline{h}_i^2 \|u\|^2_{H^2(\triangle\Omega_i)}$$

which is the desired inequality. □

If we choose the optimal values for $\overline{h}_i$ such that the right hand side of the error estimate (73) is minimized given $\tilde{N}$, the resulting optimal $\overline{h}_i$ will depend on $\|u\|_{H^2(\triangle\Omega_i)}$ which is not desirable as evaluating or estimating this before having the solution u is practically not possible.

Our approach is to estimate

$$\|u - \hat{u}\|^2_{H^1(\Omega_1)} \leq C \frac{1 + \gamma_1^2}{\nu_1} \sum_{i=1}^{\ell} \kappa_i \overline{h}_i^2 \|u\|^2_{H^2(\triangle\Omega_i)} \leq C \frac{1 + \gamma_1^2}{\nu_1} \|u\|^2_{H^2(\Omega)} \max_{i=1,\dots,\ell} \{\kappa_i \overline{h}_i^2\}$$

and to minimize this error estimate while keeping the number $\tilde{N}$ defined in (70) constant. The number $\tilde{N}$ is an upper bound for the number of simplices N in our triangulation by the estimate (71).

Let us denote

$$E = E(\overline{h}_1, \dots, \overline{h}_\ell) = \Big(\max_{i=1,\dots,\ell} \{\kappa_i \overline{h}_i^2\} \Big)^{\frac{1}{2}}.$$

LEMMA 4. *The values*

$$\overline{h}_i = \Big(\frac{\Gamma_{opt}}{\tilde{N}}\Big)^{\frac{1}{d}} \frac{1}{\kappa_i^{\frac{1}{2}}} \quad (75)$$

are optimal values for $\overline{h}_i$ *to minimize* E *given* $\tilde{N}$ *where*

$$\Gamma_{opt} = \sum_{i=1}^{\ell} |\triangle\Omega_i| \kappa_i^{\frac{d}{2}}$$

and the following relationship between $\tilde{N}$ *and* E *holds*

$$E = \Big(\frac{\Gamma_{opt}}{\tilde{N}}\Big)^{\frac{1}{d}}. \tag{76}$$

PROOF. The optimization problem of minimizing E given $\tilde{N}$, might be formulated as

$$\begin{cases} E^2(\overline{h}_1, \dots, \overline{h}_\ell) = \max\limits_{i=1,\dots,\ell} \{\kappa_i \overline{h}_i^2\} \longrightarrow min \\ \tilde{N}(\overline{h}_1, \dots, \overline{h}_\ell) = \sum\limits_{i=1}^{\ell} \dfrac{|\triangle\Omega_i|}{\overline{h}_i^d}. \end{cases}$$

Let us consider the new variables

$$r_i = \frac{\Gamma_{opt}}{\tilde{N}} \frac{1}{\kappa_i^{\frac{d}{2}} \overline{h}_i^d} \tag{77}$$

then we have

$$E^2 = \max_{i=1,\dots,\ell} \Big\{ \Big(\frac{\Gamma_{opt}}{\tilde{N}}\Big)^{\frac{2}{d}} \frac{1}{r_i^{\frac{2}{d}}} \Big\} = \Big(\frac{\Gamma_{opt}}{\tilde{N}}\Big)^{\frac{2}{d}} \frac{1}{(\min_{i=1,\dots,\ell} r_i)^{\frac{2}{d}}} \tag{78}$$

and

$$\tilde{N} = \sum_{i=1}^{\ell} |\triangle\Omega_i| \big\{ \frac{\tilde{N}}{\Gamma_{opt}} \kappa_i^{\frac{d}{2}} r_i \big\} = \frac{\tilde{N}}{\Gamma_{opt}} \sum_{i=1}^{\ell} |\triangle\Omega_i| \kappa_i^{\frac{d}{2}} r_i.$$

Then the optimization problem is

$$\begin{cases} \min\limits_{i=1,\dots,\ell} r_i \longrightarrow max \\ \sum\limits_{i=1}^{\ell} |\triangle\Omega_i| \kappa_i^{\frac{d}{2}} r_i = \sum\limits_{i=1}^{\ell} |\triangle\Omega_i| \kappa_i^{\frac{d}{2}}. \end{cases}$$

The values $r_i = 1$ satisfy the constraint and $\min_{i=1,\dots,\ell} r_i = 1$. To achieve a higher value for $\min_{i=1,\dots,\ell} r_i$ all of r_i should be strictly larger than 1 which will contradict with the constraint. So the values $r_i = 1$ are the optimal values and by (77) the optimal values for $\overline{h}_i$ are given by (75). By the equation (78) the equation (76) follows. □

To see what is exactly the gain when we coarsen the triangulation optimally let us compare the error estimate for the two cases, when we coarsen the triangulation optimally or when the uniform triangulation is used.

LEMMA 5. *If we choose the* $\overline{h}_i$ *to be constant equal to some* $\overline{h}$ *then we have the following relation between* E *and* $\tilde{N}$

$$E = \Big(\frac{\Gamma_{unf}}{\tilde{N}}\Big)^{\frac{1}{d}} \tag{79}$$

where

$$\Gamma_{unf} = |\Omega| \big(\max_{i=1,\dots,\ell} \kappa_i\big)^{\frac{d}{2}}.$$

PROOF. If $\overline{h}_i = \overline{h}$ for $i = 1, \dots, \ell$ then we have

$$E^2 = \max_{i=1,\dots,\ell} \{\kappa_i \overline{h}_i^{-2}\} = \overline{h}^{-2} \max_{i=1,\dots,\ell} \kappa_i$$

and

$$\tilde{N} = \sum_{i=1}^{\ell} \frac{|\triangle\Omega_i|}{\overline{h}_i^d} = \frac{1}{\overline{h}^d} \sum_{i=1}^{\ell} |\triangle\Omega_i| = \frac{|\Omega|}{\overline{h}^d}$$

so we may compute

$$\tilde{N}^{\frac{1}{d}} E = \frac{|\Omega|^{\frac{1}{d}}}{\overline{h}} \overline{h} \big(\max_{i=1,\dots,\ell} \kappa_i\big)^{\frac{1}{2}} = |\Omega|^{\frac{1}{d}} \big(\max_{i=1,\dots,\ell} \kappa_i\big)^{\frac{1}{2}} = \Gamma_{unf}^{\frac{1}{d}}$$

which proves the equation (79). □

As we see from the equations (76) and (79) the only difference between the optimal and uniform triangulations are the numbers Γ_{opt} and Γ_{unf}. For example the dependence of E on $\tilde{N}$ in both optimal and uniform cases are the same.

Clearly we should have $\Gamma_{opt} \le \Gamma_{unf}$ because Γ_{opt} is the result of minimizing E. But it is also trivial to check the inequality $\Gamma_{opt} \le \Gamma_{unf}$ also explicitly by the expressions that we have for these numbers as follows

$$\Gamma_{opt} = \sum_{i=1}^{\ell} |\triangle\Omega_i| \kappa_i^{\frac{d}{2}} \le \Big(\sum_{i=1}^{\ell} |\triangle\Omega_i|\Big) \max_{i=1,\dots,\ell} \kappa_i^{\frac{d}{2}} = |\Omega| \big(\max_{i=1,\dots,\ell} \kappa_i\big)^{\frac{d}{2}} = \Gamma_{unf}.$$

As we see the difference between Γ_{opt} and Γ_{unf} is that Γ_{opt} is a weighted sum of $|\triangle\Omega_i|$ by the weights $\kappa_i^{\frac{d}{2}}$ which is a decreasing sequence, hence in the case of large domains usually Γ_{opt} is very small compared to Γ_{unf} which is the advantage of non-uniform optimal triangulation.

In practice evaluation of κ_i is not straightforward, but we may make estimates of it from above and in this case we obtain an error estimate in the form of (73) and then the optimal $\overline{h}_i$ will be as they are for κ_i.

The numbers κ_i are defined by (72) which contain the values ν_i which in turn are defined by (56) and depend on the values of α_i, but α_i are defined by the equation (55) for the Neumann problem and by (66) for the Dirichlet problem. However we may notice that we may use the formula (66) for both cases by setting the values γ_i to zero in the case of Neumann problem.

By the inequality

$$1 + \gamma_i^2 \le 2 \max(1, \gamma_i)^2$$

we may estimate α_i as

$$\alpha_i = \frac{\sqrt{2(1+\gamma_i^2)}\beta_i}{\frac{\lambda}{2\Lambda} + \sqrt{2(1+\gamma_i^2)}\beta_i} \le \frac{2\max(1,\gamma_i)\beta_i}{\frac{\lambda}{2\Lambda} + 2\max(1,\gamma_i)\beta_i}$$

$$= \frac{2c\frac{\max(1,\gamma_i)}{\min(1,d_i)}}{\frac{\lambda}{2\Lambda} + 2c\frac{\max(1,\gamma_i)}{\min(1,d_i)}} = \frac{1}{1 + C\frac{\min(1,d_i)}{\max(1,\gamma_i)}} \qquad (80)$$

where $C = \frac{\lambda}{4c\Lambda}$.

Because

$$\frac{\min(1, d_i)}{\max(1, \gamma_i)} \le 1$$

by the concavity of logarithm we have the estimate

$$\ln(1+C\frac{\min(1,d_i)}{\max(1,\gamma_i)}) \geq \ln(1+C)\frac{\min(1,d_i)}{\max(1,\gamma_i)}$$

so we may estimate

$$\frac{1}{1+C\frac{\min(1,d_i)}{\max(1,\gamma_i)}} = e^{-\ln(1+C\frac{\min(1,d_i)}{\max(1,\gamma_i)})} \leq e^{-\ln(1+C)\frac{\min(1,d_i)}{\max(1,\gamma_i)}}. \tag{81}$$

Denoting

$$\upsilon = \ln(1+C)$$

from (80) and (81) we have

$$\alpha_i \leq e^{-\upsilon\frac{\min(1,d_i)}{\max(1,\gamma_i)}}$$

so we obtain the following estimate for ν_m

$$\nu_m \leq \prod_{i=1}^{m} e^{-\upsilon\frac{\min(1,d_i)}{\max(1,\gamma_i)}} = e^{-\upsilon\sum_{i=1}^{m}\frac{\min(1,d_i)}{\max(1,\gamma_i)}}$$

and by this we obtain the estimate

$$\kappa_i \leq \sum_{m=i}^{\ell} e^{-\upsilon\sum_{j=1}^{m}\frac{\min(1,d_j)}{\max(1,\gamma_j)}}. \tag{82}$$

6. Cylindrical Domains

In the case of cylindrical domains

$$\Omega = (0,\ell)\times(0,r)$$

where $r>0$ and ℓ is a positive integer, we may consider the domains

$$\Omega_i = (0,i)\times(0,r)$$

then we have

$$\triangle\Omega_i = (i-1,i)\times(0,r).$$

It is clear that for this family of domains $d_i = 1$.

The Poincaré constant for these domains with zero boundary condition on the boundary part $(i-1,i)\times\{0,r\}$ is

$$\gamma_i = \frac{r}{\pi}.$$

In the case of Neumann problem we should set $\gamma_i = 0$ but we may use the formula above and set $r = 0$.

So by (82) we have the estimate

$$\begin{aligned}\kappa_i &\leq \sum_{m=i}^{\ell} e^{-\upsilon\sum_{j=1}^{m}\frac{1}{\max(1,\frac{r}{\pi})}} = \sum_{m=i}^{\ell} e^{-\upsilon\min(1,\frac{\pi}{r})m} \\ &= \frac{e^{-\nu\min(1,\frac{\pi}{r})i} - e^{-\nu\min(1,\frac{\pi}{r})(\ell+1)}}{1-e^{-\nu\min(1,\frac{\pi}{r})}} \leq \frac{1}{1-e^{-\nu\min(1,\frac{\pi}{r})}} e^{-\nu\min(1,\frac{\pi}{r})i} \\ &\qquad = Ce^{-\upsilon\min(1,\frac{\pi}{r})i} = \hat{\kappa}_i.\end{aligned}$$

Let us compute Γ_{opt} and Γ_{unf} corresponding to these estimates of κ_i. For Γ_{opt} since $d = 2$ we have

$$\hat{\Gamma}_{opt} = \sum_{i=1}^{\ell} |\triangle\Omega_i|\hat{\kappa}_i = \sum_{i=1}^{\ell} |(i-1,i)\times(0,r)|Ce^{-\upsilon\min(1,\frac{\pi}{r})i}$$
$$= Cr\sum_{i=1}^{\ell} e^{-\upsilon\min(1,\frac{\pi}{r})i} = Cr\frac{1-e^{-\upsilon\min(1,\frac{\pi}{r})\ell}}{1-e^{-\upsilon\min(1,\frac{\pi}{r})}}e^{-\upsilon\min(1,\frac{\pi}{r})}$$

and

$$\hat{\Gamma}_{unf} = |\Omega|\max_{i=1,\dots,\ell}\hat{\kappa}_i = |(0,\ell)\times(0,r)|\max_{i=1,\dots,\ell}Ce^{-\upsilon\min(1,\frac{\pi}{r})i} = Cr\ell e^{-\upsilon\min(1,\frac{\pi}{r})}.$$

Hence we may compute the ratio of error estimate if the uniform triangulation is used compared to the optimal one

$$\left\{\frac{\hat{E}_{unf}}{\hat{E}_{opt}}\right\}^2 = \frac{\hat{\Gamma}_{unf}}{\hat{\Gamma}_{opt}} = C\ell\frac{1-e^{-\upsilon\min(1,\frac{\pi}{r})}}{1-e^{-\upsilon\min(1,\frac{\pi}{r})\ell}}. \tag{83}$$

For $x > 0$ let us prove the inequalities

$$\frac{1}{1+x} \le \frac{1-e^{-x}}{x} \le \frac{2}{1+x}. \tag{84}$$

After some transformations the first inequality is equivalent to $e^x \ge 1+x$ which holds. After some transformations the second inequality in (84) is equivalent to

$$\frac{1-x}{1+x} \le e^{-x}. \tag{85}$$

For $1 \le x$ this inequality clearly holds. For $0 < x < 1$ by convexity of the exponential function we have

$$e^x \le (e-1)x + 1$$

and by the inequality $e < 3$ we obtain

$$e^x \le 2x + 1.$$

Now it is possible to see that

$$2x + 1 \le \frac{1+x}{1-x}$$

holds for $0 < x < 1$.

Hence we have the inequality

$$e^x \le \frac{1+x}{1-x}$$

which after taking reciprocals is equivalent to (85). So we have proved the inequalities (84).

Now by (83) and (84) we obtain

$$\frac{C}{2}\frac{\frac{1}{\pi\upsilon}\max(\pi,r)+\ell}{\frac{1}{\pi\upsilon}\max(\pi,r)+1} \le \left\{\frac{\hat{E}_{unf}}{\hat{E}_{opt}}\right\}^2 \le 2C\frac{\frac{1}{\pi\upsilon}\max(\pi,r)+\ell}{\frac{1}{\pi\upsilon}\max(\pi,r)+1}$$

and hence

$$\hat{E}_{unf} \approx \hat{E}_{opt}\begin{cases} C_1 + C_2\sqrt{\ell}, & 0 \le r < \pi \\ C_1 + C_2\sqrt{\frac{\ell}{r}}, & \pi \le r. \end{cases}$$

So in the case of Neumann problem and Dirichlet problem with $r < \pi$, the error estimate in the case of optimal triangulation is smaller than the error estimate in the case of uniform triangulation by a factor of $\frac{1}{\sqrt{\ell}}$. And in the case of Dirichlet problem with $\pi \leq r$ the error estimate in the case of optimal triangulation is smaller than the error estimate in the case of uniform triangulation by a factor of $\sqrt{\frac{r}{\ell}}$.

CHAPTER 4

Elliptic Variational Inequalities

In this chapter we introduce the problem of elliptic variational inequalities with pointwise constraint on the value and derivatives of the solution. Then we study the asymptotic behavior of a single solution.

To introduce the problem studied in this and next chapter let us consider the simple case of a two dimensional cylinder and its boundaries

$$\begin{aligned} \Omega_\ell &= (-\ell,\ell)\times(0,1),\\ \Gamma_\ell &= (-\ell,\ell)\times\{0,1\},\\ \Delta_\ell &= \{-\ell,\ell\}\times(0,1). \end{aligned}$$

Let us consider the closed convex set

$$\mathcal{K}_g(\Omega_\ell)=\Big\{v\in H^1(\Omega_\ell)\ \Big|\ v=0 \text{ on } \Gamma_\ell,\ v=g \text{ on } \Delta_\ell,$$
$$|v(x)|^2+|Dv(x)|_2^2\leq 1 \text{ for a.e. } x\in\Omega_\ell\Big\}$$

here the pointwise constraint on the value and gradient may change from one problem to the other. Let us consider $u_\ell\in\mathcal{K}_g(\Omega_\ell)$ the solution of the variational inequality

$$\int_{\Omega_\ell} Du_\ell\cdot D(u_\ell-v)\leq\langle f,u_\ell-v\rangle\,,\quad\forall v\in\mathcal{K}_g(\Omega_\ell)$$

where $f\in L^2(\Omega_\ell)$. We are interested in the asymptotic behavior of the solution as ℓ goes to infinity.

In this chapter we study the asymptotic of a single solution and in the next chapter we consider the asymptotic behavior when the applied force term f is periodic in the lateral direction of the cylinder Ω_ℓ or if it is defined in the cross section of the cylinder.

1. Notation and Problem Setting

For the integers d and q, such that $1\leq q\leq d-1$ let us consider the domain $\omega\subset\mathbb{R}^{d-q}$ and for $\ell\geq 0$ we define the cylinder

$$\Omega_\ell=(-\ell,\ell)^q\times\omega,$$

by Γ_ℓ we denote the lateral boundary, that is

$$\Gamma_\ell=(-\ell,\ell)^q\times\partial\omega$$

and by Δ_ℓ we denote the boundary at the ends of the cylinder, that is

$$\Delta_\ell=\partial((-\ell,\ell)^q)\times\omega.$$

We use the notation

$$|x|_\infty=\max_{i=1,\dots,d}|x_i|$$

for $x \in \mathbb{R}^d$.

We write a point $X = (x_1, \dots, x_q, x_{q+1}, \dots, x_d) \in \mathbb{R}^d$ as $X = (X_1, X_2)$ where $X_1 = (x_1, \dots, x_q) \in \mathbb{R}^q$ and $X_2 = (x_{q+1}, \dots, x_d) \in \mathbb{R}^{d-q}$.

For a function v by the notation $D_{X_1} v$ we understand the vector of derivatives of v in X_1 variables, but also we might mean the vector

$$(D_{x_1} v, \cdots, D_{x_q} v, \underbrace{0, 0, \cdots, 0}_{d-q \text{ times}})^T \in \mathbb{R}^d$$

and similarly we treat the vector $D_{X_2} v$.

Let us set

$$H_0^1(\Omega_\ell; \Gamma_\ell) = \left\{ v \in H^1(\Omega_\ell) \;\middle|\; v = 0 \text{ on } \Gamma_\ell \right\}$$

equipped with the norm

$$\|v\|_{H_0^1(\Omega_\ell;\Gamma_\ell)} = \||Dv|_2\|_{L^2(\Omega_\ell)}$$

and for $g \in H_0^1(\Omega_\ell; \Gamma_\ell)$ let us set

$$H_g^1(\Omega_\ell) = \left\{ v \in H_0^1(\Omega_\ell; \Gamma_\ell) \;\middle|\; v - g \in H_0^1(\Omega_\ell) \right\}.$$

Let us consider a closed convex set $K \subset \mathbb{R}^{d+1}$ such that $0 \in K$ and define the following set

$$\mathcal{K}_g(\Omega_\ell) = \left\{ v \in H_g^1(\Omega_\ell) \;\middle|\; (v, Dv) \in K \text{ a.e. in } \Omega_\ell \right\}.$$

This is a closed convex subset of $H_0^1(\Omega_\ell; \Gamma_\ell)$.

Let us assume that $g \in \mathcal{K}_g(\Omega_\ell)$ that is $(g, Dg) \in K$ a.e. in Ω_ℓ.

For $f \in H^{-1}(\Omega_\ell)$ we may consider the unique $u_\ell \in \mathcal{K}_g(\Omega_\ell)$ solution to the variational inequality

$$\begin{cases} \displaystyle\int_{\Omega_\ell} Du_\ell \cdot (Du_\ell - Dv) \le \langle f, u_\ell - v \rangle, \quad \forall v \in \mathcal{K}_g(\Omega_\ell) \\ u_\ell \in \mathcal{K}_g(\Omega_\ell). \end{cases} \tag{86}$$

2. Comparison Principle

As usual for two linear functionals $f_1, f_2 \in H^{-1}(\Omega_\ell)$ we say $f_1 \le f_2$ if $0 \le \langle f_2 - f_1, v \rangle$ for all $v \in H_0^1(\Omega_\ell)$ such that v is pointwise positive.

In the following lemma we prove the weak maximum comparison principle for our variational inequality.

In this section we consider the case when $K = K_0 \times K'$ where K_0 is a closed interval containing 0 and K' is a bounded, closed and convex subset of $\mathbb{R}^d$ and $0 \in K'$.

Let us note that because K' is bounded the solution to the inequality (86) is Lipschitz continuous.

Lemma 6. *If $g_1, g_2 \in H_0^1(\Omega_\ell; \Gamma_\ell)$, $f_1, f_2 \in H^{-1}(\Omega_\ell)$, $f_2 \le f_1$ and u_i for $i = 1, 2$ be the solutions to the inequality* (86) *with the corresponding g_i and f_i then*

$$\sup_{\Omega_\ell}(u_2 - u_1) \le \sup_{\Delta_\ell}(g_2 - g_1). \tag{87}$$

PROOF. Let us denote

$$M = \sup_{\Delta_\ell}(g_2 - g_1),$$

we may note that because $g_2 = g_1$ on Γ_ℓ we have $M \geq 0$.

Let us define

$$v_1 = u_1 + ((u_2 - u_1) - M)^+$$
$$v_2 = u_2 - ((u_2 - u_1) - M)^+$$

here $x^+ = \max(x, 0)$. It is clear that $v_1 \in H^1_{g_1}(\Omega_\ell)$ and $v_2 \in H^1_{g_2}(\Omega_\ell)$ and we may also check that

$$v_1 = u_1\chi_{\{u_2-u_1\leq M\}} + (u_2 - M)\chi_{\{u_2-u_1>M\}}$$
$$v_2 = u_2\chi_{\{u_2-u_1\leq M\}} + (u_1 + M)\chi_{\{u_2-u_1>M\}}$$

from here it is clear that $Dv_1, Dv_2 \in K'$ a.e. in Ω_ℓ. Let us check that also $v_1, v_2 \in K_0$ a.e. in Ω_ℓ. First for v_1, in the case $u_2 - u_1 \leq M$, $v_1 = u_1 \in K_0$ and in the case $u_2 - u_1 > M$, $v_1 = u_2 - M$ and hence $u_1 < v_1 \leq u_2$ which shows that $v_1 \in K_0$ a.e. in Ω_ℓ. For v_2, in the case $u_2 - u_1 \leq M$, $v_2 = u_2 \in K_0$ and in the case $u_2 - u_1 > M$, $v_2 = u_1 + M$ and hence $u_1 \leq v_2 < u_2$ which shows that $v_2 \in K_0$ a.e. in Ω_ℓ.

So because $K = K_0 \times K'$ we have $v_1 \in \mathcal{K}_{g_1}(\Omega_\ell)$ and $v_2 \in \mathcal{K}_{g_2}(\Omega_\ell)$.

Testing (86) for u_1 with v_1 and for u_2 with v_2 we obtain

$$-\int_{\Omega_\ell} Du_1 \cdot D((u_2 - u_1) - M)^+ \leq -\big\langle f_1, ((u_2 - u_1) - M)^+\big\rangle$$
$$\int_{\Omega_\ell} Du_2 \cdot D((u_2 - u_1) - M)^+ \leq \big\langle f_2, ((u_2 - u_1) - M)^+\big\rangle.$$

Summing these two inequalities we obtain

$$\int_{\Omega_\ell} D(u_2 - u_1) \cdot D((u_2 - u_1) - M)^+ \leq \big\langle f_2 - f_1, ((u_2 - u_1) - M)^+\big\rangle.$$

By $f_2 \leq f_1$ we have $\big\langle f_2 - f_1, ((u_2 - u_1) - M)^+\big\rangle \leq 0$ so we have

$$\int_{\Omega_\ell} D(u_2 - u_1) \cdot D((u_2 - u_1) - M)^+ \leq 0.$$

But

$$\int_{\Omega_\ell} D(u_2 - u_1) \cdot D((u_2 - u_1) - M)^+ = \int_{\Omega_\ell} D((u_2 - u_1) - M) \cdot D((u_2 - u_1) - M)^+ = \int_{\Omega_\ell} |D((u_2 - u_1) - M)^+|^2$$

and we obtain

$$\int_{\Omega_\ell} |D((u_2 - u_1) - M)^+|^2 \leq 0.$$

From here using the Poincaré inequality we obtain $((u_2 - u_1) - M)^+ = 0$ a.e. in Ω_ℓ that is by continuity of u_1 and u_2, $u_2 - u_1 \leq M$ in Ω_ℓ which is the inequality (87). □

The following corollary is the weak maximum principle for our variational inequality.

Corollary 1. *If $f \le 0$ then*

$$\sup_{\Omega_\ell} u \le \sup_{\Delta_\ell} g.$$

Proof. This follows from the lemma by taking $f_1 = 0$, $g_1 = 0$, $u_1 = 0$ and $f_2 = f$, $g_2 = g$, $u_2 = u$. □

The following corollary is the weak module maximum comparison principle for our variational inequality.

Corollary 2. *If $f_2 = f_1$ then*

$$\sup_{\Omega_\ell} |u_2 - u_1| \le \sup_{\Delta_\ell} |g_2 - g_1|. \tag{88}$$

Proof. Because of the equality $f_1 = f_2$ we have both the inequalities

$$\sup_{\Omega_\ell}(u_2 - u_1) \le \sup_{\Delta_\ell}(g_2 - g_1)$$

and

$$\sup_{\Omega_\ell}(u_1 - u_2) \le \sup_{\Delta_\ell}(g_1 - g_2)$$

and from these inequalities the corollary follows. □

3. Uniform Bound

In the asymptotic estimates we assume that the solution is uniformly bounded. This kind of estimate is possible to obtain under different mild assumptions. In the following some of the alternative assumptions are brought.

If K is bounded in the first direction then obviously the solution to the variational inequality is uniformly bounded.

If K is bounded in a direction in X_2 then because of the zero boundary condition on the lateral boundary Γ_ℓ and the boundedness of ω the solution will be uniformly bounded.

4. Asymptotic of a Single Solution

In this section for u_ℓ solution of (86) we prove results about the asymptotic behavior of it as ℓ approaches $+\infty$.

We consider only the case when $f = 0$ because the main issue is the boundary condition g. The estimates for the case when f is non zero might be obtained in a similar way.

The following theorem proves a general result which shows that the asymptotic behavior depends on how we can extend g inside the cylinder.

Theorem 4. *Let $0 \le \phi(X_1)$ be a Lipschitz function such that*

$$(g, Dg + (u_\ell - g)D_{X_1}\phi) \in K, \ \text{a.e. in } \Omega_\ell \tag{89}$$

then there exists a constant $C > 0$ depending on ω such that

$$\int_{\Omega_\ell} (1 - C|D_{X_1}\phi|)\psi|Du_\ell|^2 \le C \int_{\Omega_\ell} (1 + |D_{X_1}\phi|)\psi|Dg|^2$$

where $\psi = \exp(-\phi(X_1))$.

PROOF. Let us consider the function

$$v = \psi g + (1-\psi)u_\ell,$$

it is clear that $v \in H^1_g(\Omega_\ell)$ and we may compute

$$Dv = \big(Dg - (u_\ell - g)\frac{D_{X_1}\psi}{\psi}\big)\psi + (1-\psi)Du_\ell$$
$$= \big(Dg + (u_\ell - g)D_{X_1}\phi\big)\psi + (1-\psi)Du_\ell.$$

Hence

$$(v, Dv) = (g, Dg + (u_\ell - g)D_{X_1}\phi)\psi + (u_\ell, Du_\ell)(1-\psi)$$

now by (89) we will have $(v, Dv) \in K$ a.e. in Ω_ℓ so $v \in \mathcal{K}_g(\Omega_\ell)$.

Now by testing the inequality (86) by v we have

$$\begin{aligned}
0 &\geq \int_{\Omega_\ell} Du_\ell \cdot \big(Du_\ell - \big((Dg + (u_\ell - g)D_{X_1}\phi)\psi + (1-\psi)Du_\ell\big)\big)\\
&= \int_{\Omega_\ell} Du_\ell \cdot \big(\psi Du_\ell - \big(Dg + (u_\ell - g)D_{X_1}\phi\big)\psi\big)\\
&= \int_{\Omega_\ell} \psi Du_\ell \cdot \big(Du_\ell - \big(Dg + (u_\ell - g)D_{X_1}\phi\big)\big)\\
&= \int_{\Omega_\ell} \psi Du_\ell \cdot \big(Du_\ell - \big(Dg - gD_{X_1}\phi + u_\ell D_{X_1}\phi\big)\big)\\
&= \int_{\Omega_\ell} \psi |Du_\ell|^2 + \psi Du_\ell \cdot (-Dg + gD_{X_1}\phi) - \psi u_\ell D_{X_1}u_\ell \cdot D_{X_1}\phi
\end{aligned}$$

so we obtain

$$\int_{\Omega_\ell} \psi |Du_\ell|^2 \leq \int_{\Omega_\ell} \psi u_\ell D_{X_1}u_\ell \cdot D_{X_1}\phi + \int_{\Omega_\ell} \psi Du_\ell \cdot (Dg - gD_{X_1}\phi).$$

Using Young's inequality we get

$$\begin{aligned}
\int_{\Omega_\ell} \psi |Du_\ell|^2 &\leq \int_{\Omega_\ell} \psi |u_\ell||D_{X_1}u_\ell||D_{X_1}\phi| + \int_{\Omega_\ell} \psi |Du_\ell||Dg - gD_{X_1}\phi|\\
&\leq \frac{1}{2}\int_{\Omega_\ell} \psi |D_{X_1}\phi||u_\ell|^2 + \frac{1}{2}\int_{\Omega_\ell} \psi |D_{X_1}\phi||D_{X_1}u_\ell|^2\\
&\quad + \frac{1}{2}\int_{\Omega_\ell} \psi |Du_\ell|^2 + \frac{1}{2}\int_{\Omega_\ell} \psi |Dg|^2\\
&\quad + \frac{1}{2}\int_{\Omega_\ell} \psi |D_{X_1}\phi||Du_\ell|^2 + \frac{1}{2}\int_{\Omega_\ell} \psi |D_{X_1}\phi||g|^2.
\end{aligned}$$

Using the Poincaré inequality we get

$$\begin{aligned}
\int_{\Omega_\ell} \psi |D_{X_1}\phi||u_\ell|^2 &= \int_{(-\ell,\ell)^q} \psi(X_1)|D_{X_1}\phi(X_1)| \int_\omega |u_\ell|^2 dX_2 dX_1\\
&\leq c_\omega^2 \int_{(-\ell,\ell)^q} \psi(X_1)|D_{X_1}\phi(X_1)| \int_\omega |D_{X_2}u_\ell|^2 dX_2 dX_1\\
&= c_\omega^2 \int_{\Omega_\ell} \psi |D_{X_1}\phi||D_{X_2}u_\ell|^2
\end{aligned}$$

and similarly

$$\int_{\Omega_\ell} \psi |D_{X_1}\phi||g|^2 \le c_\omega^2 \int_{\Omega_\ell} \psi |D_{X_1}\phi||D_{X_2}g|^2.$$

So we have

$$\begin{aligned}\frac{1}{2}\int_{\Omega_\ell} \psi |Du_\ell|^2 \le \frac{c_\omega^2}{2}\int_{\Omega_\ell} \psi |D_{X_1}\phi||D_{X_2}u_\ell|^2 + \frac{1}{2}\int_{\Omega_\ell} \psi |D_{X_1}\phi||D_{X_1}u_\ell|^2 \\ + \frac{1}{2}\int_{\Omega_\ell} \psi |Dg|^2 + \frac{1}{2}\int_{\Omega_\ell} \psi |D_{X_1}\phi||Du_\ell|^2 + \frac{c_\omega^2}{2}\int_{\Omega_\ell} \psi |D_{X_1}\phi||D_{X_2}g|^2 \\ \le C\int_{\Omega_\ell} \psi |D_{X_1}\phi||Du_\ell|^2 + C\int_{\Omega_\ell} (1+|D_{X_1}\phi|)\psi |Dg|^2\end{aligned}$$

from here for some new $C > 0$ we obtain

$$\int_{\Omega_\ell} \big(1 - C|D_{X_1}\phi|\big)\psi |Du_\ell|^2 \le C\int_{\Omega_\ell} (1+|D_{X_1}\phi|)\psi |Dg|^2$$

which proves the theorem. □

In the following three lemmas we apply the theorem above to different situations.

Lemma 7. *If for some $r > 0$ we have*

$$(g, D_{X_1}g, D_{X_2}g) + (0, B_r^q(0), 0) \subset K, \ \text{a.e. in } \Omega_\ell \tag{90}$$

then there exist a constant $C_1 > 0$ depending on ω and r such that if

$$\alpha < C_1$$

then

$$\int_{\Omega_\ell} e^{-\alpha |X_1|_\infty} |Du_\ell|^2 \le C_2 \int_{\Omega_\ell} e^{-\alpha |X_1|_\infty} |Dg|^2$$

here the constant $C_2 > 0$ depends on ω, r and α.

Proof. Let us consider for $\alpha > 0$, $\phi(X_1) = \alpha |X_1|_\infty$ then

$$|D_{X_1}\phi| = \alpha$$

and by the uniform boundedness of u_ℓ and g, $|u_\ell|, |g| \le c$, we have

$$|(u_\ell - g)D_{X_1}\phi| \le 2c\alpha.$$

So if $\alpha < \frac{r}{2c}$ then by (90) the assumption (89) holds and by the previous theorem we have

$$\int_{\Omega_\ell} e^{-\alpha |X_1|_\infty} \big(1 - C\alpha\big)|Du_\ell|^2 \le C\int_{\Omega_\ell} e^{-\alpha |X_1|_\infty}(1+\alpha)|Dg|^2$$

so if also

$$\alpha < \frac{1}{C}$$

we obtain

$$\int_{\Omega_\ell} e^{-\alpha |X_1|_\infty} |Du_\ell|^2 \le \frac{C(1+\alpha)}{1 - C\alpha}\int_{\Omega_\ell} e^{-\alpha |X_1|_\infty} |Dg|^2$$

which proves the lemma. □

COROLLARY 3. *If in addition to the assumptions in the previous lemma we have for some* $\delta > 0$, $g = 0$ *in* $\Omega_{\ell-\delta}$ *then for* $0 < \ell_0 < \ell - \delta$ *we have*

$$\int_{\Omega_{\ell_0}} |Du_\ell|^2 \leq Ce^{-\alpha(\ell-\ell_0)} \int_{\Omega_\ell \setminus \Omega_{\ell-\delta}} |Dg|^2.$$

PROOF. We have

$$\int_{\Omega_{\ell_0}} |Du_\ell|^2 \leq e^{\alpha\ell_0} \int_{\Omega_\ell} e^{-\alpha|X_1|_\infty} |Du_\ell|^2 \leq C_2 e^{\alpha\ell_0} \int_{\Omega_\ell} e^{-\alpha|X_1|_\infty} |Dg|^2$$
$$\leq C_2 e^{\alpha\delta} e^{-\alpha(\ell-\ell_0)} \int_{\Omega_\ell \setminus \Omega_{\ell-\delta}} |Dg|^2$$

and this proves the corollary. □

LEMMA 8. *If* $K = K_0 \times K'$ *where* K_0 *is a closed interval containing* 0 *and* K' *is a bounded, closed and convex subset of* $\mathbb{R}^d$ *and for some* $r > 0$ *we have* $B_r^q(0) \times \{0\}^{d-q} \subset K'$ *then*

$$\int_{\Omega_{\ell_0}} |Du_\ell|^2 \leq Ce^{-\alpha(\ell-\ell_0)} \Big(C\ell^{q-1} + \int_{\Omega_\ell \setminus \Omega_{\ell-\frac{c}{r}}} |Dg|^2 \Big).$$

PROOF. Let by uniform boundedness $|u_\ell|, |g| \leq c$ and consider

$$h(X_1) = r\big(|X_1|_\infty - (\ell - \frac{c}{r})\big)^+$$

then $h = 0$ in $\Omega_{\ell-\frac{c}{r}}$, $h = c$ on Δ_ℓ and $\pm Dh = (\pm D_{X_1} h, 0) \in B_r^q(0) \times \{0\}^{d-q} \subset K'$.

Let us define

$$\tilde{g} = \min\big(\max(g, -h), h\big)$$

then $\tilde{g} \in H_0^1(\Omega_\ell; \Gamma_\ell)$, $\tilde{g} = 0$ in $\Omega_{\ell-\frac{c}{r}}$, $\tilde{g} = g$ on Δ_ℓ and $D\tilde{g}$ is equal either to Dg, Dh or $-Dh$.

Let us consider $\phi(X_1) = \alpha|X_1|_\infty$ for $|X_1|_\infty \leq \ell - \frac{c}{r}$ and $\phi(X_1) = \alpha(\ell - \frac{c}{r})$ for $\ell - \frac{c}{r} \leq |X_1|_\infty \leq \ell$.

Then in $\Omega_{\ell-\frac{c}{r}}$ because $\tilde{g} = 0$ we have

$$(\tilde{g}, D\tilde{g} + (u_\ell - \tilde{g})D_{X_1}\phi) = (0, u_\ell D_{X_1}\phi, 0)$$

and we have

$$|u_\ell D_{X_1}\phi| \leq c\alpha.$$

So if

$$\alpha < \frac{r}{c}$$

in $\Omega_{\ell-\frac{c}{r}}$ the condition (89) holds. In $\Omega_\ell \setminus \Omega_{\ell-\frac{c}{r}}$ we have $D_{X_1}\phi = 0$, so we should check if $(\tilde{g}, D\tilde{g}) \in K$. We have when $0 \leq g$, $0 \leq \tilde{g} \leq g$ and when $g \leq 0$, $g \leq \tilde{g} \leq 0$, so by the condition that $(g, Dg) \in K$ a.e. in Ω_ℓ we have $g \in K_0$ a.e. in Ω_ℓ hence $\tilde{g} \in K_0$ a.e. in Ω_ℓ. We have that $D\tilde{g}$ is a.e. either equal to Dg, Dh or $-Dh$, so $D\tilde{g} \in K'$ a.e. in Ω_ℓ. Hence also in this domain the condition (89) holds.

So by the theorem for sufficiently small α we obtain

$$\int_{\Omega_\ell} e^{-\alpha|X_1|_\infty} |Du_\ell|^2 \leq C \int_{\Omega_\ell} e^{-\alpha|X_1|_\infty} |D\tilde{g}|^2.$$

From here as in corollary 3 because $\tilde{g} = 0$ in $\Omega_{\ell-\frac{c}{r}}$, for $0 < \ell_0 < \ell - \frac{c}{r}$ we obtain

$$\int_{\Omega_{\ell_0}} |Du_\ell|^2 \leq Ce^{-\alpha(\ell-\ell_0)} \int_{\Omega_\ell \setminus \Omega_{\ell-\frac{c}{r}}} |D\tilde{g}|^2$$

and by the estimate

$$\int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} |D\tilde{g}|^2 \leq \int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} \{|Dh|^2 + |Dg|^2\}$$

$$\leq \int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} \{r^2 + |Dg|^2\} = |\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}|r^2 + \int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} |Dg|^2$$

$$\leq C\ell^{q-1} + \int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} |Dg|^2$$

the lemma is proved. □

LEMMA 9. *Let K be the closed unit ball in $\mathbb{R}^{d+1}$ with respect to the euclidean norm, then we have*

$$\int_{\Omega_{\ell_0}} |Du_\ell|^2 \leq C\ell^{q-1}e^{-\alpha(\ell-\ell_0)}.$$

PROOF. Let us consider the function

$$h(X_1) = \sin\big((|X_1|_\infty - (\ell - \frac{\pi}{2}))^+\big)$$

then $h = 0$ in $\Omega_{\ell-\frac{\pi}{2}}$, $h = 1$ on Δ_ℓ and $h^2 + |Dh|^2 = 1$.

Then as in the previous lemma we may define

$$\tilde{g} = \min\big(\max\big(g, -h\big), h\big)$$

and $\phi(X_1) = \alpha|X_1|_\infty$ for $|X_1|_\infty \leq \ell - \frac{\pi}{2}$ and $\phi(X_1) = \alpha(\ell - \frac{\pi}{2})$ for $\ell - \frac{\pi}{2} \leq |X_1|_\infty \leq \ell$.

As in the previous lemma by the fact that $\tilde{g} = 0$ in $\Omega_{\ell-\frac{\pi}{2}}$ by choosing α sufficiently small the condition (89) will hold.

Again in the domain $\Omega_\ell\backslash\Omega_{\ell-\frac{\pi}{2}}$ we have $D_{X_1}\phi = 0$ so we should check if $(\tilde{g}, D\tilde{g}) \in K$ a.e. in Ω_ℓ. We have that $(\tilde{g}, D\tilde{g})$ is either equal to (g, Dg), (h, Dh) or $-(h, Dh)$, so by the equation $h^2 + |Dh|^2 = 1$ we have $(\tilde{g}, D\tilde{g}) \in K$ a.e. in Ω_ℓ. Hence also in this domain the condition (89) holds.

So by the theorem for sufficiently small α and because $\tilde{g} = 0$ in $\Omega_{\ell-\frac{\pi}{2}}$, for $0 < \ell_0 < \ell - \frac{\pi}{2}$ we obtain

$$\int_{\Omega_{\ell_0}} |Du_\ell|^2 \leq Ce^{-\alpha(\ell-\ell_0)} \int_{\Omega_\ell\backslash\Omega_{\ell-\frac{\pi}{2}}} |D\tilde{g}|^2$$

and by the estimate

$$\int_{\Omega_\ell\backslash\Omega_{\ell-\frac{\pi}{2}}} |D\tilde{g}|^2 \leq C\ell^{q-1}$$

the lemma is proved. □

CHAPTER 5

Asymptotic of the Difference of Two Solutions

In this chapter we consider the case when the boundary condition is $g = 0$ and the force term f is periodic in the lateral direction of the cylinder or defined on the cross section of the cylinder, in these cases we show that the solution converges to a function which is itself solution of a problem in the periodic cell or defined on the cross section.

1. Notation

Let us define $v \in H^1(\Omega_\ell)$ as periodic in the directions $e_1, e_2, \ldots, e_q$ in Ω_ℓ, if for $i = 1, \ldots, q$ the trace of v on $(-\ell, \ell)^{i-1} \times \{-\ell\} \times (-\ell, \ell)^{q-i} \times \omega$ is equal to its trace on $(-\ell, \ell)^{i-1} \times \{\ell\} \times (-\ell, \ell)^{q-i} \times \omega$, and we denote

$$H^1_{per}(\Omega_\ell) = \Big\{ v \in H^1_0(\Omega_\ell; \Gamma_\ell) \;\Big|\; v \text{ is periodic as defined above.} \Big\}$$

and consider the norm of $H^1_0(\Omega_\ell; \Gamma_\ell)$ for $H^1_{per}(\Omega_\ell)$.

Let us define the periodic cell

$$Q = (0,1)^q \times \omega$$

and its lateral boundary

$$\Gamma = (0,1)^q \times \partial\omega.$$

Let us define the space

$$H^1_0(Q; \Gamma) = \Big\{ v \in H^1(Q) \;\Big|\; v = 0 \text{ on } \Gamma \Big\}$$

with the norm

$$\|v\|_{H^1_0(Q;\Gamma)} = \||Dv|_2\|_{L^2(Q)}.$$

We define $v \in H^1_0(Q; \Gamma)$ as periodic in the directions $e_1, e_2, \ldots, e_q$ in Q, if for $i = 1, \ldots, q$ the trace of v on $(0,1)^{i-1} \times \{0\} \times (0,1)^{q-i} \times \omega$ is equal to its trace on $(0,1)^{i-1} \times \{1\} \times (0,1)^{q-i} \times \omega$, and we set

$$H^1_{per}(Q) = \Big\{ v \in H^1_0(Q; \Gamma) \;\Big|\; v \text{ is periodic as defined above.} \Big\}$$

and consider the norm of $H^1_0(Q; \Gamma)$ for $H^1_{per}(Q)$.

As in the previous chapter let us consider a closed convex set $K \subset \mathbb{R}^{d+1}$ such that $0 \in K$ and define the following closed convex sets

$$\mathcal{K}_{per}(\Omega_\ell) = \Big\{ v \in H^1_{per}(\Omega_\ell) \;\Big|\; (v, Dv) \in K \text{ a.e. in } \Omega_\ell \Big\}$$

$$\mathcal{K}_{per}(Q) = \Big\{ v \in H^1_{per}(Q) \;\Big|\; (v, Dv) \in K \text{ a.e. in } Q \Big\}$$

and

$$\mathcal{K}_0(\Omega_\ell; \Gamma_\ell) = \Big\{ v \in H^1_0(\Omega_\ell; \Gamma_\ell) \;\Big|\; (v, Dv) \in K \text{ a.e in } \Omega_\ell \Big\}$$

$$\mathcal{K}(\omega) = \Big\{ v \in H_0^1(\omega) \;\Big|\; (v, \underbrace{0, 0, \cdots, 0}_{q \text{ times}}, D_{X_2} v) \in K \text{ a.e. in } \omega \Big\}.$$

2. Preliminary Analysis

In the following we bring a general definition of a force which is periodic in the directions $e_1, e_2, \dots, e_q$.

DEFINITION 1. *For a positive integer ℓ, every $f \in (H^1_{per}(Q))^*$ defines a $f_\ell \in (H^1_{per}(\Omega_\ell))^*$ because for all $v \in H^1_{per}(\Omega_\ell)$ we may define*

$$\tilde{v}(X_1, X_2) = \frac{1}{(2\ell)^q} \sum_{i \in \{-\ell, \dots, \ell-1\}^q} v(X_1 + i, X_2)$$

then $\tilde{v} \in H^1_{per}(Q)$ and we may define

$$\langle f_\ell, v \rangle = (2\ell)^q \langle f, \tilde{v} \rangle .$$

For the norm of f_ℓ we may estimate for any $v \in H^1_{per}(\Omega_\ell)$

$$\begin{aligned}
|\langle f_\ell, v \rangle| = (2\ell)^q |\langle f, \tilde{v} \rangle| &\le (2\ell)^q \|f\|_{(H^1_{per}(Q))^*} \|\tilde{v}\|_{H^1_{per}(Q)} \\
&= (2\ell)^q \|f\|_{(H^1_{per}(Q))^*} \Big(\int_Q |D\tilde{v}|^2 \Big)^{\frac{1}{2}} \\
&= (2\ell)^q \|f\|_{(H^1_{per}(Q))^*} \Big(\int_Q \Big| \frac{1}{(2\ell)^q} \sum_{i \in \{-\ell, \dots, \ell-1\}^q} Dv(X_1 + i, X_2) \Big|^2 \Big)^{\frac{1}{2}} \\
&\le \|f\|_{(H^1_{per}(Q))^*} \Big(\int_Q (2\ell)^q \sum_{i \in \{-\ell, \dots, \ell-1\}^q} \big| Dv(X_1 + i, X_2) \big|^2 \Big)^{\frac{1}{2}} \\
&= (2\ell)^{\frac{q}{2}} \|f\|_{(H^1_{per}(Q))^*} \|v\|_{H^1_{per}(\Omega_\ell)}
\end{aligned}$$

hence

$$\|f_\ell\|_{(H^1_{per}(\Omega_\ell))^*} \le (2\ell)^{\frac{q}{2}} \|f\|_{(H^1_{per}(Q))^*} .$$

REMARK 4. *If $v \in \mathcal{K}_{per}(\Omega_\ell)$ then $\tilde{v} \in \mathcal{K}_{per}(Q)$, this is because for $x = (X_1, X_2) \in Q$, $(\tilde{v}(x), D\tilde{v}(x))$ is the mean value of $(v(X_1+i, X_2), Dv(X_1+i, X_2))$ for $i \in \{-\ell, \dots, \ell-1\}^q$ so by convexity of K because for each i, $(v(X_1+i, X_2), Dv(X_1+i, X_2)) \in K$ so we have $(\tilde{v}(x), D\tilde{v}(x)) \in K$.*

In the following we bring a general definition of a force which is defined on the cross section of the cylinder.

DEFINITION 2. *For $\ell > 0$, every $f \in H^{-1}(\omega)$ defines a $f_\ell \in (H^1_0(\Omega_\ell; \Gamma_\ell))^*$ because for all $v \in H^1_0(\Omega_\ell; \Gamma_\ell)$ we may define*

$$\tilde{v}(X_2) = \frac{1}{(2\ell)^q} \int_{(-\ell, \ell)^q} v(X_1, X_2) dX_1$$

then $\tilde{v} \in H^1_0(\omega)$ and we define

$$\langle f_\ell, v \rangle = (2\ell)^q \langle f, \tilde{v} \rangle .$$

For the norm of f_ℓ one may estimate

$$\|f_\ell\|_{(H^1_0(\Omega_\ell; \Gamma_\ell))^*} \le (2\ell)^{\frac{q}{2}} \|f\|_{H^{-1}(\omega)} .$$

REMARK 5. *If $v \in \mathcal{K}_0(\Omega_\ell; \Gamma_\ell)$ then $\tilde{v} \in \mathcal{K}(\omega)$.*

Let us define the solution u_∞ for two cases of periodic force and when the force is defined on the cross section which will be the limit of the asymptotic behavior of our problems.

DEFINITION 3. *For the case of periodic force* $f \in (H_{per}(Q))^*$, $u_\infty \in \mathcal{K}_{per}(Q)$ *is the solution of the following variational inequality*

$$\int_Q Du_\infty \cdot (Du_\infty - Dv) \leq \langle f, u_\infty - v \rangle\,, \quad \forall v \in \mathcal{K}_{per}(Q) \tag{91}$$

and in the case of forces defined on the cross section $f \in H^{-1}(\omega)$, $u_\infty \in \mathcal{K}(\omega)$ *is the solution of*

$$\int_\omega D_{X_2} u_\infty \cdot (D_{X_2} u_\infty - D_{X_2} v) \leq \langle f, u_\infty - v \rangle\,, \quad \forall v \in \mathcal{K}(\omega). \tag{92}$$

In the following lemma we show that the solution u_∞ which is defined either on Q or ω might be extended to a solution in Ω_ℓ.

LEMMA 10. *In the case of periodic force, for* ℓ *a positive integer, if we extend* u_∞ *periodically in the directions* $e_1, \ldots, e_q$ *we will have* $u_\infty \in \mathcal{K}_{per}(\Omega_\ell)$ *and it is the solution of the following variational inequality*

$$\begin{cases} \displaystyle\int_{\Omega_\ell} Du_\infty \cdot (Du_\infty - Dv) \leq \langle f_\ell, u_\infty - v \rangle\,, \quad \forall v \in \mathcal{K}_{per}(\Omega_\ell) \\ u_\infty \in \mathcal{K}_{per}(\Omega_\ell) \end{cases} \tag{93}$$

and in the case of forces defined on the cross section, for $\ell > 0$ *, if we continue* u_∞ *constantly in the directions* $e_1, \ldots, e_q$ *we will have* $u_\infty \in \mathcal{K}_0(\Omega_\ell; \Gamma_\ell)$ *and it is the solution of*

$$\begin{cases} \displaystyle\int_{\Omega_\ell} Du_\infty \cdot (Du_\infty - Dv) \leq \langle f_\ell, u_\infty - v \rangle\,, \quad \forall v \in \mathcal{K}_0(\Omega_\ell; \Gamma_\ell) \\ u_\infty \in \mathcal{K}_0(\Omega_\ell; \Gamma_\ell). \end{cases} \tag{94}$$

PROOF. In the case of periodic force, as in the definition 1 and by the remark 4 we may take any $v \in \mathcal{K}_{per}(\Omega_\ell)$ and define $\tilde{v} \in \mathcal{K}_{per}(Q)$, testing the equation (91) by this test function we have

$$\int_Q Du_\infty \cdot \big(Du_\infty - \frac{1}{(2\ell)^q} \sum_{i \in \{-\ell, \ldots, \ell-1\}^q} Dv(X_1 + i, X_2)\big)$$
$$\leq \big\langle f, u_\infty - \frac{1}{(2\ell)^q} \sum_{i \in \{-\ell, \ldots, \ell-1\}^q} v(X_1 + i, X_2) \big\rangle.$$

By the periodicity of u_∞ we have

$$u_\infty(x) = \frac{1}{(2\ell)^q} \sum_{i \in \{-\ell, \ldots, \ell-1\}^q} u_\infty(X_1 + i, X_2)$$

so we have

$$\int_Q Du_\infty \cdot D\big(\frac{1}{(2\ell)^q} \sum_{i \in \{-\ell, \ldots, \ell-1\}^q} (u_\infty - v)\,(X_1 + i, X_2)\big)$$
$$\leq \big\langle f, \frac{1}{(2\ell)^q} \sum_{i \in \{-\ell, \ldots, \ell-1\}^q} (u_\infty - v)\,(X_1 + i, X_2) \big\rangle.$$

By the change of variable on the left hand side and the definition of f_ℓ on the right hand side we obtain

$$\sum_{i\in\{-\ell,\dots,\ell-1\}^q} \int_{Q+i} Du_\infty \cdot (Du_\infty - Dv) \le \langle f_\ell, u_\infty - v\rangle$$

that is

$$\int_{\Omega_\ell} Du_\infty \cdot (Du_\infty - Dv) \le \langle f_\ell, u_\infty - v\rangle$$

and this proves the lemma for the case of periodic force.

In the case of forces defined on the cross section, as in definition 2 and by the remark 5 we may take any $v \in \mathcal{K}_0(\Omega_\ell;\Gamma_\ell)$ and define $\tilde{v} \in \mathcal{K}(\omega)$, testing the equation (92) by this test function we have

$$\int_\omega D_{X_2}u_\infty \cdot \big(D_{X_2}u_\infty - \frac{1}{(2\ell)^q}\int_{(-\ell,\ell)^q} D_{X_2}v(X_1,X_2)dX_1\big)dX_2$$
$$\le \big\langle f, u_\infty - \frac{1}{(2\ell)^q}\int_{(-\ell,\ell)^q} v(X_1,X_2)dX_1\big\rangle.$$

Because u_∞ does not depend on X_1 we have

$$u_\infty = \frac{1}{(2\ell)^q}\int_{(-\ell,\ell)^q} u_\infty dX_1$$

so we have

$$\int_\omega Du_\infty \cdot \big(\frac{1}{(2\ell)^q}\int_{(-\ell,\ell)^q} D\left(u_\infty - v\right)(X_1,X_2)dX_1\big)dX_2$$
$$\le \big\langle f, \frac{1}{(2\ell)^q}\int_{(-\ell,\ell)^q} \left(u_\infty - v\right)(X_1,X_2)dX_1\big\rangle.$$

By Fubini's theorem on the left hand side and the definition of f_ℓ on the right hand side we obtain

$$\int_{\Omega_\ell} Du_\infty \cdot (Du_\infty - Dv) \le \langle f_\ell, u_\infty - v\rangle$$

and this finishes the proof of the lemma. □

In the case of periodic force we consider a $f \in (H^1_{per}(Q))^*$ and by definition 1 we have the corresponding $f_\ell \in (H^1_{per}(\Omega_\ell))^*$. We have that $H^1_0(\Omega_\ell)$ is a subspace of $H^1_{per}(\Omega_\ell)$, so $f_\ell \in H^{-1}(\Omega_\ell)$ and we consider u_ℓ to be the solution of the variational inequality (86) with $g = 0$ and this periodic f_ℓ, that is

$$\begin{cases} \displaystyle\int_{\Omega_\ell} Du_\ell \cdot (Du_\ell - Dv) \le \langle f_\ell, u_\ell - v\rangle\,, \quad \forall v \in \mathcal{K}_0(\Omega_\ell) \\ u_\ell \in \mathcal{K}_0(\Omega_\ell). \end{cases}$$

In the case of forces defined on the cross section we consider $f \in H^{-1}(\omega)$ and by definition 2 we have the corresponding $f_\ell \in (H^1_0(\Omega_\ell;\Gamma_\ell))^*$. We have that $H^1_0(\Omega_\ell)$ is a subspace of $H^1_0(\Omega_\ell;\Gamma_\ell)$, so $f_\ell \in H^{-1}(\Omega_\ell)$ and again as above we consider u_ℓ to be the solution of the variational inequality (86) with $g = 0$ and this f_ℓ.

3. $K = K_0 \times K'$ and $f \geq 0$

In this section we study the asymptotic behavior of the solution u_ℓ when $f \geq 0$, $K = K_0 \times K'$ where K_0 is a closed interval containing 0 and K' is a bounded, closed and convex subset of $\mathbb{R}^d$ and for some $r > 0$ we have $B_r^q(0) \times \{0\}^{d-q} \in K'$.

LEMMA 11. *In both cases of periodic force and forces defined on the cross section we have*

$$0 \leq u_\infty.$$

PROOF. In both cases testing the inequality (91) or (92) by $v = u_\infty^+$ and doing similar estimates as we did in lemma 6 we prove this lemma. □

In the following lemma using the comparison principle 6 we prove inequalities showing that u_ℓ is positive, monotone growing in ℓ on a fixed domain, and bounded by u_∞.

LEMMA 12. *In both cases of periodic force and forces defined on the cross section we have*

(1)
$$0 \leq u_\ell$$

(2)
$$u_\ell \leq u_{\ell'} \text{ in } \Omega_\ell \text{ for } \ell' \geq \ell$$

(3)
$$u_\ell \leq u_\infty$$

PROOF. We bring the proof for the case of forces defined on the cross section. For the case of periodic forces the proof is similar.

Let us notice that because $f \geq 0$ then $-f_\ell \leq 0$, so by the corollary 1 we obtain $-u_\ell \leq 0$ because $g = 0$, so $0 \leq u_\ell$.

We may consider for $\ell' \geq \ell$, $u_{\ell'}$ as the solution to the variational inequality (86) in Ω_ℓ with the Dirichlet boundary condition $u_{\ell'}$ on Δ_ℓ. By this consideration and lemma 6 taking $u_1 = u_{\ell'}$, $u_2 = u_\ell$ and $f_1 = f_2 = f_\ell$, because on Δ_ℓ, $u_\ell = 0$ and $u_{\ell'} \geq 0$ we have $u_\ell - u_{\ell'} \leq 0$ on Δ_ℓ, hence $u_\ell - u_{\ell'} \leq 0$ in Ω_ℓ, that is $u_\ell \leq u_{\ell'}$ in Ω_ℓ.

By the previous lemma we have that $0 \leq u_\infty$. Considering the inequality (94) for u_∞, by lemma 6 taking $u_1 = u_\infty$, $u_2 = u_\ell$ and $f_1 = f_2 = f_\ell$, because on Δ_ℓ, $u_\ell = 0$ and $u_\infty \geq 0$ we have $u_\ell - u_\infty \leq 0$ on Δ_ℓ hence $u_\ell - u_\infty \leq 0$ in Ω_ℓ, that is $u_\ell \leq u_\infty$ in Ω_ℓ. □

LEMMA 13. *There exists a* $\tilde{u}_\infty \in H^1_{loc}(\mathbb{R}^q \times \omega)$ *such that* $(\tilde{u}_\infty, D\tilde{u}_\infty) \in K$ *a.e. in* $\mathbb{R}^q \times \omega$ *,*$\tilde{u}_\infty = 0$ *on* $\mathbb{R}^q \times \partial\omega$*,* $\tilde{u}_\infty \leq u_\infty$ *and for all* $\ell_0 > 0$

$$u_\ell \nearrow \tilde{u}_\infty \text{ a.e. in } \Omega_{\ell_0},\ u_\ell \to \tilde{u}_\infty \text{ in } L^2(\Omega_{\ell_0}),\ Du_\ell \rightharpoonup D\tilde{u}_\infty \text{ in } L^2(\Omega_{\ell_0}).$$

PROOF. Let us define

$$\tilde{u}_\infty(x) = \sup_\ell u_\ell(x)$$

then by the previous lemma the sequence of functions u_ℓ is positive and bounded by u_∞ so $0 \leq \tilde{u}_\infty \leq u_\infty$ and u_ℓ monotonically increasing converges to $\tilde{u}_\infty$. By the inequalities $0 \leq \tilde{u}_\infty \leq u_\infty$ we have $\tilde{u}_\infty \in L^2(\Omega_{\ell_0})$ and for fixed $\ell_0 > 0$, because $Du_\ell \in K'$ a.e. and K' is bounded we have that u_ℓ is a uniformly bounded sequence of $H^1_0(\Omega_{\ell_0}; \Gamma_\ell)$ thus for any sequence ℓ_k, there is a subsequence ℓ_{k_n} such that the sequence $u_{\ell_{k_n}}$ converges weakly in $H^1_0(\Omega_{\ell_0};\Gamma_\ell)$ to a function in that space, but this function can only be the function $\tilde{u}_\infty$, hence $\tilde{u}_\infty \in H^1_{loc}(\mathbb{R}^q \times \omega)$ and

$$u_\ell \rightharpoonup \tilde{u}_\infty \text{ in } H^1_0(\Omega_{\ell_0};\Gamma_\ell),\; u_\ell \to \tilde{u}_\infty \text{ in } L^2(\Omega_{\ell_0}).$$

Since $\mathcal{K}_0(\Omega_{\ell_0};\Gamma_\ell)$ is a closed convex subset of $H^1_0(\Omega_{\ell_0};\Gamma_\ell)$, it is weakly closed hence $\tilde{u}_\infty$ belongs to this convex set, so $(\tilde{u}_\infty, D\tilde{u}_\infty) \in K$ a.e. in $\mathbb{R}^q \times \omega$. □

Let us note that in the lemma above actually one may show convergence in smaller spaces.

LEMMA 14. *In the case of periodic force the function $\tilde{u}_\infty$ is periodic and in the case of a force defined on the cross section the function $\tilde{u}_\infty$ is independent of X_1.*

PROOF. We bring the proof for the case of forces defined on the cross section, the proof for the case of periodic forces is similar.

We claim that for $h > 0$, and $i = 1, \dots, q$

$$u_\ell(X_1 - he_i, X_2) \le u_{\ell+h}(X_1, X_2). \tag{95}$$

Let us denote $\sigma^i_h v = v(X_1 - he_i, X_2)$, that is the function v shifted to the right in the direction e_i.

For any $v \in H^1(\Omega_\ell + he_i)$ with the boundary values equal to $u_{\ell+h}$, because $\Omega_\ell + he_i \subset \Omega_{\ell+h}$ if we continue v by $u_{\ell+h}$ on $\Omega_{\ell+h} \backslash (\Omega_\ell + he_i)$ we will have

$$\int_{\Omega_\ell + he_i} Du_{\ell+h} \cdot D(u_{\ell+h} - v) \le \left\langle f, \int_{(-\ell,\ell)^q + he_i} u_{\ell+h} - v \right\rangle.$$

Then by a change of variable for any $v \in H^1_0(\Omega_\ell;\Gamma_\ell)$ with boundary values $v = \sigma^i_{-h} u_{\ell+h}$ on Δ_ℓ we have

$$\int_{\Omega_\ell} D(\sigma^i_{-h} u_{\ell+h}) \cdot D(\sigma^i_{-h} u_{\ell+h} - v) \le \left\langle f, \int_{(-\ell,\ell)^q} \sigma^i_{-h} u_{\ell+h} - v \right\rangle$$
$$= \left\langle f_\ell, \sigma^i_{-h} u_{\ell+h} - v \right\rangle.$$

Now by lemma 6 taking $u_1 = \sigma^i_{-h} u_{\ell+h}$, $u_2 = u_\ell$ and $f_1 = f_2 = f_\ell$, because on Δ_ℓ, u_ℓ is zero and $\sigma^i_{-h} u_{\ell+h} \ge 0$ we obtain $u_\ell \le \sigma^i_{-h} u_{\ell+h}$ in Ω_ℓ. This gives us the inequality (95).

In a similar way we get

$$u_\ell(X_1 + he_i, X_2) \le u_{\ell+h}(X_1, X_2). \tag{96}$$

Passing to the limit in the inequality (95) and the inequality (96) we obtain for all $h > 0$

$$\tilde{u}_\infty(X_1 - he_i, X_2) \le \tilde{u}_\infty(X_1, X_2) \le \tilde{u}_\infty(X_1 - he_i, X_2)$$

which proves that $\tilde{u}_\infty$ is independent of X_1. Let us note that in the case of periodic forces the ℓ and h should be integers. □

LEMMA 15. *In both cases of periodic forces and forces defined on the cross section we have $u_\infty = \tilde{u}_\infty$.*

PROOF. We claim that

$$\int_{\Omega_\ell} |D(u_\ell - u_\infty)|^2 \le C\ell^{q-1} \tag{97}$$

where C is a constant independent of ℓ. Let us note that to prove this inequality we do not use the positivity of f. Again here we bring the proof for the case of forces defined on the cross section, the proof for periodic forces is similar.

Let us define

$$d(X_1) = r\,\mathrm{dist}\big(X_1, \partial((-\ell,\ell)^q)\big)$$

then we have

$$|D_{X_1} d(X_1)| = r\big|D_{X_1}\,\mathrm{dist}\big(X_1, \partial((-\ell,\ell)^q)\big)\big| \leq r$$

hence we have $Dd \in K'$.

Let us consider

$$v_\ell = \min\Big(\max\big(\frac{u_\ell + u_\infty}{2}, -d(X_1)\big), d(X_1)\Big)$$

because $d(X_1) = 0$ on Δ_ℓ we have $v_\ell = 0$ on Δ_ℓ, and because u_ℓ and u_∞ are 0 on Γ_ℓ we have v_ℓ is 0 on Γ_ℓ.

If $v_\ell \leq 0$ then $\frac{u_\infty + u_\ell}{2} \leq v_\ell \leq 0$ and if $0 \leq v_\ell$ then $0 \leq v_\ell \leq \frac{u_\infty + u_\ell}{2}$, so $v_\ell \in K_0$ a.e. in Ω_ℓ. The gradient Dv_ℓ is equal to $\frac{1}{2}(Du_\ell + Du_\infty)$, $-Dd(X_1)$ or $Dd(X_1)$, so we have $Dv_\ell \in K'$ a.e. in Ω_ℓ, hence $v_\ell \in \mathcal{K}_0(\Omega_\ell)$.

By the equality

$$z - \min(\max(z, -d), d) = (z-d)^+ - (z+d)^-$$

we may compute

$$\begin{aligned} u_\infty - v_\ell &= \Big\{u_\infty - \frac{1}{2}(u_\ell + u_\infty)\Big\} \\ &\quad + \Big\{\frac{1}{2}(u_\ell + u_\infty) - \min\Big(\max\big(\frac{u_\ell + u_\infty}{2}, -d(X_1)\big), d(X_1)\Big)\Big\} \\ &= -\frac{1}{2}(u_\ell - u_\infty) + \big(\frac{1}{2}(u_\ell + u_\infty) - d(X_1)\big)^+ - \big(\frac{1}{2}(u_\ell + u_\infty) + d(X_1)\big)^- \end{aligned}$$

and similarly

$$u_\ell - v_\ell = \frac{1}{2}(u_\ell - u_\infty) + \big(\frac{1}{2}(u_\ell + u_\infty) - d(X_1)\big)^+ - \big(\frac{1}{2}(u_\ell + u_\infty) + d(X_1)\big)^-.$$

For simplicity let us denote

$$s = \big(\frac{1}{2}(u_\ell + u_\infty) - d(X_1)\big)^+ - \big(\frac{1}{2}(u_\ell + u_\infty) + d(X_1)\big)^-.$$

Testing the inequality satisfied by u_∞ by v_ℓ we obtain

$$-\frac{1}{2}\int_{\Omega_\ell} Du_\infty \cdot D(u_\ell - u_\infty) \leq -\frac{1}{2}\langle f_\ell, u_\ell - u_\infty\rangle - \int_{\Omega_\ell} Du_\infty \cdot Ds + \langle f_\ell, s\rangle.$$

Similarly testing the inequality satisfied by u_ℓ by v_ℓ we obtain

$$\frac{1}{2}\int_{\Omega_\ell} Du_\ell \cdot D(u_\ell - u_\infty) \leq \frac{1}{2}\langle f_\ell, u_\ell - u_\infty\rangle - \int_{\Omega_\ell} Du_\ell \cdot Ds + \langle f_\ell, s\rangle.$$

Summing these two inequalities we obtain

$$\frac{1}{2}\int_{\Omega_\ell} |Du_\ell - Du_\infty|^2 \leq -\int_{\Omega_\ell} \big(Du_\ell + Du_\infty\big) \cdot Ds + 2\langle f_\ell, s\rangle. \tag{98}$$

Let $c > 0$ denote the uniform bound of u_ℓ and u_∞.

In $\Omega_{\ell - \frac{c}{r}}$ we have

$$\mathrm{dist}\big(X_1, \partial((-\ell,\ell)^q)\big) \geq \frac{c}{r}$$

therefore in $\Omega_{\ell-\frac{c}{r}}$

$$d(X_1) \geq c$$

and then

$$s = \big(\frac{1}{2}(u_\ell + u_\infty) - d(X_1)\big)^+ - \big(\frac{1}{2}(u_\ell + u_\infty) + d(X_1)\big)^- = 0.$$

Now let us estimate the terms on the right hand side of the inequality (98). For the first term by the constraint on the gradient because K' is bounded we have Du_ℓ, Du_∞ and Ds are bounded so

$$\int_{\Omega_\ell} \big(Du_\ell + Du_\infty\big) \cdot Ds = \int_{\Omega_\ell \backslash \Omega_{\ell-\frac{c}{r}}} \big(Du_\ell + Du_\infty\big) \cdot Ds \leq C|\Omega_\ell \backslash \Omega_{\ell-\frac{c}{r}}| \leq C\ell^{q-1}.$$

For the second term we have

$$\langle f_\ell, s\rangle = \big\langle f, \int_{(-\ell,\ell)^q \backslash (-(\ell-\frac{c}{r}),\ell-\frac{c}{r})^q} s\big\rangle$$
$$\leq \|f\|_{H^{-1}(\omega)} \Big\| \int_{(-\ell,\ell)^q \backslash (-(\ell-\frac{c}{r}),\ell-\frac{c}{r})^q} s\Big\|_{H_0^1(\omega)}$$

and we may estimate

$$\Big\| \int_{(-\ell,\ell)^q \backslash (-(\ell-\frac{c}{r}),\ell-\frac{c}{r})^q} s\Big\|^2_{H_0^1(\omega)}$$
$$\leq \int_\omega \Big\{ \int_{(-\ell,\ell)^q \backslash (-(\ell-\frac{c}{r}),\ell-\frac{c}{r})^q} |D_{X_2} s| dX_1 \Big\}^2 dX_2$$
$$\leq C|((-\ell,\ell)^q \backslash (-(\ell-\frac{c}{r}),\ell-\frac{c}{r})^q|^2 \leq C(\ell^{q-1})^2$$

so we have

$$\langle f_\ell, s\rangle \leq C\ell^{q-1}.$$

Using these estimates for the right hand side terms in the inequality (98) we obtain the inequality (97).

In the case $f \geq 0$ the inequality (97) shows that $u_\infty = \tilde{u}_\infty$, indeed by the Poincaré inequality for the domain ω we have

$$\int_{\Omega_\ell} |u_\ell - u_\infty|^2 \leq c_\omega^2 \int_{\Omega_\ell} |D(u_\ell - u_\infty)|^2 \leq C\ell^{q-1} \tag{99}$$

and by the lemmas 12 and 13 we have

$$u_\ell \leq \tilde{u}_\infty \leq u_\infty$$

hence

$$0 \leq u_\infty - \tilde{u}_\infty \leq u_\infty - u_\ell \Longrightarrow |\tilde{u}_\infty - u_\infty| \leq |u_\ell - u_\infty|$$

so by the inequality (99) we have

$$\int_{\Omega_\ell} |\tilde{u}_\infty - u_\infty|^2 \leq C\ell^{q-1}. \tag{100}$$

Now by lemma 14, $\tilde{u}_\infty$ is independent of X_1 hence

$$\int_{\Omega_\ell} |\tilde{u}_\infty - u_\infty|^2 = (2\ell)^q \int_\omega |\tilde{u}_\infty - u_\infty|^2$$

by this equality and the inequality (100) we obtain

$$\int_\omega |\tilde{u}_\infty - u_\infty|^2 \leq \frac{C}{\ell}$$

which implies that $u_\infty = \tilde{u}_\infty$. □

THEOREM 5. *For all $\ell_0 > 0$*

$$u_\ell \nearrow u_\infty \text{ a.e. in } \Omega_{\ell_0},\ u_\ell \to u_\infty \text{ in } L^2(\Omega_{\ell_0}),\ Du_\ell \rightharpoonup Du_\infty \text{ in } L^2(\Omega_{\ell_0}).$$

PROOF. The proof follows by lemmas 13 and 15. □

4. $K = K_0 \times K_1 \times K_2$ and $f = f(X_2)$

In this section we consider the case when the force is defined in the cross section, $K = K_0 \times K_1 \times K_2$ where K_0, K_1 and K_2 are respectively closed and convex subsets of $\mathbb{R}, \mathbb{R}^q$ and $\mathbb{R}^{d-q}$, $0 \in K_0$, $0 \in K_2$ and for some $r > 0$, $B^q_r(0) \in K_1$.

THEOREM 6. *There exists $\alpha > 0$ depending on r, ω and the uniform bound on u_ℓ and u_∞ such that*

$$\int_{\Omega_{\frac{1}{2}\ell}} |D(u_\ell - u_\infty)|^2 \leq C(1 + \|f\|^2_{H^{-1}(\omega)})e^{-\alpha\ell}.$$

PROOF. Let $c > 0$ be the uniform bound of u_ℓ and u_∞. Let us consider

$$h(X_1) = r(|X_1|_\infty - (\ell - \frac{c}{r}))^+$$

then $h = 0$ in $\Omega_{\ell - \frac{c}{r}}$, $h = c$ on Δ_ℓ and $D_{X_1}h \in B^q_r(0) \subset K_1$.

Let us define

$$g_\infty = \min\big(\max\big(u_\infty, -h\big), h\big)$$

then $g_\infty \in H^1_0(\Omega_\ell; \Gamma_\ell)$ and $g_\infty = 0$ in $\Omega_{\ell - \frac{c}{r}}$, $g_\infty = u_\infty$ on Δ_ℓ, $D_{X_1}g_\infty$ is equal either to 0, $D_{X_1}h$ or $-D_{X_1}h$ and $D_{X_2}g_\infty$ is either equal to $D_{X_2}u_\infty$ or 0.

Let us define

$$\rho(X_1) = \begin{cases} \frac{1}{2}e^{-\alpha(\frac{\ell}{2} - |X_1|_\infty)} & , |X_1|_\infty \leq \frac{\ell}{2} \\ 1 - \frac{1}{2}e^{-\alpha(|X_1|_\infty - \frac{\ell}{2})} & , \frac{\ell}{2} < |X_1|_\infty \leq \ell - \frac{c}{r} \\ 1 - \frac{1}{2}e^{-\alpha((\ell - \frac{c}{r}) - \frac{\ell}{2})} & , \ell - \frac{c}{r} < |X_1|_\infty \leq \ell. \end{cases}$$

Clearly we have

$$D_{X_1}\rho(X_1) = \begin{cases} \frac{1}{2}\alpha e^{-\alpha(\frac{\ell}{2} - |X_1|_\infty)} D_{X_1}(|X_1|_\infty) & , |X_1|_\infty \leq \frac{\ell}{2} \\ \frac{1}{2}\alpha e^{-\alpha(|X_1|_\infty - \frac{\ell}{2})} D_{X_1}(|X_1|_\infty) & , \frac{\ell}{2} < |X_1|_\infty \leq \ell - \frac{c}{r} \\ 0 & , \ell - \frac{c}{r} < |X_1|_\infty \leq \ell \end{cases}$$

so because $|D_{X_1}(|X_1|_\infty)| \leq 1$ we have

$$|D_{X_1}\rho(X_1)| \leq \begin{cases} \alpha \min(\rho(X_1), 1 - \rho(X_1)) & , |X_1|_\infty \leq \ell - \frac{c}{r} \\ 0 & , \ell - \frac{c}{r} < |X_1|_\infty \leq \ell. \end{cases} \tag{101}$$

Let us define

$$v_1 = (1-\rho)(u_\infty - g_\infty) + \rho u_\ell$$
$$v_2 = (1-\rho)u_\ell + \rho u_\infty.$$

Clearly we have $v_1 \in H_0^1(\Omega_\ell)$ and $v_2 \in H_0^1(\Omega_\ell; \Gamma_\ell)$.

Let us show that $v_1 \in \mathcal{K}_0(\Omega_\ell)$, to do this we should show that $(v_1, Dv_1) \in K$ a.e. in Ω_ℓ.

In the domain $\Omega_{\ell-\frac{c}{r}}$ we have $g_\infty = 0$ so

$$v_1 = (1-\rho)u_\infty + \rho u_\ell. \tag{102}$$

By (102) clearly $v_1 \in K_0$ in $\Omega_{\ell-\frac{c}{r}}$.

Let us compute the X_1 derivatives

$$D_{X_1}v_1 = \rho D_{X_1}u_\ell + (u_\ell - u_\infty)D_{X_1}\rho = \rho D_{X_1}u_\ell + (1-\rho)(u_\ell - u_\infty)\frac{D_{X_1}\rho}{1-\rho}. \tag{103}$$

By (101) we may estimate

$$|(u_\ell - u_\infty)\frac{D_{X_1}\rho}{1-\rho}| \le 2c\alpha$$

so if

$$\alpha \le \frac{r}{2c} \tag{104}$$

then

$$(u_\ell - u_\infty)\frac{D_{X_1}\rho}{1-\rho} \in B_r^q(0) \subset K_1.$$

So by (103) we have $D_{X_1}v_1 \in K_1$ a.e. in $\Omega_{\ell-\frac{c}{r}}$.

Let us compute the X_2 derivatives

$$D_{X_2}v_1 = (1-\rho)D_{X_2}u_\infty + \rho D_{X_2}u_\ell$$

so $D_{X_2}v_1 \in K_2$ a.e. in $\Omega_{\ell-\frac{c}{r}}$. Hence a.e. in $\Omega_{\ell-\frac{c}{r}}$, $(v_1, Dv_1) \in K$.

Now let us consider the domain $\Omega_\ell \backslash \Omega_{\ell-\frac{c}{r}}$. It is easy to see that if $u_\infty \ge 0$ then $0 \le u_\infty - g_\infty \le u_\infty$ and if $u_\infty \le 0$ then $u_\infty \le u_\infty - g_\infty \le 0$. So because $0 \in K_0$ we have $v_1 \in K_0$ a.e. in $\Omega_\ell \backslash \Omega_{\ell-\frac{c}{r}}$.

In the domain $\Omega_\ell \backslash \Omega_{\ell-\frac{c}{r}}$, ρ is constant. Computing the X_1 derivatives we have

$$D_{X_1}v_1 = (1-\rho)(-D_{X_1}g_\infty) + \rho D_{X_1}u_\ell$$

and because $-D_{X_1}g_\infty$ is equal to 0, $-D_{X_1}h$ or $D_{X_1}h$ we have $-D_{X_1}g_\infty \in B_r^q(0)$, so $D_{X_1}v_1 \in K_1$ a.e. in $\Omega_\ell \backslash \Omega_{\ell-\frac{c}{r}}$.

We may compute the X_2 derivatives

$$D_{X_2}v_1 = (1-\rho)D_{X_2}(u_\infty - g_\infty) + \rho D_{X_2}u_\ell$$

we have $D_{X_2}(u_\infty - g_\infty)$ is either equal to 0 or $D_{X_2}u_\infty$, so $D_{X_2}v_1 \in K_2$ a.e. in $\Omega_\ell \backslash \Omega_{\ell-\frac{c}{r}}$. Hence a.e. in $\Omega_\ell \backslash \Omega_{\ell-\frac{c}{r}}$ we have $(v_1, Dv_1) \in K$.

So a.e. in Ω_ℓ we have $(v_1, Dv_1) \in K$ and this shows that $v_1 \in \mathcal{K}_0(\Omega_\ell)$.

Now let us show that $v_2 \in \mathcal{K}_0(\Omega_\ell; \Gamma_\ell)$, to do this we should show that $(v_2, Dv_2) \in K$ a.e. in Ω_ℓ. It is clear that $v_2 \in K_0$ a.e. in Ω_ℓ.

In $\Omega_{\ell-\frac{c}{r}}$, we may compute the X_1 derivatives

$$\begin{aligned} D_{X_1}v_2 &= (1-\rho)D_{X_1}u_\ell + (u_\infty - u_\ell)D_{X_1}\rho \\ &= (1-\rho)D_{X_1}u_\ell + \rho(u_\infty - u_\ell)\frac{D_{X_1}\rho}{\rho}. \end{aligned} \tag{105}$$

By (101) we obtain

$$|(u_\infty - u_\ell)\frac{D_{X_1}\rho}{\rho}| \le 2c\alpha$$

then by (104) we have

$$(u_\infty - u_\ell)\frac{D_{X_1}\rho}{\rho} \in B_r^q(0) \subset K_1.$$

So by (105) we have $D_{X_1}v_2 \in K_1$ a.e. in $\Omega_{\ell-\frac{c}{r}}$.

We may compute the X_2 derivatives

$$D_{X_2}v_2 = (1-\rho)D_{X_2}u_\ell + \rho D_{X_2}u_\infty$$

so $D_{X_2}v_2 \in K_2$ a.e. in $\Omega_{\ell-\frac{c}{r}}$. Hence a.e. in $\Omega_{\ell-\frac{c}{r}}$, $(v_2, Dv_2) \in K$.

In the domain $\Omega_\ell \backslash \Omega_{\ell-\frac{c}{r}}$ because ρ is constant we may compute

$$Dv_2 = (1-\rho)Du_\ell + \rho Du_\infty$$

so a.e. in $\Omega_\ell \backslash \Omega_{\ell-\frac{c}{r}}$ we have $(v_2, Dv_2) \in K$.

So a.e. in Ω_ℓ we have $(v_2, Dv_2) \in K$ and this shows that $v_2 \in \mathcal{K}_0(\Omega_\ell; \Gamma_\ell)$.

Testing the inequality (94) by v_2 we have

$$\int_{\Omega_\ell} Du_\infty \cdot (Du_\infty - Dv_2) \le \langle f_\ell, u_\infty - v_2\rangle$$

and testing the inequality (86) by v_1 we have

$$\int_{\Omega_\ell} Du_\ell \cdot (Du_\ell - Dv_1) \le \langle f_\ell, u_\ell - v_1\rangle .$$

Summing these two inequalities we obtain

$$\int_{\Omega_\ell} Du_\ell \cdot (Du_\ell - Dv_1) + \int_{\Omega_\ell} Du_\infty \cdot (Du_\infty - Dv_2) \le \langle f_\ell, u_\ell + u_\infty - (v_1 + v_2)\rangle$$

and by some computation we obtain

$$\begin{aligned}\frac{1}{2}\int_{\Omega_\ell}&(Du_\ell - Du_\infty)\cdot((Du_\ell - Du_\infty) + (Dv_2 - Dv_1))\\ &\le -\frac{1}{2}\int_{\Omega_\ell}(Du_\infty + Du_\ell)\cdot((Du_\ell + Du_\infty) - (Dv_1 + Dv_2))\\ &\qquad + \langle f_\ell, u_\ell + u_\infty - (v_1 + v_2)\rangle .\end{aligned}$$

We compute

$$\begin{aligned}u_\ell + u_\infty - (v_1 + v_2) &= u_\ell + u_\infty - \big((1-\rho)(u_\infty - g_\infty) + \rho u_\ell + (1-\rho)u_\ell + \rho u_\infty\big)\\ &= u_\ell + u_\infty - (u_\ell + u_\infty - (1-\rho)g_\infty) = (1-\rho)g_\infty\end{aligned}$$

and

$$\begin{aligned}u_\ell - u_\infty + (v_2 - v_1) &= u_\ell - u_\infty + \big((1-\rho)u_\ell + \rho u_\infty - (1-\rho)(u_\infty - g_\infty) - \rho u_\ell\big)\\ &= u_\ell - u_\infty + \big((1-2\rho)(u_\ell - u_\infty) + (1-\rho)g_\infty\big)\\ &= 2(1-\rho)(u_\ell - u_\infty) + (1-\rho)g_\infty.\end{aligned}$$

So we have

$$\frac{1}{2}\int_{\Omega_\ell}(Du_\ell - Du_\infty)\cdot D(2(1-\rho)(u_\ell-u_\infty)+(1-\rho)g_\infty)$$
$$\leq -\frac{1}{2}\int_{\Omega_\ell}(Du_\infty + Du_\ell)\cdot D((1-\rho)g_\infty) + \langle f_\ell,(1-\rho)g_\infty\rangle$$

from here after some computations we obtain

$$\begin{aligned}\int_{\Omega_\ell}|Du_\ell - Du_\infty|^2(1-\rho) &\leq \int_{\Omega_\ell}(D_{X_1}u_\ell - D_{X_1}u_\infty)\cdot D_{X_1}\rho(u_\ell-u_\infty)\\ &\quad -\frac{1}{2}\int_{\Omega_\ell}(Du_\ell - Du_\infty)\cdot D((1-\rho)g_\infty)\\ &\quad -\frac{1}{2}\int_{\Omega_\ell}(Du_\infty + Du_\ell)\cdot D((1-\rho)g_\infty)+\langle f_\ell,(1-\rho)g_\infty\rangle\\ &= \int_{\Omega_\ell}(D_{X_1}u_\ell - D_{X_1}u_\infty)\cdot D_{X_1}\rho(u_\ell-u_\infty)\\ &\quad -\int_{\Omega_\ell}Du_\ell\cdot D((1-\rho)g_\infty)+\langle f_\ell,(1-\rho)g_\infty\rangle\,.\end{aligned}$$

Let us estimate the first term on the right hand side of the inequality above. First we estimate

$$\int_{\Omega_\ell}(D_{X_1}u_\ell - D_{X_1}u_\infty)\cdot D_{X_1}\rho(u_\ell-u_\infty)$$
$$\leq \int_{\Omega_\ell}|D_{X_1}u_\ell - D_{X_1}u_\infty||D_{X_1}\rho||u_\ell-u_\infty| \tag{106}$$

then by the inequality $|D_{X_1}\rho|\leq\alpha(1-\rho)$ we obtain

$$\int_{\Omega_\ell}|D_{X_1}u_\ell - D_{X_1}u_\infty||D_{X_1}\rho||u_\ell-u_\infty|$$
$$\leq \alpha\int_{\Omega_\ell}|D_{X_1}u_\ell - D_{X_1}u_\infty||u_\ell-u_\infty|(1-\rho). \tag{107}$$

Now by the Young inequality we have

$$\int_{\Omega_\ell}|D_{X_1}u_\ell - D_{X_1}u_\infty||u_\ell-u_\infty|(1-\rho)$$
$$\leq \frac{1}{2}c_\omega\int_{\Omega_\ell}|D_{X_1}u_\ell - D_{X_1}u_\infty|^2(1-\rho)+\frac{1}{2c_\omega}\int_{\Omega_\ell}|u_\ell-u_\infty|^2(1-\rho) \tag{108}$$

and by the Poincaré inequality for the domain ω we have

$$\int_{\Omega_\ell}|u_\ell-u_\infty|^2(1-\rho)\leq c_\omega^2\int_{\Omega_\ell}|D_{X_2}u_\ell - D_{X_2}u_\infty|^2(1-\rho). \tag{109}$$

By (106),(107) ,(108) and (109) we have

$$\begin{aligned}\int_{\Omega_\ell} (D_{X_1} u_\ell - D_{X_1} u_\infty) \cdot D_{X_1} \rho (u_\ell - u_\infty) \\ \leq \frac{1}{2} \alpha c_\omega \int_{\Omega_\ell} |D_{X_1} u_\ell - D_{X_1} u_\infty|^2 (1 - \rho) \\ + \frac{1}{2} \alpha c_\omega \int_{\Omega_\ell} |D_{X_2} u_\ell - D_{X_2} u_\infty|^2 (1 - \rho).\end{aligned}$$

And by the equality

$$|D_{X_1} u_\ell - D_{X_1} u_\infty|^2 + |D_{X_2} u_\ell - D_{X_2} u_\infty|^2 = |Du_\ell - Du_\infty|^2$$

we obtain

$$\int_{\Omega_\ell} (D_{X_1} u_\ell - D_{X_1} u_\infty) \cdot D_{X_1} \rho (u_\ell - u_\infty) \leq \frac{1}{2} \alpha c_\omega \int_{\Omega_\ell} |Du_\ell - Du_\infty|^2 (1 - \rho).$$

So we have

$$\begin{aligned}(1 - \frac{1}{2} \alpha c_\omega) \int_{\Omega_\ell} |Du_\ell - Du_\infty|^2 (1 - \rho) \\ \leq - \int_{\Omega_\ell} Du_\ell \cdot D((1 - \rho) g_\infty) + \langle f_\ell, (1 - \rho) g_\infty \rangle . \quad (110)\end{aligned}$$

Now let us estimate the terms on the right hand side of the inequality above. We estimate

$$\begin{aligned}- \int_{\Omega_\ell} Du_\ell \cdot D((1 - \rho) g_\infty) &= - \int_{\Omega_\ell \setminus \Omega_{\ell - \frac{c}{r}}} Du_\ell \cdot D((1 - \rho) g_\infty) \\ &\leq \int_{\Omega_\ell \setminus \Omega_{\ell - \frac{c}{r}}} |Du_\ell| (|D\rho| |g_\infty| + (1 - \rho) |Dg_\infty|) \\ &\leq \int_{\Omega_\ell \setminus \Omega_{\ell - \frac{c}{r}}} |Du_\ell| (\alpha |g_\infty| + |Dg_\infty|)(1 - \rho) \\ &\leq Ce^{-\frac{1}{2} \alpha \ell} \int_{\Omega_\ell \setminus \Omega_{\ell - \frac{c}{r}}} |Du_\ell| (\alpha |g_\infty| + |Dg_\infty|).\end{aligned}$$

We have $|g_\infty| \leq h \leq c$ and

$$|Dg_\infty| \leq \max(|D_{X_1} h|, |D_{X_2} u_\infty|) \leq r + |D_{X_2} u_\infty|$$

so we have

$$
\begin{aligned}
-\int_{\Omega_\ell} Du_\ell \cdot D((1-\rho)g_\infty) &\le Ce^{-\frac{1}{2}\alpha\ell}\int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} |Du_\ell|\big(\alpha c + r + |D_{X_2}u_\infty|\big) \\
&\le Ce^{-\frac{1}{2}\alpha\ell}\Big\{\frac{1}{2}\int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} |Du_\ell|^2 + \frac{1}{2}\int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} \big(\alpha c + r + |D_{X_2}u_\infty|\big)^2\Big\} \\
&\le Ce^{-\frac{1}{2}\alpha\ell}\Big\{\frac{1}{2}\int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} |Du_\ell|^2 + \int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} \big((\alpha c + r)^2 + |D_{X_2}u_\infty|^2\big)\Big\} \\
&\le Ce^{-\frac{1}{2}\alpha\ell}\Big\{(\alpha c + r)^2\big|\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}\big| + \int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} |Du_\ell|^2 + |D_{X_2}u_\infty|^2\Big\} \\
&\le Ce^{-\frac{1}{2}\alpha\ell}\Big\{C\ell^{q-1} + \|f_\ell\|^2_{H^{-1}(\Omega_\ell)} + \|f_\ell\|^2_{(H^1_0(\Omega_\ell;\Gamma_\ell))^*}\Big\} \\
&\le Ce^{-\frac{1}{2}\alpha\ell}\Big\{C\ell^{q-1} + 2\|f_\ell\|^2_{(H^1_0(\Omega_\ell;\Gamma_\ell))^*}\Big\} \\
&\le Ce^{-\frac{1}{2}\alpha\ell}\Big\{\ell^{q-1} + \ell^q\|f\|^2_{H^{-1}(\omega)}\Big\} \\
&\le C\ell^{q-1}\max(1,\ell)\big(1 + \|f\|^2_{H^{-1}(\omega)}\big)e^{-\frac{1}{2}\alpha\ell}. \qquad (111)
\end{aligned}
$$

By the definition of f_ℓ we have

$$
\langle f_\ell, (1-\rho)g_\infty\rangle \le \|f_\ell\|_{(H^1_0(\Omega_\ell;\Gamma_\ell))^*}\Big\{\int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} |D((1-\rho)g_\infty)|^2\Big\}^{\frac{1}{2}}
$$

and we estimate

$$
\begin{aligned}
\int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} |D((1-\rho)g_\infty)|^2 &\le \int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} (\alpha|g_\infty| + |Dg_\infty|)^2(1-\rho)^2 \\
&\le Ce^{-\alpha\ell}\int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} (\alpha|g_\infty| + |Dg_\infty|)^2 \\
&\le Ce^{-\alpha\ell}\int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} (\alpha c + r + |D_{X_2}u_\infty|)^2 \\
&\le Ce^{-\alpha\ell}\int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} 2\big((\alpha c + r)^2 + |D_{X_2}u_\infty|^2\big) \\
&\le Ce^{-\alpha\ell}\Big\{(\alpha c + r)^2\big|\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}\big| + \int_{\Omega_\ell\backslash\Omega_{\ell-\frac{c}{r}}} |D_{X_2}u_\infty|^2\Big\} \\
&\le Ce^{-\alpha\ell}\Big\{C\ell^{q-1} + \|f_\ell\|^2_{(H^1_0(\Omega_\ell;\Gamma_\ell))^*}\Big\} \\
&\le Ce^{-\alpha\ell}\Big\{\ell^{q-1} + \ell^q\|f\|^2_{H^{-1}(\omega)}\Big\} \\
&\le C\ell^{q-1}\max(1,\ell)\big(1 + \|f\|^2_{H^{-1}(\omega)}\big)e^{-\alpha\ell}.
\end{aligned}
$$

So we have

$$\langle f_\ell, (1-\rho)g_\infty\rangle \le C\ell^{\frac{1}{2}q}\|f\|_{H^{-1}(\omega)}\Big\{\ell^{q-1}\max(1,\ell)\big(1+\|f\|^2_{H^{-1}(\omega)}\big)e^{-\alpha\ell}\Big\}^{\frac{1}{2}}$$
$$= C\ell^{q-\frac{1}{2}}\max(1,\ell)^{\frac{1}{2}}\|f\|_{H^{-1}(\omega)}\big(1+\|f\|^2_{H^{-1}(\omega)}\big)^{\frac{1}{2}}e^{-\frac{1}{2}\alpha\ell}$$
$$\le C\ell^{q-1}\max(1,\ell)\big(1+\|f\|^2_{H^{-1}(\omega)}\big)e^{-\frac{1}{2}\alpha\ell}. \quad (112)$$

From (110) if

$$\alpha < \frac{2}{c_\omega}$$

then by the estimates (111) and (112) we have

$$\int_{\Omega_\ell} |Du_\ell - Du_\infty|^2(1-\rho) \le C\ell^{q-1}\max(1,\ell)\big(1+\|f\|^2_{H^{-1}(\omega)}\big)e^{-\frac{1}{2}\alpha\ell}. \quad (113)$$

We have in $\Omega_{\frac{1}{2}\ell}$ the estimate

$$\frac{1}{2} \le 1-\rho$$

and this together with the estimate (113) prove the theorem. □

5. $K = K_0 \times K'$ and $q = 1$.

In this section we consider the case when $q = 1$, $K = K_0 \times K'$ where K_0 is a closed interval containing 0 and K' is a closed and convex subset of $\mathbb{R}^d$ such that for some $r > 0$, $B_r^q(0) \times \{0\}^{d-q} \subset K'$ and in the case $d \ge 3$, K' is bounded in the X_2 directions, that is for some $R > 0$, $K' \subset \mathbb{R}^q \times B_R^{d-q}(0)$.

For ease of notation in the following let us denote $w_\ell = u_\ell - u_\infty$.

LEMMA 16. *For $0 \le \ell' < \ell'' \le \ell$ we have*

$$\int_{\Omega_{\ell'}} |Dw_\ell|^2 dx \le \frac{1}{\ell''-\ell'}\int_{\Omega_{\ell''}\setminus\Omega_{\ell'}} |Dw_\ell||w_\ell| + \frac{1}{r}\frac{E_\ell}{\ell''-\ell'}\sup_{\Delta_{\ell''}}|w_\ell| \quad (114)$$

where

$$E_\ell = \left(\langle f_\ell, u_\ell\rangle - \int_{\Omega_\ell}|Du_\ell|^2\right) + \left(\langle f_\ell, u_\infty\rangle - \int_{\Omega_\ell}|Du_\infty|^2\right). \quad (115)$$

PROOF. For $0 \le \ell' < \ell'' \le \ell$ let us consider the cut-off function $\rho : (-\ell, \ell) \to \mathbb{R}$ such that $\rho = 1$ in $(-\ell', \ell')$, $\rho = 0$ in $(-\ell, -\ell'')\cup(\ell'', \ell)$ and on $(-\ell'', -\ell')$ and (ℓ', ℓ''), ρ interpolates between interval end values linearly. Then we have $0 \le \rho \le 1$ and

$$|D_{X_1}\rho| = |\rho'| \le \frac{1}{\ell''-\ell'}. \quad (116)$$

Now let us consider the functions

$$v_1 = \delta\,(u_\infty + w_\ell\rho)$$
$$v_2 = \delta\,(u_\ell - w_\ell\rho)$$

where $0 \le \delta \le 1$ is a constant to be chosen later appropriately. It is clear that both these functions are 0 on Γ_ℓ so $v_1, v_2 \in H_0^1(\Omega_\ell; \Gamma_\ell)$. We have $v_2 = \delta u_\ell = 0$ on Δ_ℓ so $v_2 \in H_0^1(\Omega_\ell)$. We also have $v_1 = \delta u_\infty$ on Δ_ℓ, hence in the case of periodic force we have $v_1 \in H_{per}(\Omega_\ell)$.

Now we choose the constant δ such that the gradients of v_1 and v_2 satisfy the constraint, to do this we compute the gradients

$$Dv_1 = \delta\big((1-\rho)\,Du_\infty + \rho Du_\ell\big) + \delta w_\ell D_{X_1}\rho$$
$$= \delta\big((1-\rho)\,Du_\infty + \rho Du_\ell\big) + (1-\delta)\big(\frac{\delta}{1-\delta} w_\ell D_{X_1}\rho\big)$$

$$Dv_2 = \delta\big((1-\rho)\,Du_\ell + \rho Du_\infty\big) - \delta w_\ell D_{X_1}\rho$$
$$= \delta\big((1-\rho)\,Du_\ell + \rho Du_\infty\big) + (1-\delta)\big(-\frac{\delta}{1-\delta} w_\ell D_{X_1}\rho\big)$$

so because $B^q_r(0) \times \{0\}^{d-q} \subset K'$, if

$$\frac{\delta}{1-\delta} w_\ell D_{X_1}\rho \in B^q_r(0) \tag{117}$$

then $Dv_1, Dv_2 \in K'$.

By corollary 2 we estimate

$$\big|\frac{\delta}{1-\delta} w_\ell D_{X_1}\rho\big| \leq \frac{\delta}{1-\delta}\frac{1}{\ell''-\ell'}|w_\ell|\chi_{\Omega_{\ell''}\setminus\Omega_{\ell'}} \leq \frac{\delta}{1-\delta}\frac{1}{\ell''-\ell'}\sup_{\Delta_{\ell''}}|w_\ell|$$

so taking

$$\delta = \frac{1}{1+\frac{1}{r}\frac{1}{\ell''-\ell'}\sup_{\Delta_{\ell''}}|w_\ell|} \tag{118}$$

we will have the statement (117). So we have $v_2 \in \mathcal{K}_0(\Omega_\ell)$, $v_1 \in \mathcal{K}_0(\Omega_\ell;\Gamma_\ell)$ and in the case of periodic force we have also $v_1 \in \mathcal{K}_{per}(\Omega_\ell)$.

Testing the inequality (93) or (94) by v_1 and the inequality (86) by v_2 we obtain the inequalities

$$-\delta\int_{\Omega_\ell} Du_\infty \cdot D\,(w_\ell\rho) \leq (1-\delta)\left(\langle f_\ell, u_\infty\rangle - \int_{\Omega_\ell}|Du_\infty|^2\right) - \delta\,\langle f_\ell, w_\ell\rho\rangle$$
$$\delta\int_{\Omega_\ell} Du_\ell \cdot D\,(w_\ell\rho) \leq (1-\delta)\left(\langle f_\ell, u_\ell\rangle - \int_{\Omega_\ell}|Du_\ell|^2\right) + \delta\,\langle f_\ell, w_\ell\rho\rangle\,.$$

Summing these inequalities together we have

$$\delta\int_{\Omega_\ell} Dw_\ell \cdot D\,(w_\ell\rho) \leq (1-\delta)\,E_\ell.$$

Dividing by δ and computing $D\,(w_\ell\rho) = \rho Dw_\ell + w_\ell D\rho$ we have

$$\int_{\Omega_\ell}|Dw_\ell|^2\rho \leq -\int_{\Omega_\ell} w_\ell Dw_\ell\cdot D\rho + \frac{1-\delta}{\delta}E_\ell$$

from (118) computing $\frac{1-\delta}{\delta}$ and using the estimate (116) we obtain the inequality (114). □

Definition 4. *Let us define $\sigma \geq 2$ as follows, if $d = 2$ then $\sigma = 2$ and if $d \geq 3$ then for some $\epsilon > 0$, $\sigma = d-1+\epsilon$.*

Lemma 17. *For $0 \leq \tilde{\ell} \leq \ell$ we have*

$$\sup_{\Delta_{\tilde{\ell}}}|w_\ell| \leq C\Big(\int_{\Delta_{\tilde{\ell}}}|Dw_\ell|^2 dX_2\Big)^{\frac{1}{\sigma}} \tag{119}$$

here in the case $d \geq 3$ we have $C \to \infty$ as $\epsilon \to 0$.

PROOF. By the Sobolev inequality for the domain ω taking into consideration that the dimension of ω is $d-1$ and that $\Delta_{\tilde{\ell}} = \{-\tilde{\ell}, \tilde{\ell}\} \times \omega$, for $\epsilon > 0$ we have

$$\sup_{\Delta_{\tilde{\ell}}} |w_\ell| \leq C\Big(\int_{\Delta_{\tilde{\ell}}} |D_{X_2} w_\ell|^{d-1+\epsilon} dX_2\Big)^{\frac{1}{d-1+\epsilon}}. \tag{120}$$

In the case $d = 2$ by taking $\epsilon = 1$ we obtain the result. In the case $d \geq 3$ by the assumption, K' is bounded in the X_2 directions, hence

$$|D_{X_2} w_\ell| \leq 2R$$

and we may estimate

$$|D_{X_2} w_\ell|^{d-1+\epsilon} = |D_{X_2} w_\ell|^{d-3+\epsilon} |D_{X_2} w_\ell|^2 \leq (2R)^{d-3+\epsilon} |D_{X_2} w_\ell|^2$$

and this proves the lemma. □

COROLLARY 4. *For $0 \leq \ell' < \ell'' \leq \ell$ we have*

$$\sup_{\Delta_{\ell'}} |w_\ell| \leq C\Big(\frac{1}{\ell'' - \ell'} \int_{\Omega_{\ell''} \setminus \Omega_{\ell'}} |Dw_\ell|^2\Big)^{\frac{1}{\sigma}}. \tag{121}$$

PROOF. Raising the inequality (119) to the power σ we have

$$\big\{\sup_{\Delta_{\tilde{\ell}}} |w_\ell|\big\}^\sigma \leq C^\sigma \int_{\Delta_{\tilde{\ell}}} |Dw_\ell|^2 dX_2$$

and then integrating from ℓ' to ℓ'' in the variable $\tilde{\ell}$ we have

$$\int_{\ell'}^{\ell''} \big\{\sup_{\Delta_{\tilde{\ell}}} |w_\ell|\big\}^\sigma d\tilde{\ell} \leq C^\sigma \int_{\ell'}^{\ell''} \int_{\Delta_{\ell'}} |Dw_\ell|^2 dX_2 d\tilde{\ell} = C^\sigma \int_{\Omega_{\ell''} \setminus \Omega_{\ell'}} |Dw_\ell|^2. \tag{122}$$

Now by corollary 2, for $\ell' \leq \tilde{\ell} \leq \ell''$ we have

$$\sup_{\Delta_{\ell'}} |w_\ell| \leq \sup_{\Omega_{\tilde{\ell}}} |w_\ell| \leq \sup_{\Delta_{\tilde{\ell}}} |w_\ell|.$$

Using this estimate we have

$$\int_{\ell'}^{\ell''} \big\{\sup_{\Delta_{\tilde{\ell}}} |w_\ell|\big\}^\sigma d\tilde{\ell} \geq \big\{\sup_{\Delta_{\ell'}} |w_\ell|\big\}^\sigma \int_{\ell'}^{\ell''} d\tilde{\ell} = (\ell'' - \ell')\big\{\sup_{\Delta_{\ell'}} |w_\ell|\big\}^\sigma.$$

Using this estimate in the left hand side of (122) we obtain

$$(\ell'' - \ell')\big\{\sup_{\Delta_{\ell'}} |w_\ell|\big\}^\sigma \leq C^\sigma \int_{\Omega_{\ell''} \setminus \Omega_{\ell'}} |Dw_\ell|^2$$

dividing this inequality by $\ell'' - \ell'$ and raising the inequality to the power $\frac{1}{\sigma}$ we obtain the inequality (121). □

LEMMA 18. *For each $0 < \gamma < 1$ there exists a constant $C > 0$ such that*

$$|||Dw_\ell|||^2_{L^2(\Omega_{\gamma\ell})} \leq C(1 + \frac{E_\ell}{\ell})^{\frac{\sigma}{\sigma-1}} \frac{1}{\ell^{\frac{1}{\sigma-1}}}. \tag{123}$$

PROOF. Let us take $0 < \ell' < \ell'' < \ell$ and $\bar{\ell} = \frac{1}{2}(\ell' + \ell'')$, then by lemma 16 we have

$$\int_{\Omega_{\ell'}} |Dw_\ell|^2 \leq \frac{1}{\bar{\ell} - \ell'} \int_{\Omega_{\bar{\ell}} \setminus \Omega_{\ell'}} |Dw_\ell||w_\ell| + \frac{1}{r} \frac{E_\ell}{\bar{\ell} - \ell'} \sup_{\Delta_{\bar{\ell}}} |w_\ell|. \tag{124}$$

Let us estimate the terms on the right hand side of this inequality.

For the first term in the case $d = 2$ we may estimate

$$\int_{\Omega_{\bar\ell}\setminus\Omega_{\ell'}} |Dw_\ell||w_\ell| \le \Big(\int_{\Omega_{\bar\ell}\setminus\Omega_{\ell'}} |Dw_\ell|^2\Big)^{\frac12}\Big(\int_{\Omega_{\bar\ell}\setminus\Omega_{\ell'}} |w_\ell|^2\Big)^{\frac12}$$
$$\le 2c|\Omega_{\bar\ell}\setminus\Omega_{\ell'}|^{\frac12}\Big(\int_{\Omega_{\bar\ell}\setminus\Omega_{\ell'}} |Dw_\ell|^2\Big)^{\frac12}$$
$$= C(\bar\ell-\ell')\Big(\frac{1}{\bar\ell-\ell'}\int_{\Omega_{\bar\ell}\setminus\Omega_{\ell'}} |Dw_\ell|^2\Big)^{\frac1\sigma}.$$

For the case $d \ge 3$ we may estimate

$$\int_{\Omega_{\bar\ell}\setminus\Omega_{\ell'}} |Dw_\ell||w_\ell| \le \Big(\int_{\Omega_{\bar\ell}\setminus\Omega_{\ell'}} |w_\ell|^2\Big)^{\frac12}\Big(\int_{\Omega_{\bar\ell}\setminus\Omega_{\ell'}} |Dw_\ell|^2\Big)^{\frac12-\frac1\sigma}\Big(\int_{\Omega_{\bar\ell}\setminus\Omega_{\ell'}} |Dw_\ell|^2\Big)^{\frac1\sigma}$$
$$\le 2c(2R)^{2(\frac12-\frac1\sigma)}|\Omega_{\bar\ell}\setminus\Omega_{\ell'}|^{1-\frac1\sigma}\Big(\int_{\Omega_{\bar\ell}\setminus\Omega_{\ell'}} |Dw_\ell|^2\Big)^{\frac1\sigma}$$
$$= C(\bar\ell-\ell')\Big(\frac{1}{\bar\ell-\ell'}\int_{\Omega_{\bar\ell}\setminus\Omega_{\ell'}} |Dw_\ell|^2\Big)^{\frac1\sigma}.$$

For the second term on the right side of (124) by corollary 4 we have the estimate

$$\sup_{\Delta_{\bar\ell}} |w_\ell| \le C\Big(\frac{1}{\ell''-\bar\ell}\int_{\Omega_{\ell''}\setminus\Omega_{\bar\ell}} |Dw_\ell|^2\Big)^{\frac1\sigma}.$$

Now by the inequality (124) and these estimates we have

$$\int_{\Omega_{\ell'}} |Dw_\ell|^2 \le C\Big(\frac{1}{\bar\ell-\ell'}\int_{\Omega_{\bar\ell}\setminus\Omega_{\ell'}} |Dw_\ell|^2\Big)^{\frac1\sigma} + \frac{CE_\ell}{\bar\ell-\ell'}\Big(\frac{1}{\ell''-\bar\ell}\int_{\Omega_{\ell''}\setminus\Omega_{\bar\ell}} |Dw_\ell|^2\Big)^{\frac1\sigma}.$$

By the equality $\ell''-\ell' = 2(\bar\ell-\ell') = 2(\ell''-\bar\ell)$ we estimate

$$\int_{\Omega_{\ell'}} |Dw_\ell|^2$$
$$\le C\Big(\frac{2}{\ell''-\ell'}\int_{\Omega_{\ell''}\setminus\Omega_{\ell'}} |Dw_\ell|^2\Big)^{\frac1\sigma} + \frac{2CE_\ell}{\ell''-\ell'}\Big(\frac{2}{\ell''-\ell'}\int_{\Omega_{\ell''}\setminus\Omega_{\ell'}} |Dw_\ell|^2\Big)^{\frac1\sigma}$$
$$\le C\big(1+\frac{E_\ell}{\ell''-\ell'}\big)\Big(\frac{1}{\ell''-\ell'}\int_{\Omega_{\ell''}\setminus\Omega_{\ell'}} |Dw_\ell|^2\Big)^{\frac1\sigma}$$

so we have

$$\int_{\Omega_{\ell'}} |Dw_\ell|^2 \le C\Big(1+\frac{E_\ell}{\ell''-\ell'}\Big)\Big(\frac{1}{\ell''-\ell'}\int_{\Omega_{\ell''}\setminus\Omega_{\ell'}} |Dw_\ell|^2\Big)^{\frac1\sigma}. \tag{125}$$

Now our goal is to iterate this inequality. To do this we consider $0 < \gamma < 1$, $\ell_1 = \gamma\ell$ and for $i = 1, 2, \dots, n-1$ define recursively

$$\ell_{i+1} = \ell_i + \frac{1}{s_n 2^i}(\ell-\ell_1)$$

where

$$s_n = \sum_{i=1}^{n-1}\frac{1}{2^i}.$$

By this choice we have $\ell_n = \ell$.

Denoting

$$I_i = \int_{\Omega_{\ell_i}} |Dw_\ell|^2$$

by the inequality (125) we have

$$I_i \le C(1 + \frac{E_\ell}{\triangle\ell_i})(\frac{1}{\triangle\ell_i})^{\frac{1}{\sigma}}(I_{i+1} - I_i)^{\frac{1}{\sigma}} \le C(1 + \frac{E_\ell}{\triangle\ell_i})(\frac{1}{\triangle\ell_i})^{\frac{1}{\sigma}} I_{i+1}^{\frac{1}{\sigma}}$$

here $\triangle\ell_i = \ell_{i+1} - \ell_i$.

We estimate

$$\frac{1}{\triangle\ell_i} = \frac{s_n 2^i}{1-\gamma}\frac{1}{\ell} \le \frac{2^i}{1-\gamma}\frac{1}{\ell}$$

and

$$1 + \frac{E_\ell}{\triangle\ell_i} \le C2^i\big(1 + \frac{E_\ell}{\ell}\big).$$

So we have the inequality

$$I_i \le C2^{(1+\frac{1}{\sigma})i}\Big((1 + \frac{E_\ell}{\ell})^\sigma \frac{1}{\ell}\Big)^{\frac{1}{\sigma}} I_{i+1}^{\frac{1}{\sigma}}.$$

Iterating this inequality for $i = 1, 2, \ldots, n-1$ we obtain

$$I_1 \le C^{\{\sum_{i=0}^{n-2}\frac{1}{\sigma^i}\}} 2^{\{(\sigma+1)\sum_{i=1}^{n-1}\frac{i}{\sigma^i}\}}\Big((1 + \frac{E_\ell}{\ell})^\sigma\frac{1}{\ell}\Big)^{\sum_{i=1}^{n-1}\frac{1}{\sigma^i}} I_n^{\frac{1}{\sigma^{n-1}}}.$$

We have the convergence of the series

$$\sum_{i=0}^{\infty}\frac{1}{\sigma^i}, \quad \sum_{i=1}^{\infty}\frac{i}{\sigma^i} < \infty$$

so we have

$$I_1 \le C\Big((1 + \frac{E_\ell}{\ell})^\sigma\frac{1}{\ell}\Big)^{\sum_{i=1}^{n-1}\frac{1}{\sigma^i}} I_n^{\frac{1}{\sigma^{n-1}}}.$$

By considering that

$$\sum_{i=1}^{n-1}\frac{1}{\sigma^i} = \frac{1 - \frac{1}{\sigma^{n-1}}}{\sigma - 1}$$

and $\ell_n = \ell$ we obtain

$$I_1 \le C\Big((1 + \frac{E_\ell}{\ell})^\sigma\frac{1}{\ell}\Big)^{\frac{1-\frac{1}{\sigma^{n-1}}}{\sigma-1}} \Big(\int_{\Omega_\ell}|Dw_\ell|^2\Big)^{\frac{1}{\sigma^{n-1}}}$$

passing to the limit as $n \to \infty$ we obtain the result. □

Lemma 19. *For each $0 < \gamma < 1$ there exists a constant $C > 0$ such that*

$$\|w_\ell\|_{L^\infty(\Omega_{\gamma\ell})} \le C\big(1 + \frac{E_\ell}{\ell}\big)^{\frac{1}{\sigma-1}}\frac{1}{\ell^{\frac{1}{\sigma-1}}}. \tag{126}$$

Proof. Let us take $\gamma' > 0$ such that $\gamma < \gamma' < 1$ then by lemma 18 we have

$$\int_{\Omega_{\gamma'\ell}} |Dw_\ell|^2 \le C\big(1 + \frac{E_\ell}{\ell}\big)^{\frac{\sigma}{\sigma-1}}\frac{1}{\ell^{\frac{1}{\sigma-1}}}$$

now using the corollary 2 and lemma 4 we may estimate

$$\sup_{\Omega_{\gamma\ell}} |w_\ell| \le \sup_{\Delta_{\gamma\ell}} |w_\ell| \le C\Big(\frac{1}{(\gamma'\ell-\gamma\ell)}\int_{\Omega_{\gamma'\ell}} |Dw_\ell|^2\Big)^{\frac{1}{\sigma}}$$
$$\le C\Big(\frac{1}{\gamma'-\gamma}\Big)^{\frac{1}{\sigma}}\Big(\big(1+\frac{E_\ell}{\ell}\big)^{\frac{\sigma}{\sigma-1}}\frac{1}{\ell}\frac{1}{\ell^{\frac{1}{\sigma-1}}}\Big)^{\frac{1}{\sigma}}$$
$$= C\big(1+\frac{E_\ell}{\ell}\big)^{\frac{1}{\sigma-1}}\frac{1}{\ell^{\frac{1}{\sigma-1}}}$$

and this proves the lemma. □

Using the lemmas 18 and 10 we obtain the following asymptotic behavior result for the cases of periodic forces and forces defined in the cross section.

Theorem 7. *For both cases of periodic force and forces defined in the cross section with the assumptions in this section for each* $0<\gamma<1$ *there exists a constant* C *such that*

$$\||Du_\ell - Du_\infty|\|^2_{L^2(\Omega_{\gamma\ell})}, \|u_\ell-u_\infty\|_{L^\infty(\Omega_{\gamma\ell})} \le \frac{C}{\ell^{\frac{1}{\sigma-1}}}.$$

Proof. Let us show the growth estimate

$$E_\ell \le C\ell. \tag{127}$$

By testing the inequality (86) by $v=0$ we have

$$\int_{\Omega_\ell} |Du_\ell|^2 \le \|f_\ell\|^2_{H^{-1}(\Omega_\ell)}.$$

Using this inequality, in the case of periodic forces we may estimate

$$E_\ell = \Big(\langle f_\ell,u_\ell\rangle - \int_{\Omega_\ell}|Du_\ell|^2\Big) + \Big(\langle f_\ell,u_\infty\rangle - \int_{\Omega_\ell}|Du_\infty|^2\Big)$$
$$\le |\langle f_\ell,u_\ell\rangle| + \int_{\Omega_\ell}|Du_\ell|^2 + |\langle f_\ell,u_\infty\rangle| + \int_{\Omega_\ell}|Du_\infty|^2$$
$$\le \|f_\ell\|_{H^{-1}(\Omega_\ell)}\|u_\ell\|_{H^1_0(\Omega_\ell)} + \int_{\Omega_\ell}|Du_\ell|^2$$
$$+ \|f_\ell\|_{(H^1_{per}(\Omega_\ell))^*}\|u_\infty\|_{H^1_{per}(\Omega_\ell)} + \int_{\Omega_\ell}|Du_\infty|^2$$
$$\le 2\|f_\ell\|^2_{H^{-1}(\Omega_\ell)} + \|f_\ell\|_{(H^1_{per}(\Omega_\ell))^*}\Big(\sqrt{2\ell}\||Du_\infty|_2\|_{L^2(Q)}\Big) + 2\ell\||Du_\infty|_2\|^2_{L^2(Q)}$$
$$= C\|f_\ell\|^2_{H^{-1}(\Omega_\ell)} + C\|f_\ell\|_{(H^1_{per}(\Omega_\ell))^*}\sqrt{\ell} + C\ell$$
$$\le C\|f_\ell\|^2_{(H^1_{per}(\Omega_\ell))^*} + C\|f_\ell\|_{(H^1_{per}(\Omega_\ell))^*}\sqrt{\ell} + C\ell$$
$$\le C\|f_\ell\|^2_{(H^1_{per}(Q))^*}\ell + C\|f_\ell\|_{(H^1_{per}(Q))^*}\ell + C\ell = C\ell$$

which shows the inequality (127) in the case of periodic forces.

In the case of forces defined in the cross section the inequality (127) is proved similarly.

Now using the estimate (127) in the inequalities (123) and (126) we prove the theorem. □

Bibliography

[BG07] Brighi, B.; Guesmia, S., Asymptotic behavior of solution of hyperbolic problems on a cylindrical domain. Discrete Contin. Dyn. Syst. 2007, Dynamical Systems and Differential Equations. Proceedings of the 6th AIMS International Conference, suppl., 160–169.

[BG08] Brighi, B.; Guesmia, S., On elliptic boundary value problems of order $2m$ in cylindrical domain of large size. Adv. Math. Sci. Appl. 18 (2008), no. 1, 237–250.

[BS94] Brenner, Susanne C.; Scott, L. Ridgway, *The Mathematical Theory of Finite Element Methods*, Springer, 1994.

[C02] Chipot, M., *ℓ goes to plus infinity*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks] Birkhäuser Verlag, Basel, 2002. viii+181 pp.

[C78] Ciarlet, Philippe G., *The Finite Element Method for Elliptic Problems*, North Holland, 1978.

[CER07] Chipot, M.; Elfanni, A.; Rougirel, A., Eigenvalues, eigenfunctions in domains becoming unbounded. Hyperbolic problems and regularity questions, 69–78, Trends Math., Birkhäuser, Basel, 2007.

[CG09] Chipot, M.; Guesmia, S., On the asymptotic behavior of elliptic, anisotropic singular perturbations problems. Commun. Pure Appl. Anal. 8 (2009), no. 1, 179–193.

[CM08] Chipot, M.; Mardare, S., Asymptotic behaviour of the Stokes problem in cylinders becoming unbounded in one direction. J. Math. Pures Appl. (9) 90 (2008), no. 2, 133–159.

[CR01a] Chipot, M.; Rougirel, A., Remarks on the asymptotic behaviour of the solution to parabolic problems in domains becoming unbounded. Proceedings of the Third World Congress of Nonlinear Analysts, Part 1 (Catania, 2000). Nonlinear Anal. 47 (2001), no. 1, 3–11.

[CR01b] Chipot, M.; Rougirel, A., On the asymptotic behaviour of the solution of parabolic problems in cylindrical domains of large size in some directions. Discrete Contin. Dyn. Syst. Ser. B 1 (2001), no. 3, 319–338.

[CR02] Chipot, M.; Rougirel, A., On the asymptotic behaviour of the solution of elliptic problems in cylindrical domains becoming unbounded. Commun. Contemp. Math. 4 (2002), no. 1, 15–44.

[CR08] Chipot, M.; Rougirel, A., On the asymptotic behaviour of the eigenmodes for elliptic problems in domains becoming unbounded. Trans. Amer. Math. Soc. 360 (2008), no. 7, 3579–3602.

[CX04a] Chipot, M.; Xie, Y., On the asymptotic behaviour of elliptic problems with periodic data. C. R. Math. Acad. Sci. Paris 339 (2004), no. 7, 477–482.

[CX04b] Chipot, M.; Xie, Y., On the asymptotic behaviour of the p-Laplace equation in cylinders becoming unbounded. Nonlinear partial differential equations and their applications, 16–27, GAKUTO Internat. Ser. Math. Sci. Appl., 20, Gakko-tosho, Tokyo, 2004.

[CX05] Chipot, M.; Xie, Y., Asymptotic behavior of nonlinear parabolic problems with periodic data. Elliptic and parabolic problems, 147–156, Progr. Nonlinear Differential Equations Appl., 63, Birkhäuser, Basel, 2005.

[CX06] Chipot, M.; Xie, Y., Elliptic problems with periodic data: an asymptotic analysis. J. Math. Pures Appl. (9) 85 (2006), no. 3, 345–370.

[CX08] Chipot, M.; Xie, Y., Some issues on the p-Laplace equation in cylindrical domains. Tr. Mat. Inst. Steklova 261 (2008), Differ. Uravn. i Din. Sist., 293–300. ISBN: 978-5-7846-0106-3

[CY08] Chipot, M.; Yeressian, K., Exponential rates of convergence by an iteration technique. C. R. Math. Acad. Sci. Paris 346 (2008), no. 1-2, 21–26.

[G06] Guesmia, S., Etude du comportement asymptotique de certaines équations aux dérivées partielles dans des domaines cylindriques, Thèse université Haute Alsace, December 2006.

[G08a] Guesmia, S., Asymptotic behavior of elliptic boundary-value problems with some small coefficients. Electron. J. Differential Equations 2008, No. 59 13 pp.

[G08b] Guesmia, S., Some results on the asymptotic behavior for hyperbolic problems in cylindrical domains becoming unbounded. J. Math. Anal. Appl. 341 (2008), no. 2, 1190–1212.

[G09] Guesmia, S., Some convergence results for quasilinear parabolic boundary value problems in cylindrical domains of large size. Nonlinear Anal. 70 (2009), no. 9, 3320–3331.

[GT83] Gilbarg, D.; Trudinger, N.S., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1983.

[H89] Horgan, Cornelius O., Recent developments concerning Saint-Venant's principle: an update. AMR 42 (1989), no. 11, part 1, 295–303.

[H96] Horgan, Cornelius O., Recent developments concerning Saint-Venants principle: a second update, Applied Mechanics Reviews, 48, 1996.

[HK83] Horgan, Cornelius O.; Knowles, James K., Recent developments concerning Saint-Venant's principle. Adv. in Appl. Mech. 23 (1983), 179–269.

[HPW84] Horgan C.O., Payne L.E. and Wheeler L.T., Spatial decay estimates in transient heat conduction. Quart. Appl. Math. 42, 119-127 (1984).

[O96] Oleinik, O., *Some asymptotic problems in the theory of partial differential equations.* Lezioni Lincee. [Lincei Lectures] Cambridge University Press, Cambridge, 1996. x+203 pp.

[OY77] Oleinik, O. A.; Yosifian, G. A., Boundary value problems for second order elliptic equations in unbounded domains and Saint-Venant's principle. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 4 (1977), no. 2, 269–290.

[DL88] Dautray, R.; Lions, J. L., *Mathematical Analysis and Numerical Methods for Science and Technology*, Springer-Verlag, 1988.

[T65] Toupin, R. A., Saint-Venant's principle. Arch. Rational Mech. Anal. 18 1965 83–96.

[W91] Wahlbin L.B., Local behavior in finite element methods; in *Handbook of Numerical Analysis* (Ciarlet P.G. and Lions J.L., eds.), vol. II (part 1), North Holland, 1991, pp. 353-522.

[X06] Xie, Y., On Asymptotic Problems in Cylinders and Other Mathematical Issues, Thesis University of Zürich, May 2006.